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Note on graphs with irreducible characteristic polynomials

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ABSTRACT

Let G be a connected simple graph with characteristic polynomial $P_G(x)$. The irreducibility of $P_G(x)$ over rational numbers $\mathbb{Q}$ has a close relationship with the automorphism group, reconstruction and controllability of a graph. In this paper we derive three methods to construct graphs with irreducible characteristic polynomials by appending paths $P_{2^{n+1}-2}$ $(n \geq 1)$ to certain vertices; union and join K_1 alternately and corona. These methods are based on Eisenstein's criterion and field extensions. Concrete examples are also supplied to illustrate our results.

1. Introduction

Throughout this paper, let G be a simple connected graph with vertex set $V = \{v_1, ..., v_n\}$ and adjacency matrix A. The walk matrix of G is defined as $W = [e, Ae, \ldots, A^{n-1}e]$ where e is the all ones vector. A graph is *controllable* if its walk ma-

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trix is invertible. The complement $\overline{G}$ of a graph G has adjacency matrix $\overline{A}$ and the same vertex set as G, where vertices v_i and v_j are adjacent in $\overline{G}$ if and only if they are not adjacent in G. The *characteristic polynomial of graph* G is defined as that of its adjacency matrix and denoted by $P_G(x) = \det(xI - A)$ where I is the identity matrix.

As usual, P_n denotes the path on n vertices. The *union* of disjoint graphs G and H is defined as $G \cup H$. The *join* $G \nabla H$ of disjoint graphs G and H is the graph obtained by joining each vertex of G to each vertex of H. The *corona* $G \circ H$ is the graph obtained from G and H by joining the i-th vertex of G to each vertex in the i-th copy of H $(i = 1, 2, ..., n)$. $G - u$ denotes the graph obtained from G by deleting the vertex u and all edges incident with u. An *automorphism* of a graph G is a permutation on V that maps edges to edges and non-edges to non-edges. The set of all automorphisms of G forms a group under the composition of maps, which is called the *automorphism group* of G and denoted by $\mathrm{Aut}(G)$.

A polynomial $f(x)$ with coefficients from field $\mathbb{F}$ is called *irreducible over* $\mathbb{F}$ if $f(x)$ cannot be expressed as a product $g(x)h(x)$ where $g(x)$ and $h(x)$ are two polynomials with coefficients from $\mathbb{F}$ whose degrees are both positive and lower than that of $f(x)$. Irreducible polynomial is important in the study of polynomial factorization and algebraic field extensions. The determination of the irreducibility of a polynomial in a given algebraic field is a fundamental and challenging problem in polynomial theory requiring different concepts and tools from many fields in algebra. There are some useful known irreducibility criteria such as Eisenstein's criterion, Dumas's criterion, Perron's criterion, Pólya theorem and so on [9]. And some methods to determine the irreducibility of a polynomial over rational numbers $\mathbb{Q}$ by using its function value is prime number are given in [1,8]. Irreducible polynomials have an extensive range of applications in some disciplines such as algebra, number theory, coding theory and cryptography [12].

Irreducible polynomial is closely related to some topics about graph theory, for instance, the automorphism group; graph reconstruction; controllability etc. Early in 1971 Mowshowitz [7] showed that if the characteristic polynomial of a graph is irreducible over the integers, its automorphism group is trivial. Teranishi [13] established an equality among the number of orbits of $\mathrm{Aut}(G)$ acts on V, $V \times V$ and the number of distinct eigenvalues of a graph when its characteristic polynomial is irreducible over $\mathbb{Q}$. The most well-known result about Graph Reconstruction Conjecture due to Tutte [14] states that a graph with irreducible characteristic polynomial over $\mathbb{Q}$ is reconstructible. Later Godsil and McKay [5] extended the Tutte's result to the algebra generated by the matrices I, A and $\overline{A}$. Wang [15] proved that the spectral version of the reconstruction conjecture about a graph is not true when its characteristic polynomial has only two irreducible factors. In addition, a method of constructing counterexamples is also provided. Godsil [6] showed the irreducibility of the characteristic polynomial of a graph always implies its controllability. Moreover, he also conjectured that almost all graphs are controllable which was confirmed to be true by O'Rourke and Touri [10]. Unlike to the construction of controllable graphs [11], there is no paper, as far as we know, considering the building of graphs with irreducible characteristic polynomials.

In this paper, we focus on the construction of graphs whose characteristic polynomials are irreducible over $\mathbb{Q}$. Three methods involving graph operations such as: appending paths $P_{2^{n+1}-2}$ ($n \geq 1$) to certain vertices of a graph; union and join K_1 alternately; corona are provided which are based on Eisenstein's criterion and field extensions. Concrete examples are also supplied to illustrate our results.

Recently, Eberhard [3] pointed out that the characteristic polynomial of a random symmetric matrix (including for example the adjacency matrix of a random graph) is irreducible with high probability under some unproved facts. For example, there are 4697820 graphs [4] with irreducible characteristic polynomials on 10 vertices of which the number of connected graphs is 11716571. Paradoxically, no method is known to construct graphs with irreducible characteristic polynomials, this motivates us to consider the problem. In Section 2, we cite some known results and give some necessary conditions for the graphs with irreducible characteristic polynomials. In Section 3, we describe the methods of constructing families of graphs whose characteristic polynomials are irreducible over $\mathbb{Q}$.

2. Preliminaries and necessary conditions

Here are some known results that we will use later.

2.1. Preliminaries

Given a graph G, the partition $\pi = V_1 \dot{\cup} V_2 \dot{\cup} \cdots \dot{\cup} V_k$ of V is an *equitable partition* if every vertex in V_i has b_{ij} neighbours in V_j, for all $i, j \in \{1, 2, \ldots, k\}$. The $k \times k$ matrix $B_\pi = [b_{ij}]$ is called the *divisor matrix* of π. Equitable partitions represent a powerful tool in spectral graph theory.

Theorem 1. *[2, p. 85] The characteristic polynomial of any divisor matrix of a graph divides the characteristic polynomial of the graph.*

Eisenstein's irreducibility criterion is a sufficient condition assuring that an integer polynomial $f(x)$ is irreducible in rational field $\mathbb{Q}$.

Theorem 2 *(Eisenstein's Criterion). [9, p. 50] Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial with integer coefficients such that the coefficient a_n is not divisible by a prime p, while the coefficients $a_0, \ldots, a_{n-1}$ are divisible by p but a_0 is not divisible by p^2. Then $f(x)$ is irreducible over $\mathbb{Q}$.*

Sachs' Coefficient Theorem relates the coefficients of the characteristic polynomial of a graph to its elementary subgraphs. An *elementary subgraph* is a graph in which each component is P_2 or a cycle.

Theorem 3 *(Sachs' Coefficient Theorem). [2, p. 36] Let $P_G(x) = x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$, and let $\mathcal{H}_i$ be the set of elementary subgraphs of G with i vertices. Then*

$$c_i = \sum_{H \in \mathcal{H}_i} (-1)^{p(H)} 2^{c(H)} \quad (i = 1, 2, ..., n),$$

where $p(H)$ denotes the number of components of H and $c(H)$ denotes the number of cycles in H.

The next theorem gives a formula for computing the characteristic polynomial of a graph with a cut edge.

Theorem 4. *[2, p. 31] Let G, H be two graphs with characteristic polynomials $P_G(x)$ and $P_H(x)$ respectively. Let Γ be a graph obtained from $G \cup H$ by adding an edge joining the vertex u of G to the vertex v of H. Then the characteristic polynomial of Γ is*

$$P_\Gamma(x) = P_G(x)P_H(x) - P_{G-u}(x)P_{H-v}(x).$$

The following theorem gives characteristic polynomial of the join of two graphs.

Theorem 5. *[2, p. 27] Let G, H be graphs with n_1, n_2 vertices respectively. The characteristic polynomial of the join $G\nabla H$ is given by the relation*

$$\begin{aligned} P_{G\nabla H}(x) = (-1)^{n_2} P_G(x) P_{\overline{H}}(-x-1) + (-1)^{n_1} P_H(x) P_{\overline{G}}(-x-1) \\ - (-1)^{n_1+n_2} P_{\overline{G}}(-x-1) P_{\overline{H}}(-x-1). \end{aligned}$$

2.2. Some necessary conditions

In this subsection, we collect some known necessary conditions for irreducibility of characteristic polynomials of graphs in the aspect of the properties of graphs and groups. In order to make this paper readable and self-contained, we briefly give the proofs.

Theorem 6. *Let G be a graph whose characteristic polynomial $P_G(x)$ is irreducible over $\mathbb{Q}$. Then*

(1) *G is connected.*
(2) *G is irregular.*
(3) *G is nonsingular.*
(4) *G has only trivial equitable partition.*
(5) *G is controllable [6].*
(6) *G has only trivial automorphism group [7].*

Proof. The statements (1)–(3) are obvious and we only prove (4)–(6) of Theorem 6.

If G has a nontrivial equitable partition π, then B_π is of order less than n, by Theorem 1, $P_{B_\pi}(x)$ is a proper factor of $P_G(x)$, a contradiction.

If G is not controllable, then G has non-main eigenvalues $\mu_1, \ldots, \mu_s$ $(1 \le s \le n-1)$, $\prod_{i=1}^{s}(x-\mu_i) \in \mathbb{Q}[x]$ is a proper factor of $P_G(x)$. Thus $P_G(x)$ is reducible over $\mathbb{Q}$, a contradiction.

If G has the nontrivial automorphism group $\mathrm{Aut}(G)$, then $\mathrm{Aut}(G)$ acts on V has at least one nontrivial orbits. Moreover, the orbit partition of V is necessarily an equitable partition, this reduces (6) to (4) of Theorem 6. $\square$

3. Constructing graphs with irreducible characteristic polynomials

In this section, we present three methods to build graphs with irreducible characteristic polynomials over $\mathbb{Q}$.

3.1. Construction by appending paths

The following theorem shows that small graph whose characteristic polynomial satisfying Eisenstein's criterion can provide the foundation for constructing infinitely many graphs with irreducible characteristic polynomials.

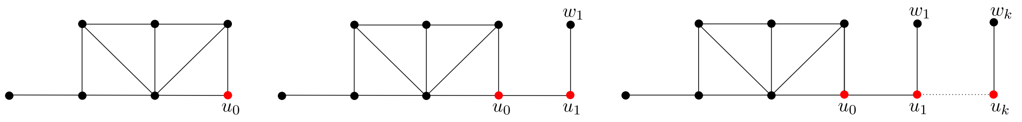
Theorem 7. *Let G_0 be a graph of order n_0 and with a fixed vertex u_0. Let u_iw_i be a path P_2 $(i = 1, 2, \ldots)$. Construct a family of graphs G_i by adding an edge between the vertex u_{i-1} and u_i $(i = 1, 2, \ldots)$. Suppose $P_{G_0}(x) = x^{n_0} + a^0_{n_0-2}x^{n_0-2} + \cdots + a^0_1 x + a^0_0$ and $P_{G_0-u_0}(x) = x^{n_0-1} + b^0_{n_0-3}x^{n_0-3} + \cdots + b^0_1 x + b^0_0$ be the characteristic polynomials of G_0 and $G_0 - u_0$, respectively. If $P_{G_0}(x)$ and $P_{G_0-u_0}(x)$ satisfy the following two conditions*

(1) $a^0_s \equiv 0 \pmod 2$ $(s = 1, ..., n_0 - 2)$, $a^0_0 \equiv 2 \pmod 4$;
(2) $b^0_t \equiv 0 \pmod 2$ $(t = 0, 1, ..., n_0 - 3)$;

then all characteristic polynomials of G_i are irreducible over $\mathbb{Q}$.

Proof. We prove this theorem by induction on the subscript G_i. For the sake of simplicity, denote by $a^1_{n_0} = a^0_{n_0-2} - 2$, $a^1_{n_0-1} = a^0_{n_0-3}$, $a^1_j = a^0_{j-2} - a^0_j - b^0_{j-1} (j = 2, \ldots, n_0 - 2)$, $a^1_1 = -(a^0_1 + b^0_0)$, $a^1_0 = -a^0_0$. By Theorem 4, the characteristic polynomial of G_1 is

$$
\begin{aligned}
P_{G_1}(x) &= P_{G_0}(x)P_{P_2}(x) - P_{G_0-u_0}(x)P_{P_2-u_1}(x) \\
&= P_{G_0}(x)(x^2-1) - xP_{G_0-u_0}(x) \\
&= x^{n_0+2} + a^0_{n_0-2}x^{n_0} + a^0_{n_0-3}x^{n_0-1} + a^0_{n_0-4}x^{n_0-2} + \cdots + a^0_1 x^3 + a^0_0 x^2 \\
&\quad - (x^{n_0} + a^0_{n_0-2}x^{n_0-2} + \cdots + a^0_2 x^2 + a^0_1 x + a^0_0) \\
&\quad - (x^{n_0} + b^0_{n_0-3}x^{n_0-2} + \cdots + b^0_1 x^2 + b^0_0 x) \\
&= x^{n_0+2} + (a^0_{n_0-2} - 2)x^{n_0} + a^0_{n_0-3}x^{n_0-1} + (a^0_{n_0-4} - a^0_{n_0-2} - b^0_{n_0-3})x^{n_0-2} +
\end{aligned}
$$



(a) G_0 (b) G_1 (c) G_k

Fig. 1. Graphs constructed by Theorem 7.

$$
\begin{aligned}
&\cdots \\
&+ (a_0^0 - a_2^0 - b_1^0)x^2 - (a_1^0 + b_0^0)x - a_0^0 \\
:=\ & x^{n_0+2} + a_{n_0}^1 x^{n_0} + a_{n_0-1}^1 x^{n_0-1} + \cdots + a_2^1 x^2 + a_1^1 x + a_0^1.
\end{aligned}
$$

From conditions (1) and (2), we have $a_s^1 \equiv 0 \pmod 2$ $(s = 1, ..., n_0)$ and $a_0^1 = (-1)^1 a_0^0 \equiv 2 \pmod 4$. Therefore the characteristic polynomial of G_1 is irreducible over $\mathbb{Q}$ by Theorem 2.

Suppose the result holds for $i \le k$, then for $i = k+1$, we have

$$
\begin{aligned}
&P_{G_{k+1}}(x) \\
&= P_{G_k}(x)P_{P_2}(x) - P_{G_k - u_k}(x)P_{P_2 - u_{k+1}}(x) \\
&= P_{G_k}(x)P_{P_2}(x) - xP_{G_{k-1}}(x)P_{P_1}(x) \\
&= P_{G_k}(x)(x^2-1) - x^2 P_{G_{k-1}}(x) \\
&= x^{n_0+2k+2} + a_{n_0+2k-2}^k x^{n_0+2k} + a_{n_0+2k-3}^k x^{n_0+2k-1} + \cdots + a_1^k x^3 + a_0^k x^2 \\
&\quad - (x^{n_0+2k} + a_{n_0+2k-2}^k x^{n_0+2k-2} + \cdots + a_1^k x + a_0^k) \\
&\quad - (x^{n_0+2k} + a_{n_0+2k-4}^{k-1} x^{n_0+2k-2} + \cdots + a_1^{k-1} x^3 + a_0^{k-1} x^2) \\
&= x^{n_0+2k+2} + (a_{n_0+2k-2}^k - 2)x^{n_0+2k} + a_{n_0+2k-3}^k x^{n_0+2k-1} + \\
&\quad (a_{n_0+2k-4}^k - a_{n_0+2k-2}^k - a_{n_0+2k-4}^{k-1})x^{n_0+2k-2} + \cdots + (a_0^k - a_2^k - a_0^{k-1})x^2 - a_1^k x - a_0^k \\
&:= x^{n_0+2k+2} + a_{n_0+2k}^{k+1} x^{n_0+2k} + a_{n_0+2k-1}^{k+1} x^{n_0+2k-1} + \cdots + a_2^{k+1} x^2 + a_1^{k+1} x + a_0^{k+1},
\end{aligned}
$$

where $a_{n_0+2k}^{k+1} = a_{n_0+2k-2}^k - 2$, $a_{n_0+2k-1}^{k+1} = a_{n_0+2k-3}^k$, $a_j^{k+1} = a_{j-2}^k - a_j^k - a_{j-2}^{k-1} (j = 2, \ldots, n_0 + 2k - 2)$, $a_1^{k+1} = -a_1^k$, $a_0^{k+1} = -a_0^k = (-1)^{k+1} a_0^0$ $(k = 1, 2, 3, \ldots)$. Applying induction hypothesis, we have $a_s^{k+1} \equiv 0 \pmod 2$ $(s = 1, ..., n_0 + 2k)$, $a_0^{k+1} \equiv 2 \pmod 4$. And $P_{G_{k+1}}(x)$ is also irreducible over $\mathbb{Q}$ by Theorem 2. This completes the proof. $\square$

Example 1. Let G_0 be the graph depicted in Fig. 1 and u_0 be a fixed vertex. It is easy to see that the two conditions of Theorem 7 are satisfied since the characteristic polynomials of G_0 and $G_0 - u_0$ are $P_{G_0}(x) = x^7 - 10x^5 - 8x^4 + 16x^3 + 14x^2 - 4x - 2$ and $P_{G_0 - u_0}(x) = x^6 - 8x^4 - 6x^3 + 8x^2 + 6x$, respectively. Therefore, we can construct an infinite family of graphs G_i by attaching an edge to vertex u_{i-1} $(i = 1, 2, \ldots)$ by Theorem 7. For example, the first two graphs G_1, G_2 have the characteristic polynomials

$P_{G_1}(x) = x^9 - 12x^7 - 8x^6 + 34x^5 + 28x^4 - 28x^3 - 22x^2 + 4x + 2$ and $P_{G_2}(x) = x^{11} - 14x^9 - 8x^8 + 56x^7 + 44x^6 - 78x^5 - 64x^4 + 36x^3 + 26x^2 - 4x - 2$. They are irreducible over $\mathbb{Q}$ by Theorem 2.

Theorem 8. *Let G_0 be a graph of order n_0 and with a fixed vertex u_0, $v_{i_1} \cdots v_{i_6}$ be a path P_6 and denote by $u_i := v_{i_3} (i = 1, 2, 3, ...)$. Construct a family of graphs G_i by adding an edge between the vertex u_{i-1} and v_{i_1} $(i = 1, 2, 3, ...)$. Suppose $P_{G_0}(x) = x^{n_0} + a^0_{n_0-2}x^{n_0-2} + \cdots + a^0_1 x + a^0_0$, $P_{G_0-u_0}(x) = x^{n_0-1} + b^0_{n_0-3}x^{n_0-3} + \cdots + b^0_1 x + b^0_0$ be the characteristic polynomials of G_0 and $G_0 - u_0$, respectively. If $P_{G_0}(x)$ and $P_{G_0-u_0}(x)$ satisfy the following two conditions*

(1) $a^0_s \equiv 0 \pmod 2$ $(s = 0, 1, ..., n_0 - 2)$, $a^0_0 \equiv 2 \pmod 4$;
(2) $b^0_t \equiv 0 \pmod 2$ $(t = 0, 1, ..., n_0 - 3)$;

then all characteristic polynomials of G_i $(i = 1, 2, ...)$ are irreducible over $\mathbb{Q}$.

Proof. We prove this theorem by induction on the subscript G_i. According to Theorem 4, the characteristic polynomial of G_1 is

$$\begin{aligned} P_{G_1}(x) &= P_{G_0}(x)P_{P_6}(x) - P_{G_0-u_0}(x)P_{P_6-v_{1_1}}(x) \\ &= P_{G_0}(x)(x^6 - 5x^4 + 6x^2 - 1) - P_{G_0-u_0}(x)(x^5 - 4x^3 + 3x) \\ &= (x^{n_0} + a^0_{n_0-2}x^{n_0-2} + \cdots + a^0_1 x + a^0_0)(x^6 - 5x^4 + 6x^2 - 1) \\ &\quad - (x^{n_0-1} + b^0_{n_0-3}x^{n_0-3} + \cdots + b^0_1 x + b^0_0)(x^5 - 4x^3 + 3x) \\ &:= x^{n_0+6} + a^1_{n_0+4}x^{n_0+4} + \cdots + a^1_2 x^2 + a^1_1 x + a^1_0. \end{aligned}$$

Set $f_1 = P_{G_0}(x)\left(x^6 - 5x^4 + 6x^2 - 1\right)$ and $f_2 = -P_{G_0-u_0}(x)\left(x^5 - 4x^3 + 3x\right)$, we analyze the coefficients of $P_{G_1}(x)$ according to the coefficients of f_1 and f_2.

Since the terms of $P_{G_0}(x)$ have all even coefficients except for the leading term, it is easy to verify that only the coefficients of x^{n_0+6}, x^{n_0+4} and x^{n_0} in f_1 are odd. The similar result holds for the coefficients of x^{n_0+4} and x^{n_0} in f_2. In view of $P_{G_1}(x) = f_1 + f_2$, we see that the leading term of $P_{G_1}(x)$ is obvious 1 and all the other coefficients of $P_{G_1}(x)$ are even. Moreover, the constant terms of $P_{G_1}(x)$ and $P_{G_0}(x)$ are negative to each other, we have $a^1_s \equiv 0 \pmod 2$ $(s = 1, ..., n_0 + 4)$ and $a^1_0 = -a^0_0 \equiv 2 \pmod 4$. Thus $P_{G_1}(x)$ is irreducible over $\mathbb{Q}$ by Theorem 2.

Next suppose the result holds for $i \le k$, when $i = k + 1$ we have

$$\begin{aligned} P_{G_{k+1}}(x) &= P_{G_k}(x)P_{P_6}(x) - P_{G_k-u_k}(x)P_{P_6-v_{k_1}}(x) \\ &= P_{G_k}(x)P_{P_6}(x) - P_{G_k-u_k}(x)P_{P_5}(x) \\ &= (x^{n_0+6k} + a^k_{n_0+6k-2}x^{n_0+6k-2} + \cdots + a^k_1 x + a^k_0)(x^6 - 5x^4 + 6x^2 - 1) \\ &\quad - (x^{n_0+6k-1} + b^k_{n_0+6k-3}x^{n_0+6k-3} + \cdots + b^k_1 x + b^k_0)\left(x^5 - 4x^3 + 3x\right) \end{aligned}$$

$$:= x^{n_0+6k+6} + a_{n_0+6k+4}^{k+1} x^{n_0+6k+4} + \cdots + a_2^{k+1} x^2 + a_1^{k+1} x + a_0^{k+1}.$$

To complete the proof, we need to show $P_{G_i-u_i}(x)$ $(i = 1, 2, \dots)$ preserves the condition (2). We prove this also by induction on the subscript $G_i - u_i$ for $i = 1, 2, \dots$.

$G_1 - u_1$ has two components, denote by C_1^1 the component obtained from G_0 by adding an edge between u_0 and P_2, and C_2^1 the remained component P_3. Then

$$P_{G_1-u_1}(x) = P_{C_1^1}(x)P_{C_2^1}(x) = P_{C_1^1}(x)(x^3 - 2x). \tag{1}$$

Since C_1^1 is exactly the G_1 of Theorem 7, $x^3 P_{C_1^1}(x)$ is a monic polynomial and all the other coefficients are even numbers. Furthermore, it is clear that all coefficients of $(-2x)P_{C_1^1}(x)$ are even. By Eq. (1), all coefficients of the monic polynomial $P_{G_1-u_1}(x)$ are even except for the highest term.

Suppose $P_{G_i-u_i}(x)$ satisfies condition (2) for $i \leq k-1$, when $i = k$, $G_k - u_k$ has two components, where C_2^k denotes the P_3 and C_1^k denotes the remained component, then

$$P_{G_k-u_k}(x) = P_{C_1^k}(x)P_{C_2^k}(x) = [P_{G_{k-1}}(x)(x^2 - 1) - P_{G_{k-1}-u_{k-1}}(x)x](x^3 - 2x).$$

Applying the same proof arguments for $P_{G_1}(x)$ in Theorem 7 and induction hypothesis, we have $P_{C_1^k}(x)$ meets condition (1) of Theorem 8. In analogy to the proof of $P_{G_1-u_1}(x)$, $P_{G_k-u_k}(x)$ satisfies condition (2).

Now we show that $P_{G_{k+1}}(x)$ satisfies the condition (1). By induction $P_{G_k}(x)$ meets condition (1), and we have shown $P_{G_k-u_k}(x)$ satisfies the conditions (2). Thus the terms of x^{n_0+6k+4} and x^{n_0+6k} in $P_{G_k}(x)P_{P_6}(x)$ and $P_{G_k-u_k}(x)P_{P_5}(x)$ have odd coefficients $-5, -1$ and $1, 3$, respectively. Adding them together induces $P_{G_{k+1}}(x)$ satisfies the following conditions:

$$a_s^{k+1} \equiv 0 \pmod 2 (s = 1, ..., n_0 + 6k + 4), a_0^{k+1} = (-1)^{k+1} a_0^0 \equiv 2 \pmod 4.$$

Therefore $P_{G_{k+1}}(x)$ is an irreducible polynomial over $\mathbb{Q}$ by Theorem 2. This completes the proof. □

In a similar fashion, we have the following generalized theorem and the proof is omitted.

Theorem 9. *Let G_0 be a graph of order n_0 and with a fixed vertex u_0, $v_{i_1} \cdots v_{i_{2^{n+1}-2}}$ be a path $P_{2^{n+1}-2}$ and denote by $u_i := v_{i_{2^n-1}} (n = 1, 2, 3, ...)$. Construct a family of graphs G_i by adding an edge between the vertex u_{i-1} and v_{i_1} $(i = 1, 2, 3, ...)$. Suppose $P_{G_0}(x) = x^{n_0} + a_{n_0-2}^0 x^{n_0-2} + \cdots + a_1^0 x + a_0^0$, $P_{G_0-u_0}(x) = x^{n_0-1} + b_{n_0-3}^0 x^{n_0-3} + \cdots + b_1^0 x + b_0^0$ be the characteristic polynomials of G_0 and $G_0 - u_0$, respectively. If $P_{G_0}(x)$ and $P_{G_0-u_0}(x)$ satisfy the following two conditions*

(1) $a_s^0 \equiv 0 \pmod 2$ $(s = 0, 1, ..., n_0 - 2)$, $a_0^0 \equiv 2 \pmod 4$*;*

(2) $b_t^0 \equiv 0 \pmod 2$ $(t = 0, 1, ..., n_0 - 3)$;

then all characteristic polynomials of G_i $(i = 1, 2, ...)$ *are irreducible over* $\mathbb{Q}$.

3.2. Construction by union and join

The next theorem gives another construction by the graph operations of union and join.

Theorem 10. *Let* G_0 *be a connected graph with* n_0 *vertices and* $\overline{G_0}$ *be its complement. Suppose* $P_{G_0}(x) = x^{n_0} + a^0_{n_0-2}x^{n_0-2} + \cdots + a^0_1 x + a^0_0$ *be the characteristic polynomial of* G_0 *and* $P_{\overline{G_0}}(-x-1) = (-1)^{n_0}x^{n_0} + c^0_{n_0-1}x^{n_0-1} + c^0_{n_0-2}x^{n_0-2} + \cdots + c^0_1 x + c^0_0$. *If* $P_{G_0}(x)$ *and* $P_{\overline{G_0}}(-x-1)$ *satisfy the following two conditions*

(1) $a^0_s \equiv 0 \pmod 2$ $(s = 1, ..., n_0 - 2)$, $a^0_0 \equiv 2 \pmod 4$;
(2) $c^0_t \equiv 0 \pmod 2$ $(t = 0, 1, ..., n_0 - 2)$, $c^0_{n_0-1} \equiv 1 \pmod 2$.

Construct a family of graphs G_i *by the following recurrence relation*

$$G_i = (G_{i-1} \cup K_1)\nabla K_1 \ (i = 1, 2, \dots).$$

Then all $P_{G_i}(x)$ *are irreducible over* $\mathbb{Q}$ *and* $P_{G_i}(x)$, $P_{\overline{G_i}}(-x-1)$ *satisfy conditions (1) and (2), respectively.*

Proof. The characteristic polynomial of a graph G joins K_1 can be computed from Theorem 5 by taking $H = K_1$,

$$\begin{aligned}
&P_{G\nabla K_1}(x) \\
&= (-1)^{n_2}(-x-1)P_G(x) + (-1)^{n_1}xP_{\overline{G}}(-x-1) - (-1)^{n_1+n_2}(-x-1)P_{\overline{G}}(-x-1) \\
&= (x+1)P_G(x) + (-1)^{n_1}xP_{\overline{G}}(-x-1) - (-1)^{n_1+2}(x+1)P_{\overline{G}}(-x-1) \\
&= (x+1)P_G(x) + (-1)^{n_1+1}P_{\overline{G}}(-x-1).
\end{aligned}$$

For short, denote by $n_i = n_0 + 2i$ the order of G_i. Replacing G with $G_i \cup K_1$ in the former formula

$$\begin{aligned}
P_{G_{i+1}}(x) &= P_{(G_i\cup K_1)\nabla K_1}(x) \\
&= (x+1)P_{G_i\cup K_1}(x) + (-1)^{n_i+2}P_{\overline{G_i\cup K_1}}(-x-1) \\
&= x(x+1)P_{G_i}(x) + (-1)^{n_i+2}P_{\overline{G_i\cup K_1}}(-x-1).
\end{aligned} \tag{2}$$

Moreover

$$P_{\overline{G_i\cup K_1}}(x) = P_{\overline{G_i}\nabla K_1}(x) = (x+1)P_{\overline{G_i}}(x) + (-1)^{n_i+1}P_{G_i}(-x-1),$$

then

$$P_{\overline{G_i \cup K_1}}(-x-1) = (-x)P_{\overline{G_i}}(-x-1) + (-1)^{n_i+1}P_{G_i}(x), \tag{3}$$

substitute (3) into (2) gives

$$P_{G_{i+1}}(x) = (x^2+x-1)P_{G_i}(x) + (-1)^{n_i+1}xP_{\overline{G_i}}(-x-1). \tag{4}$$

We prove this theorem by induction on the subscript of G_i. For the sake of simplicity, we denote

$$\begin{aligned}
&a^{k+1}_{n_{k+1}-2} = a^k_{n_k-2} + c^k_{n_k-1} - 1,\ a^{k+1}_{n_{k+1}-3} = a^k_{n_k-3} + a^k_{n_k-2} + c^k_{n_k-2},\\
&a^{k+1}_{n_{k+1}-j} = a^k_{n_k-j} + a^k_{n_k-j+1} - a^k_{n_k-j+2} + c^k_{n_k-j+1}\ (j = 4, \dots, n_{k+1}-2),\\
&a^{k+1}_1 = a^k_0 - a^k_1 + c^k_0,\ a^{k+1}_0 = -a^k_0 = (-1)^{k+1}a^0_0,\ n_{k+1} = n_k + 2,\ \text{for all } k = 0, 1, \dots.
\end{aligned}$$

We distinguish two cases according to the parity of n_0. Suppose n_0 is odd, put $i = 0$ in equation (4), we have

$$\begin{aligned}
P_{G_1}(x) &= (x^2+x-1)P_{G_0}(x) + xP_{\overline{G_0}}(-x-1)\\
&= x^{n_0+2} + a^0_{n_0-2}x^{n_0} + a^0_{n_0-3}x^{n_0-1} + \cdots + a^0_1x^3 + a^0_0x^2\\
&\quad + x^{n_0+1} + a^0_{n_0-2}x^{n_0-1} + \cdots + a^0_2x^3 + a^0_1x^2 + a^0_0x\\
&\quad - (x^{n_0} + a^0_{n_0-2}x^{n_0-2} + \cdots + a^0_1x + a^0_0)\\
&\quad + (-x^{n_0+1} + c^0_{n_0-1}x^{n_0} + c^0_{n_0-2}x^{n_0-1} + \cdots + c^0_1x^2 + c^0_0x)\\
&= x^{n_0+2} + (a^0_{n_0-2} - 1 + c^0_{n_0-1})x^{n_0} + (a^0_{n_0-3} + a^0_{n_0-2} + c^0_{n_0-2})x^{n_0-1}\\
&\quad + (a^0_{n_0-4} + a^0_{n_0-3} - a^0_{n_0-2} + c^0_{n_0-3})x^{n_0-2} + \cdots + (a^0_0 + a^0_1 - a^0_2 + c^0_1)x^2\\
&\quad + (a^0_0 - a^0_1 + c^0_0)x - a^0_0\\
&:= x^{n_1} + a^1_{n_1-2}x^{n_1-2} + a^1_{n_1-3}x^{n_1-3} + \cdots + a^1_2x^2 + a^1_1x + a^1_0.
\end{aligned}$$

From $a^0_s \equiv 0 \pmod 2$ $(s = 1, 2, \dots, n_0-2)$, $a^0_0 \equiv 2 \pmod 4$ and $c^0_t \equiv 0 \pmod 2$ $(t = 0, 1, ..., n_0-2)$, $c^0_{n_0-1} \equiv 1 \pmod 2$, recall $n_1 = n_0 + 2$, it immediately follows that

$$a^1_s \equiv 0 \pmod 2\ (s = 1, 2, ..., n_1-2),\ a^1_0 \equiv 2 \pmod 4.$$

Thus $P_{G_1}(x)$ is an irreducible polynomial over $\mathbb{Q}$ by Theorem 2.

For $P_{\overline{G_1}}(-x-1)$, we also denote

$$\begin{aligned}
&c^{k+1}_{n_{k+1}-1} = c^k_{n_k-1} - 2,\ c^{k+1}_{n_{k+1}-2} = c^k_{n_k-2} + c^k_{n_k-1} - 1, c^{k+1}_{n_{k+1}-3} = c^k_{n_k-3} + c^k_{n_k-2} - a^k_{n_k-2},\\
&c^{k+1}_{n_{k+1}-j} = c^k_{n_k-j} + c^k_{n_k-j+1} - a^k_{n_k-j+1} - a^k_{n_k-j+2}\ (j = 4, \dots, n_{k+1}-2),\\
&c^{k+1}_1 = c^k_0 - a^k_0 - a^k_1,\ c^{k+1}_0 = -a^k_0 = (-1)^{k+1}a^0_0,\ \text{for all } k = 0, 1, \dots.
\end{aligned}$$

Note that n_0 is odd, by the equation (3), we have

$$\begin{aligned}
P_{\overline{G_1}}(-x-1) &= P_{K_1\cup(\overline{G_0\cup K_1})}(-x-1) = x(x+1)P_{\overline{G_0}}(-x-1) + (-1)^{n_0}(x+1)P_{G_0}(x)\\
&= -x^{n_0+2} + c^0_{n_0-1}x^{n_0+1} + c^0_{n_0-2}x^{n_0} + \cdots + c^0_1x^3 + c^0_0x^2\\
&\quad + (-x^{n_0+1} + c^0_{n_0-1}x^{n_0} + c^0_{n_0-2}x^{n_0-1} + \cdots + c^0_1x^2 + c^0_0x)\\
&\quad - (x^{n_0+1} + a^0_{n_0-2}x^{n_0-1} + \cdots + a^0_1x^2 + a^0_0x)\\
&\quad - (x^{n_0} + a^0_{n_0-2}x^{n_0-2} + \cdots + a^0_1x + a^0_0)\\
&= -x^{n_0+2} + (c^0_{n_0-1} - 2)x^{n_0+1} + (c^0_{n_0-2} + c^0_{n_0-1} - 1)x^{n_0} +\\
&\quad (c^0_{n_0-3} + c^0_{n_0-2} - a^0_{n_0-2})x^{n_0-1} + (c^0_{n_0-4} + c^0_{n_0-3} - a^0_{n_0-3} - a^0_{n_0-2})x^{n_0-2}\\
&\quad + \cdots + (c^0_0 + c^0_1 - a^0_1 - a^0_2)x^2 + (c^0_0 - a^0_0 - a^0_1)x - a^0_0\\
&:= -x^{n_1} + c^1_{n_1-1}x^{n_1-1} + c^1_{n_1-2}x^{n_1-2} + \cdots + c^1_2x^2 + c^1_1x + c^1_0.
\end{aligned}$$

Due to $P_{G_0}(x)$, $P_{\overline{G_0}}(-x-1)$ satisfy (1), (2), respectively, then

$$c^1_t \equiv 0 \pmod 2\ (t = 0, 1, ..., n_1 - 2),\ c^1_{n_1-1} \equiv 1 \pmod 2,$$

so $P_{\overline{G_1}}(-x-1)$ satisfies condition (2).

Suppose $P_{G_k}(x)$, $P_{\overline{G_k}}(-x-1)$ satisfy (1), (2) for $i \le k$, when $i = k+1$, note that $n_{k+1} = n_k + 2 = n_0 + 2(k+1)$, we have

$$\begin{aligned}
P_{G_{k+1}}(x) &= (x^2 + x - 1)P_{G_k}(x) + xP_{\overline{G_k}}(-x-1)\\
&= x^{n_k+2} + a^k_{n_k-2}x^{n_k} + a^k_{n_k-3}x^{n_k-1} + \cdots + a^k_1x^3 + a^k_0x^2\\
&\quad + x^{n_k+1} + a^k_{n_k-2}x^{n_k-1} + \cdots + a^k_1x^2 + a^k_0x\\
&\quad - (x^{n_k} + a^k_{n_k-2}x^{n_k-2} + \cdots + a^k_1x + a^k_0)\\
&\quad + (-x^{n_k+1} + c^k_{n_k-1}x^{n_k} + c^k_{n_k-2}x^{n_k-1} + \cdots + c^k_1x^2 + c^k_0x)\\
&= x^{n_k+2} + \left(a^k_{n_k-2} - 1 + c^k_{n_k-1}\right)x^{n_k} + \left(a^k_{n_k-3} + a^k_{n_k-2} + c^k_{n_k-2}\right)x^{n_k-1}\\
&\quad + (a^k_{n_k-4} + a^k_{n_k-3} - a^k_{n_k-2} + c^k_{n_k-3})x^{n_k-2} + \cdots + (a^k_0 + a^k_1 - a^k_2 + c^k_1)x^2\\
&\quad + (a^k_0 - a^k_1 + c^k_0)x - a^k_0\\
&:= x^{n_{k+1}} + a^{k+1}_{n_{k+1}-2}x^{n_{k+1}-2} + a^{k+1}_{n_{k+1}-3}x^{n_{k+1}-3} + \cdots + a^{k+1}_2x^2 + a^{k+1}_1x + a^{k+1}_0.
\end{aligned}$$

Since $P_{G_k}(x)$, $P_{\overline{G_k}}(-x-1)$ satisfy (1) and (2), respectively, then

$$a^{k+1}_s \equiv 0 \pmod 2\ (s = 1, 2, ..., n_{k+1} - 2),\ a^{k+1}_0 \equiv 2 \pmod 4,$$

so $P_{G_{k+1}}(x)$ is an irreducible polynomial over $\mathbb{Q}$ by Theorem 2.

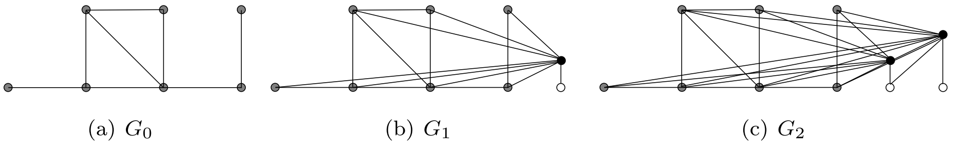


Fig. 2. Graphs constructed by Theorem 10.

And we have

$$
\begin{aligned}
&P_{\overline{G_{k+1}}}(-x-1)\\
&= P_{K_1\cup(\overline{G_k\cup K_1})}(-x-1) = x(x+1)P_{\overline{G_k}}(-x-1) + (-1)^{n_k}(x+1)P_{G_k}(x)\\
&= -x^{n_k+2} + c^k_{n_k-1}x^{n_k+1} + c^k_{n_k-2}x^{n_k} + \cdots + c^k_1x^3 + c^k_0x^2\\
&\quad + (-x^{n_k+1} + c^k_{n_k-1}x^{n_k} + c^k_{n_k-2}x^{n_k-1} + \cdots + c^k_1x^2 + c^k_0x)\\
&\quad - (x^{n_k+1} + a^k_{n_k-2}x^{n_k-1} + \cdots + a^k_1x^2 + a^k_0x)\\
&\quad - (x^{n_k} + a^k_{n_k-2}x^{n_k-2} + \cdots + a^k_1x + a^k_0)\\
&= -x^{n_k+2} + (c^k_{n_k-1} - 2)x^{n_k+1} + (c^k_{n_k-2} + c^k_{n_k-1} - 1)x^{n_k}\\
&\quad + (c^k_{n_k-3} + c^k_{n_k-2} - a^k_{n_k-2})x^{n_k-1} + (c^k_{n_k-4} + c^k_{n_k-3} - a^k_{n_k-3} - a^k_{n_k-2})x^{n_k-2}\\
&\quad + \cdots + (c^k_0 + c^k_1 - a^k_1 - a^k_2)x^2 + (c^k_0 - a^k_0 - a^k_1)x - a^k_0\\
&:= -x^{n_{k+1}} + c^{k+1}_{n_{k+1}-1}x^{n_{k+1}-1} + c^{k+1}_{n_{k+1}-2}x^{n_{k+1}-2} + \cdots + c^k_2x^2 + c^k_1x + c^k_0,
\end{aligned}
$$

since $P_{G_k}(x)$, $P_{\overline{G_k}}(-x-1)$ satisfy (1), (2), respectively, then

$$c^{k+1}_t \equiv 0 \pmod 2\ (t = 0, 1, ..., n_{k+1}-2),\ c^{k+1}_{n_{k+1}-1} \equiv 1 \pmod 2,$$

i.e., $P_{\overline{G_{k+1}}}(-x-1)$ satisfies condition (2).

The proof of $n_0 \equiv 0 \pmod 2$ is similar to $n_0 \equiv 1 \pmod 2$.

By induction, we get that for any $G_i = (G_{i-1} \cup K_1)\nabla K_1$ $(i = 1, 2, \dots)$, $P_{G_i}(x)$ and $P_{\overline{G_i}}(-x-1)$ satisfy conditions (1) and (2), respectively. Therefore $P_{G_i}(x)$ is an irreducible polynomial over $\mathbb{Q}$ and the proof is completed. □

Example 2. Let G_0 be the graph depicted in Fig. 2. Then $P_{G_0}(x) = x^7 - 8x^5 - 4x^4 + 12x^3 + 6x^2 - 4x - 2$, $P_{\overline{G_0}}(-x-1) = -x^7 - 7x^6 - 8x^5 + 16x^4 + 24x^3 - 6x^2 - 12x - 2$, they satisfy the two conditions in Theorem 10, hence all graphs $G_i = (G_{i-1} \cup K_1)\nabla K_1$ have irreducible characteristic polynomials over $\mathbb{Q}$. For instance, $P_{G_1}(x) = x^9 - 16x^7 - 20x^6 + 32x^5 + 46x^4 - 16x^3 - 24x^2 + 2$, $P_{G_2}(x) = x^{11} - 26x^9 - 52x^8 + 44x^7 + 150x^6 + 8x^5 - 122x^4 - 24x^3 + 30x^2 + 4x - 2$. They are irreducible over $\mathbb{Q}$ by Theorem 2.

3.3. Construction by corona

The following construction uses field extensions, the relation of eigenvalues between a graph and its corona.

Theorem 11. *Let G be a graph with n vertices, adjacency matrix A and characteristic polynomial $P_G(x)$. Denote by $\hat{G}$ the corona graph of G and K_1, $\hat{A}$, $P_{\hat{G}}(x)$ its adjacency matrix and characteristic polynomial, respectively. If $P_G(x)$ is irreducible over $\mathbb{Q}$, then $P_{\hat{G}}(x)$ is either irreducible over $\mathbb{Q}$ or can be factored into two irreducible factors with the same degree.*

Proof. We can label the vertices of $\hat{G}$ such that

$$\hat{A} = \begin{bmatrix} A & I \\ I & O \end{bmatrix},$$

then

$$\begin{aligned} P_{\hat{G}}(x) = \det(xI - \hat{A}) &= \det \begin{bmatrix} xI - A & -I \\ -I & xI \end{bmatrix} \\ &= \det(\begin{bmatrix} xI - A & -I \\ -I & xI \end{bmatrix} \begin{bmatrix} I & O \\ \frac{1}{x}I & I \end{bmatrix}) \\ &= \det \begin{bmatrix} (x - \frac{1}{x})I - A & -I \\ O & xI \end{bmatrix} \\ &= x^n \det((x - \frac{1}{x})I - A) \\ &= x^n P_G(x - \frac{1}{x}). \end{aligned}$$

Next we investigate the relation between the eigenvalues of G and $\hat{G}$. Suppose μ is any eigenvalue of $\hat{G}$, since $\hat{G}$ has the unique perfect matching as its spanning elementary subgraph, Theorem 3 implies the constant term of $P_{\hat{G}}(x)$ is ± 1, thus $\mu \neq 0$. Since

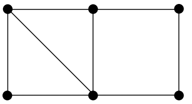
$$P_{\hat{G}}(\mu) = \mu^n P_G(\mu - \frac{1}{\mu}) = 0,$$

we see that

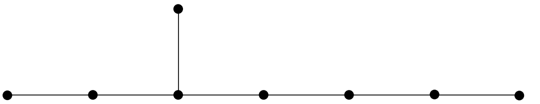
$$P_G(\mu - \frac{1}{\mu}) = 0.$$

It follows that

$$\lambda := \mu - \frac{1}{\mu} \tag{5}$$



(a) G_0



(b) T_8

Fig. 3. Graphs with irreducible characteristic polynomials over $\mathbb{Q}$.

is a root of $P_G(x)$, i.e., λ is an eigenvalue of G, which is also equivalent to

$$\mu = \frac{1}{2}(\lambda \pm \sqrt{\lambda^2 + 4}). \tag{6}$$

Let $\mathbb{K} = \mathbb{Q}(\lambda)$, since $P_G(x)$ is irreducible over $\mathbb{Q}$, the degree of the field extension $[\mathbb{K} : \mathbb{Q}] = n$. Let $\mathbb{L} = \mathbb{Q}(\mu)$, by Eq. (5), we get $\lambda \in \mathbb{L}$. Thus $\mathbb{K} \subseteq \mathbb{L}$.

On the other hand, from (6), we see that $\mu \in \mathbb{K}(\sqrt{\lambda^2 + 4})$. However, $[\mathbb{K}(\sqrt{\lambda^2 + 4}) : \mathbb{K}] = 1$ or 2, which depends on $\sqrt{\lambda^2 + 4} \in \mathbb{K}$ or not. If $\sqrt{\lambda^2 + 4} \in \mathbb{K}$, then Eq. (6) gives $\mu \in \mathbb{K}$, hence $\mathbb{L} \subseteq \mathbb{K}$ and we conclude $\mathbb{K} = \mathbb{L}$. Therefore $[\mathbb{L} : \mathbb{Q}] = n$, the minimal polynomial of μ has degree n, it follows that $P_{\hat{G}}(x)$ is the product of two factors of degree n. If $\sqrt{\lambda^2 + 4} \notin \mathbb{K}$, $[\mathbb{K}(\sqrt{\lambda^2 + 4}) : \mathbb{K}] = 2$, so $[\mathbb{K}(\sqrt{\lambda^2 + 4}) : \mathbb{Q}] = 2n$, i.e., the minimal polynomial of μ has degree $2n$. Since $P_{\hat{G}}(x)$ has degree $2n$ and $P_{\hat{G}}(\mu) = 0$, this implies $P_{\hat{G}}(x)$ is exact the minimal polynomial of μ, and thus irreducible in the case. □

It follows a corollary from the proof of Theorem 11.

Corollary 1. *Let G be a graph with irreducible characteristic polynomial $P_G(x)$. Denote by $\hat{G}$ the corona graph of G and K_1, $P_{\hat{G}}(x)$ its characteristic polynomial. Suppose λ is any a root of $P_G(x)$. If $\sqrt{\lambda^2 + 4} \notin \mathbb{Q}(\lambda)$, then $P_{\hat{G}}(x)$ is irreducible over $\mathbb{Q}$.*

We exemplify Theorem 11 and Corollary 1 by the following computation.

Example 3. Let G_0 and T_8 be the graphs depicted in Fig. 3. Their characteristic polynomials are

$$P_{G_0}(x) = x^6 - 8x^4 - 4x^3 + 9x^2 + 4x - 1,$$
$$P_{T_8}(x) = x^8 - 7x^6 + 14x^4 - 8x^2 + 1.$$

Let $\lambda(G_0)$, $\lambda(T_8)$ be any root of $P_{G_0}(x)$ and $P_{T_8}(x)$, respectively. We calculate by Mathematica 11.3 that $\sqrt{\lambda(G_0)^2 + 4}$ does not belong to the algebraic number field $\mathbb{Q}(\lambda(G_0))$. However, $\sqrt{\lambda(T_8)^2 + 4}$ can be expressed in $\mathbb{Q}(\lambda(T_8))$ as follows:

$$\sqrt{\lambda(T_8)^2 + 4} = 2\lambda(T_8)^7 - 14\lambda(T_8)^5 + 26\lambda(T_8)^3 - 9\lambda(T_8).$$

By Theorem 11 and Corollary 1, the characteristic polynomial of $\hat{G}_0$

$$P_{\hat{G}_0}(x) = x^{12} - 14x^{10} - 4x^9 + 56x^8 + 16x^7 - 87x^6 - 16x^5 + 56x^4 + 4x^3 - 14x^2 + 1,$$

is irreducible over $\mathbb{Q}$ while that of $\hat{T}_8$ can be factored as follows:

$$\begin{aligned} P_{\hat{T}_8}(x) &= x^{16} - 15x^{14} + 84x^{12} - 225x^{10} + 311x^8 - 225x^6 + 84x^4 - 15x^2 + 1 \\ &= (x^8 - 8x^6 + 14x^4 - 7x^2 + 1)(x^8 - 7x^6 + 14x^4 - 8x^2 + 1). \end{aligned}$$

Declaration of competing interest

We confirm that there is no competing interest.

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References

[1] J. Brillhart, M. Filaseta, A. Odlyzko, On an irreducibility theorem of A. Cohn, Can. J. Math. 33 (5) (1981) 1055–1059.
[2] D. Cvetković, P. Rowlinson, S. Simić, An Introduction to the Theory of Graph Spectra, Cambridge University Press, 2010.
[3] S. Eberhard, The characteristic polynomial of a random matrix, arXiv:2008.01223v1.
[4] A. Farrugia, The rank of pseudo walk matrices: controllable and recalcitrant pairs, Open J. Discrete Appl. Math. 3 (3) (2020) 41–52.
[5] C. Godsil, B.D. McKay, Spectral conditions for the reconstructibility of a graph, J. Comb. Theory, Ser. B 30 (3) (1981) 285–289.
[6] C. Godsil, Controllable subsets in graphs, Ann. Comb. 16 (4) (2012) 733–744.
[7] A. Mowshowitz, Graphs, groups and matrices, in: Proceedings of the 25th Summer Meeting Canadian Math. Congress, Congr. Num. IV. Utilitas Math., Winnipeg, 1971, pp. 509–522.
[8] M.R. Murty, Prime numbers and irreducible polynomials, Am. Math. Mon. 109 (5) (2002) 452–458.
[9] V.V. Prasolov, Polynomials, Springer–Verlag, 2004.
[10] S. O'Rourke, B. Touri, On a conjecture of Godsil concerning controllable random graphs, SIAM J. Control Optim. 54 (6) (2016) 3347–3378.
[11] Z. Stanić, Further results on controllable graphs, Discrete Appl. Math. 166 (2014) 215–221.
[12] Y. Song, Z. Li, The Construction and Determination of Irreducible Polynomials over Finite Fields, Lecture Notes in Comput. Sci., vol. 9713, 2016, pp. 618–624.
[13] Y. Teranishi, Eigenvalues and automorphisms of a graph, Linear Multilinear Algebra 57 (6) (2009) 577–585.
[14] W.T. Tutte, All the king's horses (A guide to reconstruction), in: Graph Theory and Related Topics, Academic Press, 1979, pp. 15–33.
[15] W. Wang, C.X. Xu, Some results on the spectral reconstruction problem, Linear Algebra Appl. 427 (1) (2007) 151–159.