# Algebraic Methods in Combinatorics

## Solutions 11

*The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises.* ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

**Problem 1:** We will imitate the proof of Lemma 2.37. Let $v_1, \ldots, v_n$ be an orthonormal eigenbasis such that $Av_i = \lambda_i v_i$. Let us first show that $\lambda_k \geq \min_{\dim U = k-1} \max_{x \in U^\perp \setminus \{0\}} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$. To this end, take $U = \operatorname{span}\{v_1, \ldots, v_{k-1}\}$. Clearly, $U^\perp = \operatorname{span}\{v_k, \ldots, v_n\}$. Let $x \in U^\perp \setminus \{0\}$ be arbitrary. We may write $x := \sum_{i=k}^n \alpha_{i-k+1} v_i$ for some $\alpha = (\alpha_1, \ldots, \alpha_{n-k+1}) \in \mathbb{R}^{n-k+1} \setminus \{0\}$. Then, we derive that

$$\begin{aligned}
\max_{x \in U^\perp \setminus \{0\}} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} &= \max_{\alpha \in \mathbb{R}^{n-k+1} \setminus \{0\}} \frac{(\sum_{i=k}^n \alpha_{i-k+1} v_i)^T A \sum_{i=k}^n \alpha_{i-k+1} v_i}{(\sum_{i=k}^n \alpha_{i-k+1} v_i)^T \sum_{i=k}^n \alpha_{i-k+1} v_i} \\
&= \max_{\alpha \in \mathbb{R}^{n-k+1} \setminus \{0\}} \frac{\sum_{i=1}^{n-k+1} \lambda_{i+k-1} \alpha_i^2}{\sum_{i=1}^{n-k+1} \alpha_i^2}.
\end{aligned}$$

Recall that $\lambda_k \geq \cdots \geq \lambda_n$. It holds that $\max_{x \in U^\perp \setminus \{0\}} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} = \lambda_k$, which means

$$\lambda_k \geq \min_{\dim U = k-1} \max_{x \in U^\perp \setminus \{0\}} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}.$$

Next, we show that $\lambda_k \leq \min_{\dim U = k-1} \max_{x \in U^\perp \setminus \{0\}} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}$. Let $U$ be any $(k-1)$-dimensional subspace, and $V = \operatorname{span}\{v_1, \ldots, v_k\}$. Note that $\dim V + \dim U^\perp = k + n - k + 1 > n$. This means there exists $v \in \mathbb{R}^n \setminus \{0\}$ such that $v \in V \cap U^\perp$. Write $v = \sum_{i=1}^k \beta_i v_i$ for some $\beta = (\beta_1, \ldots, \beta_k) \in \mathbb{R}^k \setminus \{0\}$. Similar to the computation above,

$$\max_{x \in U^\perp \setminus \{0\}} \frac{\langle Ax, x\rangle}{\langle x, x\rangle} \geq \frac{\langle Av, v\rangle}{\langle v, v\rangle} = \frac{(\sum_{i=1}^k \beta_i v_i)^T A \sum_{i=1}^k \beta_i v_i}{(\sum_{i=1}^k \beta_i v_i)^T \sum_{i=1}^k \beta_i v_i} = \frac{\sum_{i=1}^k \lambda_i \beta_i^2}{\sum_{i=1}^k \beta_i^2} \geq \lambda_k.$$

Due to the arbitrariness of $U$, we acquire that

$$\lambda_k \leq \min_{\dim U = k-1} \max_{x \in U^\perp \setminus \{0\}} \frac{\langle Ax, x\rangle}{\langle x, x\rangle}.$$

Altogether, it holds that

$$\lambda_k = \min_{\dim U = k-1} \max_{x \in U^\perp \setminus \{0\}} \frac{\langle Ax, x \rangle}{\langle x, x \rangle}.$$

**Problem 2 (method 1):** We will imitate the proof of Theorem 2.38, using Problem 1. We may w.l.o.g. assume that $B$ is indexed by $[m]$ and $A$ by $[n]$ (so that $B$ is the $m \times m$ submatrix of $A$ in the upper left corner of $A$). Let $e_1, \ldots, e_n$ denote the standard basis vectors. For a vector $x \in \mathbb{R}^m$, let $x' \in \mathbb{R}^n$ be obtained by appending $n-m$ zeros to $x$. For $U \subseteq \mathbb{R}^m$, let $U' := \{x' \mid x \in U\}$, so $\dim U = \dim U'$. Putting $U_1 := \operatorname{span}\{v_{m+1}, \ldots, v_n\}$, it holds that $U' \perp U_1$ and $(U^\perp)' = (U' + U_1)^\perp$.

$$\begin{aligned}
\mu_i(B) &= \min_{\substack{U \subseteq \mathbb{R}^m \\ \dim U = i-1}} \max_{x \in U^\perp \setminus \{0\}} \frac{\langle Bx, x \rangle}{\langle x, x \rangle} \\
&= \min_{\substack{U \subseteq \mathbb{R}^m \\ \dim U = i-1 \\ U_2 := U' + U_1}} \max_{x \in U_2^\perp \setminus \{0\}} \frac{\langle Ax, x \rangle}{\langle x, x \rangle} \\
&\geq \min_{\dim U_2 = n-m+i-1} \max_{x \in U_2^\perp \setminus \{0\}} \frac{\langle Ax, x \rangle}{\langle x, x \rangle} = \lambda_{n-m+i}(A),
\end{aligned}$$

Where the inequality follows since we are taking a minimum over a larger set.

**Problem 2 (method 2):** Let $A' := -A$ and $B' := -B$. Clearly, $\mu_i(B) = -\mu_{m-i+1}(B')$ and $\lambda_i(A) = -\lambda_{n-i+1}(A')$, where $\mu_1(B') \geq \cdots \geq \mu_m(B')$ are the eigenvalues of $B'$ and $\lambda_1(A') \geq \cdots \geq \lambda_n(A')$ are the eigenvalues of $A'$. Now,

$$\mu_i(B) = -\mu_{m-i+1}(B') \geq -\lambda_{m-i+1}(A') = \lambda_{n-m+i}(A).$$

where the inequality is by the first half of Theorem 2.38 on $A'$ and $B'$.

**Problem 3:**

(a) Let $S$ be an independent set in $G$ of size $\alpha(G)$. It holds that $A_{S,S}$ is a zero-matrix, meaning $\lambda_{\alpha(G)}(A_{S,S}) = 0$. By Cauchy interlacing theorem, $\lambda_{\alpha(G)}(A) \geq \lambda_{\alpha(G)}(A_{S,S}) = 0$.

(b) Let $S$ be a clique in $G$ of size $\omega(G)$. It holds that $A_{S,S} = J_S - I_S$, where $J_S \in \mathbb{R}^{S \times S}$ is the all-ones matrix and $I_S \in \mathbb{R}^{S \times S}$ is the identity matrix. This means that $\lambda_{\omega(G)}(A_{S,S}) = -1$ if $|S| > 1$ and $\lambda_{\omega(G)}(A_{S,S}) = 0$ otherwise. In either case, by Cauchy interlacing theorem, $\lambda_{\omega(G)}(A) \geq \lambda_{\omega(G)}(A_{S,S}) \geq -1$.

(c) Here, we need to assume that $\omega(G) > 1$. Similar to (b), let $S$ be a clique in $G$ of size $\omega(G)$. As $A_{S,S} = J_S - I_S$, we know that $\lambda_2(A_{S,S}) = -1$. Thus, by Cauchy interlacing theorem, $\lambda_{n-\omega(G)+2}(A) \leq \lambda_2(A_{S,S}) = -1$.

(d) Let $S$ be a vertex cover of $G$ of size $\tau(G)$, and $T := [n] \setminus S$. By definition, $T$ is an independent set of $G$, i.e. $A_{T,T} = 0$. This means $\lambda_1(A_{T,T}) = 0$. By Cauchy interlacing theorem, $\lambda_{\tau(G)+1}(A) = \lambda_{n-|T|+1}(A) \leq \lambda_1(A_{T,T}) = 0$.

**Problem 4:** For any graph $H$, we use the following convention: $V(H)$ is the set of vertices and $E(H)$ is the set of edges, and $d(H)$ is the average degree.

Letting $\mathbb{1} \in \mathbb{R}^{V(G)}$ be the all-ones vector, it holds that $\lambda_1(G) \geq \frac{\langle A\mathbb{1},\mathbb{1}\rangle}{\langle \mathbb{1},\mathbb{1}\rangle} = \frac{2|E(G)|}{|V(G)|} = d(G)$. For any $S \subseteq V(G)$, denote $G[S]$ to be the induced subgraph of $G$, i.e. the subgraph with vertex set $S$ and edge set $\{e \in E(G) : e \in \binom{S}{2}\}$. Now, we apply Cauchy interlacing theorem on the adjacency matrices of $G$ and of $G[S]$ to derive $\lambda_1(G[S]) \leq \lambda_1(G)$. By the former discussion, $d(G[S]) \leq \lambda_1(G[S]) \leq \lambda_1(G)$.

We have proved that every induced subgraph of $G$ has average degree at most $\lambda_1 = \lambda_1(G)$. This means that every induced subgraph $H$ of $G$ has some vertex with degree in $H$ at most $\lambda_1$. Putting $n = |V(G)|$, we can iteratively take a vertex from $G$ to form a new graph, resulting in a permutation of vertices $v_1, \ldots, v_n$ such that for each $i \in [n]$, $v_i$ is incident to at most $\ell := \lfloor \lambda_1 \rfloor$ vertices among $\{v_{i+1}, \ldots, v_n\}$. Now, we greedily color the vertices in the order $v_n, v_{n-1}, \ldots, v_1$ with $(\ell + 1)$ colors: color $v_n$ arbitrarily; given the coloring of $v_{i+1}, \ldots, v_n$, note that $v_i$ has at most $\ell$ neighbors have been colored (among $v_{i+1}, \ldots, v_n$), so we can color $v_i$ so that its color differs from the colors of its neighbors among $v_{i+1}, \ldots, v_n$. In the end, we find a proper coloring of $G$ with at most $(\ell + 1)$ colors, i.e. $\chi(G) \leq \ell + 1 \leq \lambda_1 + 1$.