CS711008Z Algorithm Design and Analysis

Lecture 6. Hidden Markov model and Viterbi's decoding algorithm

Dongbo Bu

Institute of Computing Technology
Chinese Academy of Sciences, Beijing, China

Outline

- The occasionally dishonest casino: an example of HMM
- Formal definition of HMM
- Finding the most probable state path: Viterbi algorithm

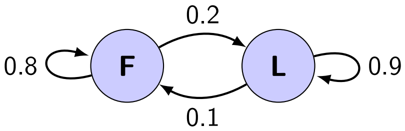
The occasionally dishonest casino: an example of HMM

The occasionally dishonest casino

- A casino have a **fair dice** and a **loaded dice**. The fair dice has identical probability $\frac{1}{6}$ for all numbers one to six while the loaded dice has probability 0.3 of a five, 0.3 of a six, and 0.1 for the numbers one to four.
- For the first roll, the casino uses the fair dice with probability $\frac{3}{5}$ and uses the loaded one with probability $\frac{2}{5}$. In the subsequent rolls, the casino switches from a fair to a loaded dice with probability 0.2 and switches back with probability 0.1. Thus the switch between dice forms a **Markov process**.

Fair dice

1	$1/6$
2	$1/6$
3	$1/6$
4	$1/6$
5	$1/6$
6	$1/6$



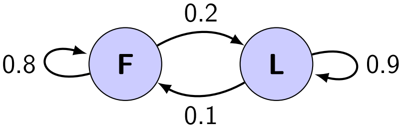
Loaded dice

1	$1/10$
2	$1/10$
3	$1/10$
4	$1/10$
5	$3/10$
6	$3/10$

The occasionally dishonest casino cont'd

Fair dice

1 : 1/6
2 : 1/6
3 : 1/6
4 : 1/6
5 : 1/6
6 : 1/6



Loaded dice

1 : 1/10
2 : 1/10
3 : 1/10
4 : 1/10
5 : 3/10
6 : 3/10

- Question: Suppose we observed a total of 10 rolls with the following outcomes:

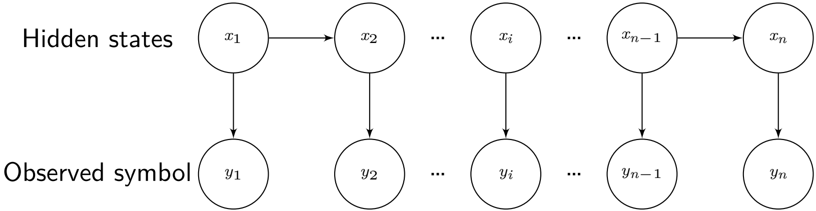
$$Y = (1, 3, 4, 5, 5, 6, 6, 3, 2, 6)$$

Could we find out the most probable state sequence, i.e. the most probable dice used for each roll?

Trial 1: Calculating log-odd score based on Markov model

- For each observed symbol, we could calculate log-odd score for a window of w rolls around it, and expect the rolls using fair dice to stand out with positive values.
- However, this is unsatisfactory since:
  - This solution depends heavily on the selection of the window size w.
  - The rolls generated using fair dice might have sharp boundaries and variable length.
- A better idea is to build a model to describe the switch between these two dice.

Trial 2: Calculating the most probable state path using HMM



- In each state of the Markov process, the outcome of a roll has different probability. Thus, the whole process forms a **hidden Markov model**. Here the **state sequence**, i.e. the dice used for each roll, is hidden.
- The essential difference between a Markov chain and a hidden Markov model is that for a HMM, there is not a one-to-one correspondence between observed symbols and states.

Formal definition of HMM

- **Transition probability**: We now distinguish the sequence of states (denoted as X) and the sequence of observed symbols (denoted as Y). The state sequence follows a simple Markov chain, so the probability of a state x_i depends only on the previous one x_{i-1}, which is characterised using transition probability:

$$a_{kl} = P(x_i = l | x_{i-1} = k)$$

- **Begin state**: To model the beginning of the process we introduce a **begin state** (denoted as state 0). The transition probability a_{0k} represents the probability of starting in state k.
- **Emission probability**: A state can generate a symbol from a distribution over all possible symbols; thus, we define **emission probability**:

$$e_k(b) = P(y_i = b | x_i = k)$$

Using HMM as a generative model

- A symbol sequence can be generated from HMM as follows:
  - Initially a state x_1 is chose according the probability a_{0k}. In this state x_i, a symbol is emitted according to the emission probability e_{x_i}.
  - Then a new state x_2 is generated according to the transition probability $a_{x_1 k}$ and so on. This way a symbol sequence $Y = (y_1, y_2, ..., y_n)$ is generated. Here we assume n is a fixed number and thus avoid defining an "end state" for simplicity.
- The joint probability of an observed symbol sequence Y and state sequence X is:

$$P(X, Y) = P(x_1 x_2 \ldots x_n, y_1 y_2 \cdots y_n) = \prod_{i=1}^{n} \left(a_{x_{i-1} x_i} e_{x_i}(y_i) \right)$$

An example

- For example, given an observed outcome of 10 rolls $Y = (1, 3, 4, 5, 5, 6, 6, 3, 2, 6)$, if $X = (\texttt{F}, \texttt{F}, \texttt{F}, \texttt{F}, \texttt{F}, \texttt{L}, \texttt{L}, \texttt{L}, \texttt{L}, \texttt{L})$, we have:

$$P(X, Y) = \frac{3}{5} \times (\frac{1}{6})^5 \times (0.8)^4 \times 0.2 \times (\frac{3}{10})^3 \times (\frac{1}{10})^2 \times 0.9^4$$

- There are a total of 2^n possible state sequence. If we are to choose just one sequence, perhaps the one with the highest joint probability should be chosen,

$$X^* = argmax_X P(X, Y)$$

Viterbi's decoding algorithm [1967]

- In 1967, Andrew Viterbi proposed a dynamic programming algorithm for decoding over noisy communication links.
- The idea can be extended for decoding in general graphical models, including Bayesian networks, Markov random fields and CRF. The extension is usually termed as **max-sum algorithm**, which aims to finding the most probable latent variables in graphical models. In these models, the **forward-backward algorithm** was generalized to **message passing** or **belief propagation**.
- A faster implementation of Viterbi's algorithm is LAZYVITERBI (J. Feldman, et al, 2002). The algorithm algorithm was built upon A^* algorithm, and it does not expand any nodes until it really needs to do so.

Viterbi's decoding algorithm: recursion

- First we rewrite $\max_X P(X, Y)$ as:

$$\max_{x_n} \max_{x_{n-1}} \ldots \max_{x_1} e_{x_n}(y_n) a_{x_{n-1} x_n} e_{x_{n-1}}(y_{n-1}) \ldots a_{x_1 x_2} e_{x_1}(y_1) a_{0 x_1}$$

- Note that we cannot build a direct recursion between $P(x_1 x_2 \ldots x_n, y_1 y_2 \cdots y_n)$ and $P(x_2 x_3 \ldots x_n, y_2 y_3 \cdots y_n)$.
- Let's consider a smaller subproblem: define $v_i(k)$ as

$$\max_{x_{i-1}} \ldots \max_{x_1} e_k(y_i) a_{x_{i-1} k} e_{x_{i-1}}(y_{i-1}) \ldots a_{x_1 x_2} e_{x_1}(y_1) a_{0 x_1}$$

We can observe the following recursion:

$$v_i(k) = e_k(y_i) \max_l (a_{lk} v_{i-1}(l))$$

- We also have

$$\max_X P(X, Y) = \max_k v_n(k)$$

Viterbi's decoding algorithm

ViterbiDecoding(Y, a, e)

Initialize $v_1(k) = a_{0k} e_k(y_1)$ for all state k;
for $i = 2$ to n **do**
  for each state k **do**
    $v_i(k) = e_k(y_i) \max_l (a_{lk} v_{i-1}(l))$;
    $ptr_i(k) = argmax_l (a_{lk} v_{i-1}(l))$;
  end for
end for
$P(X^*, Y) = max_k (v_n(k))$;
$x_n^* = argmax_k (v_n(k))$;
for $i = n - 1$ to 1 **do**
  $x_i^* = ptr_{i-1}(x_{i+1}^*)$;
end for
return X;

Time complexity: $O(nK^2)$, where K denotes the number of possible states

An example

	y_i	$v_i(\mathtt{F})$	$ptr_i(\mathtt{F})$	$v_i(\mathtt{L})$	$ptr_i(\mathtt{L})$
$i = 1$	1	$1.000 * 10^{-1}$	-	$4.000 * 10^{-2}$	-
$i = 2$	3	$1.333 * 10^{-2}$	F	$3.600 * 10^{-3}$	L
$i = 3$	4	$1.778 * 10^{-3}$	F	$3.240 * 10^{-4}$	L
$i = 4$	5	$3.370 * 10^{-4}$	F	$1.067 * 10^{-4}$	F
$i = 5$	5	$3.161 * 10^{-4}$	F	$2.880 * 10^{-5}$	L
$i = 6$	6	$4.214 * 10^{-6}$	F	$7.776 * 10^{-6}$	L
$i = 7$	6	$5.619 * 10^{-7}$	F	$2.100 * 10^{-6}$	L
$i = 8$	3	$7.492 * 10^{-8}$	F	$1.890 * 10^{-7}$	L
$i = 9$	2	$9.989 * 10^{-9}$	F	$1.701 * 10^{-8}$	L
$i = 10$	6	$1.332 * 10^{-9}$	F	$4.592 * 10^{-9}$	L