CS201B: Endsem Examination

November 27-December 2, 2021

Submission Deadline: 72 hours **Maximum Marks**: 60

Question 1. (10 marks) Consider the following argument:

> Let U be the set of all sets. Define a partial ordering on U by inclusion: $A \leq B$ iff $A \subseteq B$ for $A, B \in U$. Consider a chain C of U under this partial ordering: $C : A_1 \leq A_2 \leq A_3 \leq \cdots$. Define $B = \cup_{i \geq 1} A_i$. Clearly, $B \in U$ and it is an upper bound of the chain C. Hence, Zorn's Lemma implies that U has a maximal element, say M.

The argument is clearly wrong since M is not a maximal element: $M \subset \{M, \{M\}\} \in U$. Identify which step in the argument is wrong and why.

Question 2. (20 marks) Let $(G, \cdot)$ be a finite group with the property that there exists only one element $a_2 \in G$ such that $a_2 \neq e$ and $a_2^2 = a_2 \cdot a_2 = e$. Define a bipartite graph $H = (G, G, E)$ as follows.

> Edge $(a, b) \in E$ if $a \neq e$ and $b = a^k$ for $k > 1$, or $a = e$ and $b = a_2$.

Prove that the graph H has a perfect matching.

Question 3. (5+5+5+10+5 marks) Let R be a ring and $a \in R$. Define $(a) = \{b \cdot a \mid b \in R\}$.

- Prove that (a) is an ideal of R.

Let polynomial $C(x, y) = (x^2 + y^2 - 1) \cdot x$. The curve $C(x, y) = 0$ is unit circle plus y-axis on the plane. $(C) = \{Q(x, y) \cdot C(x, y) \mid Q(x, y) \in \mathbb{R}[x, y]\}$ is an ideal of the ring $\mathbb{R}[x, y]$, the ring of polynomials in two variables with coefficients in $\mathbb{R}$.

Define $R = \mathbb{R}[x, y]/(C)$. For any point $P \in \mathbb{R} \times \mathbb{R}$ on the plane, define $R_P = \{\frac{f}{g} \mid f, g \in R \text{ and } g(P) \neq 0\}$ and $I_P = \{\frac{f}{g} \mid f, g \in R \text{ and } g(P) \neq 0 \text{ and } f(P) = 0\}$. Prove that

- R_P is a ring.
- I_P is a maximal ideal of R_P.
- For point $P = (1, 0)$, $I_P = (y)$.
- For point $P = (0, 1)$, $(x) \subseteq I_P$.

It can be shown that $I_P \neq (x)$. Therefore, ring R_P contains information about whether curve C is *degenerate* at point P.