MA202 Project Report

Numerical analysis and simulation of a Double Pendulum

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Problem Statement

A double pendulum is a pendulum with another pendulum attached to its end. Its motion can be described as chaotic and strongly sensitive to the initial conditions. It is governed by a set of coupled ordinary differential equations. Systems exhibiting chaotic motion diverge over time if subjected to near similar initial conditions. It is also dependent on various other parameters like length of the rods, masses of the bobs etc. Systems where these parameters differ are known as complex or compound double pendulum. In this project we are/will exploring/explore numerical methods to solve the equations of motion of a double pendulum based on the parameters. The motion will be restricted to the two-dimensional plane. We will also explore the changes in motion of the double pendulum based on its parameters.

Introduction

A pendulum is a weight suspended from a pivot so that it can swing freely. The forces acting on the pendulum are the weight of its mass and tension from the string/rod. Equilibrium position of a pendulum is the position of the bob when the suspended pendulum is at rest. Releasing the pendulum will cause the pendulum to oscillate with respect to its equilibrium position. The oscillation happens because the gravity will try to pull the pendulum's bob towards the equilibrium position at all times hence gravity acts as an restoring force and thereby the system undergoes oscillation. Double pendulum is a mechanical system where another pendulum is connected to the bob of a simple pendulum. A double pendulum is used to demonstrate chaotic motion. The reason for its chaotic motion is due to its high dependence on initial position of the double pendulum and the motion is dictated by two highly coupled non-linear 2^{nd} order ODE's. We try to solve this non-linear 2^{nd} order ODE's using various numerical methods and find the trajectory of the pendulum in time.

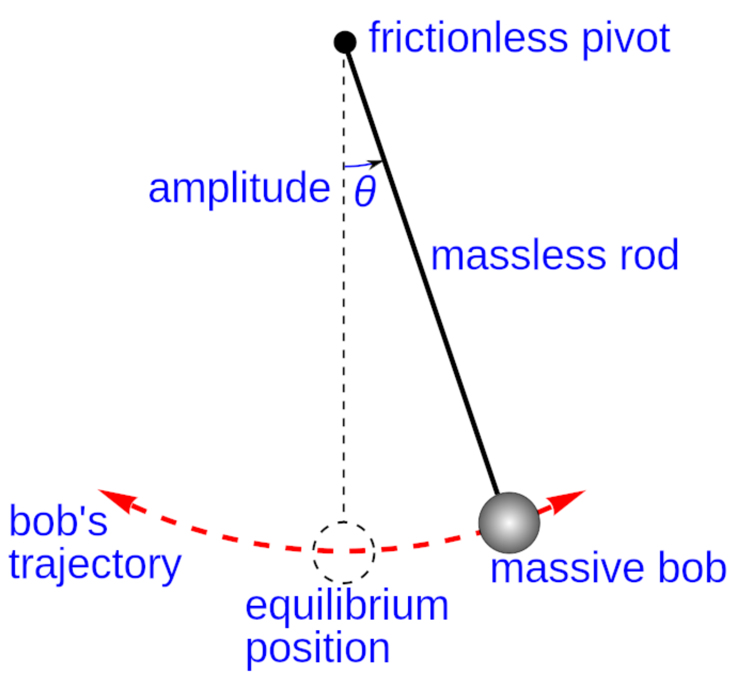

Figure 2.1: Single Pendulum

Physical model

Let us consider the two pendulum masses as point masses. The free body diagram for the both the pendulum bobs are shown in the below figure. The first pendulum bob will have the force from its weight and two tension forces acting on it. The two tension forces are due to the tension from pivot and the tension from the second pendulum bob. For the second pendulum bob there will the force due to its weight and only one tension force which will be due to the first pendulum bob. Both the weights will act vertically downwards at all times but the angle between tension forces and the vertical axis will change dynamically based on the positions of the pendulum masses.

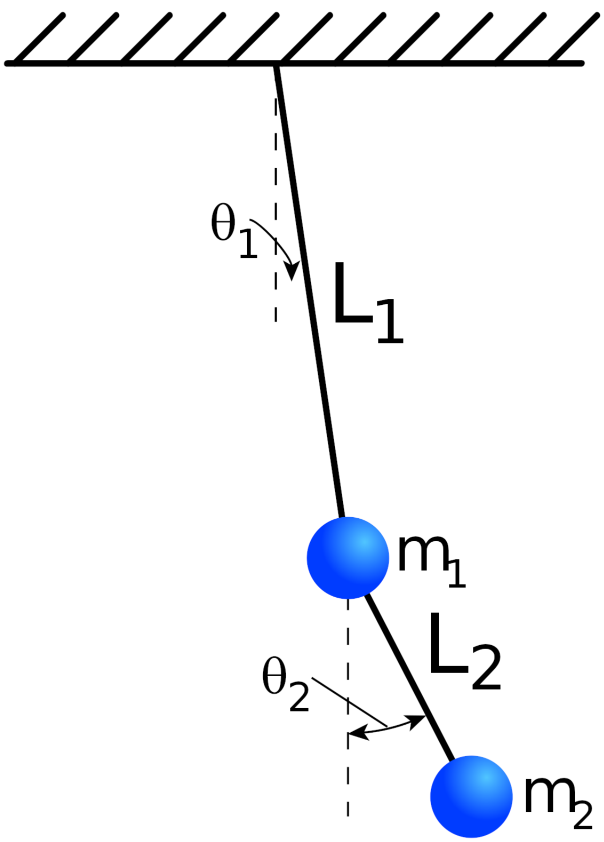

Figure 3.1: Double Pendulum

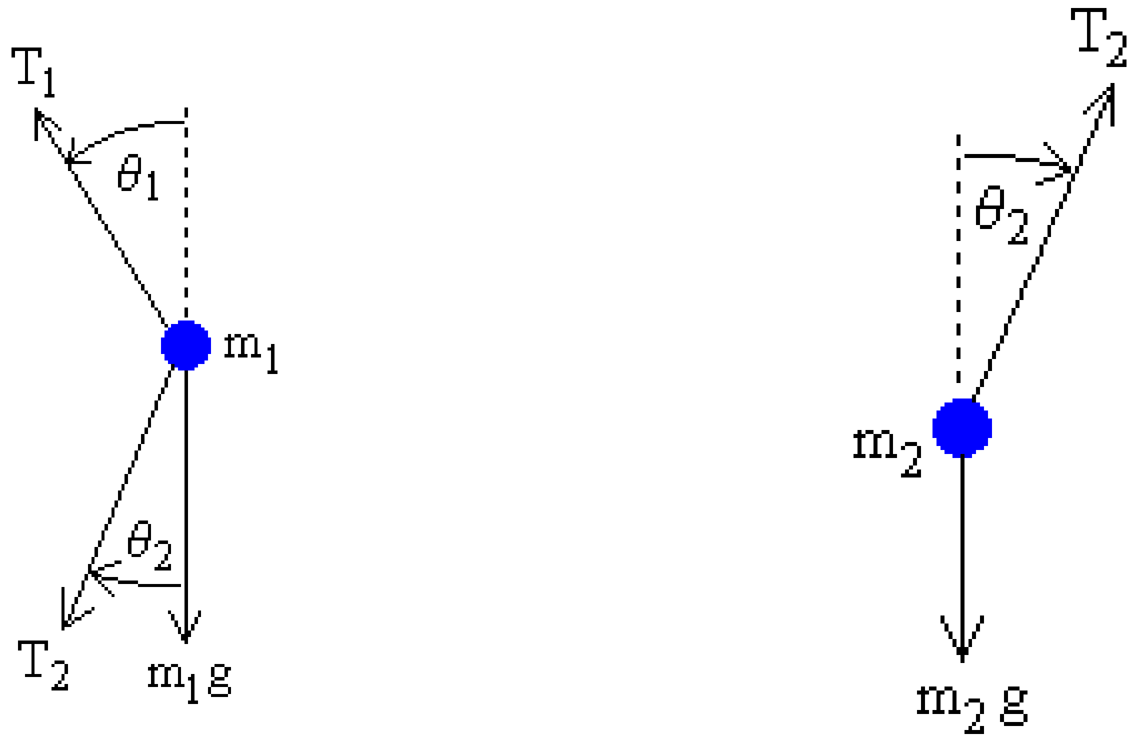

Figure 3.2: Individual bobs

3.1 Notations

1. m_1 = mass of the first pendulum bob
2. m_2 = mass of the second pendulum bob
3. θ_1 = angle between rod 1 and vertical
4. θ_2 = angle between rod d 2 and vertical
5. T_1 = tension in rod 1
6. T_2 = tension in rod 2
7. L_1 = length of rod 1
8. L_2 = length of rod 2
9. ω_1 = angular velocity of rod 1
10. ω_2 = angular velocity of rod 2
11. g = gravitational constant

Assumptions

The mathematical model for the double pendulum is created based on the following assumptions.

- All the motions are restricted in 2-D plane i.e. the system is restricted to move only in the vertical and horizontal direction
- The bobs are assumed to be point masses
- The rods are assumed to be weightless
- The rods are rigid and does not undergo any strain during the motion
- Change in value of gravity due to the change in height is neglected
- Effects of air drag are neglected

Governing Equations

To derive a mathematical model for the said problem, we will apply the conservation equations to both the zones. We will first apply mass conservation equation, in both the zones. As mass is a fundamentally conserved quantity this equation can be used in both the zones.

5.1 Kinematics

Kinematics describe the motion and the state of system without considering force. In this section we will define the state of objects without any external force.
Using Trigonometric equations to write the potions of bob 1

$$x_1 = L_1 \sin\theta_1 \tag{5.1}$$

$$y_1 = -L_1 \cos\theta_1 \tag{5.2}$$

Finding potion of bob 2 relative to bob 1 and shifting it to the origin

$$x_2 = x_1 + L_2 \sin\theta_2 \tag{5.3}$$

$$y_2 = y_1 - L_2 \cos\theta_2 \tag{5.4}$$

Derivation of positions to get velocity

$$x_1' = \theta_1' L_1 \cos\theta_1 \tag{5.5}$$

$$y_1' = \theta_1' L_1 \sin\theta_1 \tag{5.6}$$

$$x_2' = x_1' + \theta_2' L_2 \cos\theta_2 \tag{5.7}$$

$$y_2' = y_1' + \theta_2' L_2 \sin\theta_2 \tag{5.8}$$

Derivation of velocity to get acceleration

$$x_1'' = -\theta_1'^2 L_1 \sin\theta_1 + \theta_1'' L_1 \cos\theta_1 \tag{5.9}$$

$$y_1'' = \theta_1'^2 L_1 \cos\theta_1 + \theta_1'' L_1 \sin\theta_1 \tag{5.10}$$

$$x_2'' = x_1'' - \theta_2'^2 L_2 \sin\theta_2 + \theta_2'' L_2 \cos\theta_2 \tag{5.11}$$

$$y_2'' = y_1'' + \theta_2'^2 L_2 \cos\theta_2 + \theta_2'' L_2 \sin\theta_2 \tag{5.12}$$

5.2 Forces

Using newton's law of F=ma, and writing separate equations for x-axis and y-axis

For Bob-1 :-
Forces are Tension from rod 1 "T_1", Tension from rod 2 "T_2" and gravitational force "$m_1 g$", as shown in diagram in physical model section

$$m_1 x_1'' = -T_1 \sin\theta_1 + T_2 \sin\theta_2 \tag{5.13}$$

$$m_1 y_1'' = T_1 \cos\theta_1 - T_2 \cos\theta_2 - m_1 g \tag{5.14}$$

For Bob-2 :-
Forces are Tension from rod 2 "T_2" and gravitational force "$m_2 g$", as shown in diagram in physical model section

$$m_2 x_2'' = -T_2 \sin\theta_2 \tag{5.15}$$

$$m_2 y_2'' = T_2 \cos\theta_2 - m_2 g \tag{5.16}$$

Parameters

All the required parameters are listed below. The parameters used in the code can be changed by the user :

1. $m_1 = 10\ kg$
2. $m_2 = 20\ kg$
3. $L_1 = 125\ m$
4. $L_1 = 125\ m$
5. $\theta_1(initial) = \pi/3$
6. $\theta_2(initial) = \pi/3$
7. $g = 9.8\ m/s^2$
8. $damping\ ratio = 0.99$
9. $h(stepsize) = 0.1$

Analytical Solution

Now we'll perform some algebraic operations in order to find expressions for the second order differential angles in terms of zeroth and first order differential equations of the angles (θ_1, θ_2). We begin by solving the equations of Tension components of the string 2 and substitute them into the Tension components on string 1. We get,

$$m_1 x_1'' = -T_1 \sin\theta_1 - m_2 x_2'' \tag{7.1}$$

$$m_1 y_1'' = -T_1 \cos\theta_1 - m_2 y_2'' - m_2 g - m_1 g \tag{7.2}$$

Now, we multiply equation (7.1) by $\cos\theta_1$ and (7.2) with $\sin\theta_1$ and rearrange the terms, obtaining

$$T_1 \sin\theta_1 \cos\theta_1 = -\cos\theta_1 (m_1 x_1'' + m_2 x_2'') \tag{7.3}$$

$$T_1 \sin\theta_1 \cos\theta_1 = \sin\theta_1 (m_1 y_1'' + m_2 y_2'' + m_1 g + m_2 g) \tag{7.4}$$

Now, as the equations (7.3) and (7.4) have a common term we equate them to obtain,

$$-\cos\theta_1 (m_1 x_1'' + m_2 x_2'') = \sin\theta_1 (m_1 y_1'' + m_2 y_2'' + m_1 g + m_2 g) \tag{7.5}$$

Similarly, we multiply the equations containing Tension in the string 2 with $\sin\theta_2$ and $\cos\theta_2$ and rearrange them to obtain,

$$T_2 \sin\theta_2 \cos\theta_2 = -\cos\theta_2 (m_2 x_2'') \tag{7.6}$$

$$T_2 \sin\theta_2 \cos\theta_2 = \sin\theta_2 (m_2 y_2'' + m_2 g) \tag{7.7}$$

Equating equations (7.6) and (7.7) to obtain,

$$-\cos\theta_2 (m_2 x_2'') = \sin\theta_2 (m_2 y_2'' + m_2 g) \tag{7.8}$$

To solve the equations 7.5 and 7.8 for θ_1'' and θ_2'' in terms of θ_1,θ_1',θ_2, and θ_2' we referred to a particular website[2]. Finally we obtain the following equations for θ_1'' and θ_2'':

$$\theta_1'' = \frac{-g(2m_1 + m_2)\sin\theta_1 - m_2 g \sin(\theta_1 - 2\theta_2) - 2\sin(\theta_1 - \theta_2) m_2 ({\theta_2'}^2 L_2 + {\theta_1'}^2 L_1 \cos(\theta_1 - \theta_2))}{L_1(2m_1 + m_2 - m_2 \cos(2\theta_1 - 2\theta_2))} \tag{7.9}$$

$$\theta_2'' = \frac{2\sin(\theta_1 - \theta_2)({\theta_1'}^2 L_1 (m_1 + m_2) + g(m_1 + m_2)\cos\theta_1 + {\theta_2'}^2 L_2 m_2 \cos(\theta_1 - \theta_2))}{L_2(2m_1 + m_2 - m_2 \cos(2\theta_1 - 2\theta_2))} \tag{7.10}$$

Numerical Solution

It is required to solve six differential equations, in order to determine the position, velocity and acceleration of both bob of double pendulum system. Due to simplicity, using the Euler method in order to solve the differential equation.

8.1 Euler method

We are taking step size "h" = 0.1 s, for solving the equations

We already have derived the function for angular acceleration in analytical equation (7.9) and equation (7.10), which is required to use here.

$$\theta_1'' = \frac{-g(2m_1+m_2)\sin\theta_1 - m_2 g\sin(\theta_1-2\theta_2) - 2\sin(\theta_1-\theta_2)m_2({\theta_2'}^2 L_2 + {\theta_1'}^2 L_1\cos(\theta_1-\theta_2))}{L_1(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))} \tag{8.1}$$

$$\theta_2'' = \frac{2\sin(\theta_1-\theta_2)({\theta_1'}^2 L_1(m_1+m_2) + g(m_1+m_2)\cos\theta_1 + {\theta_2'}^2 L_2 m_2\cos(\theta_1-\theta_2))}{L_2(2m_1+m_2-m_2\cos(2\theta_1-2\theta_2))} \tag{8.2}$$

Using the differential equation $\frac{d\omega}{dt} = \alpha$ to find the angular velocity using Euler method.

$$\theta_1'[n] = \theta_1'[n-1] + h \times \theta_1''[n-1] \tag{8.3}$$

$$\theta_2'[n] = \theta_2'[n-1] + h \times \theta_2''[n-1] \tag{8.4}$$

Using the differential equation $\frac{d\theta}{dt} = \omega$ to find the angle using Euler method.

$$\theta_1[n] = \theta_1[n-1] + h \times \theta_1'[n-1] \tag{8.5}$$

$$\theta_2[n] = \theta_2[n-1] + h \times \theta_2'[n-1] \tag{8.6}$$

Algorithm Used

We used p5.js to code the problem. The reason behind using p5.js is that it makes visualizing/drawing part easy. The algorithm used to solve the problem is as follows :

1. Initialized all parameters.
2. Running a loop for for infinite time with step size of 0.1s.
3. Calculating angular acceleration using previous values of angular velocity and angle.
4. Using angular acceleration and Euler's formula for finding angular velocity.
5. Using angular velocity and Euler's formula for finding angle.
6. Repeating step 3,4,5 for second bob also.
7. Also at each step multiplying velocity with 0.99 for damping.

Code

```
// Coded in P5.js

let r1 = 125;
let r2 = 125;
let m1 = 10;
let m2 = 20;
let a1;
let a2;
let a1_v = 0;
let a2_v = 0;
let g = 1;
let h = 0.1;
let px2 = -1;
let py2 = -1;
let cx, cy;

let buffer;

function setup() {
  createCanvas(500, 300);
  background(110);


  // Initial angle
  a1 = PI / 3;
  a2 = PI / 3;

  cx = width / 2;
  cy = 50;
  buffer = createGraphics(width, height);
  buffer.background(175);
  buffer.translate(cx, cy);
}

function draw() {
  frameRate(100);
  background(150);
```

```
imageMode(CORNER);
image(buffer, 0, 0, width, height);

// Euler method solution

let num1 = -g * (2 * m1 + m2) * sin(a1);
let num2 = -m2 * g * sin(a1 - 2 * a2);
let num3 = -2 * sin(a1 - a2) * m2;
let num4 = a2_v * a2_v * r2 + a1_v * a1_v * r1 * cos(a1 - a2);
let den = r1 * (2 * m1 + m2 - m2 * cos(2 * a1 - 2 * a2));

let a1_a = (num1 + num2 + num3 * num4) / den;

num1 = 2 * sin(a1 - a2);
num2 = a1_v * a1_v * r1 * (m1 + m2);
num3 = g * (m1 + m2) * cos(a1);
num4 = a2_v * a2_v * r2 * m2 * cos(a1 - a2);
den = r2 * (2 * m1 + m2 - m2 * cos(2 * a1 - 2 * a2));

let a2_a = (num1 * (num2 + num3 + num4)) / den;

translate(cx, cy);
stroke(0);
strokeWeight(2);

let x1 = r1 * sin(a1);
let y1 = r1 * cos(a1);

let x2 = x1 + r2 * sin(a2);
let y2 = y1 + r2 * cos(a2);

line(0, 0, x1, y1);
fill(0);
ellipse(x1, y1, m1, m1);

line(x1, y1, x2, y2);
fill(0);
ellipse(x2, y2, m2, m2);


//angle1,angle2 velocity incremented by angle1,angle2 acceleration
a1_v += a1_a*h;
a2_v += a2_a*h;

//angle1,angle2 change by angle1,angle2 velocity
a1 += a1_v*h;
a2 += a2_v*h;
```

```
  //Dampening of angle1, angle2 velocities
  a1_v *= 0.99;
  a2_v *= 0.99;

  //For tracing the path of the lower bob

   buffer.stroke(0);
   if (frameCount > 1) {
     buffer.line(px2, py2, x2, y2);
   }

  px2 = x2;
  py2 = y2;
}
```

Results and Discussions

Some examples from the simulation :

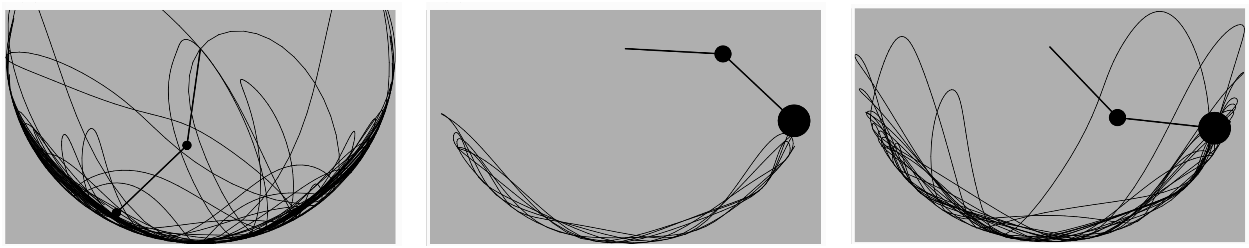

Figure 11.1: Simulation of the double pendulum using p5.js

Plots :

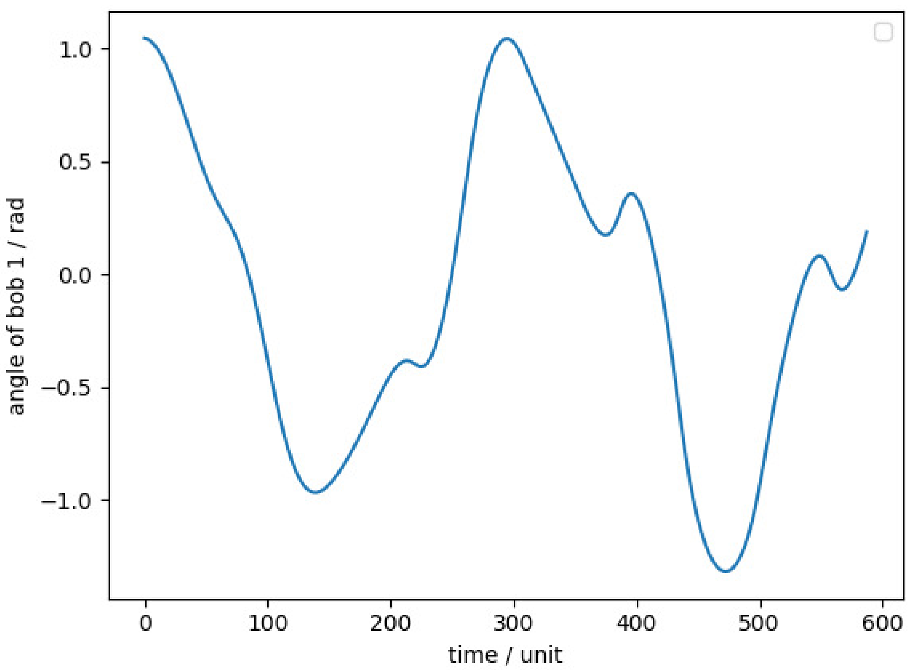

Figure 11.2: Plot btw angle of the upper bob and time

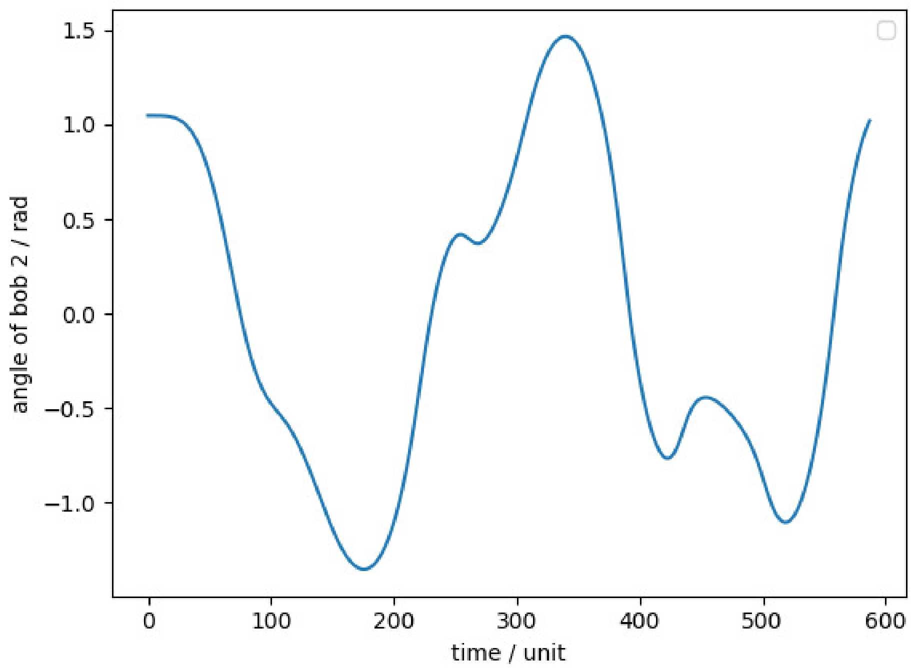

Figure 11.3: Plot btw angle of the lower bob and time

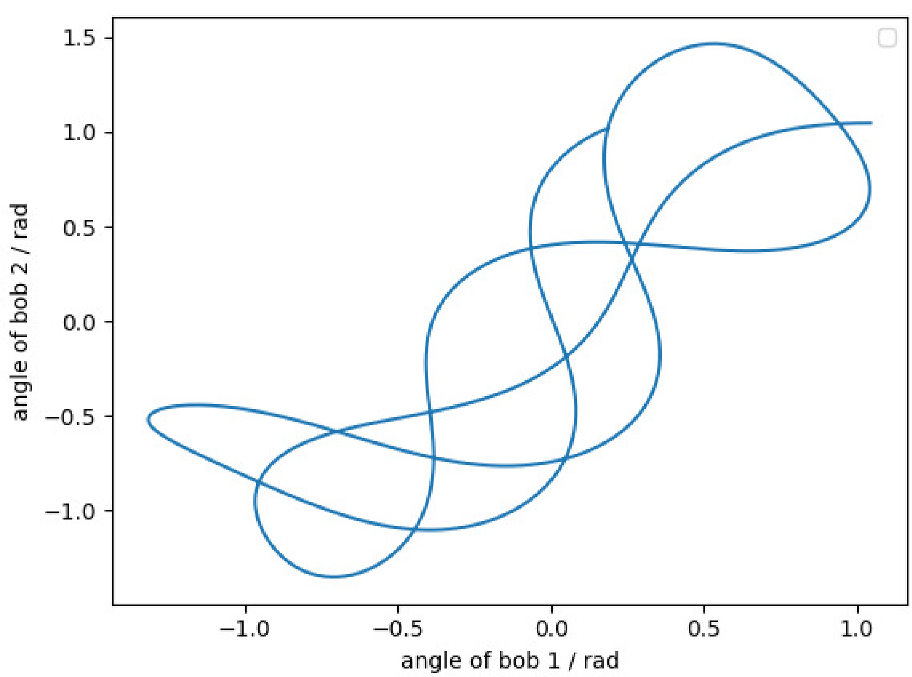

Figure 11.4: Plot btw angle of the lower and upper bob

References

1. Double Pendulum. Wikipedia. https://en.wikipedia.org/wiki/Double_pendulum
2. My Physics Lab. Eric Neumann. https://www.myphysicslab.com/pendulum/double-pendulum-en.html