Algebraic Methods in Combinatorics

Solutions 6

The aim of the homework problems is to help you understand the theory better by actively using it to solve exercises. ***Do not read the solutions*** *before you believe you have solved the problems: it ruins your best way of preparing for the exam. The purpose of this write-up is merely to provide some guideline on how solutions should look like, and to help clean up hazy arguments. For hints, feel free to consult your teaching assistant.*

Problem 1: Suppose $\mathcal{F} = \{A_1, \dots, A_m\}$. For each $i \in [n]$, denote $B_i = \{j \in [m] : i \in A_j\}$. By assumption, $|B_i \cap B_{i'}| = 1$ for all $i \neq i'$. By Fisher inequality (Theorem 1.21) on $B_1, \dots, B_n$, we know that $n \leq m$. In other words, $|\mathcal{F}| \geq n$.

Problem 2(a): Let $L_i, R_i \subseteq V$ be the subsets of V forming the bipartition of B_i. Consider the adjacency matrix $A_i \in \mathbb{F}_2^{n\times n}$ of B_i. Note that the j^{th} row of B_i equals

$$\begin{cases} 0, & \text{if } j \notin L_i \cup R_i, \\ \chi(R_i), & \text{if } j \in L_i, \\ \chi(L_i), & \text{if } j \in R_i. \end{cases}$$

It is immediate that $r(A_i) \leq 2$ (actually it is exactly 2 unless B_i is empty). Since every edge appears in an odd number of B_i's, it follows that

$$\sum_{i=1}^{m} A_i = J - I.$$

The entries on the diagonal equal 0 because each A_i has only zero entries on the diagonal. Let us show that $r(J - I) \geq n - 1$ (recall that we are working in $\mathbb{F}_2$!) This follows easily as the eigenvalues of $J - I$ are $n - 1$ (which could equal 0 in $\mathbb{F}_2$) with multiplicity 1 and $-1 = 1$ with multiplicity $n - 1$. Hence, $r(J - I) \geq n - 1$.

Finally, $n - 1 \leq r(J - I) \leq \sum_{i=1}^{m} r(A_i) \leq 2m$. It follows that $m \geq (n - 1)/2$.

Problem 2(b): Similar to 2(a), let $L_i, R_i \subseteq V$ be the subsets of V forming the bipartition of B_i. Let matrix $A_i' \in \mathbb{R}^{n\times n}$ such that $A_i'(u, v) = 1$ if $u \in L_i, v \in R_i$ and $A_i'(u, v) = 0$ otherwise. Note that $A_i' + (A_i')^T = A_i$, defined in 2(a). Putting $A' := \sum_{i=1}^{m} A_i'$, it holds that $A' + (A')^T = J - I$.

Now, suppose for contradiction that $m \leq n-2$. Consider the following equations for $x \in \mathbb{R}^n$.

$$\sum_{j=1}^{n} x_j = 0, \quad \sum_{j \in R_i} x_j = 0, \ \forall i \in \{1, \ldots, m\}.$$

As there are only $(n-1)$ equations, there exists a non-zero solution $\tilde{x} \in \mathbb{R}^n$. Consider $\tilde{x}^T A' \tilde{x}$. On one hand, $\tilde{x}^T A' \tilde{x} = \tilde{x}^T (A')^T \tilde{x}$, so

$$2\tilde{x}^T A' \tilde{x} = \tilde{x}^T A' \tilde{x} + \tilde{x}^T (A')^T \tilde{x} = \tilde{x}^T (J - I) \tilde{x} = \left(\sum_{j=1}^{n} \tilde{x}_j \right)^2 - \sum_{j=1}^{n} \tilde{x}_j^2 = - \sum_{j=1}^{n} \tilde{x}_j^2 \neq 0.$$

On the other hand, for each $i \in \{1, \ldots, m\}$, we have that

$$\tilde{x}^T A_i \tilde{x} = \sum_{j \in L_i, k \in R_i} \tilde{x}_j \tilde{x}_k = \sum_{j \in L_i} \tilde{x}_j \sum_{k \in R_i} \tilde{x}_k = 0.$$

Thus, $\tilde{x}^T A' \tilde{x} = \sum_{i=1}^{m} \tilde{x}^T A_i' \tilde{x} = 0$, contradicting $\tilde{x}^T A' \tilde{x} \neq 0$. Therefore, $m \geq n-1$.

Problem 3:

Let $\mathcal{I}_2 = \{i : 2 \nmid |A_i|\}$ and $\mathcal{I}_3 = \{i : 3 \nmid |A_i|\}$. Since none of the $|A_i|$ is divisible by 6, it follows that $|\mathcal{I}_2| + |\mathcal{I}_3| \geq m$. By the oddtown theorem, $|\mathcal{I}_2| \leq n$. The same argument shows that $|\mathcal{I}_3| \leq n$. Indeed, let $v_i \in \mathbb{F}_3^n$ be the characteristic vector of A_i. Then, for any $i, j \in \mathcal{I}_3$, we have

$$v_i \cdot v_j \begin{cases} = 0 & \text{if } i \neq j, \ (\text{as } 6 \mid |A_i \cap A_j|), \\ \neq 0 & \text{if } i = j, \ \text{as } 3 \nmid |A_i|. \end{cases}$$

Let us show the vectors $v_i, i \in \mathcal{I}$, are linearly independent. Indeed, suppose $\sum_{i \in \mathcal{I}} \alpha_i v_i = 0$. Taking the scalar product with v_j, we get $\alpha_j = 0$, for any j. Hence, $v_i, i \in \mathcal{I}_3$ are linearly independent implying $|\mathcal{I}_3| \leq n$. Hence, $m \leq 2n$.

Problem 4: Let $A = \{a_1, \ldots, a_r\}$ be an arbitrary set in $\mathcal{A}$. Suppose the intersection of all sets in $\mathcal{A}$ is empty. Then for every $i \in [r]$, there exists a set $A_i \in \mathcal{A}$ such that $a_i \notin A_i$. But then $A \cap A_1 \cap A_2 \ldots A_r = \emptyset$, a contradiction.