Algebraic Methods in Combinatorics

Instructor: Benny Sudakov

Assignment 7

To be completed by November 6th, 14:00

The solution of each problem should be no longer than one page!

Problem 1. Let $m \geq (n+1)(r-1)+1$, and let $A_1, \ldots, A_m$ be subsets of $[n]$. Show that there exist pairwise disjoint sets $I_1, \ldots, I_r \subseteq [m]$ such that the unions $\cup_{j \in I_i} A_j$ are the same for all $i \in [r]$, and similarly the intersections $\cap_{j \in I_i} A_j$ are the same for all $i \in [r]$.

Problem 2.

Let X be a subset of $\mathbb{R}^d$ of size n. An X-simplex is the convex hull of some $d+1$ points of X. Show that there is a point in $\mathbb{R}^d$ which is contained in at least $c_d \binom{n}{d+1}$ X-simplicies, where c_d is a positive constant that depends only on d.

Problem 3. Let $X \subset \mathbb{R}^d$ be finite and let $0 < \varepsilon < 1$. A set Y is a *weak ε-net for X* (with respect to convex sets) if for every convex set B such that $|B \cap X| \geq \varepsilon |X|$, B intersects Y.

Show (using the previous assignment) that there is a weak ε-net for X of size at most $C_{d,\varepsilon}$, where $C_{d,\varepsilon}$ is a constant that depends only on d and ε.

Problem 4. Let $F_1, \ldots, F_{d+1}$ be sets consisting of 2 points in $\mathbb{R}^d$ each. Show that it is possible to split each F_i into a_i and b_i in so that $conv(a_1, \ldots, a_{d+1}) \cap conv(b_1, \ldots, b_{d+1}) \neq \emptyset$.