Algebraic Methods in Combinatorics

Solutions 7

The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises. ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

Problem 1: Let v_i be the characteristic vector of A_i. By Tverberg's theorem, there is a partition $[m] = I_1 \cup \cdots \cup I_r$ and a vector y such that $y \in \text{conv}(v_i : i \in I_j)$ for every $j \in [r]$. So we can write $y = \sum_{i \in I_j} \alpha_i v_i$ for every $j \in [r]$, where $\alpha_i \geq 0$ and $\sum_{i \in I_j} \alpha_i = 1$. Now, for each $j \in [r]$, denote $I'_j := \{i \in I_j : \alpha_i > 0\}$. Observe that for every $k \in [n]$ and $j \in [r]$, we have $k \in \bigcup_{i \in I'_j} A_i$ if and only if $y_k > 0$, and $k \in \bigcap_{i \in I'_j} A_i$ if and only if $y_k = 1$. Hence, the sets $\bigcup_{i \in I'_j} A_i$ ($j \in [r]$) are all the same, and the sets $\bigcap_{i \in I'_j} A_i$ ($j \in [r]$) are all the same. Compare this with the proof of Theorem 1.10.

Problem 2: Apply Tverberg's theorem with the largest possible r. We need $(r-1)(d+1)+1 \leq n$, so $r \leq \frac{n-1}{d+1} + 1$. So take $r = \lfloor \frac{n-1}{d+1} \rfloor + 1 \geq \frac{n-1}{d+1}$. By Tverberg, there is a partition $X = X_1 \cup \cdots \cup X_r$ and a point $y \in \mathbb{R}^d$ such that $y \in \text{conv}(X_i)$ for every $i \in [r]$. For every $1 \leq i_1 < \cdots < i_{d+1} \leq r$, we can apply the colorful Carathéodory theorem with $M_j = X_{i_j}$ ($j = 1, \dots, d+1$) to deduce that there are $x_{i_j} \in X_{i_j}$ with $y \in \text{conv}(x_{i_1}, \dots, x_{i_{d+1}})$. Since there are $\binom{r}{d+1}$ choices for $1 \leq i_1 < \cdots < i_{d+1} \leq r$, we get at least $\binom{r}{d+1}$ simplices containing y. Now, $\binom{r}{d+1} \geq \binom{(n-1)/(d+1)}{d+1} \geq c_d \binom{n}{d+1}$ for a small enough constant c_d depending on d.

Problem 3: We construct the set Y greedily. Start with $Y_0 = \emptyset$. For $i \geq 1$, if Y_{i-1} is a weak ε-net for X then stop. Otherwise, there is $X' \subseteq X$, $|X'| \geq \varepsilon |X|$, such that $Y_{i-1} \cap \text{conv}(X') = \emptyset$. Apply Problem 2 to get a point y_i such that y is contained in at least $c_d \binom{|X'|}{d+1} \geq c_d \binom{\varepsilon n}{d+1}$ of the X'-simplices. Observe that no X'-simplex contains any point of Y_{i-1} because $Y_{i-1} \cap \text{conv}(X') = \emptyset$. Set $X_i := X'$ and $Y_i := Y_{i-1} \cup \{y_i\}$. We want to claim that this process eventually stops (and bound the number of steps). By definition, each X_i ($i \geq 1$) has a collection $\mathcal{S}_i$ of at least $c_d \binom{\varepsilon n}{d+1}$ X_i-simplices which contain y_i. As we saw, these simplices do not contain y_j for any $j < i$. Hence, no simplex can belong to both $\mathcal{S}_i$ and $\mathcal{S}_j$ for $i \neq j$. (Indeed, assuming $j < i$, every simplex in $\mathcal{S}_j$ contains y_j but every simpliex in $\mathcal{S}_i$ doesn't). Since the total number of

X-simplices is $\binom{n}{d+1}$, the number of different sets $\mathcal{S}_i$ (and hence the number of steps of the process) is at most $\frac{\binom{n}{d+1}}{c_d\binom{\varepsilon n}{d+1}} \leq C_{d,\varepsilon}$ for some large enough constant $C_{d,\varepsilon}$.

Problem 4: Write $F_i = \{x_i, y_i\}$. The vectors $x_i - y_i$, $i = 1, \ldots, d+1$, are linearly dependent, so there exist $\alpha_1, \ldots, \alpha_{d+1} \in \mathbb{R}$, not all 0, such that

$$\sum_{i=1}^{d+1} \alpha_i(x_i - y_i) = 0 \tag{1}$$

Let $I = \{i : \alpha_i > 0\}$, $J = [d+1] \setminus I = \{i : \alpha_i \leq 0\}$. Rearranging (1), we get

$$\sum_{i\in I} \alpha_i x_i - \sum_{i\in J} \alpha_i y_i = \sum_{i\in I} \alpha_i y_i - \sum_{i\in J} \alpha_i x_i. \tag{2}$$

Denote by z the vector that equals the LHS and RHS in (2). Let $\alpha = \sum_{i\in I} \alpha_i - \sum_{i\in J} \alpha_i$. Then $\alpha > 0$ because $\alpha_i \geq 0$ for every $i \in I$, $-\alpha_i \geq 0$ for every $i \in J$, and the α_is are not all 0. Now,

$$\frac{1}{\alpha} z = \sum_{i\in I} \frac{\alpha_i}{\alpha} x_i - \sum_{i\in J} \frac{\alpha_i}{\alpha} y_i. \tag{3}$$

and

$$\frac{1}{\alpha} z = \sum_{i\in I} \frac{\alpha_i}{\alpha} y_i - \sum_{i\in J} \frac{\alpha_i}{\alpha} x_i. \tag{4}$$

Note that (3) and (4) are both convex combinations. So it remains to take

$$a_i = \begin{cases} x_i & i \in I \\ y_i & i \in J \end{cases}$$

and

$$b_i = \begin{cases} y_i & i \in I \\ x_i & i \in J \end{cases}$$

Then $z \in \operatorname{conv}(a_1, \ldots, a_{d+1})$ and $z \in \operatorname{conv}(b_1, \ldots, b_{d+1})$.