1 A recursion formula for the coefficients

1.1 The coefficients of ϱ_G

Definition 1.1. *Let $A \in \mathbb{R}^{n\times n}$ symmetric, $v \in \mathbb{R}^n$. For arbitrary multi-indices $I = (i_1, \ldots, i_k) \in \mathbb{N}^*$ with $0 \le i_1 < i_2 < \ldots < i_k$ define the coefficients δ_I by*

$$\delta_i \coloneqq \langle v, A^i v\rangle \text{ for } i \in \mathbb{N},$$

$$\delta_{(i_1,\ldots,i_k)} \coloneqq \begin{cases} 0 & \text{if } i_j + 1 = i_{j+1} \text{ for some } j \\ \sum_{v\in\{0,1\}^{k-1}} (-1)^{|v|} \cdot \delta_{i_k-|v|} \cdot \delta_{(i_1,\ldots,i_{k-1})+v} & \text{otherwise,} \end{cases}$$

where $|v| = \sum_{i=1}^k v_i$ denotes the number of ones in v.

Claim 1.2. *Let $A \in \mathbb{R}^{n\times n}$ symmetric, $v \in \mathbb{R}^n$, let $d \coloneqq \dim Z(v, A)$. For $i = 0, \ldots, d$ define the vector $w_i^d \in \mathbb{N}^d$ by*

$$w_i^d \coloneqq (0, 2, 4, \ldots, 2(i-1), 2i+1, 2(i+1)+1, \ldots, 2(d-1)+1).$$

Then the characteristic polynomial of $A|_{Z(v,A)}$ is given by

$$p(x) = \sum_{i=0}^{d} (-1)^{|w_i^d|} \cdot \frac{\delta_{w_i^d}}{\delta_{w_d^d}} x^i,$$

where again $|w| \coloneqq \sum_{j=1}^k w_j$ denotes the L^0-norm of a vector.

Remark 1.3. If A satisfies $tr(A) \equiv 0 \mod 2$, then every δ_I for I of length k is divisible by 2^{k-1}. If furthermore n is even, every δ_I is divisible by 2^k.

Remark 1.4. For every I of length $k > d$, it holds $\delta_I = 0$.

Proof. It can be verified that the vectors

$$v_j \coloneqq \sum_{l=0}^{j} (-1)^{|w_i^d|} \delta_{w_i^d} A^i v$$

form an orthogonal basis of $Z(v, A)$. This implies, that the projection of $A^k v$ onto the subspace generated by $v, \ldots, A^{k-1}v$ has coordinates $(-1)^{|w_i^k|} \cdot \frac{\delta_{w_i^k}}{\delta_{w_k^k}}$ relative to the basis $v, \ldots, A^{k-1}v$. The assertion immediately follows. □

Hence, if G is an undirected graph with adjacency matrix A, the above approach can be used to determine the recurrence polynomial ϱ_G.

Remark 1.5. A few comments on this approach:

1. Even when using dynamic programming, the recursion formula for the δ_I still has exponential complexity.

2. The absolute value of the δ_I grows exponentially as well. Furthermore, in general the δ_I for multiindices of fixed length do not share a common factor other than 2^{k-1}.

3. The most interesting question that arises: Why is every $\delta_{w_i^d}$ divisible by $\delta_{w_d^d}$? In particular, in general the coefficient $\delta_{w_d^d}$ seems to be a large square number. However, I could not prove this and there seem to be exceptions. Furthermore, I could not find any meaning of the square root of $\delta_{w_d^d}$.

1.2 The recurrence degree of G

If one is only interested in the recurrence degree of G, the following procedure describes a way to compute it in time $\mathcal{O}(n^3)$:

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Initialize B := I_n, v := (1, ..., 1), k := n
While k > 0 do:
    w := vB
    if w = 0:
        return n - k
    pick i ∈ {1, ..., k} with w_i ≠ 0
    for j ≠ i:
        update the columns of B: B_j := w_i B_j - w_j B_i
    delete the i-th column of B
    v := Av
    k := k - 1
return n
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2 Local recurrence polynomials

2.1 General theory

There exists an isomorphism between the set of all linear recurrent sequences having characteristic polynomial p and the quotient ring $\mathbb{Z}[x]/p(x)$ (see [2]). If we identify the set of integer sequences with the ring of formal power series $\mathbb{Z}[[x]]$, this isomorphism is given by $\sigma(t) \mapsto p(t)t^{-1}\sigma(t^{-1})$. The term

$t^{-1} \cdot \sigma\left(t^{-1}\right) = \frac{r_\sigma(t)}{\varrho_\sigma(t)}$ (where r_σ and ϱ_σ are coprime polynomials) is well-defined if and only if σ is a linear recurrent sequence and $\varrho_\sigma(t)$ divides $p(t)$ if and only if p is a characteristic polynomial for σ. In this case, ϱ_σ is the least characteristic polynomial for σ.

Let $q(x)$ be the image of $\sigma \in \mathbb{Z}[[x]]$ in $\mathbb{Z}[x]/p(x)$. The sequence σ has a characteristic polynomial ϱ_σ of degree lower than p if and only if $q(x) \in \mathbb{Z}[x]/p(x)$ is a zero divisor (if and only if p and q are not coprime). The polynomial ϱ_σ is given by

$$\varrho_\sigma(x) = \frac{p(x)}{\gcd(p(x), q(x))}.$$

Hence, if $q(x)$ is known, we can compute ϱ_σ by factoring p and q (resp. using the Euclidean algorithm).

When the first $\deg(p)$ terms of σ are known, it is possible to directly compute the polynomial $p(t)t^{-1}\sigma\left(t^{-1}\right)$. However, if σ is the (local or global) walk count of a graph, this polynomial can be obtained in other ways.

2.2 Undirected graphs

Let G be an undirected graph with adjacency matrix A and let $u, v \in V$. The number of walks of length k starting in u and ending in v is denoted by w_k^{uv} and can be computed as $w_k^{uv} = \langle e_u, A^k e_v\rangle$. Denote the global walk count sequence of G by w_r and the images of w_r resp. w_r^{uv} in $\mathbb{Z}[x]/\chi_G[x]$ by r_G resp. r_{uv}. By linearity of the described isomorphism it follows that $r_G = \sum_{u,v\in V} r_{uv}$.

2.2.1 The global case

Let the complement of G be denoted by G^c. Then Cvetkovic et al. (cf. [1], p. 46) showed the following:

Theorem 2.1. *Let $w_G(x) \coloneqq \sum_{k=0}^{\infty} w_k t^k$. Then it holds that*

$$\frac{\chi_{G^c}(-1-x)}{\chi_G(x)} = (-1)^n \left(w_G\left(x^{-1}\right)x^{-1} + 1\right).$$

In our case, this immediately implies

Corollary 2.2.

$$r_G(x) = -\chi_G(x) + (-1)^n \chi_{G^c}(-1-x).$$

This provides a convenient way to compute the recurrence polynomial of a given graph.

2.2.2 The local case

We are furthermore interested in computing recurrence polynomials for the local walk counts between particular nodes u and v of the given graph. If $u = v$, this is relatively easy:

Claim 2.3. *Let G be an undirected graph with adjacency matrix A, $v \in V$. Then it holds that*

$$r_{vv}(x) = \chi_{A_v}(x),$$

where A_v is obtained from A by removing the v-th row and column.

In other words, r_{vv} is the characteristic polynomial of $G \backslash \{v\}$. In particular, it follows that ϱ_{vv} consists of all factors of the characteristic polynomial of A which are not contained in the characteristic polynomial of A_v. Furthermore, the previously described approach can be generalized to obtain that ϱ_{vv} is the characteristic polynomial of $A|_{Z(e_v,A)}$.

If $u \neq v$, it gets more complicated. We say $p \in V^{k+1}$ is a simple path of length k, if $p = (v_1, \ldots, v_{k+1})$, such that the v_i are pairwise unequal and $\{v_i, v_{i+1}\} \in E$ for all $i = 1, \ldots, k$. In particular, the simple paths of length zero are exactly the nodes of G and the simpe paths of length one are its edges (where each edge gives rise to two paths of length one). For nodes $u, v \in V$, let P_{uv} be the (finite) set of simple paths starting in u and ending in v.

Claim 2.4. *Let G be an undirected graph with adjacency matrix A, let $u, v \in V$. Then it holds, that*

$$r_{uv}(x) = \sum_{p \in P_{uv}} \chi_{G \backslash p}(x).$$

Note, that this is a generalization of the previous Claim 2.3, since $P_{vv} = \{(v)\}$. Using that $r_G = \sum_{u,v \in V} r_{uv}$, we get the following equation:

$$r_G(x) = \sum_{p \subseteq G \text{ simple path}} \chi_{G \backslash p}(x).$$

Therefore, the recurrence polynomial ϱ_G consists of all factors of the characteristic polynomial, which do not divide this sum.

Remark 2.5. This is particularly interesting in the case when G is a tree, due to the fact that for every pair of nodes u, v there exists exactly one simple path connecting these nodes. Hence, the sum r_{uv} consists of only one characteristic polynomial of the induced subgraph of G containing all nodes outside this unique path.

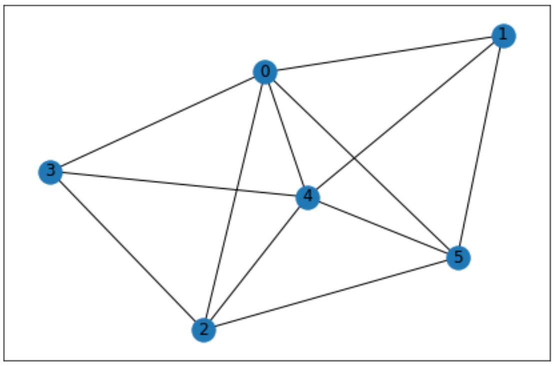

Figure 1: Every local recurrence polynomial is a proper divisor of the minimal polynomial.

Corollary 2.6. *The recurrence polynomial for the sequence of closed walks on G is equal to its minimal polynomial.*

Proof. Let the sequence of closed walks on G be denoted by w_r^{cl} and let $r_{cl}(x)$ be the image of $w^{cl}(x)$ in $\mathbb{Z}[x]/_{\chi_G(x)}$. Then it holds that

$$r_{cl}(x) = \sum_{v \in V} r_{vv}(x) = \sum_{v \in V} \chi_{A_v}(x) = \chi'_G(x).$$

Obviously, $\frac{\chi_G(x)}{\gcd\left(\chi_G(x), \chi'_G(x)\right)}$ equals the minimal polynomial of G. □

Corollary 2.7. *Let d be the minimal degree among the degrees of the factors of the characteristic polynomial of G. If G contains two nodes with (shortest path) distance $n-d$, then the characteristic polynomial of G equals its minimal polynomial.*

Indeed, if (u, v) is a pair of nodes with distance $n-d$, then the local residual r_{uv} has degree $d-1$ (since the shortest path connecting u and v contains $n-d+1$ nodes) and can therefore not contain a factor of χ_G. Hence, the local recurrence polynomial ϱ_{uv} is equal to χ_G. But since ϱ_{uv} divides the minimal polynomial of G, they have to be equal.

Remark 2.8. The local recurrence polynomials ϱ_{uv} are always a factor of the minimal polynomial of G. However, ϱ_{uv} can be a multiple of ϱ_G, a factor of ϱ_G or neither of those. This can be seen for example in the path P_4 on five vertices. It holds

$$\begin{aligned}
\varrho_G(x) &= x(x^2-3),\\
\varrho_{1,5}(x) &= x(x-1)(x+1)(x^2-3) = \chi_G(x),\\
\varrho_{1,3}(x) &= x(x^2-3),\\
\varrho_{2,3}(x) &= x^2-3,\\
\varrho_{2,4}(x) &= (x-1)(x+1)(x^2-3)
\end{aligned}$$

In general however, we can make no statement about the local recurrence polynomials. There exist graphs where every local recurrence polynomial is a proper divisor of the minimal polynomial of G (see Figure 1). On the other hand, there exist graphs where every local recurrence polynomial is equal to χ_G, while ϱ_G is a proper divisor of the characteristic polynomial (e.g. the path P_3).

Proof. Proof of Claim 2.4.
Rough idea: First use that ϱ_{uu} is equal to the characteristic polynomial of A restricted to $Z(e_u, A)$, to show the claim for closed walks.

Afterwards: Every walk from u to v can (uniquely) be decomposed as $(v_0 = u, C_0, v_1, C_1, \dots, v_{n-1}, C_{n-1}, v_n = v, C_n)$, where $(u, v_1, \dots, v_{n-1}, v)$ is a simple path and every C_i is a closed walk on v_i not passing through any v_j for $j < i$. This should lead to the desired result. $\square$

Corollary 2.9. *Let G be an undirected graph, $u \in V$, let $d := \max_{v\in V}(dist(u,v))$. Then the local recurrence polynomial ϱ_{uu} has degree at least $d+1$.*

Proof. Recall that ϱ_{uu} is the characteristic polynomial of $A|_{Z(e_u,A)}$. Since $(A^i e_u)_v = 0$ for all $i < d$, but $\left(A^d e_u\right)_v > 0$, the vector $A^d e_u$ is linearly independent of all $A^i e_u$ for all $i < d$. Hence, $\deg(\varrho_{uu}) = \dim(Z(e_u, A)) \geq d+1$. $\square$

Corollary 2.10. *Let G be an undirected graph, $u, v \in V$. Then ϱ_{uv} is a divisor of ϱ_{uu}.*

Proof. Let $\varrho_{uu} = x^d - \sum_{i=0}^{d-1} \beta_i x^i$. Then it holds that

$$A^k e_u = \sum_{i=k-d}^{k-1} \beta_i A^i e_u$$

for all $k \geq d$. In particular, this implies

$$w_k^{uv} = \langle e_u, A^d e_v\rangle = \sum_{i=k-d}^{k-1} \beta_i \langle e_u, A^i e_v\rangle = \sum_{i=k-d}^{k-1} \beta_i w_i^{uv}$$

and hence, ϱ_{uu} is a characteristic polynomial for w_r^{uv} as desired. $\square$

Unfortunately, in general ϱ_{uv} is a proper divisor of the greatest common divisor of ϱ_{uu} and ϱ_{vv}.

2.3 A collection of semi-useful equations

This section holds a collection of equations which could possibly later be of some use.

Let $G = (V, E)$ be again an undirected graph with adjacency matrix A. Denote its characteristic polynomial by $\chi_G(x)$ and define the quotient ring $B_G \coloneqq \mathbb{Z}[x]/_{\chi_G(x)}$.

Recall, that we are interested in computing the polynomials $r_{uv}(x) \in \mathbb{Z}[x]$ for $u, v \in V$. Observe, that for our application it suffices to compute the image of the polynomials $r_{uv}(x)$ in the quotient ring B_G. This is due to the fact, that the recurrence polynomial consists of exactly those factors of the characteristic polynomial, which do not divide $r_G(x)$. Furthermore, a factor of χ_G divides r_G if and only if it divides the image of r_G in B_G.

The following equations hold inside the quotient ring B_G:

1. For all $u, v \in V$: $\sum_{w \in N(u)} r_{wv}(x) = x \cdot r_{uv}(x)$.

   In other words, the costly matrix multiplication by A on a $\mathbb{Z}^n$-level can be replaced by a simple multiplication by x on the B_G-level.

2. $\sum_{\{u,v\} \in E} r_{uv}(x) = x \cdot \chi'_G(x)$.

3. $\sum_{u,v \in V} \deg(u) r_{uv}(x) = x \cdot r_G(x)$.

4. $r_{uv}(x)^2 = r_{uu}(x) \cdot r_{vv}(x)$.

5. $\sum_{u,v \in V} r_{uv}(x)^2 = \chi'_G(x)^2$.

The following equations hold absolutely (e.g. not only in B_G):

1. $r'_G(x) = \sum_{v \in V} r_{G \setminus \{v\}}(x)$.

2. $r_G(x) = (-1)^{n-1} r_{G^c}(-1 - x)$.

3. $r_{vv}(x) + (-1)^{n-1} r^c_{vv}(-1-x) = r_{G\setminus\{v\}}(x) = \sum_{u \neq v} r_{uv}(x) + (-1)^{n-1} r^c_{uv}(-1-x)$ for all $v \in V$.

   Here, r^c denotes the corresponding polynomial of G^c.

4. The number of length two walks in G and G^c differs exactly by

$$w^2_G - w^2_{G^c} = (n-1)(4m - n(n-1)).$$

References

[1] D. M. Cvetkovic et al. Spectra of graphs. theory and application. 1980.

[2] M. Hall. An isomorphism between linear recurring sequences and algebraic rings. *Transactions of the American Mathematical Society*, 44(2):196–218, 1938.