Algebraic Methods in Combinatorics

Exam — Solutions

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Problem 1 Let $q = p_1 \dots p_k$ where all the p_i are distinct primes, and let $n \geq 1$ be an integer. Let $\mathcal{A}$ be a family of subsets of $[n]$ such that the size of every set is not divisible by q, while q divides $|A_i \cap A_j|$ for all distinct $A_i, A_j \in \mathcal{A}$. Show that for every such family $\mathcal{A}$ we have $|\mathcal{A}| \leq kn$.

Solution. For $i = 1, \dots, k$, let $\mathcal{A}_i \subset \mathcal{A}$ be the family of sets that are not divisible by p_i. Then $\bigcup_{i=1}^k \mathcal{A}_i = \mathcal{A}$. Therefore, it is enough to show that $|\mathcal{A}_i| \leq n$.

For $A \in \mathcal{A}_i$, let v_A be the characteristic vector of A over $\mathbb{F}_{p_i}$. Then for every $A, B \in \mathcal{A}_i$, we have $\langle v_A, v_B \rangle = 0$ if $A \neq B$, and $\langle v_A, v_A \rangle \neq 0$. Therefore $\{v_A : A \in \mathcal{A}_i\}$ is a system of linearly independent vectors in $\mathbb{F}_{p_i}^n$. Indeed, if $\sum_{A \in \mathcal{A}_i} \lambda_A v_A = 0$, then taking the scalar product of both sides of the equation with any v_A, we get $\lambda_A = 0$. This shows that $|\mathcal{A}_i| \leq n$.

Problem 2 Suppose the Zurich sport organisation ASVZ has n members and we define a *team* as a subset of $[n]$. Let $\mathcal{A}, \mathcal{B}$ be collections of teams such that $|\mathcal{A}| + |\mathcal{B}| < n$. Suppose there is a subset $S \subseteq [n]$ such that every team in $\mathcal{A}$ contains an even number of members in S, while every every team in $\mathcal{B}$ contains an odd number of members in S. Show that there exists another subset $S' \neq S$ of $[n]$ with this property.

Solution. For $A \subset [n]$, let v_A denote the characteristic vector of A over $\mathbb{F}_2$. As $|\mathcal{A}| + |\mathcal{B}| < n$, there exists a vector $w \in \mathbb{F}_2^n$ such that $w \neq 0$ and $\langle w, v_A \rangle = 0$ for every $A \in \mathcal{A} \cup \mathcal{B}$.

Note that the conditions of the problem are equivalent to $\langle v_A, v_S \rangle = 0$ for every $A \in \mathcal{A}$, and $\langle v_B, v_S \rangle = 1$ for every $B \in \mathcal{B}$. But then v_S replaced with $v_S + w$ also satisfies these equalities. The vector $v_S + w$ is the characteristic vector of some $S' \subset [n]$, which then satisfies the desired conditions.

Problem 3 Let G be a d-regular graph on n vertices with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_n$ and let $\lambda := \max(|\lambda_2|, |\lambda_n|)$. Show that

$$\lambda \geq \sqrt{d\left(1 - \frac{d-1}{n-1}\right)}$$

Solution. Let A be the adjacency matrix of G, and note that $\lambda_1 = d$. Consider A^2. Every entry on the diagonal of A^2 is d, so $\mathrm{tr}(A^2) = dn$. On the other hand, the eigenvalues of A^2 are $\lambda_1^2, \dots, \lambda_n^2$, and therefore $\operatorname{tr}(A^2) = \sum_{i=1}^n \lambda_i^2$. Hence, $dn = \sum_{i=1}^n \lambda_i^2$, and $dn - d^2 = \sum_{i=2}^n \lambda_i^2$. If $\lambda = \max\{|\lambda_2|, |\lambda_n|\}$, then the previous implies that

$$dn - d^2 \leq (n-1)\lambda^2.$$

From this, we get the desired inequality $\lambda \geq \sqrt{d(1 - \frac{d-1}{n-1})}$.

Problem 4 Let n points be given in the plane such that each three of them can be enclosed in a circle of radius 1. Prove that all n points can be enclosed in a circle of radius 1.

Solution. Let P be the set of points, and for each $p \in P$, let C_p be the disk of radius 1 with center p. Also, let $\mathcal{C} = \{C_p : p \in P\}$. By the conditions of the problem, any three members of $\mathcal{C}$ have a nonempty intersection. As the elements of $\mathcal{C}$ are convex, we can use Helly's theorem to conclude that there is a point x contained in the intersection of all elements of $\mathcal{C}$. The circle of radius 1 with center x encloses P.

Problem 5 Let G be a graph with minimum degree δ, and with maximum eigenvalue λ_1. Let $\chi(G)$ be the number of colors needed to properly color the vertices of G.
(a) Show that $\delta \leq \lambda_1$.
(b) Show that $\chi(G) \leq \lambda_1 + 1$.

Solution. (a) Let A be the adjacency matrix of G. For every $v \in \mathbb{R}^n$, we have $||Av||_2 \leq \lambda_1 ||v||_2$. Apply this inequality in case v is the all 1 vector. The left hand side is

$$||Av||_2^2 = \left(\sum_{i=1}^n \deg(i)^2\right)^{1/2} \geq \sqrt{n}\delta,$$

while the right hand side is $\sqrt{n}\lambda_1$. From this, we get $\lambda_1 \geq \delta$.

(b) Let G' be an induced subgraph of G, and let A' be the corresponding adjacency matrix, λ_1' be the maximum eigenvalue, and δ' be the minimum degree. By the Cauchy interlacing theorem, we have $\lambda_1' \leq \lambda_1$, so $\delta' \leq \lambda_1$ by (a). As this is true for every induced subgraph G', we get that G is λ_1-degenerate. Therefore, $\chi(G) \leq \lambda_1 + 1$.

Problem 6 Let G be a graph and for each vertex v, specify a set $B(v) \subseteq \{1, \ldots, d_G(v)\}$, where $d_G(v)$ denotes the degree of v in G. Show that if $\sum_{v \in G} |B(v)| < e(G)$ where $e(G)$ is the number of edges of G, then G has a nontrivial subgraph H such that $d_H(v) \notin B(v)$ for every vertex $v \in G$.

Solution. Define the polynomial $g : \mathbb{R}^{E(G)} \to \mathbb{R}$ such that for every $x \in \mathbb{R}^{E(G)}$, we have

$$g(x) = \prod_{v \in V(G)} \prod_{b \in B(v)} \left(-b + \sum_{e \in E(G): v \in e} x(e) \right).$$

Note that the total degree of g is $\sum_{v \in V(G)} |B(v)| < e(G)$. Also, we have $g(0) \neq 0$. Let us further define the polynomial $f : \mathbb{R}^{E(G)} \to \mathbb{R}$ as follows:

$$f(x) = g(x) - g(0) \cdot \prod_{e \in E(G)} (1 - x(e)).$$

Clearly, we have $f(0) = 0$ and $f(x) = g(x)$ if $x \in \{0,1\}^{E(G)}$, $x \neq 0$. Furthermore, the degree of f is $e(G)$, and the coefficient of the term $\prod_{e \in E(G)} x_e$ is nonzero. Therefore, by the Combinatorial Nullstellensatz, there exists a vector $z \in \{0,1\}^{E(G)}$ such that $f(z) \neq 0$. Define the subgraph H of G such that we include an edge e in H if and only if $z_e = 1$. Then H is nonempty as $z \neq 0$. Also, for each $v \in V(G)$, we have $d_H(v) = \sum_{e \in E(G): v \in e} z_e \notin B(v)$. Therefore, H suffices.