1 A recursion formula for the coefficients

1.1 The coefficients of ϱ_G

Definition 1.1. *Let $A \in \mathbb{R}^{n\times n}$ symmetric, $v \in \mathbb{R}^n$. For arbitrary multi-indices $I = (i_1, \dots, i_k) \in \mathbb{N}^*$ with $0 \le i_1 < i_2 < \dots < i_k$ define the coefficients δ_I by*

$$\delta_i := \langle v, A^i v\rangle \text{ for } i \in \mathbb{N},$$

$$\delta_{(i_1,\dots,i_k)} := \begin{cases} 0 & \text{if } i_j + 1 = i_{j+1} \text{ for some } j \\ \sum_{v\in\{0,1\}^{k-1}} (-1)^{|v|} \cdot \delta_{i_k - |v|} \cdot \delta_{(i_1,\dots,i_{k-1})+v} & \text{otherwise,} \end{cases}$$

where $|v| = \sum_{i=1}^k v_i$ denotes the number of ones in v.

Claim 1.2. *Let $A \in \mathbb{R}^{n\times n}$ symmetric, $v \in \mathbb{R}^n$, let $d := \dim Z(v, A)$. For $i = 0, \dots, d$ define the vector $w_i^d \in \mathbb{N}^d$ by*

$$w_i^d := (0, 2, 4, \dots, 2(i-1), 2i+1, 2(i+1)+1, \dots, 2(d-1)+1).$$

Then the characteristic polynomial of $A|_{Z(v,A)}$ is given by

$$p(x) = \sum_{i=0}^{d} (-1)^{|w_i^d|} \cdot \frac{\delta_{w_i^d}}{\delta_{w_d^d}} x^i,$$

where again $|w| := \sum_{j=1}^k w_j$ denotes the L^0-norm of a vector.

Remark 1.3. If A satisfies $tr(A) \equiv 0 \mod 2$, then every δ_I for I of length k is divisible by 2^{k-1}. If furthermore n is even, every δ_I is divisible by 2^k.

Remark 1.4. For every I of length $k > d$, it holds $\delta_I = 0$.

Proof. It can be verified that the vectors

$$v_j := \sum_{l=0}^{j} (-1)^{|w_i^d|} \delta_{w_i^d} A^i v$$

form an orthogonal basis of $Z(v, A)$. This implies, that the projection of $A^k v$ onto the subspace generated by $v, \dots, A^{k-1}v$ has coordinates $(-1)^{|w_i^k|} \cdot \frac{\delta_{w_i^k}}{\delta_{w_k^k}}$ relative to the basis $v, \dots, A^{k-1}v$. The assertion immediately follows. □

Hence, if G is an undirected graph with adjacency matrix A, the above approach can be used to determine the recurrence polynomial ϱ_G.

Remark 1.5. A few comments on this approach:

1. Even when using dynamic programming, the recursion formula for the δ_I still has exponential complexity.

2. The absolute value of the δ_I grows exponentially as well. Furthermore, in general the δ_I for multiindices of fixed length do not share a common factor other than 2^{k-1}.

3. The most interesting question that arises: Why is every $\delta_{w_i^d}$ divisible by $\delta_{w_d^d}$? In particular, in general the coefficient $\delta_{w_d^d}$ seems to be a large square number. However, I could not prove this and there seem to be exceptions. Furthermore, I could not find any meaning of the square root of $\delta_{w_d^d}$.

1.2 The recurrence degree of G

If one is only interested in the recurrence degree of G, the following procedure describes a way to compute it in time $\mathcal{O}(n^3)$:

Initialize $B := I_n, v := (1, \dots, 1), k := n$
While $k > 0$ do:
    $w := vB$
    if $w = 0$:
        return $n - k$
    pick $i \in \{1, \dots, k\}$ with $w_i \neq 0$
    for $j \neq i$:
        update the columns of B: $B_j := w_i B_j - w_j B_i$
    delete the i-th column of B
    $v := Av$
    $k := k - 1$
return n

2 Local recurrence polynomials

2.1 General theory

There exists an isomorphism between the set of all linear recurrent sequences having characteristic polynomial p and the quotient ring $\mathbb{Z}[x]/p(x)$ (see [1]). If we identify the set of integer sequences with the ring of formal power series $\mathbb{Z}[[x]]$, this isomorphism is given by $\sigma(t) \mapsto p(t)t^{-1}\sigma\left(t^{-1}\right)$. The term

$t^{-1} \cdot \sigma\left(t^{-1}\right) = \frac{r_\sigma(t)}{\varrho_\sigma(t)}$ (where r and ϱ_σ are coprime polynomials) is well-defined if and only if σ is a linear recurrent sequence and $\varrho_\sigma(t)$ divides $p(t)$ if and only if p is a characteristic polynomial for σ. In this case, ϱ_σ is the least characteristic polynomial for σ.

Let $q(x)$ be the image of $\sigma \in \mathbb{Z}[[x]]$ in $\mathbb{Z}[x]/p(x)$. The sequence σ has a characteristic polynomial ϱ_σ of degree lower than p if and only if $q(x) \in \mathbb{Z}[x]/p(x)$ is a zero divisor (if and only if p and q are not coprime). The polynomial ϱ_σ is given by

$$\varrho_\sigma(x) = \frac{p(x)}{\gcd(p(x), q(x))}.$$

Hence, if $q(x)$ is known, we can compute ϱ_σ by factoring p and q (resp. in time $\mathcal{O}(n^2)$ with the Euclidean algorithm).

When the first $\deg(p)$ terms of σ are known, it is possible to directly compute the polynomial $p(t)t^{-1}\sigma\left(t^{-1}\right)$. However, if σ is the (local or global) walk count of a graph, this polynomial can be obtained in other ways.

2.2 Undirected graphs

Let G be an undirected graph with adjacency matrix A and let $u, v \in V$. The number of walks of length k starting in u and ending in v is denoted by w_k^{uv} and can be computed as $w_k^{uv} = \langle e_u, A^k e_v \rangle$. Denote the global walk count sequence of G by w_r and the images of w_r resp. w_r^{uv} in $\mathbb{Z}[x]/\chi_G[x]$ by r_G resp. r_{uv}. By linearity of the described isomorphism it follows that $r_G = \sum_{u,v \in V} r_{uv}$. Hence, we are interested in computing the polynomials r_{uv} for $u, v \in V$ arbitrary.

If $u = v$, this is relatively easy:

Claim 2.1. *Let G be an undirected graph with adjacency matrix A, $v \in V$. Then it holds*

$$r_{vv}(x) = \chi_{A_v}(x),$$

where A_v is obtained from A by removing the v-th row and column.

In other words, r_{vv} is the characteristic polynomial of $G \backslash \{v\}$. In particular, it follows that ϱ_{vv} consists of all factors of the characteristic polynomial of A which are not contained in the characteristic polynomial of A_v. Furthermore, the previously described approach can be generalized to obtain that ϱ_{vv} is the characteristic polynomial of $A|_{Z(e_v, A)}$.

If $u \neq v$, it gets more complicated. We say $p \in V^{k+1}$ is a simple path of length k, if $p = (v_1, \ldots, v_{k+1})$, such that the v_i are pairwise unequal and

$\{v_i, v_{i+1}\} \in E$ for all $i = 1, \ldots, k$. In particular, the simple paths of length zero are exactly the nodes of G and the simpe paths of length one are its edges (where each edge gives rise to two paths of length one). For nodes $u, v \in V$, let P_{uv} be the set of simple paths starting in u and ending in v.

Claim 2.2. *Let G be an undirected graph with adjacency matrix A, let $u, v \in V$. Then it holds*

$$r_{uv}(x) = \sum_{p \in P_{uv}} \chi_{G \backslash p}(x).$$

Note, that this is a generalization of the previous Claim 2.1. Using that $r_G = \sum_{u,v \in V} r_{uv}$, we get the following equation:

$$r_G(x) = \sum_{p \subseteq G \text{ simple path}} \chi_{G \backslash p}(x).$$

Therefore, the recurrence polynomial ϱ_G consists of all factors of the characteristic polynomial, which do not divide this sum.

Corollary 2.3. *Let d be the minimal degree among the degrees of the factors of the characteristic polynomial of G. If G contains two nodes with (shortest path) distance $n - d$, then the characteristic polynomial of G equals its minimal polynomial.*

Indeed, if (u, v) is a pair of nodes with distance $n - d$, then the local residual r_{uv} has degree $d - 1$ (since the shortest path connecting u and v contains $n - d + 1$ nodes) and can therefore not contain a factor of χ_G. Hence, the local recurrence polynomial ϱ_{uv} is equal to χ_G. But since ϱ_{uv} divides the minimal polynomial of G, they have to be equal.

Remark 2.4. The local recurrence polynomials ϱ_{uv} are always a factor of the minimal polynomial of G. However, ϱ_{uv} can be a multiple of ϱ_G, a factor of ϱ_G or neither of those. This can be seen for example in the path P_4 on five vertices. It holds

$$\begin{aligned}
\varrho_G(x) &= x(x^2 - 3), \\
\varrho_{1,5}(x) &= x(x-1)(x+1)(x^2 - 3) = \chi_G, \\
\varrho_{1,3}(x) &= x(x^2 - 3), \\
\varrho_{2,3}(x) &= x^2 - 3, \\
\varrho_{2,4}(x) &= (x-1)(x+1)(x^2 - 3)
\end{aligned}$$

In general however, we can make no statement about the local recurrence polynomials. There exist graphs where every local recurrence polynomial is

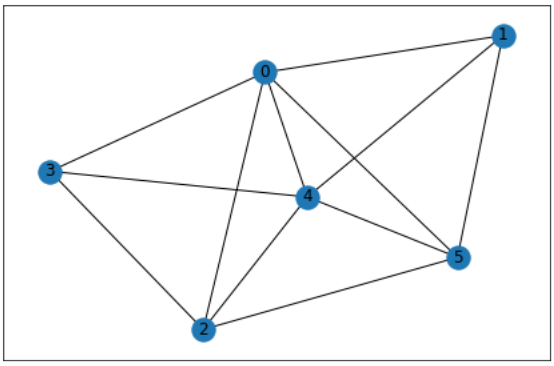

Figure 1: Every local recurrence polynomial is a proper divisor of the minimal polynomial.

a proper divisor of the minimal polynomial of G (see Figure 1). On the other hand, there exist graphs where every local recurrence polynomial is equal to χ_G, while ϱ_G is a proper divisor of the characteristic polynomial (e.g. the path P_3).

Proof. Proof of Claim 2.2.
Rough idea: First use that ϱ_{uu} is equal to the characteristic polynomial of A restricted to $Z(e_u, A)$, to show the claim for closed walks.

Afterwards: Every walk from u to v can (uniquely) be decomposed as $(v_0 = u, C_0, v_1, C_1, \ldots, v_{n-1}, C_{n-1}, v_n = v, C_n)$, where $(u, v_1, \ldots, v_{n-1}, v)$ is a simple path and every C_i is a closed walk on v_i not passing through any v_j for $j < i$. This should lead to the desired result. $\square$

Corollary 2.5. *Let G be an undirected graph, $u \in V$, let $d := \max_{v \in V}(dist(u, v))$. Then the local recurrence polynomial ϱ_{uu} has degree at least $d + 1$.*

Proof. Recall that ϱ_{uu} is te characteristic polynomial of $A|_{Z(e_u,A)}$. Since $(A^i e_u)_v = 0$ for all $i < d$, but $\left(A^d e_u\right)_v > 0$, the vector $A^d e_u$ is linearly independent of all $A^i e_u$ for all $i < d$. Hence, $\deg(\varrho_{uu}) = \dim(Z(e_u, A)) \geq d + 1$. $\square$

Corollary 2.6. *Let G be an undirected graph, $u, v \in V$. Then ϱ_{uv} is a divisor of ϱ_{uu}.*

Proof. Let $\varrho_{uu} = x^d - \sum_{i=0}^{d-1} \beta_i x^i$. Then it holds that

$$A^k e_u = \sum_{i=k-d}^{k-1} \beta_i A^i e_u$$

for all $k \geq d$. In particular, this implies

$$w_k^{uv} = \langle e_u, A^d e_v \rangle = \sum_{i=k-d}^{k-1} \beta_i \langle e_u, A^i e_v \rangle = \sum_{i=k-d}^{k-1} \beta_i w_i^{uv}$$

and hence, ϱ_{uu} is a characteristic polynomial for w_r^{uv} as desired. $\square$

Unfortunately, in general ϱ_{uv} is a proper divisor of the greatest common divisor of ϱ_{uu} and ϱ_{vv}.

References

[1] M. Hall. An isomorphism between linear recurring sequences and algebraic rings. *Transactions of the American Mathematical Society*, 44(2):196–218, 1938.