# CS203 - Mathematics For Computer Science -III

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**Problem 1** A company asks you to set up a mechanism to send single bits (0 or 1) between two points A and B. The company has n identical communication channels available. However, these channels are not perfect. Each of these channels flips the bits send through it (0 becomes 1 or 1 becomes 0) with probability p. The company gives you two options for using these channels to connect A and B: either the channels can be connected in series between A and B or they can be connected in parallel between A and B. In the series mode, the output of the channel at the receiving end is taken as the received bit, while in the parallel mode, the bit which appears in the majority of the channel outputs is taken as the received bit. Answer the following questions:

(a) If $p = \frac{1}{3}$ and $n = 5$, which mode of connecting the channels maximizes the probability that the correct bit is received at the receiving end?

**Series:** To get the correct output at the receiving end, the number of flips should be even(0 or 2 or 4).

$$P(\text{flips} = 0) = (\tfrac{2}{3})^5 = \tfrac{32}{243}$$

$$P(\text{flips} = 2) = \tbinom{5}{2} * (\tfrac{1}{3})^2 * (\tfrac{2}{3})^3 = \tfrac{80}{243}$$

$$P(\text{flips} = 4) = \tbinom{5}{4} * (\tfrac{1}{3})^4 * (\tfrac{2}{3}) = \tfrac{10}{243}$$

$$P(\text{series}) = P(\text{flips} = 0) + P(\text{flips} = 2) + P(\text{flips} = 4)$$

$$P(\text{series}) = \tfrac{122}{243}$$

**Parallel:** To get the correct output at the receiving end, the maximum bit should be the correct bit .. which means that the flips should take either at 0 , 1 or 2 channels. Flips more than this will change the value of output.

$$P(\text{flips} = 0) = (\tfrac{2}{3})^5 = \tfrac{32}{243}$$

$$P(\text{flips} = 1) = \tbinom{5}{1} * (\tfrac{1}{3}) * (\tfrac{2}{3})^4 = \tfrac{80}{243}$$

$$P(\text{flips} = 2) = \tbinom{5}{2} * (\tfrac{1}{3})^2 * (\tfrac{2}{3})^3 = \tfrac{80}{243}$$

$$P(\text{series}) = P(\text{flips} = 0) + P(\text{flips} = 2) + P(\text{flips} = 4)$$

$$P(\text{series}) = \tfrac{192}{243}$$

**So Parallel mode of connecting maximizes the probability of receiving correct bit.**

(b) Assume that $p = \frac{1}{3}$, $n = 3$ and the parallel mode is used for connecting A and B. A chooses the bit 0 with probability $\frac{2}{3}$ and the bit 1 with probability $\frac{1}{3}$ and sends this random bit to B. The majority of the bits received at B turns out to be equal to 1. Given this fact, what is the probability that the original bit sent by A was also equal to 1?

**Original Byte = OB**
**Received Byte = RB**
We need to find $P(OB = 1/RB = 1)$

**Using Bayes' theorem**

$$P(OB = 1/RB = 1) = \frac{P(RB=1/OB=1)*P(OB=1)}{P(RB=1)} \text{ —(1)}$$

**Calculating** $P(RB = 1)$
$P(RB = 1) = P(OB = 1)*P(RB = 1/OB = 1)+P(OB = 0)*P(RB = 1/OB = 0)$

$P(OB = 1) = \frac{1}{3}$

$P(RB = 1/OB = 1) = P(\text{flip}= 0) + P(\text{flip} = 1)$
$P(\text{flip}= 0) = (\frac{2}{3})^3 = \frac{8}{27}$
$P(\text{flip}= 1) = \binom{3}{1} * (\frac{2}{3})^2 * (\frac{1}{3}) = \frac{12}{27}$
$P(RB = 1/OB = 1) = \frac{20}{27}$

$P(OB = 0) = \frac{2}{3}$

$P(RB = 1/OB = 0) = P(\text{flip}= 2) + P(\text{flip} = 3)$
$P(\text{flip}= 2) = \binom{3}{2} * (\frac{1}{3})^2 * (\frac{2}{3}) = \frac{6}{27}$
$P(\text{flip}= 3) = \binom{3}{3} * (\frac{1}{3})^3 = \frac{1}{27}$
$P(RB = 1/OB = 0) = \frac{7}{27}$

Substituting the values in formula gives

$$\mathbf{P(RB = 1)} = \frac{1}{3} * \frac{20}{27} + \frac{2}{3} * \frac{7}{27}$$

$$\mathbf{P(RB = 1)} = \frac{34}{51}$$

Substituting all the values in equation 1 :

$$P(OB = 1/RB = 1) = \frac{\frac{20}{27}*\frac{1}{3}}{\frac{34}{51}} \; P(OB = 1/RB = 1) = \frac{20}{34}$$

**Problem 2** Suppose $U$ is a continuous random variable with the probability density function ($c \epsilon R$)

$$g(u) = \begin{cases} c - |u| & if, |u| \leq \frac{1}{2} \\ 0 & otherwise \end{cases}$$

(a) Find the constant $c$.

We know that the area under the probability distribution curve is equal to 1.

$$\int_{\frac{-1}{2}}^{\frac{1}{2}} (c - |u|) \, du = 1$$

$$|cu - \frac{u^2}{2}|_{\frac{-1}{2}}^{\frac{1}{2}} = 1$$

$$c - \tfrac{1}{4} = 1 => c = \tfrac{5}{4}$$

(b) The *cumulative distribution function* of a random variable $X$ is the function given by $F_X(x) = P(X \leq x)$ for every $x \epsilon R$. Find the cumulative distribution function $F_U$ of U.

**Case 1 : When** $u \leq \frac{-1}{2}$

$$F_U(u) = 0$$

**Case 2: When** $\frac{-1}{2} < u \leq 0$

$$F_U(u) = \int_{\frac{-1}{2}}^{u} (\frac{5}{4} + u) \, du$$

$$F_U(u) = \frac{5}{4} * (u + \frac{1}{2}) + (\frac{u^2}{2} - \frac{1}{8})$$

$$F_U(u) = \frac{u^2}{2} + \frac{5u}{4} + \frac{1}{2}$$

**Case 3: When** $0 < u \leq \frac{1}{2}$

$$F_U(u) = \int_{-\infty}^{0} g(u)\,du + \int_{0}^{u} (\frac{5}{4} - u)\,du$$

$$\int_{-\infty}^{0} g(u)\,du = F_U(0) = |\frac{u^2}{2} + \frac{5u}{4} + \frac{1}{2}|_{x=0} = \frac{1}{2}$$

$$F_U(u) = \frac{1}{2} + \int_{0}^{u} (\frac{5}{4} - u)\,du$$

$$F_U(u) = \frac{1}{2} + \frac{5u}{4} - \frac{u^2}{2}$$

**Case 4: When** $x > \frac{1}{2}$

$$F_U(u) = F_U(\frac{1}{2}) + \int_{\frac{1}{2}}^{\infty} 0\,du$$

$$F_U(u) = 1$$

$$F_U(u) = \begin{cases} 0 & if, u \leq \frac{-1}{2} \\ \frac{u^2}{2} + \frac{5u}{2} + \frac{1}{2} & \frac{-1}{2} < u \leq 0 \\ \frac{-u^2}{2} + \frac{5u}{2} + \frac{1}{2} & 0 < u \leq \frac{1}{2} \\ 1 & u > \frac{1}{2} \end{cases}$$

(c) Evaluate the conditional probability $P_r(\frac{1}{8} < U < \frac{2}{5} | \frac{1}{10} < U < \frac{1}{5})$.

We can write the above statement as :

$$P_r(\frac{5}{40} < U < \frac{16}{40} | \frac{4}{40} < U < \frac{8}{40}).$$

By Conditional Probability :

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

In the above equation

$$P(A \cap B) = P(\tfrac{5}{40} < U < \tfrac{8}{40})$$

$$P_r(\tfrac{5}{40} < U < \tfrac{16}{40} | \tfrac{4}{40} < U < \tfrac{8}{40}) = \frac{P(\frac{5}{40}<U<\frac{8}{40})}{P(\frac{4}{40}<U<\frac{8}{40})}$$

$$\frac{P(\frac{5}{40}<U<\frac{8}{40})}{P(\frac{4}{40}<U<\frac{8}{40})} = \frac{F_U(\frac{8}{40})-F_U(\frac{5}{40})}{F_U(\frac{8}{40})-F_U(\frac{4}{40})}$$

Substituting the values in the above derived formula of $F_U(u)$, we get,

$$P_r(\tfrac{5}{40} < U < \tfrac{16}{40} | \tfrac{4}{40} < U < \tfrac{8}{40}) = \tfrac{261}{352}$$

**Problem 3** Alice has an unbiased 5-sided die and 5 different coins with her. The probabilities of obtaining a head on tosses of these coins are $\frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}$ and $\frac{5}{6}$ respectively. She likes to observe patterns in subsequent tosses of these coins. Alice performs the following experiment. She rolls the 5-sided die and if the $i^{th}$ side turns up, she chooses the $i^{th}$ coin and starts tossing this coin repeatedly.

(a) What is the expected number of tosses required for obtaining 6 consecutive heads given that the side 1 turned up during the roll of the die?

Let $e$ be the expected number of tosses to get 6 consecutive heads.

Start tossing.All possible sequences of coin tosses can be split into 7 disjoint cases.

- If we get a tail immediately (probability $\frac{5}{6}$) then the expected number is $e + 1$.
- If we get a head then a tail (probability $\frac{1}{6} * \frac{5}{6}$), then the expected number is $e + 2$.
- If we get two heads and then a tail (probability $(\frac{1}{6})^2 * \frac{5}{6}$),then the expected number of tosses will be $e + 3$.
- If we get three heads and then a tail (probability $(\frac{1}{6})^3 * \frac{5}{6}$),then the expected number of tosses will be $e + 4$.
- If we get four heads and then a tail (probability $(\frac{1}{6})^4 * \frac{5}{6}$),then the expected number of tosses will be $e + 5$.
- If we get five heads and then a tail (probability $(\frac{1}{6})^5 * \frac{5}{6}$),then the expected number of tosses will be $e + 6$.
- If we get six heads (probability $(\frac{1}{6})^6$,then the expected number of tosses will be 6.

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So from Partition Formula:

$$E[X] = \sum_{i=1}^{n} P(B_i) * E[X/B_i]$$

$$E[tosses] = 6p^6 + \sum_{i=1}^{6}(e+i)*(p)^{i-1}*(1-p)$$

$$E[tosses] = 6p^6+(1-p)*((e+1)+p(e+2)+p^2(e+3)+p^3(e+4)+p^4(e+5)+p^5(e+6))$$

$$E[tosses] = e(1-p)(1+p+p^2+p^3+p^4+p^5)+(1-p)(1+2p+3p^2+4p^3+5p^4+6p^5)+6p^6$$

$$E[tosses] = e(1-p^6) + \frac{1-p^6}{1-p}$$

Solving this linear equation, we get

$$E[tosses] = \frac{1-p^6}{(1-p)p^6}$$

Here p = $\frac{1}{6}$

$$E[tosses] = \frac{5}{6}*(e+1)+\frac{5}{6^2}*(e+2)+\frac{5}{6^3}(e+3)+\frac{5}{6^4}*(e+4)+\frac{5}{6^5}*(e+5)+\frac{5}{6^6}(e+6)+\frac{1}{6^6}*6$$

On solving , we get

$$E[tosses] = 55986$$

(b) What is the expected number of tosses required for obtaining 6 consecutive heads while performing this random experiment?

**Answer:**In this case the space an be divided into cases according to the roll of the dice. If $E_I$ represents the expectation value when the dice rolls I, then

$$E[tosses] = \sum_{i=1}^{5}\frac{1}{5}E_I$$

On calculating $E_I$ from the above calculated equation,we get

$$E[tosses] = \frac{55986+1092+126+\frac{1995}{64}+\frac{186186}{15625}}{5} = 11449.417$$

(c) What is the probability that in the first n tosses, she obtains n consecutive heads?

P(n consecutive heads if $i^{th}$ side of die shows up) $= (\frac{i}{6})^n$

P(n consecutive heads) $= \sum_{i=1}^{5} P(dieface = i) * P(nconsecutiveheads/dieface = i)$

P(n consecutive heads)$= \sum_{i=1}^{5} (\frac{1}{5}) * (\frac{i}{6})^n$

(d) In the first n-tosses, she obtains n consecutive heads. Given this outcome, calculate the probability that the $i^{th}$ side turned up during the roll of the die (the closed form expression for arbitrary i). How does these probabilities behave as $n \to \infty$?

We need to calculate P(die face = i / n consecutive heads)

**Notations:** die face = i : DF=i
n consecutive heads : CH = n

**Using Bayes' Theorem:**

$$P(DF = i/CH = n) = \frac{P(CH=n/DF=i) * P(DF=i)}{P(CH=n)}$$

From part (c) of the problem , we know that $P(CH = n) = \sum_{i=1}^{5} (\frac{1}{5}) * (\frac{i}{6})^n$ and

$$P(CH = n/DF = i) = (\frac{i}{6})^n$$

$$P(DF = i) = \frac{1}{5}$$

Replacing the values in the first equation , we get

$$P(DF = i/CH = n) = \frac{i^n}{1^n + 2^n + 3^n + 4^n + 5^n}$$

As $n \to \infty$,

$$P(DF = i/CH = n) = \begin{cases} 0 & if, 1 \le i \le 4 \\ 1 & i = 5 \end{cases}$$

**Problem 4** In this programming exercise, let us explore the behavior of the averages, $\frac{X_1 + X_2 + X_3 + + X_n}{n}$ of independent and identically distributed random variables $X_i$ as $n \to \infty$.

(a) Give a brief account of how you implemented random sampling according to the required distributions in your program.

**Answer:** We want to generate a dataset of size $n$ such that if $n$ becomes very large (or as $n \to \infty$) , the probability of getting $X = m$ is equal to $p_m$.

For the random sampling part, we take inputs of all the $p_i$ and divide all of them by $\sum p_i$ in case the probabilities do not sum to one.

I used python **numpy.random.choice** function to generate sample. What the function essentially do is , it takes two lists

- First, the list of values which it should produce as sample(in our case [0 , m-1]).Let it be $L_1$.
- Second , the list of probabilities $(p_0, p_1, ......P_{m-1})$telling the probabilities of the above values. Here $p_i$ is the probability of picking number i.Let the list be $L_2$.

The function generates a sample such that the value from $L_1$ is picked with corresponding probability in $L_2$.
We then take the average of these samples.

(b) How does the frequency plot of the averages behave as $n \to \infty$?

**Answer:** As $n \to \infty$ the plot looks more and more like a normal distribution curve or **Bell Curve** with mean being the expectation value E where E is

$$E = \sum_{i=0}^{m-1} i * p_i$$

This is a demonstration of the Central Limit Theorem which states that the distribution of the means drawn from converges to a normal distribution as $n$ tends to infinity.

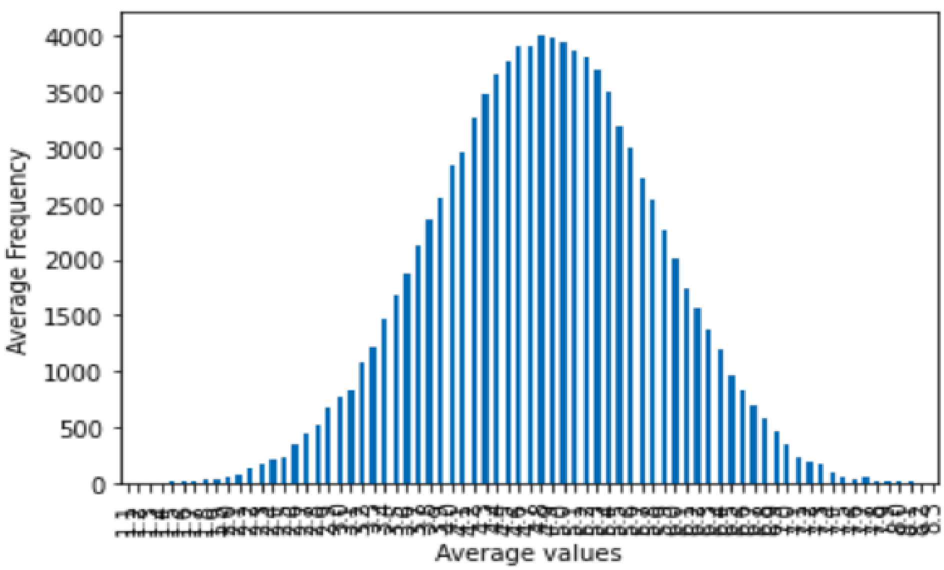


Figure : Graph

(c) Does the shape of the frequency plot change on varying m or the values of the probabilities? Can you interpret the shape of the plots for these distributions in terms of any of the concepts that were discussed in class?

**Answer:** The nature of the frequency plots does not change on varying m or the value of probabilities .It will always resemble a Bell Curve .

- Changing the probabilities modifies the expectation value, and hence the central peak location of the normal distribution. In mathematical terms, it modifies $\mu$ value in $e^{\frac{-1}{2}(\frac{x-\mu}{\sigma})^2}$.
- Increasing the value of m means that the data points are spread out over a much larger range. This means that the curve have a greater variance as m increases.