CS201B: Midsem Examination

September 24, 2021

Submission Deadline: September 27, 2021, 23:55hrs **Maximum Marks**: 50

Question 1. (10+10 marks) For any number $\ell > 0$ prove that

$$G_\ell(X) = \sum_{n \geq 0} n^\ell X^n = \frac{g(X)}{(1-X)^{\ell+1}},$$

where $g(X)$ is a polynomial of degree less than ℓ. Using the above, prove that for any numbers k and ℓ, and for any polynomial f of degree at most ℓ,

$$\sum_{i=0}^{\ell+1} (-1)^i \cdot \binom{\ell+1}{i} \cdot f(k+i) = 0.$$

Question 2. (10 marks) Derive the number of primes less than 400 using the principle of Inclusion-Exclusion.

Question 3. (20 marks) Given a set A, a $\mathbb{Z}$-*module* is defined to be a set whose elements have the form

$$\alpha = \sum_{a \in A} c_a a$$

where $c_a \in \mathbb{Z}$, the set of integers. It is denoted as $\mathbb{Z}(A)$. One can define addition of elements in $\mathbb{Z}(A)$ naturally:

$$\alpha + \beta = \sum_{a \in A} c_a a + \sum_{a \in A} d_a a = \sum_{a \in A} (c_a + d_a) a.$$

A proper subset $B \subset \mathbb{Z}(A)$ is called a *submodule* if B is closed under addition, that is, if $\alpha, \beta \in B$ then $\alpha + \beta \in B$. A submodule B is *maximal* if there is no submodule that properly contains B. Prove that $\mathbb{Z}(A)$ has a maximal submodule.