

NEIGHBORHOOD SYSTEMS: A Qualitative Theory for Fuzzy and Rough Sets

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Abstract

The theory of neighborhood systems is abstracted from the geometric notion of "near" or "negligible distances." It is a "new" theory of the classical concept of neighborhood systems within the context of advanced computing. By definition neighborhood systems include both rough sets and topological spaces as special cases. The deeper and more interesting part is in its interactions with fuzzy sets: Intuitively, qualitative fuzzy sets should be characterized by "elastic" membership functions that can tolerate "a small amount of continuous stretching with limited number of broken points." Based on neighborhood systems we develop a theory for such qualitative fuzzy sets. As illustrations fuzzy inferences and Lyapunov stability are discussed.

Keywords: membership function, neighborhood system, qualitative fuzzy set, rough set, topology

Glossary

Membership Function : A membership function (MF) is a curve that defines how each point in the input space is mapped to a membership value (or degree of membership) between 0 and 1.

Neighborhood System : A neighborhood system assigns each object a (possibly empty, finite, or infinite) family of non-empty subsets. Such subsets, called neighborhoods, represent the semantics of "near."

Qualitative Fuzzy Set : Each qualitative fuzzy set is represented by a basic (binary) neighborhood of the space of membership functions. Precise and exact grades of memberships play no role in such qualitative fuzzy sets.

Rough Set : A rough set is a formal approximation of a crisp set (i.e., conventional set) in terms of a pair of sets which give the lower and upper approximation of the original set.

Topology : General topology is the branch of topology which studies the properties of topological spaces and structures that are defined on them: open and closed sets, interior and closure, neighborhood and closeness, etc.

1 Basic Overview

First an overview of information learned will be discussed and these things are mentioned in the report below. An equivalence relation gives a partition and this can become the relational database. Binary relations are topological partitions which give center set and has a coarsening structure. In Rough Set we look at the data and see what the data means; data mining is a subset of rough set theory. An unknown concept can be defined using basic knowledge (as known as a rough set); upper and lower approximations on equivalence relations or binary granulation could be used to describe such an unknown concept.

2 Introduction

Uncertainty processing is a fact of life. It is an important part of daily routines. Various theories have been proposed. Two notable theories related to our interests are Lotfi A. Zadehs fuzzy set theory [26] and Zdzislaw Pawlaks rough set theory [20]. In this paper, we discuss these notions from a geometric point of view, namely, the theory of neighborhood systems. Roughly,

a neighborhood system assigns each object a (possibly empty, finite, or infinite) family of non-empty subsets. Such subsets, called neighborhoods, represent the semantics of "near." Using neighborhoods, one can define open sets, closed sets, and hence the interior and closure of any subset [8], [21]. If we take equivalence classes as neighborhoods, the lower and upper approximations are precisely the interior and closure respectively. From this point of view rough set theory is a special form of neighborhood system theory; it was called category [3]. In fact generalized rough sets based on modal logic are all special forms of neighborhood systems [25]. Formally neighborhoods play the most fundamental role in mathematical analysis. Informally, it is a common and intuitive notion: It is in databases [5], [18], in rough sets [20], in logic [2], in texts of genetic algorithms [6], and many others. In the context of advanced computing a systematic study of neighborhood systems was initiated by this author and his students. It was motivated from database retrieval and data mining [9], [10], [11], [12], [3], [24], [13].

In this paper we also will use neighborhood systems to formulate fuzzy sets. A few words on our philosophical view may be in order: Zadehs full agenda is a grand revolution in mathematics and science. It not only fuzzifies mathematics and science, it may fuzzify their foundations, methodologies, and philosophies. We, however, adopt a conservative strategy; we only fuzzify mathematics, but keep its foundation unfuzzified. In other words, though fuzzy theory is designed to express fuzzy and imprecise concepts, the theory itself is crispy and precise – see figure below. At this point in time our strategy probably is "correct" and effective. With unfuzzified foundations, we can apply both fuzzy and traditional mathematics to real world problems simultaneously, and find consistent solutions.

This paper aims towards applications in fuzzy controllers, so we examine the theory from topological point of view; we emphasize "continuous stretching." For full qualitative fuzzy set theory, we also need to examine it from measure (stochastic) theoretical point of view; in this paper we only touch on it slightly. For the full study we will need to develop a measure theory or theories for neighborhood systems. Interestingly, we find that neighborhood systems are the most natural data structures for belief functions; the study will be report in the near future. Our ultimate goal is an axiomatic fuzzy set theory.

Organization and Acknowledgment

The paper is divided into three parts, neighborhood systems, qualitative fuzzy sets, and applications. The numbering of sections is independent of the partitions. Definitions, theorems, propositions, and corollary are enumerated as one group within each section. This author would like to express his deepest thank to Professor Zadeh for his warm invitation to join the Berkeley Initiative in Soft Computing Group (BISC). His thanks also go to Dr. Martin Wildberger at EPRI, Electric Power Research Institute, for his generous sponsorship.

My Insight:

Lotfi Asker Zadeh is a mathematician and computer scientist who proposed, in his theory of fuzzy logic, the making of the membership function that operates over the range of real numbers $[0,1]$. He also proposed new operations of logic and showed that fuzzy logic was a generalization of the classical logic [28]. A rough set, on the other hand, is a formal approximation of a crisp set (i.e., conventional set) in terms of a pair of sets which give the lower and upper approximation of the original set. The lower and upper approximation sets themselves are crisp sets in the standard version of rough set theory [29]. A crisp set can be thought of as any collection of distinct things considered as a whole. The objects of a set are called elements or members. The elements of a set can be anything from numbers to people, or letter of the alphabet to other sets, and so on.

General topology is the branch of topology which studies the properties of topological spaces and structures that are defined on them: open and closed sets, interior and closure, neighborhood and closeness, etc. In topology, a set U is called open if you can "wiggle" or "change" any point x in U by a small amount in any direction and still be inside U . In other words, if x is surrounded only by elements of U ; it can't be on the edge of U . The concept of open sets can be formalized in various degrees of generality. A closed set, on the other hand, contains its own boundary. So, if you are "outside" a closed set and you "wiggle" a little bit, you will stay outside the set. The interior of a set S consists of all the points which are intuitively "not on the edge of S ". A point which is in the interior of S is an interior point of S . The closure of the set S consists of all the points which are intuitively "close to S ". A point which is in the closure of S is a point of closure of S . Fuzzy logic is

a superset of conventional (Boolean) logic that has been extended to handle the concept of partial truth - where the truth values are between "completely true" and "completely false." The importance of fuzzy logic derives from the fact that most modes of human reasoning and especially common sense reasoning are approximate in nature. So the essential characteristics of fuzzy logic as founded by, as we mentioned earlier, Zadeh and Lofti are as follows: [30]

1. In fuzzy logic, exact reasoning is viewed as a limiting case of approximate reasoning.
2. In fuzzy logic everything is a matter of degree.
3. Any logical system can be fuzzified
4. In fuzzy logic, knowledge is interpreted as a collection of elastic or, equivalently, fuzzy constraint on a collection of variables
5. Inference is viewed as a process of propagation of elastic constraints

Fuzzy sets are an extension of classical set theory and are used in fuzzy logic. In classical set theory the membership of elements in relation to a set is assessed in binary terms according to a crisp condition - an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in relation to a set; this is described with the aid of a membership function $\mu \rightarrow [0, 1]$. Fuzzy sets are an extension of classical set theory since, for a certain universe, a membership function may act as an indicator function, mapping all elements to either 1 or 0, as in the classical notion. [31] Fuzzy logic is derived from fuzzy set theory dealing with reasoning that is approximate rather than precisely deduced from classical predicate logic. It can be thought of as the application side of fuzzy set theory dealing with well thought out real world expert values for a complex problem.

A membership function (MF) is a curve that defines how each point in the input space is mapped to a membership value (or degree of membership) between 0 and 1. The input space is sometimes referred to as the universe of discourse, a fancy name for a simple concept. One of the most commonly used examples of a fuzzy set is the set of tall people. In this case the universe of discourse is all potential heights, say from 3 feet to 9 feet, and the word tall would correspond to a curve that defines the degree to which any person

is tall. If the set of tall people is given the well-defined (crisp) boundary of a classical set, we might say all people taller than 6 feet are officially considered tall. But such a distinction is clearly absurd. It may make sense to consider the set of all real numbers greater than 6 because numbers belong on an abstract plane, but when we want to talk about real people, it is unreasonable to call one person short and another one tall when they differ in height by the width of a hair. But if the kind of distinction shown is unworkable, then what is the right way to define the set of tall people? Much as with our plot of weekend days, the figure following shows a smoothly varying curve that passes from not-tall to tall. The output-axis is a number known as the membership value between 0 and 1. The curve is known as a membership function and is often given the designation of μ . This curve defines the transition from not tall to tall. Both people are tall to some degree, but one is significantly less tall than the other. Subjective interpretations and appropriate units are built right into fuzzy sets. If I say "She's tall," the membership function tall should already take into account whether I'm referring to a six-year-old or a grown woman. Similarly, the units are included in the curve. Certainly it makes no sense to say "Is she tall in inches or in meters?" [33].

Part I. A Theory of Neighborhood Systems

3 Neighborhood Systems

In this section, we will give a short exposition on the theory of neighborhood systems in the context of advanced computing.

3.1 The Semantics of "Near"

The notion of near is rather imprecise. Let us examine the following two examples.

Example 1 Is Santa Monica "near" Los Angeles?

Answers could vary. For local residents, answers are often "yes." For visitors who have no cars, answers may be "no."

Example 2 Is 1.73 "near" $\sqrt{3}$?

Again answers vary. Intrinsically "near" is a subjective judgment. One might wonder whether there is a scientific theory for such subjective judgments? Mathematicians have offered a nice solution. They simply include the contexts into the formalism: Given the radius of an acceptable error $\epsilon = 1/100$, is 1.73 "near" ? Similarly, if a neighborhood system has been assigned to each city in Los Angeles area, then we have a definite answer for Example 1. So a proper formulation for such a question is:

Given a neighborhood system (the context), is p near q ?

Example 3. A Context Free Answer

Is the sequence $1, 1/2, 1/3, \dots, 1/n, \dots$ "near" zero?

The answer is a context free "yes." In other words, it is a "yes" for all contexts: For any given context, $\epsilon > 0$, there is a number $N = [1/\epsilon] + 1$, such that, for all $n > N$, $1/n$ is "near" zero, where $[1/\epsilon]$ denotes the biggest integer $\leq 1/\epsilon$. For readers who familiar with the standard (ϵ, δ) -definition of limit can spot the origin of neighborhood systems. Such a contexts free answer is precisely the classical notion of limits, $\lim_{n \rightarrow \infty} 1/n = 0$. In our language, we may say that limit is the context free answers of "near." Perhaps we should also point out here that there is no context free answers for the question whether two points (any finite number of points) are near or not.

Roughly speaking mathematical analysis is started from examining infinitesimal which is, in our view, one form of uncertainty. Such a notion can be traced back to Archimedes and Eudoxus. Cauchy's (ϵ, δ) -definition provided a "definite form" answer to such infinitesimal uncertainty; A. Robinson provided a direct answer (non-standard analysis). One can viewed neighborhood systems is an adoption of (ϵ, δ) -method to the finite universe.

My Insight:

In topology, the neighborhood system $V(x)$ for a point x is the collection of all neighborhoods for the point x . A neighborhood is one of the basic concepts in topological space. Intuitively speaking, a neighborhood of a point is a set containing the point where you can "wiggle" or "move" the point a bit without leaving the set.

3.2 Fundamentals of Neighborhood Systems

Let U be the universe of discourse and p be an object in U .

1. A neighborhood, denoted by $N(p)$, or simply N , of p is a non-empty subset of U , which may or may not contain the object p . A neighborhood system of an object p , denoted by $NS(p)$, is a maximal family of neighborhoods of p . If p has no (non-empty) neighborhood, then $NS(p)$ is an empty family; in this case, we simply say that p has no neighborhood.
2. A neighborhood system of U , denoted by $NS(U)$ is the collection of $NS(p)$ for all p in U . For simplicity a set U together with $NS(U)$ is called a neighborhood system space (NS-space) or simply neighborhood system.
3. A subset X of U is open if for every object p in X , there is a neighborhood $N(p) \subseteq X$. A subset X is closed if its complement is open.
4. $NS(p)$ and $NS(U)$ are open if every neighborhood is open. $NS(U)$ is topological, if $NS(U)$ are open and U is the usual topological space [8]. In such a case both $NS(U)$ and the collection of open sets are called topology.
5. An object p is a limit point of a set E , if every neighborhood of p contains a point of E other than p . The set of all limit points of E is called derived set. E together with its derived set is a closed set.
6. $NS(U)$ is discrete, if $NS(U) = P(U)$, the power set.
7. $NS(U)$ is indiscrete, if $NS(U) = \{U\}$.
8. $NS(U)$ is serial, if $\forall p, N(p)$ is non-empty (called Frechet (V) space in [21]).
9. $NS(U)$ is reflexive, if $\forall p, p \in N(p)$.
10. $NS(U)$ is symmetric, if $\forall p \forall q, q \in N(p) \Rightarrow p \in N(q)$.
11. $NS(U)$ is transitive, if $\forall p \forall q \forall r, q \in N(p) \text{ and } r \in N(q) \Rightarrow r \in N(p)$.
12. $NS(U)$ is Euclidean, if $q \in N(p)$, and $r \in N(p) \Rightarrow r \in N(q)$.

Example 2.2

1. Let $U = \{x, y\}$ for example 1 to 5.
 $NS(x) = \{x\}, \{x, y\};$ $NS(y) = \phi$; ϕ is the empty set
 Open sets: $\phi, \{x\}, \{x, y\};$ $\{y\}$ is not open
 It is not a topological space.
2. $NS(x) = \{x\}, \{x, y\};$ $NS(y) = \{x, y\}.$
 Open sets: $\phi, \{x\}, \{x, y\};$ $\{y\}$ is not open.
 It is a topological space.
3. $NS(x) = \{x\}, \{x, y\};$ $NS(y) = \{y\}, \{x, y\};$
 Open sets: $\phi, \{x\}, \{y\}; \{x, y\}.$
 It is a discrete NS-space.
4. $NS(x) = \{x, y\};$ $NS(y) = \{x, y\};$
 Open sets: $\phi, \{x, y\};$ $\{x\}$ and $\{y\}$ are not open.
 It is an indiscrete NS-space.
5. $NS(x) = \{x\};$ $NS(y) = \{y\};$
 Open sets: $\phi, \{x\}, \{y\}, \{x, y\}.$
 It is a topological space.
6. Let U be the Euclidean plan, we assign each point p a family of neighborhoods by open solid disks: $\forall p$ and $\forall \epsilon$,

$$N(p, \epsilon) = \{x : d(x, p) < \epsilon \},$$

where d is the usual Euclidean distance. The family of such $N(p, \epsilon)$'s, $\forall p$ and $\forall \epsilon$, are the neighborhood systems of the usual topology of Euclidean plan. A topological space is a neighborhood system space, but not the converse; see next example.

7. Next, we assign each point p in the Euclidean plan U a unique neighborhood, namely, a open solid disk of radius 2:

$$N(p, 2) = \{x : d(x, p) < 2 \}.$$

Such an assignment gives U a neighborhood system, but not a topological space.

8. Let U be the point sin the Euclidean plan with integral coordinates ≤ 1 , that is the following points:

The neighborhoods are:

$$N(P) = \{A, B, C, D, E, F, G, H\}$$

$$N(A) = \{A, B, C, P, G, H\}$$

$$N(B) = \{A, B, C, P\}$$

$$N(C) = \{A, B, C, D, P, E\}$$

$$N(D) = \{C, D, P, E\}$$

$$N(E) = \{A, B, C, P, G, H\}$$

$$N(F) = \{P, E, F, G\}$$

$$N(G) = \{A, P, E, F, G, H\}$$

$$N(H) = \{A, P, G, H\}$$

Definition 2.3. Let X be a subset of U .

$$I[X] = \{ p : \text{there is a } N(p) \subseteq X \} = \text{interior of } X,$$

i.e., $I[X]$ is the largest open set contained in X ,

$$C[X] = \left\{ p : \forall N(p), X \cap N(p) \neq \emptyset \right\} = \text{closure of } X$$

i.e., $C[X]$ os the smallest closed set contains X .

$I[X]$ and $C[X]$ are precisely the lower and upper approximation in rough set theory; see below,

We will collect few simple properties of topology and neighborhood systems, and point out their differences.

Proposition 2.4.

1. A topological space is a neighborhood system space (NS-space), but not the converse.
2. Intersections and finite unions of closed sets are closed in NS-spaces.

3. In topological spaces, unions and finite intersections of open sets are open. In NS-spaces, unions is open, but intersections may not be open.
4. In a topological space $NS(U)$ determines and is determined by the collection of all open sets. This property may or may not be true for neighborhood systems.
5. A neighborhood system is a subbase (see below) of a topology. An open set in NS-space is also open in this topology, the converse may not be true.

A family B of subsets is a base for a topology, if every open set is a union of members of B . A family S of subsets is a subbase, if the finite intersections of members of S is a base. Note that the union and intersection of empty family is the whole space and empty set respectively. We can take a neighborhood system as a subbase of some topology. Note that two distinct neighborhood systems may give the same topology.

My Insight:

The term "universe of discourse" generally refers to the entire set of terms used in a specific discourse, i.e. the family of linguistic or semantic terms that are to any one area of interest. Topology is the branch of mathematics; mainly an extension of geometry. Topology builds on set theory, considering both sets of points and families of sets. In the mathematical definition, we let X be any set and let T be a family of subsets of X . Then T is a **topology** on X if:

1. Both the empty set and X are elements of T .
2. Any union of elements of T is an element of T .
3. Any intersection of finitely many elements of T is an element of T .

If T is topology on X , then X together with T is called a **topological space**. Now to describe a topological space, we use words like open and close. A set in T is called **open**. The complement of a set in T . If neither a set nor its complement is in T , then the set is neither open nor closed. A function or map from one topological space to another is called continuous if the inverse image of any open set is open. If the function maps the real

numbers to the real numbers, then this definition of continuous is equivalent to the definition of continuous in calculus. If a continuous function is one-to-one and onto and if the inverse of the function is also continuous, then the function is called a homeomorphism and the domain of the function is said to be homeomorphic to the range. Another way of saying this is that the function has a natural extension to the topology. If two spaces are homeomorphic, they have identical topological properties, and are considered to be topologically the same. The square and the circle are homeomorphic, as are the coffee cup and the doughnut. But the circle is not homeomorphic to the doughnut. [32]

A binary neighborhood system (BNS) is: To each object $p \in V$, we associate a crisp subset $Bp \subseteq U$. In notation,

$$B : V \rightarrow 2U : p \rightarrow Bp$$

The map B or the collection Bp is referred to as a binary neighborhood system for V on U . We have called Bp a basic neighborhood and B or Bp a basic neighborhood system respectively.

A binary relation (BR) is: Let $R \subseteq V \times U$ be a binary relation. For each object $p \in V$, we associate a binary subset $Np \subseteq U$, where

$$Np = \{u \mid pRu\}$$

that consists of all elements u that are related to p by R .

1. Proposition: Given a binary neighborhood system Bp for V on U , there is a binary relation $R \subseteq V \times U$ such that

$$Np = Bp$$
and vice versa.
2. If the binary relation R is an equivalence relation E , then the binary set Np is the equivalence class $[p]E$. In rough set theory an equivalence class is called an elementary set. So the binary neighborhood Np may also be called an elementary neighborhood of p .

A subset X is a definable set if it is a union of equivalence classes. So a subset X is called a definable neighborhood, if X is a union of elementary neighborhoods. If the definable neighborhood X contains elementary neighborhood Bp of p , it is a definable neighborhood of p .

A fuzzy binary neighborhood (FBNS) is: To each object $p \in V$, we associate a fuzzy subset (a clump), denoted by FBp . In other words, we have a map

$$NB : V \rightarrow FZ(U) : p \rightarrow FBp,$$

where $FZ(U)$ means all fuzzy sets on U . FBp is called an fuzzy elementary neighborhood or fuzzy binary neighborhood, FB a fuzzy binary neighborhood system.

A fuzzy binary relation (FBR) is: Let I be the unit interval $[0, 1]$. Let FR be a fuzzy binary relation whose membership function, denoted by the notation

$$FR : V \times U \rightarrow I : (p, u) \rightarrow r.$$

To each $p \in V$, we associate a fuzzy set (aclump) FNp whose membership function is $FNp : U \rightarrow I$, defined by

$$FNp(u) = FR((p, u)).$$

Proposition: Given a fuzzy binary neighborhood system FBp for V on U , there is a fuzzy binary relation FR such that

$$FNP = FBP$$

and vice versa. So from now we use FB as a fuzzy binary neighborhood system as well as a fuzzy binary relation FR .

A crisp/fuzzy neighborhood system (NS/FNS) is: To each object $p \in V$, we associate a (empty, finite or infinite) family of clumps (crisp/fuzzy subsets). The mathematical system defined by these families of clumps is called crisp/ fuzzy neighborhood system or simply neighborhood system, and these clumps associated to p are called fundamental neighborhoods of p .

3.3 Continuous Mappings and Pullbacks

Let U and V be two universes. Let $F: U \rightarrow V$ be a mapping that assigns, $\forall x \in U$, a unique object $y \in V$.

Definition 2.5. Let $NS(U)$ and $NS(V)$ be two neighborhood systems. F is said to be continuous if $\forall N(y) \exists N(x)$ such that $F(N(x)) \subseteq N(y)$. F is called a homeomorphism, if both F and its inverse F^{-1} are continuous [8].

Let O be any subset of V . We shall write

$$F^{-1}(O) = \{u : F(u) \in O\}$$

i.e., $F^{-1}(O)$ is the set of all those objects of U , which are mapped into O by F . The set $F^{-1}(O)$ is called the complete inverse image (under F) of the set O , we may simply call it the pullback of O (under F).

Let us recall some results from general topology [8]. Let $T(V)$ be the topology of V , namely, the family of all open sets in V . We will write

$T(U) = \{ F^{-1}(O) : O \in T(V) \}$ varies through all open sets in V

i.e., $T(U)$ is the set of all inverse image of all the open sets in V . It is well known that $T(U)$ is the smallest topology such that F is continuous [8]. We will call $T(U)$ the pullback of $T(V)$ under F , or simply pullback of F .

Proposition 2.6. Let U be the universe and $\{V_m\}$ be a family of topological spaces. Let $M = \{F_m : U \rightarrow V_m\}$ be a family of functions. There is a unique minimal topology $T(U)$ such that every F_m is continuous.

Such a topology will be called the minimal M -topology. Note that M -topology is not just the union of all the pullback of F_m , it is the smallest topology generated by all the pullbacks.

We have similar results in neighborhood system. Let $F(q) = p$. We will take $F^{-1}(N(p))$ as a neighborhood of q . Let $N(p)$ varies through $NS(p)$, then we have a neighborhood system at q . Let q varies through all objects in U , we have the neighborhood system,

$$NS(U) = \{ N(q) = F^{-1}(N) : F(q) = p, N(p) \in NS(V), q \in U \}.$$

We will call the procedure of forming $NS(q)$ and $NS(U)$ through the mapping F , pullback of $NS(V)$ under F , or simply pullback of F . We have an analogous results for neighborhood systems.

Proposition 2.7. Let U be the universe and $\{V_m\}$ be a family of neighborhood systems. Let $M = \{F_m\}$ be a family of functions. There is a unique minimal neighborhood system $NS(U)$ such that every F_m is continuous.

4 Generalized Rough Sets as Neighborhood Systems

Rough set theory can be viewed as an approximation theory using equivalence relations as neighborhoods. It is immediate that one can generalize the theory to any binary relations. In this section, we discuss neighborhood systems that are derived from binary relations.

My Insight: A rough set is a formal approximation of a crisp set in terms of a pair of sets which give the lower and upper approximation of the original set. The lower and upper approximation sets themselves are crisp

sets in the standard version of rough set theory, but in other variations, the approximating sets may be fuzzy sets as well [30].

4.1 Binary Relations and Basic Neighborhoods

A *minimal neighborhood* of p , denoted by $MN(p)$, is a minimal member of $NS(p)$ in the sense that $MN(p)$ contains no member of $N(p)$ as proper subsets. Note that in general such $MN(p)$ may or may not exist. The maximal family of all $MN(p)$ at p will be denoted by $MNS(p)$. The family of $MNS(p)$ for all p will be denoted by $MNS(U)$. Let $n(p)$ be the number of (distinct) $MNS(p)$'s at p . If $n(p) = n$ is a constant integer for all p , $MNS(U)$ is an n -minimal neighborhood system, and denoted by $n\text{-}MNS(U)$.

Let R be a binary relation defined on U , then

$$B(p) = \{x : pRx\}$$

is a neighborhood of p , called a *basic (binary) neighborhood*. Let $BS(U)$, called a *basic neighborhood system*, be the collection of all $B(p)$; note that $B(p)$ can be empty for some p . Note that $BS(U)$ defines and is defined by a binary relation. If R is serial (that is, $B(p)$ is non-empty for all p), then $BS(U)$ is an 1-minimal neighborhood system; this is the most interested case. From the implementation point of view, we can rephrase a basic (binary) neighborhood system as follows:

Definition 3.1. A basic neighborhood system $BS(U)$ is a data structure that assigns each datum a set (could be empty) of data.

Proposition 3.2.

1. A minimal member of $NS(p)$ may or may not exist. Even for a non-empty neighborhood system $NS(p)$, $MNS(p)$ could be empty. The neighborhood system of a real number has no minimal neighborhood; $NS(p)$ is an infinite set at for every real number p .
2. A binary relation on U defines and is defined by a basic neighborhood system $BS(U)$.

In Definition 2.1, various neighborhood systems are defined by their local properties. We summarize their relationships in the following table:

An S_4 -neighborhood system is a topological space, however, the converse is not necessary true. Similarly the n -graded binary relations [25] correspond to n -minimal neighborhood systems; we skip the details.

Binary Relation	Implications	Basic (Binary) Neighborhood
Serial	\leftrightarrow	serial
Reflexive	\leftrightarrow	reflexive
Symmetric	\leftrightarrow	symmetric
Transitive	\leftrightarrow	transitive
Euclidean	\leftrightarrow	Euclidean
reflexive, symmetric	\leftrightarrow	Brouwer's, B
reflexive, transitive	\leftrightarrow	S_4 , (see note)
Equivalence	\leftrightarrow	clopen topology, S_5

4.2 Rough Sets and Neighborhoods

Let R be an equivalence relation on U . It partitions U into equivalence classes. The partition, by abuse of notation, is denoted by R again. The pair (U, R) is called approximation space which is the universe of discourse in rough set theory. The equivalence class containing x will be denoted by $[x]$. Rough setters view R as an indiscernibility relation, that is, the elements within the equivalence class are indistinguishable by the available information. So the approximation space is a multi-set in which an element x repeats itself as many times as the number of elements in $[x]$. Taking this point of view, a subset X can not be adequately described. It can only be approximated. As usual, let ϕ be the empty set.

Definition 3.1. Let X be subset of the approximation space (U, R) .

$$\underline{R}(X) = \{x : [x] \subseteq X\} = \text{the lower approximation,}$$

$$\overline{R}(X) = \{x : [x] \cap X \neq \phi\} = \text{the upper approximation.}$$

Definition 3.2. The pair $(\underline{R}(X), \overline{R}(X))$ is called a rough set.

We should caution the readers that this is a technical definition of rough sets defined by Pawlak [20]. However, rough setters often use "rough set" as any subset X in the approximation space, where $\underline{R}(X)$ and $\overline{R}(X)$ are defined. We also would like to note that the pair $(\underline{R}(X), \overline{R}(X))$ as pair of sets (no knowledge of R) does not characterize rough set. In general, for a subset X there is possibly a subset Y such that

$$\underline{R}(X) \subseteq Y \subseteq \overline{R}(X), \text{ but } \underline{R}(X) \neq \underline{R}(Y) \text{ or/and } \overline{R}(X) \neq \overline{R}(Y).$$

As we have remarked earlier that the partition can be regarded a neighborhood system on U . That is, the equivalence class $[x]$ is a neighborhood of

$y, \forall y \exists [x]$. We will call such a $NS(U)$ clopen neighborhood system, S_5 -neighborhood system or Palwak topology [15].

Proposition 3.3 Let (U, R) be an approximation space. $NS(U)$ is a clopen topology.

Part II A Qualitative Theory of Fuzzy Sets

5 Real World Fuzzy Sets

The goal of this section is to investigate what a real world fuzzy should be.

5.1 Some Critical Questions

We will start with few fundamental questions:

Question 1 Could there be more than one membership functions for a given fuzzy set?

Question 2 (a) Do fuzzy set operations exit, if there are more than one membership functions for a fuzzy set?
(b) Do fuzzy set operations exit in the traditional sense(derived from logical connective)?

Question 3 Could a membership function represent more than one fuzzy sets?

Discussion of Question 1:

The status of answers is somewhat inconsistent in the literature. In his book [7, pp. 4], Kandel gave several membership functions for a real world fuzzy set which consists of real numbers that are much greater than 1. Similarly Zimmermann gave a definition for fuzzy numbers by two conditions. So a fuzzy number may have several membership functions [27, pp. 57].

On the other hand, Zimmermann also quoted that "a fuzzy set is represented solely by stating its membership functions" [19], [27, p12]. Moreover,

s- and t-norms are defined with implicit assumption that a membership function is unique for each fuzzy set; see the arguments in question 2.

Proposal 1. It seems that there are different types of fuzzy sets, so a taxonomy is needed. For qualitative fuzzy sets membership functions are not necessarily unique.

Discussion of Question 2:

Let us consider the operation, intersection, and assume that it is defined by an arbitrary but a fixed t-norm $t(-, -)$. Let $(U, FK) = (U, GK)$ be a fuzzy set defined by two distinct membership functions FK and GK. Let FH and GH be another pair that define the same fuzzy set $(U, FH) = (U, GH)$. The intersection, \cap , can be defined in two ways:

- (1) $(U, FK) \cap (U, FH) = (U, FZ)$,
- (2) $(U, GK) \cap (U, GH) = (U, GZ)$,

where $FZ = t(FK, FH)$ and $GZ = t(GK, GH)$. If the intersection *does exist*, we must show that the "two" intersections are *indeed* the same, i.e.,

$$(U, FZ) = (U, GZ).$$

If such an equality is proved, the intersection is said to be well-defined. However no contemporary authors have shown such a well-defined-ness. So from the literature, we may conclude that

- (1) Either a fuzzy set is defined by a unique membership function,
- (2) Or fuzzy set operations have not been established.

Even in quantitative theory (one membership function for one fuzzy set) [14], we may also want to adopt (2) for different reasons. It might be helpful to point out that rough sets have no set theoretical operations either. So we have

Proposal 2

- (a) Fuzzy set operations (for qualitative theory) may not exist,
- (b) Fuzzy set operations (even for quantitative theory) may not be from logical connectives.

Discussion of Question 3:

Let MF be the space of all *membership functions*. Qualitative fuzzy sets

are defined by subsets of MF. The natural question is: Do these subsets overlap? Or equivalently do they form a crispy partition on MF? There is no explicit answer to this question in the literature. For building a beautiful mathematical theory, it would be nice to have such a partition. However a total space of membership functions is often "continuous." It is hard to believe there is a natural and crispy partition on such a "continuous" space. We have developed both theories in next section, however, the theory is natural for positive answer, not quite natural for negative answer.

Proposal 3 Both positive and negative answers are acceptable, but positive is more natural.

Further Discussions:

We would like to caution readers that positive answer has "serious" implications: Let FX be a membership function that represents both fuzzy sets A and B. Let C be another fuzzy set with a membership function FZ. Suppose $s(FX, FY)$ is the union defined by some "clever" s-norm. Naturally one may ask which of these two fuzzy sets, $A \cup B$ or $A \cup C$, does $s(FX, FY)$ represent? At current stage, there is no clear answer. At first impression such a phenomenon may sound

"ridiculous," however, after some analysis one may conclude that it is no more surprising than classical "contradiction" (two fuzzy rules are "contradictory"). For example, a crispy set is the union of singleton, so a fuzzy set is a "weighted union" of singletons

Proposal 4 Fuzzy operations (or connectives) are weighted and multi-valued.

This proposal will be in our future study.

5.2 Characteristic of Real World Fuzzy Sets

Let the membership space M be the unit interval $[0, 1]$. Let FX be a membership function,

FX: $U \rightarrow M = [0,1]$ The subset where membership function FX taking value α will be given a geometric term, α -level-curve or level-curve when α is not explicitly given:

$$[\alpha] = \{ x: FX(x) = \alpha \} = \alpha \text{ - level - curve}$$

i.e., α -cut minus strong α -cut . there are special α -level-curves: 1-level-curve and 0-level-curve. They will be called real set (core), denoted by RFX, and

complement set, denoted by CFX respectively. We will not use the term core, because it is a special and very important notion in rough set theory.

definition 4.1. The collection of all level-curves is a partition on U, called grade partition.

The main goal of this section is to formalize the characteristics of the following intuitive statement:

An "elastic" membership function can tolerate a small amount of continuous stretching with limited number of broken points.

We need a family of membership functions to express the stretching of a fuzzy set will be called admissible functions or admissible family of membership functions. Following Kandel, membership functions of the same membership function. That is, a real world fuzzy set is defined by a family of number? To specify these contexts mathematically, we will assume two positive functions. We still need to specify two contexts. What is small? What is limited numbers ϵ and ϵ' are given. We also need a (additive or non-additive) measure μ that can estimate the "number" of points.

Now we will describe the characteristics of an admissible family of "elastic" membership functions: First let us adopt the notion of "almost everywhere" from mathematical analysis

We say that two functions $f(p)$ and $g(p)$ are equal almost everywhere iff $\mu(E) < \epsilon'$, where $E = \{ p : f(p) \neq g(p) \}$. In other words $f(p)$ and $g(p)$ are equal for almost all p , and the measure of the set of points where these two functions are not equal is less than ϵ' . All the constraints, such as "less than" and "equal to," stated below will all be understood as "less than almost everywhere" and "equal to almost everywhere" respectively

Characteristic (1) The grades of different memberships of the same point should be "near:" $FX(x)$ and $FY(x)$ should belong to the same basic neighborhood of a pre-chosen basic ϵ -neighborhood system of the unit interval. In other words, the distance between α -level-curve ($\alpha = FX(x)$) and α' -level-curve ($\alpha' = FY(x)$) and α' -level-curve ($\alpha' = FY(x)$) are less than the chosen ϵ .

When we stretch the elastic membership function slightly, we do keep some relationship unchanged.

Characteristic (2) The grade partition should be the same: $FX(x) = FX(y) \leftrightarrow FY(x) = FY(y)$ for all admissible function FY . In other words, if x and

y are in the same α -level-curve for FX , they should be in the same α' -level-curve for FY .

Characteristic (3) The order of the grades of memberships should be the same: $FX(x) < FX(y) \leftrightarrow FY(x) < FY(y)$ for all admissible function FY . In other words, if α -level-curve is lower than β -level-curve ($\alpha=FX(x)$, $\beta=FX(y)$), then α' -curve is lower than β' -level-curve ($\alpha'=FY(x)$, $\beta'=FY(y)$).

Finally, we also want the members who absolutely belong to (or do not belong to) the fuzzy set should stay in (or out of) the set absolutely

Characteristic (4) The set with absolute membership, called real set, should be the same: $FX(x) = 1 \leftrightarrow FY(x) = 1$ for all FY .

Characteristic (5) The set with absolute no membership, called complement set, should be maintained: $FX(x) = 0 \leftrightarrow FY(x) = 0$ for all FY .

These five "imprecise" characteristics may characterize what qualitative fuzzy sets should be. Each qualitative fuzzy set is represented by a basic (binary) neighborhood of the space of membership functions. Precise and exact grades of memberships play no role in such qualitative fuzzy sets.

5.3 Future Direction

In this paper, we focus on topological point of view, we only touch on measure theoretical point of view briefly. In order to have a complete account on measure theoretic point of view we need to develop a "finite type measure" theory; one can view neighborhood systems as a "finite type topological" theory.

6 Topological Aspects of Qualitative Fuzzy Sets

Based on the characteristics of the "elastic" membership functions, we will try to define qualitative fuzzy sets formally. For simplicity, we will focus on topological aspects in this section. However, if one interprets constraints properly (e.g., equal is understood as equal almost everywhere) all the results are valid for general qualitative fuzzy sets.

6.1 FUNDAMENTALS

Let ϵ be a small number selected. A sub-interval in $[0, 1]$ is said to be an ϵ -neighborhood, if the length of the sub-interval is 2ϵ (ϵ is the radius, 2ϵ is the diameter). A mapping

$h: [0, 1] \rightarrow [0, 1]$ is called an ϵ -homeomorphism, if (1) h is a homeomorphism on the closed unit interval (Definition 2.5), and (2) $\forall p$ in $[0, 1]$, $|p - h(p)| < 2\epsilon$. Let A be a fixed point set that contains at least 0, 1. h is called ϵ -homeomorphism relative to A , if (a) h is an ϵ -homeomorphism and (b) h keeps the set A pointwise fixed, that is, $p = h(p)$ for all $p \in A$. In terms of neighborhood systems, h is a ϵ -homeomorphism relative to A if (1) h is a homeomorphism, (2) $p, h(p)$ is contained in a basic ϵ -neighborhood, and (3) h keeps A pointwise fixed. Intuitively, an ϵ -homeomorphism is a "light pulling" of the string from one end to the other end, and keeping A pointwise fixed, where the string is the unit interval made of elastic material.

Definition 5.1. Membership functions F_X and F_Y are ϵ -deformable iff

1. both induce the same minimal topology on U ,
2. there exists an ϵ -homeomorphism h relative to $A = \{0, 1\}$ such that $F_X(x) = h(F_Y(x))$ for all $x \in U$, or $F_X = h \bullet F_Y$.

The binary relation so defined will be called the ϵ -deformation.

Note that ϵ -homeomorphism h keeps 0 and 1 fixed, so it is an order preserving map. The binary relation, ϵ -deformation, is reflexive and symmetric, but may not be transitive. It defines a set of admissible families of membership functions; each family is called an ϵ -deformable family of membership functions. Geometrically the family is a basic neighborhood in the space of membership functions see Section 3.1

Definition 5.2. Such an ϵ -deformable family (a basic neighborhood) of membership functions defines a ϵ -qualitative fuzzy set.

For $\epsilon = 1$ and $A = \{0, 1\}$, the ϵ -deformable relation becomes an equivalence relation; it will be called 1-equivalence relation.

Theorem 5.3. In a finite universe $U = \{u_1, u_2, \dots, u_n\}$, F_X and F_Y are ϵ -deformable iff (1) both have the same grade partition, (2) $|F_X(x) - F_Y(x)| < 2\epsilon$ for all x in U , (3) The two linear orders of grade partitions are isomorphic.

Proof : Note that $FX(U) = FX(u_1), FX(u_2), \dots, FX(u_n)$ is a finite discrete set in the closed unit interval $[0, 1]$. The grade partition (Definition 4.1.) is the pullback of these finite discrete points under FX . So the condition (1) in Theorem 5.3 and (1) in Definition 5.1 are equivalent. It is obvious that (2) of Definition 5.1 implies (2) and (3) of Theorem 5.3 (A homeomorphism keeping 0.1 fixed is an order preserving map). To see the converse, by (3) of Theorem 5.3, we can have an order preserving map h between $FY(U)$ and $FX(U)$. Then we extend h linearly to the unit interval. By (2) of Theorem 5.3, $|x - h(x)| < 2\epsilon$, such an extended map h moves points within distances. So h is an ϵ -homeomorphism relative to 0, 1. We have established (2) of Definition 5.1. Q E D.

Corollary 5.4. The binary relation -deformation induces a reflexive and symmetric basic neighborhood system on the space of all membership functions.

By taking $\epsilon = 1$, the condition (2) of Theorem 5.3 disappears and sub-intervals reduce to the total interval; we have the following:

Corollary 5.5. In a finite universe $U = u_1, u_2, \dots, u_n$, FX and FY are 1-equivalent iff

1. both have the same grade partition,
2. The two linear orders of grade partitions are isomorphic.

Corollary 5.6. In a finite universe $U = u_1, u_2, \dots, u_n$, 1- qualitative fuzzy set is characterized by

1. The grade partition,
2. The linear orders of grade partitions.

6.2 None-Finite Universe

For non-finite universe, it is much more complicated. We will treat the details in different paper. We will sketch only some results here. As usual the notion of neighborhoods of objects can be generalized to neighborhoods of subsets [8]. Let U has the minimal topology. An α -cut (strong α -cut) is a closed (open) neighborhood of the real set (core). A neighborhood system N_{sis} said to be open-closed-nested chain between two disjoint closed sets A and B , if

1. N_i is linearly ordered by inclusion,
2. $A = \bigcap N_s$ and $X \setminus B = \bigcup N_s$,
3. $I(N_s) \supseteq C(N_t)$, if $s > t$ and $N_s \neq N_t$ (Note that if $N_s = N_t$ then (3) does not hold).

The maximal chain of open-closed-nested neighborhood system between the real set RFX (core) and complement CFX will be called the maximal chain of FX .

Now we have the non-finite set version of Theorem 5.3 -Corollary 5.6.

Theorem 5.3 α Qualitative fuzzy set FX can be characterized by the following two qualitative properties (1) the minimal topologies that are pullbacks of FX and FY are the same, (2) $|FX(x) - FY(x)| < 2\epsilon$ for all x in U , (3) two maximal chains of FX and FY are isomorphic.

Corollary 5.4 α The binary relation ϵ -deformation induces a reflexive and symmetric basic neighborhood system on the space of all membership functions.

Corollary 5.5 α FX and FY are 1- equivalent iff (1) the minimal topologies that are pullbacks of FX and FY are the same, (2) two maximal chains of FX and FY are isomorphic.

Corollary 5.6 α . 1- qualitative fuzzy set is characterized by (1) a topology that is a pullback of the topology of real numbers, (2) a maximal chain.

7 P-Fuzzy Sets

Though we believe ϵ -deformable families (see comments on Question 3) should not form a partition in a membership space, we will offer a "partition theory" for references. Properties of the binary relation, ϵ -deformation, are induced from the properties of basic ϵ -neighborhood. In order to have a "partition theory," we need to have a reflexive, symmetric and transitive basic ϵ -neighborhood system. So we choose left-closed-right-open intervals as basic ϵ -neighborhoods (It will work equally well, if we have made "opposite" choice). In other words, we partition the closed unit interval into the

following sub-intervals $[0, P_1)$, $[P_1, P_2)$, $[P_2, P_3)$, ..., $[P_n, 1]$, where the length of all sub-intervals is ϵ , except the last one $[P_n, 1]$ which may or may not be shorter. Now we can modify Definition 5.1 into the following:

Definition 6.1. Membership functions FX and FY are P -equivalent iff (1) both induce the same minimal topology on U , (1a) $A = \{0, P_1, P_2, P_3, \dots, P_n, 1\}$ (2) there exists a ϵ -homeomorphism h relative to A such that

$$FX(x) = h(FY(x)) \text{ for all } x \in U, \text{ or } FX = h \bullet FY$$

It is easy to see that P -equivalence is indeed an equivalence relation. We would like to point out that the fixed point set A is a parameter of P -equivalency. Choosing such a sequence $0, P_1, P_2, P_3, \dots, P_n, 1$ is not a natural condition. This selection reflects the un-natural-ness of the crispy partition on MF , the membership function space; it is not a desirable theory.

Definition 6.2. A P -fuzzy set is a mathematical object that consists of an P -equivalence class of membership functions.

The theorem 5.3 can be reformulated as follows:

Theorem 6.3. In a finite universe $U = \{u_1, u_2, \dots, u_n\}$, FX and FY are P -equivalent iff

1. both have the same grade partition
2. $\forall x, FX(x), FY(x)$ is contained in one of the sub-intervals $[0, P_1)$, $[P_1, P_2)$, $[P_2, P_3)$, ..., $[P_n, 1]$.
3. $CFX = CFY$ and $RFX = RFY$, and two linear orders of grade partitions are isomorphic.

(1) of Theorem 6.3 follows immediately from (1) of Definition 6.1. Since A is a fixed point set, so sub-intervals are mapped into themselves by h . So (2) of

(1) of Theorem 6.3 follows immediately from (1) of Definition 6.1. Since A is a fixed point set, so sub-intervals are mapped into themselves by h . So (2) of Theorem 6.3 follows. Note that h is a homeomorphism, it is a monotonic map; (3) of Theorem 6.3 is proved. Conversely, let h be a map that maps $FY(x)$ to $FX(x)$, and A onto itself (as identity map on A). We can extend h linearly to the whole unit interval. It is easy to see that such an h is an ϵ

-homeomorphism relative to A . So we have shown that Theorem 6.3 implies Definition 6.1. QED.

PART III APPLICATIONS

8 Qualitative Fuzzy Inference on Finite Universe - Armstrong Inference

What is a fuzzy set? If we abstract away from intuition, then a fuzzy set is a mathematical object that is defined by and only by a set (can be a singleton) of membership functions. We could say fuzzy sets are merely new names for real-valued functions. So all the information about a fuzzy set should be carried by and only by membership function(s). We will take this view in this section. Under this view, fuzzy inference is the "usual inference" restricted to information that is carried by real-valued functions. Using this view we are not inventing a new inference, we merely restrict the old notion into a special circumstance. So our view, we believe, is the 'correct' view of fuzzy inference or implications.

8.1 Qualitative Fuzzy Sets as Information Carriers

We will view membership functions as some information carriers; it carries some fuzzy information on a finite universe U . We will organize a collection of membership functions into a Pawlak Information System (PIS) [20]. So Armstrong inference in relational database theory [22] can be treated as inference on information carried by membership functions. Taking this point of view, we have "fuzzy inference theory" [17]. We should caution readers that it is different from the "usual" fuzzy inference in the literature. To illustrate the idea more concretely, we take $M = FX1, FX2, FX3$ and FY be two collections of membership functions defined on $U = ob-1, ob-2, ob-3, \dots, ob-8, ob-9$. M and FY can be represented by the table below. Each column represents the values of a membership function.

Each fuzzy set gives a grade partition on U . It is clear that FY -grade-partition is coarser than each FXi -grade-partition. Let the intersection be

$$M\text{-partition} = \cap FXi\text{-grade-partition.}$$

U	FX ₁	FX ₂	FX ₃	...	FY
ob-1	.09	.06	.13		.09
ob-2	.09	.06	.13		.09
ob-3	.09	.11	.06		.09
ob-4	.18	.09	.14		.18
ob-5	.18	.09	.14		.18
ob-6	.18	.15	.09		.04
ob-7	.18	.15	.09		.04
ob-8	.18	.15	.09		.04
ob-9	.18	.15	.09		.04

In rough set language, we say knowledge FY is depended on knowledge M. In database language [22], we say "fuzzy attribute" FY is (extensional) functionally depended on "fuzzy attributes" M. Since this is a paper concerning rough and fuzzy sets, not databases, we will follow Pawla [20, pp. 45], and define:

Definition 7.1. Let M and N be two collections of membership functions, we will write

$m \rightarrow \eta$ iff η -partition is coarser than M-partition.

Intuitively, $M \Rightarrow \eta$ means that the information carried by η can be inferred from the information carried by M, and will be so read.

Proposition 7.2. $M \rightarrow FY$ iff FY-partition is coarse than M-partition iff FY is continuous on U with M -topology.

Theorem 7.3. In a finite universe,

(1) FY is a continuous function on U (with the minimal M-topology) iff (2) $M = FX_1, FX_2, \dots, FX_n \Rightarrow FY$ (Information carried by M can be inferred from information carried by FY) iff (3) FY is a polynomial over FX_1, FX_2, \dots, FX_n

First "iff" follows immediately from the definition of continuous function (Definition 2.5) and the minimal M-topology. Since the universe U is finite, functional dependency can be expressed by polynomial, i.e., FY is a polynomial over FX_1, FX_2, \dots, FX_n . Converse is obvious. The proves the second "iff"

Now, we will transform these "quantitative" results into qualitative one. A qualitative fuzzy set is continuous if its membership functions are all continuous. A qualitative fuzzy set Y is a polynomial over a family of qualitative

fuzzy sets X_i if its membership functions F_Y are polynomial over membership functions F_{X_i} . Recall that F_Y and F_{X_i} is one of the membership functions for the qualitative fuzzy set Y and X_i respectively.

Theorem 7.4. In a finite universe,

1. Y is a continuous function on U (with the minimal M -topology) iff
2. $M = \{X_1, X_2, \dots, X_n\} \Rightarrow Y$ (Qualitative fuzzy information carried by M can be inferred from qualitative fuzzy information carried by Y) iff
3. Y is a polynomial over X_1, X_2, \dots, X_n

Proof: By Theorem 5.3, the statement (2) of Theorem 7.4 is the same as (2) of Theorem 7.3. So every F_Y is continuous on U ; This proves the first "if". To show the first "only if", the statement (1) of Theorem 7.4 implies that all F_Y is continuous over the minimal M -topology. The base of the minimal M -topology is the M -partition. So the F_Y -partition is coarse than M -partition; this is true for every F_Y . This established the (2) of Theorem 7.3. These arguments complete the first "iff." Next, let us consider the second "iff." Let us assume the condition (2) of Theorem 7.4 is valid. Then every F_Y is continuous. By Theorem 7.3, every F_Y is polynomial over F_{X_i} . Since the M -topology is independent of the choices of F_{X_i} , so F_Y is polynomial over every choices of F_{X_i} ; so statement (3) of Theorem 7.4 is proved. Conversely, a polynomial is continuous if all of its "variables" are continuous. So we have (3) implies (2). QED.

From Theorem 7.3, we can easily get the following: Theorem 7.4 The inferential closure of a family M of qualitative fuzzy sets is the polynomial functions over M .

9 Fuzzy Logic Controllers-Lyapunov Stability

In classical fuzzy logic control we have

(Step 1) a set of linguistic rules.

By fuzzification, we get

(Step 2) a set of fuzzy rules.

In this step, domain experts replace each linguistic constant by a membership function. So the set of linguistic rules is transformed into a set of fuzzy rules. To be more accurate we should say a set of membership function rules, since each linguistic constant is replaced by one membership function. Classical fuzzy designers then use various inference method to transform a set of membership function rules into

(Step 3) a candidate control function. If designers are lucky, by experiments they may find the candidate is indeed (Step 4) the desired control function.

Otherwise, they should continue these 4 steps. In our approach, we keep the linguistic rules the same, and will replace membership functions by qualitative fuzzy sets, so we will get

(Step 2') a set of qualitative fuzzy set rules.

Each fuzzy set is represented by an admissible family of membership functions. Fuzzy designers apply various inference method to transform such a set of qualitative fuzzy rules into a family of candidate control functions.

(Step 3') The family forms a "virtual" tubular neighborhood.[1];

By experiments (we will use evolutionary computing), some of these candidates may turn out to be the desirable real control functions.

(Step 4') These real control functions may form a "tubular neighborhood."

If we can show further that the tubular neighborhood satisfy the bounded condition, by the definition of Lyapunov's stability [23], the solutions satisfy the stability condition [16].

10 Conclusion

In science, mathematics and engineering, approximation is indispensable. Their fundamental notion is based on the theory of topological spaces (topological neighborhood systems). In computing world such a notion is too restrictive, we propose a generalized or weaker notion, namely, neighborhood systems. It is, intuitively, a "finite type" topology. Classical topology can be viewed as modern formulation of (ϵ, δ) -definition of limit which provided a "definite form" answer to the infinitesimal uncertainty. Neighborhood systems may be an effective notion in expressing some negligible uncertainty.

Fuzziness is, we believe, one form of such an uncertainty. In this paper it seems that we have successfully expressed real world fuzzy sets in terms of neighborhood systems. Interestingly other uncertainty, such as rough sets, belief functions and others also find their tie with neighborhood systems. So the theory of neighborhood systems will well be a fundamental notion of approximation in advanced computing. We will continue to report our findings.

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My Insight:

Examples:

Example 1:

Let Z be integers. Let R denote the equivalence relation called congruence mod m . That is, $x R y$ if $x - y$ is divisible by m . Let $m = 4$. Then the equivalence classes are:

$$\begin{aligned}[0] &= \{ \dots, -8, -4, 0, 4, 8, \dots \} \\ [1] &= \{ \dots, -7, -3, 1, 5, 9, \dots \} \\ [2] &= \{ \dots, -6, -2, 2, 6, 10, \dots \} \\ [3] &= \{ \dots, -5, -1, 3, 7, 11, \dots \}\end{aligned}$$

In other words, $[0], [1], [2], [3]$ is a partition for the integers Z . The quotient set of this equivalence relation is denoted by Z . Congruence mod m is one of the few early examples of equivalence relations. It has been used as a generalized equality in number theory.

Example 2:

$$M1(c) = 0$$

$$M2(c) = 1/2$$

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$$Mn(c) = 1/n$$

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Each Mn , $n = 1, 2, \dots$ is a different fuzzy membership function. So, there are infinitely many different fuzzy subsets. This is quite a contrast to classical set theory. There are only two subsets in U , namely the empty set and whole U (singleton). These examples illustrate that there are too many functional representations of one intuitive real world fuzzy set. So there is a need to define an equivalence among the representations of an intuitive real world fuzzy set