Theory of Probability Solutions

Ryan Schumm, PhD

Durrett ed. 4, Problem 1.1.3

Let $\mathcal{S}_d = \{(a_1, b_1] \times \cdots \times (a_d, b_d] : a_i, b_i \in \mathbb{R},\ 1 \leq i \leq d\}$. Show that $\sigma(\mathcal{S}_d) = \mathcal{R}^d$ where $\mathcal{R}^d$ is the Borel subsets of $\mathbb{R}^d$.

Solution:
Let $\mathcal{U}$ be the set of all open sets of $\mathbb{R}^d$ and $S \in \mathcal{S}_d$. By definition, we can write

$$S = (a_1, b_1] \times \cdots \times (a_d, b_d],\ a_i, b_i \in (-\infty, \infty).$$

Take $\eta_i > 0$ such that $a_i + \eta_i < b_i$ for all $i = 1, \ldots, d$ and set

$$U = (a_1, b_1) \times \ldots \times (a_d, b_d), \tag{1}$$
$$V = [a_1 + \eta_1, b_1] \times \ldots \times [a_d + \eta_d, b_d] \tag{2}$$

It follows that $S = U \cup V \in \sigma(\mathcal{U})$ since the compliment of any open set is a closed set and sigma algebras are closed under unions. By definition of a generated sigma algebra and since S was arbitrary, we have $\mathcal{S}_d \subset \sigma(\mathcal{U}) \implies \sigma(\mathcal{S}_d) \subset \sigma(\mathcal{U})$.

Now let $U \in \mathcal{U}$ and define

$$\mathbb{S} = \{S = (a_1, b_1] \times \cdots \times (a_d, b_d] :\ a_i, b_i \in \mathbb{Q},\ S \subset U\}. \tag{3}$$

Note that $\mathbb{S}$ is a countable subset of $\mathcal{S}_d$ which implies that $\cup_{S \in \mathbb{S}} S \in \sigma(\mathcal{S}_d)$. For any $x \in U$, we have that

$$x \in (c_1, e_1) \times \cdots \times (c_d, e_d) \subset U \tag{4}$$

since the open sets $\mathcal{U}$ are generated by open boxes. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $a_i, b_i \in \mathbb{Q}$ such that $x_i \in (a_i, b_i] \subset (c_i, e_i)$ for all $i = 1, \ldots, d$. It follows that

$$x \in \cup_{S \in \mathbb{S}} S \implies U \subset \cup_{S \in \mathbb{S}} S \tag{5}$$

since x was arbitrary. By construction, we also have that $\cup_{S \in \mathbb{S}} S \subset U \implies U = \cup_{S \in \mathbb{S}} S$. Therefore, we have shown that $\mathcal{U} \subset \sigma(\mathcal{S}_d) \implies \sigma(\mathcal{U}) \subset \sigma(\mathcal{S}_d)$. Hence, $\sigma(\mathcal{S}_d) = \sigma(\mathcal{U}) = \mathcal{R}^d$.

Durrett ed. 4, Problem 1.1.4

A sigma field $\mathcal{F}$ is said to be countably generated if there is a countable collection of subsets $\mathcal{C} \subset \mathcal{F}$ so that $\sigma(\mathcal{C}) = \mathcal{F}$. Show that $\mathcal{R}^d$ is a countably generated.

Solution:
Let the countable set $\mathcal{C}$ be given by

$$\mathcal{C} = \{(a_1, b_1] \times \cdots \times (a_d, b_d] : \ a_i, b_i \in \mathbb{Q}\} \tag{6}$$

and observe that $\mathcal{C} \subset \mathcal{S}_d \implies \sigma(\mathcal{C}) \subset \sigma(\mathcal{S}_d) = \mathcal{R}^d$. In the previous problem, we showed that $U = \cup_{S \in \mathcal{S}} S \in \mathcal{C}$ for all $U \in \mathcal{U}$ where $\mathcal{S} \subset \mathcal{C}$ is defined in (3). It follows that $\mathcal{U} \subset \mathcal{C} \implies \sigma(\mathcal{U}) = \mathcal{R}^d \subset \sigma(\mathcal{C})$. Thus, we have shown that $\mathcal{R}^d$ is generated by the countable set $\mathcal{C}$.

Durrett ed. 4, Problem 1.2.3

Show that a distribution function at most countably many discontinuities.

Solution:
Let X be a random variable with distribution function F and $D \subset \mathbb{R}$ be the set of discontinuities of F. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and F is an increasing function with $F(d) = F(d^+)$, there exists a $q_d \in \mathbb{Q}$ such that $F(d^-) < q_d < F(d)$ for all $d \in D$. Furthermore, for any $x, y \in D$ with $x < y$, we have that $F(x^-) < F(x) \leq F(y^-) < F(y)$. Therefore, $q_x \neq q_y$. It follows that the function $f : D \to \mathbb{Q}$ where $f(d) = q_d, \ \forall d \in D$ is injective which implies $|D| \leq |\mathbb{Q}|$. Since $\mathbb{Q}$ is countable, D is at most countable.

Durrett ed. 4, Problem 1.2.5

Suppose X has continuous density f, $P(\alpha \leq X \leq \beta) = 1$ and g is a function that is strictly increasing and differentiable on (α, β). Then $g(X)$ has density $f(g^{-1}(y))/g'(g^{-1}(y))$ for $y \in (g(\alpha), g(\beta))$.

Solution:
By the definition of a distribution function and the fact that g is strictly increasing function, we can write

$$P(g(X) \leq z) = P(X \leq g^{-1}(z)) = F(g^{-1}(z)) = \int_{\alpha}^{g^{-1}(z)} f(x) dx. \tag{7}$$

Set $\xi = g(x) \implies dx = d\xi / g'(g^{-1}(\xi))$ and observe that

$$\int_{\alpha}^{g^{-1}(z)} f(x) dx = \int_{g(\alpha)}^{z} \frac{f(g^{-1}(\xi))}{g'(g^{-1}(\xi))} d\xi \tag{8}$$

which yields the desired result.