Algebraic Methods in Combinatorics

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Assignment 12

To be completed by December 11

Problem 1.

(a) Given an $n \times n$ matrix A, denote by $A^{(k)}$ the $n \times n$ matrix defined by $\left(A^{(k)}\right)_{i,j} = (A_{i,j})^k$ (i.e. entries of A are raised to the power k). Let r be the rank of A. Show that $\operatorname{rank}\left(A^{(k)}\right) \leq \binom{r+k-1}{k}$.

(b) Let A be an $n \times n$ symmetric matrix, such that $A_{i,i} = 1$ for every $1 \leq i \leq n$ and $|A_{i,j}| \leq \varepsilon$ for $1 \leq i \neq j \leq n$. Show that $\operatorname{rank}(A) \geq \frac{n}{1+(n-1)\varepsilon^2}$.

(c) A *binary code of length* d is a set $C \subseteq \{-1, 1\}^d$, i.e. a set of ± 1 vectors of length d. We say that a code C of length d is ε*-balanced* if every two vectors in C differ in at least $\frac{1-\varepsilon}{2}d$ and at most $\frac{1+\varepsilon}{2}d$ coordinates; in other words, $|\langle u, v\rangle| \leq \varepsilon d$ for every distinct $u, v \in C$.

Let $\frac{1}{\sqrt{d}} \leq \varepsilon \leq 1/2$. Show that every ε-balanced code of length d has size at most $2^{c\varepsilon^2 \log(1/\varepsilon) d}$, where c is an absolute contant.

Problem 2.

(a) Let p be prime and $n \geq d(p-1)+1$ for some integer d. Let $h(x_1, \ldots, x_n)$ be a polynomial of degree d such that $h(0, \ldots, 0) = 0$. Prove that there is a non-zero vector $e \in \{0, 1\}^n$ such that $h(e) = 0 \pmod{p}$.

(b) Let H be a hypergraph with at least $d(p-1)+1$ edges and maximum degree d. Show that there is a non-empty set of edges of H whose union has size $0 \pmod{p}$.