Module Title: Mathematical Analysis, Statistics and Probability

Competence: APPLY Mathematical Analysis, Statistics and Probability

Unit 1: Apply fundamentals of integrals

Learning Outcomes:
1. Calculate the primitive functions
2. Calculate definite integrals
3. Apply definite integrals

Learning Outcomes: 1.1: Indefinite integrals
1.1.1: Definition

An **integral** or an **anti-derivative** of function $f(x)$ is the function $F(x)$ whose derivative is equal to $f(x)$. Thus, we say $F(x)$ is an anti-derivative of $f(x)$ and write

$F'(x) = f(x)$.

The process of solving for anti-derivatives is called **anti-differentiation** (or **integration**) which is the opposite operation of differentiation (process of finding derivatives).

Example 1.

The function $F(x) = \ln x$ is the primitive of $f(x) = \frac{1}{x}$ since $(\ln x)' = \frac{1}{x}$.

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Also, $F(x)=\ln x+5$ is the primitive of $f(x)=\frac{1}{x}$ since $(\ln x+5)'=\frac{1}{x}+0=\frac{1}{x}$.

Example 2.

The primitive function of $f(x)=\cos x$ is $F(x)=\sin x+c$ since $F'(x)=(\sin x+c)'=\cos x$.

Thus, $\int f(x)dx = F(x)+C$ if and only if $F'(x)=f(x)$.

Notation

The anti-derivative of $f(x)$ is called the **indefinite integral** of $f(x)$ is denoted by

$\int f(x)dx$ so that $\int f(x)dx = F(x)+c$,

Where The symbol $\int$ is the sign of integration *and C is a constant.*

Example

a. Find $\int xdx$

Solution

$$\int xdx = \frac{x^2}{2}+c \text{. Indeed } \left(\frac{x^2}{2}+c\right)' = x$$

b. Find $\int 5dx$

Solution

$$\int 5dx = 5x + C$$

1.1.2 Properties of integrals

1) $\int kf(x)dx = k\int f(x)dx$

2) $\int [f(x)+\mathrm{g(x)}]dx = \int f(x)dx + \int g(x)dx$

3) $\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$

List of basic integration formulae

1) If k is constant, $\int kdx = kx + C$

2) $\int u^n du = \frac{1}{n+1}u^{n+1} + C$,where $n \neq -1$, n is a constant

3) If $b \neq -1$, and u a differentiable function, $\int u^b du = \frac{u^{b+1}}{b+1} + C$

4) By definition, $\int x^{-1}dx = \int \frac{1}{x}dx = \ln|x| + c$ for x nonzero

5) $\int e^x dx = e^x + c$, the integral of exponential function of base e , if $x = ax$

$$\int e^{ax}dx = \frac{e^{ax}}{a} + c$$

6) $If\ a > 0\ and\ a \neq 1, \int a^x dx, = \frac{a^x}{\ln a} + c$

If $x = kx \int a^{kx}\,dx = \frac{1}{k}\frac{a^{kx}}{lna}$

7) $\int \frac{1}{x-1}dx = \ln|x-1| + C\ If\ a \neq 0, \int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$

8) $if\ a \neq 0\ and\ n \neq -1, \int (ax+b)\ dx = \frac{(ax \quad b)}{a(n \quad\ )} + c$

Integration involving trigonometric functions

9) $\int \cos x dx = \sin x + C$ 10) $\int \sin x dx = -\cos x + C$ 11) $\int \frac{dx}{1+x^2} = Arc\tan x + C$

12) $\int \frac{dx}{\sqrt{1-x^2}} = Arc\sin x + C$ 13) $\int \frac{dx}{\cos^2 x} = \tan x + C$ 14) $\int \frac{dx}{\sin^2 x} = -\cot x + C$

15) $If\ a \neq 0, \int \sin(ax+b)dx = -\frac{1}{a}\cos(\mathrm{ax+b}) + C$ 16) $\int \sec^2 x dx = \tan x + C$

17) $\int \mathrm{cosec}^2 \mathrm{x} dx = -\cot x + C$ 18) $\int \sec^2(ax+b)dx = \frac{1}{a}\tan(\mathrm{ax+b}) + C$

19) $\int \mathrm{cosec}^2(ax+b)dx = -\frac{1}{a}\cot(\mathrm{ax+b}) + C$ 20) $\int \cos(\mathrm{ax+b})dx = \frac{1}{a}\sin(ax+b) + C$

Example 1.

Find $\int x dx$

Solution

$$\int x dx = \frac{x^2}{2} + c$$

Example 2.

Find $\int \frac{dx}{x^2 - 4}$

Solution

$$\int \frac{dx}{x^2-4}$$

$$\int \frac{dx}{x^2-4} = \int \frac{dx}{x^2-2^2}$$

But, $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

Then,

$$\int \frac{dx}{x^2-2^2} = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + c = \ln\sqrt[4]{\frac{x-2}{x+2}} + c$$

Example 3.

Find $\int \cos 3x dx$

Solution

$$\int \cos 3x dx = \frac{1}{3}\sin 3x + c$$

Example 4.

$$\int 8e^{-2x} dx$$

Solution

$$\int 8e^{-2x} dx = -4e^{-2x} + C$$

Example5.

Function $y = f(x)$ is such that $\frac{dy}{dx} = \frac{x^3-5}{x^2}$. Find the expression of $y = f(x)$ if $f(1) = \frac{1}{2}$

Solution

3) $\int \frac{x^3-5}{x^2}dx = \int (x-5x^{-2})dx = \frac{1}{2}x^2 + \frac{5}{x} + C$

$F(x) = \frac{1}{2}x^2 + \frac{5}{x} + C \,;\, F(1) = \frac{1}{2} + 5 + C = \frac{1}{2} \,;\, C = -5$

Therefore, $F(x) = \frac{1}{2}x^2 + \frac{5}{x} - 5$

Exercises

Find each of the following integrals:

1. $\int (4x-5)\,dx$
2. $\int \left(6x^2+4x+3\right)dx$
3. $\int \left(x^3+x^2+x\right)dx$
4. $\int (3x-4)^2\,dx$
5. $\int 5\,dx$

6. $\int e^{3x+1}dx$ **7.** $\int 3^x\,dx$

1.1.3 Techniques of integration

1.1.3.1 Integration by substitution or changing variables

Integration by substitution is based on rule for differentiating composite functions. The formula for integration by substitution is

$$\int f(x)\,dx = \int f\left(x(t)\right)x'(t)\,dt$$

Example 1.

Find $\int (2x+1)^4\,dx$

Solution

Let $t = 2x+1$, so $dt = 2dx \Rightarrow dx = \frac{1}{2}dt$

We have,

$$\int (2x+1)^4\,dx = \frac{1}{2}\int t^4 dt = \frac{1}{2}\left(\frac{t^5}{5}\right) + c = \frac{1}{10}(2x+1)^5 + c$$

Example 2.

Find $\int \frac{x^3}{(x-1)^2}dx$

Solution

Let $t = x-1 \Rightarrow x = t+1$, so $dt = dx \Rightarrow dx = dt$

We have,

$$\int \frac{x^3}{(x-1)^2}dx = \int \frac{(t+1)^3}{t^2}dt = \int \frac{t^3+3t^2+3t+1}{t^2}dt$$

$$= \int\left(t+3+\frac{3}{t}+\frac{1}{t^2}\right)dt = \int t\,dt + \int 3\,dt + \int \frac{3}{t}dt + \int \frac{1}{t^2}dt$$

$$= \frac{t^2}{2} + 3t + 3\ln|t| - t^{-1} + c = \frac{(x-1)^2}{2} + 3(x-1) + 3\ln|x-1| - \frac{1}{x-1} + c$$

Example 3.

Evaluate

$$\int \frac{\ln x}{x}dx$$

Solution

To integrate $\int \frac{\ln x}{x} dx$, we let $u = \ln x$.

.Then, $du = \frac{1}{x} dx$. Thus $\int \frac{\ln x}{x} dx = \int u du = \frac{1}{2} u^2 + C$

Substituting u by $\ln x$ yields $\int \frac{\ln x}{x} dx = \frac{1}{2} \ln^2 x + C$

1.1.3.2 Integration by parts

The integral of a product of two functions does not equal the product of the integrals of the two functions.
To apply the integration by parts to a given integral, we must first factor its integrand into two parts.

An effective strategy is to choose for $\frac{dv}{dx}$ the most complicated factor that can readily be integrated. Then, we differentiate the other part, u, to find $\frac{du}{dx}$.

To develop a rule, we start with the product rule for differentiation:

$$\int u dv = uv - \int v du$$

The following table is used

u	v'
Logarithmic function	Polynomial function
Polynomial function	Exponential function
Polynomial function	Trigonometric function
Exponential function	Trigonometric function
Trigonometric function	Exponential function

Example 1.

Find $\int \ln x dx$

Solution

Here we can write $\int \ln x dx = \int 1 \cdot \ln x dx$

Let $u = \ln x \Rightarrow du = \frac{1}{x} dx$ and $dv = dx \Rightarrow v = x$

Then,

$$\int \ln x\, dx = x \ln|x| - \int x \frac{dx}{x} = x \ln|x| - \int dx = x \ln|x| - x + c.$$

Example 2

Find $\int xe^x dx$

Solution

Let $u = x \Rightarrow du = dx$ and $dv = e^x dx \Rightarrow v = \int e^x dx = e^x$

Then,

$$\int xe^x dx = xe^x - \int e^x dx = xe^x - e^x + c.$$

Example 3.

Find $\int x \sin 2x dx$

Solution

Let $u = x \Rightarrow du = dx$ and $dv = \sin 2x dx \Rightarrow v = -\frac{1}{2}\cos 2x$

Then,

$$\int x \sin 2x dx = -\frac{x}{2}\cos 2x - \int -\frac{1}{2}\cos 2x dx$$

$$= -\frac{x}{2}\cos 2x + \int \frac{1}{2}\cos 2x dx = -\frac{x}{2}\cos 2x + \frac{1}{4}\sin 2x + c$$

1.1.3.3 Integration of rational functions

A function $f(x) = \frac{g(x)}{h(x)}$, where $g(x)$ *and* $h(x) \neq 0$

Case 1: Degree of the numerator is greater than or equal to the degree of the denominator

Recall that if quotient of the division $\frac{f(x)}{g(x)}$ is $q(x)$ and remainder is $r(x)$, then $\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$.

Example

Find $\int \frac{x^2}{x+1} dx$

Find $\int \frac{x+1}{x-1} dx$

Solution

$$
\begin{array}{r|l}
 & x-1 \\
\hline
x+1 & x^2 \\
 & \underline{-x^2 - x} \\
 & \quad -x \\
 & \quad \underline{x+1} \\
 & \qquad 1
\end{array}
$$

$$\Rightarrow \frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$$

Then,

Solution

$$
\begin{array}{r|l}
 & 1 \\
\hline
x-1 & x+1 \\
 & \underline{-x+1} \\
 & \qquad 2
\end{array}
$$

$$\Rightarrow \frac{x+1}{x-1} = 1 + \frac{2}{x-1}$$

Then,

$$\int \frac{x+1}{x-1} dx = \int \left(1 + \frac{2}{x-1}\right) dx$$

$$\int \frac{x^2}{x+1}dx = \int\left(x-1+\frac{1}{x+1}\right)dx$$

$$= \int xdx - \int dx + \int \frac{1}{x+1}dx$$

$$= \frac{x^2}{2} - x + \ln|x+1| + c$$

$$= \int dx + \int \frac{2}{x-1}dx$$

$$= x + 2\ln|x-1| + c$$

Case 2: Degree of the numerator is less than degree of the denominator

To each factor $ax+b$ occurring once in the denominator of a proper rational fraction, there is corresponding single partial fraction of the form $\frac{A}{ax+b}$; where A is a constant to be found. But to each factor $ax+b$ occurring n times in the denominator of a proper rational fraction, there corresponds a sum of n partial fractions of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_n}{(ax+b)^n};$$

where A_n are constants to be found.

Example

Find $\int \frac{x+3}{x^2-5x+4}dx$

Solution

We need to factorise x^2-5x+4. That is,
$x^2-5x+4=(x-4)(x-1)$

Then,

$$\int \frac{x+3}{x^2-5x+4}dx=\int \frac{x+3}{(x-4)(x-1)}dx$$

Let $\frac{x+3}{(x-4)(x-1)}=\frac{A}{x-4}+\frac{B}{x-1}$

$$\Leftrightarrow \frac{x+3}{(x-4)(x-1)}=\frac{A(x-1)+B(x-4)}{(x-4)(x-1)}$$

$\Leftrightarrow x+3=A(x-1)+B(x-4)$ $\Leftrightarrow x+3=Ax-A+Bx-4B$

$\Leftrightarrow x+3=(A+B)x-A-4B$

$$\begin{cases} A+B=1 \\ \underline{-A-4B=3} \end{cases}$$

$$-3B=4 \Rightarrow B=-\frac{4}{3}$$

And

$$\frac{x+3}{(x-4)(x-1)}=\frac{\frac{7}{3}}{x-4}+\frac{\frac{-4}{3}}{x-1}$$

Now,

$$\int\frac{x+3}{x^2-5x+4}dx=\int\left(\frac{\frac{7}{3}}{x-4}+\frac{\frac{-4}{3}}{x-1}\right)dx$$

$$=\frac{7}{3}\int\frac{1}{x-4}dx-\frac{4}{3}\int\frac{1}{x-1}dx \quad =\frac{7}{3}\ln|x-4|-\frac{4}{3}\ln|x-1|+c$$

$$=\frac{1}{3}\ln\left|(x-4)^7\right|-\frac{1}{3}\ln\left|(x-1)^4\right|+c \quad =\ln\left|\sqrt[3]{(x-4)^7}\right|-\ln\left|\sqrt[3]{(x-1)^4}\right|+c$$

$$=\ln\left|\sqrt[3]{\frac{(x-4)^7}{(x-1)^4}}\right|+c$$

Example 2.

Find $\int\frac{dx}{x^2-4}$

Solution

Since $x^2-4=(x-2)(x+2)$,

$$\frac{1}{(x-2)(x+2)}=\frac{A}{x-2}+\frac{B}{x+2}$$

$$\Leftrightarrow 1=A(x+2)+B(x-2)$$

Take $x=-2$,

$$1=A(-2+2)+B(-2-2)$$

$$\Leftrightarrow 1=-4B \Rightarrow B=\frac{-1}{4}$$

Take $x=2$,

$$1=A(2+2)+B(2-2)$$

$$\Leftrightarrow 1=4A \Rightarrow A=\frac{1}{4}$$

Then, $\frac{1}{x^2-4}=\frac{\frac{1}{4}}{x-2}+\frac{\frac{-1}{4}}{x+2}$

$$\Rightarrow \int\frac{dx}{x^2-4}=\int\frac{\frac{1}{4}}{x-2}dx+\int\frac{\frac{-1}{4}}{x+2}dx$$

$$=\frac{1}{4}\ln|x-2|-\frac{1}{4}\ln|x+2|+c=\ln\sqrt[4]{x-2}-\ln\sqrt[4]{x+2}+c=\ln\sqrt[4]{\frac{x-2}{x+2}}+c$$

Alternative method

$$\int \frac{dx}{x^2 - 4}$$

$$\int \frac{dx}{x^2 - 4} = \int \frac{dx}{x^2 - 2^2}$$

But, $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + c$

Then,

$$\int \frac{dx}{x^2 - 2^2} = \frac{1}{4} \ln \left| \frac{x-2}{x+2} \right| + c = \ln \sqrt[4]{\frac{x-2}{x+2}} + c$$

1.1.3.4 Integration of irrational functions

Integrals containing $\sqrt[n]{ax+b},\ a \neq 0$

When finding integral containing $\sqrt[n]{ax+b},\ a \neq 0$, we let $u^n = ax+b$.

Example 1.

Find $\int \sqrt[3]{3x+1}\, dx$

Let $u^3 = 3x+1 \Rightarrow u = \sqrt[3]{3x+1}$

$\Rightarrow 3u^2 du = 3dx \Leftrightarrow u^2 du = dx$

$\int \sqrt[3]{3x+1}\, dx = \int u u^2 du$

$$=\int u^3 du \quad =\frac{u^4}{4}+c \quad =\frac{u^3 u}{4}+c \quad =\frac{(3x+1)\sqrt[3]{3x+1}}{4}+c$$

Learning Outcomes: 1.2 Definite integrals

We define the definite integrals of the function $f(x)$ with respect to x from a to b to be

$$\int_a^b f(x)dx \quad =\left[F(x)\right]_a^b = F(b)-F(a)$$

Example

Evaluate the following definite integrals:

1) $\int_2^3 xdx$ 2) $\int_1^4 \left(e^x-\sqrt{x}\right)dx$ 3) $\int_0^\pi \left[\frac{1}{2}\sin x + x\right]dx$

Solution

$$1)\ \int_2^3 xdx = \left[\frac{x^2}{2}\right]_2^3 = \left(\frac{3^2}{2}-\frac{2^2}{2}\right) = \frac{9}{2}-\frac{4}{2} = \frac{5}{2}$$

$$2)\ \int_1^4 \left(e^x-2\sqrt{x}\right)dx = \int_1^4 e^x dx - 2\int_1^4 \sqrt{x}dx = \left[e^x\right]_1^4 - 2\left[\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right]_1^4 = \left(e^4-e^1\right)-2\left(\frac{4^{\frac{3}{2}}}{\frac{3}{2}}-\frac{1^{\frac{3}{2}}}{\frac{3}{2}}\right)$$

$$=\left(e^4-e\right)-2\left(\frac{16}{3}-\frac{2}{3}\right)=e^4-e-\frac{2}{3}\text{x}14=e^4-e-\frac{28}{3}$$

$$3)\ \int_0^{\pi}\left[\frac{1}{2}\sin x+x\right]dx=\int_0^{\pi}\frac{1}{2}\sin xdx+\int_0^{\pi}xdx=\frac{1}{2}\int_0^{\pi}\sin xdx+\int_0^{\pi}xdx=\frac{1}{2}\left[-\cos x\right]_0^{\pi}+\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$=\frac{1}{2}\left(-\cos\pi-(-\cos 0)+\left(\frac{\pi^2}{2}-\frac{0^2}{2}\right)\right)=\frac{1}{2}\left((1+1)+\frac{\pi^2}{2}\right)=\frac{2}{2}+\frac{\pi^2}{2}=1+\frac{\pi^2}{2}$$

Exercises

Evaluate the integrals;

1. $\int_0^3 xdx$ 2. $\int_1^2 (x^2-x)\,\mathrm{dx}$ 3. $\int_1^2 (3x^2-6x)dx$

4. $\int x^2\ln xdx$ **5.** $.\int 8x\cos xdx$

6. Find a function $F(x)$ satisfying $F'(x)=5x^2+1$ *and* $\mathrm{F}(0)=2$

Learning Outcomes: 1.3 Applications of definite integrals

1.3.1 Area of a region between two curves

We can apply the definite integrals to evaluate the area bounded by the graph of function and lines, *x =a, x=b* and y = 0 on the interval where the function is defined.

$$A=\int_a^b f(x)dx-\int_a^b g(x)dx=\int_a^b\left[f(x)-g(x)\right]dx$$

is calculated as follows

$$A = \int_a^b \left[\begin{pmatrix} upper \\ function \end{pmatrix} - \begin{pmatrix} lower \\ function \end{pmatrix} \right] dx \text{ with } f(x) \geq g(x) \text{ for } a \leq x \leq b$$

Example

Find the area of plane region M lying between the curves $f(x) = 8 - 3x^2$ *and* $g(x) = x^2 - 4x$

Solution

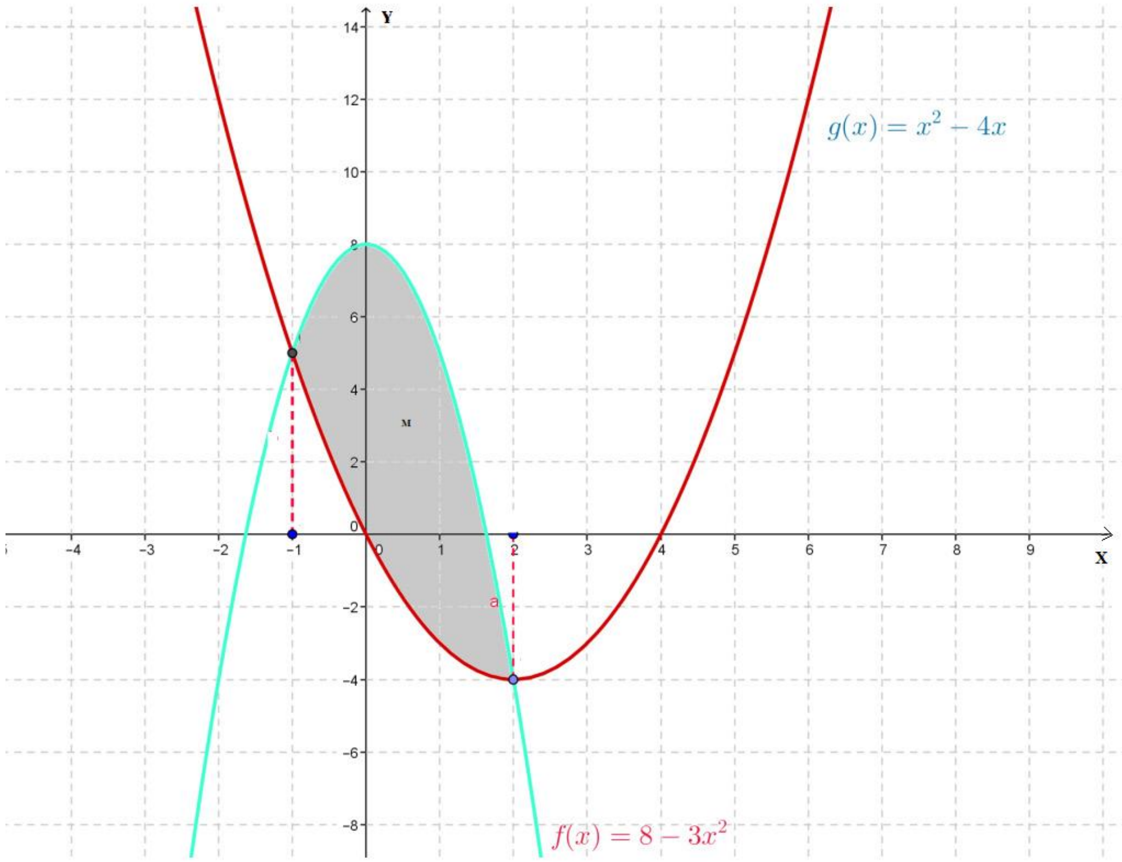
First, we have to solve the equation $f(x) = g(x)$ to find the intersections of the curves.

$$8 - 3x^2 = x^2 - 4x \ .$$

$$x^2 - x - 2 = 0$$

$$x = 2 \text{ or } x = -1.$$

The graphs of the two functions are parabola.



The bounded region M between $f(x)=8-3x^2$ *and* $g(x)=x^2-4x$ is shaded.

Since $f(x)\geq g(x)$ *for* $-1\leq x\leq 2$, the area A *of* M is given by:

$$A=\int_{-1}^{2}\left[f(x)-g(x)\right]dx=\int_{-1}^{2}\left[\left(8-3x^2\right)-\left(x^2-4x\right)\right]dx$$

$$=\int_{-1}^{2}\left(8-4x^2+4x\right)dx=\left[8x-\frac{4x^3}{3}+\frac{4x^2}{2}\right]_{-1}^{2}$$

$$=8(2)-\frac{4}{3}(8)+8-\left[-8+\frac{4}{3}+2\right]$$

$$=16-\frac{32}{3}+8-\left(-\frac{14}{3}\right)=24-\frac{18}{3}=\frac{72-18}{3}=\frac{54}{3}=18$$

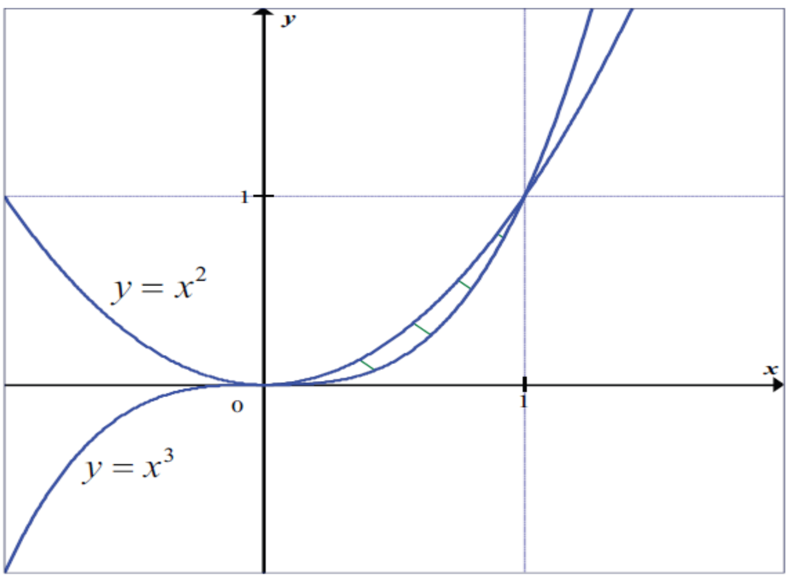
The final answer is expressed in term of surface area as: $A=18\times$square units

Example 2.

Find the area enclosed by the curves $y = x^3 \quad and\ y = x^2$.

Solution

The sketch of the two curves is as shown below:



We now need to know the intersection points of the two curves.

To do this, we solve for $x^2 = x^3$ or $x^2 - x^3 = 0$

$\Rightarrow x = 0\ \ or\ \ x = 1$

The area is given by

$$A=\int_0^1\left(x^2-x^3\right)dx=\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}\ \ sq.\ units$$

Exercises

a) Determine the area of the region K bounded by $y = 2x^2 + 10\ \ and\ y = 4x + 16$

b) Evaluate the area of the plane region bounded by the graphs of functions
$f(x) = -x^2 - 2x + 2$ *and* $g(x) = x^2 + x - 3$

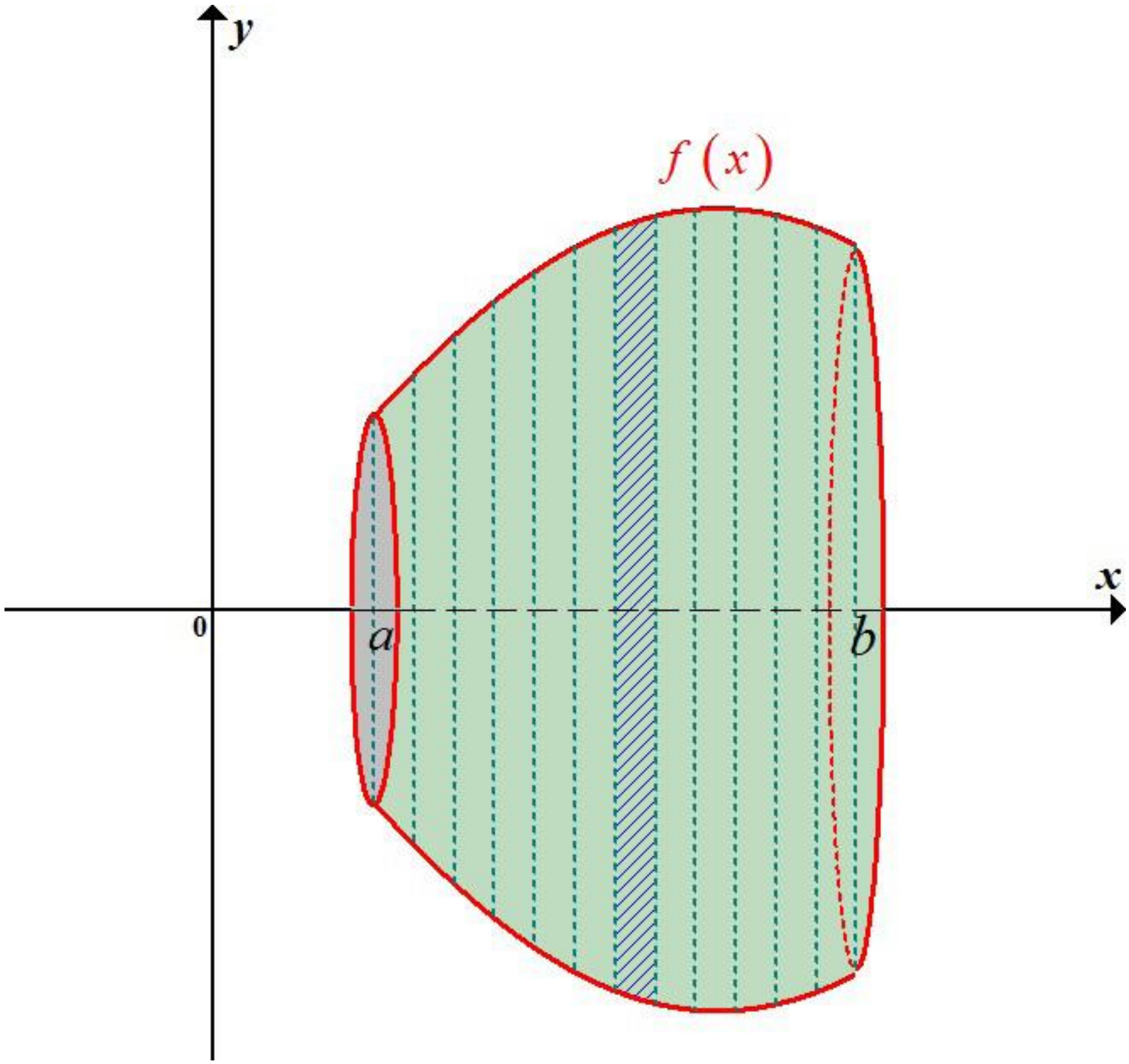
1.3.2. Calculation of volume of a solid of revolution

Consider a region of $f(x)$ between $x = a$ and $x = b$ revolving around $x - axis$.
The volume of the solid of revolution is obtained by considering the area $A(x)$ of the disc of radius $y = f(x)$ such that

$$V = \pi\int_a^b f^2(x)dx$$

$$A(x) = \pi y^2$$

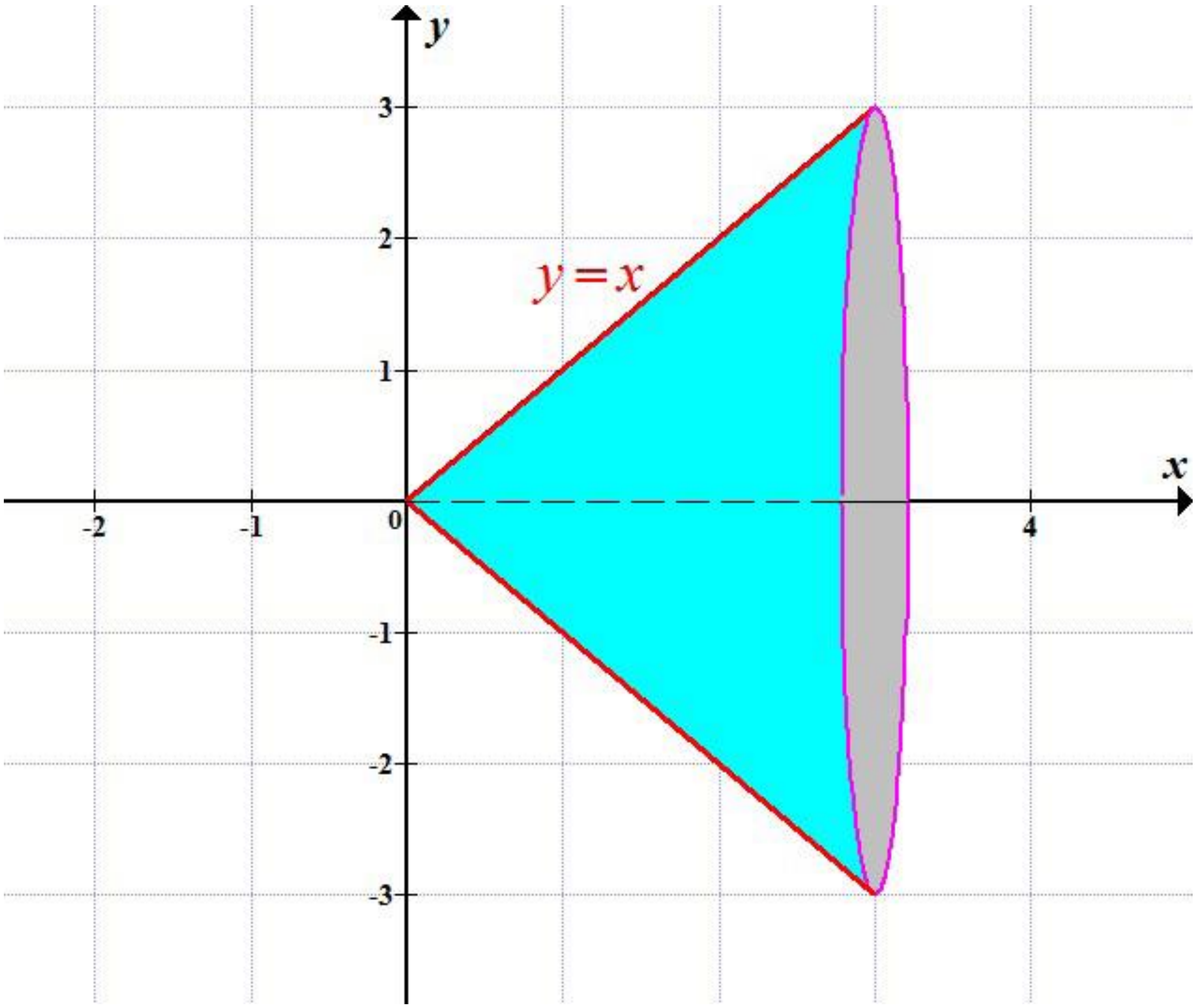
$$V = \int_a^b \pi y^2 dx = \int_a^b \pi f^2(x)dx = \pi\int_a^b f^2(x)dx.$$



Example1.

Use integration to find the volume of the solid generated when the line $y = x$ for $0 \le x \le 3$ is rotated through one revolution (360°) about the $x-axis$.

Solution



$$V = \pi \int_0^3 y^2 \, dx = \pi \int_0^3 x^2 \, dx = \pi \left[\frac{x^3}{3} \right]_0^3 = \pi \frac{3^3}{3} = 9\pi \ cubic\,units$$

Note that the above figure shows a cone with radius 3 and height 3. Thus, we could find the volume of the cone formed using the formula $\frac{\pi}{3} r^2 h$.

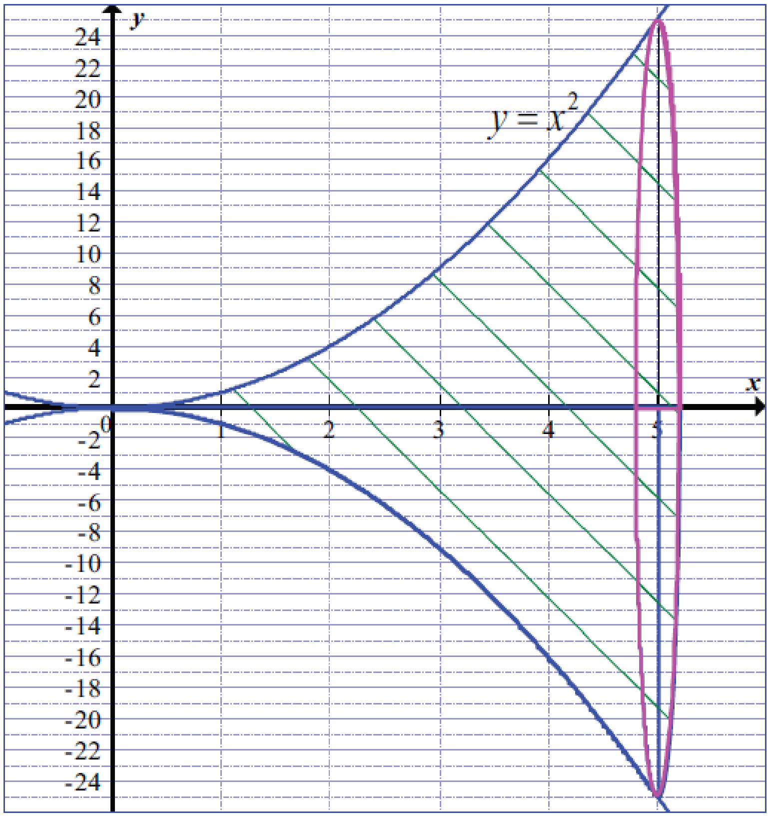
So, $V = \frac{\pi}{3} \times 3^2 \times 3 = 9\pi \ cubic\ units$ (as before).

Example 2.

Find the volume of the solid revolution formed when the area closed by the curve $y = x^2 \ for\ 0 \le x \le 5$ is revolved about the $x - axis$.

Solution

Consider the figure below:



Volume is

$$V = \pi \int_a^b f^2 dx$$

$$= \pi \int_0^5 \left(x^2\right)^2 dx$$

$$= \pi \int_0^5 x^4 dx$$

$$= \pi \left[\frac{x^5}{5}\right]_0^5$$

$$= \pi\left(625 - 0\right)$$

$$= 625\pi \quad cubic\ units$$

Volume for two defining functions (Washer method)

$$V = \pi \int_a^b \left(\left[g(x)\right]^2 - \left[f(x)\right]^2\right) dx$$

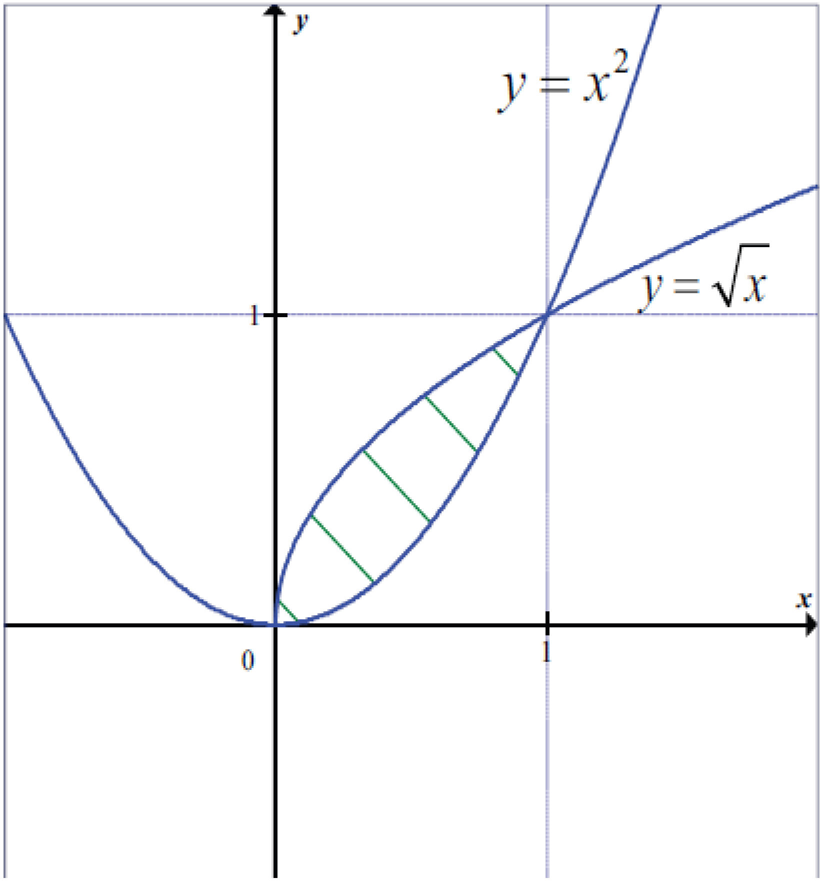
This method is called **washer method.**

Example

Find the volume of the solid of revolution generated by revolving the region enclosed by $y = \sqrt{x}$ and $y = x^2$ about the *x*-axis.

Solution

First, sketch the two functions



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Points of intersection are (0,0) and (1,1), then we take the integral between 0 and1. The function $y = \sqrt{x}$ is above the function $y = x^2$.

$$V = \pi\int_0^1 \left[\left(\sqrt{x}\right)^2 - \left(x^2\right)^2\right]dx$$

$$= \pi\int_0^1 \left(x - x^4\right)dx$$

$$= \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 \quad = \pi\left(\frac{1}{2} - \frac{1}{5} - 0 + 0\right) \quad = \frac{3\pi}{10} \text{ cubic units}$$

Unit 2. Identify the measures of dispersion and interpret bivariate data

Learning Outcomes:

1. Identify the measures of dispersion
2. Describe the measures of the bivariate data
3. Determine the regression line

Learning Outcomes :2.1 Identify the measures of dispersion

1. **Variance**

The Variance measures how far a set of numbers is spread out. A variance of zero indicates that all the values are identical. Variance is always non-negative: a small variance indicates that the data points tend to be very close to the mean and hence to each other, while a high variance indicates that the data points are very spread out around the mean and from each other.

The variance for variable *x* is

$$\sigma_x^2 = \frac{1}{n}\sum\left(x-\bar{x}\right)^2$$

Thus, the variance is also defined by:

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\bar{x}\right)^2$$

The mean is denoted and defined by:

$$\bar{x} = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

$$= \sum_{i=1}^{n} \frac{x_i}{n}$$

$$= \frac{1}{n}\sum_{i=1}^{n} x_i$$

Example 1.

Calculate the variance of the following distribution: 9, 3, 8,8, 9, 8, 9, 18

Solution

$$\bar{x}=\frac{9+3+8+8+9+8+9+18}{8}=9$$

$$\sigma^2=\frac{(9-9)^2+(3-9)^2+(8-9)^2+(8-9)^2+(9-9)^2+(8-9)^2+(9-9)^2+(18-9)^2}{8}=15$$

Example 2.
Calculate the variance of the distribution of the following grouped data:

Class	[10,20[	[20,30[	[30,40[	[40,50[	[50,60[	[60,70[	[70,80[
Frequency	1	8	10	9	8	4	2

Solution

Class	x	f	xf	x^2	$x^2 f$
[10,20[	15	1	15	225	225
[20,30[	25	8	200	625	5000
[30,40[	35	10	350	1225	12250
[40,50[	45	9	405	2025	18225
[50,60[	55	8	440	3025	24200
[60,70[	65	4	260	4225	16900
[70,80[	75	2	150	5625	11250
		$\sum x=42$	$\sum xf=1820$	$\sum x_i^2=16975$	$\sum x^2 f=88050$

$$\bar{x}=\frac{1820}{42}=43.33 \qquad \sigma^2=\frac{88050}{42}-(43.33)^2=218.94$$

Find the variance of the following set of data:

1. 1,3,2,1,2,5,4,0,2,6
2. 3,2,1,5,4,6,0,4,7,8
3. 1,5,6,7,6,4,2,6,3
4. 5,4,5,5,4,5,4,4,5,3
5. 8,7,6,8,6,5,6,4,1

2. Standard deviation

We define the **standard deviation** to be the square root of the variance.

Thus, the standard deviation is denoted and defined by;

$$\sigma = SD(X) = \sqrt{Var(X)}$$

or

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}\left(x_i - \overline{x}\right)^2}{n}} \text{ or } \sigma = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\overline{x}\right)^2}$$

Example 1.

The heights (in meters) of six children are 1.42, 1.35, 1.37, 1.50, 1.38 and 1.30. Calculate the mean height and the standard deviation of the heights.

Solution

$$\text{Mean} = \frac{1}{6}(1.42 + 1.35 + 1.37 + 1.50 + 1.38 + 1.30) = 1.39 \quad m$$

$$\text{Variance} = \frac{1}{6}(1.42^2 + 1.35^2 + 1.37^2 + 1.50^2 + 1.38^2 + 1.30^2) - 1.39^2$$
$$= 0.00386 \ m^2$$

$$\text{Standard deviation} = \sqrt{0.00386 \ m^2} = 0.0621 \ m$$

Example 2.
The number of customers served lunch in a restaurant over a period of 60 days is as follows:

Number of customers served lunch	**Number of days in the 60 day period**
20-29	6
30-39	12
40-49	16
50-59	14
60-69	8
7079	4

Find the mean and standard deviation of the number of customers served lunch using this grouped data.

Solution

We need the mid-interval values for all groups

Groups	Mid-interval values (x_i)	Frequency (f_i)	$f_i x_i$	$f_i x_i^2$
20-29	24.5	6	147	3601.5
30-39	34.5	12	414	14283.0
40-49	44.5	16	712	31684.0
50-59	54.5	14	763	41583.5
60-69	64.5	8	516	33282.0
70-79	74.5	4	298	22201.0
		$\sum = 60$	$\sum = 2850$	$\sum = 146635$

The mean is $\bar{x} = \frac{2850}{60} = 47.5$

The standard deviation is $\sigma = \sqrt{\frac{146635}{60} - 47.5^2} = 13.7$

A large standard deviation indicates that the data points can spread far from the mean and a small standard deviation indicates that they are clustered closely around the mean. Standard deviation is often used to compare real-world data against a model to test the model.

Exercises

Find the standard deviation of the following set of data

1. 202,205,207,203,205,206,207,209
2. 1009,1011,1008,1007,1012,1010,106
3. 154,158,157,156,155,154,159
4. 7804,7806,7805,7807,7808
5. 56,54,55,59,58,57,55

3. Coefficient of variation

The coefficient of variation measures variability in relation to the mean (or average) and is used to compare the relative dispersion in one type of data with the relative dispersion in another type of data. It allows us to compare the dispersions of two different distributions if their means are positive. The greater dispersion corresponds to the value of the coefficient of greater variation.

The coefficient of variation is a calculation built on other calculations: the standard deviation and the mean as follows:

$$Cv = \frac{\sigma}{\bar{x}} \times 100$$

Where:

- σ is the standard deviation.
- $\bar{x}$ is the mean.

Example

One data series has a mean of 140 and standard deviation 28.28. The second data series has a mean of 150 and standard deviation 24. Which of the two has a greater dispersion?

Solution

$$Cv_1 = \frac{28.28}{140} \times 100 = 20.2\%$$

$$Cv_2 = \frac{24}{150} \times 100 = 16\%$$

The first data series has a higher dispersion.

Exercises

Find the coefficient of variation of the following set of data

1. 2,9,8,4,7,3,2
2. 12,11,9,8,6,10,7,9
3. 5,9,8,6,0,10,8,3,14
4. 8,10,7,11,6,12,9
5. 7,6,0,9,6,12,12,9,8,6

Learning Outcomes 2.2: Describe the measures of the bivariate data

a) Covariance

The **covariance of variables** *x* **and** *y* is a measure of how these two variables change together. If covariance is zero, the variables are said to be **uncorrelated,** meaning that there is no linear relationship between them.

$$\operatorname{cov}(x, y) = \frac{1}{n} \sum (x - \bar{x})(y - \bar{y})$$

Example

Find the covariance of the following distribution

$y \backslash x$	0	2	4
1	2	1	3
2	1	4	2
3	2	5	0

Solution

Convert the double entry into a simple table and compute the arithmetic means

x_i	y_i	f_i	$x_i f_i$	$y_i f_i$	$x_i y_i f_i$
0	1	2	0	2	0
0	2	1	0	2	0
0	3	2	0	6	0
2	1	1	2	2	2
2	2	4	8	8	16
2	3	5	10	15	30
4	1	3	12	3	12
4	2	2	8	4	16
4	3	0	0	0	0
		$\sum_{i=1}^{9} f_i = 20$	$\sum_{i=1}^{9} x_i f_i = 40$	$\sum_{i=1}^{9} y_i f_i = 41$	$\sum_{i=1}^{9} x_i y_i f_i = 76$

$$\operatorname{cov}(x,y) = \frac{1}{n}\sum_{i=1}^{k} f_i x_i y_i - \bar{x}\,\bar{y} \qquad \bar{x} = \frac{1}{n}\sum_{i=1}^{k} x_i f_i, \quad \bar{y} = \frac{1}{n}\sum_{i=1}^{k} y_i f_i$$

$$\bar{x} = \frac{40}{20} = 2\ ,\ \bar{y} = \frac{41}{20} = 2.05$$

$$\operatorname{cov}(x, y) = \frac{76}{20} - 2 \times 2.05 = -0.3$$

Exercises

1. The scores of 12 students in their mathematics and physics classes are

Mathematics	2	3	4	4	5	6	6	7	7	8	10	10
Physics	1	3	2	4	4	4	6	4	6	7	9	10

Find the covariance of the distribution.

2. The values of two variables x and y are distributed according to the following table

y \ x	100	50	25
14	1	1	0
18	2	3	0
22	0	1	2

Calculate the covariance

b) Coefficient of correlation

The Coefficient of correlation is a measure of the strength of linear relationship between two variables.

The coefficient of correlation between two variables *x* and *y* is given by

$$r = \frac{\mathrm{cov}(x,y)}{\sigma_x \sigma_y}$$

Thus, $-1 \le r \le 1$

Where,

$\mathrm{cov}(x,y)$ is covariance of x and y

σ_x is the standard deviation for x

σ_y is the standard deviation for y

Example

Consider the following table:

x	y
3	6
5	9
7	12
3	10
2	7
6	8

1. Find the standard deviations σ_x, σ_y
2. Find covariance $\mathrm{cov}(x,y)$
3. Calculate the ratio $\frac{\mathrm{cov}(x,y)}{\sigma_x \sigma_y}$.

Solution

$\operatorname{cov}(x,y)=5$

$\sigma_x^2=\frac{42}{6}=7$, $\sigma_y^2=\frac{23.36}{6}=3.89$

Then, the Pearson's coefficient of correlation is

$$r=\frac{\operatorname{cov}(x,y)}{\sigma_x\sigma_y} \qquad r=\frac{5}{\sqrt{7}\sqrt{3.89}}=\frac{5}{\sqrt{27.23}}=0.96$$

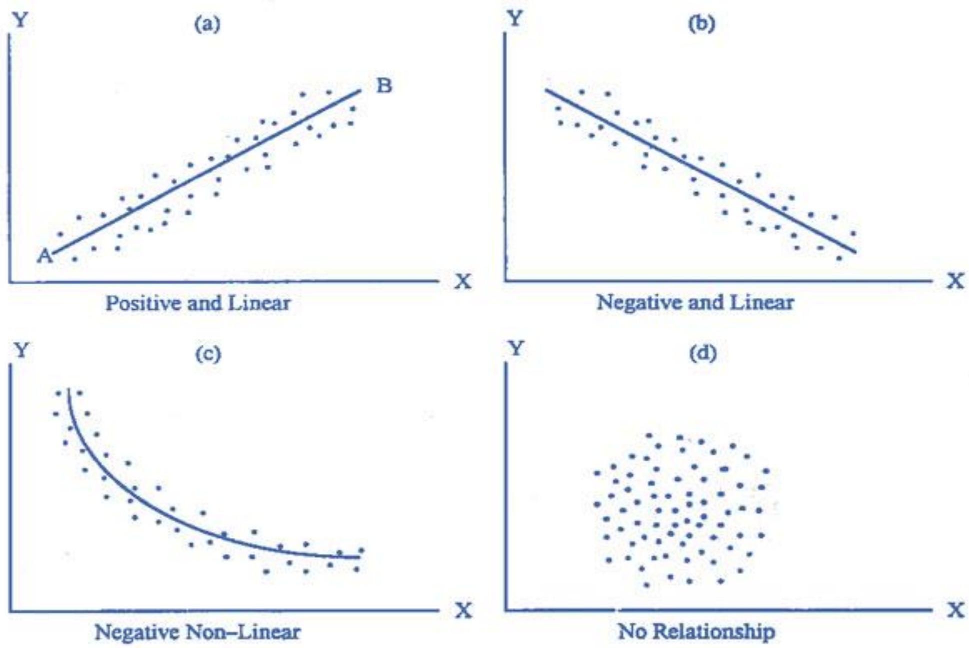
Then, there is a very strong positive linear relationship between two variables.

Learning Outcomes 2.3: Determine the regression line

a) Scatter plot

A scatter plot is a type of plot or mathematical diagram using Cartesian coordinates to display values for typically two variables for a set of data. It is used to show the relationship of two variables.

Scatter diagram



Example

Chemistry (xi)	8	7	6	9	8	9	7	8	5	6
English (yi)	7	8	7	9	8	8	7	9	7	5

Represent the data on scatter diagram.

b) Regression line

We use the regression line to **predict** a value of y for any given value of x and vice versa. The "best" line would make the best predictions: the observed y-values should stray as little as possible from the line. This straight line is the regression line from which we can adjust its algebraic expressions and it is written as $y = ax + b$.

$$Lyx \equiv y - \bar{y} = \frac{cov(x,y)}{var(x)}(x - \bar{x})$$

Or

Short cut method of finding regression line

$$\begin{cases} \sum_{i=1}^{k} f_i y_i = a \sum_{i=1}^{k} f_i x_i + b\,n \\ \sum_{i=1}^{k} f_i x_i y_i = a \sum_{i=1}^{k} f_i x_i^2 + b \sum_{i=1}^{k} f_i x_i \end{cases}$$

Example

Find the regression line of y on x for the following data and estimate the value of y for $x = 4, x = 7, x = 16$ and the value of x for $y = 7, y = 9, y = 16$.

x	3	5	6	8	9	11
y	2	3	4	6	5	8

Solution

x	y	$x - \bar{x}$	$y - \bar{y}$	$(x - \bar{x})^2$	$(y - \bar{y})^2$	$(x - \bar{x})(y - \bar{y})$
3	2	-4	-2.6	16	6.76	10.4
5	3	-2	-1.6	4	2.56	3.2
6	4	-1	-0.6	1	0.36	0.6
8	6	1	1.4	1	1.96	1.4
9	5	2	0.4	4	0.16	0.8
11	8	4	3.4	16	11.56	13.6
$\sum_{i=1}^{6} x_i = 42$	$\sum_{i=1}^{6} y_i = 28$			$\sum_{i=1}^{6} (x_i - \bar{x})^2 = 42$	$\sum_{i=1}^{6} (y_i - \bar{y})^2 = 23.36$	$\sum_{i=1}^{6} (x_i - \bar{x})(y_i - \bar{y}) = 30$

$$\bar{x}=\frac{42}{6}=7\ ,\ \bar{y}=\frac{28}{6}=4.7$$

$$\operatorname{cov}(x,y)=\frac{1}{n}\sum_{i=1}^{k}(x-\bar{x})(y-\bar{y})=\frac{30}{6}=5$$

$$\sigma_x^2=\frac{42}{6}=7\ ,\ \sigma_y^2=\frac{23.36}{6}=3.89$$

$$L_{y/x}\equiv y-\bar{y}=\frac{\operatorname{cov}(x,y)}{\sigma_x^2}(x-\bar{x})$$

$$L_{y/x}\equiv y-4.7=\frac{5}{7}(x-7)$$

Finally, the line of y on x is

$$L_{y/x}\equiv y=\frac{5}{7}x-0.3$$

Or

$$L_{y/x}\equiv y=ax+b$$

$$\begin{cases}\sum_{i=1}^{k} f_i y_i = a\sum_{i=1}^{k} f_i x_i + b\,n\\ \sum_{i=1}^{k} f_i x_i y_i = a\sum_{i=1}^{k} f_i x_i^2 + b\sum_{i=1}^{k} f_i x_i\end{cases}$$

$$\begin{cases}28=42a+6b\\ 226=336a+42b\end{cases}\Leftrightarrow\begin{cases}a=\dfrac{5}{7}\\ b=-0.3\end{cases}$$

Thus, the line of y on x is

$$L_{y/x} \equiv y = \frac{5}{7}x - 0.3$$

Exercises

1. Consider the following table:

x	y
60	3.1
61	3.6
62	3.8
63	4
65	4.1

a) Find the regression line of y on x

b) Calculate the approximate y value for the variable $x = 64$.

Given the statistical Distribution

x_i	7	8	9	11	15
y_i	33	25	17	9	6

a) Calculate the linear correlation coefficient**.**

b) Determine the equation of the regression line of y on x .

c) Draw a scatter plot of this set of the distribution and the regression line.

LU3: Apply fundamentals of probabilities

Learning Outcomes:

1. Apply the techniques of counting
2. Compute the probabilities
3. Calculate the conditional probability

Learning Outcome 3.1 Apply techniques of counting

Introduction

The word" Probability" denotes chance and the theory of probability deals with laws governing the chances of occurrence of phenomena which are unpredictable nature.
It is the chance that something will happen how likely it is that some event will happen. No engineer or scientist can conduct research and development works without knowing the probability theory.

Probability theory originated from the games of chance played by tossing coins, throwing dices, drawing cards etc.

$$\text{Probability (A)} = \text{P (A)}$$
$$= \frac{\text{Favourable outcomes for event A}}{\text{number of all possible outcomes}}$$
$$= \frac{n(A)}{n(S)}$$

Probabilities of any possible events is in the range of 0 to 1. i.e. In general, for any event A,
$0 \leq P(A) \leq 1$

Example 1.

Consider a class of 40 students where 20 students don't like pepper. Find the probability that a student selected at random likes pepper.

Solution

Define an event that a student does not like pepper

Favorable outcomes (number of events) n(x) = 20

The sample space (total number of trials) n(S) = 40

So probability that a student selected at random doesn't like pepper is

$$\textit{Probability } (x) = \frac{\text{Favourable outcomes}}{\text{the total number of trials}}$$
$$= \frac{n(x)}{n(S)} = \frac{20}{40} = \frac{1}{2}$$

1. Venn diagram

A Venn diagram refers to representing mathematical or logical sets pictorially as circles or closed curves within an enclosing rectangle (the universal set), common elements of the sets represented by intersections of the circles.

As you interpret the information, remember:

(i) And implies intersection

(ii) Or implies Union

(iii) Not implies complement.

Example

A survey involving 120 people about their preferred breakfast showed that:

55 eat eggs for breakfast.

40 drink juice for breakfast.

25 eat both eggs and drink juice for breakfast.

(a) Represent the information on a Venn diagram.

(b) Calculate the following probabilities.

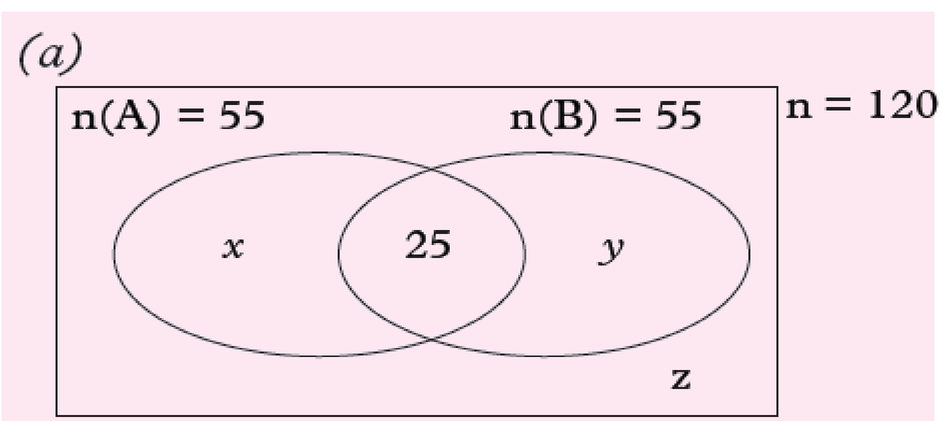
(i) A person selected at random takes only one type for breakfast.

(ii) A person selected at random takes neither eggs nor juice for breakfast.

Solution

Let A= Eggs those who eat eggs only,

B = Juice those who take juice only and z represent those who did not take any.



Here, we can now solve for the number of people who didn't take any for breakfast.

$x = 55 - 25 = 30$

So 30 people took Eggs only

Also, y = 40 – 25 = 15

So, 15 people took Juice only.

Hence 30 + 25 + 15 + z = 120

Z = 120 – (30+15+25)

Z = 120 – 70

Z = 50

The number of people who did not take anything for breakfast is 50.

(b)

(i) *Probability of those who take only one type for breakfast is the probability of those who take Eggs only or juice only.*

$$P\ (\text{Eggs or Juice}) = \frac{30 + 15}{120}$$

$$= \frac{45}{120} = \frac{3}{8}$$

(ii) *Probability of those who take neither eggs nor juice for breakfast is the probability of those who do not take anything.*

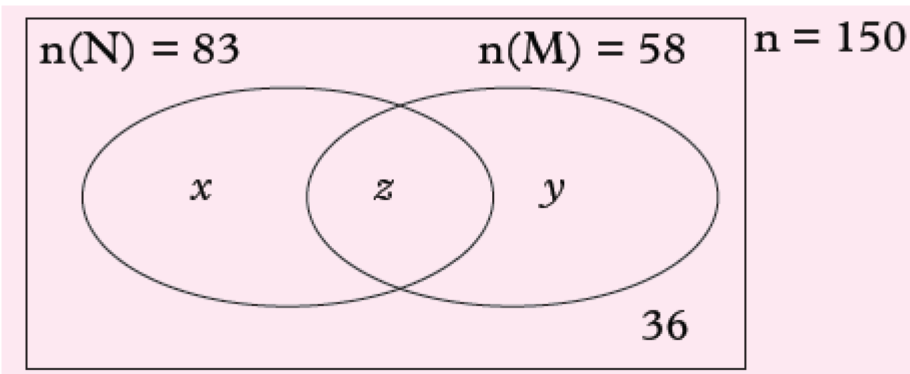
P *(neither Eggs nor Juice)*

$$= \frac{50}{120} = \frac{5}{12}$$

Example 2.
In a survey of 150 Rwandan people about which newspapers they read, 83 read the New Times, 58 read the Imvaho Nshya. 36 read neither of those two papers.

Represent the data on the Venn diagram and find the chance that a person selected at random reads both papers.

Solution
Here we can let N= New Times, M= Imvaho Nshya, x= New Times only, z= both New Times and Imvaho Nshya, y= Imvaho Nshyaonly



We can therefore calculate the value of z which is required in the question

$83 = x + z, so\ z = 83 - x$
$58 = y + z, so\ z = 58 - y$

Hence $83 - x = 58 - y$, which gives $x - y = 25$ ………………(1)
Also, $x + z + y + 36 = 150$

But x + z = 83, so we have 83 + y + 36 = 150

Solving, we get y = 150 – 36 – 83 = 31
Hence x = 25 + y = 25 + 31 = 56

Then we can solve for z = 58 – y
= 58 – 31 = 27.

We therefore get the number of people who read both New Times and Monitor as 27.

Probability of those who read three papers

$$= \frac{\textit{Number of those who read all}}{\textit{Total number of people}} = \frac{27}{150} = \frac{9}{50}$$

Exercise

In a survey of 50 people about which Hotels they patronize among Hilltop, Serena, and Lemigo. We find that 15 people eat at Hilltop, 30 people eat at Serena, 19 people eat at Lemigo 8 people eat at Hilltop and Serena, 12 people eat at Hilltop and Lemigo,7 people eat at Serena and Lemigo. 5 people eat at Hilltop, Serena, and Lemigo.

(a) What is the chance that a person selected at random eats only at Hilltop?
(b) How many eat at Hilltop and Serena, but not at Lemigo?
(c) How many people don't eat at any of these three hotels?
(d) What is the probability that a person selected at random do not eat at any of the hotels mentioned?

2. Tree Diagram

A tree diagram is a diagram used to show the total number of possible outcomes in a probability experiment.

Example

Using a tree diagram, determine all the possible outcomes when a coin is tossed once?

Solution

The tree diagram in Fig. 11.1 shows the sequence of events.

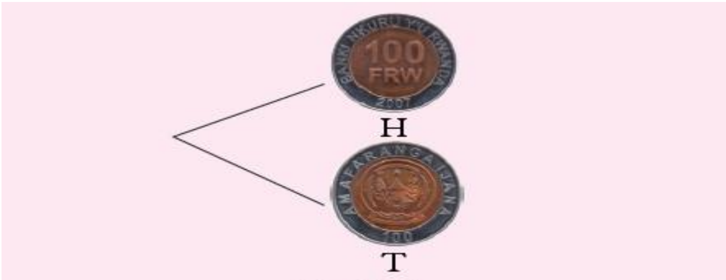


Fig 11.1

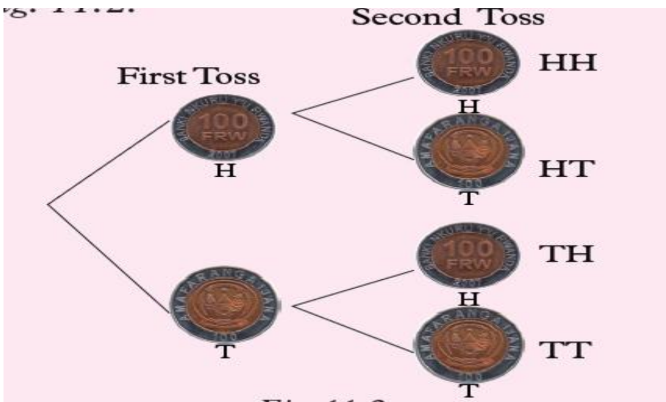
We obtain 2 outcomes from tossing a coin once i.e. head (H) and tail (T)

Example 2

Using a tree diagram, determine all the possible outcomes that can be obtained when a coin is tossed twice.

Solution

In the first toss, we get either a head (H) or a tail (T).



Therefore, we have 4 possible outcomes i.e. {HH, HT, TH, TT}

3. **Counting by using Permutation and Combination**

i. **A permutation** is an act of arranging the objects or numbers in order. Permutation is an arrangement of n objects in a specific order the total number of permutation of n objects taking r at a time is denoted by p(n,r).

$$p(n,r) = \frac{n!}{(n-r)!}$$

Permutation can be done in two ways,

- **Permutation with repetition**: The number of ways of arranging n objects, among which n_1 are alike, n_2 are alike etc.

  Is given by $\dfrac{n!}{n_1!*n_2!........*n_r!}$

- **Permutation without Repetition:** This method is used when we are asked to reduce 1 from the previous term for each time.

Example1.

Find the number of permutations if n = 9 and r = 2.

Solution:

Given n = 9 and r = 2.
Permutation = $nPr \ = \ n!/\,(n-r)!$

$$= \frac{9!}{(9-2)!} \ = \frac{9!}{7!} = 72$$

Thus, the number of permutations = 72

Example2.

a. Find $p(14{,}2)$
b. Find $p(10{,}4)$

Solution

a. $p(14{,}2) = \frac{14!}{(14-2)!} = \frac{14!}{12!} = \frac{14*13*12!}{12!} = 14*13 = 182$
b. $p(10{,}4) = \frac{10!}{(10-4)!} = \frac{10!}{6!} = \frac{10*9*8*7*6!}{6!} = 10*9*8*7 = 5040$

Exercise

A maths debating team consists of 4 speakers.

a) In how many ways can all 4 speakers be arranged in a row for a photo?
b) How many ways can the captain and vice-captain be chosen?

Permutation with repetition

Examples:

i. Find the number of words can be organized with the letters of the word MATHEMATICS by regrouping them.

Solution:

Here we can observe that there are 2 M's, 2 A's and 2T's this is the example of permutation with repetition = $\frac{n!}{(P!\text{q}!r!)}$

Required number of ways are $=\frac{11!}{(2!2!2!)} = 4989600$

ii. In how many ways can we permute the letters of word MISSISSIPPI.

Solution

we have $n = 11, M = 1, I = 4, S = 4, P = 2$

Thus we get $\frac{11!}{1!4!4!2!} = 34650$

ii. Combinations are the way of selecting the objects or numbers from a group of objects or collection, in such a way that the order of the objects does not matter.

It is denoted by $C(n,r)$ and given by $C(n,r) = \frac{p(n,r)}{r!} = \frac{n!}{r!\,(n-r)!}$.

Examples

1. Consider $C(5,3) = \frac{5!}{3!(5-3)!} = \frac{5!}{3!2!} = \frac{5*4*3!}{3!2!} = \frac{20}{2} = 10$

2. How many ways can a basketball team of 5 players be chosen from 8 players?
   Solution
   $C(8,5) = \frac{8!}{5!(8-5)!} = \frac{8!}{5!3!} = \frac{8*7*6*5!}{5!3*2*1} = \frac{8*7*6}{6} = 56.$

Combination without repetition

Examples:

1. How many ways did 2 green and 2 black balls come out of a bag containing 7 green and 8 black balls?

Solution:

Here bag contains 7 green from that we have to choose 2 so it is 7 CHOOSE 2 problem and 8 black balls from that we have to choose 2 so it is 8 CHOOSE 2 problem.

Hence the Required number $= 7C2 * 8C2 = \left\{\frac{7!}{2!(7-2)!}\right\} * \left\{\frac{8!}{2!(8-2)!}\right\} = 21 * 28 = 588$

so in 588 ways we can select 2 green and 2 black from that bag.

2. Find the numeral of ways in which a 6-member cabinet can be set up from 8 gentlemen and 4 ladies so that the cabinet consists of at least 3 ladies.

Solution:

For forming the committee, we can choose from 3 each men and women and 2 men 4 women so the problem includes 8 CHOOSE 3, 4 CHOOSE 3, 8 CHOOSE 2 and 4 CHOOSE 4.

Two types of cabinet can be formed

(i) Having 3 men and 3 ladies

(ii) Having 2 men and 4 ladies

Possible no. of ways $= (8C3 * 4C3) + (8C2 * 4C4) = \left\{\frac{8!}{3!(8-3)!}\right\} * \left\{\frac{4!}{3!(4-3)!}\right\} + \left\{\frac{8!}{2!(8-2)!}\right\} * \left\{\frac{4!}{4!(4-4)!}\right\} = 56 * 4 + 28 * 1 = 252$

So in 252 ways we can form such cabinet.

3. A bag contains 6 blue balls, 5 green balls and 4 red balls. Three balls are selected at random without replacement. Find the probability that

a) they are all blue

b) 2 are blue and 1 is green

c) there is one of each colour.

Solution

The number of all possible outcomes is $\binom{15}{3} = \frac{15!}{3! \times 12!} = \frac{15 \times 14 \times 13}{3!} = 455$

a) The number of ways of obtaining 3 blue balls is

$$\binom{6}{3} \times \binom{5}{0} \times \binom{4}{0} = \frac{6!}{3!3!} \times 1 \times 1 = \frac{6 \times 5 \times 4}{3!} = \frac{120}{6} = 20$$

Thus, P (all are blue) $= \frac{20}{455} = \frac{4}{91}$

b) The probability of obtaining 2 blue balls and one green ball is:

$$\binom{6}{2} \times \binom{5}{1} \times \binom{4}{0} = \frac{6!}{2!4!} \times 5 \times 1 = \frac{6 \times 5}{2!} \times 5 = \frac{150}{2} = 75$$

Thus, P(2 blue and 1 green) $= \frac{75}{455} = \frac{15}{91}$

c) The probability of obtaining 1 blue ball, 1 green ball and 1 red ball is:

$$\binom{6}{1} \times \binom{5}{1} \times \binom{4}{1} = \frac{6!}{1!5!} \times \frac{5!}{1!4!} \times \frac{4!}{1!3!} = 6 \times 5 \times 4 = 120$$

Thus, $P\,(1 \text{ blue, } 1 \text{ green and } 1 \text{ red}) = \frac{120}{455} = \frac{24}{91}$

Exercises

1. For five men and 4 women, a group of 6 will be formed. In how many ways this can be done so that the group has more men.
2. Find how many ways you can rearrange letters of the word “BANANA” all at a time
3. In a hand of poker, 5 cards are dealt from a regular pack of 52 cards.

i. What is the total possible number of hands?

ii. In how many of these hands are there:

a) 4 Kings?
b) 2 Clubs and 3 Hearts?
c) all Hearts?
d) all the same colour?
e) four of the same kind?
f) 3 Aces and two Kings?

Learning Outcome 3.2: Compute probabilities

Definitions of terminologies and probabilities

a. Random Experiments:

In the study of probability, any process of observation is referred to as an experiment. The results of an observation are called the outcomes of the experiment. An experiment is called a random experiment if its outcome cannot be predicted. Random experiments are those experiment whose results depend on chance.

Typical examples of a random experiment are the roll of a die, the toss of a coin, drawing a card from a deck.

b. Sample Space:

The set of all possible outcomes of a random experiment is called the sample space (or universal set), and it is denoted by S. An element in S is called a sample point. Each outcome of a random experiment corresponds to a sample point.

Example:

Find the sample space for the experiment of tossing a coin:

(a) once and (b) twice.

Solution

(a) There are two possible outcomes, heads or tails. Thus

S = {H, T)

where H and T represent head and tail, respectively.

(b) There are four possible outcomes. They are pairs of heads and tails. Thus

S = (HH, HT, TH, TT)

c. Event:

An event is a set consisting of possible outcomes of an experiment with the desired qualities. It is a subset of a sample space.

Examples of events are:

1. Getting a tail when tossing a coin.

2.When two dice are thrown the event "total 8 points" is composite.

E$= \{2{,}6), (6{,}2), (3{,}5), (5{,}3), (4{,}4)\}$

F= event "total 12 point"

$= \{(6{,}6)\}$ is elementary

d. Inclusive event

Inclusive events are events that can happen at the same time

If two events, A and B, are inclusive, then that means that if A occurs, B could also occur, and vice versa.

then the probability that either A or B occurs is P (A or B) = P(A) + P(B) – P (A and B)

E. Equally likely

Have equal probabilities.

Example: When a fair coin is tossed, there are only two possible outcomes H or T. Then P(H) = P(T).

F. Mutually exclusive events

Events A and B are said to be mutually exclusive if the events A and B are disjoint.

i.e. A and B cannot occur at the same time.

For mutually exclusive events, $A \cap B = \varnothing$. $P(A \cap B) = P(\varnothing) = 0$; and so the addition law reduces to $P(A \cup B) = P(A) + P(B)$.

Example1.

A card is drawn from a pack of 52. A is the event of drawing an ace and B is the event of drawing a spade. Find P(A), P(B), $P(A \cap B)$ and $P(A \cup B)$.

Solution

$$P(A) = P(\text{an ace}) = \frac{4}{52} = \frac{1}{13}$$

$$P(B) = P(\text{a spade}) = \frac{13}{52} = \frac{1}{4}$$

$$P(A \cap B) = P(\text{the ace of spades}) = \frac{1}{52}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Example 2.

A marble is drawn from an urn containing 10 marbles of which 5 are red and 3 are blue. Let A be the event: the marble is red; and let B be the event: the marble is blue. Find P(A), P(B) and $P(A \cup B)$.

Solution

$P(A) = \frac{5}{10} = \frac{1}{2}$, $P(B) = \frac{3}{10}$ and since the marble cannot be both red and blue,

A and B are mutually exclusive so $P(A \cup B) = P(A) + P(B) = \frac{5}{10} + \frac{3}{10} = \frac{8}{10} = \frac{4}{5}$

Example 3.

Given that $P(A) = \frac{1}{3}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = x$. Find x if A and B are

(a) independent

(b) mutually exclusive.

Solution

(a) If the events are independent, then $P(A \cap B) = P(A) \times P(B) = \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{2}{5} - \frac{2}{15} = \frac{5+6-2}{15} = \frac{9}{15} = \frac{3}{5}$$

Thus, $x = \frac{3}{5}$

(b) If the events are mutually exclusive, then $A \cap B = \varnothing$ which gives $P(A \cap B) = 0$

We know that $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$x = \frac{1}{3} + \frac{2}{5} - 0 = \frac{5+6}{15} = \frac{11}{15}$$

Thus, $x = \frac{11}{15}$

Exercises

1. A factory runs two machines, A and B. Machine A operates for 80% of the time while machine B operates for 60% of the time and at least one machine operates for 92% of the time. Do these machines operate independently?

1. If A and B are any two events with $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{12}$, and $P(A \cap B) = \frac{1}{4}$. Find $P(A \cup B)$.

3. If A and B are independent events such that $P(A) = \frac{3}{4}$ and $P(B) = \frac{5}{6}$, find
   a) $P(A \cap B)$; b) $P(A \cup B)$; c) $P(A \cap B')$;

4. A die is biased so that the probability of throwing a six is $\frac{1}{3}$. If the die is thrown twice, find the probability of obtaining

two sixes

Learning Outcome 3.3: Conditional probability

1. **probability by using Tree diagram**

A tree diagram is a means which can be used to show the probabilities of certain outcomes occurring when two or more trials take place in succession.

The outcome is written at the end of the branch and the fraction on the branch gives the probability of the outcome occurring. For each trial, the number of branches is equal to the number of possible outcomes of that trial. In the diagram there are two possible outcomes, A and B, of each trial.

Example 1

A coin is tossed twice.

(a) Represent the outcomes on a tree diagram.

(b) Determine the following probabilities.

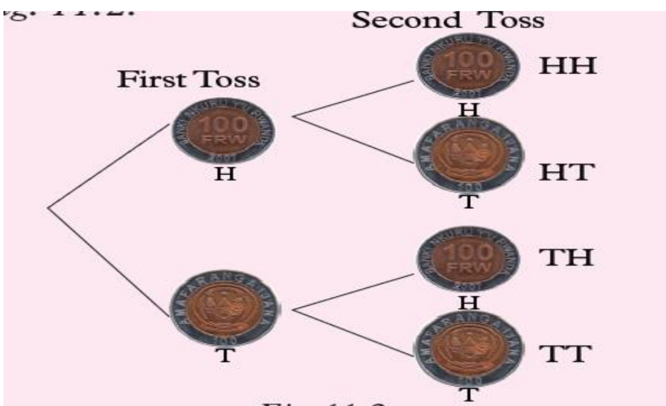
(i) Getting H followed by T

(ii) Getting two heads

(iii) Getting head and tail irrespective of order.

Solution

a.



(b) $P(HT)$

$$= \frac{\text{Number of events where } H \text{ is followed by } T}{\text{Total number of possible outcomes}}$$

$$= \frac{1}{4}$$

$$(c)\ P(HH) = \frac{\text{Number of ways of getting two heads}}{\text{Total number of possible outcomes}} = \frac{1}{4}$$

(d) There are two ways of getting a head and tail without caring about the order in which they follow one another i.e. HT or TH.

We determine this probability as follows;

$$\frac{\text{Number of ways of getting HT or TH}}{\text{Total number of possible outcomes}} = \frac{2}{4} = \frac{1}{2}$$

Example 2.

A bag contains 8 balls of which 3 are red and 5 are green. One ball is drawn at random, its color is noted and the ball replaced in the bag. A ball is again drawn from the bag and its color is noted. Find the probability that the ball drawn will be

a) red followed by green,

b) red and green in any order,

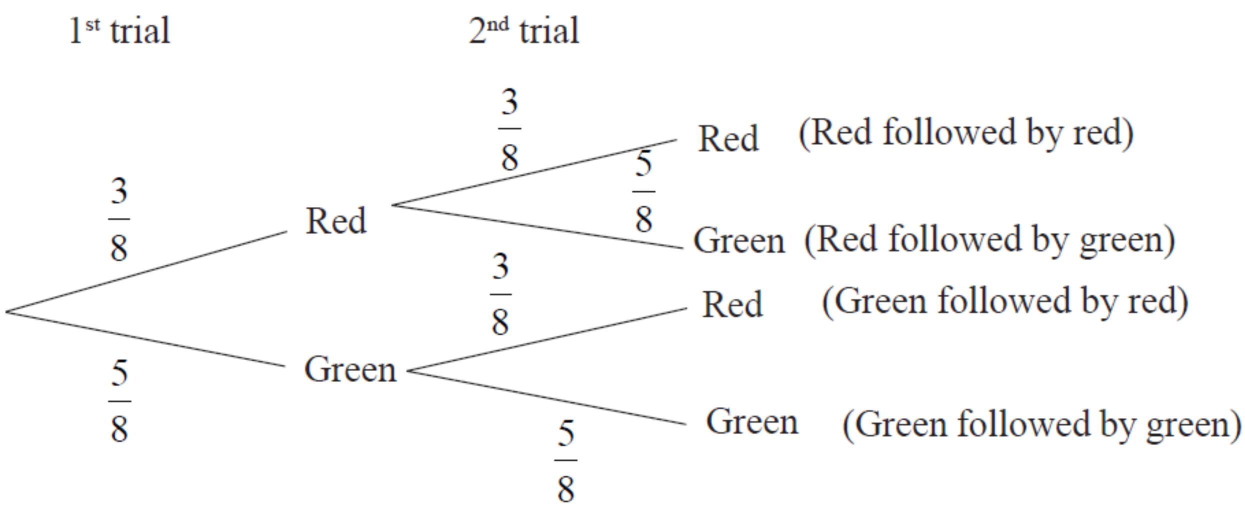
c) of the same color.

Solution

Since there are 3 red balls and 5 green balls, for the

1st trial, the probability of choosing a red ball is $\frac{3}{8}$ and probability of choosing a green ball is $\frac{5}{8}$ and since after the 1st trial, the ball is replaced in the bag, for the second trial the probabilities are the same as in the first trial.

Draw a tree diagram showing the probabilities of each outcome of the two trials.



a) $P(\text{Red followed by green}) = \frac{3}{8} * \frac{5}{8} = \frac{15}{64}$

b) $P(\text{Red and green in any order}) = \frac{3}{8} * \frac{5}{8} + \frac{5}{8} * \frac{3}{8} = \frac{15}{32}$

c) $P(\text{both of the same colors}) = \frac{3}{8} * \frac{3}{8} + \frac{5}{8} * \frac{5}{8} = \frac{17}{32}$

2. Independent events (multiplication law)

If probability of event B is not affected by the occurrence of event A, events A and B are said to be independent and $\boldsymbol{P(A \cap B) = P(A) * P(B)}$**.**

This rule is the simplest form of the **multiplication law of probability**.

Example 1.

A die is thrown twice. Find the probability of obtaining a 4 on the first throw and an odd number on the second throw.

Solution

$S = \{1,2,3,4,5,6\}$

Let A be the event: "a 4 is obtained on the first throw", then A= {4}

$$P(A) = \frac{n}{N} = \frac{1}{6}$$

Let B be the event: "an odd number is obtained on the second throw".

$B = \{1,3,5\}$

Since the result on the second throw is not affected by the result on the first throw, A and B are independent events.

There are 3 odd numbers, then

$P(B) = \frac{n}{N} = \frac{3}{6} = \frac{1}{2}$

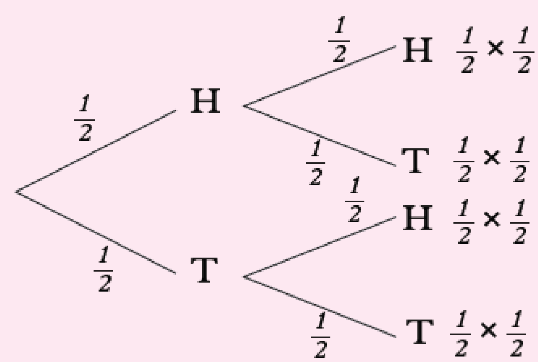
Therefore,

$P(A \cap B) = P(A) * P(B) = \frac{1}{6} * \frac{1}{2} = \frac{1}{12}$

Example 2.

A coin is tossed twice. What is the probability of getting a tail in both tosses?

Solution



The probability of getting two tails is

P (T and T) = P (T) × P (T)

$= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

Example 3.

Three different machines in a factory have different probabilities of breaking down during a shift as shown in table below:

Machine	*Probability of breaking*
A	$\frac{4}{15}$
B	$\frac{3}{10}$
C	$\frac{2}{11}$

Find:

a. The probability that all machines will break down during one shift.

b. The probability that none of the machines will break down in a particular shift.

Solution

(a) P (A and B and C breaking) = P (A breaking) × P (B breaking) × P(C breaking)
P(A and B and C breaking)

$$= \frac{4}{15} \times \frac{3}{10} \times \frac{2}{11}$$

$$= \frac{24}{1650} = \frac{4}{275}$$

(b) P(none of machines A, B and C break) = P(A and B and C do not break)
= P (A not breaking) × P (B not breaking) × P(C not breaking)

P (A does not break down)
= 1 – P(A breaks down)

$$= 1 - \frac{4}{15} = \frac{11}{15}$$

P (B does not break down)
= 1 – P(B breaks down)

$$= 1 - \frac{3}{10} = \frac{7}{10}$$

P(C does not break down)
= 1 – P(C breaks down)

$$= 1 - \frac{2}{11} = \frac{9}{11}$$

Hence P (none of machines A, Band C break) $= \frac{11}{15} \times \frac{7}{10} \times \frac{9}{11} = \frac{693}{1650} = \frac{21}{50}$

Example 4.

A factory runs two machines. The first machine operates for 80% of the time while the second machine operates for 60% of the time and at least one machine operates for 92% of the time. Do these two machines operate independently?

Solution

Let the first machine be M_1 and the second machine be M_2.

$P(M_1) = 80\% = 0.8$, $P(M_2) = 60\% = 0.6$ and $(M_1 \cup M_2) = 92\% = 0.92$.

Now, $P(M_1 \cup M_2) = P(M_1) + P(M_2) - P(M_1 \cap M_2)$

$$P(M_1 \cap M_2) = P(M_1) + P(M_2) - P(M_1 \cup M_2)$$
$$= 0.8 + 0.6 - 0.92$$
$$= 0.48 = 0.8 * 0.6$$
$$= P(M_1)P(M_2)$$

Thus, the two machines operate independently.

Exercises

A bag A contains 5 red balls and 3 green balls. The second bag B contains 4 red and 6 green balls. A bag is selected at random and two balls are picked from it one after the other without replacement.

a) Represent the information on the Tree diagram.

b) Find the probability of the following events;

i) Both balls are red from bag A.

ii) Both balls are of different colours from different bags.

3. Conditional probability

The probability of an event B given that event A has occurred is called the conditional probability of B given A and is written P(B | A) . Here P(B|A) is the probability that B occurs considering A as the sample space, and since the subset of A in which B occurs is $A \cap B$, then

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}.$$

From this result, we have general statement of the multiplication law:

$$P(A \cap B) = P(A) \times P(B \mid A).$$

If A and B are independent, then the probability of B is not affected by the occurrence of A and so P(B | A) = P(B) giving

$$P(A \cap B) = P(A) \times P(B)$$

Example1.

A die is tossed. Find the probability that the number obtained is a 5 given that the number is greater than 3.

Solution

Let A be the event that the number is a 5 and B the event that the number is greater than 3. Then $A \cap B = \{5\}$ and and $B = \{4, 5, 6\}$. We require

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{6} \times \frac{6}{3} = \frac{1}{3}$$

Example 2.

A coin is tossed twice in succession. Let A be the event that the first toss is heads and let B be the event that the second toss is heads. Find:

(a) P(A), (b) P(B), (c) $P(B \cap A)$, (d) $P(B|A)$.

Solution

(a) $P(A) = \frac{1}{2}$ (b) $P(B) = \frac{1}{2}$

(c) $P(B \cap A) = P(HH) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

(d) $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} \times \frac{2}{1} = \frac{2}{4} = \frac{1}{2}$

Example 3.

At a middle school, 18% of all students play football and basketball, and 32% of all students play football. What is the probability that a student who plays football also plays basketball?

Solution

Let A be a set of students who play football and B a set of students who play basketball; then the set of students who play both games is $A \cap B$. We have P(A) = 32% = 0.32 ,

$P(A \cap B)$ =18% = 0.18. We need the probability of B known that A has occurred.

Therefore,

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$
$$= \frac{0.18}{0.32}$$
$$= 0.5625$$
$$= 56\%$$

Exercises

1. A die is tossed. Find the probability that the number obtained is a 4 given that the number is greater than 2.
2. Suppose a study of speeding violations and drivers who use car phones produced the following fictional data:

	Speeding violation in the last year	No speeding violation in the last year	Total
Car phone user	**25**	**280**	**305**
Not a car phone user	**45**	**405**	**450**
Total	**70**	**685**	**755**

Calculate the following probabilities using the table:

a) P(person is a car phone user).

b) P(person had no violation in the last year).

c) P(person had no violation in the last year AND was a car phone user).

d) P(person is a car phone user OR person had no violation in the last year).

e) P(person is a car phone user GIVEN person had a violation in the last year).

f) P(person had no violation last year GIVEN person was not a car phone user).