Algebraic Methods in Combinatorics

Instructor: Benny Sudakov

Assignment 9

To be completed by November 20th, 14:00

Problem 1. Let G be a graph and let A be its adjacency matrix. Show that the (i, j)-th entry in A^k is the number of walks of length k from vertex i to vertex j.

Problem 2. Let G be a graph on n vertices; denote its eigenvalues by $\lambda_1 \geq \ldots \geq \lambda_n$. Prove the following statements.

(a) $\lambda_1 \geq -\lambda_n$.

(b) Suppose that G is connected. Then G is bipartite if and only if $\lambda_1 = -\lambda_n$.

(c) G is bipartite if and only if its spectrum is *symmetric*, i.e. $\lambda_i = -\lambda_{n+1-i}$ for every $i \in [n]$. (Ideally, give two proofs of the "if" direction: one using induction on the number of components, and one using Problem 1.)

Problem 3. Let G be a k-regular graph in which any two adjacent vertices have a unique neighbour and any two non-adjacent vertices have exactly 2 neighbours. Show that:

(a) G has $1 + k^2/2$ vertices.

(b) There are at most 6 different values that k can take.