Algebraic Methods in Combinatorics

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Assignment 5

To be completed by October 23

Problem 1. Let $n \geq 2k$. Find an explicit k-colouring of the edges of $K_{\binom{n}{2k-1}}$ such that there is no monochromatic K_{n+1} (recall that K_m is the complete graph on m vertices).

Problem 2. Let $\mathbf{f} = (f_1, \ldots, f_m)$ be a sequence of polynomials in variables $x_1, \ldots, x_n$ over a field $\mathbb{F}$. The *zero pattern* of $\mathbf{f}$ at $c \in \mathbb{F}^n$ is the sequence $(\sigma_1, \ldots, \sigma_m)$ where $\sigma_i = 0$ if $f_i(c) = 0$ and otherwise $\sigma_i = *$. Let $Z(\mathbf{f})$ be the set of zero patterns of $\mathbf{f}$ at c, where c ranges over $c \in \mathbb{F}^n$, and denote the size of the set $Z(\mathbf{f})$ by $z(\mathbf{f})$.

For example, let $\mathbf{f} = (f_1, f_2)$, where $f_1(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 1)^2$ and $f_2(x_1, x_2) = x_1 + x_2 - 1$. Then the zero pattern of $\mathbf{f}$ at $(1, 2)$ is $(0, *)$, at $(1/2, 1/2)$ it is $(*, 0)$ and at $(3, 4)$ it is $(*, *)$. The pattern $(0, 0)$ is not a zero pattern of any point in $\mathbb{R}^2$. Hence $Z(\mathbf{f}) = \{(0, *), (*, 0), (*, *)\}$ and $z(\mathbf{f}) = 3$.

In this exercise our goal is to give upper bounds on $z(\mathbf{f})$ in term of the degrees of the polynomials in $\mathbf{f}$.

(a) Show that the number of zero patterns is at most 2^m.

If $m \leq n$, show that there is a choice of *linear* polynomials $f_1, \ldots, f_m$ in n variables such that $z(\mathbf{f}) = 2^m$, where $\mathbf{f} = (f_1, \ldots, f_m)$.

From now on, we assume that $m \geq n$. Let d_i be the degree of f_i.

(b) Show that $z(\mathbf{f}) \leq \binom{n + \sum_{i=1}^m d_i}{n}$.

The *support* of a zero pattern $(\sigma_1, \ldots, \sigma_m)$ is the set of i's such that $\sigma_i = *$. For the remaining parts, suppose that $d_i \leq d$ for every $i = 1, \ldots, m$.

(c) Show that the number of zero patterns of $\mathbf{f}$ whose support has size at most $m - n$ is at most $\binom{md-(d-1)n}{n}$.

(d) Show that the number of zero patterns whose support has size larger than $m - n$ is at most $\sum_{i=0}^{n-1} \binom{m}{i}$.

(e) Deduce that $z(\mathbf{f}) \leq \binom{md-(d-2)n}{n}$.