Algebraic Methods in Combinatorics

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Assignment 13

To be completed by December 18

Problem 1. For any prime p, any graph $G = (V, E)$ with average degree strictly larger than $2p - 2$ and maximum degree at most $2p - 1$ contains a p-regular subgraph.

Problem 2. Let $x_1, \dots, x_n \in \mathbb{Z}_n$. Show that there is a non-empty set $I \subseteq [n]$ such that $\sum_{i \in I} x_i = 0$.

Problem 3. Let p be prime, and let $v_i = (a_i, b_i)^T$, where $a_i, b_i \in \mathbb{F}_p$ and $i \in [3p]$, be such that $\sum_{i \in [3p]} v_i = 0$. Show that there is a set $I \subset [3p]$ of size p such that $\sum_{i \in I} v_i = 0$.

Problem 4. Let A and B be non-empty sets in $\mathbb{F}_p$ (where p is prime), and let

$$X = \{a + b : a \in A,\, b \in B,\, ab \neq 1\}.$$

Show that $|X| \geq \min\{|A| + |B| - 3, p\}$.

Problem 5.

(a) Let $x_1, \dots, x_n$ be reals. The *Vandermonde* matrix is an $n \times n$ matrix M defined by $M_{i,j} = x_i^{j-1}$, i.e.

$$M = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{pmatrix}$$

Show that $\det(M) = \prod_{i>j}(x_i - x_j)$.

(b) Let $p \geq 3$ be prime, and let A and B be subsets of $\mathbb{F}_p$, each of size n. Prove that there exist orderings $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$ of A and B, respectively, such that the sums $a_i + b_i$ are distinct for $i \in [n]$.