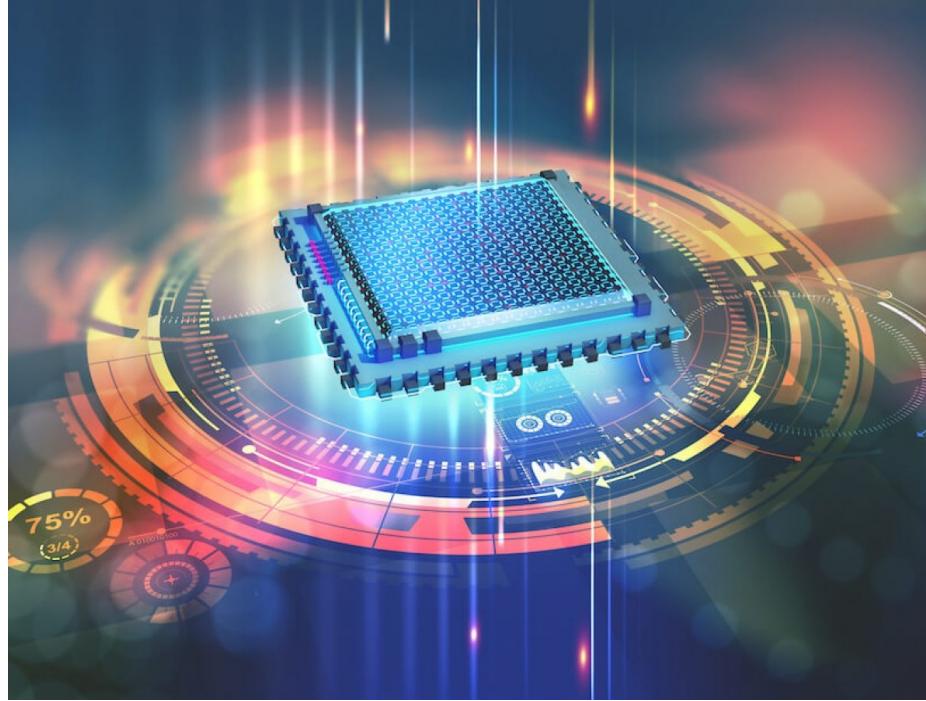


Introduction to Quantum Computing



We start from bits

- (but things will get more complex quickly)
- A bit is anything which can be in one of two states
- We denote these two states as 0 and 1
- How many possibilities if we have 3 bits? And if we have n bits?

Representing bit-strings as vectors

- One bit: 0 or 1
- We will represent them as vectors since this easily generalizes to qubits
- $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- With 2 bits we have 4 possibilities:

$$|00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$|10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$|11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Tensor products

- All the possible combinations, with the product of values

$$\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} \otimes \begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_0 y_0 \\ x_0 y_1 \\ x_1 y_0 \\ x_1 y_1 \end{bmatrix} \qquad |00\rangle = |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- For now, values can be just 0 and 1
- Observation: exactly one 1 in each vector on the LHS implies exactly one 1 in the result
- The operator can be generalized to n vectors
- Tensor product corresponds to compose states

Manipulating bits

- One bit operation: NOT
- $\text{NOT}(x) = 1$ if $x=0$
0 if $x=1$
- NOT just flips the bit
- Notation for NOT(x): \bar{x}

Manipulating bits as vectors

- x can be either $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- More in general $x = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$
- We use matrices to represent operators, and matrix multiplication to apply them
- NOT is denoted as X , and represented by the matrix

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Matrix multiplication

- We can multiply a matrix $n \times m$ and a matrix $m \times p$ and obtain a matrix $n \times p$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,p} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m,1} & b_{m,2} & \cdots & b_{m,p} \end{bmatrix}$$

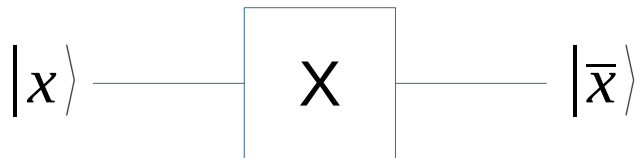
$$C = \begin{bmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,p} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{n,1} & c_{n,2} & \cdots & c_{n,p} \end{bmatrix}$$

$$\begin{aligned} c_{i,j} &= a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,m}b_{m,j} = \\ &= \sum_{k=1}^m a_{i,k}b_{k,j} \end{aligned}$$

Element obtained by multiplying
row by column

Flipping a bit

- Circuit diagram:



- Ket notation:

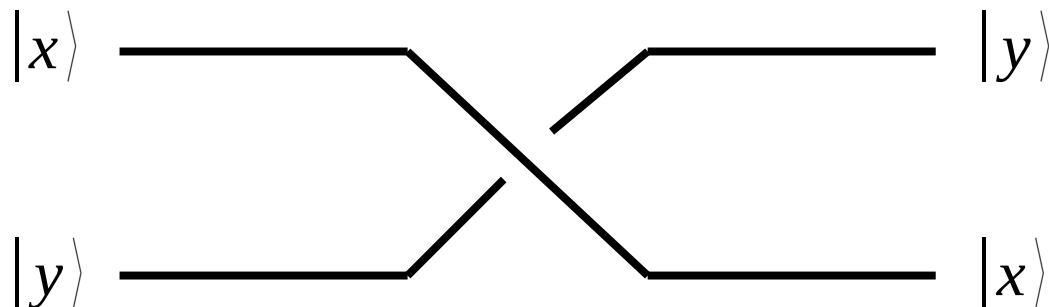
$$|\bar{x}\rangle = X|x\rangle$$

Manipulating one bit

- If someone hands you a bit, there are two ways you can process it to obtain another bit:
 - 1) Keep it as it is (identity)
 - 2) Flip it (NOT)
- That's it. If we want more interesting computations, we'd better have more bits
- We could also turn it into constant 0 or 1, but these are not *reversible*

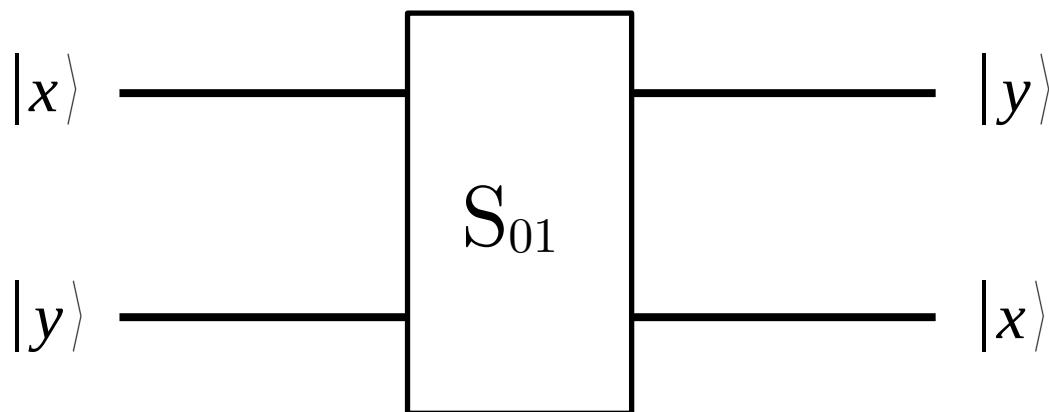
Swapping two bits

- Given two bits, what can we do?
For instance, swap them!



Swapping two bits

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Swapping two bits

- Given two bits, what can we do?
For instance, swap them!

$$S_{01}|x\rangle|y\rangle = |y\rangle|x\rangle$$

We assume tensor product
between vectors, and matrix
multiplication between matrices
and vectors

- As a matrix:

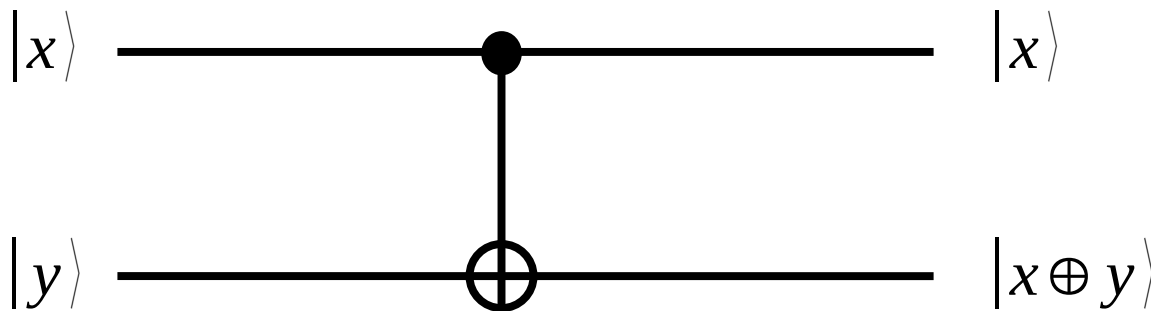
$$S_{01} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Controlled NOT

- NOT on the second bit, if the first is 1, identity otherwise

$$CNOT|x\rangle|y\rangle=|x\rangle|x\oplus y\rangle$$

where $x\oplus y=(x+y)\bmod 2$



Controlled NOT: test

- What matrix represents CNOT?

$$CNOT|x\rangle|y\rangle = |x\rangle|x \oplus y\rangle$$

$$(A) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(C) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$(B) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(D) \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Controlled NOT: test solution

- What matrix represents CNOT?

$$CNOT|x\rangle|y\rangle = |x\rangle|x \oplus y\rangle$$

$$(A) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(B) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(C) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$(D) \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Controlled NOT: test explanation

$$CNOT |x\rangle|y\rangle = |x\rangle|x \oplus y\rangle$$

$$\begin{bmatrix} 00 \\ 01 \\ 10 \\ 11 \end{bmatrix}$$

CNOT swaps 10 and 11

$$00 \rightarrow 00, 01 \rightarrow 01, 10 \rightarrow 11, 11 \rightarrow 10$$

- The matrix doing this is
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$