Algebraic Methods in Combinatorics

Instructor: Benny Sudakov

Assignment 1

To be completed by September 25, 8pm

The solution of each problem should be no longer than one page!

Starred problems are typically harder. Don't worry if you cannot solve them.

Problem 1. Let q be a power of a prime number, and let $\mathcal{A}$ be a family of subsets of $[n]$ such that $|A|$ is *not* divisible by q for every $A \in \mathcal{A}$, and $|A \cap B|$ *is* divisible by q for every distinct $A, B \in \mathcal{A}$. Prove that $|\mathcal{A}| \leq n$.

Problem 2. Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of $[n]$.

(a) Suppose that $|A \cap B|$ is *odd* for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Prove that $|\mathcal{A}||\mathcal{B}| \leq 2^{n-1}$.

(b) How large can $|\mathcal{A}||\mathcal{B}|$ be if, instead, $|A \cap B|$ is *even* for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$?

Problem 3. Let $\mathcal{A}$ be a family of subsets of $[n]$ such that $|A|$ and $|A \cap B|$ are even for every $A, B \in \mathcal{A}$.

(a) Show that $|\mathcal{A}| \leq 2^{\lfloor n/2 \rfloor}$.

(b) Show that the bound $2^{\lfloor n/2 \rfloor}$ is best possible.

(c)* In fact, the restriction on the size of the elements does not change much.

Show that if $\mathcal{A}$ is a family of subsets of $[n]$ such that $|A \cap B|$ is even for every distinct $A, B \in \mathcal{A}$, then $|\mathcal{A}| \leq 2^{\lfloor n/2 \rfloor} + 1$.

Problem 4. Let $\mathcal{A}$ be a family of subsets of $[n]$. Suppose that every element in $[n]$ belongs to one of the sets in $\mathcal{A}$. Furthermore, suppose that $\mathcal{A}$ is *not* 2-*colourable*, i.e. for every 2-colouring of $[n]$, at least one of the sets in $\mathcal{A}$ is monochromatic; but $\mathcal{A} \setminus \{A\}$ is 2-colourable for every $A \in \mathcal{A}$. Prove that $|\mathcal{A}| \geq n$.