Prime Numbers

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1 Properties of Prime Number

- A number p is said to be a prime number if it is divisible by 1 or p itself.
- Every natural number can be written as a product of prime numbers.

2 Proofs for Number of Primes are infinite

In all the subsequent proofs, we will prove the following.

$$P = \text{set of all prime numbers}$$

$$|P| \to \infty$$

2.1 Euclid's Proof

Let us assume that the number of prime numbers are finite where p_n is the largest prime number.

$$P = \{p_1, p_2, p_3, ..., p_n\}$$

Define a number N as follows

$$N = (p_1 * p_2 * p_3 ..... * p_n) + 1$$

Some observations about N :

- $N > p_n$
- All the elements in set P does not divide N.

Since N is a natural number, there can be two possibilities for the above behavior.

- Either N is a prime number.
- Or $\exists$ prime number $q > p_n$ that divides N.

Either of the point proves that there exists a prime number greater than p_n.
It is a contradiction to our above claim that p_n is the largest prime number. Hence proved.

2.2 Proof 2

If the number of primes are finite, then suppose that p is the largest prime number.
Consider the following natural number $m = 2^p - 1$. Since it is natural number, there exists a prime number $q \leq p$ which divides the above number.

$$2^p - 1 = 0 \text{ mod } q$$
$$2^p = 1 \text{ mod } q$$

By **Fermat Little Theorem** we know,

$$a^{q-1} = 1 \text{ mod } q\text{, where q is a prime number, q and a are co-prime.}$$
$$2^{q-1} = 1 \text{ mod } q\text{, where q is a prime number.}$$

Since p is the prime number, it should be smallest such number. This implies that $(q-1)|p$, which means that $q > p$, hence the contradiction.

2.3 Erdos Proof of Infinity of Primes

The proof by Erdos actually proves something more stronger out of which we can comment that number of primes are infinite.

Theorem: Is P is set of all primes, then the following series diverges.

$$P = \{p_1, p_2, p_3, ..., p_n\}$$

$$\frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_n} \to \infty$$

Proof: Proof by contradiction. We will assume that the series converges.
A series is said to be convergent if its partial sum converges.

$$S_n = \sum_{i=1}^{n} a_i$$

$$n \to \infty, S_n \to L$$

If a series is divergent, then it should have infinitely many positive terms to add up to the summation.So, if we show that the series of sum of $\frac{1}{p_i}$ is divergent, we will conclude that there are infinite prime numbers.

We have assumed that the sum

$$S_n = \frac{1}{p_1} + \frac{1}{p_2} + ... + \frac{1}{p_n}$$
$$\lim_{n\to\infty} S_n = L$$

Since the partial sum converges, we can say that

$$\exists k, \sum_{n \geq k+1} \frac{1}{p_n} \leq \frac{1}{2}$$

We will divide prime numbers into two sets.

$$\text{Small\_Prime} = \{p_1, p_2, ..., p_k\}$$
$$\text{Large\_Prime} = \{p_{k+1}, p_{k+2}......\}$$

We will consider all natural numbers between 1 to N. Let us introduce two new terms.

$$N_{big} = \text{Count of Numbers between 1 to } N \text{ which are divisible by Large\_Primes.}$$
$$N_{small} = \text{Count of Numbers between 1 to } N \text{ which are divisible by Small\_Prime.}$$

Total numbers are equal N.

$$N_{big} + N_{small} = N$$

We need to bound N_{big} and N_{small}. Total numbers between 1 to N which are divisible by p = $\frac{N}{p}$.

2.3.1 Bound on N_{big}

$$N_{big} = \sum_{i \geq k+1} \frac{N}{p_i}$$

$$N_{big} = N \sum_{i \geq k+1} \frac{1}{p_i}$$

Using the converge of the series.

$$N_{big} = N \sum_{i \geq k+1} \frac{1}{p_i} \leq \frac{N}{2}$$

2.3.2 Bound on N_{small}

Any natural number $n \leq N$ can be written as:

$$n = a_n * b_n^2$$

where a_n is the square free part. Since n only has small prime divisors, the term a_n is just a product of distinct small primes. Since there are k many small primes, this means that there are 2^k different square-free parts.

$$a_n \leq 2^k$$
$$b_n \leq \sqrt{N}$$

$$N_{small} \leq 2^k * \sqrt{N}$$

We can find some k, such that

$$N_{small} \leq N/2$$

This implies

$$N_{big} + N_{small} \leq N$$

which is a contradiction. Hence proved.

3 References

https://www.math.ucdavis.edu/ hunter/intro_analysis_pdf/ch4.pdf
https://en.wikipedia.org/wiki/Fermat's_little_theorem