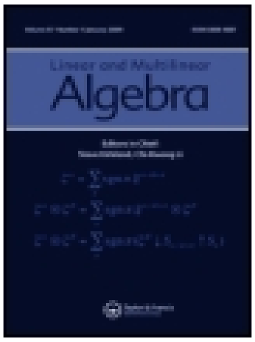

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Yasuo Teranishi

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# Main Eigenvalues of a Graph

YASUO TERANISHI*

*Graduate School of Mathematics, Nagoya University, Nagoya, 464-8602, Japan*

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An eigenvalue of a graph is said to be a main eigenvalue if it has an eigenvector not orthogonal to the main vector $j = (1, 1, \ldots, 1)$. In this paper we shall study some properties of main eigenvalues of a graph.

*Keywords*: Main eigenvalue; Spectra; Quotient graph

## 1. INTRODUCTION

For a graph $G$, let $A(G)$ be the adjacency matrix of $G$, and $P_G(t)$ the characteristic polynomial of $A(G)$.

The walk generating function $H_G(t)$ of $G$ is defined by

$$H_G(t) = \sum_{k=0}^{\infty} N_k t^k,$$

where $N_k$ stands for the number of walks of length $k$ in $G$.

Let $G$ be a graph with $n$ vertices and $r$ main eigenvalues $\mu_1, \mu_2, \ldots, \mu_r$. Then it is known that the walk generating function $H_G(t)$ is a rational function of the form

$$H_G(t) = \sum_{i=1}^{r} \frac{c_i}{1 - t\mu_i}, \tag{1.1}$$

where $c_i$ are positive numbers (*cf.* p. 46 [1]).

*e-mail: teranish@math.nagoya-u.ac.jp

Let $G^c$ be the complement of a graph $G$. According to a result of D. M. Cvetković (Theorem 1.11 [1]),

$$\frac{P_{G^c}(-1-t)}{P_G(t)} = (-1)^n(H_G(t^{-1})t^{-1}+1). \tag{1.2}$$

Then it follows from (1.1) and (1.2) that the rational function $P_{G^c}(-1-t)/P_G(t)$ has poles of order 1 at all main eigenvalues of $G$ and is holomorphic at other points.

PROPOSITION 1.1 *If the number of distinct main eigenvalues of a graph $G$ is $r$, then its complement $G^c$ has exactly $r$ distinct main eigenvalues.*

*Moreover if $\mu_1, \mu_2, \ldots, \mu_r$ (resp. $\mu'_1, \mu'_2, \ldots, \mu'_r$) are the main eigenvalues of $G$ (resp. $G^c$), we have*

$$\frac{P_{G^c}(t)}{P_G(-1-t)} = (-1)^n \prod_{i=1}^{r} \frac{t-\mu'_i}{t+1+\mu_i}, \tag{1.3}$$

*where $n$ denotes the number of vertices of $G$.*

*Proof* Note that the polynomial $(-1)^n P_{G^c}(-1-t)$ is a monic polynomial of degree $n$ and the rational function $P_{G^c}(-1-t)/P_G(t)$ has simple poles at $\mu_1, \mu_2, \ldots, \mu_r$. So we obtain

$$(-1)^n \frac{P_{G^c}(-1-t)}{P_G(t)} = \prod_{i=1}^{r} \frac{t-\lambda_i}{t-\mu_i}, \tag{1.4}$$

for some $r$ complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_r$, with $\lambda_i \neq \mu_j$, $1 \leq i, j \leq r$. If we replace $t$ by $-1-t$, we have

$$(-1)^n \frac{P_G(-1-t)}{P_{G^c}(t)} = \prod_{i=1}^{r} \frac{t+1+\mu_i}{t+1+\lambda_i}.$$

This implies that $\lambda_1, \lambda_2, \ldots, \lambda_r$ are all distinct and the main eigenvalues of the complement $G^c$ are $-1-\lambda_1, -1-\lambda_2, \ldots, -1-\lambda_r$.

This, together with (1.4), completes the proof. ∎

COROLLARY *Let $\mu_1 > \mu_2 > \cdots \mu_r$ (resp. $\mu'_1 > \mu'_2 > \cdots > \mu'_r$) be main eigenvalues of a graph $G$ (resp. $G^c$). Then*

$$\mu'_i + \mu_{r-i+1} > -1, \quad \textit{for } i = 1, 2, \ldots, r,$$
$$\textit{and} \quad \mu'_i + \mu_{r-i+2} < -1, \quad \textit{for } i = 2, 3, \ldots, r.$$

*Proof* From (1.1), (1.2) and Proposition 1.1 it follows that there must be a zero of $\Pi_i(t+\mu_i+1)$ between $\mu'_k$ and $\mu'_{k+1}$ for each $k$, $k=1,2,\ldots,r$. Here we set $\mu'_{r+1}=\infty$, which implies the result. ∎

For a real number $\lambda$, set

$$m_G(\lambda) = \begin{cases} \text{the multiplicity of } \lambda, & \text{if } \lambda \text{ is an eigenvalue of a graph } G, \\ 0, & \text{otherwise,} \end{cases}$$

and if $\lambda$ is an eigenvalue of $G$, denote by $E_G(\lambda)$ the corresponding eigenspace.

PROPOSITION 1.2 *Let $\lambda$ be an eigenvalue of a graph $G$ then*

(1) $E_G(\lambda) \supset E_{G^c}(-1-\lambda)$ *and* $m_G(\lambda) = m_{G^c}(-1-\lambda)+1$, *if $\lambda$ is a main eigenvalue of $G$,*
(2) $E_G(\lambda) = E_{G^c}(-1-\lambda)$, *if $\lambda$ is not a main eigenvalue of $G$ and $-1-\lambda$ is not a main eigenvalue of $G^c$,*
(3) $E_G(\lambda) \subset E_{G^c}(-1-\lambda)$ *and* $m_G(\lambda) = m_{G^c}(-1-\lambda)-1$, *if $\lambda$ is not a main eigenvalue of $G$ but $-1-\lambda$ is a main eigenvalue of $G^c$.*

*Proof* Let $W$ be the subspace of $E_G(\lambda)$ orthogonal to the main vector $\boldsymbol{j}$. Then for any $\boldsymbol{v} \in W$,

$$\begin{aligned} A(G^c)\boldsymbol{v} &= J\boldsymbol{v} - \boldsymbol{v} - A(G)\boldsymbol{v} \\ &= (-1-\lambda)\boldsymbol{v}, \end{aligned}$$

where $J$ is the all-ones matrix and $I$ is the identity matrix. Hence we have $W \subset E_{G^c}(-1-\lambda)$. In particular (2) follows from this. If $\lambda$ is a main eigenvalue of $G$ then there exists an eigenvector $\boldsymbol{v}$ such that $E_G(\lambda) = \mathbb{R}\boldsymbol{v} + W$, On the other hand from Proposition 1.1 we see that $m_G(\lambda) = m_{G^c}(-1-\lambda)+1$, and hence $E_G(\lambda) \supset E_{G^c}(-1-\lambda)$. From (1) and (2) we obtain (3). ∎

From Proposition 1.2 we see that an eigenvalue of $G$ is a main eigenvalue if and only if

$$m_G(\lambda) > m_{G^c}(-1-\lambda).$$

A graph is said to be self-complementary if it is isomorphic to its complement.

PROPOSITION 1.3 *If $G$ is a self-complementary graph with $n$ vertices and $r$ main eigenvalues $\mu_1, \mu_2, \ldots, \mu_r$, then $n-r$ is an even integer and the characteristic polynomial of the graph $G$ takes the form*

$$P_G(t) = \prod_{i=1}^{r}(t-\mu_i) \prod_{i=1}^{(n-r)/2} (t-\lambda_i)(t+\lambda_i+1).$$

*Proof* Let us define a polynomial $F_G(t)$ by

$$F_G(t) = P_G(t) \Big/ \prod_{i=1}^{r}(t-\mu_i).$$

Then according to Proposition 1.1 we have

$$F_G(t) = (-1)^{n-r} F_G(-1-t). \tag{1.5}$$

If we put $t=-1/2$ then we get

$$F_G(-1/2) = (-1)^{n-r} F_G(-1/2).$$

If $n-r$ is an odd integer we have $F_G(-1/2)=0$. But this contradicts to the fact that eigenvalues of graphs are algebraic integers. If $\lambda$ is a root of the equation $F_G(t)=0$ then, according to (1.5), $-1-\lambda$ is also a root. Moreover by the same reason as above we see that $\lambda \neq -1-\lambda$, and hence we obtain the desired result. ∎

If $G$ is a connected regular graph with degree $d$ then $d$ is a main eigenvalue of $G$ and other eigenvalues are all non-main. In this case Propositions 1.1 and 1.3 are results of H. Sachs (*cf.* [1] Chap. 3).

## 2. NUMBER OF MAIN EIGENVALUES, HANKEL MATRIX, AND GRAPH EQUATIONS

We denote by $N_k$ the number of walks with length $k$ in $G$ and consider the Hankel matrix associated with the sequence $\{N_i;\ i=0,1,2,\ldots\}$:

$$H = \begin{pmatrix} N_0 & N_1 & N_2 & \cdots \\ N_1 & N_2 & \cdots & \cdots \\ N_2 & \vdots & \ddots & \\ \vdots & \vdots & & \end{pmatrix}$$

It follows from (1.1) that the number of main eigenvalues of $G$ is equal to the number of poles of the rational function

$$H_G(t^{-1})t^{-1} = \sum_{k=0}^{\infty} \frac{N_k}{t^{k+1}} \tag{2.1}$$

By a well known property of the Hankel matrix ([3] Chap. XV, Theorem 8), we then obtain the following

PROPOSITION 2.1 *The number of main eigenvalues of a graph $G$ is equal to the rank of the Hankel matrix $H$.*

If the number of main eigenvalues of a graph $G$ is $r$, we introduce a quadratic form in $r$ variables $t_0, t_1, \ldots, t_{r-1}$ by

$$F(t_0, \ldots, t_{r-1}) = \sum_{0 \le j,k \le r-1} N_{j+k} t_j t_k.$$

Then by (1.1), we can write $N_k$ in the form

$$N_k = \sum_{i=1}^{r} c_i \mu_i^k \quad (c_i > 0).$$

Therefore we have

$$\begin{aligned} F(t_0, \ldots, t_{r-1}) &= \sum_{0 \le j,k \le r-1} N_{j+k} t_j t_k \\ &= \sum_{i=1}^{r} c_i \left( \sum_{j,k} \mu_i^{j+k} t_j t_k \right) \\ &= \sum_{i=1}^{r} c_i \left( \sum_{k=0}^{r-1} \mu_i^k t_k \right)^2. \end{aligned}$$

Since $c_i$ are all positive this implies that $F(t_0, \ldots, t_{r-1}) = 0$, for some real numbers $t_0, \ldots, t_{r-1}$, if and only if $t_0, t_1, \ldots, t_{r-1}$ satisfy a linear homogeneous equation:

$$\sum_{k=0}^{r-1} \mu_i^k t_k = 0 \quad \text{for } i = 1, 2, \ldots, r.$$

Since $\mu_1, \ldots, \mu_r$ are all distinct, we have $t_k = 0$ for $k = 0, 1, \ldots, r-1$. Thus we have proved that the quadratic form $F(t_0, \ldots, t_{r-1})$ is positive definite. Accordingly we obtain the following theorem:

THEOREM 2.2 *Let $M_i$ be the i by i principal minor of the Hankel matrix:*

$$M_i = det((N_{k+l})_{0 \le k,l \le i-1}) \quad for\ i = 1, 2, \ldots.$$

*Then $M_i$ is a nonnegative integer and the number of main eigenvalues of a graph G is not greater than r if and only if $M_{r+1} = 0$.*

*Example*

(1) $N_0 N_2 - N_1^2 \ge 0$ for any graph and equality holds if and only if the number of main eigenvalues is 1.

(2)

$$\det \begin{pmatrix} N_0 & N_1 & N_2 \\ N_1 & N_2 & N_3 \\ N_2 & N_3 & N_4 \end{pmatrix} \ge 0$$

for any graph and equality holds if and only if the number of main eigenvalues is not greater than 2.

THEOREM 2.3

(1) *The number of main eigenvalues of a graph G is r if and only if there exist rational numbers $a_1, \ldots, a_r$ such that*

$$N_s = \sum_{i=1}^{s} a_i N_{s-i} \quad for\ s = r, r+1, \ldots, \tag{2.2}$$

*and r is the smallest positive integer with this property.*

(2) *If $\mu_1, \mu_2, \ldots, \mu_r$ are the main eigenvalues of G, then*

$$\prod_{i=1}^{r} (t - \mu_i) = t^r - \sum_{i=1}^{r} a_i t^{r-i}. \tag{2.3}$$

*Proof* According to Chap. XV, Theorem 7 in [4], (1) follows from Proposition 2.1. We now prove the part (2).

From (2.2) we get

$$\left(t^r - \sum_{i=1}^{r} a_i t^{r-i}\right) t^{-1} H_G(1/t)$$
$$= \sum_{s=0}^{\infty} \left(N_s - \sum_{i=1}^{s} a_i N_{s-i}\right) t^{r-s-1}$$
$$= \sum_{s=0}^{r-1} \left(N_s - \sum_{i=1}^{s} a_i N_{s-i}\right) t^{r-s-1},$$

and hence $(t_r - \Sigma a_i t_{r-1}) t^{-1} H_G(1/t)$ is a polynomial. Since the rational function $t^{-1} H_G(1/t)$ has its poles at $\mu_1, \mu_2, \ldots, \mu_r$, we have

$$\mu_i^r - \sum_{i=1}^{t} a_i \mu_i^{r-i} = 0 \quad (i = 1, 2, \ldots, r),$$

and we obtain (2.3). ■

By a similar argument as in the proof of Theorem 2.3, we obtain the following

THEOREM 2.4 *Let $\mu_1, \mu_2, \ldots, \mu_r$ be the main eigenvalues of a graph $G$ and suppose that there exist real numbers $a_1, a_2, \ldots, a_m$ such that*

$$N_s = \sum_{i-1}^{m} a_i N_{s-i} \quad (s = m, m+1, \ldots).$$

*Then the polynomial $\prod_{i=1}^{r}(t - \mu_i)$ divides the polynomial $t^m - \sum_{i=1}^{m} a_i t^{m-i}$.*

As an application of Theorem 2.3 and Theorem 2.4 we now prove the following theorem:

THEOREM 2.5 *Let $G$ be a graph with exactly $r$ main eigenvalues $\mu_1, \mu_2, \ldots, \mu_r$ and with adjacency matrix $A$. Then the ideal*

$$\{f(t) \in \mathbb{Q}[t]; f(A)J = 0\}$$

*is generated by the polynomial*

$$g(t) = \prod_{i=1}^{r} (t - \mu_i).$$

*Proof* Let $j = (1, \ldots, 1)$ be the main vector. Then we have $N_k = jA^k j^t$ for $k = 0, 1, 2, \ldots,$, where $j^t$ stands for the transposed vector of $j$. Therefore this theorem follows from Theorem 2.3 and Theorem 2.4. ■

*Remark* The coefficients of $g(t)$ are integers, since the coefficients of $g(t)$ are all rational numbers and the eigenvalues of a graph are algebraic integers.

PROPOSITION 2.6 *Let $\mu_1, \mu_2, \ldots, \mu_r$ be the main eigenvalues of a graph $G$ with adjacency matrix $A$ and let*

$$f(t) = t^m - \left(\sum_{i=1}^{m} a_i t^{m-i}\right)$$

*be a polynomial with rational coefficients such that the matrix $f(A)$ is negative semidefinite. Then*

$$N_m \leq \sum_{i=1}^{m} a_i N_{m-i}$$

*where equality holds if and only if the polynomial $q(t) = \Pi(t - \mu_i)$ divides $f(t)$.*

*Proof* The first part of this proposition is evident since $f(A)$ is negative semidefinite and $N_k = jA^k j^t$. The equality holds if and only if $j\, f(A)\, j^t = 0$. Since $f(A)$ is negative semidefinite, we see that the vector $j$ is an eigenvector of $f(A)$ corresponding to the eigenvalue 0. Thus we get $f(A) j^t = 0$, and hence the second part of the theorem follows from Theorem 2.5. ■

By the Perron-Frobenius theorem all entries of an eigenvector corresponding to the largest eigenvalue $\mu$ of a graph $G$ are non-negative, and hence the largest eigenvalue is a main eigenvalue of $G$. According to Theorem 2.5, the number of main eigenvalues of $G$ is 1 if and only if the adjacency matrix $A$ satisfies $AJ = \mu J$ and hence $G$ is a regular graph. Moreover if $f(t)$ is a polynomial with only one real root and $f(t)$ satisfies $f(A)J = 0$ then it folllows from (1) in Theorem 2.5 that $G$ is a regular graph.

For example, since the polynomial $t^m-\lambda$($m$ is an odd positive integer and $\lambda$ a positive number) has only one real root, if the adjacency matrix $A$ of a graph $G$ satisfies $(A^m-\lambda I)J=0$ then $G$ is a regular graph. This is a result of M. Syslo [7]. Moreover M. Syslo [7] proved that if the adjacency matrix $A$ of a connected graph $G$ satisfies the matrix equation $(A^m-rI)J=0$ with $m$ even integer and $r$ positive integer, then $G$ is either regular or semiregular graph.

Let $G$ be a connected bipartite graph with the largest eigenvalue $\mu$. Suppose now that $-\mu$ is a main eigenvalue of $G$ and all eigenvalues except $\pm\mu$ are non-main. Then by Theorem 2.5, the adjacency matrix $A$ of $G$ satisfies $(A^2-\mu^2 I)J=0$ and hence $G$ is a semiregular graph. According to Theorem 2.5 we then obtain a generalization of Syslo's theorem:

PROPOSITION 2.7 *Let $G$ be a graph with adjacency matrix $A$. Then we have*:

(1) *if there exists a polynomial with real coefficients $f(t)$ such that it has only one real root $\mu$ and moreover it satisfies a matrix equation $f(A)J=0$, then $G$ is a regular graph with the largest eigenvalue $\mu$,*
(2) *if there exists a polynomial $f(t)$ with real coefficients such that it has only two real roots $\mu$, $-\mu(\mu>0)$ and moreover it satisfies a matrix equation $f(A)J=0$. Then $G$ is a regular graph or semiregular graph with the largest eigenvalue $\mu$.*

*Example* Let $G$ be a graph with adjacency matrix $A$ and let $\mu$ be the largest eigenvalue of $G$. Then for any positive integer $k$, the matrix $A^k-\mu^k I$ is negative semidefinite. According to Proposition 2.6, we have

$$N_k \leq \mu^k N_0, \quad \text{for } k=0,1,2,\ldots,$$

where for any even number $k$, equality holds if and only if $G$ is regular, and for any odd number $k$, equality holds if and only if $G$ is regular or semiregular.

Let $n$ and $m$ be the vertex number and edge number of $G$ respectively and let $V(G)=\{v_1,\ldots,v_n\}$ be the vertices of $G$, $E(G)$ the set of edges of $G$. We denote $d_i$ the degree of the vertex $v_i$, $1\leq i\leq n$.

Note that $N_0 = n$, $N_1 = 2m$,

$$N_2 = \sum_{i=1}^{n} d_i^2 \quad \text{and}$$

$$N_3 = 2\sum_{i \sim j} d_i d_j$$

where the sum is over all pairs $(v_i, v_j)$ of adjacent vertices of $G$.

The inequality above for $k = 1, 2$ and 3 yields the following inequalities:

(1) (L. Collatz, U. Sinogowitz (*cf.* [1] Theorem 3.8))
$2m \leq \mu n$, equality holds if and only if $G$ is regular,
(2) (M. Hofmeister [6])
$\sum_{i=1}^{n} d_i^2 \leq \mu^2 n$, equality holds if and only if $G$ is regular or semi-regular,
(3) $2\sum_{i \sim j} d_i d_j \leq \mu^3 n$, equality holds if and only if $G$ is regular.

## 3. MAIN EIGENVALUES OF QUOTIENT GRAPHS

Let $\pi = (V_1, \ldots, V_k)$ be an equitable partition of the vertex set $V(G)$ of a graph $G$ (*i.e.*, for all $i$ and $j$, the number $d_{ij}$ of edges from any vertex in $V_i$ to the cell $V_j$ is independent of the choice of vertex in $V_i$). The quotient graph $G/\pi$ of $G$ with respect to $\pi$ is a directed graph with the cells of $\pi$ as its vertex set, and with $d_{ij}$ arcs from a vertex $V_i$ to a vertex $V_j$. The adjacency matrix $A(G/\pi)$ of the quotient graph $G/\pi$ is the $k \times k$ matrix whose $ij$ entry is $d_{ij}$. For an equitable partition $\pi$, let $j(\pi) = (n_1, \ldots, n_k)$ with $n_j = |V_j|$ for $1 \leq j \leq k$. We now define main eigenvalues of a quotient graph $G/\pi$.

Definition 3.1 For an equitable partition $\pi = (V_1, \ldots, V_k)$ of a graph $G$, an eigenvalue $\lambda$ of the quotient graph $G/\pi$ is said to be a main eigenvalue of $G/\pi$ if $\lambda$ satisfies the following two conditions:

(1) $\lambda$ has a right eigenvector which is not orthogonal to the vector $j(\pi)$.
(2) $\lambda$ has a left eigenvector which is not orthogonal to the vector $j$.

If $\pi$ is the discrete partition, this definition is equivalent to that of main eigenvalue of a graph.

It is known (*cf.* Theorem 3 from [3]) that if an eigenvalue of $G$ is main then it is an eigenvalue of $G/\pi$. In Theorem 3.4 we shall prove that an eigenvalue of a graph with a equitable partition $\pi$ is a main eigenvalue if and only if it is a main eigenvalue of $G/\pi$.

PROPOSITION 3.2 *For a graph $G$ with an equitable partition $\pi = (V_1, \ldots, V_k)$ let*

$$B(G/\pi) = \begin{pmatrix} 0 & n_1 & n_2 & \cdots & n_k \\ 1 & & & & \\ 1 & & & & \\ \vdots & & A(G/\pi) & & \\ 1 & & & & \end{pmatrix}.$$

*Then the walk generating function $H_G(t)$ is given by*

$$t^{-1}H_G(t^{-1}) = -\frac{P_{B(G/\pi)}(t)}{P_{A(G/\pi)}(t)} + t. \tag{3.1}$$

*Proof* The number of walks in $G$ with length $m$ from a vertex in $V_i$ to the vertices in $V_j$ does not depend on the choice of a vertex in $V_i$ and it is equal to the $ij$ entry of the matrix $A(G/\pi)^m$. Hence we see that the number $N_m$ of walks of length $m$ is given by

$$N_m = j(\pi)A(G/\pi)^m j^t.$$

Therefore we obtain

$$\begin{aligned} H_G(t) &= \sum_m N_m t^m \\ &= j(\pi)\left(\sum_m (A(G/\pi)^m t^m) j^t\right) \\ &= j(\pi)(I - tA(G/\pi))^{-1} j^t, \end{aligned}$$

and from this we get

$$\begin{aligned} t^{-1}H_G(t^{-1}) &= j(\pi)(tI - A(G/\pi))^{-1} j^t \\ &= (j(\pi)\operatorname{adj}(tI - A(G/\pi))j^t)P_{G/\pi}(t)^{-1} \\ &= -\frac{P_{B(G/\pi)}(t)}{P_{A(G/\pi)}(t)} + t. \end{aligned}$$

∎

If a graph $G$ has an equitable partition $\pi$, it is also equitable for the complement $G^c$. The corresponding quotient adjacency matrix is given by

$$A(G^c/\pi) = J(\pi) - A(G/\pi) - I,$$

where $I$ is the $k \times k$ identity matrix and

$$J(\pi) = \begin{pmatrix} n_1 & \cdots & n_k \\ \vdots & & \vdots \\ n_1 & \cdots & n_k \end{pmatrix}.$$

Now we generalize the Cvetković formula (1.2).

THEOREM 3.3 *Let $\pi$ be an equitable partition with $k$ cells of a graph $G$. If $G$ has $n$ vertices then*

(1) $P_{G^c/\pi}(-1-t)/P_{G/\pi}(t) = (-1)^k(H_G(t^{-1})t^{-1}+1)$,
(2) $(-1)^n P_{G^c}(-1-t)/P_G(t) = (-1)^k P_{G^c/\pi}(-1-t)/P_{G/\pi}(t)$.

*Proof* If $\pi = (V_1, \ldots, V_k)$ with $n_i = |V_i|$ for $1 \le i \le k$, we find that the number of all walks with length $m$ from the vertices in $V_i$ to the vertices in $V_j$ is equal to the $ij$-entry of the matrix $n_i A(G/\pi)^m$. Denoting by $w_j(t)$ the generating function of the number of walks to the cell $V_j$, we have

$$\begin{aligned} J(\pi)(I - tA(G/\pi))^{-1} &= J(\pi)\sum_{m=0}^{\infty} A(G/\pi)^m t^m \\ &= \begin{pmatrix} w_1(t) & \ldots & w_k(t) \\ \vdots & & \vdots \\ w_1(t) & \ldots & w_k(t) \end{pmatrix}. \end{aligned}$$

Therefore we get

$$\begin{aligned} &(-1)^k \frac{P_{G^c/\pi}(-1-t^{-1})}{P_{G/\pi}(t^{-1})} \\ &= \frac{\det(I + t(J(\pi) - A(G/\pi))}{\det(I - t(A(G/\pi))} \\ &= \det(tJ(\pi)(I - tA(G/\pi))^{-1} + I) \end{aligned}$$

$$\begin{aligned}&= \det(tW(t) + I)\\ &= t\sum_{i=1}^{k} w_i(t) + 1\\ &= tH_G(t) + 1,\end{aligned}$$

where $W(t)$ is the matrix with $w_{ij}(t)$ as the $ij$-entry.

This completes part (1). Part (2) follows from part (1). ■

The following result is due to Cvetković (Theorem 3 [3]):

COROLLARY 1 *Let $G$ be a graph with an equitable partition $\pi$. If $\lambda$ is a main eigenvalue of $G$ then it is an eigenvalue of $G/\pi$.*

*Proof* The assertion follows readily from (1) in Theorem 3.3. ■

*Example* If $G$ is a regular graph of degree $r$ with $n$ vertices, by Theorem 3.3 (2), we obtain the following formula due to $H$. Sachs (see Theorem 2.8 in [1]):

$$\frac{P_{G^c}(t)}{P_G(-1-t)} = (-1)^n \frac{t-n+r+1}{t+r+1}.$$

Recall that the complements of two cospectral graphs are not necessary cospectral. From Theorem 3.3 we obtain the following result:

COROLLARY 2 *Let $G_1$ and $G_2$ are cospectral graphs with equitable partitions. If their quotient graphs have the same adjacency matrices then the complements $G_1^c$* and *$G_2^c$ are cospectral.*

*Example* ([5] *cf.* Chap. 5) For two vertices u and v of a distance-regular graph $G$, $G-u$ and $G-v$ are cospectral graphs with cospectral complements.

THEOREM 3.4 *Let $G$ be a graph with an equitable partition $\pi$. Then an eigenvalue of $G$ is a main eigenvalue if and only if it is a main eigenvalue of the quotient graph $G/\pi$.*

*Proof* First of all we claim that an eigenvalue $\lambda$ with multiplicity $m$ of the quotient graph $G/\pi$ is a main eigenvalue of $G$ if and only if $\lambda$ is an eigenvalue of the matrix $B(G/\pi)$ with multiplicity $m-1$. Since the main eigenvalues of $G$ are exactly the poles of the rational function $H(t^{-1})t^{-1}$, this claim follows from Proposition 3.2.

Let $\lambda$ be a main eigenvalue of $G$. Then $\lambda$ is an eigenvalue of $G/\pi$. We now prove that $\lambda$ is a main eigenvalue of $G/\pi$.

$\lambda$ has a right eigenvector which is not orthogonal to the vector $\boldsymbol{j}(\pi)$, because otherwise the solution space of the linear equation $B(G/\pi)\boldsymbol{x}^t = \lambda\boldsymbol{x}^t$ contains a column vector of the form $\boldsymbol{x}^t = (0, \boldsymbol{y})^t$ with $\boldsymbol{y} = (x_1, \ldots, x_k)$ and $\boldsymbol{y}$ satisfies $A(G/\pi)\boldsymbol{y}^t = \lambda\boldsymbol{y}^t$.

Hence the multiplicity of $\lambda$ in $B(G/\pi)$ is not less than the multiplicity in $A(G/\pi)$.

This contradicts to the assumption that $\lambda$ is a main eigenvalue of $G$. In a similar way we find that $\lambda$ has a left eigenvector which is not orthogonal to the vector $\boldsymbol{j}$.

Let $\lambda$ be a main eigenvalue of $A(G/\pi)$ with multiplicity $m$ and let $L(\lambda)$ be the eigenspace of $B(G/\pi)$ with respect to $\lambda$. Assume that $L(\lambda)$ contains a column vector $\boldsymbol{y}^t = (y_0, y_1, \ldots, y_k)^t$ with $y_0 \neq 0$. Then $\boldsymbol{y}' = (y_1, \ldots, y_k)^t$ satisfies

$$A(G/\pi)\boldsymbol{y}' - \lambda\boldsymbol{y}' = -y_0\boldsymbol{j}.$$

This implies that all left eigenvectors of $A(G/\pi)$ associated with the eigenvalue $\lambda$ are orthogonal to the vector $\boldsymbol{j}$, and hence contradicts to the assumption that $\lambda$ is a main eigenvalue of $G/\pi$. Thus the first entry of every vector in $L(\lambda)$ is 0.

Let $\boldsymbol{y} = (0, y_1, \ldots, y_k)$ be a vector in $L(\lambda)$.

Then the column vector $\boldsymbol{y}' = (y_1, \ldots, y_k)^t$ is a right eigenvector of $A(G/\pi)$ with the eigenvalue $\lambda$ and $\boldsymbol{y}'$ is orthogonal to $\boldsymbol{j}(\pi)$.

But, since $\lambda$ is a main eigenvalue of $G/\pi$, this implies that the multiplicity of $\lambda$ in $B(G/\pi)$ is less than that of $\lambda$ in $A(G/\pi)$. From Proposition 3.2 we see that $\lambda$ is a main eigenvalue of $G$. ∎

The following proposition is a generalization of Proposition 1.2 for quotient graphs and it is a direct consequence of Proposition 1.2 and Theorem 3.3.

PROPOSITION 3.5 *Let $G$ be a graph with an equitable partition $\pi$ and with exactly $r$ main eigenvalues. Let $\mu_1, \mu_2, \ldots, \mu_r$ be the main eigenvalues of $G$ and $\mu'_1, \mu'_2, \ldots, \mu'_r$ the main eigenvalues of $G^c$. Then*

$$\frac{P_{G^c/\pi}(t)}{P_{G/\pi}(-1-t)} = (-1)^k \prod_{i=1}^{r} \frac{t - \mu'_i}{t + 1 + \mu_i}.$$

From Proposition 3.5 we obtain the following

PROPOSITION 3.6 *An eigenvalue $\lambda$ of a quotient graph $G/\pi$ is a main eigenvalue of $G$ if and only if one of the following equivalent conditions* (1) *and* (2) *holds*:

(1) $m_{G/\pi}(\lambda) = m_{G^c/\pi}(-1-\lambda) + 1$,
(2) $m_{G/\pi}(\lambda) > m_{G^c/\pi}(-1-\lambda)$.

COROLLARY *Let $G$ be a graph with an equitable partition $\pi$. Then the polynomials $P_{G/\pi}(t)$ and $P_{G^c/\pi}(-1-t)$ are coprime if and only if every eigenvalue $\lambda$ of $G/\pi$ is a main eigenvalue of $G$ and $m_{G/\pi}(\lambda) = 1$.*

*Proof* The assertion readily follows from Proposition 3.6. ■

The join $G_1{}^*G_2$ of graphs $G_1$ and $G_2$ is the graph constructed from the direct sum of $G_1$ and $G_2$ by joining every vertex in $G_1$ with every vertex in $G_2$. From Theorem 3.3 we can easily prove the following

PROPOSITION 3.7 *For each $i = 1,2,\ldots,k$, if $\pi_i = (V_{i,j},\ 1 \leq j \leq m_i)$ is an equitable partition of a graph $G_i$,* then $(V_{ij}, 1 \leq i \leq k, 1 \leq j \leq m_i)$ *is an equitable partition of the graph*

$$G = G_1 * G_2 * \cdots * G_k.$$

*Let $B$ denote the matrix associated with this partition. Then the characteristic polynomial of the graph $G$ is*

$$P_G(t) = P_B(t) \prod_{i=1}^{k} \frac{P_{G_i}(t)}{P_{G_i/\pi_i}(t)},$$

*where $P_B(t)$ is the characteristic polynomial of $B$.*

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