Algebraic Methods in Combinatorics HS19 Exam Solutions

Problem 1. For any prime p, any graph $G = (V, E)$ with average degree strictly larger than $2p-2$ and maximum degree at most $2p-1$ contains a p-regular subgraph.

Proof. The problem is a generalisation of Theorem 3.6 and the proof follows exactly along the lines of that theorem. Let $n = |V|$. Average degree condition implies $e(G) = \frac{1}{2}\sum_{v\in V(H)} d(v) > (p-1)n$ and the max degree condition $\Delta(G) \leq 2p-1$. Now, for each vertex $v \in V(G)$, we define the polynomial $p_v : \mathbb{Z}_p^{E(G)} \to \mathbb{Z}_p$ by $(x_e)_{e\in E(G)} \mapsto \sum_{e\ni v} x_e^{p-1}$. Since $\sum_{v\in V(G)} \deg p_v = (p-1)n$ and the number of variables is $e(G) > (p-1)n$, we may apply the Chevalley–Warning Theorem to conclude that the number of simultaneous solutions $(x_e)_{e\in E(G)}$ to all the p_v is divisible by p. There is a trivial solution given by $x_e = 0$ for all $e \in E(G)$, so there must exist another solution in which not all variables are 0.

Note that for any $x \in \mathbb{Z}_p$, $x^{p-1} = 1$ in $\mathbb{Z}_p$ if and only if $x \neq 0$. Thus if $\sum_{e\in v} x_e^{p-1} = 0$ in $\mathbb{Z}_p$ and $|\{e \ni v : x_e \neq 0\}| > 0$, then since $d(v) \leq 2p-1$, we must have $|\{e \ni v : x_e \neq 0\}| = p$. It follows that the subgraph having edge set $\{e \in E(G) : x_e \neq 0\}$ is p-regular and nonempty. □

Problem 2. Let $A \subseteq \mathbb{Z}_p$, with p prime and let $X = \{a+b : a, b \in A, a \neq b, ab \neq 2\}$. Show that $|X| \geq \min(2|A|-5, p)$.

Proof. Given A, B, we define $X_{AB} = \{a+b : a \in A, b \in B, a \neq b, ab \neq 2\}$. We will show that $|X_{AB}| \geq \min(|A|+|B|-4, p)$ for $A, B \neq \emptyset$ provided $|A| \neq |B|$. From this, we can solve the problem as follows: If $|A| < 2$, the statement is trivial. Otherwise, choose some arbitrary $a \in A$, let $B = A\backslash\{a\}$ and apply the above to obtain

$$|X_{AA}| \geq |X_{AB}| \geq \min(|A|+|B|-4, p) = \min(2|A|-5, p).$$

Suppose now that $A, B \neq \emptyset$ and $|A| \neq |B|$. If $|A|+|B| \geq p+4$ then we may remove some vertices from A and/or B to obtain $A' \subseteq A, B' \subseteq B$ such that $|A'|+|B'| = p+4$. So if we show the claim for A', B' we get $|X_{AB}| \geq |X_{A'B'}| \geq p$, as desired. So we may assume $|A|+|B| \leq p+4$.

Suppose for the sake of contradiction that $|X_{AB}| \leq |A|+|B|-5$. Then we may choose $C \supset X_{AB}$ such that $|C| = |A|+|B|-5$. Now define the polynomial $f(x,y) = (xy-2)(x-y)\prod_{c\in C}(x+y-c)$ over $\mathbb{Z}_p$ and observe that $f = 0$ on $A \times B$ and $\deg f = |C|+3 = (|A|-1)+(|B|-1)$. Moreover, observe that the coefficient of the term $x^{|A|-1}y^{|B|-1}$ in f is exactly $\binom{|C|}{|A|-3} - \binom{|C|}{|A|-2}$ (mod p). (We always have to choose

xy from the first term. If we choose x from the second term, we have to choose $|A|-3$ further x's from the $|C|$ terms, and if we choose $-y$ from the second term then we have to choose $|A|-2$ further x's.)

Now note that

$$\binom{|A|+|B|-5}{|A|-3} - \binom{|A|+|B|-5}{|A|-2} = \frac{(|A|-|B|)(|A|+|B|-5)!}{(|A|-2)!(|B|-2)!} \not\equiv 0 \pmod{p},$$

since the numerator of the above expression is a product of positive integers of size at most $|A|+|B|-5 < p$, in addition to the integer $|A|-|B|$, which satisfies $|A|-|B| \neq 0$ and $-p+1 \leq |A|-|B| \leq p-1$. Thus we may apply Nullstellensatz to reach a contradiction. □

Problem 3. Let $L = \{\ell_1, \ldots, \ell_s\}$ and $\mathcal{F}$ be an r-uniform L-intersecting family of subsets of an n-element set. Suppose, furthermore, that the greatest common divisor of $\ell_1, \ldots, \ell_s$ does not divide r. Prove $|\mathcal{F}| \leq n$.

Proof. Let $m = |\mathcal{F}|$, and let $v_1, \ldots, v_m \in \{0,1\}^n$ be the characteristic vectors of the sets in $\mathcal{F}$. We show that $v_1, \ldots, v_m$ are linearly independent over $\mathbb{Q}$, which then implies $m \leq n$.

Suppose that this is not the case, then there exist $\alpha_1, \ldots, \alpha_m \in \mathbb{Q}$, not all zero, such that

$$\sum_{i=1}^{m} \alpha_i v_i = 0. \tag{1}$$

Here, we can assume that $\alpha_1, \ldots, \alpha_m$ are integers with greatest common divisor 1, otherwise, we can multiply (1) with the common denominator of $\alpha_1, \ldots, \alpha_m$, or divide by the greatest common divisor.

Now for $q = 1, \ldots, m$, take the scalar product of (1) with v_s. Then we get $\sum_{i=1}^m \alpha_i \langle v_i, v_q \rangle = 0$, where $\langle v_q, v_q \rangle = r$ and $\langle v_i, v_q \rangle \in L$ for $i \neq q$.

There are two ways to finish the proof from this point.

1. By the prime factorization theorem, there must exist a prime power p^k such that $p^k \mid \gcd(\ell_1, \ldots, \ell_s)$ and $p^k \nmid r$ otherwise $\gcd(\ell_1, \ldots, \ell_s) \mid r$. This means that $p^k \mid \langle v_i, v_q \rangle$ whenever $i \neq q$, and $p^k \nmid r = \langle v_q, v_q \rangle$. But then $p | \alpha_q$, otherwise $\sum_{i=1}^m \alpha_i \langle v_i, v_q \rangle$ is not divisible by p^k. As this is true for every $q = 1, \ldots, m$, we get that $p \mid \gcd(\alpha_1, \ldots, \alpha_m)$, contradiction.

2. Let t be the smallest integer such that $\gcd(\ell_1, \ldots, \ell_s) \mid tr$. Then $t \mid \alpha_q$, otherwise $\gcd(\ell_1, \ldots, \ell_s)$ does not divide $\sum_{i=1}^m \alpha_i \langle v_i, v_q \rangle$. As this is true for every $q = 1, \ldots, m$, we get that $t \mid \gcd(\alpha_1, \ldots, \alpha_m)$, contradiction.

□

Problem 4. Let $n \geq 3$ and $\mathcal{A}$ be a family of subsets of $[2n]$. Assume that for distinct $A, B \in \mathcal{A}$, $A \cup B$ has one of only two possible sizes.

(a) Show that $|\mathcal{A}| \leq 1 + 2n + \binom{2n}{2}$.

(b) Show that if additionally for some fixed $s \geq 2$ we have $|A \cap [n]| = s$ for all $a \in \mathcal{A}$ then $|\mathcal{A}| \leq 1 + \binom{2n}{2}$.

(c) Show that the bound from part (b) can be improved to $|\mathcal{A}| \leq \binom{2n}{2}$.

Proof. (a) Let $\bar{A}$ denote the complement $[n] \setminus A$ of A. Note that $A \cup B = [n] \setminus \bar{A} \cap \bar{B}$. In particular intersection of any two sets in the family $\bar{\mathcal{A}} = \{\bar{A} | A \in \mathcal{A}\}$ have one of two possible sizes. By Theorem 1.38 $|\bar{\mathcal{A}}| \leq 1 + \binom{2n}{1} + \binom{2n}{2}$ as desired.

(b) Let ℓ_1, ℓ_2 denote the possible intersection sizes. We follow along the lines of the proof of Theorem 1.38 and 1.39. Let v_i be the characteristic vector of F_i and define $f_i : \mathbb{R}^{2n} \to \mathbb{R}$ by $f_i(x) = \prod_{j:\ell_j < |F_i|} (x \cdot v_i - \ell_j)$. We replace each instance of x_i^2 with x_i. Then we observe that $f_i(v_i) \neq 0$ and $f_i(v_k) = 0$ if $k < i$. For every $i \in [2n]$ let us define polynomials $g_i : \mathbb{R}^{2n} \to \mathbb{R}$ by $g_i(x) = (x_1 + \ldots + x_n - s)x_i$ and x_i^2 replaced by x_i. Then $g_i(v_j) = 0$ for all i, j by the condition of the problem. We now show $f_1, \ldots, f_{|\mathcal{A}|}, g_1, \ldots, g_{2n}$ are linearly independent. Assume there are α_i's and β_j's, not all zero, such that $\sum_{i=1}^{|\mathcal{A}|} \alpha_i f_i + \sum_{j=1}^{2n} \beta_j g_j = 0$. Plugging in v_i gives $\alpha_i f_i(v_i) = 0$ which implies $\alpha_i = 0$. Plugging in the i-th basis vector e_i, using the fact that $s \neq 1$, which gives a contradiction, completing the proof.

(c) We reuse notation from the previous part but add another polynomial $g_0(x) = x_1 + \ldots + x_n - s$. Evaluating at a zero vector shows $\beta_0 = 0$ in the above argument.

$\square$

Problem 5. Let G be a k-regular graph in which any two adjacent vertices have a unique neighbour and any two non-adjacent vertices have exactly 2 neighbours. Show that:

(a) G has $1 + \frac{k^2}{2}$ vertices.

(b) There are at most 6 different values that k can take.

Proof. We follow along the lines of the proof of Theorem 2.18.

(a) Let $A = A(G)$. Observe that sum of elements in row i counts the number of walks of length 2 in G starting in i. There are k walks which return to i there are k such walks ending in a neighbor of i and $2(n - k - 1)$ such walks to a non-neighbor of i, giving a total of $2n - 2$ such walks. This means that all 1's vector is an eigenvector of A^2 with eigenvalue $2n - 2$ as well as k^2 since G is k-regular. This gives $k^2 = 2n - 2$ as desired.

(b) The condition of the problem tells us that $A^2 = (k-2)I + 2J - A$, where J is the all ones matrix. Rearranging this we get $A^2 + A - (k-2)I = 2J$. If we take an orthonormal eigenbasis of A it is also an eigenbasis of $A^2 + A - (k-2)I$ so also of J. This means eigenvalues of A are k with multiplicity 1 and all other eigenvalues satisfy $\lambda^2 + \lambda - (k-2) = 0$. So are equal to $\lambda_{\pm} = \frac{-1 \pm \sqrt{4k-7}}{2}$ and let's say λ_+ appears s times and λ_- appears $n-1-s$ many times. We know also that trace of A is 0 so $0 = k + s\lambda_+ + (n-1-s)\lambda_- = k - k^2/4 + (s - k^2/4)\sqrt{4k-7}$, where we used part (a). Since k is even let $k = 2\ell$ and we know that $8\ell - 7$ divides $(\ell^2 - 2\ell)^2$. It also divides $(64\ell^2 - 128\ell)^2 = ((8\ell-7+7)^2 - 16(8\ell-7+7))^2$ so $8\ell - 7$ divides $(49 - 16 \cdot 7)^2 = 49 \cdot 81$. There are 15 positive divisors of $49 \cdot 81$, only 6 of which have form $8\ell - 7$, showing the claim.

□

Problem 6. Let $F_1, \ldots, F_{d+1}$ be sets consisting of 2 points in $\mathbb{R}^d$ each. Show that it is possible to split each F_i into a_i and b_i in such a way that $conv(a_1, \ldots, a_{d+1}) \cap conv(b_1, \ldots, b_{d+1}) \neq \emptyset$.

Proof. Let us denote $F_i = \{a_i, b_i\}$ arbitrarily. There are $d+1$ vectors $a_i - b_i \in \mathbb{R}^d$ so they must be linearly dependent, i.e. there exist λ_i, not all zero such that $\sum_{i=1}^{d+1} \lambda_i(a_i - b_i) = 0$. We may rescale λ's so that $\sum_i \lambda_i = 1$. Now if $\lambda_i < 0$ we switch a_i and b_i to make $\lambda_i \geq 0$. This means $\sum_i \lambda_i a_i = \sum_i \lambda_i b_i := u$ meaning $u \in conv(a_1, \ldots, a_{d+1}) \cap conv(b_1, \ldots, b_{d+1})$, as desired. □

Grade	Min	Max
6	51	60
5,75	46	50
5,5	41	45
5,25	35	40
5	29	34
4,75	24	28
4,5	20	23
4,25	16	19
4	12	16
3,75	11	11
3,5	10	10
3,25	7	8
3	5	6
2,5	4	4
2	3	3
1,5	2	2
1,25	1	1
1	0	0