Walk counts via linear recurrences

Sven Kosub
Department of Computer & Information Science
University of Konstanz
Box 67, D-78457 Konstanz, Germany
Sven.Kosub@uni-konstanz.de

Contributors Welcome
*
*
*
*

Draft of Oct 24, 2021; LaTeXed on 14:31, Oct 24, 2021.

Abstract

We report on explicit formulae for the number of walks in graphs based on algebraic methods for linear recurrent sequences.

1 Expressing walk counts as a linear recurrence relation

Walk counts. Let $G = (V, E)$ be a directed or an undirected graph with vertex set $V = [n]$. A walk of length r in G is a finite sequence $(v_0, v_1, \dots, v_r)$ of adjacent vertices, i.e., $(v_{k-1}, v_k) \in E$ for $k \in [n]$; an (i, j)-walk of length r is a walk such that $v_0 = i$ and $v_r = j$. For non-empty sets $S, T \subseteq V$, let $W_r[S, T] = W_r(G)[S, T]$ denote the number of all (i, j)-walks of length r in G such that $i \in S$ and $j \in T$. The following number of walks are usually considered:

- local walk count of length r at vertex pair (i, j): $w_r(i, j) =_{\text{def}} W_r[\{i\}, \{j\}]$
- (total) closed-walk count of length r: $cw_r =_{\text{def}} \sum_{i \in V} w_r(i, i)$
- (total) walk count of length r: $w_r =_{\text{def}} W_r[V, V]$

Linear recurrence relations for walk counts. Let $A = A(G)$ be the adjacency matrix of G. We make use of the fact that $w_r(i, j) = (A^r)_{ij}$. The following is reminiscent of [5, 10]:

Theorem 1. *Let $G = (V, E)$ be a directed or undirected graph, $V = [n]$. Let $S, T \subseteq V$ be non-empty subsets of vertices of G. There exists a monic polynomial $p \in \mathbb{Z}[x]$ of degree $d \leq n$, i.e., $p(x) = \sum_{k=0}^{d} a_k x^{d-k}$ where $a_k \in \mathbb{Z}$ and $a_0 = 1$, such that for all $r \geq d$,*

$$\sum_{k=0}^{d} a_k W_{r-k}[S, T] = 0 \tag{1}$$

In other words, the integer sequence $W_r[S,T]$ can be characterized as a homogeneous linear recurrent sequence of degree d

$$W_r[S,T] = -\sum_{k=1}^{d} a_k W_{r-k}[S,T] \qquad \textit{for } r \geq d$$

with initial values $W_0[S,T], \ldots, W_{d-1}[S,T]$.

Proof. Let $p \in \mathbb{Z}[x]$ be the monic polynomial of least degree such that $p(A) = 0$ (where 0 is the null matrix). Since the (monic) characteristic polynomial $\chi_G(x) = \det(xI - A)$ of G satisfies $\chi_G(A) = 0$ (Cayley-Hamilton), the polynomial p is well-defined. Let d be the degree of p and let $p(x) = \sum_{k=0}^{d} a_k x^{d-k}$ be the expansion of p. It holds that $A^{r-d}p(A) = 0$ for all $r \geq d$. Thus, for vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we obtain

$$0 = \mathbf{x}^T A^{r-d} p(A)\ \mathbf{y} = \mathbf{x}^T A^{r-d} \left(\sum_{k=0}^{d} a_{d-k} A^k \right) \mathbf{y} = \sum_{k=0}^{d} a_{d-k}\ \mathbf{x}^T A^{r-d+k} \mathbf{y}$$

Choosing $\mathbf{x} = \sum_{i \in S} \mathbf{e}_i$ and $\mathbf{y} = \sum_{j \in T} \mathbf{e}_j$ for sets $S, T \subseteq V$, we find

$$0 = \sum_{k=0}^{d} a_{d-k} W_{r-d+k}[S,T] = \sum_{k=0}^{d} a_k W_{r-k}[S,T]$$

This shows the theorem. □

We comment on Theorem 1:

1. The theorem states that for each pair of sets, there is a polynomial which specifies a linear recurrence relation. The proof shows that there is one polynomial for all pairs. In fact, the minimum polynomial of the adjacency matrix of graph G is the unique least-degree polynomial which works for all pairs. However, there are graphs where walk count sequences for certain pairs of vertex sets are specified by polynomials of lower degree. Such polynomials do not meet the Cayley-Hamilton condition.

2. The theorem can be extended to any linear combination of $W_r(S,T)$ and $W_r(S',T')$. So, the sequence of closed-walk counts cw_r is specifiable by a linear recurrence relation, too.

3. The theorem can be reformulated for sequences of powers of Laplace or Seidel matrices of graphs or digraphs, when meaningful.

2 Linear recurrent sequences over fields

We analyze Eq. (1) in terms of linear recurrent sequences (see, e.g., [8, 1, 12, 4]).

Algebraic theory. We summarize basic algebraic facts on sequences over some field K of characteristic 0, e.g., $\mathbb{R}$, $\mathbb{C}$. (An extension to sequences over rings is given in [4] from where we borrow notation.)

Let K^∞ denote the set of all infinite sequences σ over K, i.e., $\sigma = (\sigma_r)_{r\geq 0}$ such that $\sigma_r \in K$. Let $p \in K[x]$ be be a monic polynomial of degree $n > 0$, $p(x) = x^n - \sum_{k=1}^{n} a_k x^{n-k}$ where $a_k \in K$. Given a vector $\mathbf{s} = (s_0, \ldots, s_{n-1}) \in K^n$ let $F(\mathbf{s}; p) = F(s_0, \ldots, s_{n-1}; p)$ denote the homogeneous linear recurrent sequence with characteristic polynomial p and initial values $s_0, \ldots, s_{n-1}$ that is defined as

$$F_r(\mathbf{s}; p) = \begin{cases} s_r & \text{for } 0 \leq r \leq k-1 \\ \sum_{i=1}^{n} a_i F_{r-i}(\mathbf{s}; p) & \text{for } r \geq k \end{cases}$$

It is known that for all $r \in \mathbb{N}$,

$$F_r(\mathbf{s}; p) = \sum_{i=1}^{n} s_i F_r(\mathbf{e}_i; p) \tag{2}$$

A sequence $\sigma \in K^\infty$ is a linear recurrent sequence iff there is a monic polynomial p (of degree n) such that

$$\sigma_r = F_r(\sigma_0, \ldots, \sigma_{n-1}; p)$$

for all $r \geq 0$; in this case, we say that p is a characteristic polynomial of σ.

Let $\mathcal{H}_K(p)$ denote the set of all linear recurrent sequences in K^∞ having characteristic polynomial p. Note that $\mathcal{H}_K(1) = \emptyset$ and that $\mathcal{H}_K(x)$ consists of sequences σ such that $\sigma_r = 0$ for $r \geq 1$. Let $\mathcal{H}_K \subseteq K^\infty$ denote the set of all linear recurrent sequences. In [9], an action of $K[x]$ on K^∞ was defined by $(x^k)\sigma_n = \sigma_{n+k}$ extended by linearity. It immediately follows that $\sigma \in \mathcal{H}_K(p)$ if and only if $p(x)\sigma = 0$.

Lemma 2. (cf. [9, 2, 13]) *Let $\sigma, \varrho \in \mathcal{H}_K$ be sequences and let $p, q \in K[x]$ be polynomials.*

1. *$\mathcal{I}(\sigma) = \{\ p \in K[x] \mid p(x)\sigma = 0\ \}$ is a principal ideal of principal ideal domain $K[x]$.*
2. *$\mathcal{H}_K(p)$ is a linear space over K.*
3. *If $\sigma \in \mathcal{H}_K(p)$ and $\varrho \in \mathcal{H}_K(q)$ then $\sigma + \varrho \in \mathcal{H}_K(pq)$.*
4. *$\mathcal{H}_K$ with Hadamard product (component-wise product) is a K-algebra, i.e., if $\sigma \in \mathcal{H}_K(p)$ and $\varrho \in \mathcal{H}_K(q)$ then $\sigma \cdot \varrho \in \mathcal{H}_K(h)$ for some $h \in K[x]$.*

Analytic theory. We summarize basic analytic facts on ordinary generating functions and explicit formulae for linear recurrent sequences over $\mathbb{C}$.

Lemma 3. (cf. [11]) *Let $p \in \mathbb{C}[x]$ be a monic polynomial of degree n with roots $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$ of multiplicities $n_1, \ldots, n_m$ such that $n_1 + \cdots + n_m = n$. Then, the n sequences satisfying*

$$\sigma_r = r^k \alpha_i^r \qquad \text{for } r \geq 0,$$

where $1 \leq i \leq m$ and $k \in \{0, \ldots, n_i - 1\}$, form a basis in linear space $\mathcal{H}_{\mathbb{C}}(p)$.

3 Linear recurrent sequences on undirected graphs

Explicit formulae for walk counts. Using Cramer's rule we find an explicit formula for walk counts of undirected graphs which shows a clear decoupling of initial components and spectral components. Let $\langle \cdot, \cdot \rangle$ denote the standard scalar product.

Theorem 4. *Let G be an undirected, non-empty graph with n vertices and $m \leq n$ distinct eigenvalues $\lambda_1, \dots, \lambda_m$. Then, there exists a regular matrix $C = C(\lambda_1, \dots, \lambda_m) \in \mathbb{R}^{m \times m}$ such that for all vertices $u, v \in V$ and all $r \in \mathbb{N}$,*

$$w_r(u,v) = \langle \bar{w}(u,v), C\bar{\lambda}^r \rangle \tag{3}$$

where $\bar{w}(u,v) = (w_0(u,v), \dots, w_{m-1}(u,v))^T$ and $\bar{\lambda}^r = (\lambda_1^r, \dots, \lambda_m^r)^T$. In fact, $C = V(\lambda_1 \dots, \lambda_m)^{-1}$ where $V(\lambda_1, \dots, \lambda_m)$ is a square Vandermonde matrix and the entries c_{ij} of C are explicitly given by

$$c_{ij} = \frac{\det(V(\lambda_1, \dots, \lambda_m)_{ij})}{\det(V(\lambda_1, \dots, \lambda_m)}$$

where $V(\lambda_1, \dots, \lambda_m)_{ij}$ is the matrix obtained from $V(\lambda_1, \dots, \lambda_m)$ by replacing i-th row with j-th unit row vector $\mathbf{e}_j$.

Proof. Let $G = (V, E)$ be an undirected graph such that $|E| \geq 1$ and $|V| \geq 2$. So, G has $m \geq 2$ distinct eigenvalues. The roots of characteristic polynomial χ_G are the eigenvalues of the graph G. Since $A(G)$ is symmetric, all eigenvalues $\lambda_1, \dots, \lambda_n$ (with multiplicities) belong to $\mathbb{R}$. The minimum polynomial μ_G and the characteristic polynomial χ_G of G have the same set of roots; but, since $A(G)$ is symmetric, all $m \leq n$ roots of μ_G are simple roots. Let $u, v \in V$ be any vertices. By Eq. (1), the sequence $w(u,v) = (w_r(u,v))_{r \geq 0}$ belongs to $\mathcal{H}_{\mathbb{R}}(\mu_G)$. As follows from Lemma 3, the m sequences $(\lambda_i^r)_{r \geq 0}$ form a basis in $\mathcal{H}_{\mathbb{R}}(\mu_G)$. It remains to find $\beta_i \in \mathbb{R}$ such that $w_r(u,v) = \sum_{i=1}^m \beta_i \cdot \lambda_i^r$ for all $0 \leq r \leq m-1$, or equivalently, $(w_0(u,v), \dots, w_0(u,v)) = (\beta_1, \dots, \beta_m) \cdot V(\lambda_1, \dots, \lambda_m)$. This is solved by inverting the Vandermonde matrix $V(\lambda_1, \dots, \lambda_m)$. Note that $\det(V(\lambda_1, \dots, \lambda_m)) \neq 0$. This proves the theorem. $\square$

Two comments on linearities involved in Theorem 4:

1. By the linearity of the scalar product, any weighted aggregation of local walk counts admits a formula corresponding to Eq. (3) with the same spectral component. In particular, for walk counts w_r and closed-walk counts cw_r, we obtain

$$w_r = \langle \bar{w}, C\bar{\lambda}^r \rangle, \qquad cw_r = \langle \bar{cw}, C\bar{\lambda}^r \rangle$$

   where $\bar{w} = (w_0, \dots, w_{m-1})^T$ and $\bar{cw} = (cw_0, \dots, cw_{m-1})^T$ are the initial segments.

2. By the linearity of the determinant in one row, it follows that Theorem 4 can be explicitly written as

$$w_r(u,v) = \sum_{i=1}^{m} \frac{\det(V(\lambda_1, \dots, \lambda_m)_i)}{\det(V(\lambda_1, \dots, \lambda_m))} \cdot \lambda_i^r \tag{4}$$

   where $V(\lambda_1, \dots, \lambda_m)_i$ is the matrix obtained from $V(\lambda_1, \dots, \lambda_m)$ by replacing i-th row with the initial segment $\bar{w}(u,v)$.

Main part of the spectrum and the recurrence polynomial. We compare Theorem 4 to the standard spectral approach to counting walks in a graph (cf. [10, 6]). Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of a graph G (listed according to their multiplicities) and let $\Lambda_G = \{\lambda_1, \ldots \lambda_m\}$ be the set of distinct eigenvalues (without multiplicities). Then, it is known that

$$w_r = \sum_{i=1}^{n} \beta_i \cdot \lambda_i^r = \sum_{\lambda \in \Lambda} \beta_\lambda \cdot \lambda^r \tag{5}$$

where $\beta_i = \nu_i^2 \geq 0$ and ν_i is the coordinate of the all-one vector $\mathbf{1}_n$ with respect to the i-th eigenvector in the orthonormal basis of n eigenvectors; β_λ are just the aggregates for a certain eigenvector $\lambda \in \Lambda$. By equating coefficients, β_λ is equal to the value given in Eq. (4) for the corresponding eigenvalue and the initial segment of $(w_r)_{r\geq 0}$. Thus, by avoiding eigenvectors as intermediates, Eq. (4) provides in many cases a more efficient way to compute the coefficients.

Inspecting Eq. (5), it is clear that not all eigenvalues must contribute to the sum. Consequently, in [5], the *main part of the spectrum* was defined as the set $\Lambda_G^+ =_{\text{def}} \{ \lambda \in \Lambda_G \mid \beta_\lambda > 0 \}$. It is easily seen that $\lambda \notin \Lambda_G^+$ if and only if the all-one vector $\mathbf{1}_n$ is orthogonal to the eigenspace of λ (cf. [10]).

We give another algebraic characterization of Λ_G^+. Define the *recurrence polynomial* $\varrho_G \in \mathbb{Z}[x]$ of a graph G to be the monic polynomial of least degree such that $(w_r)_{r\geq 0} \in \mathcal{H}_{\mathbb{R}}(\varrho_G)$. By Lemma 2, the recurrence polynomial ϱ_G is well-defined.

Lemma 5. *$\varrho_G(x) = \prod_{\lambda \in \Lambda_G^+}(x - \lambda)$.*

Proof. Since ϱ_G is the generator of the principal ideal $\mathcal{I}((w_r)_{r\geq 0})$, ϱ_G divides all characteristic polynomials of $(w_r)_{r\geq 0}$. Therefore, all roots of ϱ_G are simple and all belong to Λ_G^+. Without loss of generality, $\Lambda_G^+ = \{\lambda_1, \ldots, \lambda_k\}$ and let $\lambda_1, \ldots, \lambda_\ell$ with $\ell \leq k$ be the roots of ϱ_G. Assume that $\ell < k$. Applying Lemma 3 to ϱ_G, we obtain $w_r = \sum_{j=1}^{k} \hat{\beta}_j \cdot \lambda_j^r$ with $\hat{\beta}_j = 0$ for $\ell < j \leq k$. By Theorem 4,

$$0 = \sum_{i=1}^{k} (\hat{\beta}_i - \beta_i) \cdot \lambda_i^k$$

Since $V(\lambda_1 \ldots, \lambda_k)$ is invertible, it follows that $\hat{\beta}_i = \beta_i$ for all $i \in \{1, \ldots, k\}$. Hence, $\beta_i = 0$ for $\ell < i \leq k$, which is a contradiction. This proves the lemma. □

Lemma 5 implies the most efficient total walk count version of Theorem 4 (i.e., involving the least number of eigenvalues).

Corollary 6. *Let G be an undirected graph and let ϱ_G be the recurrence polynomial of G (of degree k) with distinct roots $\lambda_1, \ldots, \lambda_k$. Then, there exists a regular matrix $C = C(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^{k\times k}$ such that for all $r \in \mathbb{N}$*

$$w_r = \langle \bar{w}, C\bar{\lambda}^r \rangle$$

where $\bar{w} = (w_0, \ldots, w_{k-1})^T$, $\bar{\lambda}^r = (\lambda_1^r, \ldots, \lambda_k^r)^T$, and $C = V(\lambda_1, \ldots, \lambda_k)^{-1}$.

Example 7. *The undirected path graph P_3 (with 4 vertices) has characteristic (and minimal) polynomial $x^4 - 3x^2 + 1 = (x^2 - x - 1)(x^2 + x - 1)$ and the 4 simple eigenvalues $\pm\frac{1}{2} \pm \frac{\sqrt{5}}{2}$. Are all necessary for Eq. (5)? By checking Vandermonde determinants according to Eq. (4), we find that the two eigenvalues $-\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ can be excluded. However, it is also easy to observe that the sequence of walk counts $w = (w_r)_{r\geq 0}$ of P_3 follows the Fibonacci recurrence relation*

$$w_r = w_{r-1} + w_{r-2} \quad \text{for } r \geq 2 \tag{6}$$

with initial values $w_0 = 4$ and $w_1 = 6$. The eigenvalues $\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ are just the roots of the recurrence polynomial $x^2 - x - 1$ of G; thus, $\Lambda^+_{P_3} = \left\{\frac{1}{2} + \frac{\sqrt{5}}{2}, \frac{1}{2} - \frac{\sqrt{5}}{2}\right\}$.

Determining recurrence polynomials of graphs. How can we construct the recurrence polynomial of a graph in an efficient way?

By Lemma 2, the following sufficient criterion is immediate.

Lemma 8. *Let G be an undirected graph. If $(w_r)_{r\geq 0} \in \mathcal{H}_{\mathbb{R}}(p)$ and p is irreducible over $\mathbb{Z}$ then p is the recurrence polynomial of G.*

However, recurrence polynomials are in general, reducible over $\mathbb{Z}$.

Figure 1: The Saltire pair.

Example 9. *The Saltire pair (see Fig. 1) consists of two non-isomorphic isospectral graphs with common characteristic polynomial $x^5 - 4x^3 = x^3(x-2)(x+2)$ and the minimal polynomial $x(x-2)(x+2)$. The graph on the left-hand side has recurrence polynomial $x^2 - 4 = (x-2)(x+2)$, the graph on the right-hand side has recurrence polynomial $x(x-2)$. Observe that both graphs differ in the number of length-2 walks.*

Lemma 10. $\varrho_G(\lambda_{\max}) = 0$

Proof. Assume to the contrary that $\varrho_G(\lambda_{\max}) \neq 0$. By Lemma 5, $\lambda_{\max} \notin \Lambda^+_G$. Note that $\lambda_{\max} \geq |\lambda_i|$ for all $\lambda_i \in \Lambda_G$. First, consider the case that $-\lambda_{\max} \in \Lambda^+_G$. Then, $|-\lambda_{\max}| > |\lambda|$ for all $\lambda \in \Lambda^+_G$ different to $-\lambda_{\max}$. Hence, there exists an odd $r \in \mathbb{N}$ large enough such that $w_r = \sum_{\lambda\in\Lambda^+_G} \beta_\lambda \cdot \lambda^r < 0$ which cannot be. It remains to consider the case $-\lambda_{\max} \notin \Lambda^+_G$. Since it is known that $cw_r = \sum_{i=1}^n \lambda_i^r$ (with multiplicities) for each graph, there exists an even $r \in \mathbb{N}$ large enough such that

$$w_r = \lambda^r_{\max} \sum_{\lambda\in\Lambda^+_G} \beta_\lambda \cdot \left(\frac{\lambda}{\lambda_{\max}}\right)^r < \lambda^r_{\max} \leq cw_r \leq w_r,$$

again a contradiction. This shows the lemma. $\square$

Proposition 11. *1. Let G be an undirected graph with minimum polynomial $\mu_G \in \mathbb{Z}[x]$ which factors into irreducible polynomials $p_1, \dots, p_k$ over $\mathbb{Z}$. Then, the irreducible polynomial p_i with root $\lambda_{\max}$ is a divisor of ϱ_G.*

2. If G is not connected then ϱ_G is the least common multiple (in $\mathbb{Z}[x]$) of the recurrence polynomials of all components of G.

Recurrent graphs. It is natural to classify graphs regarding the degrees of their recurrence polynomials. A graph G is said to be *k-recurrent* if and only if its recurrence polynomial has degree k.

Proposition 12. *The 1-recurrent graphs are exactly the regular graphs.*

Proof. Obviously, each regular graph is 1-recurrent. Now, let G be a 1-recurrent graph. By Lemma 10, $\varrho_G(x) = x - \lambda_{\max}$ for the largest eigenvalue of G. Hence, $w_1 = \lambda_{\max} \cdot w_0$, or equivalently, $\lambda_{\max} = w_1/w_0 = \bar{d}(G)$ where $\bar{d}(G)$ is the average degree of G. Thus, G is regular. □

Conjecture 13. *A 2-recurrent graph G is a connected non-regular bipartite graph or a disjoint union of a k-regular and an ℓ-regular graph for $k \neq \ell$.*

Evidences: A 2-recurrent graph is not regular. Write the recurrence polynomial as $\varrho_G(x) = (x - \alpha_1)(x - \alpha_2)$ with $\alpha_1 = \lambda_{\max}$. If $\alpha_2 = -\lambda_{\max}$ then G is connected and bipartite (cf., e.g., [3]). So suppose, $|\alpha_2| < \alpha_1$. Since ϱ_G is monic, roots are either integers or irrational numbers. If α_1 and α_2 are natural numbers, the recurrence polynomial can be realized by the disjoint union of an α_1-regular and an α_2-regular graph. How does P_3 fit into this?

Note that there are connected non-regular bipartite graphs that are k-recurrent for $k > 2$, like, e.g., the path graph P_5 which is 3-recurrent as can be seen by its irreducible recurrence polynomial $x^3 - x^2 - 2x + 1$.

Applications. We interpret Eq. (4) in terms of deviations from the geometric progression with the average degree $\bar{d}$ of a graph as common ratio. We define for $r \in \mathbb{N}$:

$$\Delta_r = \frac{w_r}{w_0} - \bar{d}^r$$

It holds that $\Delta_0 = \Delta_1 = 0$ and $\Delta_r \geq 0$ (cf., e.g., [7]). Using factorizations of Vandermonde determinants we obtain the following formula.

Corollary 14. *Let G be an undirected graph with recurrence polynomial ϱ_G of degree m. Let $\lambda_1, \dots, \lambda_m$ be the roots of ϱ_G. Then, the coefficient β_i of λ_i in Eq. (4) is given by*

$$\beta_i = w_0 \cdot \frac{\prod_{k=1,k\neq i}^{m} (\bar{d} - \lambda_k) + \sum_{k=3}^{m} (-1)^{k+m} \Delta_{k-1} S_{m-k,i}(\lambda_1, \dots, \lambda_m)}{\prod_{k=1,k\neq i}^{m} (\lambda_i - \lambda_k)}$$

where $S_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} x_{i_j}$ is the k-th elementary symmetric polynomial and $S_{k,\ell}(x_1, \dots, x_n) = S_k(x_1, \dots, x_{\ell-1}, 0, x_{\ell+1}, \dots, x_n)$.

Note that there are similar formulae for geometric progressions with common ratio $\sqrt[k]{w_k/w_0}$.

Proposition 15. *Let G be a graph with recurrence polynomial ϱ_G of degree at most 2. Then,*

$$\varrho_G(\bar{d}) \leq 0 \iff w_r w_s \leq w_0 w_{r+s} \text{ for all } r, s \in \mathbb{N}$$

Proof. The equivalence is obvious for 1-recurrent graphs. Let λ_1, λ_2 be the roots of ϱ_G for a 2-recurrent graph G. Let λ_1 be the largest eigenvalue of G, $\lambda_1 \geq |\lambda_2|$. After rearranging terms, we obtain from Eq. (5) and Corollary 14

$$w_0 w_{r+s} - w_r w_s = \beta_1 \beta_2 (\lambda_1^r - \lambda_2^r)(\lambda_1^s - \lambda_2^s) = -w_0^2 \frac{\varrho_G(\bar{d})}{(\lambda_1 - \lambda_2)^2} (\lambda_1^r - \lambda_2^r)(\lambda_1^s - \lambda_2^s)$$

which is non-negative for all $r, s \in \mathbb{N}$ if and only if $\varrho_G(\bar{d}) \leq 0$. This proves the proposition. □

References

[1] D. Andrica, O. Bagdasar. *Recurrent Sequences. Key Results, Applications, and Problems.* Problem Books in Mathematics. Springer, Cham, 2020.

[2] B. Benzaghou. Algèbres de Hadamard. *Bulletin de la Société Mathématique de France*, 98:209–252, 1970.

[3] A. E. Brouwer, W. H. Haemers. *Spectra of Graphs.* Springer, New York, NY, 2012.

[4] U. Cerruti, F. Vaccarino. *R*-algebras of linear recurrent sequences. *Journal of Algebra*, 175:332–338, 1995.

[5] D. M. Cvetković. The generating function for variations with restrictions and paths of the graph and self-complementary graphs. *Univerzitet u Beogradu, Publikacije Elektrotehničkog fakulteta. Serija Matematika i Fizika*, 320/328:27–34, 1970.

[6] A. Dress, I. Gutman. The number of walks in a graph. *Applied Mathematics Letters*, 16(5):797–801, 2003.

[7] P. Erdős, M. Simonovits. Compactness results in extremal graph theory. *Combinatorica*, 2(3):275–288, 1982.

[8] G. Everest, A. van der Poorten, I. Shparlinski, T. Ward. *Recurrence Sequences.* Mathematical Surveys and Monographs #104. AMS, Providence, RI, 2003.

[9] M. Hall, Jr. An isomorphism between linear recurring sequences and algebraic rings. *Transactions of the American Mathematical Society*, 44(2):196–218, 1938.

[10] F. Harary, A. J. Schwenk. The spectral approach to determining the number of walks in a graph. *Pacific Journal of Mathematics*, 80(2):443–449, 1979.

[11] N. V. Ivanov. Linear recurrences. Preprint, Michigan State University, Department of Mathematics, East Lansing, MI, January 2008.

[12] N. V. Ivanov. Algebra of linear recurrence relations in arbitrary characteristic. Technical report arXiv:1604.04947 [math.CO], April 2016.

[13] R. G. Larson, E. J. Taft. The algebraic structure of linearly recursive sequences under Hadamard product. *Israel Journal of Mathematics*, 72(1–2):118–132, 1990.