Algebraic Methods in Combinatorics

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Assignment 11

To be completed by December 4

Problem 1. Let A be a real symmetric matrix; denote its eigenvalues by $\lambda_1 \geq \ldots \geq \lambda_n$. Show that the following holds for every $1 \leq k \leq n$ (this is the second part of the variational definition of eigenvalues of symmetric matrices, which was not proved in the lecture).

$$\lambda_k = \min_{\dim U = k-1} \max_{x \in U^\perp,\, x \neq 0} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$$

Problem 2. Show that if A is a symmetric matrix with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, and B is a principal submatrix of A with eigenvalues $\mu_1 \geq \cdots \geq \mu_m$, then $\mu_i \geq \lambda_{i+n-m}$ for every $1 \leq i \leq m$. (This is the second half of the Cauchy interlacing theorem.)

Problem 3. Let G be a graph; denote its eigenvalues by $\lambda_1 \geq \ldots \geq \lambda_n$. Recall that $\alpha(G)$ is the size of the largest independent set in G and $\omega(G)$ is the size of the largest complete subgraph of G. A *vertex cover* of G is a set of vertices U such that every edge in G has at least one vertex in U; let $\tau(G)$ be the size of the smallest vertex cover of G. Prove the following statements.

(a) $\lambda_{\alpha(G)} \geq 0$.

(b) $\lambda_{\omega(G)} \geq -1$.

(c) $\lambda_{n-\omega(G)+2} \leq -1$.

(d) $\lambda_{\tau(G)+1} \leq 0$.

Problem 4. Let G be a graph with maximum eigenvalue λ_1 and chromatic number $\chi(G)$. Show that $\chi(G) \leq \lambda_1 + 1$.