Algebraic Methods in Combinatorics

Partial Solutions of Assignment 12

The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises. ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

Problem 1: (a) Let $\mathbf{c}_1, \ldots, \mathbf{c}_n$ be the column vectors of A and let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be a basis for the column space of A. For an arbitrary column $\mathbf{c}_i$, there are α_j such that $\mathbf{c}_i = \alpha_1 \mathbf{v}_1 + \ldots + \alpha_r \mathbf{v}_r$. Let $\mathbf{c}_i^{(k)} := (\mathbf{c}_i(1)^k, \ldots, \mathbf{c}_i(n)^k)$ denote the i-th column of $A^{(k)}$. Note that

$$\mathbf{c}_i(j)^k = (\alpha_1 \mathbf{v}_1(j) + \ldots + \alpha_r \mathbf{v}_r(j))^k = \sum_f f(\alpha_1 \mathbf{v}_1(j), \ldots, \alpha_r \mathbf{v}_r(j)) = \sum_f f(\alpha_1, \ldots, \alpha_r) f(\mathbf{v}_1(j), \ldots, \mathbf{v}_r(j))$$

where the sum is over all monomials f in r variables of degree exactly k. Now, let us define, for each such monomial f, the vector $\mathbf{u}_f := (f(\mathbf{v}_1(1), \ldots, \mathbf{v}_r(1)), \ldots, f(\mathbf{v}_1(n), \ldots, \mathbf{v}_r(n)))$. Then, the above equality implies that $\mathbf{c}_i(j)^k = \sum_f f(\alpha_1, \ldots, \alpha_r)\mathbf{u}_f$. Hence, the rank of $A^{(k)}$ is at most the number of such monomials f. Finding the number of monomials of degree k in r variables is exactly the balls in bins problems, which gives $\binom{r+k-1}{k}$ such f. $\square$

(b) By Lemma 2.36 in the notes, since A is symmetric we have that $rk(A) \geq \frac{(trA)^2}{trA^2}$. By assumption, A is 1 on the diagonal and so, $trA = n$. Further, since A is symmetric, we have that $0 \leq trA^2 \leq n(1 + (n-1)\max_{i \neq j} |A_{i,j}|^2) \leq n(1 + (n-1)\epsilon^2)$. Hence, $rkA \geq \frac{n}{1+(n-1)\epsilon^2}$. $\square$

(c) Let $C = \{v_1, \ldots, v_N\}$ and define A to be an $N \times N$ matrix by letting $A_{i,j} = \frac{1}{d}\langle v_i, v_j \rangle$. Note that since A is the normalized Gram matrix of the vectors of C, which live in $\{-1, 1\}^d$, we have $rk(A) \leq d$. Then, part (a) implies that $rk(A^{(k)}) \leq \binom{d+k}{k}$. Note also that $A^{(k)}$ is symmetric, has 1 entries in the diagonal and is such that $|A_{i,j}^{(k)}| \leq \epsilon^k$ for $i \neq j$. Hence, $rk(A^{(k)}) \geq \frac{N}{1+\epsilon^{2k}N}$ by part(b). Therefore,

$$\binom{d+k}{k} \geq \frac{N}{1+\epsilon^{2k}N}.$$

Now, using that $\binom{x}{y} \leq (ex/y)^y$, we get that

$$(e(d+k)/k)^k \geq \frac{N}{1+\epsilon^{2k}N} \Leftrightarrow N(k^k - (e\epsilon^2(d+k))^k) \leq (e(d+k))^k.$$

Now, take $k := \lceil \frac{1.1e\epsilon^2 d}{1-1.1e\epsilon^2} \rceil$. Note that since $1/\sqrt{d} \leq \epsilon \leq 1/2$, we have that $k = O(\epsilon^2 d)$ and $k^k \geq (1.1e\epsilon^2(d+k))^k \geq (e\epsilon^2(d+k))^k + (0.1e\epsilon^2(d+k))^k$. Thus, the above inequality implies

$$N \cdot (0.1e\epsilon^2(d+k))^k \leq (e(d+k))^k \Leftrightarrow N \leq (10\epsilon^{-2})^k \leq 2^{c\epsilon^2 \log(1/\epsilon) d}$$

for some constant c, as desired.

Problem 2: We intend to use the Chevalley-Warning Theorem. Define the polynomial g as $g(x_1, \ldots, x_n) := h(x_1^{p-1}, \ldots, x_n^{p-1})$. Note that g has degree at most $(p-1)d < n$ and therefore, by the Chevalley-Warning Theorem the number of solutions to $g = 0$ is divisible by p. Since $g(\mathbf{0}) = 0$, there must be a non-zero solution $\mathbf{x}$ so that $g(\mathbf{x}) = 0$. But note that then $h(x_1^{p-1}, \ldots, x_n^{p-1}) = 0$ and $(x_1^{p-1}, \ldots, x_n^{p-1}) \in \{0,1\}^n \setminus \{\mathbf{0}\}$. Thus, $e := (x_1^{p-1}, \ldots, x_n^{p-1})$ is as desired. $\square$

(b) Let x_e denote a variable for a given edge $e \in H$. Define the polynomial on these variables as

$$h(x_e, e \in \mathcal{H}) := \sum_{v \in V(H)} \left(1 - \prod_{v \in e}(1 - x_e)\right)$$

Note that $h(\mathbf{0}) = 0$ and that it has degree at most the maximum degree of $\mathcal{H}$ which is d. Therefore, since the number of variables is the number of edges in H which is at least $d(p-1)+1$, the previous part (a) implies that there is a non-zero vector $\mathbf{x} \in \{0,1\}^n$ such that $h(\mathbf{x}) \equiv 0 \pmod p$. Consider the set $S \subseteq H$ of edges $e \in H$ with $x_e = 1$. Note then that $h(\mathbf{x})$ counts the number of vertices in the union of the edges in S (indeed, if a vertex is not in the union, then its term in the sum will be 0, and if it is, its term will be 1). Hence, S is a non-empty set of edges whose union has size $0 \pmod p$.

$\square$