1 An approach using $\mathbb{Z}[x]/_{\chi_G(x)}$

Let $G = (V, E)$ be an undirected graph with adjacency matrix A. Denote its characteristic polynomial by $\chi_G(x)$ and define the quotient ring $B_G := \mathbb{Z}[x]/_{\chi_G(x)}$.

Recall, that we are interested in computing the polynomials $r_{uv}(x) \in \mathbb{Z}[x]$ for $u, v \in V$. Observe, that for our application it suffices to compute the image of $r_G(x) := \sum_{u,v \in V} r_{uv}(x)$ in the quotient ring B_G. This is due to the fact, that the recurrence polynomial consists of exactly those factors of the characteristic polynomial, which do not divide $r_G(x)$. Furthermore, a factor of χ_G divides r_G if and only if it divides the image of r_G in B_G.

The goal is to calculate the recurrence polynomial of G without having to calculate large matrix powers of the adjacency matrix. In these notes I will state some observations/lemmas (mostly without proof), which could maybe lead to the desired result.

Lemma 1.1. *It holds that*

$$\sum_{v \in V} r_{vv}(x) = \chi'_G(x).$$

We can view $\mathbb{Z}^n$ as a $\mathbb{Z}[x]$-module, where the action of x is defined by $x \cdot v := A \cdot v$. Furthermore,

$$\varphi \colon \mathbb{Z}^n \times \mathbb{Z}^n \to B_G, (e_u, e_v) \mapsto r_{uv}$$

with $\mathbb{Z}$-linear continuation defines a map of $\mathbb{Z}[x]$-modules (B_G is a $\mathbb{Z}[x]$-module via the usual multiplication with x). The next lemma states, that φ is actually a $\mathbb{Z}[x]$-module homomorphism (Consequently, φ can be seen as an inner product on the module $\mathbb{Z}^n$). This allows to replace the costly matrix multiplication on $\mathbb{Z}^n$ by a simple multiplication by x on the level of polynomials. The major drawback is that by transitioning into the ring B_G, we lose exact information on the adjacency matrix A. In particular, there exist non-isomorphic graphs with distinct recurrence polynomials, but same characteristic polynomial. Hence, restricting ourselves to B_G (without making use of the adjacency matrix A itself) can not lead to a solution of our main problem.

Lemma 1.2. *The map φ, defined as above, is a bilinear $\mathbb{Z}[x]$-module homomorphism.*

The following corollaries show, how this lemma can be applied in our setting.

Corollary 1.3. *In B_G, for all $u, v \in V$ it holds that*

$$\sum_{w \in N(u)} r_{wv}(x) = x \cdot r_{uv}(x).$$

This immediately implies:

Corollary 1.4. *In B_G, it holds that*

$$\sum_{\{u,v\} \in E} r_{uv}(x) = x \cdot \chi'(x)$$

Consequently, computing $\sum_{\{u,v\} \notin E, u \neq v}$ would solve our problem, but is unfortunately not as easy.

Corollary 1.5. *In B_G, it holds that*

$$\sum_{u,v \in V} \deg(u) r_{uv}(x) = x \cdot r_G(x).$$

The following corollaries are interesting in the sense, that the sum over all $r_{uv}(x)^2$ can be computed (efficiently) when only knowing the characteristic polynomial of G (without knowledge of the adjacency matrix). In particular, this sum has to be equal (in B_G) for all, even non-isomorphic, graphs with the same characteristic polynomial.

Corollary 1.6. *In B_G, for all $u, v \in V$ it holds that*

$$r_{uv}(x)^2 = r_{uu}(x) \cdot r_{vv}(x).$$

Immediate consequence:

Corollary 1.7. *In B_G, it holds that*

$$\sum_{u,v \in V} r_{uv}(x)^2 = \chi'_G(x)^2.$$

A similar result features the number of closed walks.

Corollary 1.8. *The number of closed walks on G of length k is given by the coefficient of x^n in the residual of $x^k \cdot \chi'_G(x)$ in B_G.*

Another strange, probably useless equation:

Corollary 1.9. *Denote the coefficient of* x^k *in* $r_{uv}(x)$ *by* r_{uv}^k*. Furthermore, let* p_k *be the truncation of* χ_G *up to level* k *(i.e.,* $\chi_G(x) = x^k \cdot p_k(x) + \mathcal{O}(x^{k-1})$*). Then for all* k*, it holds in* B_G *that*

$$p_k \cdot \chi'_G(x) = \sum_{u,v \in V} r_{uv}^k \cdot r_{uv}(x).$$

Remark 1.10. All equations also work on divisors of G (resp. equitable partitions). For example, if P is an equitable partition of G, then

$$\sum_{p \in P} \varphi(e_p, e_p) = \chi'_P(x) \cdot \frac{\chi_G(x)}{\chi_P(x)}.$$

The next equation provides an efficient formula for the derivative of $r_G(x)$. Unfortunately, the recurrence polynomial of G obviously depends on the constant coefficient of $r_G(x)$. However, a direct computation involves computing the number of walks of length n on G.

Lemma 1.11. *Let* G^c *be the complement of* G*. Then it holds that*

$$r'_G(x) = (-1)^{n+1} \chi'_G(x) - \chi'_{G^c}(-1-x).$$

2 A more algebraic approach

Let again $G = (V, E)$ be an undirected graph with adjacency matrix A. We define the centrality $c(v)$ of a node $v \in V$ as the limit of the probability, that a walk of length k starts in node v. It is known, that the centrality of v corresponds to the v-th entry of the (unique) eigenvector of A to the largest eigenvalue of A.

Lemma 2.1. *It holds that* $c(u) = c(v)$ *if and only if* u *and* v *belong to the same set in the coarsest equitable partition of* G*.*

It is known that the characteristic polynomial of the coarsest equitable partition of G is a divisor of the characteristic polynomial of G. Furthermore, all main eigenvalues of G are also zeros of the characteristic polynomial of every equitable partition of G. In some cases, the characteristic polynomial of the coarsest equitable partition and the recurrence polynomial of G actually coincide. This, however, does not hold true for all graphs. In particular, if the coefficients of the eigenvector to the largest eigenvalue of G fulfill more linear equations than in Lemma 2.1, the recurrence polynomial of G is a proper divisor of the characteristic polynomial of the coarsest equitable partition of G.

Lemma 2.2. *Let v be the eigenvector of A for the largest eigenvalue λ. Let the orthogonal complement of the kernel of the map*

$$\mathbb{Z}^n \to \mathbb{R}, u \mapsto \langle u, v \rangle$$

be generated by the vectors $u_1, \ldots, u_k$ ($\mathbb{R}$ can be replaced by the splitting field of the main polynomial of G). Define the matrix $P \in \mathbb{Z}^{n \times k}$ as $P := (u_1, \ldots, u_k)$ and denote its pseudo-inverse by $P^+ \in \mathbb{Q}^{k \times n}$. Then the recurrence polynomial of G is the characteristic polynomial of $P^+ \cdot A \cdot P$.

In particular, the degree of the recurrence polynomial of G is equal to the rank of the lattice generated by the coefficients of v in the splitting field of χ_G (interpreted as $\mathbb{Z}$-module) and a basis of this lattice would directly lead to a formula for the recurrence polynomial of G. There could exist an approach making use of the Hamiltonian normal form of A, but lattice theory is complicated...