Collection of Equations

Let $G = (V, E)$ be a graph on $n \coloneqq |V|$ nodes. Denote its complement by G^c (i.e. G^c has the same nodes as G and two nodes are adjacent in G^c if and only if they are not adjacent in G). Recall that

$$r_{uv}^G(x) \coloneqq \chi_G(x) \cdot x^{-1} \cdot H_{uv}(x),$$

where $H_{uv}(x)$ is the generating function of the sequence of walks from node $u \in V$ to node $v \in V$ (actually, the map $\sigma(x) \mapsto \chi_G(x) \cdot x^{-1} \cdot \sigma(x)$ defines an isomorphism from the set of linear recurrence sequences having characteristic polynomial $\chi_G(x)$ into the quotient ring $\mathbb{Z}[x]/_{\chi_G(x)}$). If the graph in question is unambiguous, we will suppress it in the notation of $r_{uv}^G(x)$ and simply write $r_{uv}(x)$ instead.

Here is a collection of facts/equations that arise in this context. Proofs can be provided if necessary.

1. This definition extends by linearity to the more general setting of walks from a subset $S \subseteq V$ to a subset $T \subseteq V$ (in particular, it holds that $r_{S,T}(x) = \sum_{u \in S} \sum_{v \in T} r_{uv}(x)$). Furthermore, define $r_G(x) \coloneqq r_{V,V}(x) = \sum_{(u,v) \in V^2} r_{uv}(x)$.

2. The degree of the polynomial $r_{uv}(x)$ is given by $\deg(r_{uv}) = n - 1 - \operatorname{dist}(u, v)$.

3. The recurrence polynomial (i.e. least characteristic polynomial) of the sequence of walks from S to T equals

$$\varrho_{S,T}(x) = \frac{\chi_G(x)}{\gcd(\chi_G(x), r_{S,T}(x))}.$$

   Therefore, ϱ can be seen as some kind of waste product of χ and r, since they seem to be better controllable.

4. The leading coefficient of $r_{uv}(x)$ equals the number of shortest paths from u to v (unfortunately this does not generalize to the lower coefficients).

5. Let P_{uv} denote the number of paths from u to v (i.e. walks visiting each node only once). Then it holds that

$$r_{uv}(x) = \sum_{p \in P_{uv}} \chi_{G \setminus p}(x)$$

   for all $u, v \in V$.

6. In particular, this implies

$$r_{vv}(x) = \chi_{G\setminus\{v\}}(x)$$

for all $v \in V$.

7. The global polynomial $r_G(x)$ can be computed as

$$r_G(x) = -\chi_G(x) + (-1)^n \chi_{G^c}(-1-x).$$

Therefore, it furthermore holds that

$$r_G(x) = (-1)^{n-1} r_{G^c}(-1-x).$$

8. Consequently, we can compute the derivatives of $\chi_G(x)$ and $r_G(x)$ as follows:

$$\chi'_G(x) = \sum_{v\in V} r_{vv}(x) = \sum_{v\in V} \chi_{G\setminus\{v\}}(x),$$
$$r'_G(x) = \sum_{v\in V} r_{G\setminus\{v\}}(x).$$

9. Let G_1 and G_2 be undirected graphs and denote their join by $G_1 \nabla G_2$ (i.e. $G_1 \nabla G_2 = (G_1^c \cup G_2^c)^c$). Then the following holds:

$$\chi_{G_1 \cup G_2}(x) = \chi_{G_1}(x) \cdot \chi_{G_2}(x),$$
$$r_{G_1 \cup G_2}(x) = r_{G_1}(x) \cdot \chi_{G_2}(x) + \chi_{G_1}(x) \cdot r_{G_2}(x),$$
$$\chi_{G_1 \nabla G_2}(x) = \chi_{G_1}(x) \cdot \chi_{G_2}(x) - r_{G_1}(x) \cdot r_{G_2}(x),$$
$$r_{G_1 \nabla G_2}(x) = -\chi_{G_1}(x) \cdot \chi_{G_2}(x) + \chi_{G_1}(x) \cdot r_{G_2}(x) + r_{G_1}(x) \cdot \chi_{G_2}(x) + 2 \cdot r_{G_1}(x) \cdot r_{G_2}(x)$$

In particular, from this we can derive that the spectrum of $G_1 \nabla G_2$ contains all non-main eigenvalues of G_1 and G_2, but none of the main eigenvalues of G_1 or G_2. (careful, here the same eigenvalue with multiplicity k is counted as k distinct eigenvalues). Furthermore, if λ is a non-main eigenvalue of some G_i, it is also a non-main eigenvalue of $G_1 \nabla G_2$. Nevertheless, there can exist non-main eigenvalues of $G_1 \nabla G_2$ which do not come from G_1 or G_2.

This simple example already shows that the polynomials $r_{S,T}(x)$ can get messy really fast, when the structure of the graph is slightly modified.

10. For a subset $S \subseteq V$, let G_S be the graph consisting of G with an additional node connected to the nodes in S. Then it holds that

$$\chi_{G_S}(x) = x \cdot \chi_G(x) - r_{SS}(x),$$
$$r_{G_S}(x) = (x+1) \cdot r_G(x) + \chi_G(x) + r_{SS}(x) + (-1)^n r^{G^c}_{\bar{S}\bar{S}}(-1-x).$$

If $S = \{v\}$ is a single node, then we can rewrite this as

$$\chi_{G_v}(x) = x \cdot \chi_G(x) - \chi_{G\setminus\{v\}}(x),$$
$$r_{G_v}(x) = x \cdot r_G(x) + \chi_G(x) - r_{G\setminus\{v\}}(x) + 2r_{v,G}(x).$$

If $S = N(v)$ for some node $v \in V$, this simplifies to the equations

$$x^2 \cdot \chi_{G\setminus\{v\}}(x) - 2x \cdot \chi_G(x) + \chi_{G_S}(x) = 0,$$
$$x^2 \cdot r_{G\setminus\{v\}}(x) - 2x \cdot r_G(x) + r_{G_S}(x) = 0.$$

11. Define the matrix $R_G(x) \in \mathbb{Z}[x]^{n\times n}$ to be the collection of all polynomials $r_{uv}(x)$, i.e.

$$\left(R_G(x)\right)_{u,v} := r_{uv}(x).$$

For an arbitrary polynomial $p(x) = \sum_{i=0}^{d} p_i x^i \in \mathbb{Z}[x]$, define its truncation on the j-th level as

$$\mathrm{tr}_j p(x) = \sum_{i=0}^{d-1-j} p_{i+j+1} x^i.$$

In particular, $\mathrm{tr}_{-1} p(x) = p(x)$ and $tr_{d-1} p(x) = 1$. Then it holds that

$$R_G(x) = \sum_{i=0}^{n-1} \mathrm{tr}_i \chi_G(x) A^i = \sum_{i=0}^{n-1} \mathrm{tr}_i \chi_G(A) x^i.$$

12. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of G, then the characteristic polynomial $\chi_{R_G(x)}(y) \in \mathbb{Z}[x, y]$ can be computed as

$$\chi_{R_G(x)}(y) = \prod_{i=1}^{n} \left(y - \frac{\chi_G(x)}{x - \lambda_i} \right) = \frac{y^n \chi_G\left(x - \frac{\chi_G(x)}{y}\right)}{\chi_G(x)}.$$

In particular, this shows that the eigenvalues of $R_G(x)$ are exactly the polynomials $\frac{\chi_G(x)}{x-\lambda}$ for the eigenvalues λ of G. Furthermore, it holds that $\det\left(R_G(x)\right) = \chi_G(x)^{n-1}$, $\mathrm{tr}\left(R_G(x)\right) = \chi'_G(x)$ and $\mathrm{tr}\left(R_G(x)^2\right) =$

$\chi_G'(x)^2 - \chi_G''(x)\chi_G(x)$ (local version of the last equation: $\sum_{(u,v)\in V^2} r_{uv}(x)^2 = \chi_G'(x)^2 - \chi_G''(x)\chi_G(x)$).

One can even show that the eigenvectors of G with respect to the eigenvalue λ coincide with the eigenvectors of $R_G(x)$ with respect to the eigenvalue $\frac{\chi_G(x)}{x-\lambda}$.

13. This induces another (rather impractical) way to compute $R_G(x)$: If P diagonalizes the adjacency matrix A (i.e. $P^\top AP = D$), then

$$R_G(x) = P \cdot \operatorname{diag}\left(\frac{\chi_G(x)}{x-\lambda_1}, \ldots, \frac{\chi_G(x)}{x-\lambda_n}\right) \cdot P^\top.$$

14. The matrix $R_G(x)$ admits some kind of linearity: For an arbitrary polynomial $p(x) \in \mathbb{Z}[x]$ of degree d, it holds that

$$p(A) \cdot R_G(x) = R_G(x) \cdot p(A) = p(x)R_G(x) - \chi_G(x) \cdot \sum_{i=0}^{d-1} \operatorname{tr}_i p(x) A^i.$$

Note that this implies linearity over the quotient ring $\mathbb{Z}[x]/_{\chi_G(x)}$. In particular, one consequence of this is an easier local version. For $u, v \in V$ arbitrary, we have the following equality:

$$r_{N(u),v}(x) = r_{u,N(v)}(x) = \begin{cases} x \cdot r_{uu}(x) - \chi_G(x), & u = v \\ x \cdot r_{uv}(x), & u \neq v \end{cases}.$$

15. There exists a direct way to compute $r_{uv}(x)$ for particular $u \neq v \in V$. In fact, it holds that

$$r_{uv}(x)^2 = \chi_{G\setminus\{u\}}(x) \cdot \chi_{G\setminus\{v\}}(x) - \chi_{G\setminus\{u,v\}}(x) \cdot \chi_G(x).$$

16. This can be carried out to the polynomial $r_G(x)$ instead of $\chi_G(x)$, i.e. we define a matrix $M_G(x) \in \mathbb{Z}[x]^{n\times n}$ by

$$(M_G(x))_{u,v} := \begin{cases} -r_{G\setminus\{u\}}(x), & u = v \\ \sqrt{r_{G\setminus\{u\}}(x) \cdot r_{G\setminus\{v\}}(x) - r_{G\setminus\{u,v\}}(x) \cdot r_G(x)}, & u \neq v \end{cases}.$$

Then (in analogy to point 7) we can write $M_G(x)$ as

$$\begin{aligned} M_G(x) &= R_G(x) + (-1)^n \cdot R_{G^c}(-1-x), \\ M_G(x) &= (-1)^n \cdot M_{G^c}(-1-x). \end{aligned}$$

Note that, apart from the sign change on the main diagonal, the matrix $M_G(x)$ is obtained from $r_G(x)$ in the same way as $R_G(x)$ from $\chi_G(x)$ (maybe this generalizes to some kind of morphisms $\mathcal{G} \to \mathbb{Z}[x]$, where $\mathcal{G}$ denotes the set of all undirected graphs).

Since $(1, \ldots, 1) \cdot M_G(x) \cdot (1, \ldots, 1)^\top = 0$, this construction cannot be carried out further.