

# 1 An approach using $\mathbb{Z}[x]/\chi_G(x)$

Let  $G = (V, E)$  be an undirected graph with adjacency matrix  $A$ . Denote its characteristic polynomial by  $\chi_G(x)$  and define the quotient ring  $B_G := \mathbb{Z}[x]/\chi_G(x)$ .

Recall, that we are interested in computing the polynomials  $r_{uv}(x) \in \mathbb{Z}[x]$  for  $u, v \in V$ . Observe, that for our application it suffices to compute the image of  $r_G(x) := \sum_{u,v \in V} r_{uv}(x)$  in the quotient ring  $B_G$ . This is due to the fact, that the recurrence polynomial consists of exactly those factors of the characteristic polynomial, which do not divide  $r_G(x)$ . Furthermore, a factor of  $\chi_G$  divides  $r_G$  if and only if it divides the image of  $r_G$  in  $B_G$ .

The goal is to calculate the recurrence polynomial of  $G$  without having to calculate large matrix powers of the adjacency matrix. In these notes I will state some observations/lemmas (mostly without proof), which could maybe lead to the desired result.

**Lemma 1.1.** *It holds that*

$$\sum_{v \in V} r_{vv}(x) = \chi'_G(x).$$

We can view  $\mathbb{Z}^n$  as a  $\mathbb{Z}[x]$ -module, where the action of  $x$  is defined by  $x \cdot v := A \cdot v$ . Furthermore,

$$\varphi: \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow B_G, (e_u, e_v) \mapsto r_{uv}$$

with  $\mathbb{Z}$ -linear continuation defines a map of  $\mathbb{Z}[x]$ -modules ( $B_G$  is a  $\mathbb{Z}[x]$ -module via the usual multiplication with  $x$ ). The next lemma states, that  $\varphi$  is actually a  $\mathbb{Z}[x]$ -module homomorphism (Consequently,  $\varphi$  can be seen as an inner product on the module  $\mathbb{Z}^n$ ). This allows to replace the costly matrix multiplication on  $\mathbb{Z}^n$  by a simple multiplication by  $x$  on the level of polynomials. The major drawback is that by transitioning into the ring  $B_G$ , we lose exact information on the adjacency matrix  $A$ . In particular, there exist non-isomorphic graphs with distinct recurrence polynomials, but same characteristic polynomial. Hence, restricting ourselves to  $B_G$  (without making use of the adjacency matrix  $A$  itself) can not lead to a solution of our main problem.

**Lemma 1.2.** *The map  $\varphi$ , defined as above, is a  $\mathbb{Z}[x]$ -module homomorphism.*

The following corollaries show, how this lemma can be applied in our setting.

**Corollary 1.3.** *In  $B_G$ , for all  $u, v \in V$  it holds that*

$$\sum_{w \in N(v)} r_{vw}(x) = x \cdot r_{uv}(x).$$

This immediately implies:

**Corollary 1.4.** *In  $B_G$ , it holds that*

$$\sum_{\{u,v\} \in E} r_{uv}(x) = x \cdot \chi'(x)$$

Consequently, computing  $\sum_{\{u,v\} \notin E, u \neq v}$  would solve our problem, but is unfortunately not as easy.

**Corollary 1.5.** *In  $B_G$ , it holds that*

$$\sum_{u,v \in V} \deg(u) r_{uv}(x) = x \cdot r_G(x).$$

The following corollaries are interesting in the sense, that the sum over all  $r_{uv}(x)^2$  can be computed (efficiently) when only knowing the characteristic polynomial of  $G$  (without knowledge of the adjacency matrix). In particular, this sum has to be equal (in  $B_G$ ) for all, even non-isomorphic, graphs with the same characteristic polynomial.

**Corollary 1.6.** *In  $B_G$ , for all  $u, v \in V$  it holds that*

$$r_{uv}(x)^2 = r_{uu}(x) \cdot r_{vv}(x).$$

Immediate consequence:

**Corollary 1.7.** *In  $B_G$ , it holds that*

$$\sum_{u,v \in V} r_{uv}(x)^2 = \chi'_G(x)^2.$$

Another strange, probably useless equation:

**Corollary 1.8.** *Denote the coefficient of  $x^k$  in  $r_{uv}(x)$  by  $r_{uv}^k$ . Furthermore, let  $p_k$  be the truncation of  $\chi_G$  up to level  $k$  (i.e.,  $\chi_G(x) = x^k \cdot p_k(x) + \mathcal{O}(x^{k-1})$ ). Then for all  $k$ , it holds in  $B_G$  that*

$$p_k \cdot \chi'_G(x) = \sum_{uv} r_{uv}^k \cdot r_{uv}(x).$$

*Remark 1.9.* All equations also work on divisors of  $G$  (resp. equitable partitions). For example, if  $P$  is an equitable partition of  $G$ , then

$$\sum_{p \in P} \varphi(e_p, e_p) = \chi'_P(x) \cdot \frac{\chi_G(x)}{\chi_P(x)}.$$

The next equation provides an efficient formula for the derivative of  $r_G(x)$ . Unfortunately, the recurrence polynomial of  $G$  obviously depends on the constant coefficient of  $r_G(x)$ . However, a direct computation involves computing the number of walks of length  $n$  on  $G$ .

**Lemma 1.10.** *Let  $G^c$  be the complement of  $G$ , Then it holds that*

$$r'_G(x) = (-1)^{n+1} \chi'_G(x) - \chi'_{G^c}(-1 - x).$$

## 2 A more algebraic approach

Let again  $G = (V, E)$  be an undirected graph with adjacency matrix  $A$ . We define the centrality  $c(v)$  of a node  $v \in V$  as the limit of the probability, that a walk of length  $k$  starts in node  $v$ . It is known, that the centrality of  $v$  corresponds to the  $v$ -th entry of the (unique) eigenvector of  $A$  to the largest eigenvalue of  $A$ .

**Lemma 2.1.** *It holds that  $c(u) = c(v)$  if and only if  $u$  and  $v$  belong to the same set in the coarsest equitable partition of  $G$ .*

It is known that the characteristic polynomial of the coarsest equitable partition of  $G$  is a divisor of the characteristic polynomial of  $G$ . Furthermore, all main eigenvalues of  $G$  are also zeros of the characteristic polynomial of every equitable partition of  $G$ . In some cases, the characteristic polynomial of the coarsest equitable partition and the recurrence polynomial of  $G$  actually coincide. This, however, does not hold true for all graphs. In particular, if the coefficients of the eigenvector to the largest eigenvalue of  $G$  fulfill more linear equations than in Lemma 2.1, the recurrence polynomial of  $G$  is a proper divisor of the characteristic polynomial of the coarsest equitable partition of  $G$ .

**Lemma 2.2.** *Let  $v$  be the eigenvector of  $A$  for the largest eigenvalue  $\lambda$ . Let the orthogonal complement of the kernel of the map*

$$\mathbb{Z}^n \rightarrow \mathbb{R}, u \mapsto \langle u, v \rangle$$

be generated by the vectors  $u_1, \dots, u_k$  ( $\mathbb{R}$  can be replaced by the splitting field of the main polynomial of  $G$ ). Define the matrix  $P \in \mathbb{Z}^{n \times k}$  as  $P := (u_1, \dots, u_k)$  and denote its pseudo-inverse by  $P^+ \in \mathbb{Q}^{k \times n}$ . Then the recurrence polynomial of  $G$  is the characteristic polynomial of  $P^+ \cdot A \cdot P$ .

In particular, the degree of the recurrence polynomial of  $G$  is equal to the rank of the lattice generated by the coefficients of  $v$  in the splitting field of  $\chi_G$  (interpreted as  $\mathbb{Z}$ -module) and a basis of this lattice would directly lead to a formula for the recurrence polynomial of  $G$ . There could exist an approach making use of the Hamiltonian normal form of  $A$ , but lattice theory is complicated...