HBA Lecture Notes in Mathematics
CMI Lecture Notes in Mathematics

T. Parthasarathy
Sujatha Babu

Stochastic Games and Related Concepts

HINDUSTAN BOOK AGENCY

Springer

HBA Lecture Notes in Mathematics

CMI Lecture Notes in Mathematics

Volume 2

The *CMI Lecture Notes in Mathematics* series is a subseries of the *HBA Lecture Notes in Mathematics* series. This subseries publishes high-quality lecture notes of Chennai Mathematical Institute, India. Undergraduate and graduate students, research scholars, and teachers of mathematics and mathematical physics would find this book series useful. The volumes are carefully written as teaching aids and highlight characteristic features of the theory. The books in this series are co-published with Hindustan Book Agency, New Delhi, India.

More information about this subseries at http://www.springer.com/series/16504

T. Parthasarathy · Sujatha Babu

Stochastic Games and Related Concepts

HINDUSTAN BOOK AGENCY

Springer

T. Parthasarathy
Chennai Mathematical Institute
Chennai, Tamil Nadu, India

Indian Statistical Institute
Chennai, Tamil Nadu, India

Sujatha Babu
Chennai, Tamil Nadu, India

ISSN 2509-8063 ISSN 2509-8071 (electronic)
HBA Lecture Notes in Mathematics
ISSN 2662-7264 ISSN 2662-7272 (electronic)
CMI Lecture Notes in Mathematics
ISBN 978-981-15-6578-6 ISBN 978-981-15-6577-9 (eBook)
https://doi.org/10.1007/978-981-15-6577-9

This work is a co-publication with Hindustan Book Agency, New Delhi, licensed for sale in all countries in electronic form only. Sold and distributed in print across the world by Hindustan Book Agency, P-19 Green Park Extension, New Delhi 110016, India. ISBN: 978-93-80250-65-6

Jointly published with Hindustan Book Agency
ISBN of the Co-Publisher's edition: 978-93-86279-81-1

Mathematics Subject Classification: 91-XX, 91Axx, 62C20, 60-XX, 91A15, 91A10, 91A05, 91A12

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface

This set of lecture notes were prepared based on ten lectures given from September 2015 to November 2015 at Chennai Mathematical Institute, India. Topics covered include minimax theorem on unit square, a square root game, stochastic games, and product solutions for simple games. Most of the work discussed/covered in this set of lectures include those done by Parthasarathy and his collaborators.

It is next to impossible to cover all the results related to stochastic games and other topics for lack of time and space. However, we have given enough references so that interested readers can consult them.

We want to express our special thanks to the following students for their active participation during the lecture: Aditya Aradhye, Purba Das, Miheer Dewaskar, Dharini Hingu, Sanath Kumar, and Sanjukta Roy. We would like to thank Miheer Dewaskar and Dharini Hingu for providing review comments on the preliminary draft(s) of these notes. We would also like to thank Sagnik Sinha for some review comments. The second author would like to thank the Indian Institute of Technology Madras, Chennai, India, where she was a doctoral student at the Department of Management Studies when this course was being offered. Finally, we wish to thank Chennai Mathematical Institute, India, and Indian Statistical Institute, Chennai, India, for providing the facilities and atmosphere necessary and conducive for such activity.

Chennai, India

T. Parthasarathy
Sujatha Babu

Contents

About the Authors

T. Parthasarathy is Adjunct Faculty at Chennai Mathematical Institute and Indian Statistical Institute, Chennai, India. He is also Former Professor at Indian Statistical Institute, New Delhi, India. His research interests include stochastic games, linear complementarity theory, and univalence. He is an author of 3 books and more than 60 research articles published in international journals of repute. He also guided six PhD students. An elected fellow of the Indian National Science Academy, New Delhi, India, and the Indian Academy of Sciences, Bengaluru, India, Prof. Parthasarathy was awarded the Shanti Swarup Bhatnagar Prize for Science and Technology in 1986. He is also a life member of the Indian Mathematical Society and the Indian Society for Probability and Statistics.

Sujatha Babu is a visiting fellow at Chennai Mathematical Institute, Chennai, India. She completed her Ph.D. at the Indian Institute of Technology Madras, Chennai, India and M.B.A. from the Indian Institute of Management Calcutta, Kolkata, India. Her research interests include theoretical aspects of game theory, stochastic games, and evolutionary games. Furthermore, her work explores the practical applications of game theory by modeling logistics management scenarios using game theory and mechanism design constructs.

Chapter 1
Matrix Games

Consider a game played between two players where each player simultaneously selects an action to play from a given set of actions. Based on the action selected by the players, each player receives a payoff. Each player tries to either maximize or minimize their expected payoff based on the nature of the game and their role in the game. When the payoff for the game is represented as a matrix, the game is referred to as a matrix game. Matrix games between two players can be classified as zero-sum matrix games or bimatrix games based on the relation between the payoff matrices of the two players.

In this introductory chapter on matrix games, we commence with some basic definitions and theorems related to matrix games. We look at properties specific to completely mixed games, symmetric equilibrium, and orderfield property. We also look at some game classes such as games on the unit square, continuous games, games of timing, games without value, and square root game.

1.1 Zero-Sum Matrix Games

In the following section, we will state (without proof) the fundamental theorem for matrix games in game theory popularly referred to as the "Minimax theorem", and also define optimal strategy and the value of the game.

We start by first describing a two-player zero-sum matrix game.

Consider a matrix game with two players. Let player-1 and player-2 simultaneously select from a set of m and n actions, respectively. Let the payoff to the two players be represented by matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$, respectively. Let S_1 and S_2 denote the set of all actions available to player-1 and player-2, respectively. Here the payoff relating to player-1's choice of action is indicated by the rows of the payoff matrices. Similarly, the payoff relating to player-2's choice of action is indicated by the columns of the payoff matrix. The two players are sometimes referred

T. Parthasarathy and S. Babu, *Stochastic Games and Related Concepts*,
HBA Lecture Notes in Mathematics 2,
https://doi.org/10.1007/978-981-15-6577-9_1

to as the “row chooser” or “row player” and “column chooser” or “column player,” respectively.

Consider the scenario where player-1 is a row chooser and a maximizer, and player-2 is a column chooser and a minimizer. That is, player-1 tries to maximize his gains while player-2 tries to minimize his losses. If $A = -B$, then the matrix game is referred to as a zero-sum game.

Consider a two-player zero-sum matrix game with payoff matrix $(A, -A)$. The two players can either play a specific row/column, or they can select a mixed strategy by associating a probability vector with the rows/columns. Indicate the mixed strategy for player-1 by $x = (x_1, \ldots, x_m)$ where $\sum_{i=1}^{m} x_i = 1$ and $x_i \geq 0$ for all i.

Similarly, indicate the mixed strategy for player-2 by $y = (y_1, \ldots, y_n)$ where $\sum_{j=1}^{n} y_j = 1$ and $y_j \geq 0$ for all j.

The expected income for the game is given by

$$E(x, y) = \sum_i \sum_j a_{ij} x_i y_j.$$

While player-1 (the maximizer) tries to maximize the payoff that he receives for any action that he selects, player-2 tries to minimize the payoff that he pays to player-1. Similarly, while player-2 (the minimizer) tries to minimize the payoff that he pays to player-1, player-1 chooses an action that allows him to get the maximum payoff. Here *min max* $E(x, y)$ is the *upper value* U and *max min* $E(x, y)$ is the *lower value* L of the zero-sum matrix game. The *upper value* U is what player-1 can get at most and the *lower value* L is what player-1 can get at least.

It is known, in general, that

$$\min_y \max_y x E(x, y) \geq \max_x \min_y E(x, y).$$

This leads us to state the famous minimax theorem due to von Neumann. Refer to Appendix A for a constructive proof of the minimax theorem. The minimax theorem shows that the above inequality becomes an equality for matrix games.

Theorem 1.1.1 (von Neumann [100, Minimax Theorem]) *For a matrix game as described above, the value of the game v is given by*

$$v = \min_y \max_x E(x, y) = \max_x \min_y E(x, y).$$

■

Remark 1.1.1 While von Neumann [100] provided a proof of the above using fixed point results in 1928, Dantzig [13] formulated the matrix game as a linear programming problem and provided a constructive proof for the minimax theorem using the simplex method. A proof of the minimax theorem using linear programming is provided in Appendix A.

Definition 1.1.1 Pure strategy: A pure strategy $x = (x_1, \ldots, x_m)$ of a player is a single action choice of the player. That is, $x_i = 1$ for some i and $x_j = 0$ for all $j \neq i$.

Definition 1.1.2 Mixed strategy: A mixed strategy of a player given by $x = (x_1, \ldots, x_m)$ is a probability distribution over the set of actions available to the player. That is,

$$x_i \geq 0 \text{ for all } i, \text{ and } \sum_{i=1}^{m} x_i = 1.$$

Definition 1.1.3 Completely Mixed Strategy: A mixed strategy $x = (x_1, \ldots, x_m)$ is completely mixed if $x_i > 0$, for all i.

Definition 1.1.4 Saddle point: Consider a two-player zero-sum matrix game with a $m \times n$ payoff matrix A. Assume that the row player is a maximizer and the column player is a minimizer. A cell (i_0, j_0) is a pure saddle point if it is both the largest in its column and the smallest in its row. That is,

$$a_{ij_0} \leq a_{i_0 j_0} \leq a_{i_0 j} \text{ for all } i \text{ and } j.$$

Example 1.1.1 Consider the following payoff matrix for a two-player zero-sum game where player-1 is the maximizer and player-2 is the minimizer.

$$\begin{pmatrix} 2 & 8 & 1 \\ 11 & 6 & 3 \\ 9 & 7 & 6 \end{pmatrix}$$

Here cell $(3, 3)$ is a saddle point as its value is the largest in the third column and smallest in the third row. ▲

Remark 1.1.2 Not all matrix games have pure saddle points as seen in the example of the matrix $\begin{pmatrix} 4 & 2 \\ 1 & 8 \end{pmatrix}$.

Remark 1.1.3 There can be more than one pure saddle point.

Remark 1.1.4 If (i_0, j_0) is a pure saddle point, then $a_{i_0 j_0}$ is called the minmax value of the matrix game.

Exercise 1 Give an example of a matrix game that has more than one saddle point.

Exercise 2 Consider a 2×2 matrix game $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that does not have a saddle point. Show that the value of the game is given by $\dfrac{ad - bc}{a + d - b - c}$.

Exercise 3 Examine whether the following games have pure saddle points. If yes, identify the pure saddle points.

(1) $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 7 \end{pmatrix}$

(2) $\begin{pmatrix} 3 & 7 & 4 & 6 \\ 4 & 8 & 3 & 3 \end{pmatrix}$

(3) $\begin{pmatrix} 1 & 1 & 2 \\ 1 & \frac{1}{2} & 1 \end{pmatrix}$

[Hint: The last game has more than one pure saddle point.]

Exercise 4 If (i_0, j_0) is a pure saddle point, show that the minmax value of the matrix game is equal to $a_{i_0 j_0}$.

Definition 1.1.5 Row domination: Consider a $m \times n$ payoff matrix A. If $a_{ij} \geq a_{kj}$ for each $j = 1$ to n, then row i dominates row k. If $a_{ij} > a_{kj}$ for each $j = 1$ to n, then row i strictly dominates row k.

Definition 1.1.6 Column domination: Consider a $m \times n$ payoff matrix A. If $a_{ij} \geq a_{ik}$ for each $i = 1$ to m, then column j dominates column k. If $a_{ij} > a_{ik}$ for each $i = 1$ to m, then column j strictly dominates column k.

Exercise 5 If a game with a 2×3 payoff matrix has a pure saddle point, show that there is row domination or column domination (or both).

Definition 1.1.7 Optimal strategy: Consider a two-player zero-sum matrix game with payoff matrix $A \in \mathbb{R}^{m \times n}$ and value v. Then $x^* = (x_1^*, \ldots, x_m^*)$ is an optimal strategy for player-1 if $\sum_{i=1}^{m} a_{ij} x_i^* \geq v$ for all j. That is, against all possible choices of player-2, player-1 will get at least v by choosing x^*. Similarly $y^* = (y_1^*, \ldots, y_n^*)$ is an optimal strategy for player-2 if $\sum_{j=1}^{n} a_{ij} y_j^* \leq v$ for all i.

Optimal strategies may be pure, mixed, or completely mixed.

Example 1.1.2 Consider a zero-sum two-player matrix game where player-1 is the maximizer and player-2 is the minimizer. Indicate player-1's action by Up ("U") and Down ("D"), and player-2's action by Left ("L") and Right ("R"). Let the payoff matrix for this game be as follows:

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} U \\ D \end{array} & \begin{pmatrix} 1 & 0 \\ 3 & 4 \end{pmatrix}. \end{array}$$

Then the optimal strategy for player-1 is to play "D" and for player-2 is to play "L", indicated as (D, L). Here the optimal strategy (D, L) is a pure optimal strategy. ▲

Exercise 6 Let x^* and y^* indicate optimal strategy for player-1 and player-2, respectively. Let v indicate the value of the game. Then show that $\min_y E(x^*, y) = \max_x E(x, y^*) = v$. NOTE: Here, v is the value of the game.

Exercise 7 Show that the value of the matrix game is unique, but the optimal strategy need not be unique.

Exercise 8 Show that the lower value of the game is less than or equal to the upper value of the game. That is, $L \leq U$.

Exercise 9 Let the payoff matrix A be as follows:

$$A = \begin{pmatrix} 1 & -2 & -5 \\ 0 & 7 & 5 \\ -5 & 0 & 15 \end{pmatrix}$$

Given $v = \frac{5}{11}$, find optimal strategies for player-1 and player-2.

Exercise 10 Let the payoff matrix A be as follows:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & c & d \end{pmatrix}$$

where a, b, c, d are real numbers. What is the value of the game? Give an optimal strategy for player-1 and for player-2. Is it unique?

Exercise 11 Find the value and a pair of optimal strategies when the payoff matrix is as follows:

$$A = \begin{pmatrix} d & 2d & \frac{1}{3} & \dots & 2d & \frac{1}{2n+1} \\ d & \frac{1}{2} & 2d & \dots & \frac{1}{2n} & 2d \end{pmatrix}$$

where d is any real number.

Sometimes, the payoff for the players is indicated using a payoff function that is dependent on the strategies used by the players. For instance, in a two-player game, the payoff function for player-1 and player-2 is given by $u(f, g)$ and $v(f, g)$, respectively, where $f \in S_1$ and $g \in S_2$.

Definition 1.1.8 Nash equilibrium: Consider a two-player game (indicated by G) with action space S_1 and S_2 and payoff function u. A strategy profile (f^*, g^*) is a Nash equilibrium of the game G if neither player-1 nor player-2 has an incentive to deviate from f^* and g^*, respectively. That is,

$$u(f', g^*) \leq u(f^*, g^*) \leq u(f^*, g'), \text{ for each } f' \in S_1, g' \in S_2$$

Definition 1.1.9 Two-Player symmetric game: Consider a two-player game with action space S_1 and S_2 and payoff function u_1 and u_2 for each player, respectively. Such a game is symmetric if each player has the same set of actions to choose from (that is, $S_1 = S_2$). Further when player-1 and player-2 choose action i and j, respectively, then $u_1(i, j) = u_2(j, i)$.

Definition 1.1.10 Two-player zero-sum symmetric game: A two-player zero-sum game with square payoff matrix is symmetric if the payoff matrix is skew symmetric. That is, $A = -A^t$.

Example 1.1.3 Consider the classic zero-sum rock-scissors-paper game played by two players. Each player indicates whether they have selected rock (R), scissors (S), or paper (P). If both players choose the same object to display, the payoff to each player is 0. If they select different objects, then scissors beats paper, paper beats rock, and rock beats scissors. If the payoff upon winning is 1 and losing is -1, the payoff matrix can be represented as follows:

$$\begin{array}{c} \\ R \\ S \\ P \end{array} \begin{array}{c} \begin{array}{ccc} R & S & P \end{array} \\ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{array}$$

This square matrix is skew symmetric. Hence, the game is symmetric. ▲

Exercise 12 Consider two players who simultaneously choose a number between 1 and 5. If both players choose the same number, the payoff is 0. The player who selects a number one larger than the opponent's selection wins 1. The player who selects a number two or more larger than the opponent's selection loses 2. Write down the payoff matrix for this game. Specify the value of the game and optimal strategy for both players. (Hint: The game is a symmetric game.)

We now define an n-player symmetric game.

Definition 1.1.11 n-Player symmetric game: Consider a game with a finite set of players $I = \{1, 2, \ldots, n\}$, the pure strategy space S_i for player $i \in I$, and a payoff function $u_i : S \to \mathbb{R}$ for player-i where $S = S_1 \times \cdots \times S_n$. Define the permutation π on the set of players as a bijection $\pi : I \to I$. The game is symmetric if for every permutation π, we have

- $S_i = S_{\pi(i)}$ for all i
- $u_{\pi(i)}(s_1, \ldots, s_n) = u_i(s_{\pi(1)}, \ldots, s_{\pi(n)})$ for all $s \in S$ and $i \in I$.

Exercise 13 (*Moulin [62, Exercise 7.16]*) Consider a symmetric three-player game where each player has two actions (say, A and B). Let the mixed strategy of player-1, player-2, and player-3 be $(p, 1-p)$, $(q, 1-q)$, and $(r, 1-r)$, respectively. Suppose that the payoff to player-1 is given by

$$u_1(p, q, r) = pqr + 3pq + pr + qr - 2q - p$$

Find the mixed equilibrium of this game.

1.2 Bimatrix Games

We now provide a brief overview of bimatrix games in this section.

Definition 1.2.1 Bimatrix game: Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. A two-player non-zero-sum game with payoff matrices A and B is called a bimatrix game, where the payoff matrices for player-1 and player-2 are A and B, respectively.

The following is an example of a bimatrix game.

Example 1.2.1 Consider a bimatrix game where the payoffs for the two players are given by matrices A and B as follows:

$$A = \begin{array}{c} \\ U \\ D \end{array}\begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \end{array}, \quad B = \begin{array}{c} \\ U \\ D \end{array}\begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} 1 & 3 \\ 0 & 6 \end{pmatrix} \end{array}$$

Sometimes the two payoff matrices are represented as a single matrix as follows:

$$\begin{array}{c} \\ U \\ D \end{array}\begin{array}{c} \begin{array}{cc} L & \quad R \end{array} \\ \begin{pmatrix} (3,1) & (2,3) \\ (1,0) & (4,6) \end{pmatrix} \end{array}$$

Here the first entry in each cell stands for player-1's payoff and the second entry for player-2's payoff. ▲

In a bimatrix game, let player-1 and player-2 choose actions i and j, respectively. Then the payoffs to player-1 and player-2 are a_{ij} and b_{ij}, respectively. Let p_i and

q_j be the probability that player-1 (the row player) and player-2 (the column player) choose the i-th row and j-th column, respectively. Then, the expected payoff to player-1 is as follows:

$$p^t Aq = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{ij} q_j$$

The expected payoff to player-2 is as follows:

$$p^t Bq = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i b_{ij} q_j$$

Here, $p = (p_1, p_2, \ldots, p_m)$ and $q = (q_1, q_2, \ldots, q_n)$.

Definition 1.2.2 Nash equilibrium: Given a bimatrix game (A, B) where both players are maximizers, (x^o, y^o) is a Nash equilibrium strategy pair if

$$x^t Ay^o \leq x^{ot} Ay^o, \text{ for all strategies } x \text{ of player-1},$$

and

$$x^{ot} By \leq x^{ot} By^o, \text{ for all strategies } y \text{ of player-2}.$$

Here, $x^{ot} Ay^o$ and $x^{ot} By^o$ are Nash equilibrium payoffs of player-1 and player-2, respectively, corresponding to the Nash equilibrium strategy pair (x^o, y^o).

Furthermore, the equilibrium payoffs may not be unique across Nash equilibrium strategies.

Example 1.2.2 Consider the bimatrix game in Example 1.2.1. Let both players be maximizers. In this game, the equilibrium is attained when player-1 and player-2 play action “D” and “R”, respectively. This Nash equilibrium strategy pair is indicated by $(x^o, y^o) = ((0, 1), (0, 1))$. The Nash equilibrium payoffs to player-1 and player-2 are 4 and 6, respectively. ▲

Definition 1.2.3 Pure Nash equilibrium: The Nash equilibrium (x^o, y^o) of a bimatrix game is a pure equilibrium if both x^o and y^o are pure strategies for player-1 and player-2, respectively.

Example 1.2.3 Consider the following bimatrix game where both players are maximizers:

$$A = \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} (6, 4) & (0, 1) \\ (1, 2) & (3, 5) \end{pmatrix} \end{array}$$

This bimatrix game has two pure Nash equilibria given by (U, L) and (D, R). The equilibrium payoffs are not unique across the Nash equilibrium strategies in this example. ▲

Definition 1.2.4 Mixed Nash equilibrium: The Nash equilibrium (x^o, y^o) of a bimatrix game is a mixed equilibrium if both x^o and y^o are mixed strategies for player-1 and player-2, respectively.

Example 1.2.4 Consider the following 2×2 bimatrix game with no pure Nash equilibrium.

$$A = \begin{array}{c} \\ U \\ D \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{pmatrix} (2,2) & (3,3) \\ (0,3) & (4,1) \end{pmatrix} \end{array}.$$

Let player-1 choose strategy U and D with probability p and $1-p$ where $0 \leq p \leq 1$. Similarly, let player-2 choose strategy L and R with probability q and $1-q$ where $0 \leq q \leq 1$. Then the payoff to the two players indicated by π_1 and π_2, respectively, is given by

$$\pi_1(p,q) = 2pq + 3p(1-q) + 0 + 4(1-p)(1-q)$$
$$\pi_2(p,q) = 2pq + 3p(1-q) + 3(1-p)q + (1-p)(1-q)$$

Consider the system of equations

$$\begin{aligned} \frac{\delta \pi_1}{\delta p} &= 0 \\ \frac{\delta \pi_2}{\delta q} &= 0 \end{aligned} \tag{1.1}$$

Any solution to the system of equations in Eq. 1.1 gives the mixed strategy equilibrium for the game. In this example, we get $p = \frac{2}{3}$ and $q = \frac{1}{3}$.

Thus, $((\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3}))$ is the mixed strategy Nash equilibrium for the bimatrix game. ▲

Definition 1.2.5 Interchangeable Nash equilibrium: Nash equilibrium pairs (x^*, y^*) and (x^0, y^0) are said to be interchangeable if (x^*, y^0) and (x^0, y^*) are also equilibrium pairs.

Example 1.2.5 Consider the following 3×3 bimatrix game:

$$\begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \end{array} \begin{array}{c} \begin{array}{ccc} b_1 & b_2 & b_3 \end{array} \\ \begin{pmatrix} (3,7) & (2,3) & (3,7) \\ (1,2) & (4,5) & (1,1) \\ (3,7) & (0,0) & (3,7) \end{pmatrix} \end{array}$$

The four pure equilibria (a_1, b_1), (a_3, b_1), (a_1, b_3), and (a_3, b_3) are all interchangeable Nash equilibrium. ▲

Exercise 14 Consider a bimatrix game where each player has two actions. Let (x^*, y^*) and (x^0, y^0) be two equilibrium pairs. Give an example where the equilibrium pairs are not interchangeable.

Exercise 15 (difficulty = 2) (*Chin et al. [11]*) Consider a two-player mixed game G where player-1 and player-2 have m and n actions each. Show that the Nash equilibrium pairs of G are interchangeable if and only if the set of all Nash equilibria of G is a convex subset of $\mathbb{R}^m \times \mathbb{R}^n$.

Exercise 16 Consider the following bimatrix game. Let player-1 have three actions —Top *(T)*, Middle *(M)*, and Bottom *(B)*. Let player-2 have three actions—Left *(L)*, Middle *(M)*, and Right *(R)*. The payoff matrix is as follows:

$$\begin{array}{c} \\ T \\ M \\ B \end{array} \begin{array}{c} \begin{array}{ccc} L & M & R \end{array} \\ \begin{pmatrix} (4,5) & (5,3) & (7,4) \\ (2,3) & (5,3) & (3,3) \\ (8,2) & (5,1) & (4,3) \end{pmatrix} \end{array}.$$

Find the Nash equilibrium of this game if the dominated strategies are not eliminated. Find the Nash equilibrium if the dominated strategies are eliminated. Explain the reason why this difference arises. (Hint: Check if the dominated strategies are strictly dominated or not.)

Exercise 17 Consider a first price sealed bid auction game with two players bidding for an object. Let the value of the object to player-1 and player-2 be a_1 and a_2, respectively. Suppose that $a_1 > a_2$. Each player simultaneously submits their bid (say) b_1 and b_2, respectively. The highest bidder wins the object and pays his bid amount as the price. In case of a tie, assume that the first player wins the bid. The first price auction game can be formulated as a strategic game. Formulate the game as a strategic game. Show that the Nash equilibria for this game is such that player-1 gets the object at a price p where $a_1 \geq p \geq a_2$. Further show that $p \in [a_1, a_2]$.

Remark 1.2.1 Exercise 17 can be easily extended to the scenario of n-players bidding for the object in a first price auction.

1.3 Completely Mixed Matrix Games

Here are some results pertaining to completely mixed matrix games (both zero-sum and bimatrix games).

Definition 1.3.1 Completely mixed matrix Game: The matrix game is completely mixed if every optimal strategy for either player is completely mixed.

Exercise 18 Show that the following games are completely mixed:

$$\text{a.} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \qquad \text{b.} \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

In the following example, we show that the presence of row or column domination in the payoff matrices results in the game not being completely mixed.

Example 1.3.1 Consider the matrix game associated with the following payoff matrix:

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

The matrix game is not completely mixed due to the presence of row (and column) domination. ▲

Exercise 19 Consider a matrix game with payoff matrix A and optimal strategy (p^*, q^*). If there exists row domination, show that the matrix game is not completely mixed. (Hint: Consider the case where row 1 is strictly dominated by a convex combination of the remaining rows. Show that $p_1^* = 0$ in this case.)

Exercise 20 Let $A = \begin{pmatrix} 5 & 4 \\ 10 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 10 \\ 4 & 3 \end{pmatrix}$. Assume that both players try to minimize their expected payoffs. Compute all equilibrium points for this bimatrix game.

For a completely mixed two-player zero-sum game, Kaplansky [34] provided the necessary and sufficient condition for the game to be completely mixed (and specifically for a symmetric game with payoff matrix of order 5). Before we discuss the results of Kaplansky, we recall the definition of two terms relating to matrices, namely, minor and cofactor of a cell in a matrix.

Definition 1.3.2 Minor of a cell in a matrix: The minor of the cell a_{ij} of a $m \times n$ matrix A is the determinant of the submatrix of A formed by deleting the i-th row and j-th column of A. It is indicated by M_{ij}.

Definition 1.3.3 Cofactor of a cell in a matrix: The cofactor of cell a_{ij} of a $m \times n$ matrix A is given by $(-1)^{i+j} M_{ij}$ where M_{ij} is the minor of a_{ij}. The cofactor is indicated by C_{ij}.

Here are a few classic results of Kaplansky on completely mixed matrix games, presented without proof.

Theorem 1.3.1 (Kaplansky [34, Theorem 5]) *A game with payoff matrix A and value* 0 *is completely mixed if and only if*

1. *Its matrix is square ($m = n$) and has rank $n - 1$.*
2. *All cofactors are different from zero and have the same sign.*

■

Theorem 1.3.2 (Kaplansky [34, Theorem 6]) *The value v of a completely mixed matrix game with payoff matrix A is given by*

$$v = \frac{det(A)}{\sum_{i,j} A_{ij}}$$

where A_{ij} is the cofactor of a_{ij}. The denominator is always different from zero. Also, if $v \neq 0$, then $\det A \neq 0$. ■

Exercise 21 If the given matrix game A is completely mixed, show that $\sum_i \sum_j A_{ij} \neq 0$.

Exercise 22 Consider the classic zero-sum rock-scissors-paper game as described in Example 1.1.3. The payoff matrix is as follows:

$$\begin{array}{c c} & \begin{array}{ccc} R & S & P \end{array} \\ \begin{array}{c} R \\ S \\ P \end{array} & \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{array}$$

Confirm the validity of Theorem 1.3.1.

Exercise 23 (*Moulin [62, Exercise 7.8]*) Give an example of a 2×2 zero-sum matrix game where player-1 has a completely mixed strategy x^* such that the expected payoff to player-1 is independent of player-2's strategy though x^* is not optimal in the mixed game. Is the same true when player-1 has pure strategies only?

Exercise 24 (*Kaplansky [34, Sect. 4]*) Consider a two-player zero-sum game with a skew-symmetric payoff matrix A of even order n. Show that the game can never be completely mixed.

Kaplansky [35] later extended the above result to odd-ordered skew-symmetric payoff matrices using the notion of a Pfaffian and provided the necessary and sufficient condition for the game to be completely mixed.

Let us first define the notion of a Pfaffian. Consider a skew-symmetric matrix A of order $2n \times 2n$. The determinant of A can be expressed as the square of a polynomial in terms of the matrix entries. The square root of the determinant (with a convention for the sign) is referred to as the Pfaffian of matrix A and indicated by $Pf(A) = \pm\sqrt{det(A)}$ (with a convention for the sign). Trivially, the Pfaffian of an odd-ordered skew-symmetric odd order matrix is 0. The following is a formal definition of a Pfaffian.

Definition 1.3.4 Pfaffian of a Matrix: Consider a skew-symmetric matrix A of order $2n \times 2n$ given by

$$A = \begin{pmatrix} 0 & a_{1,2} & \dots & a_{1,2n} \\ -a_{2,1} & 0 & \dots & a_{2,2n} \\ \vdots & \vdots & & \vdots \\ -a_{2n,1} & -a_{2n,2} & \dots & 0 \end{pmatrix}$$

Consider the set of all permutations π on the set $\{1, 2, \dots, 2n\}$ of the form

$$\pi = \begin{bmatrix} 1 & 2 & 3 & 4 & \dots & 2n \\ i_1 & j_1 & i_2 & j_2 & \dots & j_n \end{bmatrix}$$

for $i_1 < i_2 < \dots < i_n$ and $i_k < j_k$. Let $sgn(\pi)$ indicate the sign of the permutation. Define the sign of the permutation to be 1 if the number of inversions in π is even and -1 otherwise. The Pfaffian of the $2n \times 2n$ skew-symmetric matrix A is defined to be

$$Pf(A) = \sum_{\pi} sgn(\pi) a_{i_1,j_1} a_{i_2,j_2} \dots a_{i_n,j_n}$$

Example 1.3.2 Consider the 2×2 skew-symmetric matrix given by $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$. Then the determinant of A can be expressed as a polynomial in terms of its cell entries as $\det A = a^2$. This polynomial is referred to as the Pfaffian. In our example of the 2×2 skew-symmetric matrix A, the Pfaffian of A is a. Thus, the Pfaffian of an even-ordered skew-symmetric matrix is given by the square root of the determinant, with a convention for the sign. ▲

Example 1.3.3 Consider the 3×3 skew-symmetric matrix given by

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

Then the determinant of A is $\det A = 0$. Thus, $pf(A) = 0$. ▲

Example 1.3.4 Consider the 4×4 skew-symmetric matrix given by

$$A = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{pmatrix}$$

Then the determinant of A is $\det A = (af - be + dc)^2$. Thus, $pf(A) = af - be + dc$. ▲

Definition 1.3.5 Principal Pfaffian of a matrix: Let A be an odd-ordered skew-symmetric matrix. The Pfaffian of the new matrix obtained by deleting the i-th row and i-th column of A is called the i-th principal Pfaffian p_i of A. Here, $p_i = \sqrt{A_{ii}}$ and is well defined as $A_{ii} \geq 0$.

Example 1.3.5 Consider the 3×3 skew-symmetric matrix given by

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

Then the determinant of A is $\det A = 0$. Thus, $pf(A) = 0$.

Deleting the first row and first column yields the following submatrix whose determinant is c^2.

$$\begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$$

Thus, the first principal Pfaffian of A is given by $p_1 = c$.

Deleting the second row and second column yields the following submatrix whose determinant is b^2.

$$\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}.$$

Thus, the second principal Pfaffian of A is given by $p_2 = b$. Along the same lines, the third principal Pfaffian of A is given by $p_3 = a$. ▲

Theorem 1.3.3 (Kaplansky [35, Theorem 1]) *Let A be a real $n \times n$ skew-symmetric matrix, n odd. Let G be the two-player zero-sum game attached to A. Then G is completely mixed if and only if the principal Pfaffians $p_1, p_2, \ldots, p_n$ are all non-zero and alternate in sign. In that case, the unique good strategy for each player is proportional to $p_1, -p_2, p_3, -p_4, \ldots$.* ■

Lastly, we will look at the mixed extension of a two-player matrix game where the game notations are as follows. Consider a two-player bimatrix game with strategy sets S_1 and S_2 and payoff functions $u_1(s_1, s_2)$ and $u_2(s_1, s_2)$ for player-1 and player-2, respectively, where $s_1 \in S_1$ and $s_2 \in S_2$. Denote the mixed strategies of the players by $\sigma_i \in \Delta S_i$ for $i = 1, 2$. A mixed strategy profile is given by $\sigma \equiv \Pi_{i \in 1,2} \Delta S_i$.

Definition 1.3.6 Mixed extension of a matrix game: Consider a two-player game G as defined above. The mixed extension of this game is again a two-player game G_m with strategy sets ΔS_1 and ΔS_2 and payoff functions U_1 and U_2, where for all $\sigma \equiv \Pi_{i \in 1,2} \Delta S_i$,

$$U_1(\sigma) = \sum_{s\equiv(s_1,s_2)\in S} u_i(s)\sigma_1(s_1)\sigma_2(s_2)$$
$$U_2(\sigma) = \sum_{s\equiv(s_1,s_2)\in S} u_2(s)\sigma_1(s_1)\sigma_2(s_2)$$

So in the mixed extension G_m of the matrix game G, player-i's strategy is a probability distribution σ_i over S_i for $i = 1, 2$. We demonstrate this with a simple example.

Example 1.3.6 Consider a two-player matrix game where player-1 has two actions—T and B, player-2 has two actions—L and R. Let the payoff matrix be given by

$$\begin{array}{cc} & \begin{array}{cc} L & \quad R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} (3,1) & (0,0) \\ (0,0) & (1,3) \end{pmatrix} \end{array}$$

Suppose that player-1 selects action T and action B with probability $\frac{3}{4}$ and $\frac{1}{4}$, respectively. Let player-2 select action L and action R with probabilities $\frac{1}{4}$ and $\frac{3}{4}$, respectively. Then the mixed strategy profile is given by

$$\begin{aligned}\sigma &\equiv (\sigma_1, \sigma_2)\\ &= ((\sigma_1(T), \sigma_1(B)), (\sigma_2(L), \sigma_2(R)))\\ &= ((\frac{3}{4}, \frac{1}{4}), (\frac{1}{4}, \frac{3}{4})).\end{aligned}$$

The payoffs for the mixed extension of the matrix game are as follows:

$$\begin{aligned}U_1(\sigma) &= u_1(T,L)\sigma_1(T)\sigma_2(L) + u_1(T,R)\sigma_1(T)\sigma_2(R)\\ &\quad + u_1(B,L)\sigma_1(B)\sigma_2(L) + u_1(B,R)\sigma_1(B)\sigma_2(R)\\ &= 3 * \frac{3}{4} * \frac{1}{4} + 0 + 0 + 1 * \frac{1}{4} * \frac{3}{4} = \frac{3}{4}.\end{aligned}$$

Along the same lines, we get $U_2(\sigma) = \dfrac{3}{4}$. ▲

Exercise 25 (*Moulin [62, Exercise 7.10]*) Consider the finite two-player zero-sum game G and its mixed extension G_m which is made only of the optimal strategies of G. If the original game G has optimal pure strategies, show that any mixed strategy in G_m is optimal in G_m. Show via an example where the new game G_m may have more optimal strategies.

1.4 Completely Mixed Bimatrix Games

Definition 1.4.1 Completely Mixed Bimatrix Game: A bimatrix game is completely mixed if every equilibrium is completely mixed.

The following is an extension of Theorem 1.3.1 to bimatrix games.

Theorem 1.4.1 (Raghavan [78, Theorem 4]) *Let the bimatrix game (A, B) be completely mixed. Then the matrices A and B are square matrices and the bimatrix game has only one equilibrium strategy.* ■

Raghavan [78] extended Kaplansky's [34] results to non-zero-sum bimatrix games. Sujatha et al. [97] showed the necessary and sufficient conditions for a bimatrix game with odd-ordered skew-symmetric payoff matrices to be completely mixed. Oviedo [66] shows the conditions under which the set of all equilibrium strategies is completely mixed in a bimatrix game. The results by Oviedo [66] that are used in the proof of Sujatha et al. [97] are as follows.

Theorem 1.4.2 (Oviedo [66, Theorem 1]) *In a bimatrix game (A, B), let ε be the set of all pairs of equilibrium strategies. If ε is completely mixed and $x_0^t A y_0 = x_0^t B y_0 = 0$, then there exists an i $(1 \le i \le n)$ such that Cofactor(A_{i1}), Cofactor(A_{i2}), ..., Cofactor(A_{in}) are different from zero and have the same sign. Similarly, there exists a j $(1 \le j \le n)$ such that Cofactor(B_{j1}), Cofactor(B_{j2}), ..., Cofactor(B_{jn}) are different from zero and have the same sign.* ■

Theorem 1.4.3 (Oviedo [66, Corollary 1]) *If ε is completely mixed and $(x_0, y_0) \in \varepsilon$, then*

$$x_0^t A y_0 = \frac{det(A)}{\sum_{i,j} A_{ij}}, \textit{ and}$$

$$x_0^t B y_0 = \frac{det(B)}{\sum_{i,j} B_{ij}},$$

where the denominators are always different from 0. ■

Theorem 1.4.4 (Oviedo [66, Proposition 1]) *Suppose there exists constants v_1 and v_2 such that for any $(x, y) \in \varepsilon$, $Ay = v_1 e$ and $x^t B = v_2 e^t$. Suppose, moreover, that both A and B are square matrices of rank $n - 1$. Then the bimatrix game (A, B) is completely mixed.* ■

Exercise 26 Consider a bimatrix game with skew-symmetric payoff matrices A and B, both of even order n. Show that the bimatrix game can never be completely mixed.

Using the above results of Oviedo [66] and Kaplansky ([34, 35]), we now state and prove the result of Sujatha et al. [97] on completely mixed bimatrix game with odd-ordered skew-symmetric payoff matrices.

Theorem 1.4.5 (Sujatha et al. [97, Theorem 3]) *Consider a bimatrix game (A, B) with odd-ordered skew-symmetric payoff matrices. Let* ε *be the set of all equilibrium points. For every* $(x, y) \in \varepsilon$*, let there exist* v_1 *and* v_2 *such that* $Ay = v_1 e$ *and* $x^t B = v_2 e^t$*, where e is the* $n \times 1$ *column vector with every element equal to 1. Then the game is completely mixed if and only if for some i, cofactors* $A_{i1}, A_{i2}, \ldots, A_{in}$ *and for some* j*, cofactors* $B_{j1}, B_{j2}, \ldots, B_{jn}$ *are all non-zero and are of the same sign. That is, the principal Pfaffians of both payoff matrices are all non-zero and alternate in sign. Let* $p_1, -p_2, \ldots, p_n$ *and* $q_1, -q_2, \ldots, q_n$ *be the principal Pfaffians of matrices A and B, respectively. In that case,* $(p_1, -p_2, \ldots, p_n)$ *and* $(q_1, -q_2, \ldots, q_n)$ *will be proportional to the unique equilibrium pair.*

Proof Let the set of all equilibrium strategies ε be completely mixed. Let x and y be the strategies used by player-1 and player-2, respectively. Since the matrices are odd-ordered skew symmetric, $det(A) = det(B) = 0$. Then, $x^t Ay = x^t By = 0$ by Oviedo [66, Corollary 1].

Further by Oviedo [66, Theorem 1] and without loss of generality, we can assume that Cofactor$(A_{ij}) > 0$ for all i and for all j. Let m_{ij} is the sub-determinant obtained by deleting the ith row and jth column of the matrix A. Then, $(-1)^{i+j} m_{ij} > 0$. By Kaplansky [35, Theorem 1], this implies that $(-1)^{i+j} p_i p_j > 0$ where p_i and p_j are the ith and jth principal Pfaffians of A.

If i and j are both even or both odd, then the above equation implies that either both p_i and p_j are greater than 0 or both are lesser than 0. If i is even and j is odd or vice versa, then either ($p_i > 0$, $p_j < 0$) or ($p_i < 0$, $p_j > 0$). Thus, all the principal Pfaffians of A are non-zero and alternate in sign. The same holds for all the principal Pfaffians of matrix B.

Conversely, let the principal Pfaffians of matrix A be non-zero and alternate in sign. Without loss of generality, we can assume that $p_i > 0$ where i is odd and $p_j < 0$ where j is even.

Then, $(-1)^{i+j} p_i p_j > 0$. That is, $(-1)^{i+j} m_{ij} > 0$, where m_{ij} is the sub-determinant obtained by deleting the ith row and jth column of matrix A. This implies that $c_{ij} > 0$, where c_{ij} is the cofactor of a_{ij}.

Hence, all cofactors of A (and similarly of B) are non-zero and have the same sign. The rank of both these matrices is $n - 1$ since the minors of order $n - 1$ are non-zero. Consider a strategy $(x_0, y_0) \in \varepsilon$. Then there exists v_1 and v_2 such that $Ay_0 = v_1 e$ and $(x_0)^t B = v_2 e^t$. By Oviedo [66, Proposition 1], ε is completely mixed. ■

Exercise 27 For the conditions detailed in Theorem 1.4.5, show that v_1 and v_2 are unique. Also, the proof of the theorem assumes that Cofactor$(A_{ij}) > 0$ for all i, for all j. Why is this assumption valid?

1.5 Symmetric Equilibrium

In two-player games, the equilibrium strategy for both players may be different. When they are the same for both players, the equilibrium is termed as symmetric equilibrium. Symmetric equilibrium is of importance in games where players have the same set of actions, such as single population games in evolutionary game theory [31]. We define the notion of symmetric equilibrium formally in this section.

Definition 1.5.1 Symmetric equilibrium in matrix games: Consider a matrix game with payoff matrix $A \in \mathbb{R}^{n \times n}$. We say that (x^*, x^*) is a symmetric equilibrium if the same strategy x^* is optimal for both players.

Exercise 28 Consider the classic rock-scissors-paper game given in Exercise 22. As the game is symmetric, it has a symmetric optimal strategy. Find the symmetric optimal strategy for this game.

Bimatrix games are of great interest in two population games in evolutionary game theory. We now look at the definition of symmetric equilibrium in bimatrix games.

Definition 1.5.2 Symmetric equilibrium in bimatrix games: Consider a bimatrix game (A, B) where $A, B \in \mathbb{R}^{n \times n}$. We say that (x^*, x^*) is a symmetric equilibrium if

$$x^{*t} A x^* \geq x^t A x^* \ for \ all x$$

and

$$x^{*t} B x^* \geq x^{*t} B y \ for \ all \ y.$$

Exercise 29 Show that the bimatrix game (A, A^t) always has a symmetric equilibrium.

Theorem 1.5.1 (Gale [22]) *For a finite zero-sum game with a skew-symmetric matrix, the value of the game is 0 and any strategy that is optimal for one player is also optimal for the other player.* ■

Exercise 30 (difficulty = 2) Prove von Neumann's minimax theorem using Gale's theorem (Theorem 1.5.1). [Hint: Refer to Gale [22] on how to convert a zero-sum matrix game into a skew-symmetric matrix game.]

Exercise 31 (difficulty = 1) Let $U = [-1, 1]$, $V = \{-1, 1\}$. Let $g(u, v) = uv$. Examine whether $\min_v \max_u g(u, v) = \max_u \min_v g(u, v)$. [Hint: Here U is a convex set whereas V is not. So, the result may or may not hold.]

Exercise 32 Consider the infinite two-player game where the payoff matrix is given by

$$H(x, y) = \begin{cases} 1, & \text{if } x > y, \\ 0, & \text{if } x = y, \\ -1, & \text{if } x < y \end{cases}$$

where $x = 1, 2, \ldots$ and $y = 1, 2, \ldots$. Does the minimax theorem hold in this case? Specifically, show that $\inf \sup K = 1 > \sup \inf K = -1$.

1.6 Shapley-Snow Theorem

Shapley-Snow theorem [88] provides a characterization of the extreme optimal solution of matrix games. It shows that the set of optimal strategies of either player is a bounded polyhedral set, i.e., a finite intersection of closed half spaces. Though a similar concept exists in linear programming, the difference is that the solution space may be unbounded in linear programming while it is bounded here as we are considering probability vectors. Thus, any arbitrary $m \times n$ matrix game A whose value is not zero may be solved by selecting an appropriate square submatrix B and checking that the optimal strategies for B are optimal for the original matrix A.

Definition 1.6.1 Half spaces: Let $u \in \mathbb{R}^l$ be a non-zero vector and α be any scalar. The non-trivial hyperplane $u^t z = \alpha$ divides $\mathbb{R}^l$ into two half spaces, namely, $\{z \in \mathbb{R}^l : u^t z \leq \alpha\}$ and $\{z \in \mathbb{R}^l : u^t z \geq \alpha\}$.

Definition 1.6.2 Extreme optimal strategy: An optimal strategy is extreme if it is not in the interior of the line segment joining two other optimal strategies.

Theorem 1.6.1 (Shapley-Snow Theorem [88, Theorem 2]) *Let $A \in \mathbb{R}^{m \times n}$ and $v \neq 0$. Let the optimal strategies for player-1 and player-2 be $x^0 = (x_1^0, \ldots, x_p^0, 0, \ldots, 0)$ and $y^0 = (y_1^0, \ldots, y_q^0, 0, \ldots, 0)$, respectively. Let $I, J \subseteq \{1, \ldots, n\}$ where $I \supseteq \{1, \ldots, p\}$ and $J \supseteq \{1, \ldots, q\}$. Then $(x^0, y^0) \in \mathbb{R}^m \times \mathbb{R}^n$ form an extreme optimal pair if and only if there exists a non-singular submatrix $B = (a_{ij})_{i \in I, j \in J}$ such that*

$$\begin{aligned} \sum_{i \in I} a_{ij} x_i^0 &= v \; for \; all \; j \in J, \\ \sum_{j \in J} a_{ij} y_j^0 &= v \; for \; all \; i \in I, \\ x_i^0 &= 0 \; if \; i \notin I, and \\ y_j^0 &= 0 \; if \; j \notin J \end{aligned}$$

Proof Let (x^0, y^0) form an extreme optimal pair. Without loss of generality, let $v \neq 0$. If $v = 0$, it is possible to add a positive quantity to A without causing a change in the optimal strategy set.

Let

$$\sum_{i \in I} a_{ij} x_i^0 = v for j = 1, 2, \ldots, q, \ldots, \bar{q}$$

and

$$\sum_{j \in J} a_{ij} y_j^0 = v for i = 1, 2, \ldots, p, \ldots, \bar{p}.$$

Let us restrict ourselves to the matrix $C \in \mathbb{R}^{\bar{p} \times \bar{q}}$ given by

$$C = \begin{pmatrix} a_{11} & \cdots & a_{1q} & \cdots & a_{1\bar{q}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{p1} & \cdots & a_{pq} & \cdots & a_{p\bar{q}} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{\bar{p}1} & \cdots & a_{\bar{p}q} & \cdots & a_{\bar{p}\bar{q}} \end{pmatrix}$$

In order to find a submatrix B of C that satisfies the theorem, we will first show that p rows of C are linearly independent, along the lines of the proof for linear programming.

If possible, let the first p rows not be linearly independent. Hence, we can consider a non-zero vector $(\tau_1, \ldots, \tau_p, 0, \ldots, 0)$ where $\sum_{i=1}^{\bar{p}} a_{ij} \tau_i = 0$ for all j and at least one $\tau_i \neq 0$.

As (x^0, y^0) are optimal strategies, $\sum_{j=1}^{n} a_{ij} y_j^0 = v$ for all $i = 1, \ldots, \bar{p}$. Thus,

$$\sum_{i=1}^{\bar{p}} \tau_i \sum_{j=1}^{n} a_{ij} y_j^0 = v \sum_{i=1}^{\bar{p}} \tau_i = 0$$

Hence,

$$\sum \tau_i = 0 \text{ since } \sum_{i=1}^{\bar{p}} a_{ij} \tau_i = 0 \text{ and } v \neq 0$$

Construct the vector $x^0 \pm \epsilon\tau$. Then $\sum a_{ij} x_i^0 \pm \epsilon \sum a_{ij} \tau_i = v \pm \epsilon \times 0 = v$ for all $j = 1, \ldots, \bar{q}$. For sufficiently small ϵ, we can treat $x^0 \pm \epsilon\tau$ as a probability vector since $\sum \tau_i = 0$ and x^0 is a probability vector. It is possible that $a_{ij} x_i^0 > v$ for all $j = \bar{q} + 1, \ldots, n$, i.e., we can always find a sufficiently small ϵ such that $a_{ij} x_i^0 \pm \epsilon \sum a_{ij} \tau_i^0 > v$. Hence, we have produced two optimal strategies $x^0 - \epsilon\tau$ and $x^0 + \epsilon\tau$ whose midpoint x^0 is also an optimal. This contradicts the fact that x^0 is an extreme optimal pair. Hence, the first p rows are linearly independent. Similarly, the first q columns are linearly independent.

Thus, $\text{Rank}(C) \geq \max\{p, q\}$. So there is always a non-singular matrix B of C that satisfies $\sum_{i \in I} a_{ij} x_i^0 = v$ and $\sum_{j \in J} a_{ij} y_j^0 = v$.

The converse can be proved similarly. ■

1.7 Some Game Classes

In this section, we look at some special game classes, namely, games without value, games on the unit square, continuous games, games of timing, and the square root game. Important results relating to these game classes are also presented.

1.7.1 Games Without Value

Sion and Wolfe [93] provide an example along the lines of Shiffman [90] to show that the value of a game does not exist when there are discontinuities. The example they provide is a game on the unit square. A game on the unit square is a two-player game where the strategy set for each player is from the interval [0, 1]. Another example of a game on the unit square is the square root game described in Sect. 1.7.5. It is interesting to note that von Neumann's result holds under certain conditions for such games. Additional details for games on the unit square are provided in Sect. 1.7.2.

The following is an example of a game that does not have a value when there are discontinuities.

Example 1.7.1 (*Sion and Wolfe [93, Sect. 2]*) Consider a two-player zero-sum game where player-1 and player-2 choose numbers $x \in [0, 1]$ and $y \in [0, 1]$, respectively. Let $\phi(x)$ be any continuous function. Let $K(x, y)$ be the payoff function defined as follows:

$$K(x, y) = \begin{cases} -1, & \text{if } x < y < \phi(x), \\ 0, & \text{if } x = y \text{ or } y = \phi(x), \\ 1, & \text{otherwise.} \end{cases}$$

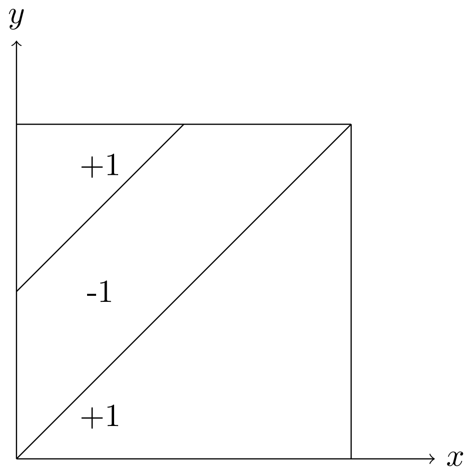
Consider the case when $\phi(x) = x + \frac{1}{2}$. If (x, y) is interpreted as a point on the unit square (Fig. 1.1), the function $K(x, y)$ is continuous except at $y = x$ or $y = x + \frac{1}{2}$.

Let each player play a mixed strategy by using the probability distribution μ and λ, respectively. Given any μ, it is possible to select a suitable y such that $\int K(x, y) d\mu(x) \leq \frac{1}{3}$. Selecting a distribution μ concentrated at three points (namely, 0, $\frac{1}{2}$, and 1) as $\frac{1}{3}$ each yields $\frac{1}{3}\Big[K(0, y) + K(\frac{1}{2}, y) + K(1, y)\Big]$ which is greater than $\frac{1}{3}$ for all y.

Sion and Wolfe showed that $\sup\limits_{\mu} \inf\limits_{\lambda} \iint K(x, y) = \frac{1}{3}$ and $\inf\limits_{\lambda} \sup\limits_{\mu} \iint K(x, y) = \frac{3}{7}$ and they are not equal. Thus, a game with discontinuities does not have a value even if both the players play a mixed strategy. ▲

Kakutani showed that the fixed point theorem applied in a Euclidean n-space implied the minimax theorem for finite games. Glicksberg [27] showed that Kakutani's fixed

Fig. 1.1 Game square for a game without value



point theorem may be extended to convex linear topological spaces, and implies the minimax theorem for continuous games (refer to Definition 1.7.2) with continuous payoffs. He also showed the existence of Nash equilibrium points. In particular, Glicksberg showed that if $K(x, y)$ is continuous or upper/lower semicontinuous, then the game has a minmax value. In Example 1.7.1, the payoff function is not semicontinuous and hence does not have a value.

The following example given in Parthasarathy and Raghavan [75] is a variation of the classic continuous Blotto game due to Shapley.

Example 1.7.2 (*Parthasarathy and Raghavan [75, Sect. 5.4]*) Consider a game played between two players who need to decide what percentage of force be assigned to attack two passes. Player-1 can assign force x to attack the first pass and $1 - x$ to attack the second pass. Player-2 can assign a force y to defend the first pass and $1 - y$ for the second pass. In addition, there is an extra stationary force of $\frac{1}{2}$ assigned to the second pass. A player receives a payment of 1 unit from the other player if his force at a pass exceeds his opponent's. He receives nothing if the forces are equally deployed. That is,

$$B(x, y) = sgn(x - y) + sgn((1 - x) - (\frac{3}{2} - y)).$$

Here, we define $sgn(x - y)$ as follows:

$$sgn(x - y) = \begin{cases} 1, & \text{if } x > y, \\ 0, & \text{if } x = y, \\ -1, & \text{if } x < y \end{cases}$$

We can similarly define $sgn((1-x)-(\frac{3}{2}-y))$. Then we have $K(x,y) = 1 + B(x,y)$. ▲

1.7.2 Game on the Unit Square

Definition 1.7.1 Game on the unit square: A two-player game where the strategy set for each player is from the interval [0, 1] is referred to as a game on the unit square.

Example 1.7.3 (*Maschler et al. 2013 [53, Example 4.14.1]*) Consider a two-player zero-sum game on the unit square, i.e., the strategy set for each player is from the interval [0, 1]. Define the payoff function as $u(x, y) = 4xy - 2x - y + 3$, for all $x \in [0, 1]$, for all $y \in [0, 1]$. Then, for each x

$$\begin{aligned}\min_{y\in[0,1]} u(x,y) &= \min_{y\in[0,1]} (4xy - 2x - y + 3)\\ &= \min_{y\in[0,1]} (y(4x-1) - 2x + 3)\end{aligned}$$

The point at which the minimum is attained is determined by the slope $4x - 1$. Hence, the minimum is attained at $y = 0$ when the slope is positive; at $y = 1$, when the slope is negative; and constant in y when the slope is 0. Note that

$$\min_{y\in[0,1]} u(x,y) = \begin{cases} 2x+2, & \text{if } x \le \frac{1}{4},\\ -2x+3, & \text{if } x \ge \frac{1}{4}.\end{cases}$$

The above attains a unique maximum at $x = \frac{1}{4}$ with value $2\frac{1}{2}$. Hence,

$$\max_{x\in[0,1]} \min_{y\in[0,1]} u(x,y) = 2\frac{1}{2}$$

Similarly, for each y, the function attains a unique minimum at $y = \frac{1}{2}$ with value $2\frac{1}{2}$. Hence,

$$\min_{y\in[0,1]} \max_{x\in[0,1]} u(x,y) = 2\frac{1}{2}$$

This shows that the above game on the unit square has a unique pure optimal strategy where player-1 chooses $\frac{1}{4}$ and player-2 chooses $\frac{1}{2}$ with probability 1. Also the game has value $2\frac{1}{2}$. ▲

As shown in Sect. 1.7.1 earlier, Sion and Wolfe [93] give an example of a game on the unit square that does not have a value when there are discontinuities. Parthasarathy [72] extends Wald's result [103] to prove that a particular class of games does have a value if the payoffs are continuous except along certain curves, provided that at least one of the players is forbidden to use a pure strategy, i.e., the player uses strategies that are restricted to absolutely continuous distributions with respect to the Lebesgue measure.

Theorem 1.7.1 (Parthasarathy [72, Theorem 1]) *Let $K(x, y)$ be bounded on the unit square $0 \le x, y \le 1$. Let all points of discontinuity of $K(x, y)$ lie on a finite number of curves of the form $y = \phi_k(x)$, $k = 1, \ldots, n$, where $\phi_k(x)$ are continuous functions. Further suppose that*

$$K(\mu, \lambda) = \int_0^1 \int_0^1 K(x, y) d\mu(x) d\lambda(y) = \int_0^1 \int_0^1 K(x, y) d\lambda(y) d\mu(x).$$

Let M be the set of all probability distribution on $[0, 1]$ *which is compact and metrizable in the weak topology. Let A denote the class of all absolutely continuous distributions in* $[0, 1]$. *Then,*

$$\max_M \inf_A K(\mu, \lambda) = \inf_A \max_M K(\mu, \lambda).$$

Proof The outline of the proof is as follows.

Let $\lambda \in A$ and $\phi(x) = \int_0^1 K(x, y) d\lambda(y)$. Let $\mu \in M$. Since λ is absolutely continuous with respect to the Lebesgue measure, it can be shown that $\phi(x)$ is continuous on $[0, 1]$ and is hence bounded. Hence, $K(\mu, \lambda)$ is continuous in μ for every fixed $\lambda \in A$. Since A is convex, M is convex and compact in the weak topology. It then follows from the general minimax theorem that

$$\max_M \inf_A K(\mu, \lambda) = \inf_A \max_M K(\mu, \lambda).$$

■

Theorem 1.7.2 (Karlin [36]) *Consider the game on unit square with separable payoff kernel given by $K(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} f_i(x) g_j(y)$. Here f_i and g_j are continuous real-valued functions on* $[0, 1]$. *Then the players have optimal strategies concentrated at finite points in the unit interval.*

Proof The outline of the proof is as follows. Consider the probability pair (μ, λ). Let $f_i(\mu) = \int f_i(x)d\mu(x)$ and $g_j(\lambda) = \int g_j(y)d\lambda(y)$. Then,

$$\begin{aligned} K(\mu, \lambda) &= \int\int K(x, y)d\mu(x)d\lambda(y) \\ &= \int\int \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} f_i(x)g_j(y)d\mu(x)d\lambda(y) \\ &= \sum_{j=1}^{m} a_{ij} f_i(\mu)g_j(\lambda). \end{aligned}$$

Define the following:

$$F = \{\tilde{f} \in \mathbb{R}^n\colon \text{There exists } \mu \text{ such that } \tilde{f}_i = \int f_i(x)d\mu(x), i \in \{1, \ldots n\}\}$$

$$G = \{\tilde{g} \in \mathbb{R}^m\colon \text{There exists } \lambda \text{ such that } \tilde{g}_j = \int g_j(y)d\lambda(y), j \in \{1, \ldots m\}\}.$$

Here F and G are convex and compact. Let $f \in F$ and $g \in G$. Define the payoff as

$$A(f, g) = f^T Ag = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{ij} f_i g_j.$$

Thus, the mixed game $K(\mu, \lambda)$ is a finite dimensional game given by $\{F, G, A = (a_{ij})\}$. Hence,

$$\begin{aligned} \min_{g\in G}\max_{f\in F} f^T Ag = \max_{f\in F}\min_{g\in G} f^T Ag &= \min_{\lambda}\max_{\mu} K(\mu, \lambda) \\ &= \max_{\mu}\min_{\lambda} K(\mu, \lambda). \end{aligned}$$

■

Any $\tilde{f} \in F$ can be emulated using a finite support. That is, as $\tilde{f}$ is in the convex hull of $Img(f)$, $\tilde{f} = \sum_{i=1}^{l} \alpha_i f(t_i)$ where $0 \le \alpha_i \le 1$, $\sum \alpha_i = 1$. Hence, if $\mu = \sum_{i=1}^{l} \alpha_i \delta_{t_i}$, then $\tilde{f} = (\int f_1(x)d\mu(x), \ldots, \int f_n(x)d\mu(x))$. Thus, the optimals (μ^*, λ^*) can also be assumed to have finite support as it is always possible to find optimals with finite support preserving $(f(\mu^*), g(\lambda^*))$.

Glicksberg [26, 28] shows that the game has a value when the function is continuous or upper/lower semicontinuous. Refer to Definitions B.2.2 and B.2.3 for the definitions of upper and lower semicontinuous functions. So if $K(x, y)$ is upper semicontinuous in both variables, then minmax value exists.

In an unpublished work, Himmelberg, Parthasarathy, and Vleck (1978) attempted to show that there exists a sequence of continuous $K_n(x, y)$ such that it converges to $K(x, y)$ where $K(x, y)$ is upper semicontinuous in x for each fixed y, bounded and jointly measurable. Let Q be a countable dense set (such as the set of rationals in [0, 1]). While $\max\limits_{0 \le x \le 1} u(x) = \sup\limits_{r_i \in \mathbb{Q}} u(r_i)$ when u is continuous in [0, 1], it fails for upper semicontinuous functions. In other words, the maximum of a continuous function in [0, 1] is equal to the supremum of the function over a countable dense set. Such a result is, however, not valid when the function is upper semicontinuous.

Further details regarding general minimax theorem can be found in the book by Parthasarathy and Raghavan [75, Chap. 5].

1.7.3 Continuous Games

Definition 1.7.2 Continuous game: A continuous game is a game with a finite set of n players with action set A_i (non-empty compact metric spaces) and continuous payoff function $u_i : A \to \mathbb{R}$, for all $i = 1, \ldots, n$ where $A = \prod\limits_{i=1}^{n} A_i$.

Definition 1.7.3 Equalizer: A strategy for one player that makes the outcome of the game independent of the other player's action is referred to as the equalizer or equalizing strategy.

Example 1.7.4 given below is of a continuous two-player game with payoff function $K(x, y)$ where the players play a distribution μ and η. This example showcases the concept of equalizer. In other words, μ_0 is called an equalizer if $K(\mu_0, \eta) \equiv c$ for $0 \le \eta \le 1$ and some constant c (Karlin [36]).

Example 1.7.4 Consider a two-player zero-sum game where player-1 and player-2 choose any number $x \in [0, 1]$ and $y \in [0, 1]$, respectively. Without loss of generality, assume player-1 to be the maximizer. Consider the payoff function for player-1 to be the standard Euclidean distance given by $K(x, y) = |x - y|$. Now player-1 can play a distribution μ_0 instead of playing a number, where μ_0 is given by the distribution where he plays 0 with probability $\frac{1}{2}$ and 1 with probability $\frac{1}{2}$. Hence, $K(\mu_0, y) = \frac{1}{2}y + \frac{1}{2}(1 - y) = \frac{1}{2}$, for all y. As the payoff function is symmetric, $K(x, \mu_0) = \frac{1}{2}$, for all x. Here μ_0 is called an equalizer as neither player can get more than $\frac{1}{2}$. This is also the value of the game. ▲

Definition 1.7.4 Support of a strategy: We say that x_o is in the support of a probability measure μ on the real line if for every $\epsilon > 0$,

$$F(x_o + \epsilon) - F(x_o - \epsilon) > 0,$$

where $F(y)$ is the distribution function for μ. In other words, $F(y) = \mu[-\infty, y]$.

Definition 1.7.5 Completely mixed strategy for continuous games: Consider a continuous game on the unit square. A strategy is completely mixed if its support is [0, 1], i.e., the support contains the entire space.

While there is a unique optimal strategy pair for completely mixed matrix games (Theorem 1.3.1), continuous two-player completely mixed games may have more than one optimal strategy pair as shown in the following example.

Example 1.7.5 (*Chin et al. [11, Example 1]*) Consider two density functions in [0, 1] given by

$$f(x) = 2x$$
$$g(x) = 2(1 - x).$$

Let μ_n and μ'_n be the moments of the distributions where $\mu_n = \int x^n f(x)dx$ and $\mu'_n = \int x^n g(x)dx$. Choose constants a_n and b_n such that $a_n + b_n\mu_1 = \mu_n$ and $a_n + b_n\mu'_1 = \mu'_n$. Since $det \begin{bmatrix} 1 & \mu_1 \\ 1 & \mu'_1 \end{bmatrix}$ is non-vanishing, a_n and b_n are unique for each n.

Construct the payoff function for the game to be the following:

$$K(x, y) = \sum_{n=1}^{\infty} \frac{a_n + b_n x - x^n}{M_n} \left(y^n - \frac{1}{n+1}\right) \frac{1}{2^n}, \tag{1.2}$$

where $M_n = \max_{0 \le x \le 1} |a_n + b_n x - x^n|$.

$K(x, y)$ is jointly continuous as the partial sums are continuous and converge uniformly on x and y. Further

$$\int_0^1 K(x, y)dy \equiv 0$$

$$\int_0^1 K(x, y) f(x)dx \equiv 0$$

$$\int_0^1 K(x, y)g(x)dx \equiv 0.$$

Hence, 0 is the equalizer of the game and the two strategy pairs $(f, 1)$ and $(g, 1)$ are optimal.

Since $f(x)$ is a completely mixed optimal for player-1, then for any optimal g^* of player-2, $\int_0^1 K(x, y)dg^*(y) \equiv 0$. Let $v_n = \int y^n dg^*(y)$. Then Eq. 1.2 leads to

$$K(x,y) = \sum_{n=1}^{\infty} \frac{a_n + b_n x - x^n}{M_n}(v_n - \frac{1}{n+1})\frac{1}{2^n} \equiv 0 \text{ in } x.$$

Hence, $v_n - \dfrac{1}{n+1} = 0$ for all n. Note that the moments of the Lebesgue measure coincide with the moments of g^*. Hence, the Lebesgue measure is the only optimal for player-2. It can also be shown that every optimal for player-1 is a convex combination of f and g. Hence, the game is completely mixed with optimal strategy pair $(f, 1)$ and $(g, 1)$. ▲

1.7.4 Games of Timing

Games of timing (Shiffman [90]) are symmetric games that can be viewed as duels where the players have the same set of strategies. The pure strategy of each player represents the time at which they can act. Each player would like to delay his action as long as possible (not infinite) provided his action is prior to his opponent's action.

Thus, in a symmetric game of timing, the kernel $K(x, y)$ satisfies the following conditions:

(1) $K(x,y) = \begin{cases} A(x,y), & \text{if } x < y, \\ 0, & \text{if } x = y, \\ -A(y,x), & \text{if } x > y, \end{cases}$
where $A(x, y)$ is continuous in $x \le y$.

(2) $A(x, y)$ is strictly increasing in x for a fixed y and strictly decreasing in y for a fixed x.

(3) $A(x, y)$ has continuous first-order derivatives on $x \le y$. Also the set of points where the derivative has value 0 contains no linear intervals.

Other than the two trivial pure optimal strategy, Shiffman [90, Sect. 6] shows that an optimal strategy exists and is unique. This optimal strategy is characterized either as a density from some point $a \in [0, 1]$ to 1, or as a density from a to 1 combined with a jump at 0.

1.7.5 Square Root Game

The square root game (Blackwell [8]) is as follows. Consider a payoff function $K(x, y) = |x - y|^{\frac{1}{2}}$. Blackwell computed the value of the game to be 0.59907 (up to 5 decimal places). When he posed this problem to his students, some of them discretized the problem as follows. First they looked at the discrete set $\{0, 1\}$. The 2×2 payoff matrix is as follows and has value 0.5.

$$\begin{array}{cc} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}$$

A further refinement led them to the discrete set $\{0, \frac{1}{2}, 1\}$ yielding the following 3×3 payoff matrix:

$$\begin{array}{cc} & \begin{array}{ccc} 0 & \frac{1}{2} & 1 \end{array} \\ \begin{array}{c} 0 \\ \frac{1}{2} \\ 1 \end{array} & \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 1 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 1 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \end{array}$$

Continuing along the same lines, they arrived at a 21×21 matrix whose value was 0.599070117367796... (up to 21 decimal places). This matched the value arrived at by Blackwell (0.59907 up to 5 decimal places).

According to them, if either of the players used the beta strategy (0.25, 0.25), then player-1's expected income was always 0.599070117367796... up to 21 decimal places, irrespective of the other player's actions.

This leads to Blackwell's theorem [8].

Theorem 1.7.3 (Blackwell [8, Theorem 1]) *Consider a general metric* $K(x, y) = |x - y|^a$, $0 < a < 1$. *Then* $\int_0^1 |x - y|^a p(x)dx \equiv c$, *for all* y, *where the* β*-density function* $p(x) = c(x(1-x))^{-\frac{(1+a)}{2}}$.

Proof Fix a in $(0, 1)$. For $0 < t < 1$, let

$$\begin{aligned} f(t) &= \int_0^1 |x - t|^a p(x)dx \\ &= \int_0^t (t - x)^a p(x)dx + \int_t^1 (x - t)^a p(x)dx \end{aligned}$$

Let us differentiate $f(t)$ with respect to t in order to show that $f(t)$ is a constant.

$$f'(t) = a\Big[\int_0^t (t-x)^{a-1}p(x)dx - \int_t^1 (x-t)^{a-1}p(x)dx\Big]$$

$$= a\Big[\int_0^t (t-x)^{a-1}p(x)dx - \int_0^{1-t} (1-t-u)^{a-1}p(u)du\Big]$$

through change of variable $u = 1 - x$

$$= a[F(t) - F(1-t)], \text{ where } F(t) = \int_0^t (t-x)^{a-1}p(x)dx$$

Applying a fractional transformation $z = \dfrac{t-x}{t(1-x)}$, we have

$$F(t) = (t(1-t))^{\frac{a-1}{2}} \int_0^1 (1-z)^{\frac{-(a+1)}{2}} z^{a-1} dz$$
$$= F(1-t)$$

Thus, $f(t)$ is a constant independent of t, i.e., player-1 has an equalizer strategy, namely, a beta distribution. ■

Remark 1.7.1 Since the spectrum of a beta distribution is [0, 1], it is a completely mixed strategy.

The following is a specific case of Theorem 1.7.3.

Theorem 1.7.4 (Blackwell [8, Theorem 1]) *Let the payoff metric $K(x, y)$ be as follows:*

$$K(x, y) = |x - y|^{\frac{1}{2}}$$

where $0 \le x, y \le 1$. Then $\int_0^1 |x-y|^{\frac{1}{2}} p(x)dx \equiv c$ for all y when the beta-density function $p(x) = c(x(1-x))^{-\frac{3}{4}}$. Also, due to symmetry in the payoff function, $\int_0^1 |x - y|^{\frac{1}{2}} p(y)dy \equiv c$ for all x. The value of the game is given by c and the beta distribution is optimal for both players. ■

Remark 1.7.2 Putting $y = 0$, we can find c. It turns out that c is approximately equal to $0.59\ldots$

Remark 1.7.3 It is not clear whether there are any other optimals for the square root game.

1.8 Algorithmic Aspects of Matrix Games

We now look at orderfield property of matrix games to determine whether finite step algorithms can be found to solve matrix games.

Definition 1.8.1 Orderfield property: A game with payoffs from an ordered field is said to exhibit orderfield property if there exists a pair of optimal strategies whose coordinates lie in the same ordered field. The value also lies in the same ordered field.

Orderfield property is an indicator that it is possible to find a finite step algorithm for the game if the order field is the set of rational numbers. In our discussion, we consider the field of rationals only. Weyl (1950) showed that matrix games possess the orderfield property in general. However, games with more than two players may not always have rational equilibrium as shown by Nash [63] for a three-player poker game. We now reproduce a three-player game with irrational Nash equilibrium.

Example 1.8.1 Consider a three-player game where player-1 has two actions: Top (T) and Bottom (B); player-2 has two actions: Left (L) and Right (R); player-3 has two actions: First (F) and Second (S). Let the payoff matrices for this game be given by the following two matrices, where the first payoff matrix specifies the payoffs when player-3 chooses action F and the second payoff matrix specifies the payoffs when player-3 chooses action S.

$$
\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} (0,0,1) & (1,0,0) \\ (1,1,0) & (2,0,8) \end{pmatrix} \end{array}
\qquad
\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{pmatrix} (2,0,9) & (1,1,1) \\ (0,1,1) & (1,0,0) \end{pmatrix} \end{array}
$$

. It can be seen that the unique Nash equilibrium of the game is given by

$$
\left(\frac{30-2\sqrt{51}}{29}, \frac{-1+2\sqrt{51}}{29}\right), \left(\frac{-6+2\sqrt{51}}{21}, \frac{27-2\sqrt{51}}{21}\right) \text{ and } \left(\frac{9-\sqrt{51}}{12}, \frac{3+\sqrt{51}}{12}\right).
$$

▲

For zero-sum matrix games, there exists polynomial time algorithm to check if the game is completely mixed and to check for pure optimal strategies. For a bimatrix game (A, B) where A and B are square matrices, it can be checked if the bimatrix game has a pure equilibrium in polynomial time. However, it is not known if there is a polynomial time algorithm to check if the bimatrix game is completely mixed. Das and Roy [14] are attempting to answer it, though it is still not completely answered.

1.9 Statistical Decision Theory and Game Theory

The unification of statistical decision theory and game theory is detailed in this section.

Wald [104] unifies statistical decision theory and game theory by treating statistical problems as special cases of zero-sum two-player games. He extends the result of von Neumann [100] to statistical decision theory where he considers the two players to be nature and the statistician. In zero-sum games, the two players are antagonistic as one player's gain is another player's loss. However, when nature is one of the players, this issue does not arise.

Consider a game on the unit square ($0 \leq x, y \leq 1$). Let $K(x, y)$ be a bounded measurable payoff function. Without assuming continuity and assuming that the players are not playing an arbitrary probability distribution, Wald tries to prove the minimax theorem. Players chose only those strategies that are continuous with respect to an absolute Lebesgue measure on [0, 1]. That is,

$$\Lambda_c^{(1)} = \{F : F' = f(x) \text{ density function on } [0,1] \text{ such that } f \leq c, \text{ for all } x \in [0,1]\}$$
$$\Lambda_c^{(2)} = \{G : G' = g(y) \text{ density function on } [0,1] \text{ such that } g \leq c, \text{ for all } y \in [0,1]\}$$

That is, the players chose a density function on their mixed strategies. Wald shows that the game has a value on the unit square, i.e., $\sup_F \inf_G K(F, G) = \inf_G \sup_F K(F, G)$.

1.10 Numerical Geometry and Game Theory

Section 1.10 provides an understanding of how the concept of rendezvous value for compact connected metric spaces links to the value of a game.

Gross [29] introduced the concept of rendezvous value for a compact connected metric space and uses game theory as a tool to prove some results relating to rendezvous value.

Theorem 1.10.1 (Gross [29, Theorem 1]) *Consider a compact connected metric space (X, d). There exists a unique constant c such that given any finite collection $\{x_1, \ldots, x_k\} \subseteq X$, there exists a $y \in X$ such that $\frac{1}{k}\sum_{i=1}^{k} d(x, y) = c$ (the rendezvous value).*

Proof The above statement can be viewed as a two-player zero-sum game where each player chooses a point in the metric space and the payoff is given by the distance between them. As per Glicksberg [26, 28], the game has a value c and optimal strategies. Let $\{x_1, \ldots, x_k\} \subseteq X$ and consider the mixed strategy by the minimizer that assigns weight $\frac{1}{k}$ to each of these points. Irrespective of whether the strategy is optimal or not, $\max_y \frac{1}{k}\sum_{i=1}^{k} d(x_i, y) \geq c$. Due to symmetry, $\min_y \frac{1}{k}\sum_{i=1}^{k} d(x_i, y) \leq c$. Since the range of a real-valued continuous function on a connected set is connected, there exists a y where equality is achieved, i.e., $\frac{1}{k}\sum_{i=1}^{k} d(x_i, y) = c$.

To show the uniqueness, if possible let there be another constant c^* such that $c^* > c$. For a game in a compact space with continuous payoff, one can approximate to an optimal strategy by a finite mixture. Hence, given $\epsilon > 0$, there exists a finite collection of points $\{x_1, \ldots, x_k\}$ and weights $\{\lambda_1, \ldots, \lambda_k\}$ with $\lambda_i > 0$ for all i and $\sum_{i=1}^{k} \lambda_i = 1$ such that $\sum_{i=1}^{k} \lambda_i d(x_i, y) \leq c + \epsilon$ for all y.

As the metric space is connected, there are infinitely many points in every neighborhood of x_i. If λ_i were rational numbers with common denominator K and numerator k_i, it is possible to select K different points $\{x'_1, \ldots, x'_k\}$ with k_i of them clustered around x_i. As the metric is continuous, it is possible to get an arithmetic average approximating an optimal strategy. As the rational numbers are dense in real, given any $\epsilon > 0$, there exists a finite collection $\{x_1, \ldots, x_k\}$ such that $\frac{1}{k}\sum_{i=1}^{k} d(x_i, y) \leq c + \epsilon$. Since $c^* > c$, it is possible to choose ϵ sufficiently small such that the value c^* cannot be achieved by any point y in the collection. The same can be shown for the case when $c^* < c$. Hence, c is the unique value. Thus, the rendezvous value is the same as the value of the game. That is,

$$\frac{1}{k}\sum d(x, y) = \min \max \int \int d(x, y) d\mu(x) d\lambda(y)$$

where μ and λ are probability measures on x and y, respectively. ∎

The following is a simple example on how to compute the rendezvous value explicitly. In general, it is difficult to compute the rendezvous value and is a challenging open problem.

Example 1.10.1 Given $\{x_1, \ldots, x_k\} \subseteq [0, 1]$, construct a function $H(z) = \frac{1}{k}\sum_{i=1}^{k} d(x_i, z)$, $z \in [0, 1]$ where the following holds:

$$H(0) = \frac{1}{k}\sum x_i = \bar{x},\ H(1) = \frac{1}{k}\sum (1 - x_i) = 1 - \bar{x},\ \text{and}$$
$$H(0) + H(1) = 1.$$

This implies that $H(0) \leq \frac{1}{2} \leq H(1)$ or $H(1) \leq \frac{1}{2} \leq H(0)$. Since $H(z)$ is a continuous function, using intermediate value theorem yields $c = \frac{1}{2}$.

Hence, when $d(x, y) = |x - y|$, player-1 can toss a fair coin and select 0 when heads and 1 when tails thus yielding $\frac{1}{2}$. In fact, it is an equalizer rule for both players. Hence, the value of the game is $\frac{1}{2}$ and the equalizer rule is optimal for both players. ▲

Remark 1.10.1 The value of y depends on the metric and the finite collection. The value of c depends on X and d.

In general, the rendezvous value is difficult to compute. The key trick for calculating the rendezvous value in some compact connected metric space is to find a probability measure μ_0 on X such that the value of $\int_X d(x, y) d\mu_0(x)$ is independent of the choice of y. This was extended by Wolf [106] as follows.

Theorem 1.10.2 (Wolf [106, Corollary 1]) *Consider the metric space (X,d). There exists a probability measure μ_0 on X such that $\int_X d(x, y) d\mu_0(x)$ is independent of the choice of y if and only if the rendezvous value $c(X, d) = \sup_\mu \int \int d(x, y) d\mu(x)$ $d\mu(y)$.* ■

A generalization of Gross theorem [29] was given by Stadje [94], who replaced the metric d with a continuous symmetric function f.

OPEN QUESTION 1.10.1 Let X have at least two distinct points. Let the diameter of X be the maximum distance between two points. Define $m(X, d)$ (often referred to as the magic number or dispersion number) as follows:

$$m(X, d) = \frac{a(X, d)}{\text{diameter of } X}$$

Since $\mathbb{R}^n$ has diameter ∞, consider a closed and bounded subset of $\mathbb{R}^n$. Define $g(\mathbb{R}^n)$ as follows:

$$g(\mathbb{R}^n) = sup\{m(X, d) : X \subset \mathbb{R}^n, X \text{ is a closed and bounded set in } \mathbb{R}^n\}.$$

When $n = 1$, any connected set is an interval in $\mathbb{R}^1$. Hence $g(\mathbb{R}^1) = \frac{1}{2}$. Cleary et al. [12] considered a sphere in $\mathbb{R}^n$ with diameter 1 and computed the value of the game. However, computing the value of $g(\mathbb{R}^n)$ for higher powers of n is not easy and remains unanswered.

1.11 Summary

The primary focus in this chapter has been on matrix games—both zero-sum and non-zero-sum (also called bimatrix) games. The minimax theorem shows that zero-sum matrix games have optimal strategy and value. The optimal strategy may be a pure strategy, mixed strategy, or completely mixed strategy. If the matrix game is completely mixed, then each player has only one optimal strategy. Also a zero-sum game with skew-symmetric payoff matrix of even order is never completely mixed. If the skew-symmetric payoff matrix is of odd order, then the game is completely mixed if and only if one of the columns of the cofactor matrix is all non-zero and is of the same sign. It is interesting to note that not all games have a value (such as games with discontinuities). Some results and examples pertaining to games with continuous strategy set and continuous payoffs were also stated. The importance of orderfield property vis-a-vis the computational complexity of matrix games has also been briefly touched upon. Other topics such as extensive form games and n-player games have not been covered here, but make for interesting reading.

Chapter 2
Finite Stochastic Games

First defined in a seminal paper by Shapley [84], a finite zero-sum stochastic game can be described as follows. Consider a two-player zero-sum game with a finite state space S, and finite action space A_1 and A_2 for player-1 and player-2, respectively. Let the two players observe the state of the system $s \in S$ and select an action simultaneously (say, $a \in A_1$ and $b \in A_2$, respectively). Let the immediate reward for player-1 be $r(s, a, b)$, where $r(.)$ is a continuous and bounded function on $S \times A_1 \times A_2$. The game now moves to the state $s' \in S$ based on the transition probability given by $q(s'|s, a, b)$. We assume q to be Markovian. The game is played at discrete points of time (say, once a day) over the infinite future in this manner. It may happen that player-1's accumulated income $\sum\limits_{n=0}^{\infty} r(s_n, a_n, b_n)$ may not converge. To avoid this, the nth day income is discounted by β^{n-1} where the discount factor β is a preassigned number in [0, 1). This automatically makes the ∞-series converge. As in the matrix game, player-1 wants to maximize his/her discounted total (expected) income while player-2 wants to minimize the same. Note that this total income will depend on the initial state $s \in S$.

Stochastic games are typically represented as $(S, A_1, A_2, r, q, \beta)$. Shapley [84] proved the existence of the minimax value and a pair of optimal stationary strategies for the two players for the finite stochastic game. This result also extends to stochastic games when the state space is countable.

Remark 2.0.1 Shapley [84] looked at stochastic games with stopping probabilities instead of discount factor.

Example 2.0.1 Consider a two-player zero-sum discounted stochastic game with state space $S = \{1, 2\}$ and discount factor $\beta = \frac{1}{2}$. Let the payoff matrix for each of the states be as follows.

$$\text{state } 1 : \left(\begin{array}{c|c} 1 & 0 \\ (1,0) & (0,1) \\ \hline 0 & 1 \\ (0,1) & (1,0) \end{array}\right), \text{ state } 2 : \left(\begin{array}{c} 0 \\ (0,1) \end{array}\right).$$

T. Parthasarathy and S. Babu, *Stochastic Games and Related Concepts*,
HBA Lecture Notes in Mathematics 2,
https://doi.org/10.1007/978-981-15-6577-9_2

In state 1, each player has two actions. In state 2, each player has only one action. In each cell of the above matrix, the number in the first line represents the payoff that player-1 receives and the second line indicates the transition probability. So the first cell in state 1 indicates that player-1 receives 1 unit from player-2 and the game remains in state 1 with probability 1 and never moves to state 2. The second cell in state 1 indicates that player-1 receives 0 unit from player-2 and the game moves to state 2 with probability 1, and so on. Once the game reaches state 2, it continues to remain in that state forever. ▲

For further reading, Krishnamurthy and Parthasarathy [41] also provide a short and comprehensive introduction to stochastic games.

This chapter deals with finite discounted stochastic games. In particular, we look at the conditions under which the stochastic game has symmetric equilibrium and orderfield property. We also highlight the conditions under which the stochastic game is completely mixed. We describe certain classes of stochastic games based on specific conditions on the payoff function and transition probability. These include single-player control stochastic game, switching control stochastic game, perfect information stochastic game, separable reward-state independent transition (SER-SIT) stochastic game, and additive reward-additive transition (ARAT) stochastic game. We also look at the conditions under which stochastic games containing one or more of the above class of games has the orderfield property and stationary equilibrium property. Chapters 3 and 4 deal with infinite stochastic games and undiscounted stochastic games.

We begin with some definitions related to stochastic games.

Definition 2.0.1 Stationary strategy: Let S be the state space and P_{A_1} be the set of probability distributions on player-1's action set A_1. A stationary strategy for player-1 is a Borel measurable mapping $f : S \to P_{A_1}$ that is independent of the history that led to the state $s \in S$. Similarly, we define a stationary strategy for player-2 as a Borel measurable mapping $g : S \to P_{A_2}$ that is independent of the history that led to the state $s \in S$.

Definition 2.0.2 Behavioral strategy: Consider a general policy π given by $\pi = (\pi_1, \pi_2, \ldots \pi_n, \ldots)$ where π_n is the strategy set for day n and $\pi_n(\ldots|h_{n-1})$ is the conditional distribution dependent on the history h_{n-1}. Such a strategy is a behavioral strategy.

For discounted finite stochastic games, it is sufficient to look at stationary strategies.

Definition 2.0.3 Absorbing state: A state s in a stochastic game is an absorbing state if the game remains in state s forever once it reaches s.

In Example 2.0.1, state 2 is an absorbing state.

Definition 2.0.4 Discounted payoff: Consider the non-zero-sum discounted stochastic game $\Gamma = (S, A_1, A_2, r_1, r_2, q, \beta)$. Let $r_1^{(n)}(s_0, f, g)$ and $r_2^{(n)}(s_0, f, g)$ be the expected immediate reward at the n-th stage to player-1 and player-2, respectively, when the game starts at state s_0 and they use stationary strategies f and g, respectively. Then the discounted payoffs (also known as the β-discounted payoff) for player-1 and player-2 are as follows:

$$I_\beta^{(1)}(f, g)(s_0) = \sum_{n=0}^{\infty} \beta^n r_1^{(n)}(s_0, f, g)$$
$$I_\beta^{(2)}(f, g)(s_0) = \sum_{n=0}^{\infty} \beta^n r_2^{(n)}(s_0, f, g)$$

For the zero-sum stochastic game (that is, when $r_1 = -r_2 = r$, say), the β-discounted payoff for player-1 is given by

$$I_\beta(f, g)(s_0) = \sum_{n=0}^{\infty} \beta^n r^{(n)}(s_0, f, g).$$

We briefly define undiscounted payoffs. Stochastic games with undiscounted payoffs are called undiscounted stochastic games. Refer to Chap. 4 for more details.

Definition 2.0.5 Undiscounted payoffs: For a non-zero-sum two-player stochastic game with starting state s_0, let (f, g) be a pair of stationary strategies for players-1 and players-2, respectively. Then the undiscounted or limiting average payoff is given as follows:

$$[\Phi^{(1)}(f, g)](s_0) = \liminf_{T\uparrow\infty} \left[\left(\frac{1}{T+1}\right) \sum_{t=0}^{T} r_t^{(1)}(s_0, f, g) \right], \text{ for player-1} \tag{2.3}$$

$$[\Phi^{(2)}(f, g)](s_0) = \liminf_{T\uparrow\infty} \left[\left(\frac{1}{T+1}\right) \sum_{t=0}^{T} r_t^{(2)}(s_0, f, g) \right], \text{ for player-2.} \tag{2.4}$$

Definition 2.0.6 Optimal strategies and value: A pair of stationary strategies (f^o, g^o) is optimal in the zero-sum discounted stochastic game if for all $s \in S$

$$I_\beta(f, g^o)(s) \le I_\beta(f^o, g^o)(s) \le I_\beta(f^o, g)(s) \tag{2.5}$$

for all $f \in P_{A_1}^S$ and for all $g \in P_{A_2}^S$. Here P_{A_1} and P_{A_2} are the set of probability distributions on player-1 and player-2's action sets A_1 and A_2, respectively.

In other words,

$$\begin{aligned} I_\beta(f^o, g^o)(s) &= \sup_f \inf_g I_\beta(f, g)(s) = \inf_g I_\beta(f^o, g)(s) \\ &= \sup_f I_\beta(f, g^o)(s) = \inf_g \sup_f I_\beta(f, g)(s), \text{ for all } s \in S. \end{aligned} \tag{2.6}$$

Under assumptions of a finite state and action space, Shapley [84] proved the existence and uniqueness of the value v_β and a pair of optimal strategies (f^o, g^o). The value for all $s \in S$ is given by

$$\begin{aligned} v_\beta(s) &= \sup_f \inf_g I_\beta(f, g)(s) = \inf_g \sup_f I_\beta(f, g)(s) \\ &= \inf_g I_\beta(f^o, g)(s) = \sup_f I_\beta(f, g^o)(s) \\ &= I_\beta(f^o, g^o)(s) \end{aligned} \tag{2.7}$$

While the value is unique, the optimal strategies may not be unique.

For a non-zero-sum two-player finite bimatrix game, (x^*, y^*) is a Nash equilibrium pair if

$$\begin{aligned} &x^{*T} Ay^* \geq x^T Ay^* \text{ for all strategies } x \text{ of player} - 1\text{, and} \\ &x^{*T} By^* \geq x^T By \text{ for all strategies } y \text{ of player} - 2 \end{aligned}$$

For the non-zero-sum discounted stochastic game with two players, a pair of stationary strategies (f^o, g^o) is an equilibrium pair for the minimization problem if

$$\begin{aligned} I_\beta^{(1)}(f^o, g^o)(s) \leq I_\beta^{(1)}(f, g^o)(s) \\ I_\beta^{(2)}(f^o, g^o)(s) \leq I_\beta^{(2)}(f^o, g)(s) \end{aligned} \tag{2.8}$$

for all f, for all g, and for all s.

Definition 2.0.7 Matrix (bimatrix) game restricted to a state: For a two-player finite zero-sum (discounted or undiscounted) stochastic game, consider the matrix $R(s) = (r(s, i, j))_{m_1 \times m_2}$ restricted to state $s \in S$. That is, for a fixed $s \in S$, the (i, j)-th element of $R(s)$ is the immediate reward to player-1 and player-2 when they choose actions i and j, respectively. Then, the one-shot game where the payoff matrix of player-1 is $R(s)$ is referred to as the matrix game restricted to state s.

Similarly, for a two-player finite non-zero-sum stochastic game, for a given $s \in S$, we call the bimatrix game $(R_1(s), R_2(s))$ as the bimatrix game restricted to state s.

Definition 2.0.8 Auxiliary game: Consider a discounted stochastic game. Let $v_\beta(s')$ indicate the value of the game when the initial state is s'. Then, the game with payoff matrix $\mathcal{A}(s)$ whose (i, j)-th element is $(r(s, i, j) + \beta \sum_{s' \in S} v_\beta(s') q(s'|s, i, j))$ is called the auxiliary game at state s (or starting at state s).

Shapley (1953) showed that the value of the game with payoff matrix $\mathcal{A}(s)$ is $val\mathcal{A}(s) = v_\beta(s)$. This gives a fixed point equation for the value vector $(v(s))_{s \in S}$.

Theorem 2.0.1 (Shapley [84, Theorems 1 and 2]) *Let A and B be the action space for player-1 and player-2, respectively. A β-discounted zero-sum stochastic game $\Gamma = (S, A, B, r, q, \beta)$ has a stationary optimal for the two players and a unique value of the game when S, A, and B are finite. The value of the game is the unique solution of the equation*

$$v(s) = \text{val}[R(s, v)]$$

where

$$R(s, u) = (r(s, i, j) + \beta \sum_{s' \in S} u(s') q(s'|s, i, j)) \, for \; any \; vector \; u. \quad \blacksquare$$

Definition 2.0.9 Nash equilibrium (in the stochastic game): A pair of stationary strategies (f^o, g^o) constitutes a Nash equilibrium for the maximization problem in the discounted non-zero-sum stochastic game if for all $s \in S$

$$I_\beta^{(1)}(f^o, g^o)(s) \geq I_\beta^{(1)}(f, g^o)(s) \text{for all } f \in P_{A_1}^S \text{ , and}$$

$$I_\beta^{(2)}(f^o, g^o)(s) \geq I_\beta^{(2)}(f^o, g)(s) \text{ for all } g \in P_{A_2}^S$$

assuming that both players want to maximize their payoffs.

Nash [63] showed the existence of equilibrium strategies for a bimatrix game. The existence of equilibrium strategies in a finite non-zero-sum n-player stochastic games was shown by Fink [20] and Takahashi [98]. We state the theorem for the two-player stochastic game.

Theorem 2.0.2 *Every two-player non-zero-sum discounted stochastic game has a pair of stationary strategies (f^*, g^*) that constitutes a Nash equilibrium. Assume that the players are maximizers. Then for each $s \in S$ and for all strategies $f \in P_1$ and $g \in P_2$, we have*

$$I_\beta^{(1)}(f^*, g^*)(s) = \sup_f I_\beta^{(1)}(f, g^*)(s) \; and$$
$$I_\beta^{(2)}(f^*, g^*)(s) = \sup_g I_\beta^{(2)}(f^*, g)(s)$$

$\blacksquare$

Theorem 2.0.3 (Fink [20, Theorem 1]) *Any n-player non-zero-sum discounted stochastic game with finite state space and finite action space has an equilibrium strategy.* $\blacksquare$

Example 2.0.2 Consider Example 2.0.1. For sake of brevity, indicate $v_{\frac{1}{2}}(s)$ by $v(s)$. As state 2 is absorbing, the value of the game starting at state 2 is $v(2) = 0$. By Theorem 2.0.1, the value of the game starting at state 1 is given by

$$v(1) = val \begin{bmatrix} 1 + \frac{1}{2}v(1) & 0 \\ 0 & 1 + \frac{1}{2}v(1) \end{bmatrix}.$$

In the above equation, the matrix is completely mixed. Hence by Theorem 1.3.2, $v(1) = \dfrac{2}{3}$ and $(\dfrac{1}{2}, \dfrac{1}{2})$ is optimal for the two players in state 1. ▲

2.1 Symmetric Optimal (Equilibrium) in Stochastic Games

This section details the conditions under which symmetric optimal and symmetric equilibria exist in zero-sum and non-zero-sum stochastic games, respectively.

Symmetric games are typically used to describe single population games in evolutionary game theory where the players in the population have the same set of strategies. In this context, the only relevant equilibrium is the symmetric Nash equilibrium which is an evolutionarily stable strategy (Maynard Smith and Price [54]). This serves as a motivation to look at symmetric equilibrium in bimatrix and stochastic games.

Definition 2.1.1 Symmetric optimal (or equilibrium) strategy: A pair of optimal (or equilibrium) strategies (f^*, g^*) is called a symmetric optimal (or equilibrium) strategy if both players use the same strategy in equilibrium, i.e., $f^* = g^*$.

Gale [22] showed the existence of a symmetric optimal strategy in finite zero-sum games with skew-symmetric payoff matrices. Symmetric equilibria have been shown to exist in discontinuous symmetric games as well under certain conditions (Dasgupta and Maskin [15]; Reny [82]). Sujatha et al. [97] extended Gale's result to show the existence of symmetric optimal for discounted stochastic games with skew-symmetric payoff matrices.

Theorem 2.1.1 (Sujatha et al. [97, Theorem 5]) *Consider a finite discounted zero-sum stochastic game where $R(s)$ is skew symmetric for all $s \in S$. Then the value of the stochastic game is* 0 *and the stochastic game has symmetric optimal stationary strategies independent of the discount factor and the transition probabilities.*

Proof Let the finite discounted zero-sum stochastic game have N states. Further let each player have n actions to select from. Let $f^o(s)$ be an optimal strategy for player-1 in the matrix game restricted to state s, $R(s)$. Gale [22] showed that for a finite zero-sum game with a skew-symmetric matrix, the value of the game is 0 and any strategy that is optimal for one player is also optimal for the other player. Hence, if we view $R(s)$ as a finite zero-sum game with skew-symmetric payoff matrix, then the value of the stochastic game is 0. Let g be any strategy for player-2 and let f^o be an optimal for player-1. We have $r(s, f^o(s), g(s)) \geq 0$ where

$$r(s, f^o(s), g(s)) = \sum_{j=1}^{n} \sum_{i=1}^{n} r(s, i, j) f^o(s, i) g(s, j).$$

Here $f^o(s, i)$ is the i-th coordinate of $f^o(s)$ while $g(s, j)$ is the j-th coordinate of $g(s)$.

Indicate $r(f, g)$ by the following:

$$\begin{pmatrix} r(1, f(1), g(1)) \\ r(2, f(2), g(2)) \\ \vdots \\ r(N, f(N), g(N)) \end{pmatrix}.$$

The total expected β-discounted income for player-1 is given by

$$\begin{aligned} I_\beta(f^o, g)(s) &= \sum_{t=0}^{\infty} \beta^t q^{(t)}(s'|s, f^o, g) r(s, f^o, g) \\ &= [I - \beta Q(f^o, g)]^{-1} r(f^o, g)|_{s^{th}\text{coordinate}}, \end{aligned}$$

where $Q(f^o, g)$ is a $N \times N$ matrix with the (s, s')th element representing the transition probability $q(s'|s, f^o, g)$. Since $[I - \beta Q(f, g)]^{-1}$ is a non-negative matrix and $r(f^o, g)$ is a non-negative vector, it follows that for all g and for all s

$$I_\beta(f^o, g)(s) \geq 0. \tag{2.9}$$

Similarly, let g^o be an optimal stationary strategy for player-2. The total expected β-discounted income for player-2 is given by

$$I_\beta(f, g^o)(s) = [I - \beta Q(f, g^o)]^{-1} r(f, g^o)|_{s^{th}\text{coordinate}}. \tag{2.10}$$

By Gale's result (1960), any strategy that is optimal for one player is also optimal for the other player. That is, $g^o = f^o$. Hence, $I_\beta(f, f^o)(s) = [I - \beta Q(f, f^o)]^{-1} r(s, f, f^o)$ (from Eq. 2.10). Hence for all s, for all f,

$$I_\beta(f, f^o)(s) \leq 0. \tag{2.11}$$

Comparing Eqs. 2.9 and 2.11, we have $I_\beta(f^o, g)(s) = I_\beta(f, f^o)(s) = 0$. That is, the value of the stochastic game starting in state s is 0. Note that the auxiliary game and the matrix game restricted to state s coincide for all discount factors. Thus, the stochastic game has symmetric optimal stationary strategies independent of the discount factor β and the transition probability $q(s'|s, f, g)$. ■

2.2 Completely Mixed Stochastic Games

Definition 2.2.1 Completely mixed stochastic game: Consider a two-player stochastic game $(S, A_1, A_2, r_1, r_2, q, \beta)$. If every optimal stationary strategy for either player assigns a positive probability to every action in every state, then the stochastic game is said to be completely mixed. Such strategies are referred to as completely mixed strategies of the stochastic game.

Example 2.2.1 Consider a stochastic game where only one of the players controls the transitions. For example, if player-2 controls the transitions, then $q(s'|s, i, j) = q(s'|s, j)$ for all $s, s' \in S$, and for all i, j. Such games are called single-player controlled stochastic game (refer to Definition 2.4.1). Let the game have positive payoffs. Let F^o and G^o denote the set of all optimal stationary strategies for player-1 and player-2, respectively. Then the stochastic game is completely mixed if for all $(f^o, g^o) \in F^o \times G^o$, $f^o(s, i)$ and $g^o(s, j)$ are strictly positive for all i, j, and s. ▲

2.3 Orderfield Property for Stochastic Games

We had seen the relevance of orderfield property in matrix games in Section 1.8. Section 2.3 looks at orderfield property in stochastic games.

As seen in Definition 1.8.1, a game whose payoffs are from an ordered field exhibits orderfield property if an optimal strategy pair (and hence the value) also lies in the same ordered field. That is, each coordinate of some optimal pair lies in the same ordered field. Note that not all stochastic games exhibit orderfield property in the field of rational numbers, as shown in this often quoted example.

Example 2.3.1 Consider a discounted two-player zero-sum stochastic game with two states and discount factor $\beta = \frac{1}{2}$.

$$s_1 : \begin{bmatrix} \begin{matrix} 1 \\ (1,0) \end{matrix} & \begin{matrix} 0 \\ (0,1) \end{matrix} \\ \\ \begin{matrix} 0 \\ (0,1) \end{matrix} & \begin{matrix} 3 \\ (1,0) \end{matrix} \end{bmatrix}, s_2 : \begin{bmatrix} \begin{matrix} 0 \\ (0,1) \end{matrix} \end{bmatrix}.$$

Let $v(1)$ and $v(2)$ be the value for the stochastic game in states s_1 and s_2, respectively. As state s_2 is absorbing, $v(2) = 0$.

$$
\begin{aligned}
v(1) &= val \begin{bmatrix} 1 + \frac{1}{2}v(1) & 0 + \frac{1}{2}v(2) \\ \\ 0 + \frac{1}{2}v(2) & 3 + \frac{1}{2}v(1) \end{bmatrix} \\
&= val \begin{bmatrix} 1 + \frac{1}{2}v(1) & 0 \\ \\ 0 & 3 + \frac{1}{2}v(1) \end{bmatrix} \\
&= \frac{(1 + \frac{1}{2}v(1))(3 + \frac{1}{2}v(1))}{4 + v(1)} \\
&\qquad \text{(using Kaplansky's theorem—Theorem 1.3.2)} \\
&= \frac{-4 + 2\sqrt{13}}{3}.
\end{aligned}
$$

▲

The absence of orderfield property in the field of rationals for a stochastic game indicates that it may not be possible to find a finite step algorithm to solve the game. As even simple stochastic games may not exhibit orderfield property in the field of rationals, the next few sections (Sects. 2.4–2.8) look at some special class of stochastic games that do exhibit orderfield property in the field of rationals.

Raghavan and Filar [79] posed an open question on finding a more comprehensive characterization of the orderfield property for stochastic games. This has been answered in an archived paper by Avrachenkov et al. [2] for both discounted and limiting average stochastic games over the orderfield of real algebraic numbers. The result for discounted stochastic games is given in Theorem 2.3.1.

Theorem 2.3.1 (Avrachenkov et al. [2, Theorem 3.1]) *Consider a discounted stochastic game Γ_β with all data lying in the field of algebraic numbers. Then the value vector v of Γ_β has entries v_s that are algebraic numbers for each $s = 1, 2, \ldots, N$.* ■

We refer to Avrachenkov et al. [2] for a detailed proof and examples illustrating the existence of orderfield when the data lies in the field of real algebraic numbers. They also provide a non-constructive proof based on Tarski's principle. *Tarski's principle*[99] states that an elementary sentence which is valid over one real closed field is valid over every real closed field. Shapley's result is considered to be a valid elementary sentence over the field of reals (Bewley and Kohlberg [3]). Since the field of reals is closed, it immediately follows that the orderfield property holds over the field of real algebraic numbers.

Sections 2.4–2.8 detail certain classes of stochastic games based on specific conditions on the payoff function and transition probability.

2.4 Single-Player Control Stochastic Games

Definition 2.4.1 Single-player control stochastic game (Parthasarathy and Raghavan [76]): In these stochastic games (also referred to as single-controller stochastic game or one-player control stochastic game), only one of the players controls the transitions. For example, if player-2 controls the transitions, then $q(s'|s, i, j) = q(s'|s, j)$ for all $s, s' \in S$, and for all i, j. Similarly, if player-1 controls the transitions, then $q(s'|s, i, j) = q(s'|s, i)$ for all $s, s' \in S$, and for all i, j.

Remark 2.4.1 As only one player controls the transition, the single-player control stochastic game behaves like a Markov decision process with respect to the controlling player. Thus, the game can be expressed as a primal-dual pair of linear programs.

Example 2.4.1 Consider the following zero-sum discounted stochastic game with two states and the following payoff matrix:

$$s_1 : \begin{bmatrix} \begin{matrix} 0 \\ (.3, .7) \end{matrix} & \begin{matrix} 2 \\ (.4, .6) \end{matrix} \\ & \\ \begin{matrix} 4 \\ (.3, .7) \end{matrix} & \begin{matrix} 1 \\ (.4, .6) \end{matrix} \end{bmatrix}, s_2 : \begin{bmatrix} \begin{matrix} 2 \\ (.5, .5) \end{matrix} & \begin{matrix} -2 \\ (1, 0) \end{matrix} \\ & \\ \begin{matrix} -1 \\ (.5, .5) \end{matrix} & \begin{matrix} 1 \\ (1, 0) \end{matrix} \end{bmatrix}.$$

This is a single-player control stochastic game where player-2 is the controlling player because the transition probability is the same in every cell of a given column. ▲

Theorem 2.4.1 (Parthasarathy and Raghavan [76, Theorem 6.1]) *A discounted single-player control stochastic game has orderfield property.*

Proof Consider a single-player controlled discounted stochastic game where player-2 controls the transitions. Let the state space be $S = \{1, \ldots, k\}$. Let the payoff matrix in each state be of order $m \times n$. Assume $r(s, i, j) \geq 0$ for all s, i, j. Such games can be formulated as a linear program as follows:

$$\begin{aligned} &\max \sum_{s=1}^{k} v(s) \\ &\text{subject to } \sum_{i=1}^{m} \Big[r(s, i, j) + \beta \sum_{s' \in S} v(s') q(s'|s, j) \Big] \xi_i^s \geq v(s), \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for all } s, \text{ for all } j \\ &\sum_i \xi_i^{(s)} = 1, \text{ for all } s \\ &\xi_i^{(s)} \geq 0, \text{ for all } s \\ &v(s) \geq 0. \end{aligned}$$

The first equation in the feasibility condition is equivalent to

$$\sum_{i=1}^{n}\left[r(s,i,j)\xi_i^s\right]+\left[\beta\sum_{s'\in S}v(s')q(s'|s,j)\right]\geq v(s),\ \text{for all } s,\ \text{for all } j.$$

This is a linear inequality where the unknowns are v and ξ. The linear program solution cannot be unbounded as the feasibility set is closed and bounded. Also ξ is bounded as it is a probability vector. Here $r(s, f(s), j) = \sum_{i=1}^{m} \xi_i^{(s)} r(s, i, j)$ where $f(s) = (\xi_1^{(s)}, \ldots, \xi_m^{(s)})$. Hence,

$$r(s, f(s), j) + \beta \sum v(s')q(s'|s, j) \geq v(s)\ \text{for all}\ s,\ \ \text{for all}\ j.$$

This is valid for any $g(s)$ where $g(s)$ is a probability vector for player-2 given by $(\eta_1^{(s)}, \ldots, \eta_n^{(s)})$. Let $q(s'|s, g(s)) = \sum_{j=1}^{n} q(s'|s, j)\eta_j^{(s)}$. Then,

$$r(s, f(s), g(s)) + \beta v(s')q(s'|s, g(s)) \geq v(s)\ \text{for all}\ s.$$

Given a pair (f, g), define a map L_{fg} from $\mathbb{R}^k \to \mathbb{R}^k$ whose s-th coordinate is given by

$$(L_{fg}v)(s) = r(s, f(s), g(s)) + \beta \sum_{s'=1}^{k} v(s')q(s'|s, g(s)).$$

Thus $(L_{fg}v)(s) \geq v(s)$ for all s. As L_{fg} is a contraction mapping, apply it repeatedly to get $L_{fg}^n v \geq v$ for all n. As $n \to \infty$, $L_{fg}^n v \to I_\beta(f, g)$. This implies that $I_\beta(f, g)(s) \geq v(s)$.

Let $M = \max |r(s, i, j)|$. Then,

$$\begin{aligned}
I_\beta(f, g)(s) &= r(f, g) + \beta \sum_{s'\in S} v(s')q(s'|s, j) + \beta^2 \sum_{s'\in S} v(s')q(s'|s, j) + \ldots \\
&\leq M + \beta M + \beta^2 M + \ldots \\
&\leq \frac{M}{1-\beta}.
\end{aligned}$$

Hence, $I_\beta(f, g)(s)$ is bounded by $\dfrac{M}{1-\beta}$. The solution $v^*(s) = v_\beta(s)$ of the above linear programming problem is the value of the stochastic game. The strategies $\xi^{*(s)}$ and $\eta^{*(s)}$ corresponding to v^* are optimal for player-1 and player-2, respectively. ■

Remark 2.4.2 As single-player control games can be expressed as a linear program problem, Parthasarathy and Raghavan [76] provide an algorithm for solving it.

We now look at some other properties of single-player controlled games. We will use the following results from Kaplansky [34] along with Theorem 1.3.2 for proving some results in stochastic games.

Theorem 2.4.2 (Kaplansky [34, Theorem 1]) *Consider a two-player matrix game with payoff matrix $A \in \mathbb{R}^{m\times n}$. Suppose player-1 has a completely mixed optimal strategy. If y^o is any optimal strategy for player-2, then $\sum\limits_{j=1}^{n} a_{ij}y_j^o \equiv v$ for $i = 1, \ldots, m$ where v is the value of the matrix game.* ■

Theorem 2.4.3 (Kaplansky [34, Theorem 4]) *Let $A \in \mathbb{R}^{n\times n}$ be the payoff matrix of a two-player matrix game. Every optimal strategy of player-1 is completely mixed if and only if every optimal strategy of player-2 is also completely mixed. That is, if one player has an optimal that is not completely mixed, then the other player also has an optimal that is not completely mixed.* ■

We will also require the following lemmas.

Lemma 2.4.1 (Parthasarathy and Raghavan [76, Lemma 4.1]) *Consider a non-singular matrix $C = (c_{ij})_{n\times n}$ where $c_{ij} = a_{ij} + b_j$ with $a_{ij} > 0$ for all i, j. Suppose $Cx = \alpha e$ where x is a probability vector, α is a scalar, and $e^t = (1, \ldots, 1)^t$. Then the matrix $A = (a_{ij})$ is non-singular and $Ax = \beta e$ for some scalar β.*

Proof If possible, let A be singular. Thus, there exists $\pi \neq 0$ such that $A\pi = 0$.

$$\begin{aligned} C\pi &= A\pi + (\sum_{j=1}^{k} b_j\pi_j)e \\ &= A\pi + \delta e \text{ where } \delta = b^t\pi \\ &= \delta e. \end{aligned}$$

Since $C\pi \neq 0$, we have $\delta \neq 0$. $C\pi = \delta e$ implies that $C^{-1}e = \dfrac{\pi}{\delta}$. Also $Cx = \alpha e$ implies that $C^{-1}e = \dfrac{x}{\alpha}$. Thus, $\pi = \dfrac{\delta}{\alpha}x$.

As $\dfrac{\delta}{\alpha} \neq 0$, π must be unisigned. Hence, $A\pi \neq 0$ since $a_{ij} > 0$ for all i, j. This contradicts our assumption that $A\pi = 0$. Hence A is non-singular. Also it is clear that $Ax = \beta e$ for some scalar β. ■

Remark 2.4.3 Lemma 2.4.1 plays an important part in the Shapley equation for auxiliary games in single-player control games since $val[r(s, i, j) + \beta \sum\limits_{s'\in S} v_\beta(s')$ $q(s'|s, j)] = v_\beta(s)$.

Lemma 2.4.2 (Sujatha et al. [96, Lemma 2]) *Consider a finite discounted zero-sum single-player controlled stochastic game where player-1 is the controlling player. That is, $q(s'|s, i, j) = q(s'|s, i)$ for all $s \in S$. Let $R(s)$ be symmetric for each*

$s, s' \in S$. Let (f^o, g^o) be a completely mixed optimal stationary strategy pair for the stochastic game. Then, $f^o(s)$ is a symmetric completely mixed optimal strategy pair for $R(s)$ for all $s \in S$. Further, for all $s \in S$, let $R(s)$ be non-singular. Then $f^o(s)$ is the unique optimal. ■

Theorem 2.4.4 (Sujatha et al. [96, Theorem 3]) *Consider a finite discounted zero-sum single-player controlled stochastic game where player-1 is the controlling player. Let $R(s)$ be symmetric for each $s \in S$. Further, suppose there exists a completely mixed optimal stationary strategy pair for the stochastic game. Then the following are equivalent:*

1. *The discounted stochastic game is completely mixed.*
2. *The matrix game $R(s)$ is completely mixed for every s.*

Proof Without loss of generality, assume that $r(s, i, j) > 0$ for every tuple (s, i, j).

We will first show that *2* follows from *1*. Let the discounted stochastic game be completely mixed. This means that the auxiliary game $\mathcal{A}(s)$ is completely mixed for every s. The (i, j)-th element of $\mathcal{A}(s)$ is given by $r(s, i, j) + \beta \sum_{s' \in S} v_\beta(s') q(s'|s, i)$.

Since $r(s, i, j) > 0$, it follows that $v_\beta(s) > 0$. Also, since $\mathcal{A}(s)$ is completely mixed, from Shapley's equation, we have

$$v_\beta(s) = \frac{|\mathcal{A}(s)|}{\sum \sum \mathcal{A}(s)_{ij}}.$$

Thus, $det(\mathcal{A}(s)) \neq 0$. That is, $\mathcal{A}(s)$ is non-singular for all $s \in S$. From Lemma 2.4.1, it follows that $R(s)$ is non-singular for all $s \in S$.

Now, let (f^o, g^o) be a completely mixed optimal stationary strategy pair for the stochastic game. By Lemma 2.4.2 , $(f^o(s), f^o(s))$ is an optimal strategy pair for $R(s)$. $(f^o(s), f^o(s))$ is also completely mixed as f^o is a completely mixed strategy of player-1. Since $R(s)$ is non-singular for all $s \in S$, it follows by Lemma 2.4.2 that $R(s)$ is completely mixed for every $s \in S$.

Conversely, suppose $R(s)$ is completely mixed for all $s \in S$. Let (f^o, g^o) be a completely mixed optimal stationary strategy pair for the discounted stochastic game. For all $s \in S$, since $r(s, i, j) > 0$ and $R(s)$ is completely mixed, we have $R(s)$ is non-singular by Kaplansky [34]. The (i, j)-th element of the auxiliary game $\mathcal{A}(s)$ starting at state s is given by $r(s, i, j) + \beta \sum_{s' \in S} v_\beta(s') q(s'|s, i)$. By Lemma 2.4.1, $\mathcal{A}(s)$ is non-singular for all $s \in S$.

By Lemma 2.4.2 , $(f^o(s), f^o(s))$ is the unique symmetric optimal strategy pair for $R(s)$. Then, $f^o(s)^t R(s) = v(s) e^t$.

If possible, let $\mathcal{A}(s_0)$ not be completely mixed for some $s_0 \in S$. Then by Kaplansky [34], there exists an optimal strategy μ^* of player-1 at s_0 that is not completely mixed. Let $\mu^* = (\mu_1^*, \mu_2^*, \ldots, \mu_n^*)$. Thus,

$$\sum_{i=1}^{n} \left[(r(s_0, i, j) + \beta \sum_{s' \in S} v_\beta(s') q(s'|s_0, i)) \mu_i^* \right] = v_\beta(s_0).$$

Also,

$$\mathcal{A}(s_0)f^o(s_0) = \sum_{i=1}^{n}\Big[(r(s_0,i,j) + \beta\sum_{s'\in S} v_\beta(s')q(s'|s_0,i))f^o(s_0,i)\Big] = v_\beta(s_0).$$

Since $\mathcal{A}(s_0)$ is non-singular, μ^* and $f^o(s_0)$ must coincide. Thus, every optimal strategy for player-1 is completely mixed. Hence, the single-player controlled stochastic game is completely mixed. ■

The following example depicts a stochastic game where the matrix game restricted to each state is completely mixed, but the auxiliary game is not completely mixed.

Example 2.4.2 (*Sujatha et al. [96, Example 2]*) In this example, $R(s)$ is completely mixed for every s but the discounted stochastic game is not completely mixed at s_1.

Consider a discounted zero-sum player-2 controlled stochastic games with three states, namely, s_1, s_2, and s_3, and discount factor $\beta = 1/2$.

$$s_1 : \begin{bmatrix} \begin{matrix} 0 \\ (0,1,0) \end{matrix} & \begin{matrix} 2 \\ (0,0,1) \end{matrix} \\ & \\ \begin{matrix} 2 \\ (0,1,0) \end{matrix} & \begin{matrix} 1 \\ (0,0,1) \end{matrix} \end{bmatrix}, s_2 : \begin{bmatrix} \begin{matrix} 1 \\ (0,1,0) \end{matrix} & \begin{matrix} -1 \\ (0,1,0) \end{matrix} \\ & \\ \begin{matrix} -1 \\ (0,1,0) \end{matrix} & \begin{matrix} 1 \\ (0,1,0) \end{matrix} \end{bmatrix},$$
$$s_3 : \begin{bmatrix} \begin{matrix} 2 \\ (0,0,1) \end{matrix} & \begin{matrix} 0 \\ (0,0,1) \end{matrix} \\ & \\ \begin{matrix} 0 \\ (0,0,1) \end{matrix} & \begin{matrix} 2 \\ (0,0,1) \end{matrix} \end{bmatrix}.$$

The auxiliary game starting at s_1 is given by

$$\mathcal{A}(s_1) = \begin{bmatrix} 0 & 3 \\ & \\ 2 & 2 \end{bmatrix}.$$

The auxiliary game is not completely mixed. However, the matrix game $R(s_1)$ is completely mixed since $(1/3, 2/3)$ is the unique symmetric optimal strategy for $R(s_1)$. In fact, $R(s)$ is completely mixed for all s. ▲

2.5 Switching Control Stochastic Games

During the Game Theory Workshop at Cornell University, Maschler (1978) first suggested the following generalization of the single-player control game called switching control stochastic game.

Definition 2.5.1 Switching control stochastic game: In a switching control stochastic game, one player controls the transition probabilities in some states, while the other player controls these transitions in the remaining states. Let S_1 and S_2 be the subset of states where player-1 and player-2, respectively, control transitions, and $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \phi$.

Example 2.5.1 Consider a zero-sum two-player stochastic game with two states and the following payoff matrix:

$$s_1 : \begin{bmatrix} 0 & 2 \\ (.3, .7) & (.3, .7) \\ & \\ 4 & 1 \\ (.4, .6) & (.4, .6) \end{bmatrix}, s_2 : \begin{bmatrix} 2 & -2 \\ (.5, .5) & (1, 0) \\ & \\ -1 & 1 \\ (.5, .5) & (1, 0) \end{bmatrix}.$$

As is apparent, player-1 controls the transitions in state s_1 and player-2 in state s_2. Hence, this is an example for a switching control stochastic game. ▲

Filar [18] shows that switching control stochastic games (both zero-sum discounted and undiscounted) have a value and at least one pair of optimal stationary strategies exist. He further shows that such games possess orderfield property.

Theorem 2.5.1 (Filar [18, Theorem 4.1]) *Let Γ_β be a discounted switching control stochastic game. Then the following results hold:*

(1) $v_\beta(s)$ is a rational function of β for all s if β is sufficiently near 1.
*(2) If $s \in S_1$, player-*1 *has a uniformly discounted optimal strategy $f^o(s)$ (i.e., optimal for all β sufficiently near* 1*), while player-*2 *has a uniformly discounted optimal strategy $g^o(s)$ for $s \in S_2$.*

Proof Let the game have finitely many states indicated by $S = \{1, \ldots, n\}$. Consider the payoff matrix $\mathcal{A}_\beta(s)$ for the auxiliary game for each state $s \in S$ for every $\beta \in (0, 1)$, where the (i, j)-th element of the payoff matrix is given by

$$\begin{aligned} &r(s, i, j) + \beta \sum_{s' \in S} v_\beta(s') q(s'|s, i, j) \\ &= r(s, i, j) + \beta \sum_{s' \in S_1} v_\beta(s') q(s'|s, i) + \beta \sum_{s' \in S_2} v_\beta(s') q(s'|s, j). \end{aligned}$$

Shapley [84] showed that if $(f_\beta(s), g_\beta(s))$ is an optimal strategy pair for the matrix game $\mathcal{A}_\beta(s)$, then the optimal stationary strategy for the stochastic game is given by $f_\beta = (f_\beta(1), \ldots, f_\beta(n))$ and $g_\beta = (g_\beta(1), \ldots, g_\beta(n))$, respectively.

By applying Shapley-Snow theorem 1.6.1 to the matrix games $\mathcal{A}_\beta(s)$ for each $s \in S$ and any $\beta \in (0, 1)$, there exists a non-singular submatrix $\mathcal{A}'_\beta(s)$ of $\mathcal{A}_\beta(s)$ and a pair of extreme optimal strategies $(f_\beta(s), g_\beta(s))$ that satisfy

$$f_\beta(s)\mathcal{A}'_\beta(s) = v_\beta(s)e \text{ and}$$
$$\mathcal{A}'_\beta(s)g_\beta(s) = v_\beta(s)e,$$

where $e = (1, \ldots, 1)^t$.

Each $\mathcal{A}_\beta(s)$ has only finitely many non-singular submatrices which define probability vectors (the strategy pair). Let these be numbered as $A_1^s, \ldots, A_{k_s}^s$ for each $s \in S$. Now let us consider all permutations of the form $\kappa = (k(1), \ldots, k(S))$ where $k(s) \in \{1, \ldots, k_s\}$. Thus, there are $\mu = \prod_{s=1}^{S} k_s$ permutations which can be labeled based on some ordering as $\kappa_1, \ldots, \kappa_\mu$. Let l correspond to κ_l. Consider a selection of submatrix, one for each state given by

$$(A_{l(1)}(1), \ldots, A_{l(n)}(n)).$$

This selection determines player-1's strategy in states $s \in S_1$ and player-2's strategy in states $s \in S_2$. Define a stationary strategy $f_l(s)$ and $g_l(s)$ for player-1 and player-2, respectively, corresponding to $\kappa_l = (l_1, \ldots, l_n)$. Thus, we have μ stationary strategy pairs (f_l, g_l).

Indicate the transition probability matrix determined by the paid (f_l, g_l) by Q_l. Then for each $\beta \in (0, 1)$, the matrix $I - \beta Q_l$ is non-singular.

Thus, for each s,

$$[I - \beta Q(f_l, g_l)(s)]^{-1} r(f_l, g_l)_{s\text{-th coordinate}} = v_\beta|_{s\text{-th coordinate}},$$

where $v_\beta = \{v_\beta(1), \ldots, v_\beta(n)\}$. Note that v_β is continuous in β.

Filar [18, Lemma 3.1] states the following. Let v_β is a continuous function for $\beta \in (0, 1)$ and $u_{j,\beta}$ are k component-wise rational functions of β. If v_β coincides with one of these rational functions for each $\beta \in (0, 1)$, then there exists some $\beta_0 \in (0, 1)$ such that $v_\beta \equiv u_{j,\beta}$ for all $\beta \in (\beta_0, 1)$. Thus, there exists some $\beta_0 \in (0, 1)$ and some fixed l_0 such that $v_\beta \equiv u_{l_0,\beta}$ for all $\beta \in (\beta_0, 1)$. Thus, $v_\beta(s)$ is a rational function of β for all s if β is sufficiently near 1.

Also for all $\beta \in (\beta_0, 1)$, $f_\beta(s) = f_{l_0}(s)$ when $s \in S_1$ and $g_\beta(s) = g_{l_0}(s)$ when $s \in S_2$. Hence, player-1 and player-2 have a uniformly discounted optimal strategy $f^o(s)$ $s \in S_1$ and $g^o(s)$ for $s \in S_2$, respectively. ■

Theorem 2.4.4 can be extended to switching control stochastic games also.

Theorem 2.5.2 (Sujatha et al. [96, Theorem 3]) *Consider a finite discounted zero-sum switching control stochastic game where S_1 and S_2 are the set of states where player-1 and player-2 are, respectively, the controlling players, $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \phi$. Let $R(s)$ be symmetric for each $s \in S$. Further, suppose there exists a completely mixed optimal stationary strategy pair for the stochastic game. Then the following are equivalent:*

(1) The discounted stochastic game is completely mixed.

(2) The matrix game $R(s)$ is completely mixed for every s.

Proof For switching control stochastic games, Lemma 2.4.2 is restated as follows: For a finite discounted zero-sum switching control stochastic game where $R(s)$ is symmetric for each s, let (f^*, g^*) be a completely mixed optimal stationary strategy pair for the stochastic game. Then, $f^*(s)$ is the unique symmetric optimal strategy for $R(s)$ for all $s \in S_1$, and $g^*(s)$ is the unique symmetric optimal strategy for $R(s)$ for all $s \in S_2$.

The proof for Theorem 2.4.4 can then be mimicked for S_1 and S_2 separately. ■

2.6 Perfect Information Stochastic Games

Definition 2.6.1 Perfect information stochastic game (Shapley [84]): In a perfect information stochastic game, one player is a dummy player in every state, not necessarily the same player. Hence, it is a subclass of switching control stochastic games.

Example 2.6.1 Consider a zero-sum two-player discounted stochastic game with four states as shown below:

$$s_1 : \begin{bmatrix} 0 \\ (.3, 0, .7, 0) \\ 4 \\ (.4, .6, 0, 0) \end{bmatrix}, s_2 : \begin{bmatrix} 5 \\ (1, 0, 0, 0) \\ 2 \\ (.2, .2, .4, .2) \\ 3 \\ (.2, .1, 0, .7) \end{bmatrix},$$

$$s_3 : \begin{bmatrix} 2 & -2 \\ (.5, 0, .5, 0) & (1, 0, 0, 0) \end{bmatrix}, s_4 : \begin{bmatrix} -1 & 1 \\ (.5, .5, 0, 0) & (0, 0, 1, 0) \end{bmatrix}.$$

In states s_1 and s_2, player-1 controls the game and player-2 is the dummy player. In the remaining states, player-2 controls the game and player-1 is the dummy player. ▲

Theorem 2.6.1 (Shapley [84]) *In a discounted perfect information stochastic game, pure optimal stationary strategies exist for both players, and the orderfield property holds.*

Proof Let the perfect information stochastic game start in state s. Consider the auxiliary game $A(s)$ whose $(i,)$-th element is given by $r(s, i) + \beta \sum v(s')q(s'|s, i)$. Since this is a single-player control game, it will attain its value $v_\beta(s) = \max_i [r(s, i) + \beta \sum v(s')q(s'|s, i)]$. Hence, the optimal strategies are pure for both players. Hence, the orderfield property holds. ■

2.7 Separable Reward-State Independent Transition (SER-SIT) Stochastic Games

Definition 2.7.1 SER-SIT stochastic games (Parthasarathy et al. [77]) In a SER-SIT stochastic game, the following conditions hold:

(1) Separable reward function: The reward function can be expressed as a sum of two functions—one that depends on the state alone, and another that depends on the actions alone, i.e., $r_1(s, i, j) = c_1(s) + a_1(i, j)$ for player-1 and $r_2(s, i, j) = c_2(s) + a_2(i, j)$ for player-2, for all $s \in S$, for all $i \in A_1$, and for all $j \in A_2$.
(2) State independent transition function: The transition function does not depend on the state that the game transitions from, i.e., $q(s'|s, i, j) = q(s'|i, j)$ for all $s, s' \in S$, for all $i \in A_1$, and for all $j \in A_2$.

Example 2.7.1 (*Filar and Vrieze [19, Example 6.5.2]*) One of the most often quoted examples for SER-SIT stochastic games is the machine sharing scenario. Each day two players determine whether to make 1 or 2 units of product on a shared machine. If the total number of products made on a machine on a certain day exceeds two units, then the machine is deemed to be in a bad state the next day (state s_2), else it is considered to be in a good state (state s_1). Let $c(1)$ and $c(2)$ be the running cost of the machine in state s_1 (good state) and s_2 (bad state), respectively, where $c(1) < c(2)$. Let each product be worth $w(k)$ for each player where $k = 1, 2$. Then the following are the payoff matrices for this non-zero-sum SER-SIT game:

$$s_1 : \begin{bmatrix} \begin{matrix} w(1) - c(1), w(2) - c(1) \\ (1, 0) \end{matrix} & \begin{matrix} w(1) - c(1), 2w(2) - c(1) \\ (0, 1) \end{matrix} \\ & \\ \begin{matrix} 2w(1) - c(1), w(2) - c(1) \\ (0, 1) \end{matrix} & \begin{matrix} 2w(1) - c(1), 2w(2) - c(1) \\ (0, 1) \end{matrix} \end{bmatrix}$$

$$s_2 : \begin{bmatrix} \begin{matrix} w(1) - c(2), w(2) - c(2) \\ (1, 0) \end{matrix} & \begin{matrix} w(1) - c(2), 2w(2) - c(2) \\ (0, 1) \end{matrix} \\ & \\ \begin{matrix} 2w(1) - c(2), w(2) - c(2) \\ (0, 1) \end{matrix} & \begin{matrix} 2w(1) - c(2), 2w(2) - c(2) \\ (0, 1) \end{matrix} \end{bmatrix}.$$

If $a(1) = 5, a(2) = 4, c(1) = 1, c(2) = 3$, the payoff matrices are as follows:

$$s_1 : \begin{bmatrix} \begin{matrix} 4, 3 \\ (1, 0) \end{matrix} & \begin{matrix} 4, 7 \\ (0, 1) \end{matrix} \\ & \\ \begin{matrix} 9, 3 \\ (0, 1) \end{matrix} & \begin{matrix} 9, 7 \\ (0, 1) \end{matrix} \end{bmatrix}, s_2 : \begin{bmatrix} \begin{matrix} 2, 1 \\ (1, 0) \end{matrix} & \begin{matrix} 2, 5 \\ (0, 1) \end{matrix} \\ & \\ \begin{matrix} 7, 1 \\ (0, 1) \end{matrix} & \begin{matrix} 7, 5 \\ (0, 1) \end{matrix} \end{bmatrix}$$

. ▲

In the following theorem, Parthasarathy et al. [77] showed that, given a two-player finite non-zero-sum SER-SIT game, a state independent stationary equilibrium strategy pair can be easily found by just solving a single bimatrix game (E, F), where the (i, j)th entries of E and F for discounted SER-SIT games are

$$a_1(i, j) + \beta \sum_{s' \in S} c_1(s') q(s'|i, j)$$

and

$$a_2(i, j) + \beta \sum_{s' \in S} c_2(s') q(s'|i, j),$$

respectively. Here a_1, a_2, c_1, and c_2 are as per Definition 2.7.1.

Orderfield property holds for both discounted and undiscounted SER-SIT games as shown by Parthasarathy et al. [77].

Theorem 2.7.1 (Parthasarathy et al. [77, Theorem 3.3]) *Let Γ be a zero-sum SER-SIT game. Then Γ possesses the orderfield property for both discounted and undiscounted SER-SIT games. Further, in both cases, both players have optimal state independent strategies that lie in the same orderfield as the game parameters.* ∎

Theorem 2.7.2 (Parthasarathy et al. [77, Theorem 4.1]) *Let Γ_β be a non-zero-sum SER-SIT game with n states. Let player-1 and player-2 have m_1 and m_2 actions, respectively. Let (x^*, y^*) be an equilibrium point of the $m_1 \times m_2$ bimatrix game whose (i, j)-th entry is given by*

$$(a_{1_{ij}} + \beta \sum_{s' \in S} q(s'|i, j)\, c_1(s'),\; a_{2_{ij}} + \beta \sum_{s' \in S} q(s'|i, j)\, c_2(s'))$$

for $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$. Then, $((\underbrace{x^*, x^*, \ldots, x^*}_{n-times}), (\underbrace{y^*, y^*, \ldots, y^*}_{n\text{-}times}))$ is an equilibrium pair for the β-discounted game Γ_β. Further, the β-discounted game Γ_β has the orderfield property. ∎

In Krishnamurthy et al. [42], the authors discuss pure strategy equilibria and show that pure strategy equilibria of the SER-SIT game and of the bimatrix game (E, F) correspond. In general, equilibria of the SER-SIT game and of the bimatrix game (E, F) may not correspond. Parthasarathy et al. [77] give an example where the SER-SIT game has better equilibrium points than the bimatrix game to which it has been reduced.

Example 2.7.2 (*Parthasarathy et al. [77, Example 5.3]*) Consider a discounted SER-SIT stochastic game with two states s_1 and s_2. Let the payoff matrices be as follows:

$$s_1 : \begin{bmatrix} 1,100 & 0,0 & 0,5 \\ (1,0) & (1,0) & (1,0) \\ \\ 5,0 & 100,1 & 0,5 \\ (0,1) & (0,1) & (1,0) \\ \\ 5,0 & 0,0 & 9,9 \\ (0,1) & (0,1) & (0,1) \end{bmatrix}, s_2 : \begin{bmatrix} 5,104 & 4,4 & 4,9 \\ (1,0) & (1,0) & (1,0) \\ \\ 9,4 & 104,5 & 4,9 \\ (0,1) & (0,1) & (1,0) \\ \\ 9,4 & 4,4 & 13,13 \\ (0,1) & (0,1) & (0,1) \end{bmatrix}.$$

The unique state independent equilibrium point is $((3, 3), (3, 3))$ with payoff $\left(\frac{9+4\beta}{1-\beta}, 4 + \frac{9+4\beta}{1-\beta}\right)$ for both players. Consider the pair of stationary strategies $((2, 1), (2, 1))$. This is also an equilibrium pair with payoff $\left(\frac{100+5\beta}{2}, \frac{5+100\beta}{2}\right)$ and $\left(\frac{1+104\beta}{2}, \frac{104+1\beta}{2}\right)$ for player-1 and player-2, respectively. For $\beta \in (\frac{1}{2}, 1)$, this second equilibrium point yields a payoff for each state that is bigger than the payoff of the state independent equilibrium point for both the players. Thus, there are better equilibrium points for both players in the SER-SIT game than in the bimatrix game that it has been reduced to. ▲

Exercise 33 In Example 2.7.1, find the equilibrium strategies and payoff to each player.

To demonstrate the conditions under which a non-zero-sum SER-SIT stochastic game has a completely mixed equilibrium, consider the notations as defined below.

Lemma 2.7.1 (Sujatha et al. [96, Lemma 5]) *Let Γ_β be a two-player finite discounted non-zero-sum SER-SIT game. Let player-1 and player-2 have m_1 and m_2 actions, respectively. Let the reward functions of the players be given by $r_k(s, i, j) = c_k(s) + a_k(i, j)$, $k = 1, 2$, $s \in S$, $i = 1, \ldots, m_1$ and $j = 1, \ldots, m_2$. Let the transition probabilities be given by $q(s'|s, i, j) = q(s'|i, j)$, $s, s' \in S$, $i = 1, \ldots, m_1$ and $j = 1, \ldots, m_2$. Let (E, F) be the $m_1 \times m_2$ bimatrix game*

$$[a_1(i, j) + \beta \sum_{s'} c_1(s')q(s'|i, j), a_2(i, j) + \beta \sum_{s'} c_2(s')q(s'|i, j)],$$

where $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$. If (E, F) has a completely mixed equilibrium, then Γ_β has a completely mixed equilibrium. In fact, if Γ_β is a completely mixed game, so is (E, F).

Proof Let (x^*, y^*) be a completely mixed equilibrium of the bimatrix game (E, F). Define $f^*(s) \equiv x^*$ and $g^*(s) \equiv y^*$ for all $s \in S$. By Theorem 2.7.2 , (f^*, g^*) is an equilibrium pair for the discounted SER-SIT game Γ_β, and by construction, (f^*, g^*) is completely mixed.

By Theorem 2.7.2 , for any equilibrium of (E, F), we can construct an equilibrium of Γ_β. Therefore, if (E, F) has an equilibrium which is not completely mixed, so does Γ_β. In other words, if Γ_β is a completely mixed game, (E, F) is a completely mixed game too. ■

Remark 2.7.1 From the result of Parthasarathy et al. [77] for two-player finite undiscounted non-zero-sum SER-SIT games, Lemma 2.7.1 can be proved for two-player finite undiscounted non-zero-sum SER-SIT games as well.

2.8 Additive Reward-Additive Transition (ARAT) Stochastic Games

Definition 2.8.1 ARAT stochastic games (Raghavan et al. [81]) In an ARAT stochastic game $\Gamma_\beta = (S, A_1, A_2, r_1, r_2, \beta)$, the following conditions hold:

(1) Additive reward function: The reward function can be expressed as a sum of two functions—one that depends on the state and action of player-1 alone and another that depends on the state and action of player-2 alone. That is, for all $s \in S$, for all $i \in A_1$, and for all $j \in A_2$

$$r_1(s, i, j) = r_1'(s, i) + r_1''(s, j)\ for\ player - 1$$
$$r_2(s, i, j) = r_2'(s, i) + r_2''(s, j)\ for\ player - 2.$$

(2) Additive transition function: The transition function can be expressed as a sum of two functions—one that depends on the state and action of player-1 alone and another that depends on the state and action of player-2 alone. That is, for all $s, s' \in S$, for all $i \in A_1$, and for all $j \in A_2$

$$q(s'|s, i, j) = q_1(s'|s, i) + q_2(s'|s, j).$$

Example 2.8.1 (*Filar and Vrieze [19, Example 6.4.1]*) An often quoted example for ARAT stochastic games is the fishery game. There are two companies that engage in fishing. Each company has two actions with reference to whether they indiscriminately fish ("I") or not ("NI"). The availability of fish represents the three states of the stochastic game—good, medium, and bad.

The availability of fish in the next year is determined by whether the companies fished indiscriminately or not in the current year and is shown through the transition probability matrix. Thus, the transition probability is additive as both companies contribute to the change in state.

Let us first look at the transition probability as shown in the matrices below. The contribution of each company to the transition probability is indicated outside the matrix for each state. For instance, in every state, company-1 influences the transition probability by a total of 0.6 and company-2 by 0.4. So if each company indiscriminately fishes (action "I") when the game is in state "medium," company-1 contributes 0.1, 0.4, and 0.1 for the game to transition to states good, medium, and bad. Similarly, company-2 contributes 0.1, 0.2, and 0.1 for the game to transition to the respective states. Hence, under the indiscriminate actions of both companies together, the game transitions from state medium to good, medium, or bad with a combined probability of 0.2, 0.6, and 0.2.

Good :		I (0.3, 0.1, 0)	NI (0.2, 0.1, 0.1)
I	(0.5, 0.1, 0)	(0.8, 0.2, 0)	(0.7, 0.2, 0.1)
NI	(0.2, 0.3, 0.1)	(0.5, 0.4, 0.1)	(0.4, 0.4, 0.2)

Medium :		I (0.1, 0.2, 0.1)	NI (0, 0.2, 0.2)
I	(0.1, 0.4, 0.1)	(0.2, 0.6, 0.2)	(0.1, 0.6, 0.3)
NI	(0, 0.3, 0.3)	(0.1, 0.5, 0.4)	(0, 0.5, 0.5)

Bad :		I (0, 0.2, 0.2)	NI (0, 0, 0.4)
I	(0.1, 0.3, 0.2)	(0.1, 0.5, 0.4)	(0.1, 0.3, 0.6)
NI	(0, 0, 0.6)	(0, 0.2, 0.8)	(0, 0, 1)

The payoffs for the companies are dependent on their cost and revenue. The cost to the company depends on factors that are particular to the company (such as efficiency, equipment cost, etc.). The revenue to the company depends on the price of the fish, which in turn is dependent on the total availability of the fish, which in turn is a representation of the state of the system. For instance, in "good" state, fish is available in plenty but the abundance can drive the price down. Thus, the payoff is additive.

An example of this is shown in the matrices below. As in the case of the transition probability matrix, the contribution of each company to the total rewards is indicated outside the matrix for each state. Each cell indicates the total contribution by both companies for the actions they take.

Good :		I	NI
		0, 6	0, 9
I	8, 0	8, 6	8, 9
NI	10, 0	10, 6	10, 9

Medium :		I	NI
		0, 4	0, 6
I	7, 0	7, 4	7, 6
NI	9, 0	9, 4	9, 6

,

Bad :		I	NI
		0, 2	0, 3
I	5, 0	5, 2	5, 3
NI	6, 0	6, 2	6, 3

Hence, the complete payoff matrix for the three states can be represented as follows:

Good :

	I	NI
I	8, 6) (0.8, 0.2, 0)	8, 9 (0.7, 0.2, 0.1)
NI	10, 6 (0.5, 0.4, 0.1)	10, 9 (0.4, 0.4, 0.2)

Medium :

	I	NI
I	7, 4 (0.2, 0.6, 0.2)	7, 6 (0.1, 0.6, 0.3)
NI	9, 4 (0.1, 0.5, 0.4)	9,6 (0, 0.5, 0.5)

Bad :

	I	NI
I	5, 2 (0.1, 0.5, 0.4)	5, 3 (0.1, 0.3, 0.6)
NI	6, 2 (0, 0.2, 0.8)	6, 3 (0, 0, 1)

▲

If this were a one-shot game, indiscriminate fishing would be of advantage. However, in the case of a repeated game over an infinite horizon, this need not be true.

Theorem 2.8.1 (Raghavan et al. [81, Theorem 2.1]) *There are pure optimal stationary strategies for both players for discounted and undiscounted zero-sum ARAT stochastic games. The orderfield property also holds for discounted and undiscounted ARAT stochastic games. Further, there are pure stationary strategies for both players that are uniformly optimal for all discount factors sufficiently close to* 1. ■

Non-zero-sum ARAT stochastic games need not necessarily have pure stationary equilibria. However, if the game does possess a stationary equilibrium strategy, then there also exists a stationary equilibrium strategy which uses in each state at most two pure actions for each player. This is shown in the following theorem.

Theorem 2.8.2 (Raghavan et al. [81, Theorem 3.1]) *Consider a discounted non-zero-sum ARAT stochastic game* Γ. *If* (f^o, g^o) *form an equilibrium pair of stationary strategies, then there exists an equilibrium pair* $(\tilde{f}, \tilde{g})$ *such that* $v^1_\beta(f^o, g^o) = v^1_\beta(\tilde{f}, \tilde{g})$ *and* $v^2_\beta(f^o, g^o) = v^2_\beta(\tilde{f}, \tilde{g})$ *and such that support* $(\tilde{f}_s) \leq 2$ *and support* $(\tilde{g}_s) \leq 2$ *for each* $s \in S$. ■

Jaśkiewicz and Nowak [33] discover a class of discounted stochastic games with stationary "almost Markov" perfect equilibria. "Almost Markov" refers to the fact that the equilibrium strategy for each player depends on the current state and the previous state of the game. That is, there is a dependence on two consecutive states. This class of games has Borel state space and compact action spaces. The transition probability is a convex combination of finitely many probability measures depending on the states. The results in this paper enhance the results by Mertens and Parthasarathy [56].

2.9 Mixture Class of Stochastic Games

We have looked at special classes of stochastic games such as ARAT games, SER-SIT games, and single-player control games. New stochastic games may be created by mixing different classes of stochastic games and allowing for transitions among states of different classes. It is possible that such mixture classes of stochastic games may not exhibit orderfield property (refer to Definition 1.8.1) or stationary equilibrium property (refer to Definition 2.9.1) though the individual class of games may exhibit orderfield property or stationary equilibrium property.

2.9.1 Orderfield Property for Mixture Classes of Stochastic Games

Consider a two-player discounted stochastic game where the state space S is as follows: $S = S_1 \cup S_2 \cup S_3$ where $S_1 \cap S_2 \cap S_3 = \phi$; player-1 and player-2 control the actions in states S_1 and S_2, respectively (that is, switching control game); and S_3 is an ARAT game. Sinha [92] shows how a mixture of switching control and ARAT classes has orderfield property. In the multi-player scenario, Mohan et al. [61] show that the discounted single-player control polystochastic game has orderfield property.

However, not all mixture classes of stochastic games exhibit orderfield property. Sinha [92] considers an example of a mixture of two zero-sum SER-SIT games. It is known that SER-SIT games by themselves exhibit orderfield property. However, this new game does not possess orderfield property as shown in the following example and is left as an exercise for the reader.

Exercise 34 Let the payoff and transition matrix for a β-discounted zero-sum stochastic game with four states, namely, $S = \{s_1, s_2, s_3, s_4\}$, be as follows:

s_1:

0 (1, 0, 0, 0)	0 (0, 0, 0.5, 0.5)
0 (1, 0, 0, 0)	0 (0, 0, 0.5, 0.5)

s_2:

-1 (1, 0, 0, 0)	-1 (0, 0, 0.5, 0.5)
-1 (1, 0, 0, 0)	-1 (0, 0, 0.5, 0.5)

s_3:

-1 (1, 0, 0, 0)	-1 (0, 0, 0.5, 0.5)
0 (1, 0, 0, 0)	0 (0, 0, 0.5, 0.5)

s_4:

0 (1, 0, 0, 0)	0 (0, 0, 0.5, 0.5)
1 (1, 0, 0, 0)	1 (0, 0, 0.5, 0.5)

The state space S is partitioned into two disjoint subsets $S_1 = \{s_1, s_2\}$ and $S_2 = \{s_3, s_4\}$ where S_1 and S_2 are both SER-SIT games and exhibit orderfield property. It can be easily seen that the complete game with state space S is not SER-SIT as it does not meet the property of separable rewards. Show that the stochastic game starting at state s_1 does not exhibit orderfield property.

Krishnamurthy et al. [43] further identify some mixture class of stochastic games that exhibit orderfield property. The technique used involves defining a new stochastic game, and defining its reward and transition functions in terms of the original game's reward and transition functions. It is then shown that the new game, and hence the original game, has the orderfield property.

Theorem 2.9.1 (Krishnamurthy et al. [43, Theorem 3.1]) *Consider a finite, zero-sum β discounted stochastic game $\Gamma_\beta = (S, A_1, A_2, r, q, \beta)$ where $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \phi$. Further let the following conditions hold:*

(1) The inputs to Γ_β (namely r, q, and β) are rational.
(2) S_1 is one of the following classes of games: single-player control, perfect information, switching control, and ARAT.
(3) S_2 is a sink. That is, $\sum_{s_2 \in S_2} q(s_2|s, i, j) = 1$ for all $s \in S_2$, $i \in A_1$, $j \in A_2$.
(4) The subgame restricted to state S_2 has the orderfield property.

Then the mixed stochastic game Γ_β has the orderfield property.

Proof We provide an overview of the proof when S_1 is a single-player control game. It can be proved similarly for other classes of games mentioned in the theorem statement. Without loss of generality, assume that S_1 is controlled by player-1. Thus, for all $s \in S, s_1 \in S_1, i \in A_1, j \in A_2$, we have $q(s|s_1, i, j) = q(s|s_1, i)$

Define a new game $\Gamma'_\beta = (S', A_1, A_2, r', q', \beta)$ where $S' = S \cup \{s^*\}$ with s^* being a new absorbing state. Further, define the payoff function and transition function for the new game as follows:

$$r'(s, i, j) = r(s, i, j) + \beta \sum_{s_2 \in S_2} q(s_2|s, i) v_\beta(s_2), \forall s \in S_1, \forall i \in A_1, \forall j \in A_2 \tag{2.12}$$

$$\begin{aligned}
&r'(s, i, j) = r(s, i, j), \forall s \in S_2, \forall i \in A_1, \forall j \in A_2\\
&r'(s^*, i, j) = 0, \forall i \in A_1, \forall j \in A_2\\
&q'(s_1|s, i) = q(s_1|s, i), \forall s \in S_1, \forall s_1 \in S_1, \forall i \in A_1\\
&q'(s_2|s*, i) = 0, \forall s_2 \in S_2, \forall i \in A_1\\
&q'(s_2|s, i, j) = q(s_2|s, i, j), \forall s \in S_2, \forall s_2 \in S_2, \forall i \in A_1, \forall j \in A_2\\
&q'(s^*|s, i) = 1 - \sum_{s_1 \in S_1} q(s_1|s, i), \forall s \in S_1, \forall i \in A_1\\
&q'(s^*|s^*, i, j) = 1, \forall i \in A_1, \forall j \in A_2
\end{aligned}$$

Based on the above definitions of the new payoff function and transition probability function, it is clear that the subgame restricted to S_2 is the same in both Γ_β and Γ'_β. Thus, for all $s_2 \in S_2$, the value and optimal strategies in Γ_β and Γ'_β are the same.

Let the game start in S_1. Then for all $s \in S_1$, by applying Eq. 2.12, we have

$$
\begin{aligned}
v'_\beta(s) &= \mathrm{val}\Big(r'(s,i,j) + \beta \sum_{s_1 \in S_1} q(s_1|s,i) v'_\beta(s_1)\Big) \\
&= \mathrm{val}\Big(r(s,i,j) + \beta \sum_{s_2 \in S_2} q(s_2|s,i) v'_\beta(s_2) + \beta \sum_{s_1 \in S_1} q(s_1|s,i) v'_\beta(s_1)\Big) \\
&= \mathrm{val}\Big(r(s,i,j) + \beta \sum_{s_1 \in S} q(s_1|s,i) v'_\beta(s_1)\Big) \\
&= v_\beta(s)
\end{aligned}
$$

The last equation follows from the uniqueness of $v_\beta(s)$ from Banach's fixed point theorem. Thus, the value and optimal strategies coincide for the new and the original stochastic game. The subgame of Γ'_β restricted to $S_1 \cup \{s^*\}$ is a single-player control game and has the orderfield property. Thus, the orderfield property holds in S_1 and S_2 in both Γ_β and Γ'_β. ∎

Krishnamurthy et al. [43] provide an example of a mixture class of stochastic games that has the orderfield property.

Exercise 35 (Krishnamurthy et al. [43, Example 3.1]) Consider the following stochastic game with three states s_1, s_2, and s_3. The game is a mixture of a set of SER-SIT states $S_1 = \{s_1, s_2\}$ and a single-player control state $S_2 = \{s_3\}$.

$$
s_1 : \begin{bmatrix} \begin{matrix} 3 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) \end{matrix} & \begin{matrix} 0 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) \end{matrix} \\ \\ \begin{matrix} 0 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) \end{matrix} & \begin{matrix} 1 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) \end{matrix} \end{bmatrix}, s_2 : \begin{bmatrix} \begin{matrix} 2 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) \end{matrix} & \begin{matrix} -1 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) \end{matrix} \\ \\ \begin{matrix} -1 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) \end{matrix} & \begin{matrix} 0 \\ (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}) \end{matrix} \end{bmatrix},
$$

$$
s_3 : \begin{bmatrix} 0 \\ (1, 0, 0) \end{bmatrix}.
$$

Let v_1, v_2, and v_3 be the value for the stochastic game starting in state s_1, s_2, and s_3, respectively. It can be seen that $v_2 = v_1 - 1$ and $v_3 = \beta v_1$. Using Shapley's theorem, show that the v_1 is rational when β is rational. Also find the optimal strategies for the two players.

Theorem 2.9.1 can be extended to a mixture of more than two classes of games. This can be proved by applying the proof for Theorem 2.9.1 inductively.

Also, Theorem 2.9.1 still holds for two-player non-zero-sum stochastic games if the orderfield property in condition 4 is replaced with the following: "The sub-game restricted to S_2 has a pair of equilibrium strategies (f^*, g^*) with rational coordinates." For other mixture classes of stochastic games, refer to Krishnamurthy et al. [43].

2.9.2 Stationary Equilibrium Property for Mixture Classes of Stochastic Games

Definition 2.9.1 Stationary equilibrium property of a game: A stochastic game $(S, A_1, A_2, r_1, r_2, q, \beta)$ has the stationary equilibrium property if the game has a stationary equilibrium pair (f^o, g^o) satisfying the following conditions $\forall s \in S$:

(1) For player-1:

$$\begin{aligned} u_1(s) &= \max_{\mu \in A_1} \left[r_1(s, \mu, g^o(s)) + \beta \int u_1(s')\, dq(s', \mu, g^o(s)) \right] \\ &= r_1(s, f^o(s), g^o(s)) + \beta \int u_1(s')\, dq(s', f^o(s), g^o(s)) \; and \end{aligned} \tag{2.13}$$

(2) For player-2:

$$\begin{aligned} u_2(s) &= \max_{\lambda \in A_2} \left[r_2(s, f^o(s), \lambda) + \beta \int u_2(s')\, dq(s', f^o(s), \lambda) \right] \\ &= r_2(s, f^o(s), g^o(s)) + \beta \int u_2(s')\, dq(s', f^o(s), g^o(s)) \end{aligned} \tag{2.14}$$

Unlike the uniqueness of the optimal value in the zero-sum game, the Nash equilibrium payoffs may not be unique.

Remark 2.9.1 From the above, it is clear that $u_k(s) = I_\beta^{(k)}(f^o, g^o)(s)$, for $k = 1, 2$.

Krishnamurthy et al. [44] look at mixture classes of games that exhibit the stationary equilibrium property. The technique used in the proof is similar to the technique demonstrated in Theorem 2.9.1.

Theorem 2.9.2 (Krishnamurthy et al. [44, Theorem 1]) *Consider a mixture class of discounted non-zero-sum stochastic game given by $\Gamma_\beta = (S, A_1, A_2, r_1, r_2, q, \beta)$ where*

(1) S_1 and S_2 are subsets of a complete, separable metric space S such that $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \phi$.
(2) S_1 is finite or countable. S_2 is an absorbing, complete, and separable metric space that has a stationary equilibrium pair. Note that S_2 is an absorbing state space if $\sum_{s' \in S_2} q(s'|s, i, j) = 1$ for all s, i, j where $s \in S_2$ whenever S_2 is finite or countable. Further, let (f_2, g_2) be a stationary equilibrium pair for S_2.

Then the mixture game Γ_β has a stationary equilibrium pair.

Proof Since S_2 is an absorbing state space, we have $\forall s \in S_2, \forall i \in A_1, \forall j \in A_2$

$$\int_{s' \in S_2} q(s'|s, i, j) = 1$$

Let (f_2, g_2) be a stationary equilibrium pair for the subgame Γ_β restricted to S_2. Let the corresponding discounted expected payoff for player-k be $I_k(f_2, g_2)(s)$, where $k = 1, 2$.

Define a new game $\Gamma'_\beta = (S \cup \{s^*\}, A_1, A_2, r'_1, r'_2, q', \beta)$ where s^* is a new absorbing state. That is, $\forall i \in A_1, \forall j \in A_2$, we have

$$q'(s^*|s^*, i, j) = 1 \tag{2.15}$$

In this new game, $\forall i \in A_1, \forall j \in A_2, k = 1, 2$, let

$$r'_k(s, i, j) = r_k(s, i, j) + \beta \int_{S_2} I_k(f_2, g_2)(s')\, dq(s'|s, i, j), \forall s \in S_1 \tag{2.16}$$

$$r'_k(s^*, i, j) = 0 \tag{2.17}$$

$$r'_k(s, i, j) = r_k(s, i, j), \forall s \in S_2 \tag{2.18}$$

$$q'(s^*|s, i, j) = 1 - \int_{s' \in S_1} dq(s'|s, i, j), \forall s \in S_1 \tag{2.19}$$

$$q'(s'|s, i, j) = q(s'|s, i, j), \forall s, s' \in S_2 \tag{2.20}$$

$$q'(s'|s, i, j) = 0, \forall s \in S_2, \forall s' \in S_1 \tag{2.21}$$

It is apparent that the subgame restricted to S_2 is the same independent stochastic game in both Γ_β and Γ'_β. Hence, a stationary equilibrium pair for the subgame restricted to S_2 in Γ'_β remains (f_2, g_2).

The subgame restricted to $S_1 \cup \{s^*\}$ is a finite or countable non-zero-sum discounted stochastic game, and hence has an equilibrium pair, say (f_1, g_1) (Fink [20] and Takahashi [98] for finite games; Federgruen [17] for countable state space).

Define a strategy pair (f', g') for the new game Γ' as follows:

$$f'(s) = \begin{cases} f_1(s), s \in S_1 \\ f_2(s), s \in S_2 \end{cases} \text{ for player} - 1 \textit{ and} \tag{2.22}$$

$$g'(s) = \begin{cases} g_1(s), s \in S_1 \\ g_2(s), s \in S_2 \end{cases} \text{ for player} - 2. \tag{2.23}$$

Note that $f'(s^*)$ can be any arbitrary element in P_{A_1} as s^* is an absorbing state. Similarly, $g'(s^*)$ can be any arbitrary element in P_{A_2} as s^* is an absorbing state.

To show that (f', g') is a stationary equilibrium pair for Γ_β, let the new game Γ'_β start in state space S_2. Since S_2 is an absorbing state space, the game never moves to S_1 from S_2.

$\Rightarrow \forall s \in S_2,\ f'(s) = f_2(s)$, and $g'(s) = g_2(s)$.
$\Rightarrow (f', g') = (f_2, g_2)$ is a stationary equilibrium pair trivially.

For player-1, $\forall s \in S_2$

$$\begin{aligned} &I_1(f_2, g_2)(s) \\ &= \max_{\mu} \Big[r_1(s, \mu, g_2(s)) + \beta \int_{S_2} I_1(f_2, g_2)(s')\, dq(s'|s, \mu, g_2(s)) \Big] \\ &= \max_{\mu} \Big[r_1(s, \mu, g_2(s)) + \beta \int_{S_1 \cup S_2} I_1(f_2, g_2)(s')\, dq(s'|s, \mu, g_2(s)) \Big] \\ &\qquad\qquad\qquad\qquad \text{(as } S_2 \text{ is an absorbing state)} \end{aligned} \tag{2.24}$$

$$\begin{aligned} &= \max_{\mu} \Big[r_1(s, \mu, g_2(s)) + \beta \int_{S_1 \cup S_2} I_1(f', g')(s')\, dq(s'|s, \mu, g_2(s)) \Big] \\ &\qquad\qquad\qquad\qquad \text{(using Eqs. 22 and 23)} \\ &= u_2(s) \text{ (say)}. \end{aligned}$$

Now let the new game Γ'_β start in $S_1 \cup \{s^*\}$. Then $\forall s \in S_1 \cup \{s^*\}$

$$\begin{aligned} u_1(s) &= \max_{\mu} \Big[r'_1(s, \mu, g'(s)) + \beta \int_{S_1 \cup \{s^*\}} I_1(f', g')(s')\, dq(s'|s, \mu, g'(s)) \Big], \\ &= \max_{\mu} \Big[r'_1(s, \mu, g'(s)) + \beta \int_{S_1} I_1(f', g')(s')\, dq(s'|s, \mu, g'(s)) \Big] \\ &= \max_{\mu} \Big[r_1(s, \mu, g_1(s)) + \beta \int_{S_2} I_1(f_2, g_2)(s')\, dq(s'|s, \mu, g_1(s)) \\ &\qquad + \beta \int_{S_1} I_1(f', g')(s')\, dq(s'|s, \mu, g'(s)) \Big] \text{(using Eq. 16)} \\ &= \max_{\mu} \Big[r_1(s, \mu, g_1(s)) + \beta \int_{S_1 \cup S_2} I_1(f', g')(s')\, dq(s'|s, \mu, g_1(s)) \Big]. \\ &\qquad\qquad\qquad\qquad \text{(using Eqs. 2.22 and 2.23)} \end{aligned} \tag{2.25}$$

Similarly, $v_1(s)$ and $v_2(s)$ can be specified for player-2. Define

$$\begin{aligned} u^*(s) &= \begin{cases} u_1(s), s \in S_1 \\ u_2(s), s \in S_2 \end{cases}, \text{for player} - 1 \\ and v^*(s) &= \begin{cases} v_1(s), s \in S_1 \\ v_2(s), s \in S_2 \end{cases}, \text{for player} - 2. \end{aligned} \tag{2.26}$$

It is easily seen that $u^*(s^*) = 0$ and $v^*(s^*) = 0$. Using the definition of $u^*(s)$, Eqs. 2.24 and 2.25 can be combined to yield

$$\begin{aligned} u^*(s) &= \max_{\mu}\left[r_1(s, \mu, g'(s)) + \beta \int\limits_{S_1 \cup S_2} u^*(s')\, dq(s'|s, \mu, g'(s))\right] \\ &= r_1(s, f'(s), g'(s)) + \beta \int\limits_{S_1 \cup S_2} u^*(s')\, dq(s'|s, f'(s), g'(s)). \end{aligned} \tag{2.27}$$

By Banach fixed point theorem, the value of $u^*(s)$ is unique. Proceeding along the same lines for player-2, we get

$$\begin{aligned} v^*(s) &= \max_{\lambda}\left[r_2(s, f'(s), \lambda) + \beta \int\limits_{S_1 \cup S_2} v^*(s')\, dq(s'|s, f'(s), \lambda)\right] \\ &= r_2(s, f'(s), g'(s)) + \beta \int\limits_{S_1 \cup S_2} v^*(s')\, dq(s'|s, f'(s), g'(s)), \end{aligned} \tag{2.28}$$

where $v^*(s)$ is unique by Banach fixed point theorem.

Hence, (f', g') is a stationary equilibrium pair for the mixture class of game, and (u^*, v^*) is the equilibrium payoff for the mixture class of game. ■

Refer to Krishnamurthy et al. [44] for other mixture classes of stochastic games which have stationary equilibrium, such as S_1 is a SER-SIT or ARAT or SIT game, or the game is a mixture of two classes of SER-SIT games. Further these results can be extended to more than two classes of games.

2.10 Algorithmic Aspects of Finite Stochastic Games

Knowing which classes of stochastic games exhibit orderfield property allows us to look at algorithms for stochastic games. Raghavan and Filar [79] provide a comprehensive survey on algorithms for stochastic games. Not all stochastic games possess orderfield property, even in some discounted zero-sum stochastic games (Shapley [84]).

Krishnamurthy [40] shows that orderfield property holds for specific types of stochastic games such as single-player control stochastic games, switching control stochastic games, perfect information stochastic games, and non-zero-sum stochastic games. He also mentions connection with planar graphs and poses some open questions. The question still remains whether there are any natural classes of games that satisfy the orderfield property.

Open Question 2.10.2 There is no efficient algorithm for finding the value and set of optimal strategies for switching control stochastic games and perfect information stochastic games. Raghavan and Syed [80] provide a policy improvement algorithm for solving zero-sum two-player perfect information games. However, an efficient general algorithm for perfect information stochastic games still does not exist. Policy improvement algorithm works for control problems but it is not clear how it works for two-player games.

2.11 Summary

When the state space and action space are finite in stochastic games, the game is termed as a finite stochastic game. The stochastic game transitions between states based on the transition probability determined by the initial state and the actions of the players. This gives rise to two types of strategies: stationary strategies and behavioral strategies. Restricting ourselves to stationary strategies and finite stochastic games, we look at the conditions under which the stochastic game and the various class of stochastic games have stationary optimal (or equilibrium strategy). The concept of completely mixed stochastic game is defined akin to completely mixed matrix games, except that the optimal strategy must be completely mixed for every state for every player. The orderfield property is analyzed for the various classes of stochastic games to shed light on the algorithmic aspects.

Chapter 3
Infinite Stochastic Games

When the action space and/or state space are not finite in a stochastic game, the game is called an *infinite stochastic game*. Similar to the finite stochastic game, the two-player zero-sum infinite stochastic game is played as follows. Indicate the state space by $S \subseteq [0, 1]$ and action space by A_1 and A_2 for each player, respectively. Let S, A_1, and A_2 be non-empty Borel subsets of a Polish space. Let the two players observe the state of the system $s \in [0, 1]$ and select an action simultaneously (say, $a \in A_1$ and $b \in A_2$, respectively). Let the immediate reward for player-1 be $r(s, a, b)$, where $r(.)$ is a bounded measurable function on $S \times A_1 \times A_2$. The game now moves to the state $s' \in S$ based on a probability measure $q(s'|s, a, b)$, referred to as the transition probability. The game is played at discrete points of time (say, once a day) over the infinite future in this manner. It may happen that player-1's accumulated income $\sum\limits_{n=0}^{\infty} r(s_n, a_n, b_n)$ may not converge. To make it convergent, the nth day income is discounted by β^{n-1} where the discount factor is given by $0 \le \beta < 1$.

Definition 3.0.1 Stationary strategy: Let A_1 be the action set of player-1. Let P_{A_1} be the space of probability distributions on A_1 where P_{A_1} is metrizable on weak topology (Parthasarathy [68]). A Borel measurable function $f : S \to P_{A_1}$ is called a stationary strategy for player-1 . Similarly, a Borel measurable function $g : S \to P_{A_2}$ is a stationary strategy for player-2. The set of stationary strategies is indicated by $P^S_{A_1}$ and $P^S_{A_2}$, respectively.

T. Parthasarathy and S. Babu, *Stochastic Games and Related Concepts*,
HBA Lecture Notes in Mathematics 2,
https://doi.org/10.1007/978-981-15-6577-9_3

3.1 Auxiliary Game

Consider a zero-sum two-player β-discounted infinite stochastic game. Let the payoff at the end of day 1 be $r(s, a, b)$ and the game transition to state s' based on the transition probability $q(s'|s, a, b)$. On day 2, the player gets a terminal reward $v(s')$ and the game ends. The payoff at the end of day 2 is $r(s, a, b) + \beta \int v(s')dq(s'|s, a, b)$ where $v : [0, 1] \to \mathbb{R}$ is a bounded measurable function.

Assume $r(s, a, b)$ is continuous over $S \times A_1 \times A_2$. Further assume that the transition function $q(.)$ is continuous. That is, if $(s_n, a_n, b_n) \to (s_0, a_0, b_0)$, then $q(.|s_n, a_n, b_n)$ converges weakly to $q(.|s_0, a_0, b_0)$.

Then $r(s, a, b) + \beta \int v(s')dq(s'|s, a, b)$ is continuous on $S \times A_1 \times A_2$ provided A_1 and A_2 are finite.

Now, consider a distribution (μ, λ) on a and b. For a fixed s, consider

$$r(s, \mu, \lambda) + \beta \int v(s')dq(s'|s, \mu, \lambda). \tag{3.29}$$

We assume Eq. 3.29 is continuous on $P_{A_1} \times P_{A_2}$ for every fixed s.

When A_1 and A_2 are compact Hausdorff pure strategy spaces, Glicksberg [27] showed that minmax result holds. That is,

$$\begin{aligned} &\min_{\lambda} \max_{\mu} [r(s, \mu, \lambda) + \beta \int v(s')dq(s'|s, \mu, \lambda)] \\ &= \max_{\mu} \min_{\lambda} [r(s, \mu, \lambda) + \beta \int v(s')dq(s'|s, \mu, \lambda)] \end{aligned} \tag{3.30}$$

3.2 Optimal Stationary Strategies for Infinite Stochastic Games

For a finite discounted stochastic game, Shapley [84] showed that the value of the game exists and that there is an optimal stationary strategy for both players. Maitra and Parthasarathy [50] proved that the discounted infinite stochastic game has a value and both players have optimal stationary strategies under certain conditions.

Theorem 3.2.1 (Maitra and Parthasarathy [50, Theorem 3.1]) *Consider a discounted infinite stochastic game. Let g^* be a fixed stationary strategy for player-2 for the stochastic game. Then*

$$\sup_f I_\beta(f, g^*)(s) = \sup_\pi I_\beta(\pi, g^*)(s),$$

where the supremum is taken over all the strategies for player-1.

Proof The following is an overview of the proof. Let $\tilde{\pi} = (f_1, \ldots, f_n, \ldots)$ be a behavioral strategy for player-1. If player-2 is allowed to use only the strategy

g^* while player-1's choice remains unrestricted, the stochastic game reduces to a dynamic programming problem. For one-player control games, Blackwell [6] shows that $\sup_\pi I_\beta(\pi, g^*)(s) = \sup I_\beta(f, g^*)(s)$.

Then, given any π of the game problem, one can always associate an equivalent $\tilde{\pi}$ such that $I_\beta(\pi, g^*)(s) = I_\beta(\tilde{\pi}, g^*)(s)$ using induction and dominated convergence theorem. For a complete proof, see Maitra and Parthasarathy [50]. ■

Theorem 3.2.2 (Maitra and Parthasarathy [50, Theorem 4.1]) *Consider the discounted infinite stochastic game where S, A_1, and A_2 are compact metric spaces. Let r be a continuous function on $S \times A_1 \times A_2$. Assume that whenever $s_n \to s_0$, $a_n \to a_0$, and $b_n \to b_0$, $q(.|s_n, a_n, b_n)$ weakly converges to $q(.|s_0, a_0, b_0)$. Then the stochastic game has a value and both players have optimal stationary strategies.*

Proof Let $f : S \to P_{A_1}$ and $g : S \to P_{A_2}$ be stationary strategies for player-1 and player-2, respectively. Now, $I_\beta(f, g)(s) = \sum_{n=0}^{\infty} \beta^n r_n(s, f, g)$. Let $U_\beta(s)$ and $L_\beta(s)$ indicate the upper and lower values of the game given by

$$U_\beta(s) = \inf_g \sup_f I_\beta(f, g)(s)$$
$$L_\beta(s) = \sup_f \inf_g I_\beta(f, g)(s)$$

It is easy to see that $U_\beta(s) \geq L_\beta(s)$ for all s.

Let u be any continuous measurable function $u : S \to \mathbb{R}$. For a fixed s, consider the auxiliary game $\mathcal{A}(s)$ whose (μ, λ)-th entry is given by $r(s, \mu, \lambda) + \beta \int u(s')dq(s'|s, \mu, \lambda)$. Let $T : C[0, 1] \to C[0, 1]$ where

$$\begin{aligned} Tu(s) &= \min_\lambda \max_\mu [r(s, \mu, \lambda) + \beta \int u(s')dq(s'|s, \mu, \lambda)] \\ &= \max_\mu \min_\lambda [r(s, \mu, \lambda) + \beta \int u(s')dq(s'|s, \mu, \lambda)] \\ &= \text{ value of the auxiliary game } \mathcal{A}(s) \end{aligned}$$

Then T is a contraction mapping with contraction coefficient $\beta \in [0, 1)$. That is,

$$||Tu - Tv|| \leq \beta||u - v|| \text{ where } ||u|| = \max_{s \in [0,1]} |u(s)|.$$

Hence, by Banach fixed point theorem, there is a unique fixed point of the map T, i.e., $Tu^* = u^*$.

$$\begin{aligned} Tu^*(s) &= \min_\lambda \max_\mu [r(s, \mu, \lambda) + \beta \int u^*(s')dq(s'|s, \mu, \lambda)] \\ &= \max_\mu \min_\lambda [r(s, \mu, \lambda) + \beta \int u^*(s')dq(s'|s, \mu, \lambda)]. \end{aligned} \tag{3.31}$$

Now $\max_{\mu}[r(s,\mu,\lambda)+\beta\int u^*(s')dq(s'|s,\mu,\lambda)]=\psi(s,\lambda)$ is continuous on $S\times P_{A_2}$. Hence, by Dubin-Savage theorem (Theorem B.2.1 in Appendix), there exists a measurable $g^*: S\to P_{A_2}$ such that $\min_\lambda \psi(s,\lambda)=\psi(s,g^*(s))$. Similarly, there exists a measurable $f^*: S\to P_{A_1}$ such that $\max_{\mu}\omega(s,\mu)=\omega(s,f^*(s))$.

From Eq. 3.31, (f^*, g^*) is a saddle point as shown below:

$$\begin{aligned}&\max_{\mu}[r(s,\mu,g^*(s))+\beta\int u^*(s')dq(s'|s,\mu,g^*(s))]\\&=r(s,f^*(s),g^*(s))+\beta\int u^*(s')dq(s'|s,f^*(s),g^*(s))\\&=\min_{\lambda}[r(s,f^*(s),\lambda)+\beta\int u^*(s')dq(s'|s,f^*(s),\lambda)]\\&=u^*(s).\end{aligned}$$

Now $Tu^*(s)$ is a continuous function where u^* depends on β. Also, $u^*(s)=I_\beta(f^*,g^*)(s)$. Let f be any arbitrary stationary strategy. Let $M(S)$ be the space of real-valued bounded measurable functions. Define $(L_{fg}u)(s)$ as $L_{fg}: M(S)\to M(S)$. From Eq. 3.31,

$$u^*(s)\geq r(s,f(s),g^*(s))+\beta\int u^*(s')dq(s'|s,f(s),g^*(s))=(L_{fg^*}u^*)(s).$$

Thus $L_{fg^*}u^*\geq L^2_{fg^*}u^*$. But $u^*\geq L_{fg^*}u^*\geq L^2_{fg^*}u^*$. Repeating this, $u^*\geq L_{fg^*}u^*\geq L^2_{fg^*}u^*\geq\cdots\geq L^n_{fg^*}u^*\geq\ldots$.

From the definition of $(L_{fg}u)(s)$, L_{fg} is a contraction map and has a unique fixed point. Say $L^n_{fg^*}u^*\to L^\infty_{fg^*}u^*$. Then, $u^*\geq L^\infty_{fg^*}u^*$. Hence, as $n\to\infty$, $L^n_{fg^*}u^*\to I_\beta(f,g^*)$ implies that $u^*\geq lim_{n\to\infty}L^n_{fg^*}u^*=I_\beta(f,g^*)$, i.e., $I_\beta(f^*,g^*)\geq I_\beta(f,g^*)$ for all f.

$L^n_{fg^*}u^*$ can be viewed as follows. For the first n-days, players use (f,g^*). On $n+1^{st}$ day, there is a terminal reward u^*. Hence, as $n\to\infty$, the game is as if the players play (f,g^*) every day. Similarly, $I_\beta(f^*,g)\geq I_\beta(f^*,g^*)$. Hence, (f^*,g^*) is a saddle point among stationary strategies.

Thus, u^* is the value of the discounted stochastic game, and the value function is continuous. Also, (f^*,g^*) is an optimal pair for both players within the class of stationary strategies. ■

Remark 3.2.1 (1) If S was just a Borel subset of a Polish space, then stronger conditions are required for Theorem 3.2.2 to hold. Namely, $r(s,\mu,\lambda)$ and $\int v(.)dq(.|s,\mu,\lambda)$ must be continuous on $S\times P_{A_1}\times P_{A_2}$.

(2) The value function u^* is continuous but the strategies need not be continuous.

(3) The value is obtained when both players use stationary strategies. Behavioral strategies may not increase the payoff for discounted stochastic games. That is, $I_\beta(f^*,g)\geq I_\beta(\pi,g)$ where π is any strategy.

Such results are also available for non-zero-sum discounted infinite stochastic games. For more details, refer to Mertens and Parthasarathy [57]. However, stationary equilibrium need not exist, as shown through a counter-example by Levy [47].

3.3 Summary

Infinite stochastic games arise when either the state space and/or action space is not finite. Instead we may consider them to be non-empty Borel subsets of a Polish space. We consider the action space to be compact. Further the reward function and transition function are also continuous. Similar to finite discounted games, the discounted infinite stochastic game has a value and optimal stationary strategies for both players under certain conditions.

Chapter 4
Undiscounted Stochastic Games

Consider a two-player stochastic game with finite state space S and finite action sets A and B. First described by Gillette [24], the undiscounted stochastic game is as follows. Let the income for player-1 for each day be $a_0, a_1, \ldots$. Create a new averaging sequence as follows:

$$x_0 = a_0, x_1 = \frac{a_0 + a_1}{2}, x_2 = \frac{a_0 + a_1 + a_2}{3}, \ldots, x_n = \frac{a_0 + a_1 + \cdots + a_n}{n+1}, \ldots$$

Assume $\{x_n\}$ to be a bounded sequence. We define $\limsup x_n$ (or $\liminf x_n$) as the undiscounted payoff for the stochastic game. Let (f, g) be a stationary strategy pair.

In general, let Q be a $k \times k$ transition matrix whose (s, s') element is $q(s'|s, f(s), g(s))$. Let

$$r(f, g) = \begin{bmatrix} r_1(f(1), g(1)) \\ \vdots \\ r_k(f(k), g(k)) \end{bmatrix} \tag{4.32}$$

Then, $\Phi(f, g) = \left[\lim \left(\frac{1}{n}(I + Q + Q^2 + \cdots + Q^{n-1})\right)\right] r(f, g)$.
Define Q^* as follows:

$$Q^* = \lim \left(\frac{1}{n}(I + Q + Q^2 + \cdots + Q^{n-1})\right),$$

where $Q^l = QQ \ldots Q$ (that is, the matrix multiplication of Q with itself taken l times). For a stochastic matrix, the limit Q^* exists, $QQ^* = Q^*$ and it is idempotent. That is, $Q^*Q^* = Q^*$. Hence, it is easy to solve for Q^*.

So when (f, g) are stationary strategies for the undiscounted stochastic game, it is sufficient to define $\Phi(f, g)(s) = Q^* r(f, g)_{|s\text{-th coordinate}}$. Here, $r(f, g)$ is the column vector in Eq. 4.32.

T. Parthasarathy and S. Babu, *Stochastic Games and Related Concepts*,
HBA Lecture Notes in Mathematics 2,
https://doi.org/10.1007/978-981-15-6577-9_4

Definition 4.0.1 Irreducible transition matrix: Given a starting state s, if the game can transition to s' in a finite number of steps with positive probability irrespective of the strategy used, then the transition matrix Q is said to be irreducible. If $q(s'|s, .) > 0$ for all $s, s' \in S$, then the transition matrix Q is irreducible.

Gillette [25] gave an example of a zero-sum undiscounted stochastic game called the "Big Match" for which the limiting average value would not exist when both the players restrict themselves to stationary strategies. Let the upper and lower values of the game be indicated by U and L. Gillette showed that $\inf_g \sup_f \Phi(f, g)(s) > \sup_f \inf_g \Phi(f, g)(s)$ for some $s \in S$.

The "Big Match" game is described below.

Example 4.0.1 The Big Match (Gillette [25])

Consider a zero-sum undiscounted stochastic game with two players and three states, namely, s_1, s_2, and s_3. Let the game start at state s_1. In state s_1, player-2 chooses a number 1 or 2. Player-1 tries to predict player-2's choice. If player-1 predicts correctly, his payoff is 1. Else his payoff is 0. If he ever predicts 2 and is incorrect, the game moves to state s_2. If he predicts 2 and is correct, the game moves to state s_3. Else the game remains in state s_1. Further, let s_2 and s_3 be absorbing states. The payoff and transition are shown below:

$$s_1 : \begin{bmatrix} \begin{matrix} 1 \\ (1, 0, 0) \end{matrix} & \begin{matrix} 0 \\ (1, 0, 0) \end{matrix} \\ & \\ \begin{matrix} 0 \\ (0, 1, 0) \end{matrix} & \begin{matrix} 1 \\ (0, 0, 1) \end{matrix} \end{bmatrix}, s_2 : \begin{bmatrix} \begin{matrix} 0 \\ (0, 1, 0) \end{matrix} \end{bmatrix}, s_3 : \begin{bmatrix} \begin{matrix} 1 \\ (0, 0, 1) \end{matrix} \end{bmatrix}$$

Player-1 will postpone predicting 2 as long as possible. Let us consider only stationary strategies. Player-2 has an optimal stationary strategy (for instance, toss a fair coin. If the coin shows head, choose 1 else choose 2 every day). The value of the game is at most $\frac{1}{2}$. However, there is no optimal strategy for player-1.

Thus, the Big Match does not have a limiting average value when only stationary strategies are considered. ▲

Blackwell and Ferguson [9] showed that there are limiting average ϵ-optimal behavioral strategies for player-1 in the Big Match game. They showed that player-1 can guarantee a limiting average reward as close as possible to $\frac{1}{2}$ using a suitable behavioral strategy.

Here are the conditions under which certain classes of undiscounted stochastic games have optimal stationary strategies and value.

Theorem 4.0.1 (Gillette [25, Theorem 1]) *Perfect information undiscounted stochastic games have a value and both players have pure optimal strategies.* ■

Theorem 4.0.2 (Gillette [25, Sect. 4]) *If the transition matrix Q is irreducible, then the undiscounted stochastic game has a value and both players have optimal stationary strategies.* ■

Kohlberg [38] proved that all repeated zero-sum games with absorbing states have a value. Mertens and Neyman [55] showed that every zero-sum undiscounted stochastic game has a value though optimal strategies may not exist. Maitra and Sudderth [51] proposed an alternative proof using descriptive theory to show the existence of value in the undiscounted zero-sum stochastic game where the state space can be countable or uncountable.

Let the history of the stochastic game that starts at initial state s_0 be $a_0, b_0, s_1, a_1, b_1, s_2, \ldots$. Here (a_t, b_t, s_{t+1}) indicates transition to state s_{t+1} when players play action a_t and b_t, respectively, after observing s_t. Let $\mathcal{H}$ be the set of all histories given the initial state. Let $f : \mathcal{H} \to \mathbb{R}$ be defined as a bounded, measurable function for both discounted and undiscounted stochastic games with finite action and state space. Let $Z = A \times B \times S$. Given s_0, $(a_0, b_0, s_1) \in Z$, we have $h = z^{(\infty)}$, i.e., any h can be identified with $z^{(\infty)}$. Give a discrete topology to z. So it is possible to define a product topology to H.

OPEN QUESTION 4.0.3 Consider the $m \times n$ matrix game (a_{ij}) where A_1 and A_2 represent the immediate payoff for player-1 and player-2, respectively. Consider the Blackwell game (infinite games of imperfect information) $Z = A_1 \times A_2$, where neither player knows what the other player does on day i but know the history of the game. Assume $A_2 = -A_1$. Blackwell [7] showed that the game $f : Z^\infty \to \mathbb{R}$ has a value if f is the indicator function of a G_δ set. If $f(z_1, z_2, \ldots, z_n, \ldots) = g(z_1)$, it can be treated as the usual matrix game. Martin [52] used a result by Gale and Stewart [23] on perfect information games to show that every Borel function f has a value. Further Maitra and Sudderth [51] adapted this to a stochastic game. The question remains whether a similar result can be proven for non-cooperative games where $f : Z^\infty \to \mathbb{R}$ and $g : Z^\infty \to \mathbb{R}$.

OPEN QUESTION 4.0.4 Is it possible to extend the result of Martin [52] for general bimatrix games?

OPEN QUESTION 4.0.5 If the reward and transition probabilities are symmetric for an undiscounted stochastic game, does the game have a symmetric equilibrium? There is a partial result by Flesch et al. [21] who show that if the transition probability is irreducible, then symmetric equilibrium holds. The existence of symmetric equilibria is important as it is a necessary condition for the existence of evolutionary stable strategies in single population games.

Kaplansky's result for completely mixed matrix games has been extended to a special case of undiscounted stochastic games as follows.

Theorem 4.0.3 (Sujatha et al. [97, Theorem 8]) *Consider a finite undiscounted zero-sum stochastic game Γ with skew-symmetric payoff matrices that are odd ordered. Suppose $R(s)$ is completely mixed for all $s \in S$. Then the stochastic game has value* 0 *and has a completely mixed optimal strategy.*

Proof As all payoff matrices are skew symmetric, the value of the stochastic game is 0. Let $(f^o(s), g^o(s))$ be the completely mixed optimal strategy for $R(s)$, for each

$s \in S$. Then (f^o, g^o) is a completely mixed stationary optimal for the discounted stochastic game with the same payoff matrices and transition probabilities as Γ. As this is true for any value of β (in particular, β near 1), (f^o, g^o) is a completely mixed stationary optimal for the undiscounted stochastic game too. ■

OPEN QUESTION 4.0.6 It is an open question as to whether the undiscounted stochastic game is completely mixed for the conditions listed in Theorem 4.0.3.

4.1 *p*-Equilibrium

For an undiscounted two-player stochastic game with uncountable state space and finite action space, proving the existence of Nash equilibrium in such a setting was an open problem. Strauch [95] first introduced the concept of p-equilibrium.

Definition 4.1.1 p-equilibrium strategy: Let p be a probability distribution on [0, 1]. Let the state space be indicated by $S = [0, 1]$. Suppose

$$p\{s \in [0,1] : I_\beta^{(1)}(f^*, g^*)(s) \geq I_\beta^{(1)}(f, g^*)(s), I_\beta^{(2)}(f^*, g^*)(s) \quad \geq I_\beta^{(2)}(f^*, g)(s)\} = 1$$

Then we say (f^*, g^*) is a p-equilibrium pair for the two-player discounted stochastic game.

Remark 4.1.1 If (f^*, g^*) is an equilibrium pair, then it is a p-equilibrium pair for every probability distribution p on [0, 1].

For the infinite state space stochastic game, the optimal stationary strategies $f : S \to P_{A_1}$ and $g : S \to P_{A_2}$ can be obtained by using selection theorem. In order to use fixed point theorem, upper semicontinuous condition is required. This will not, however, work in the case of uncountable state space. Through a counter-example, Levy [47, 48] showed that the existence of p-equilibrium strategy need imply the existence of a stationary strategy in an uncountable state space discounted stochastic game. Levy [47]'s result is based on the results of Kohlberg and Mertens [39]. This is the most general best result and is obtained by imposing conditions on the reward and transition function. In general, it is assumed that $q \ll p$. Levy [47]'s counter-example is valid even when $q \ll p$.

Parthasarathy [74] showed that the existence of p-equilibrium stationary strategies implied the existence of equilibrium stationary strategies in stochastic games with uncountable state space under certain conditions, namely, reward functions and transition functions are continuous.

Theorem 4.1.1 (Parthasarathy [74, Theorem 1]) *Consider a discounted stochastic game with uncountable state space $S = [0, 1]$, finite action spaces A and B. Let the reward functions $r_1(s, a, b)$ and $r_2(s, a, b)$ be continuous over $S \times A \times B$. Further let $q(.|s, a, b)$ be strongly continuous over $S \times A \times B$. Let p be a probability distribution over* [0, 1] *with $q(.|s, a, b)$ absolutely continuous with respect to p. If there* *exists a p-equilibrium stationary pair for the discounted stochastic game, then there exists an equilibrium pair.* ■

Further when the reward and transition functions are both separable, Parthasarathy [74] showed the existence of equilibrium stationary strategies as stated below.

Theorem 4.1.2 (Parthasarathy [74, Theorem 2]) *Consider a discounted stochastic game with uncountable state space $S = [0, 1]$, finite action spaces A and B. Let the reward functions be separable as follows:*

$$r_i(s, a, b) = l_i(s, a) + m_i(s, b), i = 1, 2,$$

where l_i and m_i are continuous functions in s. Let the transition functions also be separable as follows:

$$q(.|s, a, b) = \frac{1}{2}\Big(q'(.|s, a) + q''(.|s, b)\Big),$$

where q' and $q^{''}$ are probability measures which are strongly continuous in s for each a, b. Suppose q' and $q^{''}$ are absolutely continuous with respect to Lebesgue measure. Then the discounted stochastic game has a pair (f^*, g^*) of equilibrium stationary strategies. Further, $I_1(f^*, g^*)(s)$ and $I_2(f^*, g^*)(s)$ are Borel measurable in s. ■

Also refer to Himmelberg et al. [30] for more details.

OPEN QUESTION 4.1.7 Blackwell's counter-example to show that there is no stationary optimal when the state space is uncountable is very easy to understand. It is based on the following: As per results from descriptive set theory, there exists a Borel set B over $[0, 1] \times [0, 1]$ containing no graphs, and the projection of B on to the X-axis is $[0, 1]$. Blackwell also provides a game theoretic proof for this descriptive set theory result. Levy [47]'s counter-example is unfortunately not easy to understand. It will be interesting to find a simpler example that can be used in lieu of Levy [47]'s counter-examples.

4.2 Reformulation of Riesz Representation Theorem

Let μ be a probability measure on a compact metric space X. Let $C(X)$ be the space of continuous functions on X and let $h \in C(X)$. Define a linear function $\wedge(h) = \int_X h(x) d\mu(x)$.

As μ is a probability distribution and $h(x)$ is non-negative, $\wedge(h)$ is non-negative. In infinite dimension, the space of probability measure becomes compact and metrizable leading us to look at the Riesz representation theorem. Given a non-negative linear function, we can always associate a measure μ.

Warga [105] reformulated the Riesz representation theorem by looking at the following space:

$$m_1 = \{f : f : S \to P_{A_1}\},$$

where $f(s)$ is a probability measure for all s. Similarly, m_2 can be defined for player-2. Let $\phi(s)$ be an integrable Lebesgue measure where $\phi(s) = \max_a \phi(s, a)$ and $||\phi|| = \int_0^1 \max |\phi(s, a)|ds$. Let $\varphi \in \mathcal{B}$ where $\varphi : S \times A_1 \to \mathbb{R}$ and $|\varphi| \leq \phi(s)$. It can be easily shown that $\mathcal{B}$ is a Banach space.

Theorem 4.2.1 *Riesz Representation Theorem (Warga [105]) Let S be a compact metric space. For any $f : S \to P_{A_1}$, there exists a linear function $\wedge_f(\phi)$ on $\mathcal{B}$ defined by*

$$\wedge_f(\phi) = \int \left(\sum_{a \in A_1} f(s, a)\phi(s, a) \right) ds$$

Proof The unit sphere in $\mathcal{B}^*$ is compact. Since a linear function can be induced on $\mathcal{B}$, any element in the space of stationary strategies can be identified with an element of $\mathcal{B}^*$. That is, $\wedge_f \in \mathcal{B}^*$. The following can be easily verified:

$$| \wedge_f (\phi)| = | \int_0^1 \left(\sum f(s, a)\phi(s, a)\right) ds \leq \int_0^1 \max_a |\phi(s, a)|ds = ||\phi||$$

It can be seen that $|| \wedge_f || \leq 1$ since $|| \wedge_f || = \sup_{||\phi||=1} | \wedge_f (\phi)|$.

Hence, m_1 is compact and metrizable in the weak-* topology. Thus, m_1 can be identified with the unit sphere in $\mathcal{B}^*$. Along the same lines, m_2 can also be identified with the unit sphere in $\mathcal{B}^*$. Hence, every stationary strategy can be identified with an element in $\mathcal{B}^*$. ■

Lemma 4.2.1 *Let $\tau : m_1 \times m_2 \to 2^{m_1 \times m_2}$. Define the map $\tau(f, g)$ as follows:*

$$\begin{aligned} \tau(f, g) = \{(f', g') :\ & u_g(s) = r_1(f'(s), g(s)) + \beta \int u_g(.)dq(.|s, f'(s), g(s)), \\ & v_f(s) = r_2(f(s), g'(s)) + \beta \int v_f(.)dq(.|s, f(s), g'(s)) \\ & \textit{almost everywhere}\} \end{aligned}$$

Then τ is upper semicontinuous. ■

Remark 4.2.1 Lemma 4.2.1 is used to prove Theorem 4.1.2. For additional details, refer to Parthasarathy [74].

4.3 Orderfield Property for Undiscounted Stochastic Games

Section 2.3 looked at orderfield property for finite discounted stochastic games. This section looks at orderfield property for undiscounted stochastic games. Orderfield property may hold for all $\beta \in [0, 1)$. It has been shown by Parthasarathy and Raghavan [76] that orderfield property holds for single-player control undiscounted stochastic games. Their proof depends on the following theorem.

Theorem 4.3.1 (L'Hospital rule) *Let f and g be two differentiable functions in (a, b) where $-\infty \leq a < b \leq \infty$. Suppose $g'(s) \neq 0$ for all $x \in (a, b)$. Also let $\dfrac{f'(x)}{g'(x)} \to A$ as $x \to a$. Further assume that $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, or $g(x) \to \infty$ as $x \to a$, or $g(x) \to -\infty$ as $x \to a$. Then $\dfrac{f(x)}{g(x)} \to A$ as $x \to a$.* ■

Remark 4.3.1 A need not be finite. For a proof, refer to Rudin [83].

Lemma 4.3.1 (Parthasarathy and Raghavan [76, Lemma 4.2]) *Let $r(t) = \dfrac{p(t)}{q(t)}$ be defined over $a < t < 1$. Assume $r(t)$ takes rational values whenever t is a rational number. Then $r(t) = \dfrac{\tilde{p}(t)}{\tilde{q}(t)}$ where $\tilde{p}$ and $\tilde{q}$ are polynomials with rational coefficients.*

Proof We prove using induction. Without loss of generality, let us assume that $q(t)$ is a constant, and that $p(t)$ and $q(t)$ are non-zero. It is enough to prove that the theorem is true for a polynomial of degree n when one of the functions (say, $q(t)$) is a constant.

Consider the case when $r(t) = p(t)$. Let $p(t) = a_0 + a_1 t + \cdots + a_n t^n$. Construct a non-singular matrix R as follows:

$$\begin{pmatrix} 1 & r_1 & r_1^2 & \cdots & r_1^n \\ 1 & r_2 & r_2^2 & \cdots & r_2^n \\ \vdots & & & & \\ 1 & r_{n+1} & r_{n+1}^2 & \cdots & r_{n+1}^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Here b_i $(i = 1, \ldots, n)$ is rational. If R is rational (i.e., r_i is rational and distinct for all i) and b_i is rational for all i, then $a_0, \ldots, a_n$ are also rational. As there are infinitely many rationals, we can always find such a matrix R.

Assume that the result is true for max{deg (p),deg (q)} $\leq n - 1$. Without loss of generality, assume that $n = \deg(p) \geq \deg(q)$. Choose any rational t^* in $(a, 1)$ such that t^* is not a root of $q(t)$.

Case 1: $n = \deg(p) > \deg(q)$

$$
\begin{aligned}
r(t) - r(t^*) &= \frac{p(t)q(t^*) - q(t)p(t^*)}{q(t)q(t^*)} \\
&= \frac{\bar{p}(t)}{\bar{q}(t)}(t - t^*)
\end{aligned}
$$

Let $\bar{r} = \dfrac{\bar{p}(t)}{\bar{q}(t)}$. Since $\max\{deg(\bar{p}), deg(\bar{q})\} \leq n - 1$, $\bar{r}$ satisfies the conditions in Lemma 4.3.1. Thus, by induction $\bar{p}$ and $\bar{q}$ can be assumed to be polynomials with rational coefficients.

Case 2: $\deg(p) = \deg\,(q) = n$

Along the same lines, we can reduce the degree of p or q. Thus considering an appropriate function $\bar{r}$ will yield the result. ■

Remark 4.3.2 Hou and Hou [32] give a nice argument by induction to show that the confluent Vandermonde matrix is non-singular. Thus, it is interesting to note the following:

$$
\det\begin{pmatrix} 1 & r_1 \\ 1 & r_2 \end{pmatrix} = r_2 - r_1 \neq 0
$$

$$
\det\begin{pmatrix} 1 & r_1 & r_1^2 \\ 1 & r_2 & r_2^2 \\ 1 & r_3 & r_3^2 \end{pmatrix} = (r_1 - r_2)(r_2 - r_3)(r_3 - r_1) \neq 0.
$$

Lemma 4.3.2 (Parthasarathy and Raghavan [76, Lemma 4.3]) *Let $f(\beta)$ be a bounded rational function in β where $\beta \in (a, 1)$. Assume $f(\beta)$ to be rational whenever β is rational. Then $\lim_{\beta \uparrow 1} f(\beta)$ is a rational number.*

Proof Let $f(\beta) = \dfrac{p(\beta)}{q(\beta)}$ where $f(\beta)$ is bounded. Also $p(\beta)$ and $q(\beta)$ do not vanish. Hence, $p(\beta)$ and $q(\beta)$ maintain the same sign resulting in $f(\beta)$ to be monotonic. Therefore, $\lim_{\beta \uparrow 1} f(\beta)$ exists.

Choose $p(\beta)$ and $q(\beta)$ such that they have rational coefficients. The limit of these two functions exists. If $\lim_{\beta \uparrow 1} q(\beta) \to 0$, then $\lim_{\beta \uparrow 1} p(\beta) \to 0$ also due to L'Hospital rule (Theorem 4.3.1). If not, $f(\beta)$ becomes unbounded.

Hence, consider $\dfrac{p'(\beta)}{q'(\beta)}$. If $\lim_{\beta \uparrow 1} q'(\beta)$ is finite, then it is trivial. If $\lim_{\beta \uparrow 1} q'(\beta) \to 0$, then $\lim_{\beta \uparrow 1} p'(\beta) \to 0$ by L'Hospital rule (Theorem 4.3.1). Proceeding similarly, Lemma 4.3.1 yields the desired result. ■

Lemmas 4.3.1 and 4.3.2 lead to the following result by Parthasarathy and Raghavan [76].

Theorem 4.3.2 (Parthasarathy and Raghavan [76, Theorem 3.1]) *Let* (S, A_1, A_2, r, q) *be a finite undiscounted single-player controlled stochastic game where player-*2 *is the controlling player. Let* $\Phi(f, g)(s)$ *denote the undiscounted reward for player-*1 *when the two players use* f *and* g*, respectively, and the game starts in state* s*. Assume that the data comes from rational fields. Then a solution exists in the rational field. That is, the value of the game is rational and the coordinates of a pair of optimal strategy for both players are rational.* ■

Remark 4.3.3 Though the result was proven theoretically, no algorithm was provided by the authors. Vrieze [102] modeled it as a linear program and provided an algorithm for it.

Avrachenkov et al. [2] show the existence of orderfield property over the field of real algebraic numbers for the limiting average stochastic game. We state the theorem without proof and refer the readers to the manuscript for the proof.

Theorem 4.3.3 (Avrachenkov et al. [2, Theorem 4.1]) *Consider a limiting average stochastic game with all data being in the field of algebraic numbers. Then the value vector* v *of the game has entries* v_s *that are algebraic numbers for each* $s = 1, 2, \ldots, N$. ■

4.4 Summary

For undiscounted stochastic games, we look at limiting average payoffs. Undiscounted stochastic games may not have a value if both the players use stationary strategies, but the value exists if both use behavioral strategies as shown in the example of the “Big match.” The undiscounted stochastic game has optimal stationary strategies if the transition matrix is irreducible. The notion of p-equilibrium can be used to prove the existence of Nash equilibrium for the undiscounted stochastic game when the state space is uncountable.

Chapter 5
N-Player Cooperative Games

The earlier chapters looked at non-cooperative games (both matrix games and stochastic games). In this chapter, we look at cooperative games where two or more players may form coalitions that provide them with better payoff than by playing alone. We will explore the various solution concepts for cooperative games.

5.1 Coalitions and Stability

We start by exploring the notion of stability.

Let $N = \{1, \ldots, n\}$ be a finite set of players. One or more players can form coalitions to cooperate with each other by coordinating strategies and sharing payoffs. The cardinality of the set of all possible coalitions is 2^N. The coalition formed by all N players is referred to as the *grand coalition*.

Let $v : 2^N \to \mathbb{R}$ be a characteristic function from the set of all possible coalitions of players (the power set) to a set of payments where $v(\phi) = 0$, $v(S)$ is the *worth of a coalition* $S \subseteq N$ and $v(N)$ is the *worth of the grand coalition*. Without loss of generality, assume that $v(\{i\}) = 0$. Such a v is called a *characteristic function* associated with $N = \{1, 2, \ldots, n\}$ player set.

Example 5.1.1 Consider a cooperative game involving three players $N = \{1, 2, 3\}$. There are 2^3 possible coalitions as follows: $\{1\}, \{2\}, \{3\}$, $\{1, 2\}, \{1, 3\}, \{2, 3\}$, $\{1, 2, 3\}$, and the empty set. The coalition indicated by $\{1, 2, 3\}$ is termed as the grand coalition. ▲

Definition 5.1.1 Winning coalition: If $v(S) \in \{0, 1\}$ for every $S \subseteq N$, then $T \subseteq N$ is termed as a winning coalition if $v(T) = 1$.

Example 5.1.2 Consider the cooperative game with $N = \{1, 2, 3\}$. Let the worth of the coalitions be as follows:

T. Parthasarathy and S. Babu, *Stochastic Games and Related Concepts*,
HBA Lecture Notes in Mathematics 2,
https://doi.org/10.1007/978-981-15-6577-9_5

$$v(\phi) = v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$$
$$v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = 1$$
$$v(\{1, 2, 3\}) = 1.$$

Then $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$ and the grand coalition are the winning coalitions. ▲

Definition 5.1.2 Simple game: A cooperative game where the value of any coalition is either 0 or 1 is called a simple game, i.e., $v : 2^N \to \{0, 1\}$.

Example 5.1.3 A *voting game* is an example of a simple game. Consider a council with three members, i.e., $N = \{1, 2, 3\}$. Any bill presented to the council is passed if two or more members vote in favor of it. Thus

$$v(S) = \begin{cases} 1, & \text{if } |S| \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

▲

Definition 5.1.3 Proper simple game: A proper simple game is a simple game such that when $v(S) = 1, v(T) = 1$ for every $T \supseteq S$. Further, if $v(S) = 1$, then $v(N|S) = 0$.

Definition 5.1.4 Imputation space: For a general n-player cooperative game, the imputation space (I) is the set of vectors in $\mathbb{R}^n$ where

$$I = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, \ldots, n, \sum_{i \in N} x_i = v(N)\}.$$

For a simple cooperative game, the imputation space I is the set of probability vectors in $\mathbb{R}^n$ where

$$I = \{x \in \mathbb{R}^n : x_i \geq 0 \text{ for } i = 1, \ldots, n, \sum_{i=1}^{n} x_i = 1\}.$$

Definition 5.1.5 Dominance order in imputation space: Let $x, y \in I$ where I denotes the imputation space. We say that $x \succ y$ (x dominates y) if there exists a non-empty coalition $S \subseteq N$ such that $x_i > y_i$ for all $i \in S$ and $\sum_{i \in S} x_i \leq v(S)$. For a simple game, $x \succ y$ if there exists a non-empty coalition $S \subseteq N$ such that $x_i > y_i$ for all $i \in S$ and $\sum_{i \in S} x_i \leq 1$.

Example 5.1.4 Let $N = \{1, 2, 3\}$ and $v(S) = 1$ if $|S| \geq 2$. Let $x = (0.5, 0.2, 0.3)$ and $y = (0.7, 0.1, 0.2)$. Then $x \succ y$ via $S = \{2, 3\}$. ▲

Definition 5.1.6 Undominated imputation: An imputation $x \in I$ is undominated if no other imputation dominates it.

Definition 5.1.7 Domination: Let $X \subseteq I$ and $X \neq \phi$. Let $\text{dom}(X) = \{y \in I :$ There exists $x \in X$ such that $x \succ_S y\}$.

Definition 5.1.8 Internal stability: A set of imputations X is internally stable if no imputation in X is dominated by any other imputation in X.

Definition 5.1.9 External stability: A set of imputations X is externally stable if every imputation not in X is dominated by some imputation in X.

Definition 5.1.10 Stable set (solution set): Let $X \subseteq I$ and $X \neq \phi$. X is a stable set (also known as solution set) for the given game if the following holds:

(1) $I = X \cup \text{dom}(X)$ and
(2) $X \cap \text{dom}(X) = \phi$.

That is, the set X satisfies internal stability and external stability.

von Neumann and Morgenstern [101] show that a simple game may not have an unique stable set through the following example.

Example 5.1.5 (von Neumann and Morgenstern [101]) Consider a simple game with $N = \{1, 2, 3\}$. Define the following for the game:

$$v(S) = \begin{cases} 1, & \text{if } |S| \geq 2, \\ 0, & \text{otherwise} \end{cases}$$

$I = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_1 + x_2 + x_3 = 1\}$.

The following are all the possible stable sets in this simple game:

(1) $\{(\frac{1}{2}, 0, \frac{1}{2})\}, \{(\frac{1}{2}, \frac{1}{2}, 0)\}, \{(0, \frac{1}{2}, \frac{1}{2})\}$ is a stable set.
(2) Let $c \in [0, \frac{1}{2})$. For each c, define $X_c = \{(c, x_2, x_3) : x_2, x_3 \geq 0, x_2 + x_3 = 1 - c\}$. Then $X_c = \{(c, t, 1-t) : 0 \leq t \leq 1 - c\}$ is a stable set. Similarly, other stable sets are obtained with c being in the second and third coordinates, i.e., $X_c = \{(t, c, 1-t) : 0 \leq t \leq 1 - c\}$ and $X_c = \{(t, 1-t, c) : 0 \leq t \leq 1 - c\}$ are also stable sets.

Hence, the above game has uncountably many stable sets. ▲

Remark 5.1.1 In Example 5.1.5, note that $c < \frac{1}{2}$. If $c = \frac{1}{2}$, then $X_{\frac{1}{2}} = \{(\frac{1}{2}, t, \frac{1}{2} - t) : 0 \leq t \leq \frac{1}{2}\}$. Consider $(0, \frac{1}{2}, \frac{1}{2}) \in I$ where $(0, \frac{1}{2}, \frac{1}{2}) \notin X_{\frac{1}{2}}$. Note that no element from $X_{\frac{1}{2}}$ dominates $(0, \frac{1}{2}, \frac{1}{2})$. Hence $(0, \frac{1}{2}, \frac{1}{2}) \notin \text{dom}(X_{\frac{1}{2}})$. Hence $X_{\frac{1}{2}} \cup \text{dom}(X_{\frac{1}{2}}) \neq I$. Thus, c must be strictly less than $\frac{1}{2}$.

Exercise 36 Let $N = \{1, 2, 3, 4, 5, 6\}$, $v(12) = v(13) = v(45) = v(46) = 1$, $v(123) = v(456) = 2$, $v(N) = 4$. Let $v(S) = 0$ for all other coalitions. Consider

$$I_v = \{(x_1, x_2, x_3, x_4, x_5, x_6) : \sum_{i \in N} x_i = 4, x_i \geq 0 \text{ for all } i = 1, \ldots, 6\}.$$

Is I_v a stable set?

Definition 5.1.11 Domination of an imputation: Let x and y be two imputations and S be a coalition. We say that x dominates y through the coalition S ($x \succ_S y$) if the following two conditions are satisfied:

(1) $x_i > y_i$ for all $i \in S$,
(2) $\sum_{i \in S} x_i \leq v(S)$.

Remark 5.1.2 The two conditions in Definition 5.1.11 indicate that

(1) The members of the coalition S prefer the imputation x to y.
(2) The members of the coalition S can obtain what x gives them, and S cannot get more than $v(S)$ worth of coalition S.

Example 5.1.6 Let $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and $y = (0, \frac{1}{4}, \frac{3}{4})$. For $S = \{1, 2\}$, we have $x \succ_S y$. ▲

It is important to note that domination relation may not be a partial order relation when the coalitions are different. That is, when $x \succ_{S_1} y$ and $y \succ_{S_2} z$, then this does not always imply that x dominates y, as shown in the following example.

Example 5.1.7 Let $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $y = (0, \frac{1}{4}, \frac{3}{4})$, and $z = (\frac{1}{2}, 0, \frac{1}{2})$. Then $x \succ_{S \in \{1,2\}} y$ and $y \succ_{S \in \{2,3\}} z$. However $x \not\succ z$. ▲

Nash and Shapley [64] catalogued all the solutions to a simple three-player poker game. However, a cooperative game may not have a stable set as shown by Lucas [49] for a game with ten players. A high-level overview of this example is described in Example 5.1.8.

Example 5.1.8 A ten-player game with no solution (Lucas [49, Example 3])

Consider the game with $N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define the following for the game:

$$\begin{aligned}
v(N) &= 5 \\
v(\{1,3,5,7,9\}) &= 4 \\
v(\{3,5,7,9\}) &= v(\{1,5,7,9\}) = v(\{1,3,7,9\}) = 3 \\
v(\{1,4,7,9\}) &= v(\{3,6,7,9\}) = v(\{2,5,7,9\}) = 2 \\
v(\{3,5,7\}) &= v(\{1,5,7\}) = v(\{1,3,7\}) = 2 \\
v(\{3,5,9\}) &= v(\{1,5,9\}) = v(\{1,3,9\}) = 2 \\
v(\{1,2\}) &= v(\{3,4\}) = v(\{5,6\}) = v(\{7,8\}) = v(\{9,10\}) = 1 \\
v(S) &= 0 \text{ for all other } S \subset N.
\end{aligned}$$

The set of imputations is given by

$$A = \{x : \sum_{i \in N} x_i = 5, x_i \geq 0 \text{ for all } i \in N\}.$$

Consider the following six imputations:

$$(1, 0, 1, 0, 1, 0, 1, 0, 1, 0), (0, 1, 1, 0, 1, 0, 1, 0, 1, 0), (1, 0, 0, 1, 1, 0, 1, 0, 1, 0)$$
$$(1, 0, 1, 0, 0, 1, 1, 0, 1, 0), (1, 0, 1, 0, 1, 0, 0, 1, 1, 0), (1, 0, 1, 0, 1, 0, 1, 0, 0, 1).$$

The above six imputations belong to the core C (refer to Definition 5.3.1). Hence, C is the convex hull for these six imputations. As a simplification, assume that the indices i, j, k can take one of the following values: $(i, j, k) = (1, 3, 5)$ or $(3, 5, 1)$ or $(5, 1, 3)$. Similarly, the indices p and q can take one of the following values: $(p, q) = (7, 9)$ or $(9, 7)$.

Consider the following:

$$B = \{x \in A : x_1 + x_2 = x_3 + x_4 = x_5 + x_6 = x_7 + x_8 = x_9 + x_{10} = 1\}.$$

Any $x \in B$ such that $x_1 + x_3 + x_5 + x_7 + x_9 \geq 4$ will lie in the core C. Further $C \subset B$. Thus,

$$C = \{x \in B : x_1 + x_3 + x_5 + x_7 + x_9 \geq 4\}.$$

Now consider the following subsets of B:

$$\begin{aligned}
E_i &= \{x \in B : x_j = x_k = 1, x_< 1, x_7 + x_9 < 1\} \\
E &= E_1 \cup E_3 \cup E_5 \\
F_{jk} &= \{x \in B : x_j = x_k = 1, x_7 + x_9 \geq 1\} \\
F_p &= \cup_q \{x \in B : x_p = 1, x_q < 1, x_3 + x_5 + x_q \geq 2, x_1 + x_5 + x_q \geq 2, \\
&\qquad\qquad x_1 + x_3 + x_q \geq 2\} \\
F_{79} &= \{x \in B : x_7 = x_9 = 1\} \\
F_{135} &= \{x \in B : x_1 = x_3 = x_5 = 1\} \\
F &= F_{13} \cup F_{35} \cup F_{51} \cup F_7 \cup F_9 \cup F_{79} \cup F_{135} - C.
\end{aligned}$$

The following are easy to prove (Owen [67, Lemma 11.4.1–11.4.4, 11.4.6]):

(1) $A - B$, $B - (C \cup E \cup F)$, C, E, and F form a partition of A.
(2) Every $x \in A - B$ is dominated by some $y \in C$.
(3) Every $x \in B - (C \cup E \cup F)$ is dominated by some $y \in C$.
(4) If $x \in E$, then there is no $y \in C$ that dominates x.
(5) If $x \in E$, then there is no $y \in F$ that dominates x.
(6) Let $x \in E_i$. Then there is no $y \in E_i \cup E_k$ that dominates x. However, there is some $y \in E_j$ that dominates x through $\{i, r, 7, 9\}$.

Let V be a stable set solution for v. For all games, $C \subset V$. Consider any x dominated by C. Then from the definition of internal stability, this x does not lie in V. This eliminates all x that either belong to $A - B$ or $B - (C \cup E \cup F)$. Hence, $V \subset C \subset E \cup F$.

Consider any $x \in E - V$. We now show that this must be dominated by some $y \in E \cap V$. By the definition of external stability, there must be some $y \in V$ that dominates x. But $y \notin C \cup F$. Thus $y \in E$ also. This implies, $y \in E \cap V$. Hence, $x \in E - V$ is dominated by some $y \in E \cap V$.

We will show that the game has no stable sets through contradiction. Suppose that the game has a stable set. Since $V \cap E \neq \phi$, there exists $x \in V \cap E$. Thus, $x \in E_i$. There exists $y \in E_j$ such that $y \succ x$. Such a y does not belong to V due to internal stability. Thus, $y \in E - V$. Now there exists $z \in E \cap V$ such that $z \succ y$. Thus, $z \in E_k$. Choose $w \in E_i$ such that $w \succ z$. Then $w \notin V$. This leads us to suppose that there is a $u \in E_j \cap V$ such that $u \succ w$. Thus, the following hold:

$$\begin{aligned}
u_7 > w_7 > z_7 > y_7 > x_7 \\
u_9 > w_9 > z_9 > y_9 > x_9 \\
u_i = 1 \geq x_i \\
u_r > 0 = x_r \\
u_i + u_r + u_7 + u_9 < 2.
\end{aligned}$$

Thus, $u \succ_{\{i,r,7,9\}} x$. This leads to a contradiction as both u and x belong to V. Hence, the game has no solution. ▲

Exercise 37 Prove the following in Example 5.1.8:

(1) $A - B$, $B - (C \cup E \cup F)$, C, E, and F form a partition of A.
(2) Every $x \in A - B$ is dominated by some $y \in C$.
(3) Every $x \in B - (C \cup E \cup F)$ is dominated by some $y \in C$.
(4) If $x \in E$, then there is no $y \in C$ that dominates x.
(5) If $x \in E$, then there is no $y \in F$ that dominates x.
(6) Let $x \in E_i$. Then there is no $y \in E_i \cup E_k$ that dominates x. However, there is some $y \in E_j$ that dominates x through $\{i, r, 7, 9\}$.

Lemma 5.1.1 *Any two elements within the stable set X cannot dominate each another.*

Proof This is trivial to prove. Let $a, b \in X$ such that $a \succ b$. Then $b \in \text{dom}(X)$ also. This contradicts the fact that $X \cap \text{dom}(X) = \phi$. Thus, two elements within the stable set cannot dominate one another. That is, the stable set X is internally stable. ■

Theorem 5.1.1 *Every proper simple game has at least one stable set.*

Proof Consider the game (N, W) where W is the set of all winning coalitions. Let $S \in W$. Without loss of generality, assume that S is a minimal winning coalition where the first k players form the minimal winning coalition. That is, $S = (1, 2, \ldots, k)$, $v(S) = 1$, and $v(T) = 0$ for all $T \subsetneqq S$. Define the solution space

$$X_S = \{x = (x_1, \ldots, x_k, 0, \ldots, 0) : x_i \geq 0 \text{ for } i = 1, \ldots, k; \sum_{i=1}^{k} x_i = 1\}.$$

To show that X_S is a solution, it is required to show that $X_S \cap \text{dom}(X_S) = \phi$ and $X_S \cup \text{dom}(X_S) = I$.

Let us first show that $X_S \cap \mathrm{dom}(X_S) = \phi$. If possible, let $X_S \cap \mathrm{dom}(X_S) \neq \phi$. That is, there exists $x \in X_S \cap \mathrm{dom}(X_S)$. Now, $x \in \mathrm{dom}(X_S)$ implies that there exists $y \in X_S$ such that $y \succ_S x$. This domination is via S only as S is the minimal winning coalition. However, $y, x \in X_S$ implies that $y_i > x_i$ for all $i = \{1, \ldots, k\}$. Hence $\sum y_i > \sum x_i$. However $\sum y_i = \sum x_i = 1$. This leads to a contradiction. Thus $X_S \cap \mathrm{dom}(X_S) = \phi$.

To show that $X_S \cup \mathrm{dom}(X_S) = I$, consider the following imputation:

$$y = (y_1, \ldots, y_k, y_{k+1}, \ldots, y_n)$$

such that $y \notin X_S$. That is, there is at least one $y_j > 0$ for some $j \geq k + 1$. Hence, $y_1 + \cdots y_k < 1$.

Define $x_i = \begin{cases} y_i + \epsilon & \text{for } i = 1, \ldots, k, \text{ such that } \sum_{i=1}^{k} x_i = 1, \\ 0, & \text{for } i = k+1, \ldots, n. \end{cases}$.

This shows that $x \succ_S y$ and $x \in X_S$. Hence $X_S \cup \mathrm{dom}(X_S) = I$. Hence, every proper simple game has at least one solution. ■

5.2 Product of Two Simple Games

The tensor product of two simple games, the monotonicity property, and the stability of such product games are detailed below.

Definition 5.2.1 Winning coalition: Consider the tensor product game $(N_1, W_1) \otimes (N_2, W_2)$ where $N_1 \cap N_2 = \phi$. Consider the larger game $(N_1 \cup N_2, W)$. S is a winning coalition of the game $N_1 \cup N_2$ if $S \cap N_1 \in W_1$ and $S \cap N_2 \in W_2$.

The following is an example of the tensor product of two simple games (Shapley [85]).

Example 5.2.1 Consider two games, $N_1 = \{1, 2, 3\}$ and $N_2 = 4$. Define the payoff for $N_1 = \{1, 2, 3\}$ to be as follows:

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Let W_1 and W_2 be the winning coalition for N_1 and N_2, respectively. The two games can be indicated as $(N_1, W_1) = (\{1, 2, 3\}, 12, 13, 23, 123)$ and $(N_2, W_2) = (\{4\}, 4)$.

Let $N = \{1, 2, 3, 4\}$ be the tensor product of two games given by $\{1, 2, 3\} \otimes \{4\}$. A coalition $S \subseteq N$ is winning if $S \cap N_1 \in W_1$ and $S \cap N_2 \in W_2$. Hence, the winning coalition for N is given by $\{124, 234, 134, 1234\}$. ▲

OPEN QUESTION 5.2.8 If the solution for the individual game is known, can the solution for the larger game be constructed?

5.2.1 Monotonicity Property

Consider two simple games N_1 and N_2 with winning coalitions W_1 and W_2, respectively. Let I_1 and I_2 be the imputation space for the games (N_1, W_1) and (N_2, W_2), respectively. Let $X_1 \subset I_1$ and $X_2 \subset I_2$.

Consider the tensor product game $(N_1, W_1) \otimes (N_2, W_2)$ where $N_1 \cap N_2 = \phi$. Define $I_1 \otimes I_2$ as follows:

$$I_1 \otimes I_2 = \{z : \sum_{i=1}^{N_1+N_2} z_i = 1\},$$

where $z = (z_1, \ldots, z_{n_1}, z_{n_1+1}, \ldots, z_{n_1+n_2})$.

Let $(x_1, \ldots, x_{n_1}) \in X_1$ and $(y_{n_1+1}, \ldots, y_{n_1+n_2}) \in X_2$. Let $\alpha \in [0, 1]$. Define $X_1 \times_\alpha X_2$ as follows:

$$X_1 \times_\alpha X_2 = \{x : x = \alpha(x_1, \ldots, x_{n_1}, 0, \ldots, 0) + (1 - \alpha)(0, \ldots, 0, y_{n_1+1}, \ldots, y_{n_1+n_2})\}.$$

Consider a parametric family of sets of imputations $\{X(\alpha) : \alpha \in [0, 1]\}$ where each $X(\alpha) \subseteq I$. The notions of monotonicity property for this family are as follows:

- A parametric family of sets of imputations $\{X(\alpha) : \alpha \in [0, 1]\}$ is *semi-monotonic* if for every α, β, x such that $0 \le \alpha \le \beta \le 1$ and $x \in X(\beta)$, there exists $y \in X(\alpha)$ with $\alpha y \le \beta x$ coordinate-wise.
- A semi-monotonic family is *monotonic* if the following holds: If $y \in X(\alpha)$, then there exists $x \in X(\beta)$ such that $\alpha y \le \beta x$ coordinate-wise.
- A semi-monotonic function is δ-*monotonic* ($0 \le \delta \le 1$) if the following holds: for every α, β, y such that $\delta \le \alpha \le \beta \le 1$ and $y \in X(\alpha)$, there exists $x \in X(\beta)$ such that $\alpha y \le \beta x$ coordinate-wise.

Remark 5.2.1 A monotonic family is automatically δ-monotonic, though the converse need not be true.

We now give an example for a semi-monotonic family of sets.

Example 5.2.2 (Shapley [85, Sect. 5]) Consider a simple majority game $N = \{1, 2, 3\}$. A solution for this is the following:

$$\{(\tfrac{1}{2}, \tfrac{1}{2}, 0), (\tfrac{1}{2}, 0, \tfrac{1}{2}), (0, \tfrac{1}{2}, \tfrac{1}{2})\}.$$

If $X(\alpha) \equiv X$ for every α, then it is a semi-monotonic family.

Let $X_c = \{(c, t, 1 - t - c) : 0 \le t \le 1 - c\}$ where $c \in [0, \frac{1}{2})$. Then if $X(\alpha) \equiv X_c$, it is a semi-monotonic family. ▲

Theorem 5.2.1 (Shapley [85, Theorem 6]) *Consider two games, namely,* (N_1, W_1) *and* (N_2, W_2) *where* N_1 *and* N_2 *indicate the player set, and* W_1 *and* W_2 *are the sets of winning coalitions for the two games, respectively. Further, let* $\{Y_1(\alpha) : 0 \le \alpha \le 1\}$

and $\{Y_2(\alpha) : 0 \le \alpha \le 1\}$ *be the monotonic family of solutions to the games* (N_1, W_1) *and* (N_2, W_2)*, respectively, except that* $Y_1(1)$ *and* $Y_2(1)$ *need not be externally stable. For* $i = 1, 2$*, let* $X_i(\alpha) = I_i - dom(Y_i(\alpha))$*. That is,*

$$X_i(\alpha) = \begin{cases} Y_i(\alpha) & \text{if } \alpha < 1, \\ \text{may be bigger than } Y_i(1) & \text{if } \alpha = 1. \end{cases}$$

Then $X = \cup_{0 \le \alpha \le 1}[X_1(\alpha) \times_\alpha X_2(1-\alpha)]$ *is a solution to the product game* $(N_1, W_1) \otimes (N_2, W_2)$*.* ■

Example 5.2.3 Consider the product game $(\{1, 2, 3\}, 12, 13, 23, 123) \otimes (\{4\}, 4)$. Let $Y_1(\alpha) = \{(\frac{\alpha}{2}, t, 1 - t - \frac{\alpha}{2}) : \alpha < 1, 0 \le t \le 1 - \frac{\alpha}{2}\}$. $\{Y_1(\alpha)\}$ is a monotonic family of solutions. However, $Y_1(1) = \{(\frac{1}{2}, t, \frac{1}{2} - t) : 0 \le t \le \frac{1}{2}\}$ does not dominate $(0, \frac{1}{2}, \frac{1}{2})$ and hence is not externally stable. So $Y_1(1)$ is not a solution.

Thus, $\{Y_1(\alpha)\}$ satisfies the conditions of Shapley's theorem 5.2.1. As per Theorem 5.2.1, $X = \cup[X_1(\alpha) \otimes_\alpha (1-\alpha)1]$ is a solution to $(\{1, 2, 3, 4\}, 124, 134, 234, 1234)$. That is,

$$X = \{\frac{\alpha^2}{2}, \alpha t, \alpha(1 - \frac{\alpha}{2} - t), 1 - \alpha) : 0 \le \alpha \le 1, 0 \le t \le 1 - \frac{\alpha}{2}\}.$$

Then $X \cup (0, \frac{1}{2}, \frac{1}{2}, 0)$ is a solution to the game $(\{1, 2, 3, 4\}, 124, 134, 234, 1234)$. ▲

Shapley raised the question whether Theorem 5.2.1 is true for a δ-monotonic family when δ is close to 1. And if yes, is there a solution to the product of the two simple games that need not have the property of full monotonicity. Parthasarathy [69–71] showed that the value of δ can be determined under certain conditions and provided an example of a product of two simple games that do not have the property of full monotonicity.

Theorem 5.2.2 (Parthasarathy [69, Theorem 4]) *Consider two games* (N_1, W_1) *and* (N_2, W_2) *where* N_1 *and* N_2 *indicate the player set, and* W_1 *and* W_2 *indicate the set of winning coalitions for the two games, respectively. Further, let* $\{Y_1(\alpha) : 0 \le \alpha \le 1\}$ *and* $\{Y_2(\alpha) : 0 \le \alpha \le 1\}$ *be a semi-monotonic family of solutions to the games* (N_1, W_1) *and* (N_2, W_2)*, respectively, except that* $Y_1(1)$ *and* $Y_2(1)$ *need not be externally stable. Further let the family be* δ_0*-monotonic. Then*

$$X = \cup_{0 \le \alpha \le 1}[X_1(\alpha) \times_\alpha X_2(1-\alpha)]$$

is a solution to the product game $(N_1, W_1) \otimes (N_2, W_2)$*, where* $X_2(\alpha) = A_{N_i} - dom_i Y_i(\alpha)$*.* ■

Remark 5.2.2 Here δ_0 must be chosen properly. For further details, see Parthasarathy [69].

OPEN QUESTION 5.2.9 Does Theorem 5.2.2 hold if δ is any positive number close to 1?

Further, Parthasarathy [70, 71] looked at the product game $H \times K$ where $H = V_n \otimes B_1$, V_n is the homogeneous weighted n-player majority game given by $[1, 1, \ldots, 1, n-2]$, B_1 is a 1-player pure bargaining game, and K is any arbitrary simple game.

Theorem 5.2.3 (Parthasarathy [71, Theorem 1]) *Let $\{X_1(\alpha) : 0 \leq \alpha \leq 1\}$ be any δ-monotonic family of product solutions to the game $H = V_n \otimes B_1$, except that $X_1(1)$ need not be externally stable. Then*

$$X = \cup_{0 \leq \alpha \leq 1} Z_1(\alpha) \times Z_2(1-\alpha)$$

is a solution for $H \otimes K$ where K is any arbitrary simple game, $Z_1(\alpha) = I_{n+1} - dom(X_1(\alpha))$ and $Z_2(1-\alpha) \equiv Z_2$ is any solution of K. ∎

5.3 Core of Cooperative Games

The core of a cooperative game looks at the division of the coalition's worth among its members where the payoffs cannot be bettered by another coalition. Hence, this forms an important solution concept for cooperative games. We shall first define the core of a cooperative game. We then look at the Bondareva-Shapley theorem and the application of the core as a solution concept for market games and bankruptcy game.

Consider a general cooperative game (not necessarily a simple game). Indicate the sum of the values received by each player in a coalition S by $x(S) = \sum\limits_{i \in S} x_i$ where $x \in I$.

Definition 5.3.1 Core (C_v): The core of a cooperative game is the collection of all undominated imputations. That is,

$$C_v = \{x \in I : x(S) \geq v(S), \text{ for all } S \subset N\}.$$

To find the elements of the core, the game is first formulated as a linear program as follows:

$$\begin{aligned} &\min \sum_{i=1}^{N} x_i \\ &\text{s.to } x(S) \geq v(S), \text{ for all } S \subseteq N \\ &x_i \geq 0, \text{ for all } i. \end{aligned} \tag{5.33}$$

Example 5.3.1 Consider a cooperative game with $N = 6$. Let the worth of the coalitions be $v(123456) = 4$, $v(12) = v(13) = v(45) = v(46) = 1$, $v(123) = v(456) = 2$. Let the remaining coalitions have value 0.

Then the set of all imputations is given by

$$I = \{x = (x_1, \ldots, x_6) : \sum_{i=1}^{6} x_i = v(N) = 4, x_i \geq 0 \text{ for all } i = 1, \ldots, 6\}.$$

The core is given by

$$\begin{aligned} C_v = \Big\{x = (x_1, \ldots, x_6) : x_i \geq 0 \text{ for all } i = 1, \ldots, 6, \\ \sum_{i=1}^{6} x_i = v(N) = 4, \\ x_1 + x_2 \geq v(12) = 1, \\ x_1 + x_3 \geq v(13) = 1, \\ x_4 + x_5 \geq v(45) = 1, \\ x_4 + x_6 \geq v(46) = 1, \\ x_1 + x_2 + x_3 \geq v(123) = 2, \\ x_4 + x_5 + x_6 \geq v(456) = 2\Big\}. \end{aligned}$$

Solving the above system of linear equations yields $y = (1, 1, 0, 1, 0, 1) \in C_v$. Another element in the core is $z = (2\epsilon, 1 - \epsilon, 1 - \epsilon, 2\epsilon, 1 - \epsilon, 1 - \epsilon)$ where $\epsilon \in [0, 1]$.

It is easy to see that $x \not\succ y$ and $y \not\succ x$ where $x, y \in C_v$. ▲

The core may be empty for a cooperative game as seen in the following example of a simple majority game.

Example 5.3.2 Let $N = \{1, 2, 3\}$, $v(12) = v(13) = v(23) = v(123) = 1$. Let the value for all other coalitions be 0.

The core is given by

$$\begin{aligned} C_v = \Big\{x = (x_1, x_2, x_3) : x_i \geq 0 \text{ for all } i = 1, \ldots, 3 \\ \sum_{i=1}^{3} x_i = v(N) = 1, \\ x_1 + x_2 \geq v(12) = 1, \\ x_1 + x_3 \geq v(13) = 1, \\ x_2 + x_3 \geq v(23) = 1\Big\}. \end{aligned}$$

Solving the above set of linear equations does not yield a solution and hence the core is empty. ▲

Remark 5.3.1 (1) In Definition 5.3.1, the condition $x(S) \geq v(S)$ is critical. If suppose $x(S) < v(S)$ for a coalition S, then S can be dominated by adding a small factor $x_i + \dfrac{v(S) - x(S)}{|S|}$ to each coordinate of $x(S)$. This leads to a dominated imputation.

(2) If $x \succ y$ via S, then $2 \leq |S| \leq N - 1$.

5.3.1 Necessary and Sufficient Conditions for Existence of Core

The core may not always exist for a cooperative game, as shown in Example 5.3.2. Bondareva [10] and Shapley [86] give a set of necessary and sufficient conditions for the existence of a core. Toward this, the game is first formulated as a linear program as mentioned in Eq. 5.33. If the minimum is obtained at x^* and if $\sum x_i^* \leq v(N)$, then $x^* \in C_v$ and the core is non-empty. Conversely if the core is non-empty (say $x' \in C_v$), then x' is an optimal solution for the linear program.

Further the corresponding dual for the linear program in Eq. 5.33 is obtained by attaching a scalar y_s to each $v(S)$ as follows:

$$\begin{aligned} \max \sum_{S \in 2^N} y_s v(S) \\ \text{s.to} \sum_{i \in S} y_s = 1, \text{ for all } i = 1, \ldots, N \\ y_s \geq 0. \end{aligned} \tag{5.34}$$

The feasibility condition in the dual is interpreted as follows. Player-k may belong to many coalitions. So for all coalitions that player-k belongs to, $\sum\limits_{k \in S} y_s = 1$.

Theorem 5.3.1 (Bondareva [10, Theorem 2.1]; Shapley [86]) *A cooperative game (N, v) has a non-empty core if and only if for all non-negative vectors $\{y_s \subseteq N\}$ satisfying $\sum\limits_{i \in S} y_s = 1$ for all $i \in N$, the following holds: $\sum y_s v(S) \leq v(N)$.* ■

5.3.2 Market Games and Core

Market games arise in the field of economics and model an environment with N players and a set of m continuous items that are distributed across the players arbitrarily. This distribution indicated by $A = (a_i)_{i \in N}$ is termed the initial endowment vector. Each player also has a valuation function f_i indicating their utility for possessing these items. The players are allowed to trade these items in order to increase their utility. Toward this, they may form coalitions so as to maximize the social welfare of the coalition.

If the players were to form the grand coalition, the best way to share the utility is through an allocation that is in the core. The core of such a game is guaranteed to be non-empty through the Bondareva-Shapley theorem 5.3.1. Further the subgame of a market game is itself a market game. Hence, for a cooperative game to be modeled as a market game, the cooperative game and all its subgames must have non-empty cores. Further the non-emptiness of the core for market games can be proven through other methods. For more details, refer to the book by Shubik [91] on market games.

Definition 5.3.2 Veto player: Player-i is called a veto player if $v(N\backslash\{i\}) = 0$. That is, if $v(S) = 1$, then $i \in S$. In other words, player-i must be in every winning coalition.

Example 5.3.3 Consider a simple cooperative game (N, v) with $N = 1, 2, 3$, $v(123) = v(13) = v(23) = 1$, $v(1) = v(2) = v(3) = v(12) = 0$. Here player-3 is the veto player as player-3 is present in every winning coalition. The absence of player-3 in a coalition makes it a losing coalition. ▲

Theorem 5.3.2 (Owen [67, Example 10.4.6]) *Let (N, v) be a proper simple game. Then the core C_v is non-empty if and only if the game has veto players.* ■

5.3.3 Core as a Solution

Definition 5.3.3 Solution: C_v is a solution for (N, v) if the following conditions are satisfied:

(1) $C_v \cup \text{dom}(C_v) = I$ and
(2) $C_v \cap \text{dom}(C_v) = \phi$.

Remark 5.3.2 As $C_v \cap \text{dom}(C_v) = \phi$ is always true for the core, this condition is sometimes not specified in the definition of the solution.

Theorem 5.3.3 *If C_v is a solution to (N, v), then C_v is the only solution.*

Proof If possible, let X be another solution to (N, v). Hence

$$X \cup \text{dom}(X) = I \text{ and } X \cap \text{dom}(X) = \phi.$$

Since imputations in C_v cannot be dominated, $X \supset C_v$. It suffices to show that $X = C_v$.

If possible, let $X \neq C_v$. Let there exist $x \in X\backslash C_v$. Since C_v is a solution, there exists $y \in C_v$ such that $y \succ x$. But $y \in C_v$ implies $y \in X$. Hence, $x, y \in X$ such that $y \succ x$. This, however, is a contradiction since X is a solution. Hence, the core is the only solution. ■

In the following example of a cooperative game with $N = 6$, we show when the core is the unique solution and when it is not. In Example 5.3.4, when $v(N) \geq 3$, the core is the unique solution set. However, in Example 5.3.5, where $v(N) < 3$, the core is not a solution.

Example 5.3.4 Let $N = \{1, 2, 3, 4, 5, 6\}$. Let $v(\{i\}) = 0$, $v(12) = v(13) = 1$, $v(45) = v(46) = 1$, $v(1245) = 2$, $v(123456) = 3$, and $v(S) = 0$ for all other applicable coalitions.

It is obvious that the core is not empty. $(1, 0, 0, 1, t, 1-t) \in C_v$ where $t \in [0, 1]$. Other elements of the core include $(1, t, 1-t, 1, 0, 0)$ and $(1, t, \frac{1}{2}-t, 1, t, \frac{1}{2}-t)$.

Claim: The core is the unique solution.

If possible, let $y = (y_1, \ldots, y_6) \notin C_v$ be a solution. We show that at least for some S, $v(12) \neq 1$ or $v(13) \neq 1$. If $y \notin C_v$, then there exists $S \subseteq N$ $(S \neq \phi)$ such that $0 \leq y(S) < v(S)$. Such an S must be 1 or 2. So we can always find an element in C_v such that $y \succ C_v$. This is a contradiction. Hence, the core is the unique solution. ▲

Example 5.3.5 In Example 5.3.4, if $v(123456) = 2$ and everything else remains the same, the core is non-empty since $(1, 0, 0, 1, 0, 0) \in C_v$ and is the only element of C_v. Hence, the core is not a solution. In fact, if $v(123456) = 2 + \delta$ where $\delta \in [0, 1)$, then the core is non-empty but will still not be a solution. ▲

5.3.4 *Core and the Bankruptcy Problem*

The bankruptcy problem deals with dividing an amount e among n claimants. Aumann and Maschler [1] provide a game theoretic analysis of the bankruptcy problem. Let the claim be the n claimants which can be indicated by $d_1, \ldots, d_n$, respectively, where $\sum_{i=1}^{n} d_i > e$.

Thus, a bankruptcy game can be represented by the pair (e, d) where $d = (d_1, \ldots, d_n)$ and $0 \leq e < \sum_{i=1}^{n} d_i$. Without loss of generality, assume that $0 \leq d_1 \leq \cdots \leq d_n$.

In game theoretic terms, let $N = \{1, \ldots, n\}$. Define $v(N) = e$ and $v(S) = (e - \sum_{i \in N \setminus S} d_i)^+$.[1]

Let $x = (x_1, \ldots, x_n)$ where x_i is the amount assigned to claimant i. Then x is a solution to the bankruptcy problem if $x_1 + \cdots + x_n = e$. Aumann and Maschler [1] show that the core of the bankruptcy game is non-empty, and the core is a stable set.

Example 5.3.6 Consider a game with three players. Let $e = 250$, $a_1 = 100$, $a_2 = 200$, and $a_3 = 300$. Hence $v(123) = 250$, $v(1) = [250 - (200 + 300)]^+ = [-250]^+ = 0$. Proceeding along similar lines, $v(2) = v(3) = v(12) = 0$, $v(23) = 150$, and $v(13) = 50$. The core is given as follows:

[1]For any real number a, $a^+ = max\{a, 0\}$.

$$\begin{aligned} C_v = \{(x_1, x_2, x_3) : & x_1 + x_2 + x_3 = 250, \\ & x_2 + x_3 \geq 150, \\ & x_1 + x_3 \geq 50, \\ & x_1 \geq 0, \\ & x_2 \geq 0, \\ & x_3 \geq 0\}. \end{aligned}$$

Since (100, 50, 100) is one of the elements in the core, the core is non-empty. ▲

Remark 5.3.3 Since the bankruptcy game is convex, $C_v \neq \phi$ and C_v is the unique stable set.

5.4 Balanced Collection

This section describes another solution concept called the balanced collection and indicates how the minimal balanced collection is precisely the extreme points of the feasible set corresponding to the maximization linear program problem.

Definition 5.4.1 Balanced collection: Consider a game (N, v). A collection of coalitions $\mathcal{C} = \{S_1, \ldots, S_k\}$ is termed a balanced collection if

- $\cup_{i=1}^{k} S_i = N$.
- For every $S \in \mathcal{C}$, there exists $y_S > 0$ such that $\sum_{i \in S} y_S = 1$, for $i = 1, \ldots, n$.

Here $(y_S : S \in \mathcal{C})$ is referred to as the *balancing vector*.

Example 5.4.1 Let $N = \{1, 2, 3\}$. The following are balanced collections:

- $\mathcal{C} = \{\{1\}, \{2\}, \{3\}\}$. This collection of single elements is trivially a balanced collection. The balanced vector is (1, 1, 1).
- $\mathcal{C} = \{\{1\}, \{23\}\}$ and all its permutations. The balanced vector is (1, 1).

▲

Definition 5.4.2 Minimal balanced collection: $\mathcal{C}$ is a minimal balanced collection if $\mathcal{C}$ is balanced and no proper subset of $\mathcal{C}$ is balanced.

Example 5.4.2 In Example 5.4.1, consider $\mathcal{C} = \{\{12\}, \{23\}, \{13\}\}$. This is a minimal balanced collection as $\{\{12\}, \{23\}\} \subset \mathcal{C}$ but is not balanced. The balanced vector for $\mathcal{C}$ is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. ▲

Owen [67] used the fact that the minimal balanced collection is precisely the extreme points of the feasible set to prove the following.

Theorem 5.4.1 (Owen [67, Theorem 10.5.8]) *It is enough to check if* $\sum y_S v(s) \leq v(N)$ *only for the minimal balanced collection as the balanced vectors corresponding to these collections are precisely the extreme points of the feasible set corresponding to the maximization linear program problem given in Eq. 5.34.* ■

Remark 5.4.1 In Example 5.4.1, it is thus enough to check for $\frac{1}{2}\Big(v(12) + v(13) + v(23)\Big) \leq v(123)$.

Example 5.4.3 For $N > 3$, though the linear program representation can be solved in polynomial time, there are theoretical difficulties as shown in this example. Let $N = \{1, 2, 3, 4\}$. The minimal balanced collection are as follows:

- $\mathcal{C} = \{\{123\}, \{134\}, \{124\}, \{234\}\}$. Balanced vector = $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.
- $\mathcal{C} = \{\{12\}, \{13\}, \{14\}, \{234\}\}$ and all possible permutations leading to four minimal balanced collections. Balanced vector = $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{2}{3})$.
- $\mathcal{C} = \{\{12\}, \{13\}, \{23\}, \{4\}\}$ and all possible permutations leading to four minimal balanced collections. Balanced vector = $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1)$.
- $\mathcal{C} = \{\{124\}, \{134\}, \{234\}\}$ and all possible permutations leading to six minimal balanced collections. Balanced vector = $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.

Thus, this game results in $2^4 - 1$ minimal balanced collections. In general, no nice algorithm exists to write the minimal balanced collection. ▲

5.5 Assignment Games

An assignment game is as follows. Let there be m sellers and m buyers. Each seller has one house to sell and each buyer can purchase only one house. Let a_i indicate the valuation of the house by seller i $(i = 1, \ldots, m)$. Let b_{ij} indicate the valuation of house i $(i = 1, \ldots, m)$ by buyer $m + j$ $(j = 1, \ldots, m)$. When $b_{ij} > a_i$, a sale takes place. Define the game for all 2×2 coalitions.

For every two-player coalition between the ith seller and jth buyer, the worth of the coalition is given by

$$v_{ij} = \begin{cases} b_{ij} - a_i, & \text{if } b_{ij} > a_i, \\ 0, & \text{otherwise} . \end{cases}$$

Note that for any coalition S that consists only of sellers or only of buyers, $v(S) = 0$. For those coalitions S that contain both sellers and buyers, the value of the coalition $v(S)$ is the maximum total profit obtained through the sale of the houses among the members of S.

The 2×2 coalition can be extended to any coalition S over $2k$ distinct players: $i_1, \ldots, i_k$ sellers and $j_1, \ldots, j_k$ buyers. Then, the aim of the assignment problem is to identify the optimal set of transactions that maximize the coalition's total gain given by

$$v(S) = max[v_{i_1 j_1} + v_{i_2 j_2} + \cdots + v_{i_k j_k}], \tag{5.35}$$

where seller i_k sells his house to buyer j_k.

Using linear programming techniques, it is possible to show that the assignment game has a non-empty core. Toward this, Eq. 5.35 can be recast as the linear programming problem for $v(N)$ where N is the collection of sellers and buyers. Consider the following linear program:

$$\max z = \sum_{i=1}^{m} \sum_{j=1}^{m} v_{ij} x_{ij} \tag{5.36}$$

$$\text{s.to.} \sum_{i=1}^{m} x_{ij} = 1$$

$$\sum_{j=1}^{m} x_{ij} = 1$$

$$x_{ij} \geq 0, i = 1, \ldots, m; \; j = 1 \ldots m.$$

Consider the permutation matrix $P = (p_{ij})$ consisting of 0s and 1s such that there is only a single 1 in each row and column. In terms of the assignment game, the permutation matrix entries can be viewed as $p_{ij} = 1$ if the i-th seller sells to the j-th buyer, and 0 otherwise.

It is obvious that every permutation matrix satisfies the constraints in the linear program depicted above. In fact, the extreme points of the linear program are the permutation matrices themselves. Now for a linear program, the maximum is found at the extreme points. Hence, the maximum value arises for the permutation matrix and is $v(N)$.

Shapley and Shubik [87] view the assignment game as a model for a two-sided market involving large indivisible products such as houses that are exchanged for money. Each unit of the product may not be similar, and their valuation by different participants can be different. They draw a link between the core of the assignment problem and the dual linear program of the assignment problem, as shown below.

Theorem 5.5.1 (Shapley and Shubik [87, Theorem 2]) *The core of an assignment game is precisely the set of solutions of the dual linear program of the corresponding assignment problem.*

Proof The dual of the linear program model of the assignment problem (Eq. 5.36) is as follows:

$$\begin{aligned}
\min w &= \sum_{i=1}^{m} u_i + \sum_{j=1}^{m} v_j \\
\text{s.to. } & u_i + v_j \geq v_{ij} \\
& u_i \geq 0, i = 1, \ldots, m \\
& v_j \geq 0, j = 1 \ldots m.
\end{aligned} \tag{5.37}$$

By the fundamental duality theorem, $w_{min} = z_{max}$.

Let $(u, v) = (u_1, \ldots, u_m, v_1, \ldots, v_m)$ be the solution for the dual given by Eq. 5.37. Then,

$$\sum_{i=1}^{m} u_i + \sum_{j=1}^{m} v_j = w_{min} = z_{max}.$$

Thus, (u, v) is an imputation of the assignment game. Further, $u_i + v_j \geq v_{ij}$, thus ensuring the feasibility of (u, v). Also by Eq. 5.35, for any coalition S,

$$\sum_{i \in S} u_i + \sum_{j \in S} v_j \geq v(S).$$

Thus, ensures that it cannot be improved by any coalition. Hence, it forms the core of the assignment game. ■

Remark 5.5.1 Note that Eq. 5.36 is an equality constraint. In general, dual variables will be unrestricted. But in our case, one can show it is equivalent to Eq. 5.37 with non-negativity constraints.

5.6 Symmetric Cooperative Games

When the characteristic function for the coalitions depends only on the cardinality of the coalitions and not on the actual players who make up the coalition, such games are termed as symmetric games.

Definition 5.6.1 Symmetric game: A game (N, v) is symmetric if $v(S) = v(T)$ whenever $|S| = |T|$. That is, the characteristic function depends only on the cardinality of the coalitions.

Example 5.6.1 Consider the game $N = \{1, 2, 3\}$ where $v(12) = v(13) = v(23) = 1$, $v(123) = 1$, and $v(\phi) = v(1) = v(2) = v(3) = 0$. This is a symmetric game. ▲

Example 5.6.2 Consider the game $N = \{1, 2, 3, 4\}$ where $v(123) = v(134) = v(234) = 1$, $v(1234) = 1$, and $v(S) = 0$ otherwise. This is not a symmetric game since $v(124) = 0$. ▲

Remark 5.6.1 We can indicate a symmetric game through the function $f : \{0, \ldots, n\} \to \mathbb{R}$ where $f(1)$ implies that $|S| = 1$ and so on. In general, $f(n)$ implies that $|S| = n$.

Theorem 5.6.1 (Shapley, [86]) *A symmetric game* (N, v) *has a non-empty core if and only if* $\frac{v(n)}{n} \geq \frac{v(s)}{s}$, *where* n *is the cardinality of the grand coalition and* s *refers to the coalition with cardinality* s.

Proof Let $C_v \neq \phi$. Consider the case when $n = 3$ as the proof can be generalized for all n.

Let $x \in C_v$ where $x = (x_1, x_2, x_3)$. Since the game is symmetric and based on the definition of the core, we have

$$\begin{aligned} x_1 + x_2 \geq v(2), x_1 + x_3 \geq v(2), &x_2 + x_3 \geq v(2) \\ &\Rightarrow 2(x_1 + x_2 + x_3) \geq 3v(2) \\ &\Rightarrow 2v(3) \geq 3v(2) \\ &\Rightarrow \frac{v(3)}{3} \geq \frac{v(2)}{2} \end{aligned}$$

Extending to any n, $\frac{v(n)}{n} \geq \frac{v(s)}{s}$.

Conversely, let $\frac{v(n)}{n} \geq \frac{v(s)}{s}$.

Claim: $x = (\frac{v(n)}{n}, \frac{v(n)}{n}, \ldots, \frac{v(n)}{n}) \in C_v$.

Consider any k coordinates of x. It satisfies the condition $\frac{v(n)}{n} \geq \frac{v(k)}{k}$. Hence, $x(s) \geq v(s)$, where $\sum_{i \in s} x_i = x(s)$. This implies that $x \in C_v$. ■

5.7 0 − 1 Normalization

We now look at the $0 - 1$ normalization of the cooperative game.

Definition 5.7.1 0-1 normalization of (N, v): The $0 - 1$ normalization v^{**} of a game (N, v) with $v^*(N) > 0$ is given by $v^{**}(S) = \frac{v^*(S)}{v^*(N)}$ where $v^*(S) = v(S) - \sum_{i \in S} v(\{i\})$.

Remark 5.7.1 (1) $v^*(i) = 0$.
(2) v^* is monotonic and $v^*(S) \geq 0$.
(3) If $v^*(N) > \sum v(i)$, then it is called an essential game.

Remark 5.7.2 For 0-1 normalization, Theorem 5.6.1 becomes $\frac{1}{n} \geq \frac{v(s)}{s}$.

5.8 Large Stable Set

Definition 5.8.1 Acceptable set: Let $A = \{x \in \mathbb{R}^n : x(S) \geq v(S)$, for all $S \subseteq N\}$. Then $x \in A$ is called an acceptable point and A is called the acceptable set. That is, no coalition can improve upon x.

Remark 5.8.1 (1) A is not empty.
(2) A is a closed convex set and bounded from below.
(3) $C \subseteq A$.
(4) Acceptable set differs from core in the following aspect. While $x(N)$ may be greater than equal to $v(N)$ in the acceptable set, $x(N) = v(N)$ always in the core.

Definition 5.8.2 Lower boundary point: We say x is a lower boundary point of $B \subseteq \mathbb{R}^n$ if $Q_x \cap \bar{B} = \{x\}$, where $Q_x = \{y \in \mathbb{R}^n : y \leq x$ coordinate-wise $\}$ and $\bar{B}$ is the closure of B. The set of all lower boundary points of A is denoted by $L(A)$.

Example 5.8.1 Consider the entire disk with its center at the origin in $\mathbb{R}^2$. Then the boundary of the circle that falls in the lower left quadrant is the lower boundary points of the disk. ▲

Exercise 38 Let $D = \{x \in \Re^3 : x_1^2 + x_2^2 + x_3^2 \leq 1, x_1 + x_2 - x_3 \geq -1\}$. Examine whether this is a convex set. Also check if it has a lower boundary point.

Definition 5.8.3 Largeness of core (Sharkey [89]): The core C_v is large if for every acceptable vector $y \in A$, there exists $x \in C_v$ such that $x \leq y$ coordinate-wise. That is, given any $y \in A$, there exists $x \in C_v$ such that $x \in Q_y$.

Definition 5.8.4 Minimal elements set: The set of all the lower boundary points of A given by $m_A = \{y \in A :$ If $y' \leq y$ for some $y' \in A$, then $y' = y\}$ is called the minimal elements set.

Sharkey [89] proves that if the core is large, then it is the unique stable set. In fact, he shows that this is true irrespective of whether the game is symmetric or not. Biswas et al. [4] showed the converse, namely, if the core is a unique solution to a symmetric game, then the core is large.

Theorem 5.8.1 (Sharkey [89, Theorem 3]; Biswas et al. [4, Theorem 1]) *Let* (N, v) *be a symmetric game. Then the core* C_v *is large if and only if* C_v *is the unique stable set of the given game.* ■

In Theorem 5.8.1, the condition of a symmetric game is required for the core to be large when the core is the unique stable set. This is shown in the following example of a game that is not symmetric and has a unique stable set.

Example 5.8.2 Consider the cooperative non-symmetric game in Example 5.3.4. We have shown that the core is a solution and is hence the unique stable set.

Claim: The core is not large.

Let $(0, 1, 1, 0, 1, 1) \in A$ and $(0, 1, 1, 0, 1, 1) \notin C_v$. If possible, let core be large. Then, there exists $x = (x_1, \ldots, x_6) \in C_v$ such that $x \leq (0, 1, 1, 0, 1, 1)$ coordinate-wise. Now $x_1 \leq 0$ and $x \in C_v$ implies that $x_1 = 0, x_2 = x_3 = 1$. Similarly $x_4 = 0, x_5 = x_6 = 1$. Hence $x = (0, 1, 1, 0, 1, 1)$. ▲

Remark 5.8.2 When $v(N) < 3$, the core is not stable. When $3 \leq v(N) < 4$, the core is stable but not large. When $v(N) \geq 4$, the core is large.

Lemma 5.8.1 *C_v is large if and only if $C_v = L(A)$ where $L(A)$ is the set of all the lower boundary points of A.*

Proof Let C_v be large. Suppose a point x in C_v is not a lower boundary point. Then there must be another lower boundary point $y \notin C_v$ such that $y \leq x$ but $y \neq x$. But this contradicts the definition of core. Hence, any point in C_v is a lower boundary point. This implies that $C_v \subseteq L(A)$.

Suppose that $C_v \neq L(A)$. Hence, there exists $y \in L(A) \backslash C_v$. Since C_v is large, there exists $x \in C_v$ such that $x \leq y$ coordinate-wise. Hence, $Q_y \cap A = \{x, y\}$. This contradicts the fact that y is a lower boundary point. Hence, $C_v \supseteq L(A)$ which implies that $C_v = L(A)$.

Conversely let $C_v = L(A)$ and let $y \in A$. Construct $Q_y \cap A$. If $Q_y \cap A = \{y\}$, then trivially C_v is large. Let $Q_y \cap A \supset \{y, x\}$. Consider the closed set $Q_y \cap A$. Since the non-empty set A is bounded from below, then A will contain lower boundary points (by Lemma B.1.3). Hence $L(Q_y \cap A) \neq \phi$.

Let $z \in L(Q_y \cap A)$. Thus, $Q_z \cap Q_y \cap A = \{z\}$. Now $z \in Q_y$ implies that $z \leq y$ coordinate-wise. Hence, $Q_z \cap A = \{z\}$ implies that $z \in L(A)$. Thus, $z \in C_v$ and hence C_v is large. ■

Lemma 5.8.2 *(Biswas et al. [4, Theorem 6]) Let (N, v) be a symmetric game with a non-empty core C_v. Then C_v is large if $L(A)$ is a convex set.*

Proof If $C_v = L(A)$, then the core is large by Lemma 5.8.1.

Suppose C_v is not large. Then $C_v \subset L(A)$. Hence, there exists $y \in L(A) \backslash C_v$ where $y = (y_1, \ldots, y_n)$. Now $y = (y_1, \ldots, y_n) \in L(A)$ implies that any permutation of the coordinates is also in $L(A)$ as the game is symmetric, i.e., $(y_2, \ldots, y_n, y_1) \in L(A), (y_3, \ldots, y_n, y_1, y_2) \in L(A), \ldots, (y_n, y_1, y_2, \ldots, y_{n-1}) \in L(A)$. Since $L(A)$ is a convex set, the convex combination of the above points (i.e., the arithmetic mean) is also in $L(A)$. That is, $(\bar{y}, \bar{y}, \ldots, \bar{y}) \in L(A)$.

Now $n\bar{y} = \sum y_i > v(n)$ implies that $\bar{y} > \dfrac{v(n)}{n}$. Since $y \notin C_v$, $(\bar{y}, \ldots, \bar{y}) > \left(\dfrac{v(n)}{n}, \ldots, \dfrac{v(n)}{n}\right)$.

From Lemma 5.6.1, $z = \left(\dfrac{v(n)}{n}, \ldots, \dfrac{v(n)}{n}\right) \in C_v$. Hence $Q_y \cap A \supset \{\bar{y}, z\}$. This contradicts the fact that $y \in L(A)$. Hence, the core is large if $L(A)$ is a convex set. ■

Remark 5.8.3 A solution exists for the game (N, v) implies that a solution exists for the $0 - 1$ normalization (N, v^{**}) and vice versa.

There are examples where the core is stable but not large. While Kikuta and Shapley [37] provided some extendibility condition, Biswas et al. [5] show that this condition is sufficient for the core to be large and stable only for games with five or lesser number of players.

5.9 Cooperative Games and Decision Theory

We briefly look at Wald's contribution [104] and the parallel between game theory and decision theory.

In statistical decision theory, a hypothesis is formulated and tested. While Neyman and Pearson [65] do not penalize for wrong decisions, Wald [104] includes penalties for wrong decisions and borrows from minimax theory for his results in decision theory. In Wald's decision theory [104], there are two players—nature and the statistician. Let δ_1 and δ_2 be two decision rules. Let R be the risk function and let Θ be the set of choices available to nature. Let $\theta \in \Theta$ be the choice of nature. As the statistician wants to minimize risk, $\delta_1 \succ \delta_2$ if $R(\theta, \delta_1) \leq R(\theta, \delta_2)$ for all $\theta \in \Theta$ and $R(\theta_0, \delta_1) < R(\theta_0, \delta_2)$ for some $\theta_0 \in \Theta$. In this case, $delta_1$ is better than δ_2.

The equivalence between cooperative games and Wald's decision theory on a few topics can be summarized as follows.

Ordering
Cooperative games: Let $x, y \in I$. If $x \succ y$, then there exists $S \neq \phi$ such that $x_i > y_i$ coordinate-wise for all $i \in S$ and $\sum\limits_{i \in S} x_i \leq v(S)$. Hence, there is no partial ordering.
Wald's decision theory: There is a partial ordering of relations between decision rules used by the statistician.

Undominated imputation and admissible rules
Cooperative games: An imputation that cannot be dominated is an undominated imputation.
Wald's decision theory: The equivalent is the *admissible rule*. A decision rule δ is called an admissible rule if it cannot be dominated by any other rule.

Core and admissible rules
Cooperative games: Undominated imputations correspond to the core.
Wald's decision theory: $\mathcal{A}$ is used to denote the set of all admissible rules.

External stability and complete class
Cooperative games: $X \subseteq I$ is externally stable if given any $x \notin X$, there exists $y \in X$ such that $y \succ_S x$.
Wald's decision theory: $\mathcal{C}$ is a complete class if given any rule δ outside $\mathcal{C}$, there exists $\delta' \in \mathcal{C}$ such that $\delta' \succ \delta$.

Solution
Cooperative games: Assume $C_v \neq \phi$. Then every solution $X \supseteq C_v$.
Wald's decision theory: Let (C) be a complete class. Then $(C) \supseteq \mathcal{A}$.

Minimal complete class
Cooperative games: If the core is a solution, then it is the unique solution.
Wald's decision theory: A complete class (C) is a minimal complete class if no proper subset of it is complete. If there exists such a minimal complete class, then it coincides with $\mathcal{A}$.

5.10 Summary

There are many solution concepts in cooperative games, such as stable sets, core, and balanced collections. We looked at some specific games such as assignment games, bankruptcy problem, and market games for these solution concepts. We studied the conditions under which the non-empty core exists for cooperative games.

Appendix A
Minimax Theorem

von Neumann [100] provided a proof for the minimax theorem using Brouwer's fixed point theorem. An alternate proof can be provided using linear programming principles. We first state some of the relevant linear programming notations and the duality theorem.

Definition A.0.1 Primal and dual linear program form: Consider a real $m \times n$ matrix $A = (a_{ij})$, $m \times 1$ matrix b and $n \times 1$ matrix c. The primal and dual linear programs in standard form are as follows:

$$\begin{array}{rl} \text{Primal : } & \max\ c \cdot x \\ & \text{s.to } A \leq b \\ & \quad x \geq 0 \end{array} \qquad \begin{array}{rl} \text{Dual : } & \min\ b \cdot y \\ & \text{s.to } A^T y \geq c \\ & \quad y \geq 0 \end{array}$$

Remark A.0.1 In Definition A.0.1, $c \cdot x$ and $b \cdot y$ are referred to as the objective functions for the primal and the dual, respectively.

Remark A.0.2 Any x satisfying the primal constraints is referred to as a feasible solution to the primal. Similarly, any y satisfying the dual constraints is referred to as a feasible solution to the dual.

Theorem A.0.1 (Duality Theorem, Dantzig [13]) *If the primal and dual linear programming problems have at least one feasible solution, then the two problems have optimal solutions. At any optimal solution, the value of the two objective functions coincide.* ■

In other words, if either one is unbounded, then the other one is infeasible.

An alternative proof to the minimax theorem stated below can be provided using linear programming.

T. Parthasarathy and S. Babu, *Stochastic Games and Related Concepts*,
HBA Lecture Notes in Mathematics 2,
https://doi.org/10.1007/978-981-15-6577-9

Theorem A.0.2 (von Neumann [100, Minimax Theorem]) *Consider a matrix game with payoff matrix $A = (a_{ij})$ where A is a $m \times n$ real matrix. Then there exists a pair of probability vectors $x^* = (x_1^*, \ldots, x_m^*)$ and $y^* = (y_1^*, \ldots, y_n^*)$ such that for a unique constant v, we have*

$$\begin{aligned} \sum_i a_{ij} x_i^* &\geq v, \; j = 1, \ldots, n \\ \sum_j a_{ij} y_j^* &\leq v, \; i = 1, \ldots, m. \end{aligned} \tag{A.1}$$

Equivalently if $K(x, y) = \sum_i \sum_j a_{ij} x_i y_j$, then (x^*, y^*) is a saddle point for $K(x, y)$.

That is,

$$v = \min_y \max_x K(x, y) = \max_x \min_y K(x, y).$$

Proof By adding a constant c to all the entries in the matrix A, we can assume that $a_{ij} > 0$ for all i, j and $v > 0$. Further, Eq. A.1 can be rewritten as follows:

$$\begin{aligned} \sum_i a_{ij} \Big(\frac{x_i}{v}\Big) &\geq 1, \text{ for all } j \\ \sum_j a_{ij} \Big(\frac{y_j}{v}\Big) &\leq 1, \text{ for all } i. \end{aligned}$$

Let

$$p_i = \frac{x_i}{v}; q_j = \frac{y_j}{v}. \tag{A.2}$$

Then $\sum_i p_i = \sum_j q_j = \frac{1}{v}$. Indicate $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_n)$.

Now consider the following primal and dual linear programs for a matrix $A = (a_{ij})$ with all positive entries:

$$\begin{array}{llcll} \text{Primal: max} & \sum_j q_j & \qquad & \text{Dual: min} & \sum_i p_i \\ \text{s.to} & \sum_j a_{ij} q_j \leq 1 \text{for all } i & & \text{s.to} & \sum_i a_{ij} p_i \geq 1 \text{ for all } j \\ & q_j \geq 0 \text{ for all } j & & & p_i \geq 0 \text{ for all } i \end{array}$$

Now $q = 0$ is a feasible solution for the above primal. For large N, we have $p = (N, \ldots, N)$ is a feasible solution for the dual. Thus, by Duality theorem A.0.1, the primal and dual have q^* and p^* as the optimal solution, respectively. Further by Duality theorem A.0.1, $\sum_i p_i^* = \sum_j q_j^*$.

Normalizing p^* and q^* through Eq. A.2 yields optimal x^* and y^* satisfying Eq. A.1 for the payoff matrix A. Further $v = \frac{1}{\sum_j q_j^*}$.

This completes the proof for the minimax theorem. ■

Consider a $m \times n$ payoff matrix A where m need not be equal to n. Construct a skew-symmetric $mn \times mn$ matrix M from matrix A as follows. Let $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. Consider the auxiliary game M where each player chooses (i, j). Indicate the choices of the two players by $\alpha = (i, j)$ and $\alpha' = (i', j')$. Define the payoff $m_{\alpha\alpha'} = a_{ij'} - a_{i'j}$. This constitutes the new skew-symmetric matrix M. Since $m_{\alpha\alpha'} = m_{\alpha'\alpha}$, M is skew symmetric. The following theorem by Gale [22] can also be used to prove minimax theorem.

Theorem A.0.3 (Gale [22]) *Let A be the $m \times n$ payoff matrix and let M be the skew-symmetric $mn \times mn$ matrix as specified above. Let ξ^o be an optimal strategy for the auxiliary game M. Let $x_i^o = \sum_{j=1}^{n} \xi_{ij}^o$ and $y_j^o = \sum_{i=1}^{n} \xi_{ij}^o$. Then $x^o = (x_1^o, \ldots, x_m^o)$ and $y^o = (y_1^o, \ldots, y_n^o)$ are optimal strategies for the game A. Further the value of the game A is $\sum_i \sum_j a_{ij} x_i^o y_j^o$.* ■

Appendix B
Some Analytic Concepts

Here are some relevant definitions and theorems from real analysis and functional analysis.

B.1 Convergence

Definition B.1.1 Weak convergence: Consider a sequence of probability distributions $\{\mu_n\}$ on a compact metric space A. Denote by $C[A]$ the set of all continuous functions with domain A. We say that $\{\mu_n\}$ converges to μ weakly if $\int f(a)d\mu_n(a) \to \int f(a)d\mu(a)$, for all $f \in C[A]$. If μ_n converges weakly to μ, we denote by $\mu_n \to \mu$.

Lemma B.1.1 *Let $u \in C[A \times B]$ be a jointly continuous function where A and B are compact metric spaces. If $\mu_n \to \mu$ and $\lambda_n \to \lambda$, then for all u*

$$\int \int u(a, b)d\mu_n(a)d\lambda_n(b) \to \int \int u(a, b)d\mu(a)d\lambda(b)$$

Proof Assume $u(a, b)$ is a finite linear combination of functions of a and functions of b, i.e., $u(a, b) = \sum_{j=1}^{n} \sum_{i=1}^{m} t_{ij} u_i(a) v_j(b)$, where $u_i \in C[A]$, $v_j \in C[B]$ and $t_{ij} \in \mathbb{R}$. As this is a finitely many sums, we have

T. Parthasarathy and S. Babu, *Stochastic Games and Related Concepts*,
HBA Lecture Notes in Mathematics 2,
https://doi.org/10.1007/978-981-15-6577-9

$$
\begin{aligned}
&\int\int u(a,b)d\mu_n(a)d\lambda_n(b)\\
&\qquad = \sum\sum t_{ij}\int\int u_i(a)v_j(b)d\mu_n(a)d\lambda_n(b)\\
&\qquad = \sum t_{ij}\int u_i(a)d\mu_n(a)\sum t_{ij}\int v_j(b)d\lambda_n(b)\\
&\qquad \to \sum t_{ij}\int u_i(a)d\mu_0(a)\sum t_{ij}\int v_j(b)d\lambda_0(b)\\
&\qquad = \sum\sum t_{ij}\int\int u_i(a)v_j(b)d\mu_0(a)d\lambda_0(b)\\
&\qquad = \int\int u(a,b)d\mu(a)d\lambda(b).
\end{aligned}
$$

In fact, by Stone-Weierstrass theorem, any general function u is the uniform limit of functions given above. Hence, the lemma follows. ■

Remark B.1.1 Refer to Definition B.1.1. Since P_A is compact and metrizable, given any sequence in P_A, there exists a subsequence that converges. For a proof, refer to Parthasarathy [68].

Lemma B.1.2 *Let $S = [0, 1]$ and let $r : S \times A \times B$ be a continuous real-valued function. Assume A and B are compact and metrizable. Then $r(s, \mu, \lambda)$ is continuous over $S \times A \times B$.*

Proof Define $r(s_n, \mu_n, \lambda_n) = \int\int r(s_n, a, b)d\mu_n(a)d\lambda_n(b)$. Further let $(s_n, \mu_n, \lambda_n) \to (s_0, \mu_0, \lambda_0)$ where $s_n \in [0, 1]$, $\mu_n \to \mu_0$ weakly, and $\lambda_n \to \lambda_0$ weakly. Then $r(s_n, \mu_n, \lambda_n) \to r(s_0, \mu_0, \lambda_0)$.

By Lemma B.1.1, $r(s_n, a, b) \to r(s_0, a, b)$ when $s_n \to s_0$. Hence, we have $\int\int r(s_n, a, b)d\mu_n(a)d\lambda_n(b) \to \int\int r(s_0, a, b)d\mu_0(a)d\lambda_0(b)$. ■

Lemma B.1.3 *Let $A \neq \phi$ and $A \subseteq \mathbb{R}^n$. If A is bounded from below, then it has at least one lower boundary point, i.e., $L(A) \neq \phi$.*

Proof Construct $a = \inf\{\sum x_i : x \in A\}$. This infimum is well defined as $x_i \geq 0$. This implies that $\sum x_i \geq 0$. Define $a = \lim_{k\to\infty} \sum_{i=1}^{n} x_i^{(k)}$ where $x^{(k)} \in A$. By Bolzano-Weierstrass theorem, as the sequence $\{x^{(k)}\}$ is a bounded sequence, there exists a convergent subsequence. Without loss of generality, assume that $\{x^{(k)}\}$ is itself the convergent subsequence, i.e., $x^{(k)} \to x^*$ as $k \to \infty$.

Hence $a = \lim_{k\to\infty} \sum_{i=1}^{n} x_i^{(k)} = \sum_{i=1}^{n} x_i^*$.

Claim: x^* is a lower boundary point. This is equivalent to showing that $Q_{x^*} \cap A = \{x^*\}$.

Since $x^* \in Q_{x^*}$ and $x^* \in A$, $x^* \in Q_{x^*} \cap A$. Suppose that there exists another point $y \in Q_{x^*} \cap A$ such that $y \neq x^*$. Hence $y \lneqq x^*$. This implies that $\sum y_i < \sum x_i^*$. This contradicts the fact that x^* is the infimum. Hence, $Q_{x^*} \cap A = \{x^*\}$. ■

B.2 Selection Theorems

When the state space is uncountable, the optimal strategies must be measurable. This motivates us to look at results relating to selection theorems. Michael [58–60] looks at continuous selections and provides nice characterization of paracompact spaces. His results are useful in optimization problems relating to continuous and differentiable functions. Parthasarathy [73] also refers to the work by Kuratowski and Ryll-Nardzewski [46], and Michael [58–60].

Kuratowski [45] looks at a compact metric space (Δ, ρ). Let 2^Δ be the space of all non-empty closed subsets of Δ. Hence, every element of 2^Δ is a compact subset of Δ. Let A and B be two such subsets. The distance between A and B (referred to as the Hausdorff metric on 2^Δ) is given by

$$d(A, B) = \max\{\max_{x \in A} \rho(x, B), \max_{y \in B} \rho(y, A)\},$$

where $\rho(x, B)$ is the minimum distance of x to B and $\rho(y, A)$ is the minimum distance of y to A. It can be easily verified that this does induce a metric.

Consider a sequence of elements from 2^Δ, i.e., $A_n \in 2^\Delta$. We define $\overline{\lim}$ for this sequence differently from that for sets as follows.

$$\overline{\lim}\, A_n = \{x : n_1 < n_2 < \cdots < n_k < \ldots, x_{n_k} \in A_{n_k}, x_{n_k} \to x\}.$$

Similarly, we can define $\underline{\lim}\, A_n$. We say that $lim A_n$ exists if $\overline{\lim}\, A_n = \underline{\lim}\, A_n$.

Definition B.2.1 Borel measurable function: Let X be a metric space and Δ be a compact metric space. Define a point-to-set-valued function $F : X \to 2^\Delta$. We say F is Borel measurable if $\{x : F(x) \cap B \neq \phi\}$ is a Borel set whenever B is a Borel set in Δ.

Definition B.2.2 Upper semicontinuous (u.s.c.): Let $F : X \to 2^\Delta$. Call F an u.s.c. map if whenever $x_n \to x$, then $\overline{\lim}\, F(x_n) \subseteq F(x)$.

Example B.2.1 Let $u : X \to \mathbb{R}$ be a real-valued function where X is any metric space. We say u is u.s.c. if whenever $x_n \to x$, then $\overline{\lim}\, u(x_n) \leq u(x)$. ▲

Definition B.2.3 Lower semicontinuous (l.s.c.): Let $F : X \to 2^\Delta$. Call F an l.s.c. map if whenever $x_n \to x$, then $\underline{\lim}\, F(x_n) \supseteq F(x)$.

Kuratowski [45] showed the following when Δ is a compact metric space.

(1) $(2^\Delta, d)$ is a compact metric space.
(2) $d(A_n, A) \to 0$ if and only if $lim A_n = A$.
(3) If F is u.s.c., then F is Borel measurable.

Definition B.2.4 Selector: Let X and Y be metric spaces. Define $F : X \to Y$ where for all $x \in X$, $F(x) \subseteq Y$. We call f a selector for F if $f : X \to Y$ such that $f(x) \in F(x)$ for all $x \in X$.

Definition B.2.5 Measurable selector: A selector f is a measurable selector if:

(1) $f(x) \in F(x)$ for all x.
(2) $f^{-1}(G)$ is a Borel set for all G open in Y.

Dubins and Savage [16] specify the conditions on X and Y such that f is a measurable selector.

Theorem B.2.1 (Dubins-Savage Selection Theorem [16]) *Let $u : S \times A \to \mathbb{R}$ be a bounded measurable function where S is a Borel set and A is a compact metric space. Further assume u to be u.s.c. on $S \times A$. Define $u(s^*) = \max_{a \in A} u(s, a)$. Then there exists a Borel measurable function $f : S \to A$ such that $u(s^*) = u(s, f(s))$.*

Proof Here is a high-level overview of the proof. For a complete proof, refer to Dubins and Savage [16] or Parthasarathy [73].

Define $F : S \to A$ as $F(s) = \{a' : \max_{a \in A} u(s, a) = u(s, a')\}$. Hence, $F : S \to 2^A$. As u is u.s.c, F is a Borel measurable map. It can be shown that F is u.s.c. Hence, there exists a measurable selector f of F where $f : S \to A$ such that $f(s) \in F(s)$. ■

Remark B.2.1 (1) Here is another interpretation. For each s, $u(s, a)$ can attain the maximum value at many a's. We can select one such a for every s to get a measurable graph.
(2) Kuratowski and Ryll-Nardzewski [46] only require that F is a measurable map instead of requiring continuity.

References

1. Aumann, R.J., Maschler, M.: Game theoretic analysis of a bankruptcy problem from the Talmud. J. Econ. Theory **36–2**, 195–213 (1985)
2. Avrachenkov, K., Ejov, V., Filar, J.A., Moghaddam, A.: Ordered field property for zero-sum stochastic games. ArXiv e-prints, 12 (2017)
3. Bewley, T., Kohlberg, E.: The asymptotic theory of stochastic games. Math. Oper. Res. **1–3**, 197–208 (1976)
4. Biswas, A.K., Parthasarathy, T., Ravindran, G.: Stability and largeness of core for symmetric games. Int. J. Game Theory **29–1**, 11–22 (2000)
5. Biswas, A.K., Parthasarathy, T., Ravindran, G.: Stability and largeness of the core. Games Econ. Behav. **34–2**, 227–237 (2001)
6. Blackwell, D.: Discounted dynamic programming. Ann. Math. Stat. **36–1**, 226–235 (1965)
7. Blackwell, D.: Infinite G_δ games with imperfect information. Zastosowania Matematyki **10**, 99–101 (1969)
8. Blackwell, D.: The square-root game. In: Bruss, F.T., Le Cam, L. (eds.) Game Theory, Optimal Stopping, Probability and Statistics: Papers in honor of TS Ferguson. IMS Lecture Notes - Monograph Series, pp. 35:35–37. Institute of Mathematical Statististics, Beachwood, OH (2000)
9. Blackwell, D., Ferguson, T.S.: The big match. Ann. Math. Stat. **39**, 159–163 (1968)
10. Bondareva, O.N.: Some applications of linear programming methods to the theory of cooperative games. Problemy Kibernetiki **10**, 119–139 (1963)
11. Chin, H., Parthasarathy, T., Raghavan, T.E.S.: Optimal strategy sets for continuous two person games. Sankhyā, Ser. A Indian J. Stat. **38–1**, 92–98 (1976)
12. Cleary, J., Morris, S.A., Yost, D.: Numerical geometry - numbers for shapes. Am. Math. Mon. **93–4**, 260–275 (1986)
13. Dantzig, G.B.: A proof of the equivalence of the programming problem and the game problem. In: Koopmans, T.C. (ed.) Activity Analysis of Production and Allocation, pp. 330–335. Wiley, New York (1951)
14. Das, P., Roy, S.: Some thoughts on completely mixed games. Unpublished Manuscript (2015)
15. Dasgupta, P., Maskin, E.: The existence of equilibrium in discontinuous economic games, Part I (Theory). Rev. Econ. Stud. **53–1**, 1–26 (1986)
16. Dubins, L.E., Savage, L.J.: How to Gamble if You Must. McGraw-Hill Book Company (1965)
17. Federgruen, A.: On N-person stochastic games with denumerable state space. Adv. Appl. Probab. **10–2**, 452–471 (1978)

T. Parthasarathy and S. Babu, *Stochastic Games and Related Concepts*,
HBA Lecture Notes in Mathematics 2,
https://doi.org/10.1007/978-981-15-6577-9

18. Filar, J.A.: Ordered field property for stochastic games when the player who controls transitions changes from state to state. J. Optim. Theory Appl. **34–4**, 503–515 (1981)
19. Filar, J.A., Vrieze, K.: Competitive Markov Decision Processes. Springer, New York (1996)
20. Fink, A.M.: Equilibrium in a stochastic n-person game. J. Sci. Hiroshima Univ. Ser. A-I Math. **28**, 89–93 (1964)
21. Flesch, J., Parthasarathy, T., Thuijsman, F., Uyttendaele, P.: Evolutionary stochastic games. Dyn. Games Appl. **3–2**, 207–219 (2013)
22. Gale, D.: The Theory of Linear Economic Models. McGraw Hill, New York (1960)
23. Gale, D., Stewart, F.M.: Infinite games with perfect information. Contrib. Theory of Games - II, Ann. Math. Stud. **28**, 245–266 (1953)
24. Gillette, D.: Representable infinite games. Ph.D. thesis, University of California, Berkeley (1953)
25. Gillette, D.: Stochastic games with zero stop probabilities. In: Dresher, M., Tucker, A.W., Wolfe, P. (eds.) Contributions to the Theory of Games, Vol. III, Annals of Mathematics Studies, vol. 39, pp. 179–187. Princeton University Press, Princeton, NJ (1957)
26. Glicksberg, I.R.: Minimax theorem for upper and lower semicontinuous payoffs. In: RAND Corporation Research Memorandum, 478 (1950)
27. Glicksberg, I.R.: A further generalization of the Kakutani fixed point theorem with application to Nash equilibrium points. Proc. Am. Math. Soc. **1**, 170–174 (1952)
28. Glicksberg, I.R.: A derivative test for finite solutions of games. Proc. Am. Math. Soc. **4**, 595–597 (1953)
29. Gross, O.: The rendezvous value of a metric space. Adv. Game Theory, Ann. Math. Stud. **52**, 49–53 (1964)
30. Himmelberg, C.J., Parthasarathy, T., Raghavan, T.E.S., Van Vleck, F.S.: Existence of p-equilibrium and optimal stationary strategies in stochastic games. Proc. Am. Math. Soc. **60**, 245–251 (1976)
31. Hofbauer, J., Sigmund, K.: Evolutionary game dynamics. Bull. Am. Math. Soc. **40–4**, 479–519 (2003)
32. Hou, S.H., Hou, E.: On a recursion formula related to confluent Vandermonde. Am. Math. Mon. **122–8**, 766–772 (2015)
33. Jaśkiewicz, A., Nowak, A.S.: Stationary almost Markov perfect equilibria in discounted stochastic games. Math. Oper. Res. **41–2**, 430–441 (2016)
34. Kaplansky, I.: A contribution to von Neumann's theory of games. Ann. Math. **46–3**, 474–479 (1945)
35. Kaplansky, I.: A contribution to von Neumann's theory of games II. Linear Algebra Appl. **226**, 371–373 (1995)
36. Karlin, S.: Mathematical Methods and Theory in Games, Programming, and Economics. Pergamon Press, Paris (1959)
37. Kikuta, K., Shapley, L.S.: Core stability in n-person games. In: Mimeo (1986)
38. Kohlberg, E.: Repeated games with absorbing states. Ann. Stat. **2–4**, 724–738 (1974)
39. Kohlberg, E., Mertens, J.F.: On the strategic stability of equilibria. Econometrica **54**, 1003–1037 (1986)
40. Krishnamurthy, N.: Equilibria in bimatrix games and stochastic games: Theoretical and computational aspects. Ph.D. thesis, Chennai Mathematical Institute, Chennai (2011)
41. Krishnamurthy, N., Parthasarathy, T.: Multistage Stochastic Games. Wiley Encyclopedia of Operations Research and Management Science (2011)
42. Krishnamurthy, N., Parthasarathy, T., Ravindran, G.: Communication complexity of stochastic games. In: Proceedings of the International Conference on Game Theory for Networks (GameNets'09), Istanbul, Turkey. IEEE Press, NJ, USA, pp. 411–417 (2009)
43. Krishnamurthy, N., Parthasarathy, T., Ravindran, G.: Orderfield property of mixtures of stochastic games. In: Sankhyā: Indian J. Stat. **72-A-1**, 246–275 (2010)
44. Krishnamurthy, N., Parthasarathy, T., Sujatha, B.: Existence of stationary equilibrium for mixtures of discounted stochastic games. Curr. Sci. Spec. Sect. Game Theory **103–9**, 1003–1013 (2012)

45. Kuratowski, K.: Topology, vol. I. Academic Press, Polish Scientific Publishers (1966)
46. Kuratowski, K., Ryll-Nardzewski, C.: A general theorem on selectors. Bulletin L'Academie Polonaise des Science, Serie des Sciences Mathematiques, Astronomiques et Physiques (BAPMAM) **13.1**, 397–403 (1965)
47. Levy, Y.: Discounted stochastic games with no stationary Nash equilibrium: two examples. Econometrica **81**, 1973–2007 (2013)
48. Levy, Y., McLennan, A.: Corrigendum to "discounted stochastic games with no stationary Nash equilibrium: two examples". Econometrica **83**, 1237–1252 (2015)
49. Lucas, W.F.: The proof that a game may not have a solution. Trans. Am. Math. Soc. Am. Math. Soc. **136**, 219–229 (1969)
50. Maitra, A., Parthasarathy, T.: On stochastic games. J. Optim. Theory Appl. **5**, 289–300 (1970)
51. Maitra, A., Sudderth, W.D.: Discrete Gambling and Stochastic Games. Springer, New York (1996)
52. Martin, D.A.: The determinacy of Blackwell games. J. Symb. Log. **63**, 1565–1581 (1998)
53. Maschler, M., Solan, E., Zamir, S.: Game Theory. Cambridge University Press, Cambridge (2013)
54. Maynard Smith, J., Price, G.R.: The logic of animal conflict. Nature **246**, 15–18 (1973)
55. Mertens, J.F., Neyman, A.: Stochastic games. Int. J. Game Theory **10**, 53–66 (1981)
56. Mertens, J.F., Parthasarathy, T.: Nonzero-sum stochastic games. In: Raghavan, T.E.S., Ferguson, T.S., Parthasarathy, T., Vrieze, O.J. (eds.) Stochastic Games and Related Topics, pp. 145–148. Kluwer Academic Publishers, Dordrecht, Netherlands (1991)
57. Mertens, J.F., Parthasarathy, T.: Equilibria for discounted stochastic games. In: Neyman, A., Sorin, S. (eds.) In Stochastic Games and Applications, pp. 131–172. Kluwer Academic Publishers (2003)
58. Michael, E.: Continuous selections I. Ann. Math. **63**, 361–381 (1956a)
59. Michael, E.: Continuous selections II. Ann. Math. **64**, 562–580 (1956b)
60. Michael, E.: Continuous selections III. Ann. Math. **65**, 375–390 (1957)
61. Mohan, S.R., Neogy, S.K., Parthasarathy, T.: Linear complementarity and discounted polystochastic game when one player controls transitions. In: Ferris, M.C., Pang, J.S. (eds.) Proceedings of the International Conference on Complementarity Problems, pp. 284–294. SIAM (1997)
62. Moulin, H.: Game Theory for the Social Sciences. New York University Press, New York (1986)
63. Nash, J.F.: Non-cooperative games. Ann. Math. **54**, 286–295 (1951)
64. Nash, J.F., Shapley, L.S.: A simple three-person poker game. Ann. Math. Stud. Princeton Univ. Press **24**, 105–116 (1950)
65. Neyman, J., Pearson, E.S.: On the problem of the most efficient tests of statistical hypotheses. Philos. Trans. R. Soc. A: Math. Phys. Eng. Sci. **231**, 694–706 (1933)
66. Oviedo, J.A.: Payoff matrices in completely mixed bimatrix games with zero-value. Union Math. Argent. **40**, 75–81 (1996)
67. Owen, G.: Game Theory. Academic Press, New York (1982)
68. Parthasarathy, K.R.: Introduction to Probability and Measure. McMillan Press Ltd (1977)
69. Parthasarathy, T.: A note on compound simple games. Proc. Am. Math. Soc. **17**, 1334–1340 (1966)
70. Parthasarathy, T.: Product solutions for simple games - ii. Proc. Am. Math. Soc. **20**, 107–114 (1969a)
71. Parthasarathy, T.: Product solutions for simple games - iii. Proc. Am. Math. Soc. **20**, 412–420 (1969b)
72. Parthasarathy, T.: On games over the unit square. SIAM J. Appl. Math. **19**, 473–476 (1970)
73. Parthasarathy, T.: Selection Theorems and Their Applications. Lecture Notes in Mathematics Series Volume 263. Springer, Berlin (1972)
74. Parthasarathy, T.: Existence of equilibrium stationary strategies in discounted stochastic games. Sankhyā: Indian J. Stat. Ser. A **44**, 114–127 (1982)

75. Parthasarathy, T., Raghavan, T.E.S.: Some Topics in Two-Person Games. Elsevier Publications (1971)
76. Parthasarathy, T., Raghavan, T.E.S.: An orderfield property for stochastic games when one player controls transition probabilities. J. Optim. Theory Appl. **33–3**, 375–392 (1981)
77. Parthasarathy, T., Tijs, S.J., Vrieze, O.J.: Stochastic games with state independent transitions and separable rewards. In: Hammer, G., Pallaschke, D. (eds.) "Selected Topics in Operations Research and Mathematical Economics", Karlsruhe 1983. Lecture Notes in Economics and Mathematical Systems, vol. 226, pp. 262–271. Springer, Berlin (1984)
78. Raghavan, T.E.S.: Completely mixed strategies in bimatrix games. J. Lond. Math. Soc. **2**, 709–712 (1970)
79. Raghavan, T.E.S., Filar, J.A.: Algorithms for stochastic games - a survey. ZOR - Math. Methods Oper. Res. **35–6**, 437–472 (1991)
80. Raghavan, T.E.S., Syed, Z.: A policy-improvement type algorithm for solving zero-sum two-person stochastic games of perfect information. Math. Program. Ser. A **95–3**, 513–532 (2003)
81. Raghavan, T.E.S., Tijs, S.J., Vrieze, O.J.: On stochastic games with additive reward and transition structure. J. Optim. Theory Appl. **47–4**, 451–464 (1985)
82. Reny, P.: On the existence of pure and mixed strategy Nash equilibria in discontinuous games. Econometrica **67–5**, 1029–1056 (1999)
83. Rudin, W.: Principles of Mathematical Analysis. McGraw-Hill Book Company (2002)
84. Shapley, L.S.: Stochastic games. Proc. Natl. Acad. Sci. **39**, 1095–1100 (1953)
85. Shapley, L.S.: Solutions of compound simple games. Advances in Game Theory, pp. 267–305. Princeton University Press, Princeton, N.J. (1964)
86. Shapley, L.S.: On balanced sets and cores. Naval Res. Logist. Q. **14–4**, 453–460 (1967)
87. Shapley, L.S., Shubik, M.: The assignment game I: the core. Int. J. Game Theory **1–1**, 111–130 (1971)
88. Shapley, L.S., Snow, R.N.: Basic solutions of discrete games. Ann. Math. Stud. Princeton Univ. Press **24**, 27–35 (1950)
89. Sharkey, W.W.: Cooperative games with large cores. Int. J. Game Theory **11**(3–4), 175–182 (1982)
90. Shiffman, M.: Games of timing. In: Kuhn, H.W., Tucker, A.W. (eds.) Contributions to the Theory of Games II. Annals of Mathematical Studies, vol. 28, pp. 97–123 (1953)
91. Shubik, M.: Strategy and Market Structure. Wiley, New York (1959)
92. Sinha, S.: A contribution to the theory of stochastic games. Ph.D. thesis, Indian Statistical Institute, New Delhi, India (1989)
93. Sion, M., Wolfe, P.: On a game without value. In: Dresher, M., Kuhn, H.W., Tucker, A.W. (eds.) Contributions to the Theory of Games III. Annals of Mathematical Studies, vol. 39, pp. 299–306 (1957)
94. Stadje, W.: A property of compact, connected spaces. Archiv der Mathematik **36**, 275–280 (1981)
95. Strauch, R.E.: Negative dynamic programming. Ann. Math. Stat. **4**, 871–890 (1966)
96. Sujatha, B., Krishnamurthy, N., Parthasarathy, T.: Stationary, completely mixed and symmetric optimal and equilibrium strategies in stochastic games. Int. J. Game Theory **46–3**, 761–782 (2017)
97. Sujatha, B., Parthasarathy, T., Saha, A.: Strategies in certain types of skew symmetric games. In: Mishra Shaheen, R.K., Raveendran, J., Deman, S. (eds.) Applications of Game Theory, Academic Foundation, pp. 43–50 (2014)
98. Takahashi, M.: Equilibrium points of stochastic noncooperative n-person games. J. Sci. Hiroshima University Ser. A-I Math. **28**, 95–99 (1964)
99. Tarski, A.: A Decision Method for Elementary Algebra and Geometry. University of California Press, Berkley and Los Angels (1951)
100. von Neumann, J.: Zur theorie der gesellschaftsspiele. Mathematische Annalen **100**, 295–320 (1928)
101. von Neumann, J., Morgenstern, O.: Theory of Games and Economic Behavior. Princeton University Press, Princeton (1944)

102. Vrieze, O.J.: Linear programming and undiscounted games in which one player controls transitions. OR Spectr. **3–1**, 29–35 (1981)
103. Wald, A.: Note on zero-sum two-person games. Ann. Math. **20**, 739–742 (1950)
104. Wald, A.: Statistical Decision Functions. Wiley, New York (1950)
105. Warga, J.: Functions of relaxed controls. SIAM J. Control **5–4**, 628–641 (1967)
106. Wolf, R.: On the average distance property and certain energy integrals. Arkiv för Matematik **35–2**, 387–400 (1997)

Index

T. Parthasarathy and S. Babu, *Stochastic Games and Related Concepts*,
HBA Lecture Notes in Mathematics 2,
https://doi.org/10.1007/978-981-15-6577-9