Algebraic Methods in Combinatorics

Solutions 10

The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises. ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

Problem 1(a): Let m denote the number of edges of G. By Lemma 2.14, we have $A_{L(G)} = B^T B - 2I_m$, where B is the incidence matrix of G. Observe that $B^T B$ is a positive semi-definite matrix. Indeed, for every vector $x \in \mathbb{R}^m$, we have $\left\langle B^T Bx, x \right\rangle = \langle Bx, Bx \rangle = |Bx|_2^2 \geq 0$. (Here we use the fact that $\left\langle M^T x, y \right\rangle = \langle x, My \rangle$ for every matrix M. Make sure you see why!). It follows that $\left\langle A_{L(G)}x, x \right\rangle = \left\langle B^T Bx, x \right\rangle - 2\langle x, x \rangle \geq -2\langle x, x \rangle$ for every $x \in \mathbb{R}^m$. This implies that every eigenvalue λ of $A_{L(G)}$ satisfies $\lambda \geq -2$. Indeed, if x is an eigenvector of λ, then $\left\langle A_{L(G)}x, x \right\rangle = \lambda \langle x, x \rangle$, so $\lambda \geq -2$.

Problem 1(b): As in Item (a), we have $A_{L(G)} = B^T B - 2I_m$. Recall that B is an $n \times m$ matrix, and hence rank$B \leq n$. Therefore, rank$(B^T B) \leq$ rank$(B) \leq n$ (the rank of the product of two matrices is not larger than the rank of each of the matrices). Also, $B^T B$ is an $m \times m$ matrix. Thus, if $m > n$ then $B^T B$ does not have full rank. Hence, there exists a non-zero $x \in \mathbb{R}^m$ such that $B^T Bx = 0$. This means that $A_{L(G)}x = (B^T B - 2I_m)x = -2x$, so -2 is an eigenvalue of $A_{L(G)}$.

Problem 1(c): Take G to be the cycle graph C_n. Observe that $L(G)$ is also (isomorphic to) C_n. As we saw in class (see Example 2.25 in the lecture notes), the eigenvalues of C_n are $\lambda_k = e^{\frac{2\pi ik}{n}} + e^{-\frac{2\pi ik}{n}}$, $k = 0, \dots, n-1$. Since $e^{\frac{2\pi ik}{n}}$ and $e^{-\frac{2\pi ik}{n}}$ are both on the unit circle, the only possibility to have $\lambda_k = -2$ is that both $e^{\frac{2\pi ik}{n}}$ and $e^{-\frac{2\pi ik}{n}}$ are equal to $-1 = e^{\pi i}$. This means in particular than $\frac{2\pi k}{n} = \pi$, so $n = 2k$. If n is odd then this is impossible, meaning that all eigenvalues of $L(G)$ are strictly larger than -2.

Problem 2(a):

First, let us determine the eigenvectors of J (this was done in class but we repeat it). Let 1_n denote the all-ones vector. Then v is an eigenvector of J if and only if $v \in \text{span}\{1_n\}$ or v is orthogonal to 1_n. Indeed, observe that $J = 1_n 1_n^\perp$, and hence $J \cdot 1_n = n \cdot 1_n$, and for any u such that $\langle u, 1_n \rangle = 0$, we have $Ju = 0$. In other words, 1_n is an eigenvector and spans the eigenspace of eigenvalue n, while the only other eigenvalue is 0 and its eigenspace has dimension $n-1$.

As usual, let us write $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ for the eigenvalues of A and let $v_1, \ldots, v_n$ be an orthonormal eigenbasis such that $Av_i = \lambda_i v_i$. We start by showing that 1_n is an eigenvector of A corresponding to λ_1 and that $\lambda_2 < \lambda_1$. By assumption, if v is an eigenvector of A then it is also an eigenvector of J, so $v \in \text{span}\{1_n\}$ or v is orthogonal to 1_n. Thus, if 1_n is not an eigenvector of A, then every eigenvector of A is orthogonal to 1_n, which is impossible, because the dimension of the space $(\text{span}\{1_n\})^\perp$ is $n-1$ (and A has a basis of n eigenvectors). So 1_n is an eigenvector of A. Let λ be the corresponding eigenvalue, so that $A1_n = \lambda 1_n$,. For each $i \in [n]$, we have $\lambda = (\lambda 1_n)_i = (A1_n)_i = \sum_{j\,:\,j \sim i} 1 = \deg(i)$, implying that G is λ-regular. By Problem 2(b) of Assignment 8, $\lambda_1 = \lambda$.

Next, assume by contradiction that $\lambda_2 = \lambda_1$, and consider $u = v_2 + 1_n$. Then we have $Au = \lambda_2 v_2 + \lambda_1 1_n = \lambda_1 u$, (as $\lambda_2 = \lambda_1$). But $u \notin \text{span}\{1_n\}$ and $u \not\perp 1_n$ (here we use that v_2 is orthogonal to $v_1 = 1_n$). In other words, u is an eigenvector of A but not an eigenvector of J, a contradiction. This shows that $\lambda_2 < \lambda_1$.

The desired polynomial is:

$$f(X) = n \cdot \prod_{i=2}^{n} \frac{(X - \lambda_i I_n)}{\lambda_1 - \lambda_i}.$$

Here we use that $\lambda_i \neq \lambda_1$ for every $i \geq 2$, as $\lambda_2 < \lambda_1$. To verify that $f(A) = J$, it is enough to show that $f(A) \cdot v_k = Jv_k$ for all $k \in [n]$. Recall that $Jv_1 = nv_1$ (as $v_1 = 1_n$) and $Jv_k = 0$ for every $k \geq 2$ (as v_k is orthogonal to v_1). We will use the following simple fact, describing the eigenvalues of $f(A)$ for a polynomial A.

Claim 1. *Let $\alpha_0, \alpha_1, \ldots, \alpha_d \in \mathbb{R}$ and let $A' = \sum_{i=0}^{d} \alpha_i A^i$. If v is an eigenvector of A with eigenvalue λ, then $A' \cdot v = (\sum_{i=0}^{d} \alpha_i \lambda^i)v$. Thus, v is an eigenvector of A' with eigenvalue $\sum_{i=0}^{d} \alpha_i \lambda^i$.*

Proof. By induction on d (exercise!). □

We now complete the solution of the exercise. By the claim, for each $k = 1, \ldots, n$ we have

$$f(A) \cdot v_k = n \cdot \prod_{i=2}^{n} \frac{(A - \lambda_i I_n)v_k}{\lambda_1 - \lambda_i} = n \cdot \prod_{i=2}^{n} \frac{\lambda_k - \lambda_i}{\lambda_1 - \lambda_i} \cdot v_k$$

So if $k \geq 2$ then $f(A) \cdot v_k = 0 = Jv_k$, and if $k = 1$ then $f(A) \cdot v_k = n \cdot v_k = Jv_k$, as required.

Remark: We could have only used distinct eigenvalues of A. Namely, if $\lambda_1 = \eta_1 > \eta_2 > \cdots > \eta_s$ are all distinct eigenvalues of A, then the polynomial $f(X) = n \cdot \prod_{i=2}^{s} \frac{(X - \eta_i I_n)}{\eta_1 - \eta_i}$ also works. This potentially gives us a polynomial of lower degree (if A has repeated eigenvalues).

Problem 2(b): First assume that G is d-regular and connected. By Problem 2(b) of Assignment 8, $\lambda_1 = d$. and 1_n is an eigenvector of $\lambda_1 = d$ (see the solution to 2(b) from Assignment 8). Moreover, by Problem 1(c) of Assignment 8, we have $\lambda_2 < \lambda_1$ (because G is connected). This means that $\lambda_1 = d$ has multiplicity 1. Hence, if $v_1, \dots, v_n$ is an orthogonal basis of eigenvectors for A, then $v_1 \in \operatorname{span}\{1_n\}$ and v_i is orthogonal to v_1 (and hence to 1_n) for every $i \geq 2$. Thus, every eigenvector of A is also an eigenvector to of J. So the claim follows by 2(a).

Now suppose there is a polynomial f such that $f(A) = J$. By Claim 1 above, if v is an eigenvector of A, then it is also an eigenvector of $f(A) = J$. Thus, if $v_1, \dots, v_n$ is an orthogonal eigenbasis for A, then if $v_1, \dots, v_n$ is also an orthogonal eigenbasis for J. Hence, some v_i must be in $\operatorname{span}\{1_n\}$, meaning that 1_n is an eigenvector of A. In the solution of part (a), we showed that if 1_n is an eigenvector of A then G is d-regular where d is the eigenvalue of 1_n. It remains to show that G is connected. Suppose not, and let $X \subsetneq V(G)$ be a connected component. Then, the vector $\chi(X)$ (i.e., the vector x satisfying $x_i = 1$ of $i \in X$ and $x_i = 0$ otherwise) is an eigenvector of A with eigenvalue d. Indeed, this is because $G[X]$ is d-regular, as G is d-regular. However, $\chi(X) \notin \operatorname{span}\{1_n\}$ and $\chi(X)$ is not orthogonal to 1_n, so $\chi(X)$ is not an eigenvector of J, a contradiction.

Problem 3: By Example 2.22 in the lecture notes, the chacaters of $\Gamma = (\mathbb{Z}/2\mathbb{Z})^n$ are given by $\chi_x(y) = (-1)^{\langle x,y \rangle}$, where $x \in \Gamma$. By Lemma 2.24, the eigenvalues of $G = G(\Gamma, S)$ are

$$\lambda_x = \sum_{s \in S} \chi_x(s),$$

for $x \in \Gamma$. Now, as $S = \{e_1, \dots, e_n\}$, we have

$$\sum_{s \in S} \chi_x(s) = \sum_{i=1}^{n} \chi_x(e_i) = \sum_{i=1}^{n} (-1)^{\langle x, e_i \rangle} = \sum_{i=1}^{n} (-1)^{x_i}.$$

Put $k := \#\{i \in [n] : x_i = 1\}$; namely, $x \in \{0,1\}^n$ has k ones and $n-k$ zeroes. Then

$$\sum_{i=1}^{n} (-1)^{x_i} = -k + (n-k) = n - 2k.$$

The possible values of k are $k = 0, \dots, n$. We see that for each $0 \leq k \leq n$, G has eigenvalue $n - 2k$ with multiplicity $\binom{n}{k}$ (because $\binom{n}{k}$ is the number of ways to choose a vector x with k ones). As a sanity check, note that the sum of multiplicities is $\sum_{k=0}^{n} \binom{n}{k} = 2^n = |\Gamma|$.