CS201 MIDSEM

(Jaya Gupta(200471), Harshit Bansal (200428) ,Pratyush Gupta(200717))

Q1.

In 1st part, we need to prove that,$G_l(X) = \sum_{n\geq 0} n^l X^n = \frac{g(X)}{(1-X)^{l+1}}$ where $g(X)$ is a polynomial of degree less than or equal to l.

Let us take $\sum_{n\geq 0} X^n = \frac{1}{1-X}$

Differentiating and then multiplying by X on both sides we get :

$$\Sigma_{n\geq 0} nX^{n-1} = \frac{1}{(1-X)^2}$$

$$\sum_{n\geq 0} nX^n = \frac{X}{(1-X)^2}$$ (Multiplied by X) **- (1)**

Repeat the above step again:

$$\Sigma_{n\geq 0} n^2X^{n-1} = 1/(1-X)^2 + 2X/(1-X)^3 = (1+X)/(1-X)^3$$

$$\Sigma_{n\geq 0} n^2X^n = X(1+X)/(1-X)^3$$ (Multiply by X)

As we can see the pattern is that $\Sigma_{n\geq 0} n^kX^n = \frac{P(X)}{(1-X)^{k+1}}$ **where P(X) is k degree polynomial.**

Proof by induction that the above pattern is true for any k.

Let k=1, the above statement holds. (Proved above in eq 1)

Let it be true for any k.

$$\sum_{n>=0} n^k X^n = \frac{P(X)}{(1-X)^{k+1}}$$ **where P(X) is k degree polynomial. - (2)**

$$P(X) = A_0 + A_1X + A_2X^2 + A_3X^3 .... + A_kX^k$$

Let us prove for k+1.

Differentiate eq (2) and then multiply with X.

$$\Sigma_{n\geq 0} n^{k+1} X^{n-1} = \frac{P'(X)}{(1-X)^{k+1}} + \frac{(k+1)P(X)}{(1-X)^{k+2}}$$

Multiply with X

$$\sum_{n\geq 0} n^{k+1} X^n = \frac{(1-X)XP'(X) + (k+1)XP(X)}{(1-X)^{k+2}} = \frac{Q(X)}{(1-X)^{k+2}}$$

Coefficient of X^{k+1} in $X(1-X)P'(X) + (k+1)P(X)X = (k+1)A_k - A_k = kA_k$ which is non zero hence Q(X) is a (k+1) degree polynomial .

Hence the above statement

$$\sum_{n>=0} n^k X^n = \frac{P(X)}{(1-X)^{k+1}}$$

where P(X) is a k degree polynomial is proved.

Now using the above result we need to prove that

$$\sum_{i=0}^{l+1} (-1)^i \binom{l+1}{i} f(k+i) = 0 \qquad \textbf{- (3)}$$

where $f(k+i)$ **is any polynomial of degree at most** l**.**

So let $f(X) = A_0 + A_1X + A_2X^2 + A_3X^3 .... + A_lX^l$

$$f(k+i) = A_0 + A_1(k+i) + A_2(k+i)^2 + A_3(k+i)^3 .... + A_l(k+i)^l$$

Consider term of degree m

$$\sum_{i=0}^{l+1}(-1)^i\binom{l+1}{i}(k+i)^m \qquad \textbf{--- (4)}$$

$$=\sum_{i=0}^{l+1}(-1)^i\binom{l+1}{i}\sum_{j=0}^{m}\binom{m}{j}k^{m-j}i^j$$

$$=\sum_{j=0}^{m}\binom{m}{j}k^{m-j}\sum_{i=0}^{l+1}(-1)^i\binom{l+1}{i}i^j$$

Inner Summation is zero will be proved later.

This holds for $m \le l$.

We need to prove $\sum_{i=0}^{l+1}(-1)^i\binom{l+1}{i}i^j = 0$ where $0 \le j \le m$ and $0 \le m \le l$. **--- (5)**

So basically we need to prove **(5)** for $0 \le j \le l$.

Define $g_n(X)$ is a polynomial of degree n.

$$G_\alpha(X)=\sum_{n\ge 0}n^\alpha X^n=\frac{g_\alpha(X)}{(1-X)^{\alpha+1}}$$

Multiply $(1-X)^{l+1}$ from RHS on LHS and then multiply by $(1-X)^{(l-\alpha)}$ on both sides

$$(1-X)^{l+1}\sum_{n\ge 0}n^\alpha X^n = g_\alpha(X)(1-X)^{l-\alpha}$$

RHS is a polynomial of degree at most l and hence coefficient of X^{l+1} is 0.**(Because we are applying this property the value of α varies from 0 *to* l . As we increase α to l+1 we will no longer be able to apply the coefficient of X^{l+1} is 0 property).**

Expanding $(1-X)^{l+1}$ in the above equation and finding the coefficient of X^{l+1}

$$\sum_{i\geq 0}\sum_{n\geq 0}(-i)^i\binom{l+1}{i}n^\alpha X^{n+i}$$

Where $n+i=l+1$ => $i=l+1-n$

$$\binom{l+1}{i}=\binom{l+1}{l+1-n}=\binom{l+1}{n}$$

The coefficient of X^{l+1} is

$$\sum_{n=0}^{l+1}(-1)^{l+1-n}\binom{l+1}{n}n^\alpha=0$$

$$\sum_{n=0}^{l+1}(-1)^n\binom{l+1}{n}n^\alpha=0$$

This proves eq (5), which proves that the inner summation is zero for every value of m in eq (4). Hence the equation (4) and thereby equation (3) are proved.

Ques 2:

Solution:

· Inclusion-Exclusion Principle:

$$|A_1 \bigcup A_2 \bigcup A_3 .... \bigcup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \bigcap A_j| + \sum_{1 \le i < j < k \le 1} |A_i \bigcap A \bigcap_j A_k| .... + (-1)^{n+1} |A_1 \bigcap A_2 ..... \bigcap A_n|$$

We need to find all prime numbers less than 400. We know that every prime number is divisible by 1 or itself. So we need to find all the numbers which satisfy this condition.

Since $\sqrt{400} = 20$, we will find all the prime numbers less than 20 and all the numbers which are divisible by the combination of these prime numbers and then subtract them from 400 to find no of prime numbers less than 400.

Prime numbers less than 20 : *2,3,5,7,11,13,17,19*

Let us define our sets in the following fashion:

A_1 = All the numbers which are divisible by 2 and are not prime.

A_2 = All the numbers which are divisible by 3 and are not prime.

A_3 = All the numbers which are divisible by 5 and are not prime.

A_4 = All the numbers which are divisible by 7 and are not prime.

A_5 = All the numbers which are divisible by 11 and are not prime.

A_6 =All the numbers which are divisible by 13 and are not prime.

A_7 = All the numbers which are divisible by 17 and are not prime.

A_8 = All the numbers which are divisible by 19 and are not prime.

Now we want to find the union of all these sets and subtract from 399(as 1 is not a prime number) to get the total number of primes less than 400.

Things to note:

1.) 1 is not included.
2.) When we find the number divisible by 2,3 …….17,19 then that will include these numbers themselves which will need to be subtracted. So total number divisible by 2 = [400/2] -1.

[.] denotes the greatest integer function

Let d = 400 .

$|A_1|$ = (d/2) -1 = [400/2] -1 = 199

$|A_2|$ = (d/3) -1 = [400/3] -1 = 132

$|A_3|$ = (d/5) -1 = [400/5] -1 = 79

$|A_4|$ = (d/7) -1 = [400/7] -1 = 56

|A5| = (d/11) -1 = [400/11] -1 = 35

$|A_6|$ = (d/13) -1 = [400/13] -1 = 29

$|A_7|$ = (d/17) -1 = [400/17] -1 = 22

$|A_8|$ =(d/19) -1 =[400/19] -1 = 20

Sum = 572

Now we will take two sets and find their intersection.

$$\Sigma_{1\le n<m\le 8} |A_n \cap A_m| = [\, d/(m.n) \,] \qquad - \qquad \textbf{(1)}$$

Total number of pairs will be $\binom{8}{2}$ = 28

By calculating the value of equation 1 comes out to be **324**.

Now we will take three sets and find their intersection.

$$\sum_{1\le i<j<k\le 1} |A_i \cap A_j \cap A_k| = [\, d/(i.j.k) \,] \qquad - \qquad \textbf{(2)}$$

The values of (i,j,k) possible will be

$(2,3,5)$, $(2,3,7)$, $(2,3,11)$, $(2, 3,13), (2,3,17), (2,3,19), (2,5,7), (2,5,11), (2,5,13), (2,5,17),$

$(2,5,19), (2,7,11), (2,7,13), (2,7,17), (2,7,19), (2,11,13), (2,11,17), (3,5,7), (3,5,11), (3,5,13), (3,5,17)$

$(3,5,19), (3,7,11), (3,7,13), (3,7,17), (3,7,19), (5,7,11)$

By calculating all the pairs of (i,j,k) value of equation **(2)** comes out to be **76.**

Now we will take four sets and find their intersection.

$$\sum_{1\le i<j<k<l\le 1} |A_i \cap A_j \cap A_k \cap A_l| = [\ d/(i.j.k.l)\ ] \qquad - \qquad \textbf{(3)}$$

The values of (i,j,k,l) possible will be $(2,3,5,7)$, $(2,3,5,11), (2,3,5,13)$.

By calculating all the pairs of (i,j,k,l) value of equation **(3)** comes out to be **3.**

All the further intersection values will be 0.

So by applying the inclusion-exclusion principle we get

$$|A_1 \cup A_2 \cup A_3 .... \cup A_n| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le 1} |A_i \cap A \cap_j A_k| .... + (-1)^{n+1} |A_1 \cap A_2 ..... \cap A_n|$$

$|A_1 \cup A_2 \cup A_3 .... \cup A_8|$ = 572 - 324 + 76 - 3 =321.

So the no of prime numbers less than 400 will be

$$399 - 321 = 78$$

So the number of primes less than 400 is 78.

Q.3)

Consider the set S of all the possible submodules of the set $Z(A)$. It is given to us that there is no submodule that contains all the elements of $Z(A)$.

Let B be a submodule then $B \subset Z(A)$.

Let us define a relation R on the set S as follows:

$aRb \Leftrightarrow a \subseteq b$

- aRb *and* $bRc \Rightarrow aRc$ *because* $a \subseteq b$ *and* $b \subseteq c$ *, so* $a \subseteq c$.
- aRb *and* $bRa \Rightarrow a = b$.

So R is reflexive, transitive, and anti-symmetric $\Rightarrow R$ is a partial order on S.

- Let us suppose that C is a chain in the set (S, R) .
- Let $C = \{S_\alpha : S_\alpha \epsilon S \& \alpha \epsilon T\}$ be a chain in S *under relation* R indexed by T.
- To apply Zorn's lemma we need to prove that this chain has an upper bound.
- Let us consider $S_{union} = \cup_{\alpha \epsilon T} S_\alpha$ be the union of all S_α.

We claim S_{union} is a submodule.

To prove the above-mentioned claim

- Let any $a, b \epsilon S_{union}$ then there exists $i, j \epsilon T$ such that $a \epsilon S_i$ and $b \epsilon S_j$
- Now by the property of chain C, we know that either $S_i \subseteq S_j$ or vice versa.
- Without loss of generality let us take $S_i \subseteq S_j$, so this means that

$a, b \epsilon S_j$. So $a + b \epsilon S_j$. Hence $a + b \; \varepsilon \; S_{union}$.

So S_{union} satisfies one property of submodule.

Now we need to prove that $S_{union} \subset Z(A)$.

- $S_{union} = \cup_{\alpha \epsilon T} S_\alpha$, S_{union} is defined in this way.
- We know that $S_1 \subseteq S_2 \subseteq S_3 .... \subset Z(A)$

- So $S_{union} \subset Z(A)$because it is the union of all S_α.
- **So this proves that S_{union} is a submodule of Z(A).**
- S_{union} **is also an upper bound of chain** C **as** $S_\alpha \mathbf{R} S_{union} \forall S_\alpha \epsilon S.$
- ***So the condition of Zorn's lemma holds true. Every chain has an upper bound which is the union of all sets in the chain. Hence*** S ***has a maximal element(which is also maximal submodule). Let it be*** M ***.***

Since S is the set of all possible submodules of Z(A), M is also the maximal submodule of Z(A). Hence proved.