Algebraic Methods in Combinatorics

Solutions 8

The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises. ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

Problem 1: Let A be the adjacency matrix of G. Then, $\lambda_1 = \max_{u \in \mathbb{R}^n \setminus \{\vec{0}\}} \frac{\langle Au,u\rangle}{\langle u,u\rangle} = \frac{\langle Av,v\rangle}{\langle v,v\rangle}$ (see, for example, Lemma 2.37 with $k=1$).

For (a), suppose for contradiction that $v_i = 0$ for some $i \in \{1,\dots,n\}$. Denote $Z := \{i : v_i = 0\}$. Let $v' = (v'_1,\dots,v'_n)$ where $v'_i = |v_i|$. It holds that $\langle v',v'\rangle = \sum_{i=1}^n (v'_i)^2 = \sum_{i=1}^n v_i^2 = \langle v,v\rangle$, and that $\langle Av',v'\rangle = 2\sum_{(i,j)\in E(G)} v'_i v'_j \geq 2\sum_{(i,j)\in E(G)} v_i v_j = \langle Av,v\rangle$. Now, for unspecified $x > 0$, consider vector $v'' = (v''_1,\dots,v''_n)$ where $v''_i = v'_i$ if $i \notin Z$ and $v''_i = x$ if $i \in Z$. Let $\alpha = \sum_{i\notin Z, j\in Z, (i,j)\in E(G)} v''_i$. As G is connected, there exists some edge $(i,j) \in (Z^c \times Z) \cap E(G)$. This means $\alpha > 0$ as $v''_i > 0$ for all $i \notin Z$. We have

$$\frac{\langle Av'',v''\rangle}{\langle v'',v''\rangle} = \frac{2\sum_{(i,j)\in E(G)} v''_i v''_j}{\sum_{i=1}^n (v''_i)^2} = \frac{2\sum_{(i,j)\in E(G)} v'_i v'_j + \alpha x + 2\sum_{i,j\in Z,(i,j)\in E(G)} x^2}{\sum_{i=1}^n (v'_i)^2 + x^2|Z|}.$$

When $x > 0$ is sufficiently small,

$$\frac{\alpha x + 2\sum_{i,j\in Z,(i,j)\in E(G)} x^2}{x^2|Z|} > \frac{\alpha x}{x^2|Z|} = \frac{\alpha}{x|Z|} \gg \frac{2\sum_{(i,j)\in E(G)} v'_i v'_j}{\sum_{i=1}^n (v'_i)^2},$$

so

$$\frac{\langle Av'',v''\rangle}{\langle v'',v''\rangle} > \frac{2\sum_{(i,j)\in E(G)} v'_i v'_j}{\sum_{i=1}^n (v'_i)^2} = \frac{\langle Av',v'\rangle}{\langle v',v'\rangle} \geq \frac{\langle Av,v\rangle}{\langle v,v\rangle} = \lambda_1.$$

This contradicts the fact that $\lambda_1 = \max_{u\in\mathbb{R}^n\setminus\{\vec{0}\}} \frac{\langle Au,u\rangle}{\langle u,u\rangle}$. Hence, $v_i \neq 0$ for all $i \in \{1,\dots,n\}$.

For (b), let $P := \{i : v_i > 0\}$ and $N := \{i : v_i < 0\}$. By (a), we know that $P \cup N = \{1,\dots,n\}$. Suppose for contradiction that $P, N \neq \emptyset$. Let $v' = (v'_1,\dots,v'_n)$ where $v'_i = |v_i|$. Note that

$$\langle Av',v'\rangle - \langle Av,v\rangle = \sum_{i\in P, j\in N:(i,j)\in E(G)} v'_i v'_j - v_i v_j = 2\sum_{i\in P, j\in N:(i,j)\in E(G)} |v_i v_j| > 0,$$

where the last inequality holds because there exists at least one edge between P and N (otherwise G is not connected). Then,

$$\frac{\langle Av', v'\rangle}{\langle v', v'\rangle} - \lambda_1 = \frac{\langle Av', v'\rangle}{\langle v', v'\rangle} - \frac{\langle Av, v\rangle}{\langle v, v\rangle} = \frac{\sum_{i \in P, j \in N : (i,j) \in E(G)} v_i' v_j' - v_i v_j}{\sum_{i=1}^{n} (v_i')^2} > 0,$$

contradicting the fact that $\lambda_1 = \max_{u \in \mathbb{R}^n \setminus \{\vec{0}\}} \frac{\langle Au,u\rangle}{\langle u,u\rangle}$. Hence, $P = \emptyset$ or $N = \emptyset$, i.e. all the v_i's have the same sign.

Indeed, (a) and (b) implies that if some vector $w \in \mathbb{R}^n$ satisfies $\lambda_1 = \frac{\langle Aw,w\rangle}{\langle w,w\rangle}$, then either all the entries of w are positive or all the entries of w are negative.

For (c), let $u \in \mathbb{R}^n$ be the eigenvector corresponding to λ_2. It holds that $\lambda_2 = \frac{\langle Au,u\rangle}{\langle u,u\rangle}$ and $\langle u, v\rangle = 0$. By (a) and (b), $v_i > 0$ for all $i \in \{1, \ldots, n\}$. So, $u_i < 0$ for some $i \in \{1, \ldots, n\}$. By (a) and (b), $\frac{\langle Au,u\rangle}{\langle u,u\rangle} < \lambda_1$ (otherwise, $u_i > 0$ for all i). Hence, $\lambda_1 > \lambda_2$.

For (d), without loss of generality, we assume that H contains all the n vertices of G because this will not change the largest eigenvalue. Let B be the adjacency matrix of H. Let w be the eigenvector corresponding to $\lambda_1(H)$ of matrix B. It suffices to show that $\lambda_1(H) = \frac{\langle Bw,w\rangle}{\langle w,w\rangle} < \max_{u \in \mathbb{R}^n \setminus \{\vec{0}\}} \frac{\langle Au,u\rangle}{\langle u,u\rangle} = \lambda_1(G)$. Indeed, if $w_i = 0$ for some $i \in \{1, \ldots, n\}$ (this is possible because H might contain isolated vertices), then (a) and (b) imply $\lambda_1(H) \leq \frac{\langle Aw,w\rangle}{\langle w,w\rangle} < \lambda_1(G)$. Otherwise, $w_i > 0$ for all $i \in \{1, \ldots, n\}$. This means, $\langle Aw, w\rangle - \langle Bw, w\rangle > 0$ (as $H \neq G$), so $\lambda_1(H) = \frac{\langle Bw,w\rangle}{\langle w,w\rangle} < \frac{\langle Aw,w\rangle}{\langle w,w\rangle} \leq \lambda_1(G)$. In either case, $\lambda_1(H) < \lambda_1(G)$.

For (e), by Problem 2(b), it holds that $\lambda_1 = d$. Note that $\mathbb{1}$ is a eigenvector corresponding to eigenvalue d (as $A\mathbb{1} = d\mathbb{1}$). Also, by (d), $\lambda_2 < \lambda_1 = d$, so v must be a multiple of $\mathbb{1}$. In other words, $v_1 = v_2 = \cdots = v_n$.

Problem 2: Let A be the adjacency matrix of G, and $v \in \mathbb{R}^n$ be a eigenvector corresponding to $\lambda_1(G)$ of matrix A. This means, $Av = \lambda_1(G)v$.

For (a), let $i \in \{1, \ldots, n\}$ that maximizes $|v_i|$. By considering the ith entry of $Av = \lambda_1(G)v$, we have $\lambda_1(G)v_i = \sum_{(i,j) \in E(G)} v_j$, so

$$|\lambda_1(G)||v_i| = |\lambda_1(G)v_i| = \left| \sum_{(i,j) \in E(G)} v_j \right| \leq \deg(i) \cdot |v_i|,$$

where $\deg(i)$ is for the degree of vertex i in graph G. As $v_i \neq 0$ (otherwise $v = \vec{0}$), it holds that $|\lambda_1(G)| \leq \deg(i) = d$. So, $\lambda_1(G) \leq d$.

For (b), use (a) and observe that $A\mathbb{1} = d\mathbb{1}$.

For (c), suppose for contradiction that $\lambda_1(G) = d$, so $Av = dv$. Recall from (a) that $|v_i| = \max_{j=1}^{n} |v_j|$. Let $S = \{j : v_j = v_i\}$. It suffices to show that $S = \{1, \ldots, n\}$ and $\deg(j) = d$ for all $j \in S$ because then, the contradiction follows as G is d-regular. Indeed, take any $j \in S$ (S is not empty because $i \in S$). By considering the jth entry of $Av = dv$, we have $dv_j = \sum_{(j,k)\in E(G)} v_k$, so $d|v_j| \le |\sum_{(j,k)\in E(G)} v_k| \le \deg(j)|v_j|$. This means that $\deg(j) \ge d$, which implies $\deg(j) = d$. Also, $v_k = v_j$ for all neighbor k of vertex j, because otherwise, $dv_j \ne \sum_{(j,k)\in E(G)} v_k$. In other words, if $j \in S$, then all the neighbors of j are also in S. As G is connected, all vertices are reachable from i, meaning $S = \{1, \ldots, n\}$. This finishes the proof for (c).

Problem 3: Let A be the adjacency matrix of G, and $X \cup Y$ be the bipartition of the vertices of G. It suffices to show that $\det(A) = 0$ (then, $Ax = 0$ for some nonzero x). Recall that $\det(A) = \sum_{\tau\in S_n} \operatorname{sign}(\tau) \prod_i A_{i,\tau(i)}$, where S_n is the set of permutations on $\{1, \ldots, n\}$ and $\operatorname{sign}(\tau) = \pm 1$. Suppose for contradiction that $\det(A) \ne 0$. Then, for some $\tau \in S_n$, $\prod_i A_{i,\tau(i)} \ne 0$. In other words, $(i, \tau(i)) \in E(G)$ for all $i \in \{1, \ldots, n\}$. Clearly, $\tau(i) \in Y$ for $i \in X$, and $\tau(i) \in X$ for $i \in Y$. Note that $\tau(i) \ne \tau(j)$ for distinct i, j. Thus, $|X| \le |Y|$ and $|Y| \le |X|$, which means $|X| = |Y|$. Now, consider $(i, \tau(i))$ for all $i \in X$. This is a perfect matching of G, contradicting to the assumption. Hence, $\det(A) = 0$, showing that 0 is a eigenvalue of G.