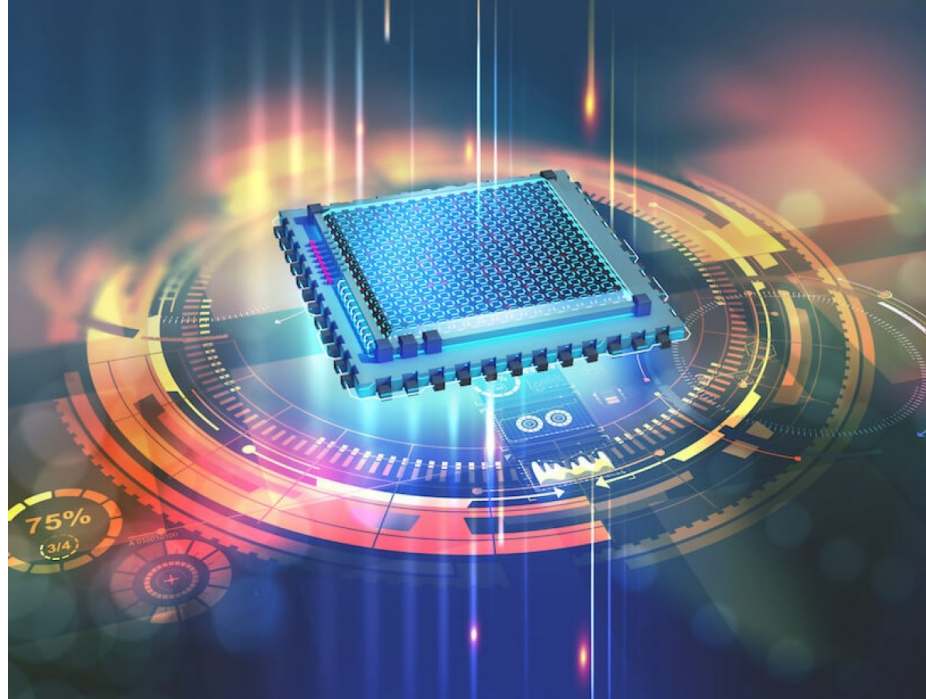


# Introduction to Quantum Computing



# Last class

- Two bit gates
- Quantum bits (qubits)
- A qubit is a system that obeys Axioms
  - State is a complex unit vector
  - Evolution is multiplication by unitary
  - Measurement is probabilistic, “collapses” state
  - Tensor product for combining systems

# Today

- We will talk more about gates and measurements on multiple qubit systems.
- We'll look at more examples of one and two qubit gates and what they can do. This will be useful for studying computation.

# Axioms of Quantum Theory

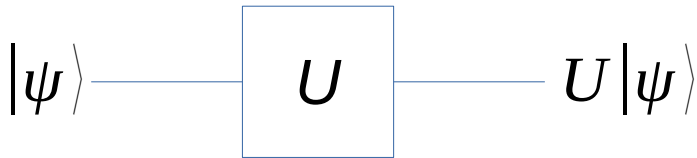
## Axiom 1 (states)

- The state of a quantum system is a complex unit vector:  
 $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ , where  $|\alpha|^2 + |\beta|^2 = 1$
- 2-dimensional vector: qubit  
d-dimensional vector: qudit

# Axioms of Quantum Theory

Axiom 2 (dynamics)

- The evolution of a closed system is described by a unitary matrix  $U$



# Axioms of Quantum Theory

## Axiom 3 (measurements)

- We can measure a system in any basis for its state space. If you measure

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$$

in the basis  $\{|0\rangle, |1\rangle\}$  the outcome is  $x$  with probability  $|\alpha_x|^2$

- Furthermore, the state of the system collapses to  $|x\rangle$
- This is called the Born rule

# Axioms of Quantum Theory

## Axiom 4 (composite systems)

If

- A has a state in  $\text{span}(V)$  for some set of vectors  $V$
  - B has a state in  $\text{span}(W)$  for some set of vectors  $W$
  - AB has a state in  $\text{span}(\{v \otimes w \mid v \text{ in } V, w \text{ in } W\})$
- 
- $\text{span}(V)$  denotes the set of all finite linear combinations of the elements of  $V$ ,  $\otimes$  is the tensor product

# Tensor products

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

$$|\phi\rangle = \beta_0|0\rangle + \beta_1|1\rangle = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

$$|\psi\rangle \otimes |\phi\rangle =$$

$$= \alpha_0\beta_0|0\rangle \otimes |0\rangle + \alpha_0\beta_1|0\rangle \otimes |1\rangle + \alpha_1\beta_0|1\rangle \otimes |0\rangle + \alpha_1\beta_1|1\rangle \otimes |1\rangle =$$

$$= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle$$

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \otimes \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix}$$



# Tensor products

$$|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

$$|\phi\rangle = \beta_0|0\rangle + \beta_1|1\rangle = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

First entry of  
first vector

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \otimes \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix}$$

Second entry of  
first vector

Two copies of the 2nd vector

# Exercise on tensor products

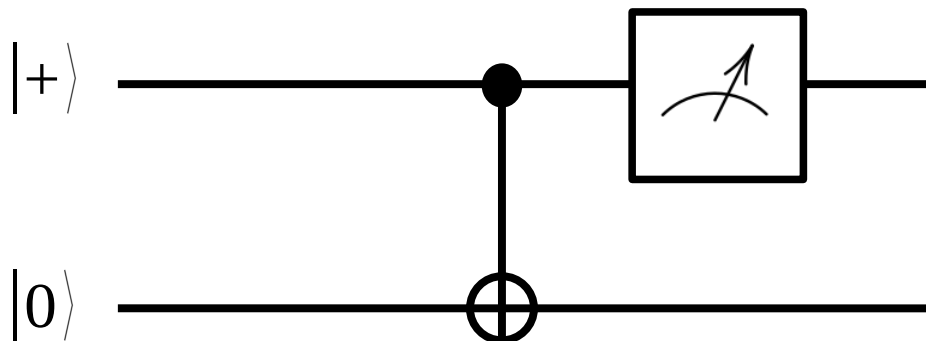
- Compute  $|\psi\rangle = |+\rangle \otimes |0\rangle$
- Remember that  $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$
- This is the state of a system made of two qubits, one in state  $|+\rangle$  and one in state  $|0\rangle$

# Solution of the exercise

$$|\psi\rangle = |+\rangle \otimes |0\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} 1 \\ \frac{1}{\sqrt{2}} 0 \\ \frac{1}{\sqrt{2}} 1 \\ \frac{1}{\sqrt{2}} 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

# More on measurement

- What if I have multiple qubits, and I measure only one of them?



- Let's refine axiom 3 to cover this case as well

# State of n qubit system

- Any state of n qubits can be written as

$$|\psi\rangle = \alpha_0 |0\rangle \otimes |\psi_0\rangle + \alpha_1 |1\rangle \otimes |\psi_1\rangle$$

- Where  $|\psi_i\rangle$  are n-1 qubit states and

$$|\alpha_0|^2 + |\alpha_1|^2 = 1$$

Phi è uno stack verticale di phi zero e phi uno, pesati ciascuno per il proprio scalare

Definizione ricorsiva che ad ogni layer aggiunge tre DOF scalari

# Axioms of Quantum Theory

Axiom 3' (partial measurement)

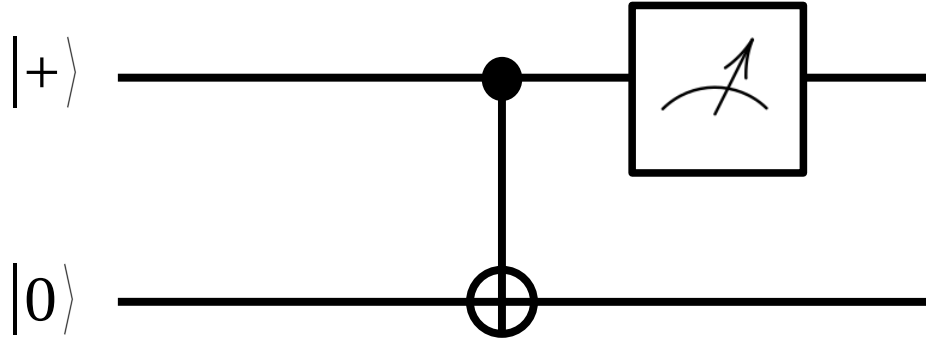
- We can measure the first qubit of the  $n$  qubit state

$$|\psi\rangle = \alpha_0|0\rangle \otimes |\psi_0\rangle + \alpha_1|1\rangle \otimes |\psi_1\rangle$$

in the basis  $\{|0\rangle, |1\rangle\}$ .

- The outcome is  $x$  with probability  $|\alpha_x|^2$
- Furthermore, the state of the system collapses to  $|x\rangle \otimes |\psi_x\rangle$
- (Generalized) Born rule

# So what happens here?



- We can compute the initial composite state
- We can compute the result after the CNOT
- We can compute the effect of the partial measurement (difficult using vectors, much better with kets)

# Arbitrary one qubit states

- We want to prepare an arbitrary qubit state  $|\psi\rangle$  starting from  $|0\rangle$  (or  $|1\rangle$ )
- That is, we want a unitary  $U$  such that  $|\psi\rangle = U|0\rangle$
- We can always find such a unitary



# Finding U

- Take  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
- From  $|\psi\rangle = U|0\rangle$  we get

$$U = \begin{bmatrix} \alpha & x \\ \beta & y \end{bmatrix}$$

for some x,y

- U needs to be unitary

# Finding U

- We show that  $U = \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix}$  is unitary hence it is a solution

$$U U^\dagger = \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} \begin{bmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} \alpha \alpha^* + \beta^* \beta & \alpha \beta^* - \beta^* \alpha \\ \beta \alpha^* - \alpha^* \beta & \beta^* \beta + \alpha^* \alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

# How to get the $|0\rangle$

- We have learned how to get any 1 qubit state from  $|0\rangle$
- We will show how to get also two qubits states
- But how to get the  $|0\rangle$  to start with?
- In some cases, it can be obtained using physical processes
  - E.g., minimal energy state, by cooling down
- Otherwise, take an unknown state, measure it so to get either  $|0\rangle$  or  $|1\rangle$ , and negate it if it is  $|1\rangle$

# Arbitrary two qubit states

- Now we want to prepare an arbitrary two qubit state  $|\psi\rangle$  starting from  $|0\rangle$ s

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

- One qubit gates are not enough

$$U|0\rangle \otimes V|0\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle)$$

- To have  $U|0\rangle \otimes V|0\rangle = |\psi\rangle$  we need

$$\frac{\alpha_{00}}{\alpha_{01}} = \frac{\alpha_{10}}{\alpha_{11}} = \frac{\beta_0}{\beta_1} \quad \text{what is not true in general}$$

# Adding a single CNOT is enough

- We want to prepare:

$$|\sigma\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

- First, note that

$$|\sigma\rangle = |0\rangle \otimes |\psi\rangle + |1\rangle \otimes |\phi\rangle$$

with  $|\psi\rangle = \alpha_{00}|0\rangle + \alpha_{01}|1\rangle$  and  $|\phi\rangle = \alpha_{10}|0\rangle + \alpha_{11}|1\rangle$

- We want to find  $U$  such that

$$U \otimes I |\sigma\rangle = |0\rangle \otimes |\psi'\rangle + |1\rangle \otimes |\phi'\rangle$$

with  $|\psi'\rangle$  and  $|\phi'\rangle$  orthogonal  
(if  $|\psi\rangle$  and  $|\phi\rangle$  are orthogonal we can choose  $U = I$ )

# Finding U

- We want to find U such that

$$U \otimes I |\sigma\rangle = |0\rangle \otimes |\psi'\rangle + |1\rangle \otimes |\phi'\rangle$$

- Let us try with

$$U = \begin{bmatrix} a & -b^* \\ b & a^* \end{bmatrix} \text{ which we know is unitary}$$

$$\begin{aligned} U \otimes I |\sigma\rangle &= (a|0\rangle + b|1\rangle) \otimes |\psi\rangle + (-b^*|0\rangle + a^*|1\rangle) \otimes |\phi\rangle = \\ &= |0\rangle \otimes |\psi'\rangle + |1\rangle \otimes |\phi'\rangle \end{aligned}$$

$$\text{where } |\psi'\rangle = a|\psi\rangle - b^*|\phi\rangle \text{ and } |\phi'\rangle = b|\psi\rangle + a^*|\phi\rangle$$

# Orthogonal vectors

- Two vectors  $|\psi\rangle, |\phi\rangle$  are orthogonal iff  $\langle\psi|\phi\rangle=0$
- This is the inner product
- Namely  $\sum_{i=1}^n \psi_i^* \phi_i$
- Note that  $\langle\psi|\phi\rangle=\langle\phi|\psi\rangle^*$
- It is linear in the right argument, antilinear in the left one
- Inner product of a unitary vector for itself is 1

# Computing the coefficients

- We want

$$|\phi'\rangle = b|\psi\rangle + a^*|\phi\rangle \text{ and } |\psi'\rangle = a|\psi\rangle - b^*|\phi\rangle$$

orthogonal

$$\begin{aligned} 0 = \langle \phi' | \psi' \rangle &= b^* a \langle \psi | \psi \rangle + a a \langle \phi | \psi \rangle - b^* b^* \langle \psi | \phi \rangle - a b^* \langle \phi | \phi \rangle = \\ &= a^2 \langle \phi | \psi \rangle - b^{*2} \langle \psi | \phi \rangle + a b^* (\langle \psi | \psi \rangle - \langle \phi | \phi \rangle) = \\ &= a^2 \langle \phi | \psi \rangle - b^{*2} \langle \psi | \phi \rangle \end{aligned}$$

- We can solve the equation to derive values for a and b



# Normalization

- We have

$$U \otimes I |\sigma\rangle = |0\rangle \otimes |\psi'\rangle + |1\rangle \otimes |\phi'\rangle$$

with  $|\psi'\rangle$  and  $|\phi'\rangle$  orthogonal

- They may not be normalized yet, we need to divide them for their module, hence we define

$$|\psi''\rangle = \frac{|\psi'\rangle}{\lambda}, |\phi''\rangle = \frac{|\phi'\rangle}{\mu}$$

- They are still orthogonal and now unitary

# Tensor product of matrices

- We want

$$(U \otimes V)(|\phi\rangle \otimes |\psi\rangle) = U|\phi\rangle \otimes V|\psi\rangle$$

- We can actually define it as follows:

$$U = \begin{bmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{bmatrix} \quad V = \begin{bmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{bmatrix}$$
$$U \otimes V = \begin{bmatrix} u_{1,1} \begin{bmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{bmatrix} & u_{1,2} \begin{bmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{bmatrix} \\ u_{2,1} \begin{bmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{bmatrix} & u_{2,2} \begin{bmatrix} v_{1,1} & v_{1,2} \\ v_{2,1} & v_{2,2} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} u_{1,1}v_{1,1} & u_{1,1}v_{1,2} & u_{1,2}v_{1,1} & u_{1,2}v_{1,2} \\ u_{1,1}v_{2,1} & u_{1,1}v_{2,2} & u_{1,2}v_{2,1} & u_{1,2}v_{2,2} \\ u_{2,1}v_{1,1} & u_{2,1}v_{1,2} & u_{2,2}v_{1,1} & u_{2,2}v_{1,2} \\ u_{2,1}v_{2,1} & u_{2,1}v_{2,2} & u_{2,2}v_{2,1} & u_{2,2}v_{2,2} \end{bmatrix}$$

# Building the circuit (1)

- Take  $V$  (unitary) such that

$$|\psi''\rangle = V|0\rangle, |\phi''\rangle = V|1\rangle$$

- We have

$$U \otimes I |\sigma\rangle = |0\rangle \otimes |\psi'\rangle + |1\rangle \otimes |\phi'\rangle$$

$$U \otimes I |\sigma\rangle = |0\rangle \otimes \lambda |\psi''\rangle + |1\rangle \otimes \mu |\phi''\rangle$$

$$U \otimes I |\sigma\rangle = |0\rangle \otimes \lambda V|0\rangle + |1\rangle \otimes \mu V|1\rangle$$

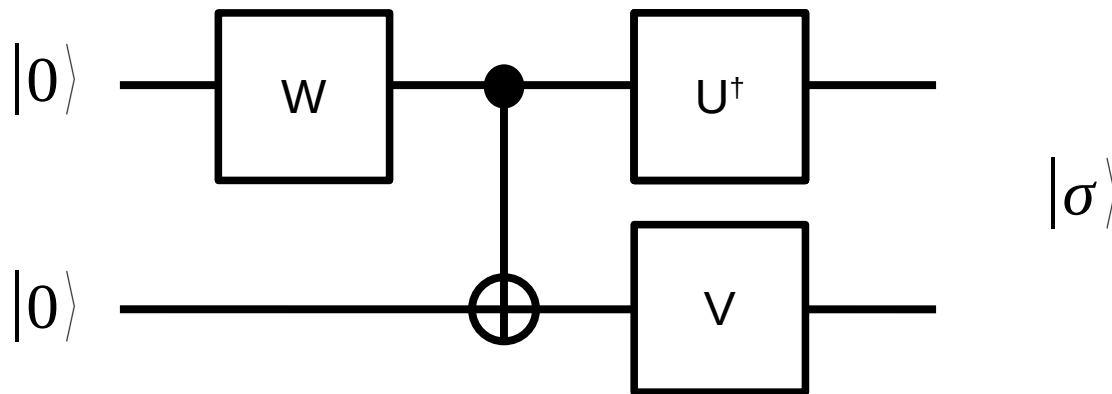
$$U \otimes I |\sigma\rangle = (I \otimes V)(\lambda |0\rangle \otimes |0\rangle + \mu |1\rangle \otimes |1\rangle)$$

$$|\sigma\rangle = (U^\dagger \otimes V)(\lambda |0\rangle \otimes |0\rangle + \mu |1\rangle \otimes |1\rangle)$$

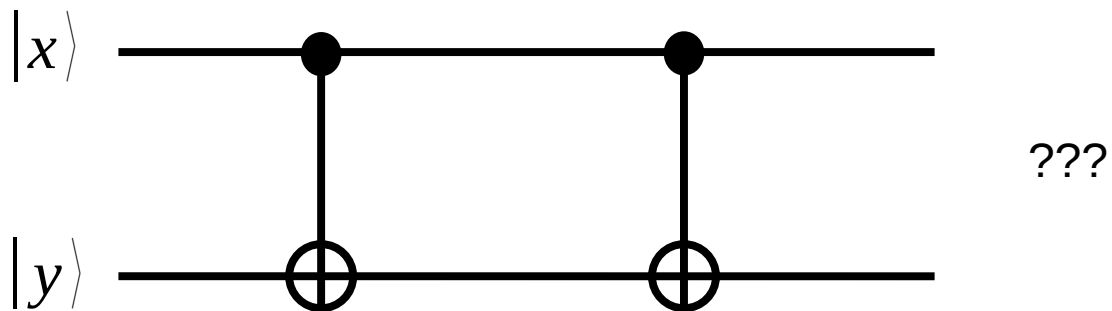
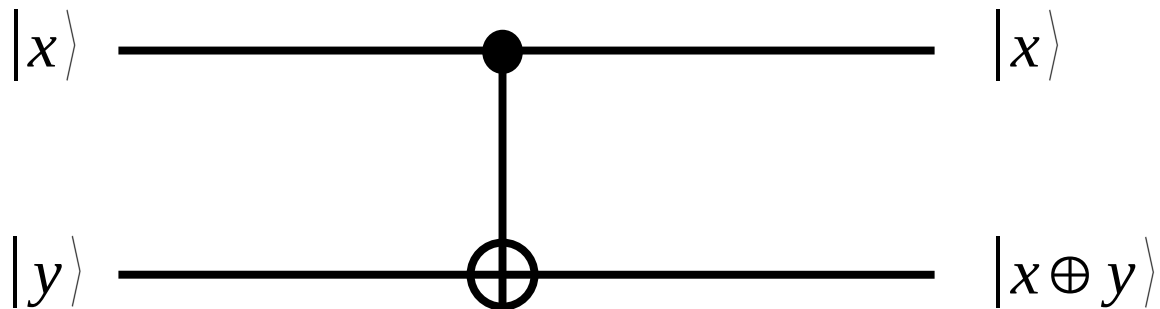
# Building the circuit (2)

- $$\begin{aligned}
 |\sigma\rangle &= (U^\dagger \otimes V)(\lambda|0\rangle \otimes |0\rangle + \mu|1\rangle \otimes |1\rangle) = \\
 &= (U^\dagger \otimes V)CNOT((\lambda|0\rangle + \mu|1\rangle) \otimes |0\rangle) = \\
 &= (U^\dagger \otimes V)CNOT((W|0\rangle) \otimes |0\rangle)
 \end{aligned}$$

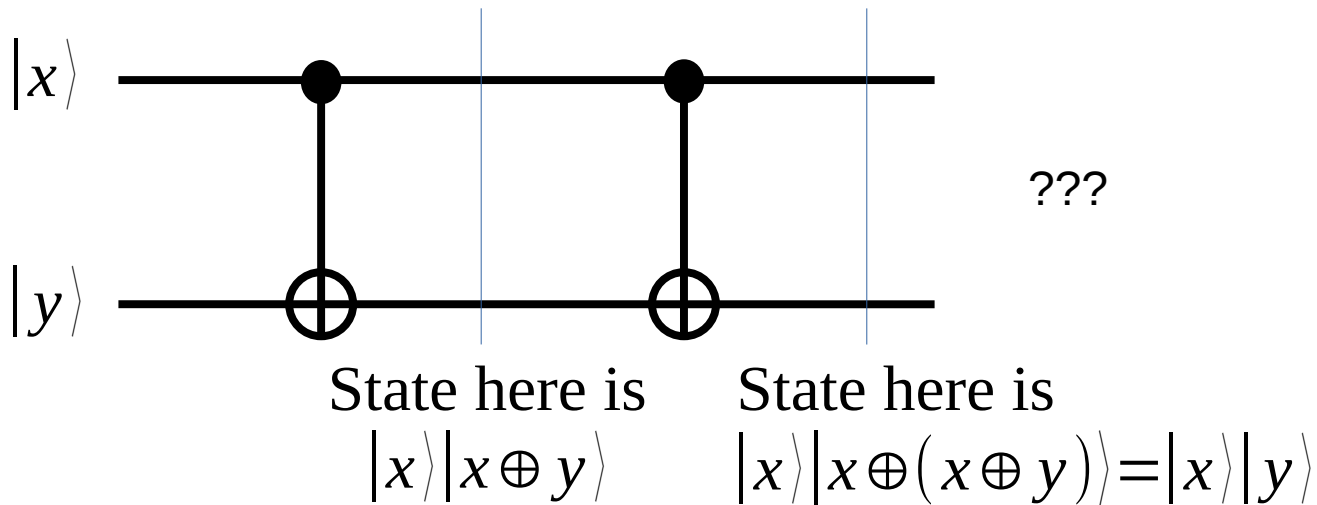
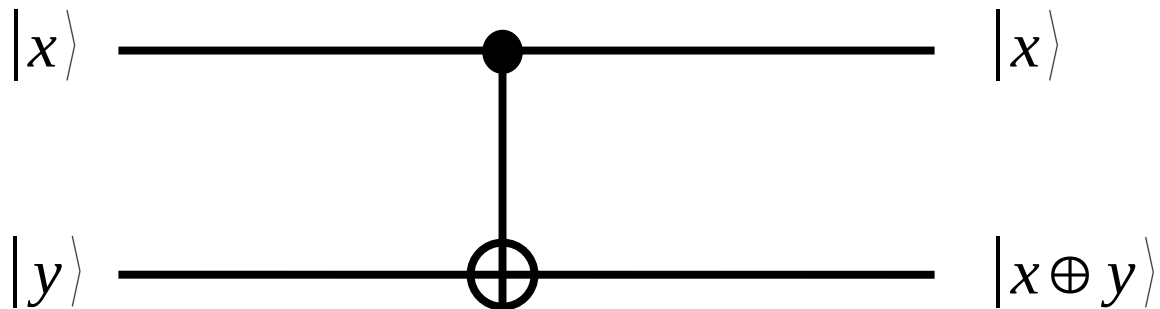
for some  $W$  unitary



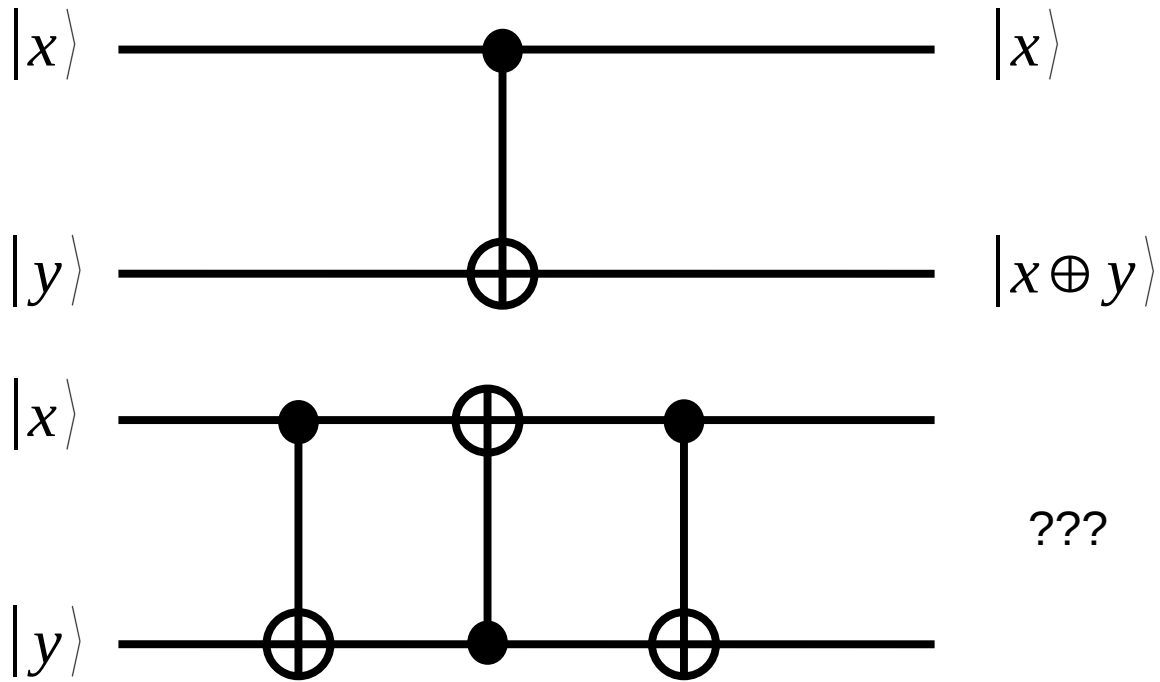
# More on CNOT



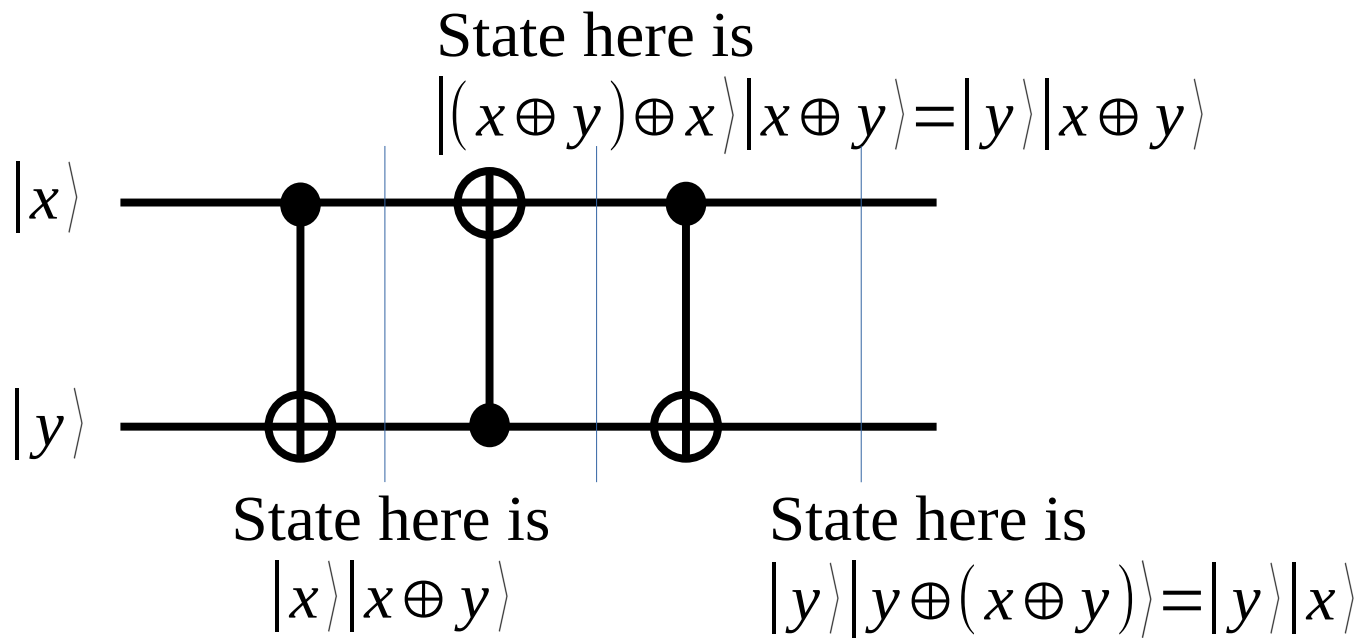
# More on CNOT



# CNOT exercise

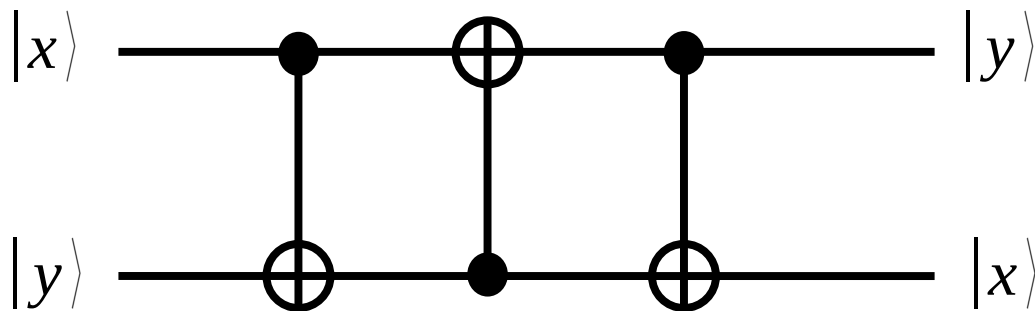


# CNOT exercise





# CNOT exercise



- This is actually a swap

# Summary

- Tensor products of vectors (to compose states) and of matrices (to compose circuits)
- I can prepare an arbitrary 1 qubit state with a 1 qubit unitary
- For 2 qubits states I need CNOT (and 3 unitaries)