Algebraic Methods in Combinatorics

Solutions 5

The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises. ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

Problem 1: Let each vertex of $K_{\binom{n}{2k-1}}$ be a representation of a unique subset of size $2k-1$ in $[n]$. We now colour the edges with colours $1,\ldots,k$. We colour the edge AB with colour k if $|A \cap B|$ is even and colour $1 \le i \le k-1$ if $|A \cap B| = 2i-1$. Suppose for sake of contradiction that there is a monochromatic K_{n+1} and let $A_1,\ldots,A_{n+1}$ denote the sets composing it. If the colour is some $i \le k-1$, then Fisher's inequality implies that this is not possible, since every pair has the same intersection size and there are more than n sets. If the colour is k, then every intersection size is even and note that each set is of odd size (namely, $2k-1$). Therefore, the Oddtown problem implies that this is also impossible, since there can only be at most n such sets.

Problem 2: , (a) Clearly, since each zero pattern lives in $\{0,*\}^m$, the number of zero patterns is at most 2^m. To show that this bound is tight when $m \le n$, take the m polynomials $f_i := x_i$. We show that the number of zero patterns of $f = (f_1,\ldots,f_m)$ is 2^m. Clearly, this also implies such an example for $n > m$. To show that f has 2^m zero patterns, note that for all $\sigma \in \{0,*\}^m$, we can take $c = (c_1,\ldots,c_m) \in \mathbb{F}^m$ to be, for each i, so that $c_i = 0$ if $\sigma_i = 0$ and $c_i = 1$ otherwise. Then, the zero pattern of f at c is σ.

(b) First, note that $\binom{n+\sum_i d_i}{n}$ is as we know the dimension of the space of polynomials with n variables and dimension at most $\sum_i d_i$. We will then, for each zero pattern, σ associate a polynomial g_σ on n variables and dimension at most $\sum_i d_i$, and show that these are linearly independent, giving the desired bound on the number of zero patterns of f. Suppose σ is the zero pattern of f at some $c_\sigma \in \mathbb{F}^n$ and let $A_\sigma \subseteq [m]$ denote the set of i such that $\sigma_i = f_i(c_\sigma) \neq 0$ - note that this set defines σ and therefore it is unique to σ. Define $g_\sigma := \prod_{i \in A_\sigma} f_i$, which is a polynomial of dimension at most $\sum_i d_i$. Then, note that $g_\sigma(c_{\sigma'}) \neq 0$ if and only if $A_\sigma \subseteq A_{\sigma'}$. We claim that this implies that the polynomials g_σ are linearly independent.

Indeed, let us order the zero patterns as $\sigma_1, \sigma_2, \ldots$, and their respective certificates $c_1 :=$

$c_{\sigma_1}, c_2 := c_{\sigma_2}, \ldots$, so that $|A_{\sigma_1}| \geq |A_{\sigma_2}| \geq \ldots$. Note now that we have that if $i < j$, then $g_{\sigma_i}(c_j) = 0$ since $A_{\sigma_i} \subseteq A_{\sigma_j}$ cannot hold; further, $g_{\sigma_i}(c_i) \neq 0$ for all i. As we have previously seen in the course, this implies that the polynomials are linearly independent.

(c) Here, we apply the proof of part (b). The crucial observation is that if each zero pattern σ has support (i.e., the set A_σ) of size at most $m-n$, then each g_σ has dimension at most $(m-n)d$ and therefore, the number of zero patterns is at most the dimension of the space of polynomials with n variables and dimension at most $(m-n)d$, which is $\binom{n+(m-n)d}{n}$ as desired.

(d) Note that the support of a zero pattern defines the zero pattern uniquely. Moreover, there are are at most $\sum_{0 \leq i \leq n-1} \binom{m}{i}$ subsets of $[n]$ of size smaller than n, which is equal to the number of subsets of $[n]$ of size larger than $m-n$. Therefore, there are at most $\sum_{0 \leq i \leq n-1} \binom{m}{i}$ possible supports of size larger than $m-n$, and so, at most that many such zero patterns.

(e) Clearly, the desired inequality is deduced if we show that if $m \leq n$, then

$$\binom{n+(m-n)d}{n} + \sum_{0 \leq i \leq n-1} \binom{m}{i} \leq \binom{2n+(m-n)d}{n}.$$

To note this, observe first that since $m \geq n$, we have that $(m-n)d + n = md - (d-1)n \geq md - (d-1)m = m$. Hence, $\binom{n+(m-n)d}{n} + \sum_{0 \leq i \leq n-1} \binom{m}{i} \leq \sum_{0 \leq i \leq n} \binom{n+(m-n)d}{i} \leq \binom{2n+(m-n)d}{n}$, as desired.(a) Clearly, since each zero pattern lives in $\{0,*\}^m$, the number of zero patterns is at most 2^m. To show that this bound is tight when $m \leq n$, take the m polynomials $f_i := x_i$. We show that the number of zero patterns of $f = (f_1, \ldots, f_m)$ is 2^m. Clearly, this also implies such an example for $n > m$. To show that f has 2^m zero patterns, note that for all $\sigma \in \{0,*\}^m$, we can take $c = (c_1, \ldots, c_m) \in \mathbb{F}^m$ to be, for each i, so that $c_i = 0$ if $\sigma_i = 0$ and $c_i = 1$ otherwise. Then, the zero pattern of f at c is σ.