Algebraic Methods in Combinatorics

Solutions 9

The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises. ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

Problem 1: The proof is by induction on k. For $k = 1$, this is immediate from the definition of the adjacency matrix A. Let now $k \geq 2$. We have $A^k = A^{k-1} \cdot A$, and therefore

$$A^k_{i,j} = \sum_{\ell=1}^{n} A^{k-1}_{i,\ell} \cdot A_{\ell,j}.$$

By induction, $A^{k-1}_{i,\ell}$ is the number of walks of length $k-1$ from i to ℓ. Also, $A_{\ell,j} = 1$ iff ℓj is an edge. Hence, $A^{k-1}_{i,\ell} \cdot A_{\ell,j}$ is exactly the number of walks of length k from i to j, such that the one-before-last vertex is ℓ. Summing over all possible ℓ, we get the number of walks of length k from i to j, as required.

Problem 2(a): Let A be the adjacency matrix of G. Recall that $\lambda_1 = \max_{v \neq 0} \frac{\langle Av,v\rangle}{\langle v,v\rangle}$. Note also that

$$\langle Av, v\rangle = \sum_{i,j=1}^{n} A_{i,j} v_i v_j = 2 \sum_{ij \in E(G)} v_i v_j$$

Now, let u be an eigenvector of λ_n, so $Au = \lambda_n u$. Let v be the vector $v_i = |u_i|$. Then

$$\langle v, v\rangle = \sum_{i=1}^{n} v_i^2 = \sum_{i=1}^{n} u_i^2 = \langle u, u\rangle$$

and

$$\langle Av, v\rangle = 2 \sum_{ij \in E(G)} |u_i u_j| \geq 2 \left| \sum_{ij \in E(G)} u_i u_j \right| = |\langle Au, u\rangle| = |\lambda_n| \langle u, u\rangle . \tag{1}$$

So $\lambda_1 \geq \frac{\langle Av,v\rangle}{\langle v,v\rangle} \geq \frac{|\lambda_n|\langle u,u\rangle}{\langle u,u\rangle} = |\lambda_n| = -\lambda_n$, as required.

Problem 2(b): For Items (b) and (c), we first prove the following:

Claim: Let G be a bipartite graph. If λ is an eigenvalue of G with multiplicity m, then $-\lambda$ is also an eigenvector of G with multiplicity m.

Proof: Let X, Y be the two parts of G, so $[n] = X \cup Y$, and every edge of G goes between X and Y. Let v be an eigenvector of λ. Define a vector u by

$$u_i = \begin{cases} v_i & i \in X, \\ -v_i & i \in Y. \end{cases}$$

For each $i \in X$, we have

$$(Au)_i = \sum_{j=1}^{n} A_{i,j} u_j = \sum_{j \,:\, ij \in E(G)} u_j = \sum_{j \,:\, ij \in E(G)} -v_j = -(Av)_i = -\lambda \cdot v_i = -\lambda u_i.$$

Here, in the third inequality all j which are neighbours of i belong to Y, because $i \in X$. In the one-before-last inequality we used that $Av = \lambda v$. Similarly, for each $i \in Y$, we have

$$(Au)_i = \sum_{j=1}^{n} A_{i,j} u_j = \sum_{j \,:\, ij \in E(G)} u_j = \sum_{j \,:\, ij \in E(G)} v_j = (Av)_i = \lambda \cdot v_i = -\lambda u_i.$$

Here we used that all neighbours of i are in X, because $j \in Y$. All in all, we got that $Au = -\lambda u$, hence $-\lambda$ is an eigenvalue of A. Also, the map f which sends v to u is an invertible linear map. We showed that this map maps the eigenspace of λ into the eigenspace of $-\lambda$. By symmetry, it also maps the eigenspace of $-\lambda$ into the eigenspace of λ. Thus, these two spaces have the same dimension, namely, λ and $-\lambda$ have the same multiplicity.

We now solve Item (b). By the claim, if G is bipartite then $-\lambda_1$ is an eigenvalue. Since $-\lambda_1 \leq \lambda_n$ by Problem 2(a), and λ_n is the smallest eigenvalue, it must be that $\lambda_n = -\lambda_1$.

In the other direction, suppose that G is connected and $\lambda_1 = -\lambda_n$. We proceed similarly to Problem 2(a). Let u be an eigenvector of $\lambda_n = -\lambda_1$, and let v be the vector $v_i = |u_i|$. As before, $\langle v, v \rangle = \langle u, u \rangle$ and $\langle Av, v \rangle \geq |\lambda_n| \langle u, u \rangle$ (see (1)). Hence, $\frac{\langle Av, v \rangle}{\langle v, v \rangle} = |\lambda_n| = \lambda_1$. On the other hand, $\frac{\langle Av, v \rangle}{\langle v, v \rangle} \leq \lambda_1$ for every v. So $\frac{\langle Av, v \rangle}{\langle v, v \rangle} = \lambda_1$. It follows that v is an eigenvector of λ_1. Indeed, one can show that $\frac{\langle Ax, x \rangle}{\langle x, x \rangle} = \lambda_1$ iff x is an eigenvector of λ_1. (Make sure that you see why; write v as a linear combination of an orthonormal eigenbasis and compute $\langle Av, v \rangle, \langle v, v \rangle$.) By Problem 1(a) in Assignment 8, as G is connected, we know that $v_i \neq 0$ for every i. Hence, $u_i \neq 0$ for every i. Now, observe that we must have equality in the inequality in (1), because otherwise we would get $\frac{\langle Av, v \rangle}{\langle v, v \rangle} > |\lambda_n| = \lambda_1$, a contradiction. We have equality iff all terms $u_i u_j$ have the same sign. Since $\langle Au, u \rangle = \lambda_n \langle u, u \rangle < 0$ and $\langle Au, u \rangle = 2 \sum_{ij \in E(G)} u_i u_j$, all terms $u_i u_j$ (for $ij \in E(G)$) must be negative. Now let $X = \{i : u_i > 0\}$ and $Y = \{i : u_i < 0\}$. Then

$[n] = X \cup Y$, and every edge of G goes between X and Y (because $u_i u_j$ is negative). It follows that G is bipartite.

Problem 2(c): The "only if" direction follows from the above claim. We give two proofs of the "if" direction. So assume that $\lambda_{n+1-i} = -\lambda_i$, and let us show that G is bipartite.

Proof 1: Induction on the number k of connected components of G. Suppose first that $k = 1$, i.e. G is connected. By assumption, $\lambda_n = -\lambda_1$. So G is bipartite by Problem 2(b). Suppose now that $k \geq 2$, and let $G_1, \dots, G_k$ be the connected components of G. Let Λ_i be the multiset of eigenvalues of G_i. As we saw in class, the eigenvalues of G are $\bigcup_{i=1}^k \Lambda_i$. Hence, there exists i such that $\lambda_n(G)$ is an eigenvalue of G_i. Then $\lambda_n(G)$ is the smallest eigenvalue of G_i (because it is the smallest eigenvalue of G). By Problem 2(a), $\lambda_1(G_i) \geq -\lambda_n(G)$. On the other hand, $\lambda_1(G_i) \leq \lambda_1(G) = -\lambda_n(G)$. So $\lambda_1(G_i) = -\lambda_n(G)$. In other words, the biggest eigenvalue of G_i equals the negative of the smallest eigenvalue of G_i. As G_i is connected, we can use Problem 2(b) to deduce that G_i is bipartite. Now remove the vertices of G_i from G, and let G' be the remaining graph. Then G' has $k-1$ connected components. Also, if $\mu_1 \geq \dots \geq \mu_m$ are the eigenvalues of G', then it still holds that $\mu_{n+1-i} = -\mu_i$. (Check! Here we use the assumption that the spectrum of G is symmetric, and also the fact that the spectrum of G_i is symmetric, which holds by the other direction of 3(c).) By the induction hypothesis, G' is bipartite. Hence, G (which is the disjoint union of G' and G_i) is bipartite.

Proof 2: By a well-known result in graph theory, a graph is bipartite iff it has no odd cycles. Also, it is well known that an odd closed walk contains an odd cycle. (A closed walk is a walk that starts and ends at the same vertex.) Thus, a graph is bipartite if and only if it has no odd closed walks. Fix an odd $k \geq 3$. By Problem 1, the number of closed walks of length k in G is

$$\sum_{i=1}^{n} A_{i,i}^k = \mathrm{tr}(A^k) = \sum_{i=1}^{n} \lambda_i^k.$$

Now, as $\lambda_{n+1-i} = -\lambda_i$, we have $\lambda_{n+1-i}^k = -\lambda_i^k$ (here we use that k is odd), so the terms λ_{n+1-i}^k and λ_i^k in the sum cancel. (Note that for odd n, the assumption $\lambda_{n+1-i} = -\lambda_i$ implies that $\lambda_{(n+1)/2} = 0$, because $\lambda_{(n+1)/2} = -\lambda_{(n+1)/2}$). So the above sum is 0. Thus, A has no closed walks of length k. As this is true for every odd k, we get that G is bipartite.

Problem 3(a): Fix a vertex v, let A be the set of neighbours of v, and let $B = V(G) \setminus (A \cup \{v\})$ be the set of vertices which are not neighbours of v. Note that $|A| = d(v) = k$, because G is k-regular. By assumption, for every $a \in A$, the vertices v, a have exactly one common neighbour. Thus, every $a \in A$ has one neighbour inside A. So the number of edges in A is

$e(A) = \frac{|A|}{2} = \frac{k}{2}$. Also, for every $b \in B$, the vertices v, b have two common neighbours, meaning that b has two neighbours in A. This means that the number of edges between A and B is $e(A, B) = 2|B| = 2(n - k - 1)$, where $n = |V(G)|$. Now, note that

$$\sum_{a \in A} d(a) = k + 2e(A) + e(A, B),$$

where the k counts the edges between v and A. This is because each edge in $E(A, B)$ is counted once by the above sum, and each edge in $E(A)$ is counted twice. On the other hand, $\sum_{a \in A} d(a) = k|A| = k^2$ (because G is k-regular). So $k^2 = k + 2e(A) + e(A, B) = 2k + 2(n - k - 1)$. Solving for n, we get $n = \frac{k^2}{2} + 1$.

Problem 3(b): Consider the matrix A^2. By Problem 1, $A^2_{i,j}$ is the number of walks of length 2 from i to j. Thus, $A^2_{i,j}$ is the number of common neighbours of i, j (in particular, $A^2_{i,i} = d(i)$ for every i). It follows that

$$A^2_{i,j} = \begin{cases} k & i = j, \\ 1 & i \sim j, \\ 2 & i \neq j,\ i \not\sim j. \end{cases}$$

So $A^2 = J + (J - I - A) + (k - 1)I = 2J - A + (k - 2)I$. Thus, $A^2 + A = 2J + (k - 2)I$. The eigenvalues of $2J + (k - 2)I$ are $2n + k - 2$ with multiplicity 1 and $k - 2$ with multiplicity $n - 1$. Now let $\lambda_1 \geq \cdots \geq \lambda_n$ be the eigenvalues of A (so $\lambda_1 = k$ because G is k-regular). Then $\lambda_i^2 + \lambda_i$ are the eigenvalues of $A^2 + A$. (As a sanity check, note that $\lambda_1^2 + \lambda_1 = k^2 + k = 2n + k - 2$, using that $n = \frac{k^2}{2} + 1$.) It follows that $\lambda_i^2 + \lambda_i = k - 2$ for every $i = 2, \ldots, n$. The equation $x^2 + x = k - 2$ has two roots: $x_1 = -\frac{1}{2} + \sqrt{k - \frac{7}{4}}$ and $x_2 = -\frac{1}{2} - \sqrt{k - \frac{7}{4}}$. Thus, $\lambda_i \in \{x_1, x_2\}$ for every $i = 2, \ldots, n$. Let s be the multiplicity of x_1 and let t be the multiplicity of x_2, so $s + t = n - 1$. Then

$$0 = \operatorname{tr}(A) = k + sx_1 + tx_2 = k - \frac{1}{2}(n - 1) + (x_1 - x_2)\sqrt{k - \frac{7}{4}}\,.$$

Rearranging and squaring, we get

$$(x_1 - x_2)^2 \left(k - \frac{7}{4}\right) = \left(\frac{1}{2}(n - 1) - k\right)^2 = \left(\frac{k^2}{4} - k\right)^2.$$

We now that $n = \frac{k^2}{2} + 1$, so k is even. To simplify, let us write $k = 2\ell$. Now the above becomes

$$(x_1 - x_2)^2 \left(2\ell - \frac{7}{4}\right) = \left(\frac{1}{2}(n - 1) - k\right)^2 = \left(\ell^2 - 2\ell\right)^2.$$

Multiplying by 4, we get $(x_1 - x_2)^2\,(8\ell - 7) = (2\ell^2 - 4\ell)^2$. So $8\ell - 7$ divides $(2\ell^2 - 4\ell)^2$. Hence, it also divides

$$(64\ell^2 - 128\ell)^2 = \big((8\ell - 7 + 7)^2 - 16(8\ell - 7 + 7)\big)^2 \equiv (7^2 - 16 \cdot 7)^2 \pmod{8\ell - 7}.$$

So $8\ell - 7$ divides $(7^2 - 16 \cdot 7)^2 = (9 \cdot 7)^2 = 3^4 \cdot 7^2$. Now, $3^4 \cdot 7^2$ has exactly $(4+1) \cdot (2+1) = 15$ divisors, and exactly 6 of them are of the form $8\ell - 7$ (i.e., congruent to 1 modulo 8). This means that there are only 6 options for ℓ, and hence for k.