Algebraic Methods in Combinatorics

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Assignment 3

To be completed by October 09

The solution of each problem should be no longer than one page!

Problem 1. Let $\mathcal{A}$ be a family of subsets of $[n]$ such that $|A|$ is odd for every $A \in \mathcal{A}$, and $|A \cap B \cap C|$ is even for every distinct $A, B, C \in \mathcal{A}$. Our aim is to show that $|\mathcal{A}| \leq n(n+1)$.

Let G be a graph, whose vertices are the sets in $\mathcal{A}$, and for distinct $A, B \in \mathcal{A}$, there is an edge between A and B if and only if $|A \cap B|$ is odd.

(a) Show that the maximum degree of G is at most n.

(b) Show that every graph on m vertices, with maximum degree at most d, has an independent set of size at least $\frac{m}{d+1}$ (recall that an *independent set* is a set of vertices with no edges between them). Deduce that G has an independent set of size at least $\frac{|\mathcal{A}|}{n+1}$.

(c) Show that every independent set in G has size at most n. Deduce that $|\mathcal{A}| \leq n(n+1)$.

Problem 2. Let $S = \{v_1, \ldots, v_m\}$ be a subset of $\mathbb{R}^n$. Let M be the *Gram matrix* corresponding to S, i.e. M is the $m \times m$ matrix M where $M_{i,j} = \langle v_i, v_j \rangle$. Let D be the $m \times m$ matrix defined by $D_{i,j} = \|v_i - v_j\|^2$.

(a) Show that $\mathrm{rank}(M) \leq n$ and $\mathrm{rank}(D) \leq n+2$.

(b) Suppose that S is a unit-distance set, i.e. $\|v_i - v_j\| = 1$ for $i \neq j$. Show that $|S| \leq n+1$.

(c)* Suppose that the distance between any two distinct points in S is either d_1 or d_2, where $d_2 > 2d_1$. Show that $|S| \leq n+2$.

Problem 3. Let $A \subseteq \mathbb{F}_3^n$ be a set such that for every distinct $a, b \in A$, where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$, there is a coordinate $i \in [n]$ such that $b_i = a_i - 1$. Show that $|A| \leq 2^n$.