Definition 0.1. *Let $A \in \mathbb{R}^{n\times n}$ and $v \in \mathbb{R}^n$. The A-cyclic subspace generated by v is the subspace which is generated by the vectors $v, Av, A^2v, \dots, A^nv$ (i.e. the smallest A-invariant subspace containing v). This subspace is denoted by $Z(v, A)$.*

Let $k \in \mathbb{N}$ be the smallest number such that $A^kv = \sum_{i=0}^{k-1} \beta_i A^i v$ is a linear combination of $v, Av, \dots, A^{k-1}v$. Then the characteristic polynomial of A restricted to $Z(v, A)$ is given by

$$p(x) = x^k - \sum_{i=0}^{k-1} \beta_i x^i.$$

Claim 0.2. *Let $A \in \mathbb{R}^{n\times n}$ symmetric, $v \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ an eigenvalue of A. Then λ is an eigenvalue of A restricted to $Z(v, A)$ if and only if there exists an eigenvector $w \in V_\lambda(A)$ with $\langle v, w\rangle \neq 0$. Furthermore, the multiplicity of each eigenvalue of A restricted to $Z(v, A)$ is equal to 1.*

Proof. Assume $\langle v, w\rangle = 0$ for all $w \in V_\lambda(A)$. Then it holds that for all $i = 0, \dots, k-1$:

$$\langle w, A^i v\rangle = \langle A^i w, v\rangle = \lambda^i \langle w, v\rangle = 0.$$

Hence, all $w \in V_\lambda(A)$ are orthogonal to all vectors in $Z(v, A)$ and in particular $Z(v, A)$ can not contain an eigenvector $w \in V_\lambda(A)$.

Conversely, assume $\langle v, w\rangle \neq 0$ for some $w \in V_\lambda(A)$. Let $k = \dim(Z(v, A))$ and write

$$A^k v = \sum_{i=0}^{k-1} \beta_i A^i v.$$

If $\lambda = 0$, it holds that for all $i \in \mathbb{N}_+$:

$$\langle w, A^i v\rangle = \langle A^i w, v\rangle = 0.$$

In particular, this implies

$$\begin{aligned} 0 &= \langle A^k v - \sum_{i=0}^{k-1} \beta_i A^i v, w\rangle \\ &= \beta_0 \langle v, w\rangle. \end{aligned}$$

Since $\langle v, w\rangle \neq 0$, this implies $\beta_0 = 0$ and hence x is a factor of the characteristic polynomial of A.
If $\lambda \neq 0$, it holds that $\langle w, A^k v\rangle = \lambda^k \langle w, v\rangle$ and hence

$$\lambda^{-k} = \frac{\langle v, w\rangle}{\langle A^k v, w\rangle}.$$

On the other hand, we can conclude

$$\langle w, A^k v\rangle = \langle w, \sum_{i=0}^{k-1} \beta_i A^i v\rangle = \sum_{i=0}^{k-1} \beta_i \lambda^i \langle v, w\rangle.$$

Therefore, it holds that

$$\sum_{i=0}^{k-1} \beta_i \lambda^i = \frac{\langle A^k, w\rangle}{\langle v, w\rangle}$$

and from this it follows that

$$\lambda^{-k} \cdot \sum_{i=0}^{k-1} \beta_i \lambda^i = 1.$$

Now, define for $i = 0, \ldots, k-1$ the coefficients α_i by

$$\alpha_i := \frac{1}{\lambda} \left(\beta_i + \alpha_{i-1}\right) \ \text{(where } \alpha_{-1} = 0\text{)}$$

and define the vector w' by

$$w' := \sum_{i=0}^{k-1} \alpha_i A^i v.$$

The vector w' is an eigenvector of A with respect to the eigenvalue λ, if and only if $\alpha_{k-1} = 1$. Indeed, from the definition of w' it follows

$$\begin{aligned} Aw' &= A \left(\sum_{i=0}^{k-1} \alpha_i A^i v \right) \\ &= \sum_{i=1}^{k-1} \alpha_{i-1} A^i v + \sum_{i=0}^{k-1} \beta_i A^i v \\ &= \sum_{i=0}^{k-1} \left(\alpha_{i-1} + \beta_i\right) A^i v \\ &= \lambda \sum_{i=0}^{k-1} \alpha_i A^i v \\ &= \lambda w'. \end{aligned}$$

From the recursive definition of the α_i, we can inductively show, that

$$\alpha_{k-1} = \lambda^{-k} \cdot \sum_{i=0}^{k-1} \beta_i \lambda^i = 1,$$

and hence the desired result follows.

It is still left to show that each eigenspace of A restricted to $Z(v, A)$ is indeed one-dimensional. This can easily be seen from the fact, that if the equation $Aw = \lambda w$ is fulfilled for some $w = \sum_{i=0}^{k-1} \alpha_i A^i v$, then the coefficients α_i are (wlog $\alpha_{k-1} = 1$) determined uniquely by

$$\alpha_i \coloneqq \frac{1}{\lambda} \left(\beta_i + \alpha_{i-1}\right) \text{ (where } \alpha_{-1} = 0),$$

if $\lambda \neq 0$, resp. by

$$\alpha_i = -\beta_{i+1}, \alpha_{k-1} = 1$$

if $\lambda = 0$.

Note that if $\alpha_{k-1} = 0$, the vector $\sum_{i=0}^{k-1} \alpha_i A^i v$ can not be an eigenvector of A.

□

Corollary 0.3. *Let G be an undirected graph with adjacency matrix A. Then the recurrence polynomial ϱ_G is exactly the characteristic polynomial of A, restricted to the A-cyclic subspace generated by $v = (1, \dots, 1)^T$.*

Corollary 0.4. *Let G be a graph with recurrence polynomial of degree at most two. Then it holds that*

$$w_r w_s \leq w_0 w_{r+s}$$

with equality if and only if the graph is 1-recurrent.

Proof. Let A be the adjacency matrix of G and let $v = (1, \dots, 1)^T$. Note that it suffices to show that $\varrho_G(\overline{d}) \leq 0$, where $\overline{d} = \frac{v^T A v}{n} = \frac{\langle v, Av \rangle}{\langle v, v \rangle}$ is the average degree of G. From the Cauchy-Schwarz inequality it follows that

$$\overline{d}^2 = \frac{\langle v, Av \rangle^2}{\langle v, v \rangle^2} \leq \frac{\langle v, v \rangle \cdot \langle Av, Av \rangle}{\langle v, v \rangle^2} = \frac{\langle v, A^2 v \rangle}{\langle v, v \rangle},$$

with equality if and only if v and Av are linearly dependent (i.e. ϱ_G is of degree 1). Let $A^2 v = \beta_0 v + \beta_1 A v$ and hence $\varrho_G(x) = x^2 - \beta_1 x - \beta_0$. Then we can conclude

$$\begin{aligned} \varrho_G(\overline{d}) &= \overline{d}^2 - \beta_1 \overline{d} - \beta_0 \\ &\leq \frac{\langle v, A^2 v \rangle}{\langle v, v \rangle} - \frac{\beta_1 \langle v, Av \rangle}{\langle v, v \rangle} - \frac{\beta_0 \langle v, v \rangle}{\langle v, v \rangle} \\ &= \frac{\langle v, A^2 v - \beta_1 A v - \beta_0 v \rangle}{\langle v, v \rangle} \\ &= 0. \end{aligned}$$

□