Algebraic Methods in Combinatorics

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Assignment 4

To be completed by October 16

The solution of each problem should be no longer than one page!

Problem 1. Let K and L be sets of non-zero integers of sizes r and s, respectively, and suppose that $k > s - r$ for every $k \in K$. Suppose that $\mathcal{A}$ is a family of subsets of $[n]$ such that $|A| \in K$ for every $A \in \mathcal{A}$, and $|A \cap B| \in L$ for every distinct $A, B \in \mathcal{A}$. Show that $|\mathcal{A}| \leq \sum_{i=s-r+1}^{s} \binom{n}{i}$.

Problem 2. Let p be a prime and let L be a subset of $\{0, \dots, p-1\}$ of size s. Suppose that $A_1, \dots, A_m$ and $B_1, \dots, B_m$ are subsets of $[n]$ such that

1. $|A_i \cap B_i| \notin L \pmod{p}$ for every $1 \leq i \leq m$.
2. $|A_i \cap B_j| \in L \pmod{p}$ for every $1 \leq i < j \leq m$.

Show that $m \leq \sum_{i=0}^{s} \binom{n}{i}$.

Problem 3. Let p be prime, $k \geq 2$, and let L be a subset of $\{0, \dots, p-1\}$ of size s. Suppose that $\mathcal{A}$ is a family of subsets of $[n]$ such that $|A| \notin L \pmod{p}$ for every $A \in \mathcal{A}$, and $|A_1 \cap \ldots \cap A_k| \in L \pmod{p}$ for every distinct $A_1, \dots, A_k \in \mathcal{A}$.

(a) Show that $|\mathcal{A}| \leq (k-1) \sum_{i=0}^{s} \binom{n}{i}$.

(b) Show that this bound is asymptotically tight for fixed p and k.