# 1 A recursion formula for the coefficients

## 1.1 The coefficients of $\varrho_G$

**Definition 1.1.** *Let $A \in \mathbb{R}^{n\times n}$ symmetric, $v \in \mathbb{R}^n$. For arbitrary multi-indices $I = (i_1, \dots, i_k) \in \mathbb{N}^*$ with $0 \leq i_1 < i_2 < \ldots < i_k$ define the coefficients $\delta_I$ by*

$$\delta_i \coloneqq \langle v, A^i v\rangle \text{ for } i \in \mathbb{N},$$

$$\delta_{(i_1,\dots,i_k)} \coloneqq \begin{cases} 0 & \text{if } i_j + 1 = i_{j+1} \text{ for some } j \\ \sum_{v\in\{0,1\}^{k-1}} (-1)^{|v|} \cdot \delta_{i_k-|v|} \cdot \delta_{(i_1,\dots,i_{k-1})+v} & \text{otherwise,} \end{cases}$$

*where $|v| = \sum_{i=1}^k v_i$ denotes the number of ones in $v$.*

**Claim 1.2.** *Let $A \in \mathbb{R}^{n\times n}$ symmetric, $v \in \mathbb{R}^n$, let $d \coloneqq \dim Z(v, A)$. For $i = 0, \dots, d$ define the vector $w_i^d \in \mathbb{N}^d$ by*

$$w_i^d \coloneqq (0, 2, 4, \dots, 2(i-1), 2i+1, 2(i+1)+1, \dots, 2(d-1)+1).$$

*Then the characteristic polynomial of $A|_{Z(v,A)}$ is given by*

$$p(x) = \sum_{i=0}^{d} (-1)^{|w_i^d|} \cdot \frac{\delta_{w_i^d}}{\delta_{w_d^d}} x^i,$$

*where again $|w| \coloneqq \sum_{j=1}^k w_j$ denotes the $L^0$-norm of a vector.*

*Remark* 1.3. If $A$ satisfies $tr(A) \equiv 0 \mod 2$, then every $\delta_I$ for $I$ of length $k$ is divisible by $2^{k-1}$. If furthermore $n$ is even, every $\delta_I$ is divisible by $2^k$.

*Remark* 1.4. For every $I$ of length $k > d$, it holds $\delta_I = 0$.

*Proof.* It can be verified that the vectors

$$v_j \coloneqq \sum_{l=0}^{j} (-1)^{|w_i^d|} \delta_{w_i^d} A^i v$$

form an orthogonal basis of $Z(v, A)$. This implies, that the projection of $A^k v$ onto the subspace generated by $v, \dots, A^{k-1}v$ has coordinates $(-1)^{|w_i^k|} \cdot \frac{\delta_{w_i^k}}{\delta_{w_k^k}}$ relative to the basis $v, \dots, A^{k-1}v$. The assertion immediately follows. □

Hence, if $G$ is an undirected graph with adjacency matrix $A$, the above approach can be used to determine the recurrence polynomial $\varrho_G$.

*Remark* 1.5. A few comments on this approach:

1. Even when using dynamic programming, the recursion formula for the $\delta_I$ still has exponential complexity.

2. The absolute value of the $\delta_I$ grows exponentially as well. Furthermore, in general the $\delta_I$ for multiindices of fixed length do not share a common factor other than $2^{k-1}$.

3. The most interesting question that arises: Why is every $\delta_{w_i^d}$ divisible by $\delta_{w_d^d}$? In particular, in general the coefficient $\delta_{w_d^d}$ seems to be a large square number. However, I could not prove this and there seem to be exceptions. Furthermore, I could not find any meaning of the square root of $\delta_{w_d^d}$.

### 1.2 The recurrence degree of $G$

If one is only interested in the recurrence degree of $G$, the following procedure describes a way to compute it in time $\mathcal{O}(n^3)$:

```
Initialize B := I_n, v := (1, ..., 1), k := n
While k > 0 do:
   w := vB
   if w = 0:
      return n − k
   pick i ∈ {1, ..., k} with w_i ≠ 0
   for j ≠ i:
      update the columns of B: B_j := w_i B_j − w_j B_i
   delete the i-th column of B
   v := Av
   k := k − 1
return n
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## 2 Local recurrence polynomials

### 2.1 General theory

There exists an isomorphism between the set of all linear recurrent sequences having characteristic polynomial $p$ and the quotient ring $\mathbb{Z}[x]/p(x)$ (see [1]). If we identify the set of integer sequences with the ring of formal power series $\mathbb{Z}[[x]]$, this isomorphism is given by $\sigma(t) \mapsto p(t)t^{-1}\sigma\left(t^{-1}\right)$. The term

$t^{-1} \cdot \sigma\left(t^{-1}\right) = \frac{r_\sigma(t)}{\varrho_\sigma(t)}$ (where $r$ and $\varrho_\sigma$ are coprime polynomials) is well-defined if and only if $\sigma$ is a linear recurrent sequence and $\varrho_\sigma(t)$ divides $p(t)$ if and only if $p$ is a characteristic polynomial for $\sigma$. In this case, $\varrho_\sigma$ is the least characteristic polynomial for $\sigma$.

Let $q(x)$ be the image of $\sigma \in \mathbb{Z}[[x]]$ in $\mathbb{Z}[x]/p(x)$. The sequence $\sigma$ has a characteristic polynomial $\varrho_\sigma$ of degree lower than $p$ if and only if $q(x) \in \mathbb{Z}[x]/p(x)$ is a zero divisor (if and only if $p$ and $q$ are not coprime). The polynomial $\varrho_\sigma$ is given by

$$\varrho_\sigma(x) = \frac{p(x)}{\gcd(p(x), q(x))}.$$

Hence, if $q(x)$ is known, we can compute $\varrho_\sigma$ by factoring $p$ and $q$ (resp. in time $\mathcal{O}(n^2)$ with the Euclidean algorithm).

When the first $\deg(p)$ terms of $\sigma$ are known, it is possible to directly compute the polynomial $p(t)t^{-1}\sigma\left(t^{-1}\right)$. However, if $\sigma$ is the (local or global) walk count of a graph, this polynomial can be obtained in other ways.

## 2.2 Undirected graphs

Let $G$ be an undirected graph with adjacency matrix $A$ and let $u, v \in V$. The number of walks of length $k$ starting in $u$ and ending in $v$ is denoted by $w_k^{uv}$ and can be computed as $w_k^{uv} = \langle e_u, A^k e_v \rangle$. Denote the global walk count sequence of $G$ by $w_r$ and the images of $w_r$ resp. $w_r^{uv}$ in $\mathbb{Z}[x]/\chi_G[x]$ by $r_G$ resp. $r_{uv}$. By linearity of the described isomorphism it follows that $r_G = \sum_{u,v \in V} r_{uv}$. Hence, we are interested in computing the polynomials $r_{uv}$ for $u, v \in V$ arbitrary.

If $u = v$, this is relatively easy:

**Claim 2.1.** *Let $G$ be an undirected graph with adjacency matrix $A$, $v \in V$. Then it holds*

$$r_{vv}(x) = \chi_{A_v}(x),$$

*where $A_v$ is obtained from $A$ by removing the $v$-th row and column.*

In other words, $r_{vv}$ is the characteristic polynomial of $G \backslash \{v\}$. In particular, it follows that $\varrho_{vv}$ consists of all factors of the characteristic polynomial of $A$ which are not contained in the characteristic polynomial of $A_v$. Furthermore, the previously described approach can be generalized to obtain that $\varrho_{vv}$ is the characteristic polynomial of $A|_{Z(e_v, A)}$.

If $u \neq v$, it gets more complicated. We say $p \in V^{k+1}$ is a simple path of length $k$, if $p = (v_1, \ldots, v_{k+1})$, such that the $v_i$ are pairwise unequal and

$\{v_i, v_{i+1}\} \in E$ for all $i = 1, \ldots, k$. In particular, the simple paths of length zero are exactly the nodes of $G$ and the simpe paths of length one are its edges (where each edge gives rise to two paths of length one). For nodes $u, v \in V$, let $P_{uv}$ be the set of simple paths starting in $u$ and ending in $v$.

**Claim 2.2.** *Let $G$ be an undirected graph with adjacency matrix $A$, let $u, v \in V$. Then it holds*

$$r_{uv}(x) = \sum_{p \in P_{uv}} \chi_{G \setminus p}(x).$$

Note, that this is a generalization of the previous Claim 2.1. Using that $r_G = \sum_{u,v \in V} r_{uv}$, we get the following equation:

$$r_G(x) = \sum_{p \subseteq G \text{ simple path}} \chi_{G \setminus p}(x).$$

Therefore, the recurrence polynomial $\varrho_G$ consists of all factors of the characteristic polynomial, which do not divide this sum.

**Corollary 2.3.** *Let $d$ be the minimal degree among the degrees of the factors of the characteristic polynomial of $G$. If $G$ contains two nodes with (shortest path) distance $n - d$, then the characteristic polynomial of $G$ equals its minimal polynomial.*

Indeed, if $(u, v)$ is a pair of nodes with distance $n - d$, then the local residual $r_{uv}$ has degree $d - 1$ (since the shortest path connecting $u$ and $v$ contains $n - d + 1$ nodes) and can therefore not contain a factor of $\chi_G$. Hence, the local recurrence polynomial $\varrho_{uv}$ is equal to $\chi_G$. But since $\varrho_{uv}$ divides the minimal polynomial of $G$, they have to be equal.

*Remark* 2.4. The local recurrence polynomials $\varrho_{uv}$ are always a factor of the minimal polynomial of $G$. However, $\varrho_{uv}$ can be a multiple of $\varrho_G$, a factor of $\varrho_G$ or neither of those. This can be seen for example in the path $P_4$ on five vertices. It holds

$$\begin{aligned}
\varrho_G(x) &= x(x^2 - 3), \\
\varrho_{1,5}(x) &= x(x-1)(x+1)(x^2 - 3) = \chi_G, \\
\varrho_{1,3}(x) &= x(x^2 - 3), \\
\varrho_{2,3}(x) &= x^2 - 3, \\
\varrho_{2,4}(x) &= (x-1)(x+1)(x^2 - 3)
\end{aligned}$$

In general however, we can make no statement about the local recurrence polynomials. There exist graphs where every local recurrence polynomial is

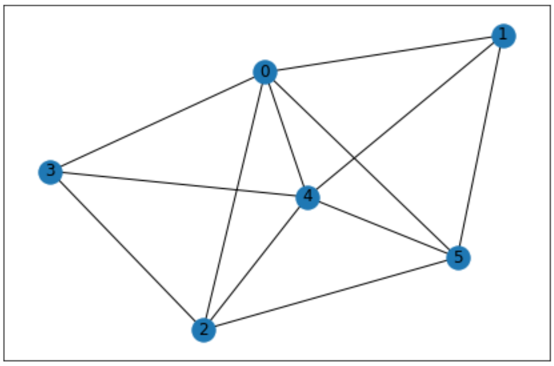


Figure 1: Every local recurrence polynomial is a proper divisor of the minimal polynomial.

a proper divisor of the minimal polynomial of $G$ (see Figure 1). On the other hand, there exist graphs where every local recurrence polynomial is equal to $\chi_G$, while $\varrho_G$ is a proper divisor of the characteristic polynomial (e.g. the path $P_3$).

*Proof.* Proof of Claim 2.2.

Rough idea: First use that $\varrho_{uu}$ is equal to the characteristic polynomial of $A$ restricted to $Z(e_u, A)$, to show the claim for closed walks.

Afterwards: Every walk from $u$ to $v$ can (uniquely) be decomposed as $(v_0 = u, C_0, v_1, C_1, \ldots, v_{n-1}, C_{n-1}, v_n = v, C_n)$, where $(u, v_1, \ldots, v_{n-1}, v)$ is a simple path and every $C_i$ is a closed walk on $v_i$ not passing through any $v_j$ for $j < i$. This should lead to the desired result. $\square$

**Corollary 2.5.** *Let $G$ be an undirected graph, $u \in V$, let $d := \max_{v \in V}(dist(u, v))$. Then the local recurrence polynomial $\varrho_{uu}$ has degree at least $d + 1$.*

*Proof.* Recall that $\varrho_{uu}$ is te characteristic polynomial of $A|_{Z(e_u,A)}$. Since $(A^i e_u)_v = 0$ for all $i < d$, but $\left(A^d e_u\right)_v > 0$, the vector $A^d e_u$ is linearly independent of all $A^i e_u$ for all $i < d$. Hence, $\deg(\varrho_{uu}) = \dim(Z(e_u, A)) \geq d + 1$. $\square$

**Corollary 2.6.** *Let $G$ be an undirected graph, $u, v \in V$. Then $\varrho_{uv}$ is a divisor of $\varrho_{uu}$.*

*Proof.* Let $\varrho_{uu} = x^d - \sum_{i=0}^{d-1} \beta_i x^i$. Then it holds that

$$A^k e_u = \sum_{i=k-d}^{k-1} \beta_i A^i e_u$$

for all $k \geq d$. In particular, this implies

$$w_k^{uv} = \langle e_u, A^d e_v \rangle = \sum_{i=k-d}^{k-1} \beta_i \langle e_u, A^i e_v \rangle = \sum_{i=k-d}^{k-1} \beta_i w_i^{uv}$$

and hence, $\varrho_{uu}$ is a characteristic polynomial for $w_r^{uv}$ as desired. $\square$

Unfortunately, in general $\varrho_{uv}$ is a proper divisor of the greatest common divisor of $\varrho_{uu}$ and $\varrho_{vv}$.

# References

[1] M. Hall. An isomorphism between linear recurring sequences and algebraic rings. *Transactions of the American Mathematical Society*, 44(2):196–218, 1938.