Chapter 4: Linear Programming
The Simplex Method

Day 1:
4.1 Slack Variables and the Pivot (text pg169-176)

In chapter 3, we solved linear programming problems graphically. Since we can only easily graph with two variables (x and y), this approach is not practical for problems where there are more than two variables involved. To solve linear programming problems in three or more variables, we will use something called "The Simplex Method."

<u>Getting Started:</u>

Variables: Use $\mathbf{x_1, x_2, x_3}$,... instead of x, y, z,...

Problems look like:

4.1 Setup!
4.2 Solving!

4.3/4.4
Look Different

Maximize $z = 3x_1 + 2x_2 + x_3$

Subject to
$$2x_1 + x_2 + x_3 \le 150$$
$$2x_1 + 2x_2 + 8x_3 \le 200$$
$$2x_1 + 3x_2 + x_3 \le 320$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$

A Standard Maximum Problem

1. z is to be maximized
2. All variables, $x_1, x_2, x_3, \ldots \ge 0$
3. All constraints are "less than or equal to" (i.e. $\le$)

<u>To Use Simplex Method:</u>

STEP 1: Convert constraints (linear inequalities) into linear equations using <u>SLACK VARIABLES.</u>

Slack variables:
s_1, s_2, s_3, etc.

For example:

If $x_1 + x_2 \le 10$

then $x_1 + x_2 + s_1 = 10$

$s_1 \ge 0$ and "takes up any slack"

Example 1: Convert each inequality into an equation by adding a slack variable.

a) $2x_1 + 4.5x_2 \le 8$

b) $x_1 + 3x_2 + 2.5x_3 \le 100$

Example 2:
a) Determine the number of slack variables needed
b) Name them
c) Use slack variables to convert each constraint into a linear equation

Maximize $z = 3x_1 + 2x_2 + x_3$

Subject to
$$\begin{aligned} 2x_1 + x_2 + x_3 &\le 150 \\ 2x_1 + 2x_2 + 8x_3 &\le 200 \\ 2x_1 + 3x_2 + x_3 &\le 320 \end{aligned}$$

with $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$

STEP 2: REWRITE the objective function so all the variables are on the left and the constants are on the right.

$z = 3x_1 + 2x_2 + x_3 \longrightarrow$

STEP 3: WRITE the modified constraints (from step 1) and the objective function (from step 2) as an augmented matrix. This is called the "simplex tableau."

There should be a row for each constraint.
The last row is the objective function.
EVERY variable used gets a column.

Example: Introduce slack variables as necessary, then write the initial simplex tableau for each linear programming problem.

Ex 3) Find $x_1 \geq 0,\ x_2 \geq 0,\ \text{and } x_3 \geq 0$ such that

$$10x_1 - x_2 - x_3 \leq 138$$
$$13x_1 + 6x_2 + 7x_3 \leq 205$$
$$14x_1 + x_2 - 2x_3 \leq 345$$

and $z = 7x_1 + 3x_2 + x_3$ is maximized.

Ex. 4) Find $x_1 \geq 0 \text{ and } x_2 \geq 0$ such that

$$2x_1 + 12x_2 \leq 20$$
$$4x_1 + x_2 \leq 50$$

and $z = 8x_1 + 5x_2$ is maximized.

Example 5: A businesswoman can travel to city A, city B, or city C. It is 122 miles to city A, 237 miles to city B, and 307 miles to city C. She can travel up to 3000 miles. Dining and other expenses are \$95 in city A, \$130 in city B, and \$180 in city C. Her expense account allows her to spend \$2000. A trip to city A will generate \$800 in sales, while a trip to city B will generate \$1300 and a trip to city C will generate \$1800. How many trips should she make to each city to maximize sales? Write the initial simplex tableau.

Day 2:
4.1 Slack Variables and the Pivot (continued)

When you are looking at a simplex tableau, you may be able to spot **basic variables**.
A **basic variable** is a variable that only has all zeros except one number in its column in the tableau.

What are the basic variables in this simplex tableau?

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z \end{array} \\ \left[\begin{array}{cccccc|c} 3 & 0 & 1 & 5 & 1 & 0 & 12 \\ 2 & 2 & 0 & 0 & 1 & 0 & 12 \\ \hline -2 & 0 & 0 & 1 & 0 & 1 & 16 \end{array}\right] \end{array}$$

One **basic feasible solution** can be found by finding the value of any basic variables and then setting all remaining variables equal to zero.

Example 6: Read a solution from the given simplex tableau.

a)
$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 3 & 0 & 1 & 5 & 1 & 0 & 12 \\ 2 & 2 & 0 & 0 & 1 & 0 & 12 \\ - & - & - & - & - & - & - \\ -2 & 0 & 0 & 1 & 0 & 1 & 16 \end{array}$$

b)
$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z \end{array} \\ \left[\begin{array}{ccccccc|c} 2 & 0 & 10 & -2 & 0 & 0 & 0 & 30 \\ 0 & 5 & 0 & 10 & 0 & 0 & 0 & -30 \\ 0 & 0 & 0 & 0 & 2 & 20 & 0 & 4 \\ \hline -6 & 0 & 0 & 8 & 0 & 2 & 3 & -39 \end{array}\right] \end{array}$$

Unfortunately, solutions read off of the initial simplex tableau are seldom optimal.

We are going to alter our matrix using some restricted row operations using one of the entries in the tableau as a **pivot**. The goal is to make all other elements in the column with the pivot equal to zero.

Remember from Ch 2:
~~1. interchange two rows~~
2. multiply the elements in a row by a nonzero constant
3. add a multiple of one row to the elements of a multiple of any other row.

Example 7: Pivot once as indicated in each simplex tableau. Read the solution from the result.

a)
$$\begin{array}{cccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & z & \\ 1 & 6 & 4 & 1 & 0 & 0 & 56 \\ 2 & \boxed{2} & 1 & 0 & 1 & 0 & 12 \\ - & - & - & - & - & - & - \\ -1 & -6 & -2 & 0 & 0 & 1 & 0 \end{array}$$

b)

$$\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \end{array} \\
\left[\begin{array}{ccccccc|c}
\boxed{2} & 1 & 1 & 1 & 0 & 0 & 0 & 150 \\
1 & 2 & 8 & 0 & 1 & 0 & 0 & 200 \\
2 & 3 & 1 & 0 & 0 & 1 & 0 & 320 \\
\hline
-3 & -2 & -1 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}$$

c)

$$\begin{array}{c}
\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \end{array} \\
\left[\begin{array}{ccccccc|c}
4 & 5 & 4 & 1 & 0 & 0 & 0 & 30 \\
0 & 2 & 5 & 0 & 1 & 0 & 0 & 15 \\
3 & \boxed{3} & 2 & 0 & 0 & 1 & 0 & 10 \\
\hline
-1 & -9 & -3 & 0 & 0 & 0 & 1 & 0
\end{array}\right]
\end{array}$$

Day 1:
4.2 Maximization Problems (text pg177-190)

Day 1: Learn to set up a linear programming problem with many variables and create a "simplex tableau."
Day 2: Learn to identify basic variables, read feasible solutions from a tableau, and "pivot" to manipulate your data.

Today – Learn to identify which variable to use as the pivot so your feasible solution gives the maximum value of the objective function.

Simplex Method Maximization Problems

Step 1: Set up simplex tableau using slack variables (Lesson 4.1, day 1)

Step 2: Locate Pivot Value

- Look for most negative indicator in last row.
- For the values in this column, divide the far right column by each value to find a "test ratio."
- The value with the *smallest non-negative* "test ratio" is your pivot.

Step 3: Pivot to find a new tableau. (Lesson 4.1, day 2)

Step 4: Repeat Steps 2 & 3 if necessary Goal: no negative indicators in the bottom row.
Repeat steps 2 & 3 until all numbers on the bottom row are positive.

Step 5: Read the solution (Lesson 4.1, day 2)

Example 1:

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 100 \\ 10 & 4 & 7 & 0 & 1 & 0 & 500 \\ - & - & - & - & - & - & - \\ -120 & -40 & -60 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Example 2:

Maximize $z = 3x_1 + 2x_2 + x_3$

subject to $2x_1 + x_2 + x_3 \leq 150$

$2x_1 + 2x_2 + 8x_3 \leq 200$

$2x_1 + 3x_2 + x_3 \leq 320$

with $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$

Example 3:

Maximize $z = 24x_1 + 36x_2$

subject to $40x_1 + 80x_2 \le 560$

$6x_1 + 8x_2 \le 72$

with $x_1 \ge 0,\ x_2 \ge 0$

Day 2:
4.2 Maximization Problems (Continued)

Example 4: Solve using the Simplex Method
Kool T-Dogg is ready to hit the road and go on tour. He has a posse consisting of 150 dancers, 90 back-up singers, and 150 different musicians and due to union regulations each performer can only appear once during the tour. A small club tour requires 1 dancer, 1 back-up singer and 2 musicians for each show while a larger arena tour requires 5 dancers, 2 back-up singer and 1 musician each night. If a club concert nets T-Dogg $175 a night while an arena show nets him $400 a night, how many of each show should he schedule so that his income is a maximum and what is that maximum income?

Example 5: Solve using the Simplex Method
The Cut-Right Knife Company sells sets of kitchen knives. The Basic Set consists of 2 utility knives and 1 chef's knife. The Regular Set consists of 2 utility knives and 1 chef's knife and 1 bread knife. The Deluxe Set consists of 3 utility knives, 1 chef's knife, and 1 bread knife. Their profit is \$30 on a Basic Set, \$40 on a Regular Set, and \$60 on a Deluxe Set. The factory has on hand 800 utility knives, 400 chef's knives, and 200 bread knives. Assuming all sets are sold, how many of set should be sold to maximize the profit. What is the maximum profit?

Day 1:
4.3 Minimization Problems & Duality (text pg 191-202)

New Matrix Term:
The **transpose** of a matrix A is found by exchanging the rows and columns. The transpose of an *m x n* matrix A is written A^T, is an *n x m* matrix.

$$A=\begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix} \qquad A^T=\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}$$

Notice the 1st row becomes the 1st column and the 2nd row becomes the 2nd column.

Example 1: Find the transpose of the given matrix.

a) $\begin{bmatrix}2 & 5\\ -4 & 1\end{bmatrix}$

b) $\begin{bmatrix}1 & 8 & 5 & 3\\ 5 & 6 & 0 & 2\\ 2 & 0 & 7 & 7\end{bmatrix}$

Getting Started:
If we are going to minimize an objective function, we have to approach the problem a little differently.

Minimize $w=8y_1+16y_2$

Subject to
$$y_1+5y_2\ge 9$$
$$2y_1+2y_2\ge 10$$

with $y_1\ge 0, y_2\ge 0$

A Standard Minimum Form

1. The objective function is to be minimized
2. All variables ≥ 0
3. All constraints are "greater than or equal to" (i.e. $\ge$)

Notice:
We use "w" instead of "z" for the objective function and we use "y" as our variable instead of x.
This is just to remind us we are doing a minimization problem, which needs to be approached differently.

Duality

Minimize $w = 8y_1 + 16y_2$

Subject to
$$y_1 + 5y_2 \ge 9$$
$$2y_1 + 2y_2 \ge 10$$

with $y_1 \ge 0, y_2 \ge 0$

There is a relationship between maximum and minimum problems.

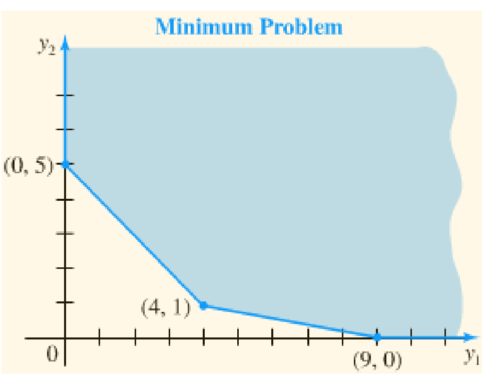
Step 1: (For the problem at right)
Without considering slack variables, write the constraints and the objective function (as is, not with negative coefficients) as an augmented matrix.

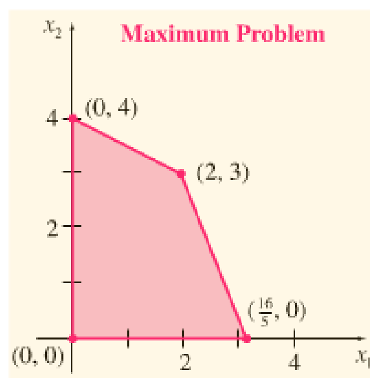
$$\begin{array}{rc} & \text{Constants} \\ & \left[\begin{array}{cc|c} 1 & 5 & 9 \\ 2 & 2 & 10 \\ \hline 8 & 16 & 0 \end{array}\right] \\ \text{Objective function} \rightarrow & \end{array}$$

Step 2: Find the transpose of the matrix.
Use this to rewrite as a maximization problem.
The maximization problem is called the "DUAL"

$$\begin{array}{rc} & \text{Constants} \\ & \left[\begin{array}{cc|c} 1 & 2 & 8 \\ 5 & 2 & 16 \\ \hline 9 & 10 & 0 \end{array}\right] \\ \text{Objective function} \rightarrow & \end{array}$$

Graphically, what's going on (think chapter 3)





Corner Point	$w = 8y_1 + 16y_2$	
(0, 5)	80	
(4, 1)	48	Minimum
(9, 0)	72	

The minimum is 48 when $y_1 = 4$ and $y_2 = 1$.

Corner Point	$z = 9x_1 + 10x_2$	
(0, 0)	0	
(0, 4)	40	
(2, 3)	48	Maximum
(16/5, 0)	28.8	

The maximum is 48 when $x_1 = 2$ and $x_2 = 3$.

So, the solution to the minimization problem Minimum = 48 when $y_1 = 4$ and $y_2 = 1$

The solution to the dual problem is Maximum = 48 when $x_1 = 2$ and $x_2 = 3$

Simplex Method

If you solve the maximization problem using simplex method:

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & \boxed{2} & 1 & 0 & 0 & 8 \\ 5 & 2 & 0 & 1 & 0 & 16 \\ \hline -9 & -10 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \begin{array}{c} \\ -R_1 + R_2 \to R_2 \\ 5R_1 + R_3 \to R_3 \end{array} & \left[\begin{array}{ccccc|c} 1 & 2 & 1 & 0 & 0 & 8 \\ \boxed{4} & 0 & -1 & 1 & 0 & 8 \\ \hline -4 & 0 & 5 & 0 & 1 & 40 \end{array}\right] \end{array}$$

$$\begin{array}{cc} & \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \begin{array}{c} -R_2 + 4R_1 \to R_1 \\ \\ R_2 + R_3 \to R_3 \end{array} & \left[\begin{array}{ccccc|c} 0 & 8 & 5 & -1 & 0 & 24 \\ 4 & 0 & -1 & 1 & 0 & 8 \\ \hline 0 & 0 & 4 & 1 & 1 & 48 \end{array}\right] \end{array}$$

The maximum for the dual problem is the same as the minimum for the original problem.

But the solution x_1 and x_2 is not the correct answer to the minimization problem.

Do you see the correct solution (4, 1) anywhere in the tableau?

AS LONG AS THE COEFFICIENT FOR Z IN THE LAST ROW OF THE MATRIX IS _____ , THE SOLUTION TO THE MINIMIZATION PROBLEM IS GIVEN BY THE COEFFICIENTS OF THE ___________________________ IN THE __________ LINE OF THE MATRIX.

Look at the following matrices. Each represents a standard minimum linear programming problem where the following steps have already taken place:

1. Standard minimization problem → converted to standard maximization problem using the dual.
2. Row operations to eliminate negative basic variables and negative numbers in final row of matrix.

READ THE SOLUTIONS FOR EACH:

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & s_1 & s_2 & s_3 & z & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & \frac{2}{9} & 0 & -\frac{1}{6} & 0 & \frac{4}{9} \\ 0 & 0 & -\frac{10}{9} & 1 & \frac{1}{3} & 0 & \frac{16}{9} \\ 1 & 0 & -\frac{1}{9} & 0 & \frac{1}{3} & 0 & \frac{7}{9} \\ \hline 0 & 0 & 16 & 0 & 6 & 1 & 104 \end{array}\right] \end{array}$$

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & s_1 & s_2 & z & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 15 & 15 & -10 & 0 & 0 \\ 3 & 0 & -6 & 5 & 0 & 3 \\ \hline 0 & 0 & 15 & 5 & 3 & 45 \end{array}\right] \end{array}$$

Example 2: State the dual problem for the linear programming problem.

Minimize $w = y_1 + 3y_2 + 7y_3$

Subject to:
$$3y_1 + 6y_2 + 9y_3 \geq -5$$
$$y_1 + 3y_2 + 9y_3 \geq 15$$

when $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$

Example 3: Use the simplex method to solve.

Minimize $w = 2y_1 + y_2 + 3y_3$

Such that
$$y_1 + y_2 + y_3 \geq 9$$
$$5y_1 + y_2 \geq 47$$

with $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$

Day 2:

4.3 Minimization Problems & Duality

Review Sections 4.1-4.3

Example1: This is the final tableau in a **standard maximum** problem. Read the solutions.

$$\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline 2 & 0 & 4 & 1 & 4 & 0 & 0 & 14 \\ 0 & 0 & 5 & 2 & 5 & 1 & 0 & 11 \\ 0 & 3 & 9 & -1 & 7 & 0 & 0 & 12 \\ \hline 0 & 0 & 2 & 9 & 2 & 0 & 2 & 40 \end{array}\right]$$

Maximum (z) = ______

when x_1 = _____ , x_2 = _____ , and x_3 = ______

Example 2: This is the final tableau in a **standard minimum** problem. Read the solutions.

$$\left[\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & z & \\ \hline 2 & 0 & 4 & 1 & 4 & 0 & 0 & 14 \\ 0 & 0 & 5 & 2 & 5 & 1 & 0 & 11 \\ 0 & 3 & 9 & -1 & 7 & 0 & 0 & 12 \\ \hline 0 & 0 & 2 & 9 & 2 & 0 & 2 & 40 \end{array}\right]$$

Minimum (w) = ______

when y_1 = _____ , y_2 = _____ , and y_3 = ______

Example 3: Set up the dual problem, and write the initial simplex tableau.

Minimize

$w = 3y_1 + 2y_2$

For

$y_1 + 2y_2 \geq 10$

$y_1 + y_2 \geq 8$

$2y_1 + y_2 \geq 12$

When

$y_1, y_2 \geq 0$

Example 4: One gram of soybean meal provides at least 2.5 units of vitamins and 5 calories. One gram of meat provides at least 4.5 units of vitamins and 3 calories. One gram of grain provides at least 5 units of vitamins and 10 calories. If a gram of soybean meal costs 7 cents, a gram of meat costs 9 cents, and a gram of grain costs 11 cents, what mixture of these three ingredients will provide at least 60 units of vitamins and 66 calories per serving at a minimum cost? What will be the minimum cost?

4.4 Nonstandard Problems – Mixture of Maximum and Minimum

Consider:

Maximize
$z = 120x_1 + 40x_2 + 60x_3$
For
$x_1 + x_2 + x_3 \le 100$
$10x_1 + 4x_2 + 7x_3 \le 500$
$x_1 + x_2 + x_3 \ge 60$
When
$x_1, x_2, x_3 \ge 0$

Minimize
$w = 40y_1 + 10y_2$
For
$y_1 + 3y_2 \ge 40$
$y_1 + y_2 \ge 32$
$14y_1 + 4y_2 \le 15$
When
$y_1, y_2 \ge 0$

Example 1:Add slack variables to convert the system of inequalities into a system of equations.

a.
$$x_1 + x_2 + x_3 \le 100$$
$$10x_1 + 4x_2 + 7x_3 \le 500$$
$$x_1 + x_2 + x_3 \ge 60$$

b.
$$y_1 + 3y_2 \ge 40$$
$$y_1 + y_2 \ge 32$$
$$14y_1 + 4y_2 \le 15$$

Another major issue with non standard problems, is a getting a negative value for a basic variable in this initial solution. If this occurs, we have to pick our pivot value a little differently than in section 4.2/4.3

If you have a negative basic variable, first
- identify the row that contains the coefficient for that variable
- SCAN LEFT: identify the <u>positive element farthest left in **that row**</u> . This tells you which <u>column</u> to choose for your pivot variable. (regardless of the negative/positive value in the last row)
- calculate the "test ratio" for each row in that column and choose the smallest corresponding coefficient for your pivot.

Example 2:
Pick the correct pivot value. Circle/Box in the value.

a)

x_1	x_2	x_3	s_1	s_2	z	
3	2	4	1	0	0	18
2	1	5	0	–1	0	8
–1	–4	–2	0	0	1	0

b)

x_1	x_2	s_1	s_2	s_3	z	
1	–1	1	0	0	0	15
2	5	0	1	0	0	75
0	1	0	0	–1	0	25
–6	–5	0	0	0	1	0

c)

x_1	x_2	x_3	s_1	s_2	s_3	z	
5	3	2	1	0	0	0	15
7	5	1	0	–2	0	0	14
9	–2	7	0	0	1	0	27
–4	–9	–5	0	0	0	1	0

To solve a "non standard problem"

PART 1: SET UP PROCEDURES

1. **If "Minimize", convert to "Maximize" by letting W = -Z**
2. **Add slack variables as needed to convert to simplex tableau.**

PART 2: PIVOT PICKING

3. **Check all "basic variables"**
   if any are negative, locate a pivot.
   - -identify the negative basic variable coefficient
   - -identify the positive element in that row farthest left, this tells you which column to choose for your pivot variable.
   - - calculate the "test ratio" for each row in that column and choose the smallest corresponding coefficient for your pivot.
   - - PIVOT on that element. Repeat step 3 as many times as necessary until all basic variables are positive.
4. When all negative "basic variables" are gone, **solve the problem using Simplex method** (Section 4.2)
   - -If there are negative #'s in your bottom row, use the most negative column & calculate the test ratio. Identify the pivot with the smallest non-negative test ratio. Pivot once.

PART 3: READ SOLUTIONS

5. If all numbers in the bottom row are positive(except possibly the VERY LAST NUMBER, VERY LAST COLUMN), **read the solution from the tableau**. If any numbers are negative, repeat step 4.

6. If you changed the minimum to a maximum, change it back using w = -z

Solve:

Example 3 –

Maximize $z = 3x_1 + 2x_2 + 2x_3$
subject to $x_1 + x_2 + 2x_3 \le 38$
$2x_1 + x_2 + x_3 \ge 24$
with $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$

Solve

Example 4:

Minimize $w = 3y_1 + 2y_2$

Such that
$$y_1 + 3y_2 \le 6$$
$$2y_1 + y_2 \ge 3$$

and $y_1, y_2 \ge 0$