$$x^2 - (a+d)x + (ad-bc)$$

$$\frac{a+d}{2} \pm \sqrt{\frac{(a+d)^2}{4} - ad + bc}$$

$$= \frac{a+d}{2} \pm \sqrt{\frac{(a-d)^2}{4} + bc}$$

$$\bar{d} = \frac{a+b + \frac{b}{c}\cdot(c+d)}{1+\frac{b}{c}}$$

$$= \frac{ac+bc+bc+bd}{c} \cdot \frac{c}{b+c}$$

$$= \frac{ac+bd+2bc}{b+c}$$

$$\bar{d} - \frac{a+d}{2} = \frac{2ac+2bd+4bc - ab - ac - bd - dc}{2(b+c)}$$

$$= \frac{ac - ab + bd - dc + 4bc}{2(b+c)}$$

$$\| \quad a(c-b) + d(b-c) + 4bc$$

$$\frac{(a-d)(c-b)+4bc}{2(b+c)} = \frac{(a-d)(c-b)}{2(b+c)} + 2\frac{bc}{b+c}$$

$$\sim \frac{(a-d)^2(c-b)^2}{4(b+c)^2} + 2\frac{(a-d)(c-b)bc}{(b+c)^2} + \underbrace{4\frac{b^2c^2}{(b+c)^2} - \frac{bc(b+c)^2}{(b+c)^2}}_{}$$

$$\frac{bc(4bc-(b^2+2bc+c^2))}{(b+c)^2}$$

$$= -bc\frac{(b-c)^2}{(b+c)^2}$$

$$=$$

$$\frac{w_1}{w_0} = \frac{7}{3}$$

$$\frac{w_2}{w_0} = \frac{17}{3} = \frac{51}{9}$$

$$\frac{w_3}{w_0} = \frac{41}{3} = \frac{123}{9}$$

$$\frac{w_4}{w_0} = 33 = \frac{99}{3} = \frac{297}{9}$$

$$\frac{w_5}{w_0} = \frac{239}{3} = \frac{717}{9}$$

$$\frac{w_6}{w_0} = \frac{577}{3}$$

$$\frac{w_{r+s}}{w_0} \geq \frac{w_r}{w_0} \cdot \frac{w_s}{w_0}$$

$$w_1 w_1 = \frac{49}{9}$$

$$w_1 w_2 = \frac{119}{9}$$

$$= \frac{(a-d)^2(c-b)^2}{4(b+c)^2} - \frac{(a-d)^2(b+c)^2}{4(b+c)^2} + 2\frac{(a-d)(c-b)bc}{(b+c)^2} - bc\frac{(b-c)^2}{(b+c)^2}$$

$$= \frac{(a-d)^2}{4(b+c)^2}\cdot(-4bc) + \ldots$$

$$= \frac{bc}{(b+c)^2}\left(-(a-d)^2 + 2(a-d)(c-b) - (b-c)^2\right)$$

$$= \frac{-bc}{(b+c)^2}\left(a-d+b-c\right)^2 \leq 0 \iff \bar{d} \leq \frac{a+d}{2} + \sqrt{\frac{(a-d)^2}{2} + bc}$$

$$\left(\frac{ac+bd+2bc}{b+c}\right)^2 - (a+d)\frac{ac+bd+2bc}{b+c} + ad-bc$$

$$= \frac{a^2c^2+b^2d^2+4b^2c^2+2abcd+4abc^2+4b^2cd}{(b+c)^2} - \frac{a^2c+abd+2abc+acd+bd^2+2bcd}{b+c}$$

$$+ \frac{(ad-bc)(b+c)^2}{(b+c)^2}$$

$$= \frac{\cancel{a^2c^2}+\cancel{b^2d^2}+4b^2c^2\cancel{+2abcd}+\overset{2}{\cancel{4}}abc^2+\overset{2}{\cancel{4}}b^2cd-a^2bc\cancel{-a^2c^2}\cancel{-ab^2d}\cancel{-abcd}-2ab^2c\cancel{-2abc^2}\cancel{-abcd}\cancel{-ac^2d}\cancel{-b^2d^2}-bcd^2\cancel{-2b^2cd}-2bc^2d+\cancel{ab^2d}+2adbc+\cancel{ac^2d}-bc(b+c)^2}{(b+c)^2}$$

$$= \frac{4b^2c^2+2abc^2+2b^2cd-a^2bc-2ab^2c-bcd^2-2bc^2d+2abcd-bc(b+c)^2}{(b+c)^2}$$

$$= -\frac{bc}{(b+c)^2}\left(-2bc-2ac-2db+a^2+2ab+d^2+2cd-2ad+b^2+d^2\right)$$

$$= -\frac{bc}{(b+c)^2}(a+b-c-d)^2 \leq 0$$ and $=0 \iff G$ is regular or disconnected

G graph, G/π division

To show: If $\lambda \in \Lambda(G/\pi) \setminus \Lambda^+(G)$

$\Rightarrow$ There exists non-trivial divisor of G/π

Let M_π be the $(n \times k)$-Matrix storing the partition for each node

$$\begin{array}{cc} & \text{partitions} \\ \text{nodes} & \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ & \vdots & \end{pmatrix} \end{array}$$

π is equitable iff. $\overbrace{A \cdot M_\pi}^{n \times k} = M_\pi \cdot B$ for some $B \in \mathbb{N}^{k \times k}$

iff.

$M_\pi \cdot B \cdot v = \lambda M_\pi \cdot v$

$A \cdot M_\pi \cdot v = \lambda \cdot M_\pi \cdot v \rightsquigarrow M_\pi \cdot v$ is EV of A with ev. λ

If v EV of A w.r.t. $\lambda \rightsquigarrow$ $Av = \lambda v \Rightarrow v^T A = \lambda v^T$

$\Rightarrow v^T \cdot M_\pi \cdot B = v^T A M_\pi = \lambda v^T M_\pi$

$\Rightarrow v^T \cdot M_\pi$ is left eigenvector of B

v EV of $B \Rightarrow M_\pi v$ EV of $A \Rightarrow v^T M_\pi^T$ EV of A
left

$\Rightarrow v^T \cdot \underbrace{M_\pi^T M_\pi}_{=\operatorname{diag}(\pi_1, \dots, \pi_k)}$ left EV of B

π_i = partition size

Let λ be eigenvalue of B, with eigenspace $\langle v_1, \dots, v_\ell \rangle$

$\Rightarrow M_\pi \cdot v_i$ are eigenvectors of A with eigenvalue λ

$\Rightarrow \forall i = 1, \dots, \ell : \langle M_\pi \cdot v_i, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \rangle = 0$

in part.: $\forall i = 1, \dots, \ell : \sum_{j=1}^{k} \pi_j \cdot (v_i)_j = 0$

$\rightsquigarrow$ all v_i are orth. to $(\pi_1, \dots, \pi_k)^T$

to show: $\exists \pi'$ s.t. $B \cdot M_{\pi'} = M_{\pi'} \cdot C$ for some C for $C = M_{\pi'}^T \cdot B \cdot M_{\pi'}$

$(v_1, \dots, v_\ell)^T \cdot (\pi_1, \dots, \pi_k)^T = 0$

<u>Claim</u>: Let v be EV for λ, $\langle v, \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_k \end{pmatrix} \rangle = 0$, let $i \in \{1, \dots, k\}$ s.t. $v_i \neq 0 \Rightarrow \exists j \neq i$ s.t. $v_j = -v_i$

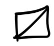
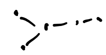
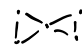
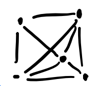
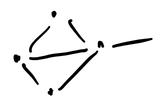
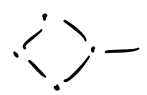
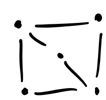
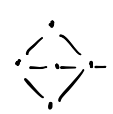
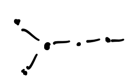
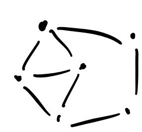
<u>Proof</u>: Assume not.

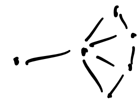
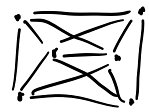
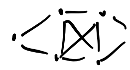
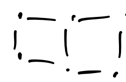
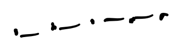
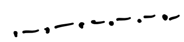
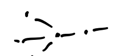
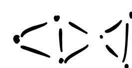
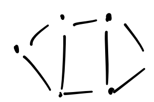
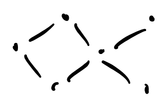
It holds that $B \cdot v = \lambda \cdot v \Rightarrow \lambda \cdot v_i = \sum_{j=1}^{k} b_{ij} v_j \Rightarrow (\lambda - b_{ii}) v_i = \sum_{j \neq i} b_{ij} v_j$

$\Downarrow$

$P_B(B) \cdot v = 0$

i.P. $g_\lambda(B) \cdot v = 0$

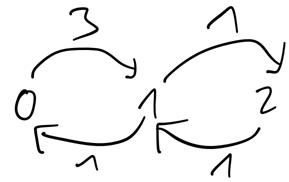
ρ_G	λ	coeff.	
x^2-3	$\sqrt{3}$	$1^{3}, \lambda^{1}$	
x^2-x-1	$\frac{1+\sqrt{5}}{2}$	$1^{2}, \lambda^{2}$	
x^2-x-4	$\frac{1+\sqrt{17}}{2}$	$1^{2}, \frac{\lambda}{2}^{2}$	
x^4-4x^2+2	$\sqrt{\sqrt{2}+2}$	$1^{1}, \lambda^2-3^{1}, \frac{-\lambda^3+4\lambda}{2}^{2}, \lambda^3-3\lambda^{1}$	
x^2-x-4	$\frac{1+\sqrt{17}}{2}$	$1^{1}, \frac{\lambda}{4}^{4}$	
x^2-2x-6	$1+\sqrt{7}$	$1^{3}, \frac{\lambda-2}{2}^{2}$	
x^4-6x^2-4x+2	$\frac{\sqrt{2}}{2}+\sqrt{\sqrt{2}+\frac{5}{2}}$	$1^{1}, \lambda^3-\lambda^2-\lambda+1^{2}, -\lambda^3+2\lambda^2+3\lambda-2^{1}, \lambda^3-\lambda^2-4\lambda-1^{1}$	
x^4-5x^2+2	$\sqrt{\frac{5+\sqrt{17}}{2}}$	$1^{1}, \lambda^{1}, \frac{\lambda^2-1}{2}^{2}, \frac{\lambda^3-3\lambda}{2}^{1}$	
x^2-6	$\sqrt{6}$	$1^{3}, \frac{\lambda}{2}^{2}$	
$x(x^2-x-3)$	$\frac{1+\sqrt{13}}{2}$	$1^{1}, \frac{\lambda}{2}^{2}, \frac{\lambda^2-\lambda}{6}^{2}=\frac{1}{2}$	
x^2-2x-4	$1+\sqrt{5}$	$1^{4}, \lambda-2^{1}$	
x^4-7x^2+3	$\sqrt{\frac{7+\sqrt{37}}{2}}$	$1^{1}, \frac{\lambda}{3}^{3}, \frac{\lambda^2}{3}^{1}, \frac{\lambda^3}{3}-2\lambda^{1}$	
$x(x^4-5x^2+5)$	$\sqrt{\frac{5+\sqrt{5}}{2}}$	$1^{1}, \lambda^2-5^{2}, -\lambda^2+4^{1}, -\lambda^3+4\lambda^{1}, \lambda^3-3\lambda^{1}$	
$(x-1)(x^2-x-5)$	$\frac{1+\sqrt{21}}{2}$	$1^{2}, \frac{2\lambda}{5}^{2}, \frac{\lambda}{5}^{2}$	

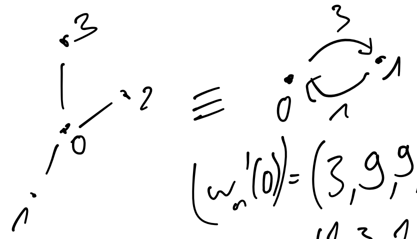
$x^4-x^3-6x^2-x+1$	$\frac{1+\sqrt{2}\sqrt{\sqrt{33}+9}+\sqrt{33}}{4}$	$1^{2}, 2\lambda^3-\lambda^2-13\lambda-7^{1}, \lambda^3-\lambda^2-5\lambda-2^{1}, -\lambda^3+\lambda^2+6\lambda+2^{2}$	
x^2-3x-6	$\frac{3+\sqrt{33}}{2}$	$1^{2}, \frac{\lambda-1}{4}^{4}$	
x^2-3x-2	$\frac{3+\sqrt{17}}{2}$	$1^{4}, \lambda-6^{2}$	
x^2-2x-1	$1+\sqrt{2}$	$1^{2}, \frac{\lambda-1}{2}^{4}$	
$x(x^2-3)$	$\sqrt{3}$	$1^{2}, \lambda^{2}, \lambda^2-1^{1}=2$	
x^4-4x^2+2	$\sqrt{\sqrt{2}+2}$	$1^{2}, \lambda^{2}, \lambda^2-1^{2}, \lambda^3-2\lambda^{1}$	
x^4-5x^2+3	$\sqrt{\frac{5+\sqrt{13}}{2}}$	$1^{3}, \lambda^{1}, \lambda^2-3^{1}, \lambda^3-4\lambda^{1}$	
$x^4-2x^3-5x^2+6x+4$	$\frac{1+\sqrt{4\sqrt{5}+13}}{2}$	$1^{2}, -\lambda^3+\lambda+\frac{5}{2}\lambda-3^{1}, \lambda^3-\lambda^2-\frac{3}{2}\lambda+2^{1}, \frac{\lambda^2-\lambda-4}{2}^{2}$	
x^2-2x-2	$1+\sqrt{3}$	$1^{4}, \lambda-2^{2}$	
x^4-6x^2+4	$\sqrt{3+\sqrt{5}}$	$1^{2}, \frac{\lambda^2-4}{2}^{2}, \frac{\lambda^3}{2}-2\lambda, -\frac{\lambda^3}{2}+3\lambda^{1}$	

$x_1 = 4x_0$

$x_3 = 2x_0$

$x_0 = 3(x_0 + 2x_3 + 4x_1)$





$x_0 = 3x_1$

$x_2 = x_1$

$x_0 + x_2 = 2x_1$

$(w_n'(0)) = (3, 9, 9, 27, \dots)$

$w_n(0) = (\frac{1}{2}, \frac{3}{4}, \frac{1}{2}, \frac{3}{4}, \dots$

$w_n(1) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \dots)$

for $n > 0$

$v \in V: \quad W_n(v) = \dfrac{\#\{\text{walks with second last} = v\}}{w_n}$

$w_n'(0) = (3, 9, 18, 36, \dots)$

$w_n(0) = (\frac{1}{4}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \dots$

$w_n'(1) = (6, 12, 24, 48, \dots)$

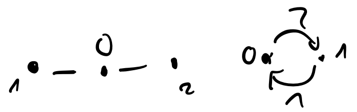
$w_n(1) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots$

$w_n'(2) = (3, 3, 6, 12, 24$

$w_n(2) = (\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \dots$

$\left(\frac{3-\sqrt{5}}{2}\right)^2 = \frac{14-2\sqrt{5}}{4}$

$= \frac{7-\sqrt{5}}{2}$



$(1,2) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$w_n'(0) = (2, 4, 4, 8, 8, \dots$

$w_n(0) = \left(\frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{2}, \dots\right)$

$w_n(1) = \left(\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \dots\right)$

$(1+\sqrt{2})^2 = 3+2\sqrt{2}$

$(1-\sqrt{2})^2 = 3-2\sqrt{2}$

$\dfrac{1}{3+}$

$1+\sqrt{2}$

$1-\sqrt{2}$

$x = \cfrac{1}{8+\cfrac{1}{4+\cfrac{1}{8+\cfrac{1}{4+1}}}} \quad \dots$

$\frac{3}{\sqrt{2}} - 2 = \frac{3-2\sqrt{2}}{\sqrt{2}} = \frac{(1-\sqrt{2})^2}{\sqrt{2}}$

$\dfrac{1}{\frac{1}{x} - 8} - 4 = x \iff \dfrac{x}{1-8x} - 4 = x$

$\iff x - 4 + 32x = x - 8x^2$

$\frac{9-6\sqrt{2}}{2} = \frac{3}{\sqrt{2}} \cdot \frac{(1-\sqrt{2})^2}{\sqrt{2}} \iff x^2 + 4x - \frac{1}{2} = 0$

$\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$

$(p_1, p_2, p_3) \cdot \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} = (p_1, p_2, p_3)$

$\frac{1}{2} p_2 = p_1$

$p_1 + p_3 = p_2$

$1/2\, p_2 = p_3$

$(1/4,\ \frac{1}{2},\ 1/4)$

$$\left(\frac{1}{2},\ \frac{1}{2}\right) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{1}{3},\ \frac{2}{3}\right)$$

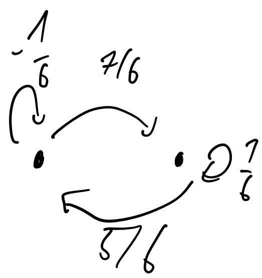
$$\left(\frac{1}{3},\ \frac{2}{3}\right) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{1}{2},\ \frac{1}{2}\right)$$

$\frac{a}{2} + \frac{c}{2} = \frac{1}{3}$ $\qquad 3a + 3c = 2$

$\frac{b}{2} + \frac{d}{2} = \frac{2}{3}$ $\qquad 3b + 3d = 4$

$\frac{a}{3} + \frac{2c}{3} = \frac{1}{2}$ $\qquad 2a + 4c = 3$

$\frac{b}{3} + \frac{2d}{3} = \frac{1}{2}$ $\qquad 2b + 4d = 3$



$\frac{3}{2} a = -\frac{1}{4}$ $\qquad a = -\frac{1}{6}$

$$\begin{pmatrix} -\frac{1}{6} & \frac{7}{6} \\ 5/6 & 1/6 \end{pmatrix}$$

$-\frac{1}{2} + 3c = 2$

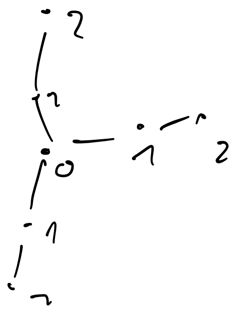
$c = \frac{5}{6}$

$$\frac{1}{6} \cdot \begin{pmatrix} -1 & 7 \\ 5 & 1 \end{pmatrix}$$

$\frac{3}{2} b = \frac{7}{4}$

$b = \frac{7}{6}$ $\qquad \frac{7}{3} + 4d = 3$

$d = \frac{1}{6}$

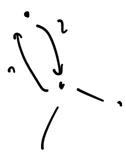


$v_1 = v_0 + v_2$ d

$\frac{2}{3} v_1 = \frac{1}{3} v_2 + v_0$

wd

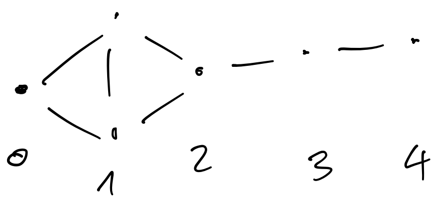
$2 v_1 = v_0 + v_2$

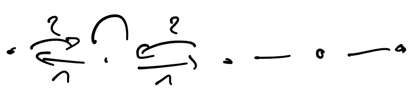


$3 v_1 = 3 v_2 + v_0$ d

d: $\frac{1}{3} v_1 + v_4 = \frac{1}{2} v_0 + \frac{1}{2} v_3$

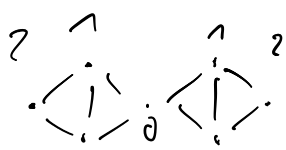
wd: $v_1 + v_4 = v_0 + v_3$





d: $\frac{1}{6} v_1 + v_4 = \frac{1}{2} v_0 + \frac{1}{2} v_3$

wd: $\frac{1}{2} v_1 + v_4 = v_0 + v_3$



d: $\frac{4}{3} v_1 = \frac{1}{2} v_0 + v_2$

wd: $2 v_1 = v_0 + v_2$

d: $v_0 + v_2 = \frac{2}{3} v_1$

wd: $2 v_0 + v_2 = v_1$

$A \cdot ($

$$\begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} -2/3 \\ 1/3 \\ 1 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 3 \end{pmatrix}}_{\begin{pmatrix} 1 & 3 \\ 1 & 1 \\ 3 & 3 \end{pmatrix}}}_{\begin{pmatrix} 1 & 2 \\ 1 & 3/2 \end{pmatrix}}$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 \\ 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \\ 3 & 9 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 1/5 & 0 & 1/5 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ 1 & 3 \\ 3 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 5/3 \\ 1 & 11/5 \end{pmatrix}$$

$(1, 2, 0 \text{———})$ $\quad$ $(\frac{1}{2}, 1, 0, \text{—})$

$(1, 0, \text{———}, 2)$ $\quad$ $(\frac{1}{2}, 0, \text{—}, 1)$

$a+d=2$
$ad-bc=0$

x^2-2x

$\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

$\mathcal{H}_K \subseteq K[[t]]$ linear recurrent sequences Ω^2

$K[t]$ acts on $\mathcal{H}_K$ via

$$p \cdot \sigma = \tilde{\pi}(p(t^{-1}) \cdot \sigma(t)) \text{ where } \tilde{\pi} : K[[t, t^{-1}]] \to K[[t]]$$

natural projection

$\mathcal{H}_K(p) \subseteq \mathcal{H}_K$ set of p-recurrent sequences

$\sigma \in \mathcal{H}_K(p)$ iff. $p \cdot \sigma = 0$

It holds $\mathcal{H}_K(p) \cong K[t]/(p)$ via

$$\sigma(t) \longmapsto p(t) \cdot t^{-1} \cdot \sigma(t^{-1})$$

$$t \cdot \frac{q(t^{-1})}{p(t^{-1})} \longleftarrow q(t)$$

If $p | q \Rightarrow \mathcal{H}_K(p) \subseteq \mathcal{H}_K(q)$

For given $\sigma \in \mathcal{H}_K(p)$, what is the smallest q with $\sigma \in \mathcal{H}_K(q)$?

$$\mathcal{H}_K(p) \cong K[t]/(p)$$

which $\sigma \in K[t]/(p)$ share common factors with p?

iff σ is a zero divisor in $K[t]/(p)$

What is multiplication on the level of power series?

$\sigma_1, \sigma_2 \in \mathcal{H}_K(p) \rightsquigarrow \sigma_1(t) \circ \sigma_2(t) := \sigma_1(t) \cdot \sigma_2(t) \cdot p(t^{-1}) \cdot t$

$v_2 (x_2 y_1 - y_2 x_1)$

$\langle x, v\rangle y - \langle y, v\rangle \cdot x = v \times (x \times y)$

v

$\langle \langle v_1, v\rangle \cdot v_2 - \langle v_2, v\rangle \cdot v_1, v\rangle = \langle v_1, v\rangle \langle v_2, v\rangle - \langle v_2, v\rangle \langle v_1, v\rangle = 0$

$(4,2,1,3,2)$

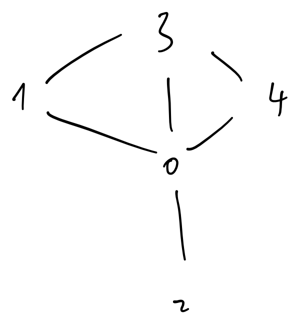
$(1,\frac{1}{2},\frac{3}{2},1)$

$\{0,1\}$ $\{0,2\}$ $\{0,3\}$ $\{0,4\}$

v_1 v_2 v_3 v_4

$v_2-\frac{3}{2}v_1$ $v_3-\frac{1}{2}v_1$ v_4-v_1

w_2 w_3 w_4



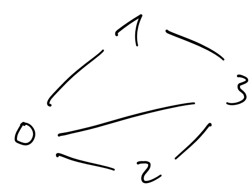
$5w_3+w_2$ w_4-0w_2

u_1 u_2

u_2+0u_1

$\vdots$

$(3,2,2,3)$



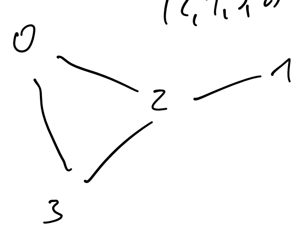
$\{0,1\}$ $\{0,2\}$ $\{0,3\}$

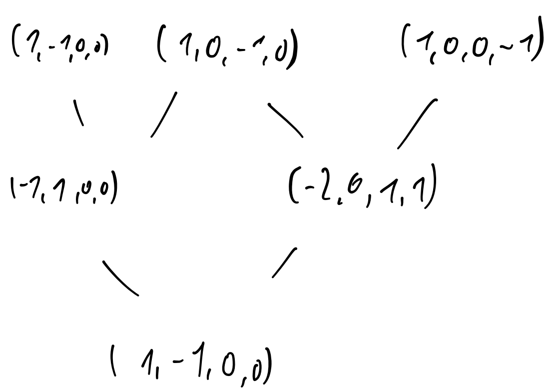
v_1 v_2 v_3

v_2-v_1 v_3

$(2,1,3,2)$

v_1 v_2 v_3





(2, 2, 3, 1)

(5, 5, 5, 3)

$$v_1, v_2 \longmapsto \langle v_1, v\rangle v_2 - \langle v_2, v\rangle v_1$$

$$\langle \langle v_1, v\rangle v_2 - \langle v_2, v\rangle v_1, A\cdot v\rangle = \langle v_1, v\rangle\langle v_2, A\cdot v\rangle - \langle v_2, v\rangle\langle v_1, A\cdot v\rangle$$

$$\langle [v_1, v_2]_v, [v_1, v_3]_v\rangle = \langle\langle v_1, v\rangle v_2 - \langle v_2, v\rangle v_1, \langle v_1, v\rangle v_3 - \langle v_3, v\rangle v_1\rangle$$

$$= \langle v_1, v\rangle^2\langle v_2, v_3\rangle - \langle v_2, v\rangle\langle v_1, v\rangle\langle v_1, v_3\rangle - \langle v_1, v\rangle\langle v_3, v\rangle\langle v_1, v_2\rangle + \langle v_2, v\rangle\langle v_3, v\rangle\langle v_1, v_1\rangle$$

definiere $[v_1, v_2]_v := \langle v_1, v\rangle v_2 - \langle v_2, v\rangle v_1$

Lie-Algebra

$$[[v_1, v_2]_v, v_3]_v = -\langle v_3, v\rangle [v_1, v_2]_v$$

$$[v_1, [v_2, v_3]_v]_v = \langle v_1, v\rangle [v_2, v_3]_v$$

$$[[v_1, v_2]_v, [v_3, v_4]_v]_v = \underbrace{\langle [v_1, v_2]_v, v\rangle}_{0} [v_3, v_4]_v$$

$$\mathbb{R}^n_v \supseteq [\mathbb{R}^n_v, \mathbb{R}^n_v]_v \supseteq [[\mathbb{R}^n_v, \mathbb{R}^n_v], [\mathbb{R}^n_v, \mathbb{R}^n_v]]_v = 0$$

$$A\cdot[v_1, v_2]_{A\cdot v} = [A\cdot v_1, A\cdot v_2]_v$$

$(4, 4, 5, 2, 3, 4, 4)$

$(16, 17, 17, 9, 13, 13, 17)$

$$\mathbb{R}^n_{A^2\cdot v} \xrightarrow{A\cdot} \mathbb{R}^n_{A\cdot v} \xrightarrow{A\cdot} \mathbb{R}^n_v$$

$(0, 1, 1, 0, 0, 0, 0) -$
$(0, 1, 0, 1, 0, 0, 0)$
$(8, -7, 0, -1, 0, 0, 0)$

$(2, 3, 0, 0, 1, 0, 0)$

$(2, 3, 0, 0, 0, 1, 0)$

$(2, 3, 0, 0, 0, 0, 1)$

$$0 \to [\mathbb{R}^n_v, \mathbb{R}^n_v]_v \to \mathbb{R}^n_v \xrightarrow{\pi} X \to 0$$

$$X = \{\lambda v \mid \lambda \in \mathbb{R}\}$$

Let $y \in v^T \Rightarrow \langle y, v\rangle = 0$

$\Rightarrow y = \langle \frac{1}{\langle v,v\rangle} v, v\rangle y - \langle y, v\rangle \frac{1}{\langle v,v\rangle} v$

iff. $A^n v \in \langle A^i v \mid i \in \{0, \dots, n-1\}\rangle$

smallest A-invariant subspace (containing v)?

$(0$

$$v,\ Av,\ A^2 v,\ A^3 v = \lambda_0 v + \lambda_1 Av + \lambda_2 A^2 v$$

$$\overset{\shortparallel}{(1, \dots, 1)} \qquad \lambda_i = ?$$

$$e_1, \dots, e_n$$

$$\to v, e_2 - e_1, \dots, e_n - e_1$$

$$\to v, Av, v_3, \dots, v_n \qquad \langle v_i \rangle = \langle v, Av \rangle^{\perp}$$

$$\begin{tikzcd}[column sep=small]
v_1 \arrow[d] & v_2 \arrow[d] & v_3 \\
v & \langle v, v_1 \rangle v_2 - \langle v, v_2 \rangle v_1 &
\end{tikzcd}$$

$$\langle v, v \rangle Av - \langle Av, v \rangle v = [v, Av]_v = \delta_0 Av - \delta_1 v$$

$$\langle [v, Av], [v, Av] \rangle = \langle v, v \rangle^2 \langle Av, Av \rangle - \langle v, v \rangle \langle Av, v \rangle^2$$

$$A^2 v = \lambda_0 v + \lambda_1 Av$$

$$\langle A^2 v, [v, Av]_v \rangle =$$

$$A^2 v = \frac{\langle A^2 v, [v, Av]_v \rangle}{\langle [v,Av],[v,Av]\rangle} [v,Av]_v + \frac{\langle A^2 v, v\rangle}{\langle v,v\rangle} v$$

$$= \frac{\langle A^2 v, \langle v,v\rangle Av - \langle Av,v\rangle v\rangle}{\langle \qquad \rangle} \left(\langle v,v\rangle Av - \langle Av,v\rangle v\right) + \frac{\langle A^2 v,v\rangle}{\langle v,v\rangle} v$$

$$= \frac{\langle v,v\rangle\langle A^2 v, Av\rangle - \langle Av,v\rangle\langle v, A^2 v\rangle}{\langle v,v\rangle^2\langle Av,Av\rangle - \langle v,v\rangle\langle Av,v\rangle^2}\left(\langle v,v\rangle Av - \langle Av,v\rangle v\right) + \dots$$

$$\lambda_1 = \frac{\langle v,v\rangle\langle A^3 v,v\rangle - \langle Av,v\rangle\langle A^2 v,v\rangle}{\langle v,v\rangle\langle A^2 v,v\rangle - \langle Av,v\rangle^2} = \frac{\delta_0\delta_3 - \delta_1\delta_2}{\delta_0\delta_2 - \delta_1^2}$$

$$\lambda_0 = \frac{-1}{\langle v,v\rangle}\left(\frac{\langle A^3 v,v\rangle\langle v,v\rangle - \langle Av,v\rangle\langle v,A^2 v\rangle}{\langle v,v\rangle\langle A^2 v,v\rangle - \langle Av,v\rangle^2}\langle Av,v\rangle - \langle A^2 v,v\rangle\right)$$

$$= \frac{1}{\delta_0}\left(\frac{\delta_1\delta_2 - \delta_0\delta_3}{\delta_0\delta_2 - \delta_1^2}\,\delta_1 - \delta_2\right)$$

$$= -\frac{\delta_1}{\delta_0}\lambda_1 - \frac{\delta_2}{\delta_0}$$

$\delta_0 = 4,\ \delta_1 = 6,\ \delta_2 = 10,\ \delta_3 = 16$

$$\lambda_1 = \frac{4\cdot 16 - 6\cdot 10}{4\cdot 10 - 36} = \frac{56-60}{4} = -1$$

$$\lambda_2 = -\frac{6}{4} \cdot (-1) - \frac{10}{4} = -1$$

$$A^2 v - \frac{\langle A^2 v, [v, Av]_v \rangle}{\langle [\quad], [\quad] \rangle} [v, Av]_v - \frac{\langle A^2 v, v \rangle v}{\langle v, v \rangle}$$

$$= A^2 v - \left(\quad \cdots \quad \right) [\quad] - \frac{\delta_2}{\delta_0} v$$

$$= A^2 v - \lambda_1 A v - \lambda_0 v$$

$$\frac{\langle A^3 v, \ldots \rangle}{\langle \ldots, \ldots \rangle} = \underline{\delta_5 - \lambda_1 \delta_4 - \lambda_0 \delta_3}$$

$$\delta_0\delta_2\delta_5 - \delta_1^2\delta_5 - \delta_0\delta_3\delta_4 + \delta_1\delta_2\delta_4 - \delta_1\delta_3^2 + \delta_2^2\delta_3$$

$$\delta_5(\delta_0\delta_2 - \delta_1^2) - \delta_4(\delta_0\delta_3 - \delta_1\delta_2) - \delta_3(\delta_1\delta_3 - \delta_2^2)$$

$$(\delta_0\delta_3 - \delta_1\delta_2)^2 = \delta_0^2\delta_3^2 - 2\delta_0\delta_1\delta_2\delta_3 + \delta_1^2\delta_2^2$$

$$(\delta_0\delta_2 - \delta_1^2)^2 = \delta_0^2\delta_2^2 - 2\delta_0\delta_1^2\delta_2 + \delta_1^4$$

$$\delta_0\delta_2\delta_4 - \delta_1^2\delta_4 - 2\delta_0\delta_3^2 + 2\delta_2^3$$

$$+ \frac{1}{\delta_0\delta_2\delta_1^2}$$

Grad von δ_G:

Start with $v_i := e_i$ for $i = 1, \dots, n$; $v := \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$ $\qquad B = I_n, \; v = \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}$

Compute $w := v^T \cdot B$ $\qquad v^T \cdot B = (\langle v, v_1\rangle, \langle v, v_2\rangle, \dots, \langle v, v_n\rangle)$

If $w = 0$:

return $n-k$

Pick $i \in \{1, \dots, n-k\}$ s.t. $w_i \neq 0$

For $j \neq i$:

$$\left.v_j := w_i v_j - w_j v_i\right\} B := B \cdot \underbrace{\begin{pmatrix} w_i & & & & 0 \\ 0 & \ddots & & & \\ -w_1 & \cdots & -w_{i-1} & -w_{i+1} & \cdots & -w_n \\ & & w_i & & \\ 0 & & & \ddots & 0 \\ & & & & w_i \end{pmatrix}}_{n \times (n-1)}$$

$v := Av$

$$v_0 = v \qquad \langle v_0, v_0 \rangle = \delta_0$$

$$\langle A^i v_0, v_0 \rangle = \delta_i$$

$$v_1 = \delta_0 A v - \delta_1 v \qquad \langle v_1, v_1 \rangle = \delta_0^2 \delta_2 - \delta_0 \delta_1^2 = \delta_0 (\delta_0 \delta_2 - \delta_1^2)$$

$$\langle A^i v_1, v_1 \rangle = \delta_0 \delta_{i+1} - \delta_1 \delta_i$$

$$v_2 = \langle v_0, v_0 \rangle \langle v_1, v_1 \rangle A^2 v - \langle v_0, v_0 \rangle \langle A^2 v_1, v_1 \rangle v_1 - \langle v_1, v_1 \rangle \langle A^2 v_1, v_0 \rangle v_0$$

$$= \delta_0^2 (\delta_0 \delta_2 - \delta_1^2) A^2 v - \delta_0 (\delta_0 \delta_3 - \delta_1 \delta_2) v_1 - \delta_0 (\delta_0 \delta_2 - \delta_1^2) \delta_2 v_0$$

$$= \delta_0^2 (\delta_0 \delta_2 - \delta_1^2) A^2 v - \delta_0^2 (\delta_0 \delta_3 - \delta_1 \delta_2) A v + \big(\delta_0 \delta_1 (\delta_0 \delta_3 - \delta_1 \delta_2) - \delta_0 \delta_2 (\delta_0 \delta_2 - \delta_1^2)\big) v$$

$$= (\delta_0 \delta_2 - \delta_1^2) A^2 v - (\delta_0 \delta_3 - \delta_1 \delta_2) A v + (\delta_1 \delta_3 - \delta_2^2) v$$

$$\langle v_2, v_2 \rangle = (\delta_0 \delta_2 - \delta_1^2)^2 \delta_4 + (\delta_0 \delta_3 - \delta_1 \delta_2)^2 \delta_2 + (\delta_1 \delta_3 - \delta_2^2)^2 \delta_0$$
$$- 2 (\delta_0 \delta_2 - \delta_1^2)(\delta_0 \delta_3 - \delta_1 \delta_2) \delta_3 + 2 (\delta_0 \delta_2 - \delta_1^2)(\delta_1 \delta_3 - \delta_2^2) \delta_2$$
$$- 2 (\delta_0 \delta_3 - \delta_1 \delta_2)(\delta_1 \delta_3 - \delta_2^2) \delta_1$$

$$= (\delta_0 \delta_2 - \delta_1^2)^2 \delta_4 \qquad + \delta_1^2 \delta_2^3$$
$$- \delta_0^2 \delta_2 \delta_3^2 \qquad + \delta_0 \delta_1^2 \delta_3^2$$
$$+ 2 \delta_0 \delta_1 \delta_2^2 \delta_3 - \delta_0 \delta_2^4 - 2 \delta_1^3 \delta_2 \delta_3$$

$$= (\delta_0 \delta_2 - \delta_1^2)^2 \delta_4 + \delta_0 \delta_2 (-\delta_0 \delta_3^2 + 2 \delta_1 \delta_2 \delta_3 - \delta_2^3) + \delta_1^2 (\delta_2^3 + \delta_0 \delta_3^2 - 2 \delta_1 \delta_2 \delta_3)$$

$$= (\delta_0\delta_2 - \delta_1^2)^2 \delta_4 - (\delta_0\delta_2 - \delta_1^2)\big(\underbrace{\delta_0\delta_3^2 - 2\delta_1\delta_2\delta_3 + \delta_2^3}_{\delta_3(\delta_0\delta_3 - \delta_1\delta_2) - \delta_2(\delta_1\delta_3 - \delta_2^2)}\big)$$

$$= (\delta_0\delta_2 - \delta_1^2)^2 \delta_4 - (\delta_0\delta_2 - \delta_1^2)\big(\delta_3(\delta_0\delta_3 - \delta_1\delta_2) - \delta_2(\delta_1\delta_3 - \delta_2^2)\big)$$

$$= (\delta_0\delta_2 - \delta_1^2)\big(\delta_4(\delta_0\delta_2 - \delta_1^2) - \delta_3(\delta_0\delta_3 - \delta_1\delta_2) + \delta_2(\delta_1\delta_3 - \delta_2^2)\big)$$

$$\langle A^i v, v_2\rangle = (\delta_0\delta_2 - \delta_1^2)\delta_{i+2} - (\delta_0\delta_3 - \delta_1\delta_2)\delta_{i+1} + (\delta_1\delta_3 - \delta_2^2)\delta_i$$

$\vdots$

$$k=1: \quad Av = \frac{\langle v_0, Av\rangle}{\langle v_0, v_0\rangle} v_0 = \frac{\delta_1}{\delta_0} v$$

$$k=2: \quad A^2 v = \frac{\langle v_0, A^2 v\rangle}{\langle v_0, v_0\rangle} v_0 + \frac{\langle v_1, A^2 v_1\rangle}{\langle v_1, v_1\rangle} v_1$$

$$= \frac{\delta_2}{\delta_0} v + \frac{\delta_0\delta_3 - \delta_1\delta_2}{\delta_0(\delta_0\delta_2 - \delta_1^2)}(\delta_0 Av - \delta_1 v)$$

$$= \frac{\delta_0\delta_3 - \delta_1\delta_2}{\delta_0\delta_2 - \delta_1^2} Av - \frac{\delta_1\delta_3 - \delta_2^2}{\delta_0\delta_2 - \delta_1^2} v = \frac{M_{013}}{M_{012}} Av - \frac{M_{113}}{M_{012}} v$$

$$k=3: \quad A^3 v = \frac{\langle v_0, A^3 v\rangle}{\langle v_0, v_0\rangle} v_0 + \frac{\langle v_1, A^3 v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \frac{\langle v_2, A^3 v_2\rangle}{\langle v_2, v_2\rangle} v_2$$

$$= \frac{\delta_3}{\delta_0} v + \frac{\delta_0\delta_4 - \delta_1\delta_3}{\delta_0(\delta_0\delta_2 - \delta_1^2)}(\delta_0 Av - \delta_1 v) + \frac{(\delta_0\delta_2 - \delta_1^2)\delta_5 - (\delta_0\delta_3 - \delta_1\delta_2)\delta_4 + (\delta_1\delta_3 - \delta_2^2)\delta_3}{(\delta_0\delta_2 - \delta_1^2)\big(\delta_4(\delta_0\delta_2 - \delta_1^2) - \delta_3(\delta_0\delta_3 - \delta_1\delta_2) + \delta_2(\delta_1\delta_3 - \delta_2^2)\big)} v_2$$

$$= \frac{\delta_3}{\delta_0} v + \frac{\mu_{0,4}}{\delta_0 \mu_{0,2}} (\delta_0 A v - \delta_1 v) + \frac{\delta_5 \mu_{0,2} - \delta_4 \mu_{0,3} + \delta_3 \mu_{1,3}}{\mu_{0,2}(\delta_4 \mu_{0,2} - \delta_3 \mu_{0,3} + \delta_2 \mu_{1,3})} \left(\mu_{0,2} A^2 v - \mu_{0,3} A v + \mu_{1,3} v\right)$$

$$= \frac{\delta_5 \cdot \mu_{0,2} - \delta_4 \cdot \mu_{0,3} + \delta_3 \cdot \mu_{1,3}}{\delta_4 \cdot \mu_{0,2} - \delta_3 \cdot \mu_{0,3} + \delta_2 \cdot \mu_{1,3}} A^2 v$$

$$+ \left(\frac{\mu_{0,4}}{\mu_{0,2}} - \mu_{0,3} \frac{\delta_5 \mu_{0,2} - \delta_4 \mu_{0,3} + \delta_3 \mu_{1,3}}{\mu_{0,2}(\delta_4 \mu_{0,2} - \delta_3 \mu_{0,3} + \delta_2 \mu_{1,3})} \right) A v$$

$$\frac{\mu_{0,4}(\delta_4 \mu_{0,2} - \delta_3 \mu_{0,3} + \delta_2 \mu_{1,3}) - \delta_5 \mu_{0,2} \mu_{0,3} + \delta_4 \mu_{0,3}^2 - \delta_3 \mu_{0,3} \mu_{1,3}}{\mu_{0,2}(\delta_4 \mu_{0,2} - \delta_3 \mu_{0,3} + \delta_2 \mu_{1,3})}$$

$$\delta_4 (\delta_0\delta_4 - \delta_1\delta_3)(\delta_0\delta_2 - \delta_1^2) - \delta_3 (\delta_0\delta_4 - \delta_1\delta_3)(\delta_0\delta_3 - \delta_1\delta_2) + \delta_2 (\delta_0\delta_4 - \delta_1\delta_3)(\delta_1\delta_3 - \delta_2^2)$$

$$- \delta_5 (\delta_0\delta_2 - \delta_1^2)(\delta_0\delta_3 - \delta_1\delta_2) + \delta_4 (\delta_0\delta_3 - \delta_1\delta_2)^2 - \delta_3 (\delta_0\delta_3 - \delta_1\delta_2)(\delta_1\delta_3 - \delta_2^2)$$

$$= \delta_0^2 \delta_2 \delta_4^2 - \delta_0 \delta_1^2 \delta_4^2 \qquad + \delta_1^3 \delta_3 \delta_4$$

$$- \delta_0 \delta_2^3 \delta_4 - \delta_1^2 \delta_2 \delta_3^2$$

$$- \delta_0^2 \delta_2 \delta_3 \delta_5 + \delta_0 \delta_1 \delta_2^2 \delta_5 + \delta_0 \delta_1^2 \delta_3 \delta_5 - \delta_1^3 \delta_2 \delta_5$$

$$- \delta_0 \delta_1 \delta_2 \delta_3 \delta_4 + \delta_1^2 \delta_2^2 \delta_4 \qquad + \delta_0 \delta_2^2 \delta_3^2$$

$$= - \delta_5 \mu_{0,2} \mu_{0,3} + \delta_0 \delta_2 (\delta_0 \delta_4^2 - \delta_2^2 \delta_4 - \delta_1 \delta_3 \delta_4 + \delta_2 \delta_3^2) - \delta_1^2 (\delta_0 \delta_4^2 - \delta_1 \delta_3 \delta_4 + \delta_2 \delta_3^2 - \delta_2^2 \delta_4)$$

$$= - \mu_{0,2} (\delta_5 \mu_{0,3} - \delta_4 \mu_{0,4} \qquad)$$

$$\frac{\delta_5 \mu_{0,3} - \delta_4 (\mu_{0,4} + \mu_{1,3}) + \delta_3 \mu_{1,4}}{\delta_4 \mu_{0,2} - \delta_3 \mu_{0,3} + \delta_2 \mu_{1,3}} \cdot A v$$

$$\frac{\delta_3}{\delta_0} \quad - \frac{\delta_1 \mu_{0,4}}{\delta_0 \mu_{0,2}} \quad + \mu_{1,3} \frac{\delta_5 \mu_{0,2} - \delta_4 \mu_{0,3} + \delta_3 \mu_{1,3}}{\mu_{0,2}(\delta_4 \mu_{0,2} - \delta_3 \mu_{0,3} + \delta_2 \mu_{1,3})}$$

$$\delta_3 \mu_{0,2} (\delta_4 \mu_{0,2} - \delta_3 \mu_{0,3} + \delta_2 \mu_{1,3}) - \delta_1 \mu_{0,4} (\delta_4 \mu_{0,2} - \delta_3 \mu_{0,3} + \delta_2 \mu_{1,3}) +$$
$$\delta_0 \mu_{1,3} (\delta_5 \mu_{0,2} - \delta_4 \mu_{0,3} + \delta_3 \mu_{1,3})$$

$$= \delta_5 \mu_{1,3} - \delta_4 \mu_{1,4} + \delta_3 \mu_{2,4}$$

$-(\delta_0 \delta_4 - \delta_1 \delta_3) + \delta_0 \delta_4 - \delta_2^2$

$-\langle w_1 A^3 v \rangle + \mu_{0,4}^{?} \qquad \delta_1 \delta_3 - \delta_2^2$

$\mu'_{0,4} = \mu_{0,4} + \mu_{1,3}$

$-\mu'_{0,4} + \mu_{1,3} = -\mu_{0,4}$

$k = 1: \; Av = \frac{\delta_1}{\delta_0} \cdot v \quad \Leftrightarrow \quad \delta_0 A v - \delta_1 v = 0$

$k = 2: \; A^2 v = \frac{\mu_{0,3}}{\mu_{0,2}} Av - \frac{\mu_{1,3}}{\mu_{0,2}} v \quad \Leftrightarrow \quad \mu_{0,2} A^2 v - \mu_{0,3} Av + \mu_{1,3} v = 0$

$k = 3: \; A^3 v = \frac{\delta_5 \cdot \mu_{0,2} - \delta_4 \cdot \mu_{0,3} + \delta_3 \cdot \mu_{1,3}}{\delta_4 \cdot \mu_{0,2} - \delta_3 \cdot \mu_{0,3} + \delta_2 \cdot \mu_{1,3}} A^2 v$

$$\frac{\delta_5 \mu_{0,3} - \delta_4 \mu'_{0,4} + \delta_3 \mu_{1,4}}{\delta_4 \mu_{0,2} - \delta_3 \mu_{0,3} + \delta_2 \mu_{1,3}} A v$$

$\delta_7 \cdot \delta_{0,3,5} - \delta_6 \cdot \delta_{1,3,5}$
$- \delta_6 \cdot \delta_{0,3,6} + \delta_5 \cdot \delta_{1,3,6}$
$\delta_{0,4,6} - \delta_4 \cdot \delta_{1,4,6}$

$\delta_0 \delta_4^2 - 2\delta_1 \delta_3 \delta_4 + \delta_1 \delta_3^2$

$\delta_5 \mu_{0,3} - \delta_4 \mu_{0,4} - \delta_4 \mu_{1,3} + \delta_3 \mu_{1,4}$

$(\delta_5 \mu_{0,3} - \delta_4 \mu_{1,3}) - (\delta_4 \mu_{0,4} - \delta_3 \mu_{1,4})$

$+\delta_5 \cdot \ldots$

$\mu_{1,2} - \mu_{0,4}' + \mu_{2,4}$

$\delta_0\delta_2 - \delta_1^2 - \delta_0\delta_4\delta_2^2 + \delta_2\delta_4^2 \delta_3^2 \;+\; \dfrac{\delta_5\,\mu_{1,3} - \delta_4\,\mu_{1,4} + \delta_3\,\mu_{2,4}}{\delta_4\,\mu_{0,2} - \delta_3\,\mu_{0,3} + \delta_2\,\mu_{1,3}}\; v$

$(\delta_7 \cdot \delta_{0,1,2,5} - \delta_6 \cdot \delta_{0,1,3,5} + \delta_5 \cdot \delta_{1,3,5}) - (\delta_6 \cdot \delta_{0,1,2,6} - \delta_5 \cdot \delta_{0,1,3,6} + \delta_4 \cdot \delta_{1,3,6})$

$\delta_1 A v - \delta_2 v$

$(\delta_{0,1,4,18} - \delta_8 \cdot \delta_{0,1,2,6})\; \mu_{0,3} A^2 v - \mu_{0,14} A v + \mu_{1,14} v$

$-(\delta_{0,1,5,17} - \ldots$

$\langle \delta_0 A^2 v - \delta_1 A v, A^2 v\rangle = \delta_0\delta_4 - \delta_1\delta_3$

$\langle \delta_1 A v - \delta_2 v, A^2 v\rangle = \mu_{1,3}$

$\delta_0\delta_3 - \delta_1\delta_2$

$\mu_{0,3} - \mu_{0,2} = \delta_0\delta_3 - \delta_1\delta_2 - \delta_0\delta_2 + \delta_1^2 \qquad \delta_0\delta_3 - \delta_2^2$

$= \delta_0(\delta_3 - \delta_2) - \delta_1(\delta_2 - \delta_1)$

$\langle \mu_{0,3} A v - \mu_{1,3} v, A^4 v\rangle$

$\mu_{1,14} - \mu_{0,13} = \delta_1\delta_4 - \delta_2\delta_3 - \delta_0\delta_3 + \delta_1\delta_2 = \delta_1(\delta_2 + \delta_4) - \delta_3(\delta_0 + \delta_2)$

$\mu_{1,13} - \mu_{0,12} = \delta_1\delta_3 - \delta_2^2 - \delta_0\delta_2 + \delta_1^2 = \delta_1(\delta_1 + \delta_3) - \delta_2(\delta_0 + \delta_2)$

$\langle A^2 v + v, A v\rangle$

$$A^3 v = \frac{\mu_{0,3}}{\mu_{0,2}} A^2 v - \frac{\mu_{1,13}}{\mu_{0,2}} A v$$

$$= \frac{\mu_{0,3}}{\mu_{0,2}}\left(\frac{\mu_{0,3}}{\mu_{0,2}} A v - \frac{\mu_{1,3}}{\mu_{0,2}} v\right) - \frac{\mu_{1,13}}{\mu_{0,2}} A v$$

$$= \ldots A v - \frac{\mu_{0,3}\mu_{1,13}}{\mu_{0,2}^2} v$$

$$= \frac{\mu_{0,3}^2 - \mu_{1,13}\mu_{0,2}}{\mu_{0,2}^2} A v$$

$(\delta_0\delta_3 - \delta_1\delta_2)^2 - (\delta_1\delta_3 - \delta_2^2)(\delta_0\delta_2 - \delta_1^2)$

$-\delta_0\delta_1\delta_2\delta_3 + \delta_0\delta_2^3$

$= \delta_0^2\delta_3^2 - 2\delta_0\delta_1\delta_2\delta_3$

$+\delta_1^3\delta_3$

$= \delta_0^2\delta_3^2 - 3\delta_0\delta_1\delta_2\delta_3 + \delta_0\delta_2^3 + \delta_1^3\delta_3$

$= \delta_0\delta_3(\delta_0\delta_3 - \delta_1\delta_2) - \delta_0\delta_2(\delta_1\delta_3 - \delta_2^2)$

$= \delta_0\delta_3(\delta_0\delta_3 - \ldots$

$-\delta_1\delta_3(\delta_0\delta_2 - \delta_1^2) \qquad -\delta_1\delta_3\mu_{0,2}$

$= \delta_0\delta_3\mu_{0,3} - \delta_0\delta_2\mu_{1,3}$

$\langle \delta_0 A v - \delta_1 v, A^3 v\rangle$

$$= \delta_0\delta_2\,\frac{\mu_{0,3}^2 - \mu_{1,13}\mu_{0,2}}{\mu_{0,2}^2}$$

$$-\delta_1^2\,\frac{\mu_{0,3}\mu_{1,3}}{\mu_{0,2}^2}$$

$$\frac{\delta_0\delta_2\mu_{0,3}^2 - \mu_{1,3}(\delta_0\delta_2\mu_{0,2} + \delta_1^2\mu_{0,3})}{\mu_{0,2}^2}$$

$(\delta_0\delta_3 - \delta_1^2)(\delta_1\delta_3 - \delta_2^2)$

$\delta_0\delta_2(\delta_0\delta_3\mu_{0,3} - \delta_0\delta_2\mu_{1,3} - \delta_1\delta_3\mu_{0,2}) - \delta_1^2\mu_{0,3}\mu_{1,3}$

$= \delta_0^3\delta_2\delta_3^2 - 3\delta_0^2\delta_1\delta_2^2\delta_3 + \delta_0^2\delta_2^4 + \delta_0\delta_1^3\delta_2\delta_3$

$-\delta_0\delta_1^3\delta_3^2 + \delta_1^5\delta_3 + \delta_0\delta_1^2\delta_2^2\delta_3 - \delta_1^4\delta_2^2 \qquad -2\delta_0\delta_2\mu_{0,2}$

$$= \delta_1\delta_3 \left(- \delta_0^2\delta_2^2 - \delta_0\delta_1^2\delta_3 + \delta_1^4 \right)$$

$$\mu_{0,2}^2 - \overbrace{2\delta_0^2\delta_2^2 + 2\delta_0\delta_1^2\delta_2 - \delta_0\delta_1^2\delta_3}$$

$$- \delta_2^2 \left(- \delta_0^2\delta_2^2 - \delta_0\delta_1^2\delta_3 + \delta_1^4 \right)$$

$$-2\delta_0^2\delta_1\delta_2^2\delta_3 + \delta_0^3\delta_2\delta_3^2 + \delta_0\delta_1^3\delta_2\delta_3$$

$$\underbrace{\delta_0\delta_3\mu_{0,3} - \delta_1\delta_3\mu_{0,2}}$$

$$= \mu_{1,3}\left(\delta_1^4 - \delta_0^2\delta_2^2 - \delta_0\delta_1^2\delta_3 \right) + \delta_0\delta_2 \left(\delta_0\delta_3^2 - 2\delta_0\delta_1\delta_2\delta_3 + \delta_1^3\delta_3 \right)$$

$$= \mu_{1,3}\left(\mu_{0,2}^2 - 2\delta_0\delta_2\mu_{0,2} - \delta_0\delta_1^2\delta_3 \right) + \delta_0\delta_2 \left(\delta_0\delta_3\mu_{0,3} - \delta_1\delta_3\mu_{0,2} \right)$$

$$= \mu_{1,3}\mu_{0,2}^2 \underbrace{-2\delta_0\delta_2\mu_{0,2}\mu_{1,3}}_{-2\mu_{1,3}\mu_{0,2}^2 - 2\delta_1^2\mu_{0,2}\mu_{1,3}} - \delta_0\delta_1^2\delta_3\mu_{1,3} + \delta_0^2\delta_2\delta_3\mu_{0,3} - \delta_0\delta_1\delta_2\delta_3\mu_{0,2}$$

$$\delta_0\delta_3 - \delta_1\delta_2 - \delta_1\delta_3 + \delta_2^2$$

$$= -\mu_{1,3}\mu_{0,2}^2 - 2\delta_1^2\mu_{0,2}\mu_{1,3} + \underbrace{\delta_0\delta_3\mu_{0,2}\mu_{1,3}}_{(\mu_{0,3}+\delta_1\delta_2)\mu_{0,2}\mu_{1,3}} + \underbrace{\delta_0^2\delta_2\delta_3(\mu_{0,3}-\mu_{1,3})}_{\delta_0\delta_2(\mu_{0,3}+\delta_1\delta_2)(\mu_{0,3}-\mu_{1,3})} \underbrace{- \delta_0\delta_1\delta_2\delta_3\mu_{0,2}}$$

$$(\mu_{0,2}+\delta_1^2)$$

$$= -\mu_{1,3}\mu_{0,2}^2 - 2\delta_1^2\mu_{0,2}\mu_{1,3} + \mu_{0,3}\mu_{0,2}\mu_{1,3} + \delta_1\delta_2\mu_{0,2}\mu_{1,3} + (\mu_{0,2}+\delta_1^2)(\mu_{0,3}+\delta_1\delta_2)(\mu_{0,3}-\mu_{1,3}) - \delta_0\delta_1\delta_2\delta_3\mu_{0,2}$$

$$= -\mu_{1,3}\mu_{0,2}^2 - 2\delta_1^2\mu_{0,2}\mu_{1,3} + \mu_{0,3}\mu_{0,2}\mu_{1,3} + \delta_1\delta_2\mu_{0,2}\mu_{1,3} + (\quad)(\quad)(\quad)$$

Für $i \geq 0$ $\delta_i := \langle v, A^i v \rangle$

Für $0 \leq i_1 < i_2 < \ldots < i_k$: $\delta_{(i_1, \ldots, i_k)} := \sum_{j=1}^{k} (-1)^{j+1} \delta_{i_{k+1-j}} \delta_{(i_1, \ldots, i_{k+1-j}, i_{k+2-j}+1, \ldots, i_k+1)}$

$\delta_{(0,2)} = \delta_2 \cdot \delta_0 - \delta_1^2$

$\delta_{(0,3,5)} = \delta_5 \, \delta_{(0,3)} - \delta_4 \cdot \delta_{(0,4)} + \delta_3 \cdot \delta_{(1,4)}$

$\delta_{(0,2,4)}$ $\delta_{(0,2,5)}$ $\delta_{(0,3,5)}$ $\delta_{(1,3,5)}$

$\delta_{(0,2,5)}$ $\delta_{(0,2,6)}$ $\delta_{(0,3,6)}$ $\delta_{(1,3,6)}$

$\delta_{(0,3,5)}$ $\delta_{(0,3,6)}$ $\delta_{(0,4,6)}$ $\delta_{(1,4,6)}$

$\delta_{(1,3,5)}$ $\delta_{(1,3,6)}$ $\delta_{(1,4,6)}$ $\delta_{(2,4,6)}$

$\delta_{(0,2,6)} = \delta_6 \cdot \delta_{(0,2)} - \delta_5 \cdot \delta_{(0,3)} + \delta_4 \cdot \delta_{(1,3)}$

$+ \delta_{(0,3,5)} - \delta_{(1,3,4)}$

$+ \delta_5 \cdot \delta_{(0,3)} - \delta_4 \cdot \delta_{(0,4)} + \delta_3 \cdot \delta_{(1,3)} - \delta_4 \cdot \delta_{(1,3)} + \delta_3 \cdot \delta_{(1,4)} - \delta_2 \cdot \delta_{(2,4)}$

$\delta_{0,2,5,7} = \delta_7 \cdot \delta_{0,2,5} - \delta_6 \cdot \delta_{0,2,6} + \delta_5 \cdot \delta_{0,3,6} - \delta_4 \cdot \delta_{1,3,6}$

$\delta'_{0,2,5,7} = \delta_7 \cdot \delta_{0,2,5} - \delta_6 (\delta_{0,3,5} + \delta_{0,2,6}) + \delta_5 \cdot (\delta_{1,3,5} + \delta_{0,3,6}) - \delta_4 \cdot \delta_{1,3,6}$

$= \delta_{0,2,5,7} - \delta_6 \cdot \delta'_{0,3,5} + \delta_5 \cdot \delta_{1,3,5}$

$\delta_6 \cdot \delta_{0,3} - \delta_5 \cdot \delta_{0,4} + \delta_4 \cdot \delta_{1,4}$

$\delta'_{0,3,5} = \delta_{0,3,5} - \delta_4 \cdot \delta_{1,3}$

$\delta'_{0,3,6} = \delta_6 \cdot \delta_{0,3} - \delta_5 (\delta_{0,4} + \delta_{1,3}) + \delta_4 \delta_{1,4}$

$\delta'_{0,3,5,7} = \delta_{0,3,5,7} - \delta_6 \delta_{1,3,5}$?

$\delta_5 \cdot \delta_{0,3} - \delta_4 \cdot \delta_{0,4} + \delta_3 \cdot \delta_{1,4}$

$\delta_7 \cdot \delta_{0,3,5} - \delta_6 \cdot \delta_{0,3,6} + \delta_5 \cdot \delta_{0,4,6} - \delta_4 \cdot \delta_{1,4,6}$

$\delta_6 \cdot \delta_{0,4} - \delta_5 \cdot (\delta_{0,5} + \delta_{1,4}) + \delta_4 \delta_{1,5}$

$\delta'_{0,3,6} = \delta_6 \cdot \delta_{0,3} - \delta_5 \cdot \delta'_{0,4} + \delta_4 \cdot \delta_{1,4}$

$= \delta_6 \cdot \delta_{0,3} - \delta_5 \cdot \delta_{0,4} - \delta_5 \cdot \delta_{1,3} + \delta_4 \cdot \delta_{1,4}$

$= \delta_{0,3,6} - \delta_5 \cdot \delta_{1,3}$

$\delta_7 \cdot \delta_{0,3,5} - \delta_6 (\delta_{0,3,6} + \delta_{1,3,5}) + \delta_5 \cdot x$

$- \delta_4 \cdot \delta_{1,4,6}$

$\delta_{0,5} + \delta_{1,4} = \delta_0 \delta_5 - \delta_2 \delta_3$

Für $i\in\mathbb{N}$: $\delta_i := \langle v, A^i v\rangle$

Für $0\le i_1<i_2<\dots<i_k \in \mathbb{N}$:

$$\delta_{(i_1,\dots,i_k)} := \begin{cases} 0, & \text{falls } i_j+1=i_{j+1} \text{ für ein } j \\ \sum\limits_{v\in\{0,1\}^{k-1}} (-1)^{|v|}\,\delta_{i_k-|v|}\cdot\delta_{(i_1,\dots,i_{k-1})+v} & \text{sonst}\end{cases}$$

wobei $|v| = \sum_i v_i$

$$\delta_{0,3,5,7} = \delta_7\cdot\delta_{0,3,5} - \delta_6\cdot(\delta_{0,3,6}+\delta_{1,3,5}) + \delta_5\cdot(\delta_{0,4,6}+\delta_{1,3,6}) - \delta_4\cdot\delta_{1,4,6}$$

$$\delta_{0,4,6} = \delta_6\cdot\delta_{0,4} - \delta_5(\delta_{0,5}+\delta_{1,4}) + \delta_4\cdot\delta_{1,5}$$

$$\delta_{1,3,5,7} = \delta_7\cdot\delta_{1,3,5} - \delta_6\cdot\delta_{1,3,6} + \delta_5\cdot\delta_{1,4,6} - \delta_4\cdot\delta_{2,4,6}$$

$$\delta_{1,3,5} = \delta_5\cdot\delta_{1,3} - \delta_4\cdot\delta_{1,4} + \delta_3\cdot\delta_{2,4}$$

$$\delta_{1,3,6} = \delta_6\cdot\delta_{1,3} - \delta_5\cdot\delta_{1,4} + \delta_4\cdot\delta_{2,4}$$

$$\delta_{1,4,6} = \delta_6\cdot\delta_{1,4} - \delta_5\cdot(\delta_{1,5}+\delta_{2,4}) + \delta_4\cdot\delta_{2,5}$$

$$\delta_{2,4,6} = \delta_6\cdot\delta_{2,4} - \delta_5\cdot\delta_{2,5} + \delta_4\cdot\delta_{3,5}$$

$y \overset{?}{=} 4x-1$

$2 \mid \delta_{i,j}$?

$\delta_{i,j} = \langle v, A^i v\rangle \langle v, A^j v\rangle - \langle v, A^{i+1} v\rangle \langle v, A^{j-1} v\rangle$

$n \cdot \langle v, A^2 v\rangle - 2 \cdot n \cdot (\dots)$

$\langle v, A^i v\rangle \bmod 2 = 0$ für $i > 0$

$\Rightarrow \delta_{i_1, \dots, i_k} \equiv 0 \mod 2^{k-1}$

3^2

45

17^2

$\vdots$

$475^2 = (5 \cdot 5 \cdot 19)^2$ $\quad 1217^2 \quad 440^2$

$S_6 = x_6 / \ldots$

$$\left.\begin{matrix} 2x(x^2-2) \\ 2x(x^2-1) \end{matrix}\right\} 4x^3-6x$$

$$\left.\begin{matrix} 2(x^2-1) \\ 2x \\ 2\cdot 1 \\ 2x^2 \\ 2x \\ 2(x^2-1) \end{matrix}\right\} 6x^2+4x-2$$

$\rightsquigarrow 4x^3+6x^2-2x-2$

x^4-3x^2-2x

$+x^2$

$+x$

x^4-2x^2

$+x^2$

$+x$

$2x^3 - 2x$

$+ x^2 - 1$

$$\chi_G = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} A_{i,j} \, \chi(A^{i,j})$$

$1,1:$ $x^2 (x+1) \, (x^3 - x^2 - 4x + 2)$

$3,3:$ $x^2 (x-1)(x+1)(x^2 - 5)$

$x^2 (x+1) \, (x^3 - x^2 - 5x + 5)$

$1,3:$ $x^2 \, (x-1)(x+1)^2$

$x^2 (x+1)(x^2 - 1)$

$2,2:$ $(x+1)(x^5 - x^4 - 7x^3 + 3x^2 + 3x - 1)$

$2,3:$ $x \, (x-1)(x+1)^2$

$x \, (x+1)(x^2 - 1)$

$x^5 - 3x^3 + x^4 - 2x^2 + 2x^3 + x^2$

$x^3 (x^2 - 3) + x^2 (x^2 - 2) + 2x^3 + x^2 = x^2 (x-1)(x+1)^2$

$0,0: \quad (x-1)(x+1)(x^2-x-4) \qquad r: (x+1)(x^3-x^2-3x+1)$

$2,2: \quad (x-1)(x+1)(x^2-x-4) \qquad r: (x+1)(x^3-x^2-3x+1)$

$0,2: \quad (x-1)(x^2-x-4) \qquad r: (x+1)^2$

x^2+2x+1

$(x^4-6x^2+4)+(x^2-2)$

$1,1: \quad x^4(x^2-2)^3(x^4-5x^2+2)$

$4,4,5,5;\ 3,3: \quad x^6(x^2-2)(x^6-9x^4+23x^2-14)$

$1,3: \quad x^5(x^2-2)^2(x^4-5x^2+2)$

$1,5: \quad x^5(x^2-2)^3$

$3,5: \quad x^6(x^2-2)^2$

$3,4: \quad x^6(x^2-2)(x^4-5x^2+2)$

$7,7: \quad x^4(x^2-2)(x^8-11x^6+39x^4-49x^2+18)$

$2,2;\overset{8,8;}{0,0}$: $x^2(x^2-2)(x^4-8x^2+6)$

$5,5;\overset{7,7;}{1,1}$: $x^2(x^2-2)(x^4-7x^2+4)$

$0,1$: $x^3(x^2-2)(x^2-5)$

$0,2$: $x^2(x^2-2)^2$

$0,5$: $3x^3(x^2-2)$

$1,5$: $2x^2(x^2-2)^2$

$1,7$: $x^2(x^2-2)(x^2+4)$

$0,8$: $6x^2(x^2-2)$

$4,4$: $x^2(x-2)(x+2)(x^2-2)^2$

$0,4$: $2x^2(x^2-2)^2$

$1,4$: $x^3(x^2-2)^2$

χ: $x^3(x^2-8)(x^2-2)^2$

$\int$: $x(x^2-8)$

$0,0$: $x(x-1)(x+1)(x^2-2)$

$1,1;2,2$: $x(x-1)(x+1)(x^2-3)$

$4,4$: $(x-2)(x-1)(x+1)^3$

$5,5$: $x(x+1)(x^3-x^2-3x+1)$

$3,3$: $(x-1)(x+1)(x^3-4x-2)$

$0,1$: $x(x+1)(x^2-2)$

$0,4$: $(x-1)^2(x+1)^2$

$0,5$: $x(x-1)(x+1)$

$0,3$: $(x-1)(x+1)$

$1,4$: $(x-1)(x+1)^2$

$1,5$: $x(x+1)$

$1,3$: $x+1$

$4,3$: $(x-2)(x+1)^2$

$4,5$: $x(x-2)(x+1)^2$

χ: $(x+1)^2(x^4-2x^3-3x^2+6x-1)$

$\int$: $x^4-2x^3-3x^2+6x-1$

$(x-2)(x+1)^2$

$(x-2)(x^2+2x+1)$

x^3-3x-2

x^3-2x

x^3-x^2-2x+2

$x^4-x^3-4x^2+2x+2$

$x^4-x^3-3x^2+x$

x^2-x-2

$\begin{smallmatrix}1\\1\\2\end{smallmatrix}$ $0-4-5-3$

$5,3: \quad (x+1)(x^3 - x^2 - 3x + 1)$

$1,2: \quad (x+1)(x^3 - 3x + 1)$

$$Ae_u = \sum_{v \in N(u)} e_v$$

$$p := \operatorname{lcm}_{v \in N(u)} (\rho_{vv}) = x^d - \sum_{i=0}^{d-1} \beta_i x^i$$

$$\Rightarrow \forall v \in N(u): \ A^d v = \sum_{i=0}^{d-1} \beta_i A^i v$$

$$\Rightarrow A^d \sum_{v \in N(u)} e_v = \sum_{v \in N(u)} \sum_{i=0}^{d-1} \beta_i A^i e_v$$

$$= \sum_{i=0}^{d-1} \beta_i A^i \sum_{v \in N(u)} e_v$$

$$\Rightarrow A^{d+1} e_u = \sum_{i=0}^{d-1} \beta_i A^{i+1} e_u$$

$$\Rightarrow \rho_{uu}(x) \mid x \cdot p(x) \quad \text{and} \quad r_{uu} = \frac{\chi_G + \sum_{v \in N(u)} r_{uv}}{x}$$

$$w_r^{uu} = \begin{cases} 1 & r = 0 \\ \sum_{v \in N(u)} w_{r-1}^{uv} & r > 0 \end{cases} \qquad \chi(t) + \sum_{v \in N(u)} r_{uv}(t)$$

In $\mathbb{Z}[x]/x \cdot \chi(x)$: $\quad r_{uu} = \dfrac{\chi(x) + \sum_{v \in N(u)} r_{uv}}{x}$? ✓

$$w_{uu}(t) = 1 + \sum_{v \in N(u)} t\, w_{uv}(t)$$

$$r_{uu}(t) = \chi(t) \cdot t^{-1} \cdot w^{uu}(t^{-1})$$

$$t^{-1} w^{uu}(t^{-1}) = \frac{r_{uu}^{1}(t)}{\rho_{uu}(t)}$$

$$\chi(t) + \sum_{v \in N(u)} r_{uv}(t) = \chi(t) \cdot w^{uu}(t^{-1})$$

$$\sum_{v \in N(u)} r_{uv}(t) = \chi(t) \cdot \left(t \cdot \frac{r_{uu}^{1}(t)}{\rho_{uu}(t)} - 1 \right)$$

$$= \underbrace{\frac{\chi(t)}{\rho_{uu}(t)}}_{\gcd(\chi_G, \chi_{G \setminus \{u\}})} \left(t \cdot r_{uu}^{1}(t) - \rho_{uu}(t) \right)$$

$$\sum_{\{u,v\} \in E} r_{uv}(t) = \chi_G(t) \sum_{u \in V} \left(t \cdot \frac{r_{uu}^{1}(t)}{\rho_{uu}(t)} - 1 \right)$$

$$r_{uu}^{1}(t) = \frac{r_{uu}(t) \rho_{uu}(t)}{\chi_G(t)}$$

$$= \chi_G(t) \left(t \cdot \sum_{u \in V} \frac{r_{uu}^{1}(t)}{\rho_{uu}(t)} - n \right)$$

$$= \chi_G(t) \left(t \cdot \frac{\sum_{u \in V} r_{uu}(t)}{\chi_G(t)} - n \right)$$

$$= t \cdot \sum_{u \in V} r_{uu}(t) - n \chi_G(t)$$

$$5x^5 - 12x^3 + 2x - 5x^5 + 20x^3 - 10x = 8x^3 - 8x$$

$$5x^4 + 8x^3 - 4x^2 - 4x$$

$$x^4 - 3x^2 + x^4 - 3x^2 + 1 + x^4 - 2x^2 + x^4 - x^2 + x^4 - 3x^2 + 1$$

$$= 5x^4 - 12x^2 + 2$$

$$r_G(t) = (t+1)\sum_{u\in V}{}' r_{uu}(t) - n\chi_G(t) + \sum_{\{u,v\}\notin E}{}' r_{uv}(t)$$

$$u,v\notin E: \quad w_r^{uv} = \begin{cases} 0 & r=0 \\ \sum'_{w\in N(u)} w_{r-1}^{vw} & r>0 \end{cases}$$

$$\Rightarrow \quad r_{uv}(t) = \frac{\sum'_{w\in N(u)} r_{vw}(t)}{t}$$

$$\sum_{u\neq v\in V} r_{uv}(t) = \frac{1}{t}\sum_{u,v\in V} \frac{(\deg(u)+\deg(v))\, r_{uv}(t)}{2}$$

$$2t\sum_{u\neq v}{}' r_{uv}(t) = \sum_{u,v}{}' (\deg(u)+\deg(v))\, r_{uv}(t)$$

$$= \sum_{u\neq v}{}' (\deg(u)+\deg(v))\, r_{uv}(t) + 2\sum_u{}' \deg(u)\, r_{uu}(t)$$

$$\sum_{u\neq v}{}' (\deg(u)+\deg(v)-2t)\, r_{uv}(t) = -2\sum_u{}' \deg(u)\, r_{uu}(t)$$

$$w_{uu}(t) = 1 + \sum_{v\in V}{}' t^{d(u,v)}\, w_{uv}(t)$$

$$t^{d(u,v)}\, w_{uv}(t) \mapsto$$

$$t^d \cdot \chi_G(t)\cdot t^{-1}\cdot t^{-d(u,v)}\cdot w_{uv}(t^{-1})$$

$$= t^{d-d(u,v)}\cdot r_{uv}(t)$$

$$d := \max_{v\in V} d(u,v)$$

$$\rightsquigarrow \text{ in } \mathbb{Z}[t]/t^d\chi_G(t): \quad t^d\cdot r_{uu}(t) = t^{d-1}\chi_G(t) + \sum_{v\in V} t^{d-d(u,v)}\cdot r_{uv}(t)$$

$$\text{in } \mathbb{Z}[t]/(t^d\chi_G(t)): \quad w_{uu}^d(t) = \sum_{\substack{v\in V\\ d(u,v)=d}} t^d\chi_G(t)\, t^{-1} t^{-d}\, w_{uv}(t^{-1})$$

$$= \sum_{\substack{v\in V\\ d(u,v)=d}} r_{uv}(t)$$

$$2(x+1)^2-2(x+1)-1$$
$$=2x^2+4x+1-2x-2-1$$
$$=2x^2+2x-1$$

$$\frac{\chi_{G^c}(-1-t)}{\chi_G(t)} = (-1)^n\Big(\underbrace{w_G(t^{-1})\,t^{-1}}_{\frac{r_G(t)}{\chi_G(t)}}+1\Big)$$

$$\Rightarrow \chi_{G^c}(-1-t) = (-1)^n\big(r_G(t)+\chi_G(t)\big)$$

$$\chi_G(t)\,\chi_{G^c}(t) = (-1)^n\chi_G(t)\big(\chi_G(-1-t)+r_G(-1-t)\big)$$

$$\Rightarrow r_G(t) = (-1)^{n+1}\, r_{G^c}(-1-t)$$

$$r^c_{uv}(t) = (-1)^n\big(r_G(-1-t)+\chi_G(-1-t)\big)\,t^{-1}\,w^c_{uv}(t^{-1})$$

$$\tilde{r}^c_{uv}(t) = (-1)^n \chi_G(t)\left(\chi_G(-1-t) + r_G(-1-t)\right) t^{-1} w^c_{uv}(t^{-1})$$
$$= r^c_{uv}(t)\,\chi_G(t)$$
$$\tilde{r}_{uv}(t) = (-1)^n \chi_G(t)\left(\chi_G(-1-t) + r_G(-1-t)\, t^{-1} w_{uv}(t^{-1})\right)$$
$$= r_{uv}(t)\,\chi_{G^c}(t)$$

$$r^c_{uu} = \frac{\chi_{G^c} + \sum_{v \in N(u)} r^c_{uv}}{t}$$

$$r^c_{uu} = \chi_{(G\setminus u)^c} = (-1)^{n-1}\left(r_{G\setminus u}(-1-t) + \underbrace{\chi_{G\setminus u}(-1-t)}_{r_{uu}(-1-t)}\right)$$

$$= (-1)^{n-1}\left(r_{G\setminus u}(-1-t) + r_{uu}(-1-t)\right)$$

$5x^4+10x^3-3x^2-12x-4$

$4x^3+4x^2-4x-4$

$4x^3+6x^2-2x-2$

$4x^3+8x^2+2x-2$

$4x^3+6x^2-2$

$4x^3+6x^2-2x-2$

$20x^3+30x^2-6x-12$

$$r_G(x)' = \sum_{v \in V} r_{G\setminus v}(x) \ ?$$

$$r_G(-1-t)' = -\sum_{v \in V} r_{G\setminus v}(-1-t)$$

$$\sum_{u\in V}' r_{uu}^c(t) = (-1)^{n-1}\left(\sum_{u\in V}' r_{uu}(-1-t) + \sum_{u\in V}' r_{G\setminus u}(-1-t)\right)$$

$$= (-1)^n \left(r_G(-1-t)' - \sum_{u\in V}' r_{uu}(-1-t)\right)$$

$$= -r_{G^c}(t)' - (-1)^n \sum_{u\in V}' r_{uu}(-1-t)$$

$$\Rightarrow \sum_{u\in V}' r_{uu}^c(t) = (-1)^{n+1}\left(r_{G^c}(t)' + \sum_{u\in V}' r_{uu}(-1-t)\right)$$

$$\boxed{\sum_{\{u,v\}\in E}' r_{uv}(t) = t\sum_{u\in V}' r_{uu}(t) - n\chi_G(t)} \Big/ \quad \sum_{\{u,v\}\notin E}' r_{uv}(t) = r_G(t) - t\sum_{u\in V}' r_{uu}(t) + n\chi_G(t)$$

$$\Rightarrow \sum_{\{u,v\}\notin E}' r_{uv}^c(t) = t\sum_{u\in V}' r_{uu}^c(t) - n\chi_{G^c}(t)$$

$$= t(-1)^{n+1}\left(r_{G^c}(t)' + \sum_{u\in V}' r_{uu}(-1-t)\right) - n\chi_{G^c}(t)$$

$$r_G(t) - t\sum_{u\in V}' r_{uu}(t) + n\chi_G(t) = t(-1)^{n+1}\left(r_{G^c}(t)' + \sum_{u\in V}' r_{uu}(-1-t)\right) - n\chi_{G^c}(t)$$

$$r_G(t)' = (-1)^{n+1}\frac{\sum_{\{u,v\}\notin E}' r_{uv}(t) + n\chi_G(t)}{t} - \sum_{u\in V}' r_{uu}^c(-1-t)$$

$$= (-1)^{n+1} \frac{t \sum'_{u \in V} r_{uu}(t) - n\chi_G(t) + n\chi_G(t)}{t} - \sum'_{u \in V} r_{uu}^c(-1-t)$$

$$= (-1)^{n+1} \sum'_{u \in V} r_{uu}(t) - \sum'_{u \in V} r_{uu}^c(-1-t)$$

$$r_{uu}^c(-1-t) = \chi_{(G\setminus u)^c}(-1-t) = (-1)^{n+1}\left(r_{G\setminus u}(t) + r_{uu}(t)\right)$$

$$= \sum'_{u \in V} (-1)^{n+1} r_{uu}(t) - r_{uu}^c(-1-t)$$

$$\boxed{\begin{aligned}
\text{Know: } \chi_G(t)' &= \sum'_{v \in V} r_{vv}(t) \\
r_G(t)' &= \sum'_{v \in V} r_{G\setminus v}(t) \\
r_G(t)' &= \sum'_{u \in V} \left[(-1)^{n+1} r_{uu}(t) - r_{uu}^c(-1-t)\right] \\
\sum'_{(u,v) \in E} r_{uv}(t) &= t \cdot \sum'_{u \in V} r_{uu}(t) - n\chi_G(t) \\
r_G(t) &= \sum'_{(u,v) \in E} r_{uv}(t) + \sum'_{\substack{(u,v) \notin E \\ u \neq v}} r_{uv}(t) + \sum'_{u \in V} r_{uu}(t)
\end{aligned}}$$

$$\sum'_{(u,v) \in E} r_{uv}^c(t) = r_{G^c}(t) - \sum'_{(u,v) \notin E} r_{uv}^c(t) - \sum'_{u \in V} r_{uu}^c(t)$$

$$= r_{G^c}(t) - t \sum'_{u \in V} r_{uu}^c(t) + n\chi_{G^c}(t) - \sum'_{u \in V} r_{uu}^c(t)$$

$$= r_{G^c}(t) + n\chi_{G^c}(t) - (t+1) \sum'_{u \in V} r_{uu}^c(t)$$

$$\Rightarrow \sum_{(u,v)\notin E}' r_{uv}(t) = r_G(t) + n\,\chi_G(t) - (t+1)\sum_{u\in V}' r_{uu}(t)$$

$$5x^5 - 25x^3 + 10x + 5x^4 + 10x^3 - 3x^2 - 4x$$

$$= 5x^5 + 5x^4 - 15x^3 - 3x^2 + 6x$$

$$\sum_{\substack{u\in V\\ u\neq v}}' r_{uv}(t) = \sum_{\substack{u\in V\\ u\neq v}}' \frac{1}{t} \sum_{w\in N(u)} r_{vw}(t)$$

$$= \frac{1}{t} \sum_{u\in V}' \deg(u)\, r_{uv}(t)$$

$$\sum_{u\in V}' \deg(u)\, r_{uv}(t) = t \sum_{u\neq V}' r_{uv}(t)$$

$$\sum_{v\in V} \deg(v) \sum_{u\in V}' r_{uv}(t)$$

$$\sum_{u\in V} \sum_{v\in V}' \deg(u)\, r_{uv}(t)$$

$$= \sum_{u\in V} \deg(u) \sum_{v\in V}' r_{uv}(t)$$

$$\sum_{\substack{(u,v)\in V\\ u\neq v}}' r_{uv}(t) = \frac{\sum_{(u,v)\in V} (\deg(u) + \deg(v))\, r_{uv}(t)}{t}$$

$$t \cdot \sum_{u\in V}' s_u(t) - r_{uu}(t) = \sum_{u\in V}' 2\deg(u)\, s_u(t)$$

$$\Rightarrow -t\,\chi_G'(t) = \sum_{u\in V}' (2\deg(u) - t)\, s_u(t)$$

$$w_{uv}(t) = \sum_{w\in N(u)}' t \cdot w_{vw}(t)$$

$$\Rightarrow r_{uv}(t) = \sum_{w\in N(u)} \chi_G(t) \cdot t^{-1} \cdot t^{-1} \cdot w_{vw}(t^{-1})$$

$$= t^{-1} \sum_{w\in N(u)} r_{vw}(t)$$

$$\chi_G(-1-t)' = -\sum_{u\in V}' r_{uu}(-1-t)$$

$$r_G(t)' = (-1)^{n+1}\chi_G(t)' + \underbrace{\chi_{G^c}(-1-t)'}_{(-1)^n(r_G(t)+\chi_G(t))}$$

$$= (-1)^{n+1}\chi_G(t)' + (-1)^n(r_G(t)' + \chi_G(t)')$$

$8x^2+10x-2$

$8x^2+12x+4$

$12x^2+12x-4$

$6x^2+12x+6$

$14x^3+28x^2-6x-4$

$16x^3+30x^2+2x-8$

$5x^3-2x^2+6x$

$$\sum_{(u,v)\in E}' r_{uv}(t) = t\cdot\chi_G'(t) - n\chi_G(t)$$

$$\sum_{(u,v)\notin E}' r_{uv}(t) = ?$$

$$r_G(t)' = (-1)^{n+1}\chi_G'(t) + \chi_{G^c}(-1-t)'$$

$$\| $$

$$\left((t+1)\chi_G'(t) - n\chi_G(t)\right)' + \sum_{(u,v)\notin E}' r_{uv}'(t)$$

$$\| $$

$$\chi_G'(t) + (t+1)\chi_G''(t) - n\chi_G'(t) + \sum_{(u,v)\notin E}' r_{uv}'(t) \qquad \underline{(-1)^n(r_G(t)' + \chi_G(t)')}$$

$$\Rightarrow \sum_{(u,v)\notin E}' r_{uv}'(t) = (n-1+(-1)^{n+1})\chi_G'(t) + (t+1)\chi_G''(t) + \chi_{G^c}(-1-t)'$$

$$r_{uw}(t) = t^{-1}\Big(r_{vw}(t) + \sum_{x\in N_{uv}}' r_{xw}(t) + \sum_{x\in N_u}' r_{xw}(t) \Big)$$

$$r_{vw}(t) = t^{-1}\Big(r_{uw}(t) + \sum_{x\in N_{uv}}' r_{xw}(t) + \sum_{x\in N_v}' r_{xw}(t) \Big)$$

$$t\, r_{uw}(t) = t^{-1}\Big(r_{uw}(t) + \sum_{x\in N_{uv}}' r_{xw}(t) + \sum_{x\in N_v} r_{xv}(t) \Big) + \sum_{x\in N_{uv}}' r_{xw}(t) + \sum_{x\in N_u}' r_{xw}(t)$$

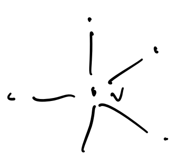
$$t^2 r_{uw}(t) - r_{uw}(t) = (t+1)\sum_{x\in N_{uv}}' r_{xw}(t) + t\cdot \sum_{x\in N_u}' r_{xw}(t) + \sum_{x\in N_v}' r_{xw}(t)$$

$$r_{uw}(t) = \frac{1}{t-1}\sum_{x\in N_{uv}}' r_{xw}(t) + \frac{1}{t^2-1}\Big(t\sum_{x\in N_u} r_{xw}(t) + \sum_{x\in N_v}' r_{xw}(t) \Big)$$

$$r_{vw}(t) - r_{uw}(t) = \frac{1}{t^2-1}\Big((t-1)\sum_{x\in N_v}' r_{xw}(t) - (t-1)\sum_{x\in N_u}' r_{xw}(t) \Big)$$

$$= \frac{1}{t+1}\Big(\sum_{x\in N_v}' r_{xw}(t) - \sum_{x\in N_u}' r_{xw}(t) \Big)$$

$$\sum_{u\neq v}' r_{uv}(t) = \frac{1}{t} \sum'^{\deg(u)} r_{uv}(t) = \frac{1}{t}\Big(\sum_{u\neq v}' \deg(u)\, r_{uv}(t) + \deg(v)\, r_{vv}(t) \Big)$$



$$\sum_{u\in N[v]}' r_{uw}(t) = r_{vw}(t) + \sum_{u\in N(u)}' r_{uw}(t) = \sum_{u\in N(u)}' \frac{t+1}{t} r_{uw}(t)$$

$$\sum_{u\in V}' (t-\deg(u)) \sum_{v\in V}' r_{uv}(t) = n\,\chi_G(t)$$

$$t \sum_{u,v}' r_{uv}(t) - \sum_{u\in V}' \deg(u) \sum_{v\in V}' r_{uv}(t) = n\,\chi_G(t)$$

$$v\in V: \sum_{\substack{u,v\notin E\\ u\neq v}}' r_{uv} = \frac{1}{t}\left(\sum_{u\neq v}' \deg(u)\, r_{uv}(t) - \sum_{u\in N_v}' \left(\sum_{w\in N_u}' r_{wv}(t)\right) + r_{uv}(t)\right)$$

$$t\cdot \chi_G'(t) - n\chi_G(t) = \sum_{u,v}' \deg(u)\, r_{uv}(t) - t\sum_{u\neq v}' r_{uv}(t)$$

$$= \sum_{u,v}' \deg(u)\, r_{uv}(t) - t\left(\sum_{u,v\in E}' r_{uv}(t) + \sum_{\substack{u,v\notin E\\ u\neq v}}' r_{uv}(t)\right)$$

$$= \sum_{v} \sum_{u}' \underbrace{\sum_{w\in N(u)}' r_{wv}(t)}$$

$t\,r_{uv}(t)$ falls $u\neq v$

$t\,r_{uv}(t) - \chi_G$ falls $u=v$

$$r_{uu} = \frac{\chi_G + \sum_{v\in N_u}' r_{uv}(t)}{t}$$

$=$

$$p_E = q - t(p_E + p_{\neg E})$$

$$(1+t)\,p_E = q - t\,p_{\neg E}$$

$$\frac{1}{t}\sum_{u\neq v}' \deg(u)\, r_{uv}(t) - \frac{1}{t}\sum_{u\in N_v}' \sum_{w\in N_u\setminus v}' r_{wv}(t) \overbrace{-\frac{1}{t}\sum_{u\in N_v}' r_{uv}(t)}^{\chi_G - r_{uu}}$$

$$t\cdot \sum_{\substack{u\neq v\\ u,v\notin E}}' r_{uv}(t) = \sum_{u\neq v}' \deg(u)\, r_{uv}(t) - \underbrace{\sum_{\substack{u\in N_v\\ w\in N_u\setminus v}}' r_{wv}(t)}_{\sum_{u\in V}' w_{uv}^2 \cdot r_{uv}(t)} - \chi_G(t) + t\,r_{uu}(t)$$

$$t\cdot \sum_{\substack{u,v\notin E\\ u\neq v}}' r_{uv}(t) = \sum_{u\neq v}' \deg(u)\, r_{uv}(t) + t\,\chi_G'(t) - n\,\chi_G(t) - \sum_{u\neq v}' w_{uv}^2\, r_{uv}(t)$$

$$t \cdot \sum_u{}' r_{uu}(t) - \sum_{u,v}{}' (t - deg(u)) r_{uv}(t) = P_E$$

$$\|$$

$$\sum_u{}' deg(u) r_{uu}(t) - \sum_{u \neq v}{}' (t - deg(u)) r_{uv}(t)$$

$$P_{\neg E} = r_G - \overset{-\chi_G'}{P_E^{\vee}} = \sum_{u,v}{}' r_{uv}(t) - \sum_u{}' deg(u) r_{uv}(t) + \sum_{u \neq v}{}' (t - deg(u)) r_{uv}(t) - \sum_u{}' r_{uu}(t)$$

$$= \sum_u{}' (1 - deg(u)) r_{uu}(t) + \sum_{u \neq v}{}' (t + 1 - deg(u)) r_{uv}(t)$$

$$= \sum_{u,v}{}' (t + 1 - deg(u)) r_{uv}(t) - t \chi_G'$$

$$= (t+1) r_G(t) - \sum_{u,v}{}' deg(u) r_{uv}(t) - t \chi_G'$$

$$r_G(t) = (t+1) \chi_G'(t) - n \chi_G(t) + (t+1) r_G(t) - \sum_{u,v}{}' deg(u) r_{uv}(t) - t \chi_G'$$

$$= n \cdot \frac{\chi_G(t)}{t} + \frac{1}{t} \sum_{u,v}{}' deg(u) r_{uv}(t) - \frac{1}{t} \chi_G'(t)$$

$$\frac{n \cdot \chi_G(t) + \sum_{u,v}{}' deg(u) r_{uv}(t)}{t} = r_G(t)$$

$$\sum_{u,v \in V}{}' (t - deg(u)) r_{uv}(t) = n \chi_G(t)$$

$$r_G(t) = \frac{1}{t} \left(n \chi_G(t) + \sum_{u,v} deg(u) r_{uv}(t) \right)$$

$$= (t+1) \chi_G'(t) - n \chi_G(t) + \sum_{\substack{(u,v) \notin E \\ u \neq v}}{}' r_{uv}(t)$$

$$t\,\chi_G'(t) - n\,\chi_G(t) + \sum_{\substack{u\neq v\\(u,v)\notin E}}' r_{uv}(t) = \frac{1}{t}\left(\sum_{u,v}' \deg(u)\, r_{uv}(t) - \sum_{(u,v)\in E}' r_{uv}(t)\right)$$

$$t\sum_{u,v} r_{uv}(t) = \sum_{u,v}' \deg(u)\, r_{uv}(t) + n\,\chi_G(t)$$

$$= \frac{1}{t}\sum_{u,v}' \deg(u) \sum_{w\in N(u)}' r_{wv}(t) \qquad + n\,\chi_G(t)$$

$$= \sum_{u}' \deg(u)\left(\sum_{v}' r_{uv}(t)\right) + n\,\chi_G(t)$$

$$= \frac{1}{t}\sum_{u}' \deg(u)\left(\sum_{v}' \deg(v)\, r_{uv}(t) + \chi_G(t)\right) \quad + n\,\chi_G(t)$$

$$= \frac{1}{t}\sum_{u,v} \deg(u)\deg(v)\, r_{uv}(t) \quad + \frac{2m}{t}\chi_G(t) + n\,\chi_G(t)$$

$$\Rightarrow t^2\, r_G(t) = \sum_{u,v}' \deg(u)\deg(v)\, r_{uv}(t) + (nt + 2m)\,\chi_G(t)$$

$$\sum_{u,v}' \deg(u)\, r_{uv}(t) = \varphi\left(\frac{w_G(t) - n}{t}\right)$$

$$= \chi_G(t)\cdot t^{-1}\cdot \frac{w_G(t^{-1}) - n}{t^{-1}}$$

$$= t\cdot\chi_G(t)\cdot t^{-1}\cdot w_G(t^{-1}) - n\,\chi_G(t)$$

$$= t\cdot r_G(t) - n\,\chi_G(t)$$

$(u,v)\notin E$: $w_{uv}(t) = t^2 \sum'_{x\in N(u)} \sum'_{y\in N(v)} w_{xy}(t)$

$$\Rightarrow \quad r_{uv}(t) = \frac{1}{t^2} \sum'_{x\in N(u)} \sum'_{y\in N(v)} r_{xy}(t)$$

$$\sum'_{u,v} \deg(u)\deg(v) r_{uv}(t) + (nt+2m)\,\chi_G(t) = t\left(\sum'_{u,v} \deg(u) r_{uv}(t) + n\chi_G(t)\right)$$

$$\Rightarrow \sum'_{u,v} \deg(u)(t-\deg(v))\, r_{uv}(t) = 2m\,\chi_G(t)$$

$$\sum'_{u,v} (t-\deg(v))\, r_{uv}(t) = n\,\chi_G(t)$$

$$n\sum'_{u,v} \deg(u)\deg(v) r_{uv}(t) - 2m\sum'_{u,v} \deg(u) r_{uv}(t) \overset{t}{\underset{\checkmark}{=}} n\sum'_{u,v} \deg(u) r_{uv}(t) - 2m\sum'_{u,v} r_{uv}(t)$$

$$t\cdot \sum'_{u,v}\left(\deg(u)-\frac{2m}{n}\right) r_{uv}(t) = \sum'_{u,v}\left(\deg(u)-\frac{2m}{n}\right)\deg(v)\, r_{uv}(t)$$

$$t\sum'_{u,v} r_{uv}(t) = \frac{1}{t}\left(\sum'_{u,v} \deg(u)\deg(v) r_{uv}(t) + 2m\,\chi_G(t)\right) + n\chi_G(t)$$

$$t^2 r_G(t) = \sum'_{u,v} \deg(u)\deg(v) r_{uv}(t) + (2m+nt)\,\chi_G(t)$$

$$t\sum'_{u,v} r_{uv}(t) = \sum'_{u,v} \deg(u) r_{uv}(t) + n\,\chi_G(t)$$

$$t\cdot\sum'_{u,v} \deg(u) r_{uv}(t) = \sum'_{u,v} \deg(u)\deg(v) r_{uv}(t) + 2m\chi_G(t)$$

$$(t-1)\, r_G(t) = \sum_{u,v}{}' (\deg(u)-1)\, r_{uv}(t) + n\chi_G(t)$$

$$w_{uv}^{k} = \sum_{x\in V}{}' \frac{1}{k+1} \sum_{i=0}^{k}{}' w_{ux}^{i}\, w_{xv}^{k-i}$$

$$w_{ux}(t)\cdot w_{xv}(t) = \left(\sum_{i=1}^{r}{}' \frac{c_i}{1-t\mu_i}\right)\left(\sum_{i=1}^{s}{}' \frac{d_i}{1-t\lambda_i}\right)$$

$$= \sum_{\mu\in E_{ux}\cap E_{xv}}{}' \frac{c_\mu d_\mu}{(1-t\mu)^2} + \sum_{\mu\neq\lambda\in E_{ux}\cup E_{vx}}{}' \frac{c_\mu d_\lambda}{(1-t\mu)(1-t\lambda)}$$

$$\int w_{ux}(t)\, w_{xv}(t)\, dt = -\sum_{\mu\in E_{ux}\cap E_{xv}}{}' \frac{c_\mu d_\mu}{\mu(1-t\mu)} + \int \sum_{\mu\neq\lambda}{}' c_\mu d_\lambda \left(\frac{\lambda}{(\lambda-\mu)(1-t\mu)} + \frac{\mu}{(\mu-\lambda)(1-t\lambda)}\right) dt$$

$$-\left(\frac{1}{\mu}\,\frac{c_\mu d_\lambda \lambda}{(\lambda-\mu)} \log(1-t\mu) + \frac{1}{\lambda}\,\frac{c_\mu d_\lambda \mu}{(\mu-\lambda)} \log(1-t\lambda)\right)$$

$$c_\mu d_\mu \sum_{n=0}^{\infty}{}' \mu^{n-1} t^n$$

$$\frac{1}{\mu}\,\frac{c_\mu d_\lambda \lambda}{\lambda-\mu} \sum_{n=1}^{\infty}{}' \frac{(t\mu)^n}{n} + \frac{1}{\lambda}\,\frac{c_\mu d_\lambda \mu}{(\mu-\lambda)} \sum_{n=1}^{\infty}{}' \frac{(t\lambda)^n}{n}$$

$$= \sum_{n=1}^{\infty}{}' \frac{c_\mu d_\lambda \lambda \mu^{n-1} - c_\mu d_\lambda \mu \lambda^{n-1}}{(\lambda-\mu)\, n}\, t^n$$

$$\lambda\mu^2 - \mu\lambda^2 = \lambda\mu(\mu - \lambda)$$

$$= \frac{c_\mu d_\lambda}{\lambda - \mu} \sum_{n=1}^{\infty} \frac{\lambda\mu^{n-1} - \mu\lambda^{n-1}}{n} t^n$$

$$\lambda\mu^3 - \mu\lambda^3 = \lambda\mu(\mu - \lambda)(\mu + \lambda)$$

$$= c_\mu d_\lambda - c_\mu d_\lambda \mu\lambda \sum_{n=3}^{\infty} \frac{\sum_{k=0}^{n-2} \lambda^k \mu^{n-2-k}}{n} t^n$$

$$\lambda\mu^4 - \mu\lambda^4 = \lambda\mu(\mu - \lambda)(\mu^2 + \mu\lambda + \lambda^2)$$

$$= c_\mu d_\lambda \left(1 - \mu\lambda t^2 \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{n} \lambda^k \mu^{n-k}\right) t^n\right)$$

$$w_{uv}(t) = \sum_{x \in V} \int w_{ux}(t) w_{xv}(t)\, dt$$

$$w_{uv}(t)' = \sum_{x \in V} w_{ux}(t) w_{xv}(t)$$

$$\int \frac{c_\mu d_\lambda}{(1-\mu t)(1-\lambda t)}\, dt = c_\mu d_\lambda \sum_{n=0}^{\infty} \frac{\mu^{n+1} - \lambda^{n+1}}{(n+1)(\mu - \lambda)} t^{n+1} = c_\mu d_\lambda \frac{\log(1-\mu t) - \log(1-\lambda t)}{\mu - \lambda}$$

Für $\mu \in E_{ux} \cap E_{xv}$: $$\frac{-c_\mu d_\mu}{\mu(1 - t\mu)} + \sum_{\substack{\lambda \neq \mu \\ \lambda \in E_{xv}}} c_\mu d_\lambda \frac{\log(1-\mu t) - \log(1-\lambda t)}{\mu - \lambda}$$

Für $\mu \in E_{ux} \setminus E_{xv}$: $$\sum_{\lambda \in E_{xv}} c_\mu d_\lambda \frac{\log(1-\mu t) - \log(1-\lambda t)}{\mu - \lambda}$$

$$= \sum_{\substack{\lambda \neq \mu \\ \lambda \in E_{xv}}} c_\mu d_\lambda \sum_{n=1}^{\infty} \frac{\mu^n - \lambda^n}{n(\mu - \lambda)} t^n$$

$$-c_\mu d_\mu \cdot \frac{1}{\mu}\sum_{n=0}^{\infty}{}' \mu^n t^n + \sum_{\lambda\neq\mu}{}' c_\mu d_\lambda \sum_{n=1}^{\infty}{}' \frac{\mu^n-\lambda^n}{n(\mu-\lambda)} t^n$$

$$= c_\mu\left(\sum_{n=1}^{\infty}{}'\left(\sum_{\lambda\neq\mu}{}' d_\lambda \frac{\mu^n-\lambda^n}{n(\mu-\lambda)} - d_\mu\mu^{n-1}\right)t^n - \frac{d_\mu}{\mu}\right)$$

$$= c_\mu\left(\sum_{n=1}^{\infty}{}'\left(\frac{1}{n}\sum_{\lambda\neq\mu}{}' d_\lambda \sum_{k=0}^{n-1}{}' \mu^k\lambda^{n-1} - d_\mu\mu^{n-1}\right)t^n - \frac{d_\mu}{\mu}\right)$$

$$= c_\mu\left(\sum_{n=1}^{\infty}{}'\left(\frac{1}{n}\sum_{k=0}^{n-1}{}' \mu^k \underbrace{\sum_{\lambda\neq\mu}{}' d_\lambda\lambda^{n-1-k}}_{w_{xv}^{n-1-k} - d_\mu\mu^{n-1-k}} - d_\mu\mu^{n-1}\right)t^n - \frac{d_\mu}{\mu}\right)$$

$$= c_\mu\left(\sum_{n=1}^{\infty}{}'\left(\frac{1}{n}\sum_{k=0}^{n-1}{}'\left(\mu^k w_{xv}^{n-1-k} - d_\mu\mu^{n-1}\right) - d_\mu\mu^{n-1}\right)t^n - \frac{d_\mu}{\mu}\right)$$

$$= c_\mu\left(\sum_{n=1}^{\infty}{}'\left(\frac{1}{n}\sum_{k=0}^{n-1}{}' \mu^k w_{xv}^{n-1-k} - 2d_\mu\mu^{n-1}\right)t^n - \frac{d_\mu}{\mu}\right)$$

$$= c_\mu\sum_{n=1}^{\infty}{}'\frac{1}{n}\left(\sum_{k=0}^{n-1}{}' \mu^k w_{xv}^{n-1-k}\right)t^n - c_\mu\sum_{n=1}^{\infty}{}' 2d_\mu\mu^{n-1}t^n - c_\mu\frac{d_\mu}{\mu}$$

$$= c_\mu\sum_{n=1}^{\infty}{}'\frac{1}{n}\left(\sum_{k=0}^{n-1}{}' \mu^k w_{xv}^{n-1-k}\right)t^n - 2c_\mu d_\mu\left(\frac{1}{2\mu} + t\sum_{n=0}{}' \mu^n t^n\right)$$

$$= c_\mu\sum_{n=1}^{\infty}{}'\frac{1}{n}\left(\sum_{k=0}^{n-1}{}' \mu^k w_{xv}^{n-1-k}\right)t^n - \frac{c_\mu d_\mu}{\mu} - 2c_\mu d_\mu\frac{t}{1-\mu t}$$

$$\sum_{\lambda\neq\mu} c_\mu d_\lambda \sum_{n=1}^{\infty} \frac{\mu^n-\lambda^n}{n(\mu-\lambda)} t^n = c_\mu \sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_{\lambda\neq\mu} d_\lambda \sum_{k=0}^{n-1} \mu^k \lambda^{n-1-k}\right) t^n$$

$$= c_\mu \sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_{k=0}^{n-1} \mu^k \underbrace{\sum_{\lambda\neq\mu} d_\lambda \lambda^{n-1-k}}_{w_{xv}^{n-1-k}}\right) t^n$$

$$= c_\mu \sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_{k=0}^{n-1} \mu^k w_{xv}^{n-1-k}\right) t^n$$

$$\int w_{xu}(t)\, w_{xv}(t)\, dt = \sum_{\mu\in E_{xu}} \left(c_\mu \sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_{k=0}^{n-1} \mu^k w_{xv}^{n-1-k}\right) t^n \right)$$

$$- \sum_{\mu\in E_{xu}\cap E_{xv}} \left(\frac{c_\mu d_\mu}{\mu} + 2 c_\mu d_\mu \frac{t}{1-\mu t} \right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(\sum_{k=0}^{n-1} w_{xv}^{n-1-k} \underbrace{\sum_{\mu\in E_{xu}} c_\mu \mu^k}_{w_{ux}^k}\right) t^n$$

$$- \ldots$$

$$= \sum_{n=1}^{\infty} \underbrace{\frac{1}{n}\left(\sum_{k=0}^{n-1} w_{xv}^{n-1-k} w_{ux}^k\right)}_{w_{uv}^{n-1}} t^n = t \cdot w_{uv}(t)$$

$$- \ldots$$

$$\sum_{x\in V}' \sum_{\mu\in E_{ux}\cap E_{xv}}' \frac{c_\mu d_\mu}{\mu} + 2 c_\mu d_\mu \frac{t}{1-\mu t} \qquad \frac{t^{-1}}{1-\mu t^{-1}} = \frac{1}{t-\mu}$$

$$\frac{c_\mu d_\mu}{\mu} + \frac{2c_\mu d_\mu}{t-\mu}$$

$$\frac{c_\mu d_\mu (t-\mu) + 2 c_\mu d_\mu \mu}{\mu(t-\mu)}$$

$$= \frac{c_\mu d_\mu}{\mu} \cdot \frac{t+\mu}{t-\mu}$$

$$\sum_{x\in V}' \sum_{\mu\in E_{ux}\cap E_{xv}}' \frac{c_\mu d_\mu}{\mu} (t+\mu)\, t^{-1} \frac{\chi_G(t)}{t-\mu}$$

$$= \sum_{x\in V}' \sum_{\substack{p(t)\mid \chi_G(t)\\ p(t)\mid r_{ux}(t)\\ p(t)\mid r_{xv}(t)}}' \frac{\chi_G(t)}{t\cdot p(t)} \cdot \sum_{\mu}' \frac{c_\mu d_\mu}{\mu}(t+\mu) \prod_{\lambda\neq\mu}' (t-\lambda)$$

$$\frac{1}{\mu} + \frac{2}{t-\mu} = \frac{t-\mu+2\mu}{\mu(t-\mu)} = \frac{t+\mu}{\mu(t-\mu)}$$

$$\frac{\chi_G(t)}{t} \cdot \sum_{\mu}' c_\mu d_\mu \frac{t+\mu}{\mu(t-\mu)} \qquad \sum_{\mu}' \frac{c_\mu d_\mu}{\mu}\left(p(t) + 2 \quad \mu \frac{p(t)}{t-\mu} \right)$$

$$= p(t) \sum_{\mu}' \frac{c_\mu d_\mu}{\mu} + \frac{2 c_\mu d_\mu}{t-\mu} \quad ?$$

$$\frac{\chi_G(t)}{t} \cdot \sum_{x \in V}' \sum'$$

$$t^2 + 2t^4 + 4t^6 + 8t^8 + \dots$$

$$r_{uv}(t) = 1$$

$$\frac{1}{t} - \frac{\chi_G(t)}{t} \cdot \sum_{x \in V}'$$

$$1 = \frac{1}{t} - (t^2 - 2)\, q(t)$$

$$\Leftrightarrow \quad t = 1 - t(t^2 - 2)\, q(t)$$

$$\Leftrightarrow \quad q(t) = \frac{t-1}{t(t^2-2)}$$

$$w_{uv}(t) = \sum_{x \in V}' \int w_{ux}(t) w_{xv}(t)\, dt$$

$$= \sum_{x \neq u,v}' \int w_{ux}(t)\, w_{xv}(t)\, dt + \int w_{uu}(t)\, w_{uv}(t)\, dt + \int w_{vv}(t) w_{uv}(t)\, dt$$

$$= \sum_{x \neq u,v}' \int v_{ux}(t)\, w_{xv}(t)\, dt + \int w_{uv}(t) \left(w_{uu}(t) + w_{vv}(t) \right) dt$$

$$p(t)\cdot t^{-1}\cdot\left(\int\sigma\right)(t^{-1}) = p(t)\cdot\ t^{-1}\cdot\left(\sum_{n=0}^{\infty}\frac{1}{n+1}\sigma_n t^{-n}\right) = p(t)\cdot\sum_{n=1}^{\infty}\frac{1}{n}\sigma_{n-1}t^{-n}$$

If p characteristic for $\sigma \Rightarrow p'$ characteristic for σ'?

$\Updownarrow$

$$t^{-1}\cdot\sigma(t^{-1}) = \frac{r(t)}{q(t)} \text{ for } q(t)\,|\,p(t)$$

$$c_0\, t\, \mu_1(t-\mu_1)^2\mu_2(t-\mu_2)^2$$
$$+\, c_1\, t\, \mu_0(t-\mu_0)^2\mu_2(t-\mu_2)^2$$
$$+\, c_2\, t\, \mu_0(t-\mu_0)^2\mu_1(t-\mu_1)^2$$

$$\sigma(t) = \sum_i \frac{c_i}{1-\mu_i t} \qquad t^{-1}\sigma(t^{-1}) = \sum_i \frac{c_i}{t-\mu_i} = \frac{\sum_i c_i \prod_{j\neq i}(t-\mu_j)}{\underbrace{\prod (t-\mu_i)}_{\rho_\sigma}}$$

$$\sigma'(t) = -\sum_i \frac{c_i}{\mu_i(1-\mu_i t)^2}$$

$$t^{-1}\sigma'(t^{-1}) = -t^{-1}\sum_i \frac{c_i}{\mu_i(1-\frac{\mu_i}{t})^2} = -\sum_i \frac{c_i t}{\mu_i(t-\mu_i)^2} = -\frac{\sum_i c_i t \prod_{j\neq i}\mu_j(t-\mu_j)^2}{\underbrace{\prod\mu_i}_{\rho_\sigma(0)}\cdot\underbrace{\prod(t-\mu_i)^2}_{\rho_\sigma^2(t)}}$$

$$+\sum_i \frac{c_i\mu_i}{\mu_i(t-\mu_i)^2} - \sum_i \frac{c_i}{(t-\mu_i)^2}$$

$$= -\sum_i \frac{c_i(t-\mu_i)}{\mu_i(t-\mu_i)^2} - \sum_i \frac{c_i}{(t-\mu_i)^2}$$

$$= -\sum_i \frac{c_i}{\mu_i(t-\mu_i)} - \frac{\sum_i c_i \prod_{j\neq i}(t-\mu_j)^2}{\rho_\sigma^2(t)}$$

$$w_{uv}(t)' = \sum_{x \in V} w_{ux}(t) w_{xv}(t)$$

in $\mathbb{Z}[t]/\chi_G(t)^2$: $\tilde{r}_{uv}(t) = \chi_G(t)^2 \cdot t^{-1} \cdot \sum_{x \in V} w_{ux}(t^{-1}) w_{xv}(t^{-1})$

$$= \sum_{x \in V} \frac{1}{t} r_{ux}(t) r_{xv}(t)$$

$$r_{uv}^{\chi^2}(t) = \chi_G(t) \cdot r_{uv}(t)$$

$$w_{uv}(t) \mapsto \chi_G^2(t) \cdot t^{-1} \cdot w_{uv}(t^{-1})$$

$$w_{uv}'(t) = \sum_{k=0}^{\infty} (k+1) w_{uv}^{k+1} t^k$$

$$\chi_G^2(t) \cdot t^{-1} \cdot \sum_{k=0}^{\infty} (k+1) w_{uv}^{k+1} t^{-k} = t \cdot \chi_G^2(t) \cdot t^{-1} \cdot \sum_{k=1}^{\infty} k w_G^k t^{-k}$$

$$\chi_G^2(t) = t^{2d} + \sum_{k=0}^{2d-1} p_k t^k \qquad = 2m\, t^{2d-1} + (2w_G^2 + p_{2d-1} w_G^1) t^{2d-2}$$

$$+ (3w_G^3 + p_{2d-1} w_G^2 + p_{2d-2} w_G^1) t^{2d-3}$$

$$w_{uu}(t) = 1 + 0 \cdot t + \deg(u)\, t^2 + \sum_{v \in V}' \sum_{k=3}^{\infty}' w_{uv}^2 \cdot w_{uv}^{k-2} t^k$$

$$= 1 + \deg(u)\, t^2 + \sum_{v \in V}' w_{uv}^2 \cdot t^2 \cdot \left(w_{uv}(t) - w_{uv}^0 \right)$$

$$= 1 + t^2 \Big(\deg(u) + \sum_{v \in V}' w_{uv}^2 \cdot w_{uv}(t) - \underbrace{w_{uu}^2}_{\deg(u)} \Big)$$

$$= 1 + t^2 \sum_{v \in V}' w_{uv}^2 \cdot w_{uv}(t)$$

$$\Rightarrow r_{uu}(t) = \chi_G(t) \cdot t^{-1} \Big(1 + t^{-2} \sum_{v \in V}' w_{uv}^2 \cdot w_{uv}(t^{-1}) \Big)$$

$$= \frac{1}{t} \chi_G(t) + \frac{1}{t^2} \sum_{v \in V}' w_{uv}^2 \cdot r_{uv}(t)$$

$$\sum_u' r_{uu}(t) = \frac{n}{t} \chi_G(t) + \frac{1}{t^2} \sum_{u,v}$$

$$x^3 - 3x - 2 \qquad 3x^3 - 15x - 12 + \frac{1}{x^2} \Big(\overbrace{x^2 + x + x^2 + x + 4x + 4}^{2x^2 + 6x + 4} \Big)$$

$$w_{uu}(t) = 1 + t^2 \sum_{v \in V}' w_{uv}^2 \cdot w_{uv}(t)$$

$$r_{uu}(t) = \frac{1}{t^2}\left(\chi_G(t) \cdot t + \sum_{v \in V}' w_{uv}^2 w_{uv}(t)\right)$$

$x^3 - 3x - 2$

$$x^3 - 5x - 4 + \frac{1}{x^2}\left(2x^3 - 6x - 4 + 4x + 4 + 2x^2 + 2x\right)$$

$$= x^3 - 5x - 4 + \frac{1}{x^2}\left(2x^3 + 2x^2\right)$$

$$= x^3 - 5x - 4 + 2x + 2)$$

$$= x^3 - 3x - 2$$

$$\frac{1}{t} + \frac{\deg(u)}{t^3} = \frac{1}{t^3}\left(t^2 + \deg(u)\right)$$

$$w_{uu}(t) = 1 + \deg(u)\, t^2 + t^3 \sum_{v \in V}' w_{uv}^3 w_{uv}(t) \Rightarrow r_{uu}(t) = \frac{1}{t^3} \chi_G(t)\left(t^2 + \deg(u)\right)$$

$$+ \frac{?}{t^3} \sum_{v \in V}' w_{uv}^3 r_{uv}(t)$$

$$w_{uu}(t) = 1 + \deg(u)\, t^2 + w_{uu}^3 \cdot t^3 + t^4 \sum_{v \in V}' w_{uv}^4 w_{uv}(t)$$

$$= tr_3\, w_{uu}(t) + t^4 \sum_{v \in V}' w_{uv}^4 w_{uv}(t)$$

$$r_{uu}(t) = \frac{1}{t^{k+1}}\left(\chi_G(t) \cdot t^k \cdot tr_k\, w_{uu}(t^{-1}) + \sum_{v \in V}' w_{uv}^{k+1} r_{uv}(t)\right)$$

$$t^{k+1} r_{uu}(t) = t^k \chi_G(t) \cdot tr_k w_{uu}(t^{-1}) + \sum_{v\in V}' w_{uv}^{k+1} r_{uv}(t)$$

$$t^{k+1} \sum_{u\in V}' r_{uu}(t) = t^k \chi_G(t)\, tr_k\Big(\sum_{u\in V}' w_{uu}(t^{-1})\Big) + \sum_{u,v\in V} w_{uv}^{k+1} r_{uv}(t)$$

$$t^k \chi_G(t)\, tr_k w_{uu}(t^{-1}) = t^{k+1} r_{uu}(t) - \tilde{r}_{uu}^k(t) + \sum_{u,v\in V}' w_{uv}^{k+1} r_{uv}(t)$$

$$3x^3 + 4x^2 - 2x - 2$$

$$w_{uu}(t^{-1}) = tr_k w_{uu}(t^{-1}) + tr^k w_{uu}(t^{-1})$$

$$t^{k+1} r_{uu}(t) = t^k \chi_G(t)\, tr_k w_{uu}(t^{-1}) + t^k \chi_G(t)\, tr^k w_{uu}(t^{-1})$$

$$\Rightarrow t^k \chi_G(t)\, tr^k w_{uu}(t^{-1}) = \sum_{v\in V}' w_{uv}^{k+1} r_{uv}(t)$$

$$\forall u: \sum_{v\in V}' w_{uv}^{k+1} r_{uv}(t) = t^k \chi_G(t)\, tr^k w_{uu}(t^{-1})$$

$$\chi_G(t)\, tr^0\, w_{uu}(t^{-1}) = \chi_G(t)\cdot t^{-1}\cdot t\cdot\left(w_{uu}(t^{-1}) - w_{uu}^0\right)$$
$$= t\, r_{uu}(t) - \chi_G(t)$$

$$t\chi_G(t)\, tr^1\, w_{uu}(t^{-1}) = \chi_G(t)\cdot t^{-1}\cdot t^2\left(w_{uu}(t^{-1}) - w_{ur}^0\right)$$
$$= t^2 r_{uu}(t) - t\,\chi_G(t)$$

$$t^2\chi_G(t)\, tr^2\, w_{uu}(t^{-1}) = \chi_G(t)\cdot t^{-1}\cdot t^3\left(w_{uu}(t^{-1}) - \deg(u)\, t^{-2} - 1\right)$$
$$\underset{\displaystyle\sum_{v\in V} w_{uv}^3\, r_{uv}(t)}{\shortparallel} \qquad = t^3 r_{uu}(t) - \deg(u)\,\chi_G(t) - t^2\chi_G(t)$$
$$= t^3 r_{uu}(t) - \chi_G(t)\left(t^2 + \deg(u)\right)$$

$$\sum_{u,v\in V} w_{uv}^2\, r_{uv}(t) = \sum_{\substack{u,v\in V\\ d(u,v)\geq 2}} w_{uv}^2\, r_{uv}(t) + \sum_{\substack{u,v\in V\\ d(u,v)=1}} w_{uv}^2\, r_{uv}(t) + \sum_{u\in V} \overset{\overset{\displaystyle w_{uu}^2}{\shortparallel}}{\deg(u)}\, r_{uv}(t)$$

$$t^k r_G(t) = \sum_{u,v} w_{uv}^k\, r_{uv}(t)$$
$$+ \sum_{u,v}\sum_{x} w_{ux}^k\, r_{xv}(t)$$

In $\mathbb{Z}[t]/\chi_G(t)$: $\displaystyle\sum_{u,v} w_{uv}^k\, r_{uv}(t) = t^k r_{uu}(t)$

$$t^k r_{uv}(t) = \sum_{x\in V} w_{ux}^k\, r_{xv}(t)$$

$$r_{uv}(t)\cdot\sum_{k=0} w_{uv}^k \qquad \Rightarrow$$

$$u,v\in V:\quad w_{uv}(t) = t^2\sum_{x\in V} w_{ux}^2\cdot w_{xv}(t)$$
$$r_{uv}(t) = \chi_G(t)\cdot t^{-1}\cdot t^{-2}\cdot\sum_{x\in V} w_{ux}^2\, w_{xv}(t^{-1})$$

$$\sum_{u,v}' \deg(u)\, r_{uv}(t) = t\cdot r_G(t) - n\,\chi_G(t) \qquad \frac{1}{t}\sum_{x\in V}' w^2_{ux}\, r_{xv}(t)$$

In $\mathbb{Z}[t]/\chi_G(t)$: $\quad t\cdot r_G(t) = \sum_{u,v} \deg(u)\, r_{uv}(t)$, $\quad t^2 r_G(t) \overset{?}{=} \sum_{u,v} \deg(u)\deg(v)\, r_{uv}(t)$

$$\chi_G'(t) = \sum_{u\in V}' r_{uu}(t) \qquad\qquad t^k r_G(t) =$$

$$\chi_G''(t) = \sum_{u\in V}' r_{uu}'(t) = \sum_{u,v\in V}$$

$$t^k r_G(t) = \sum_{u,v}'\Big(\sum_x' w^k_{ux}\Big)\, r_{uv}(t)$$

$$\underbrace{\sum_{u,v}' w^1_{uv}\, r_{uv}(t) + \sum_{u\neq v} t\, r_{u,v}(t)} = \sum_{u,v}' \deg(u)\, r_{uv}(t)$$

$$\sum_{\{u,v\}\in E}' (1+t)\, r_{uv}(t) + t\sum_{u,v\notin E} r_{uv}(t) = \sum_{u,v}' \deg(u)\, r_{uv}(t)$$

$$\sum_{u,v} \deg(u)\, r_{uv}(t) = \varphi\Big(\sum_{u,v}' \deg(u)\, w_{uv}(t)\Big)$$

$$\sum_{u,v}' \deg(u)\, w_{uv}(t) = \frac{1}{t}\big(w_G(t) - n\big) \text{ ?}$$

$$\Rightarrow \sum_{u,v} \deg(u)\, r_{uv}(t) = \chi_G(t) \cdot t^{-1} \cdot t \cdot (w_G(t^{-1}) - n)$$

$$= t \cdot r_G(t) - n \cdot \chi_G(t)$$

$$\sum_{x \in N(u)} r_{xv}(t) = \sum_{x \in N(v)} r_{xu}(t) = t \cdot r_{uv}(t)$$

$$\left(\sum_{i=0}^{n} t^i\right) \cdot \chi_G'(t) = \sum_{i=0}^{n} \sum_{u,v \in V} w_{uv}^i \cdot r_{uv}^i(t)$$

$$= \sum_{u,v \in V} \left(\sum_{i=0}^{n} w_{uv}^i\right) r_{uv}(t)$$

$$\left(\sum_{i=0}^{n} t^i\right) \cdot r_{uu}(t) = \sum_{v \in V} \left(\sum_{i=0}^{n} w_{uv}^i\right) r_{uv}(t)$$

$$\chi_G(t) \cdot \chi_G'(t) = 0$$

$$\left(\sum_{i=1}^{n} a_i t^i\right) \cdot \chi_G'(t) = \sum_{i=0}^{n} a_i\ t^i \cdot \chi_G'(t)$$

$$= \sum_{i=0}^{n} a_i \sum_{u,v} w_{uv}^i\, r_{uv}(t)$$

$$= \sum \sum^{n} (a_i w_{uv}^i)\ r_{uv}(t)$$

$u,v \qquad i=0$

$$\sum_{u,v}' w_{uv}^{n} \, r_{uv}(t) = -\sum_{u,v}' \sum_{i=0}^{n-1}{}' a_i w_{uv}^{i} \, r_{uv}(t)$$

$$\sum_{u,v}' (t-\deg(u)) r_{uv}(t) = 0$$

$$\prod_{u\in V}(t-\deg(u)) \cdot \chi_G'(t) = \sum_{v\in V}' \prod_{u\in V}(t-\deg(u)) \, r_{vv}(t)$$

$$t^2 \chi_G'(t) = \sum_{u,v}' w_{uv}^{2} \, r_{uv}(t)$$

$$\shortparallel$$

$$t \cdot \sum_{u,v}' w_{uv}^{1} \, r_{uv}(t) = \sum_{u,v,x}' w_{uv}^{1} w_{vx}^{1} \, r_{ux}(t)$$

$$t^{k+1} \chi_G'(t) = \sum_{u,v}' w_{uv}^{k+1} \, r_{uv}(t)$$

$$= \sum_{u,v,x}' w_{uv}^{k} w_{vx}^{1} \, r_{ux}(t)$$

$$= \sum_{u,v}' \left(\sum_{x}' w_{ux}^{k} w_{xv}^{1} \right) r_{uv}(t)$$

$\langle r_u, r_v \rangle = r_{uv}$

$\rightsquigarrow \chi_G'(t) = \left\langle \begin{pmatrix} 1 \\ | \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ | \\ 1 \end{pmatrix} \right\rangle$

$t^k \chi_G'(t) = (1, \ldots, 1) A^n \begin{pmatrix} 1 \\ | \\ 1 \end{pmatrix}$

$B := \mathbb{Z}[x] / \chi_G(x)$

$M := \mathbb{Z}^n$, identify e_i with $v_i \in V$

B-module via $\overset{\sum_i a_i x^i}{\overset{\shortparallel}{p}} \cdot v := \sum_i a_i A^i v$

$\langle \cdot, \cdot \rangle : M \times M \to B$, $\langle u, v \rangle \longmapsto r_{uv}(x)$ + linear expansion

It holds $\langle pS, T \rangle = p \cdot \langle S, T \rangle = \langle S, p \cdot T \rangle$ for all $p \in B$

Define $G := \sum_{u \in V} u \in M$ View M as multisets $\subseteq V$ $S, T \subseteq G$

$\sum_i \langle u, u \rangle = \chi_G'(x)$

What is $\underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ | & \ddots & | \\ 1 & \cdots & 1 \end{pmatrix}}_{E}$ in $\mathbb{Z}[x] / \chi_G(x)$?

$E \cdot S = |S| \cdot G$

What is $\langle G, G \rangle$?

$\mathcal{B} \longrightarrow M, \quad p \longmapsto p \cdot G$

orth. basis of M?

$$w_{uv}(t)^2 = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} w_{uv}^i w_{uv}^{n-i} \right) t^n$$

If $G = \sum_{i=1}^{n} s_i$, what is $\sum_{i=1}^{n} \langle s_i, s_i \rangle$?

$Q(\mathcal{B})$ total quotient ring

$\rightsquigarrow$ $M \otimes_{\mathcal{B}} Q(\mathcal{B})$ is a $Q(\mathcal{B})$-module with $\langle \cdot, \cdot \rangle : M \times M \longrightarrow Q(\mathcal{B})$
$Q(\mathcal{B})$-bilinear

If $\langle s, t \rangle = 0$, they are disconnected

If $\langle s, s \rangle = 0$, s is "empty"

If $\langle ts, s \rangle = 0$, s is isolated
$\Leftrightarrow t \langle s, s \rangle = 0$

If $\langle s, t \rangle \in Q(\mathcal{B})^\times$, s and t are "in general position"

For $s \in M$, what is $\langle s, s \rangle$ (up to multiplication by $Q(\mathcal{B})^\times$)?

ρ_G generates the annihilator ideal of $\langle G,G\rangle$

$M \longrightarrow M^*$, $s \longmapsto \langle s, \cdot \rangle$

$M \longrightarrow G$, $s \longmapsto \langle s, G\rangle$

$\langle u+v, u+v\rangle = \mu \langle G,G\rangle$ for some $\mu \in Q(B)^*$?

$\langle u-v, u-v\rangle = \mu \cdot \langle u,u\rangle \cdot \rho_G$?

$x^2 \cdot (5x^2-11)$

(x^2+x-1) $\quad$ (x^2-x-1)

$2(x^2-x-2)$

$2(x^2)$

$x(x+2)$ $\quad$ (x^2-2x)

(x^2-x)

(x^2-x-2)

$(x+1)^2$ $\quad (x^2-2x-3)$ $\quad (x-3)(x+1)$

(x^2-x) $\quad x(x-1)$

(x^2-x-2) $\quad (x-2)(x+1)$

x $\quad x^3-3x$ $\quad x(x^2-5)$

x^3-5x-4 $\quad (x+1)(x^2-x-4)$

x^3-2x $\quad x(x^2-2)$

x^3-2x $\quad x(x^2-2)$

$$\underset{\displaystyle\sum_{u\in V}\langle u,u\rangle}{\underset{\shortparallel}{\chi'_G}} = q\cdot \underset{\displaystyle\sum_{s\in P}\frac{\langle s,s\rangle}{\|s\|^2}}{\underset{\shortparallel}{\rho'_G}} + q'\cdot \rho_G$$

.

$P \in \mathbb{R}^{n\times k}_{\mathbb{Q}?}$ is a partition of G, if all column sums are equal to 1
and $P^T \cdot P$ is diagonal

P is equitable, if there exists $B \in \mathbb{Z}^{k\times k}$, s.t.

$$A \cdot P = P \cdot B$$

If P is orthogonal, it is equitable $(B = P^{-1} A P)$

Otherwise, $P^+ :=$ pseudoinverse of P

Then $B = P^+ A P$

It holds $P \cdot P^+ = I_k$

$P^+ \cdot P \in \mathbb{R}^{n\times n}$ is projection onto main eigenvectors of A

(Is it?)

If P is equitable, then

inner product on M

Euclidean scalar product

$$\frac{\chi_G}{\chi_B} \cdot \chi_B' = \sum_{i=1}^{k} \frac{\langle P_i, P_i \rangle}{|P_i|}$$

cardinality of P_i $= \langle P_i, P_i \rangle \in \mathbb{Q}/\mathbb{R}$

$$r_G(t) = \chi_G(t) \cdot t^{-1} \cdot \sigma(t^{-1})$$

$$\chi_G'(t) + t \cdot \chi_G'(t) + \Big(t^2 \cdot \chi_G'(t) - \sum_{u \in V}' \deg(u)\, r_{uu}(t) - \sum_{u,v}' w_{uv}^1 w_{uv}^2 r_{uv}(t)$$

$$- \sum_{u,v}' (w_{uv}^2 - 1)\, r_{uv}(t) \Big)$$

$$\Big(\sum_{i=0}^{k} t^i \Big) \chi_G'(t) = \sum_{u,v}' \Big(\sum_{i=0}^{k} w_{uv}^i \Big) r_{uv}(t)$$

$$\chi_G(t) = x^k \left(p_k(t) \right) + O(x^{k-1}) \qquad r_{uv}(t) = \sum_{i=0}^{d} r_{uv}^i t^i$$

$$p_k \chi_G'(t) = \sum_{u,v}' r_{uv}^k r_{uv}(t)$$

$$t^k \chi_G'(t) = \sum_{u,v}' w_{uv}^k r_{uv}(t)$$

$$\chi_G'(t)^2 = \sum_{u,v}' r_{uv}(t)^2 \quad \text{in } \mathbb{Z}[t]/\chi_G(t)$$

$$\Big(\sum_u' r_{uu}(t) \Big)^2 = \sum_u' r_{uu}(t)^2 + \sum_{u \neq v}' r_{uu}(t) r_{vv}(t)$$

$$\Rightarrow \sum_{u \neq v}' r_{uv}(t)^2 = \sum_{u \neq v}' r_{uu}(t) r_{vv}(t), \text{ resp. } r_{uv}(t)^2 = r_{uu}(t) r_{vv}(t) \text{ in } \mathbb{Z}[t]/\chi_G(t)$$

$$\sum_{k=1}^{n} p_k(x) \cdot x^{k-1} = \chi_G'(x)$$

$$R(x) = \left(r_{uv}(x)\right)_{u,v}$$

$$R(x) = \sum_{k=1}^{n} p_k(A) \cdot x^{k-1}$$

$$r_G(x) = (1, \ldots, 1) \cdot R(x) \cdot \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$