Small-dimensional Linear Programming and Convex Hulls Made Easy

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April 3rd 2025

Background

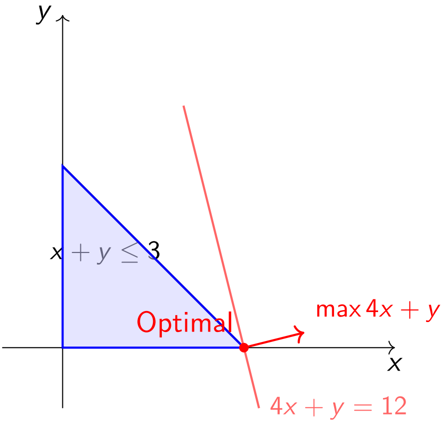
Linear programming captures one of the most canonical and influential constrained optimization problems.

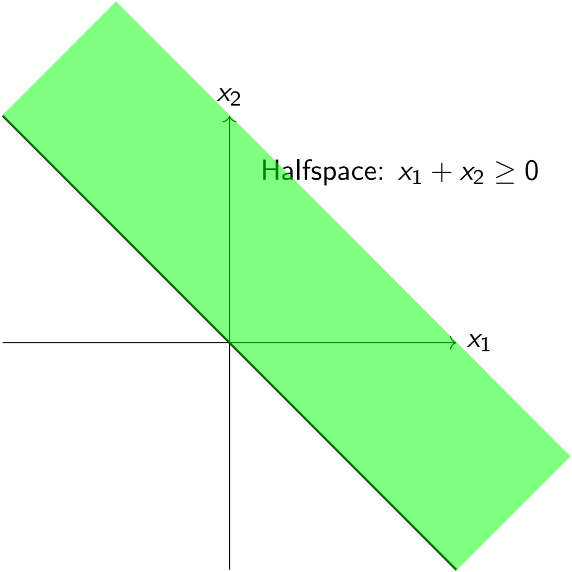
More precisely, it asks to maximize or minimize a linear objective under linear inequality and equality constraints.

$$
\begin{aligned}
\max / \min \;\; & c^T x \\
\text{subject to} \;\; & Ax \le e \\
& Bx \ge f \\
& Cx = g
\end{aligned}
$$

Background - Geometrical Interpretation

Every constraint corresponds to a hyperplane (a halfspace, precisely). The intersection of several halfspaces gives us a polyhedron.





Hyperplane: $x_1 + x_2 = 0$

Background - Simplex Method

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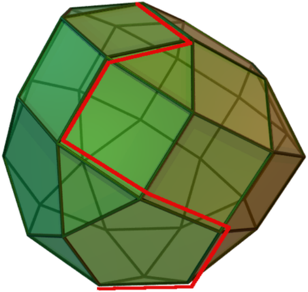


Background - Simplex Method Running Time

Algorithm	**Complexity**	**Speed**	**Path**
Simplex	Exponential	Fast	Edges of polyhedron
Ellipsoid	Polynomial	Slow	Ellipsoid shrinking
Interior Point	Polynomial	Fast	Through the interior

Table: Comparison of Famous LP Algorithms

Previous Work

When d, the dimension of LP (i.e. the number of variables), is considered to be constant, then LP can be solved in time linear in m, the number of constraints.

These methods[2] are complicated and running time superexponential to d: e.g. 3^{d^2}.

We also have a simple algorithm[1] with complexity $O(d^2 m) + (\log m) O(d)^{d/2+O(1)} + O(d^4 \sqrt{m} \log m)$ but the analysis is involved.

The presented algorithm gives a simple procedure and complexity $O(d!m)$.

Algorithm - Sketch

This algorithm makes use of a very simple idea:

Suppose we delete one constraint $\vec{a}x \geq b$, and somehow managed to solve the new LP. Then we check if the optimum x^* conforms with the deleted constraint.

- If it doesn't violate the constraint, i.e. $\vec{a}x^* \geq b$, then x^* is also the optimum for original LP, because deleting one constraint gives us a larger feasible area.
- If it violates the constraint, then the opt value lies on the hyperplane corresponding to this constraint. We can project the feasible area onto the corresponding hyperplane, thus reducing dimension by 1. Then solve the new LP problem with lower dimension.

Algorithm - Correctness 1

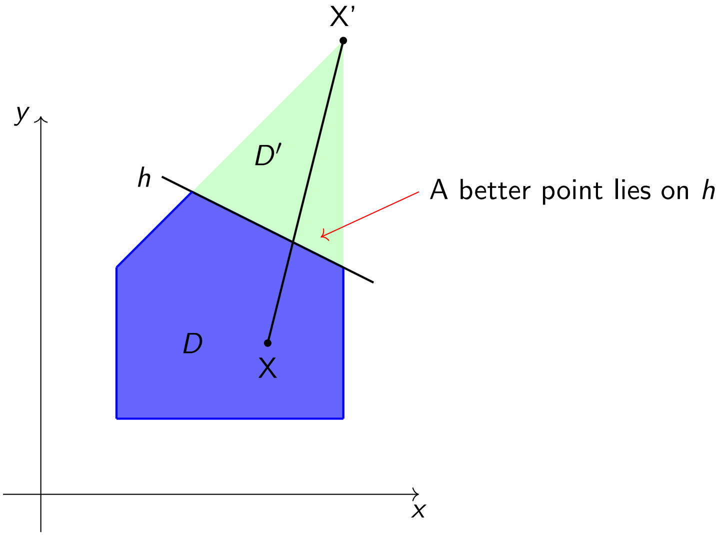
Let original feasible region be D, feasible region after deleting constraint be D', the optimum for D' be x^* . Clearly $D \subseteq D'$ and in this case $x^* \in D$.

$$u^T x^* \underset{\text{(i)}}{\leq} \max_D \underset{\text{(ii)}}{\leq} \max_{D'} = u^T x^*$$

(i): $x^* \in D$, by definition of max.

(ii): Optimizing on a larger area gives no worse optimal value than the original one.

Algorithm - Correctness 2



Algorithm - Further Problems

Some problems left to deal with:

- What if after deletion of bounding hyperplane the new problem become unbounded?
- When to stop the recurssion?
- What's next when we reduce the dimension?

Algorithm - Unboundedness and Recurssion

The first two problems can be solved by adding a bounding box, i.e. explicit lower and upper bounds for variables $-\alpha \leq x_i \leq \alpha$ for all d variables x_i.

Recurssion stops when all original hyperplanes are deleted and it's impossible to have unbounded case.

Algorithm - Reducing Dimension

Denote the set of m halfspaces as $\mathcal{H}$ and the hyperplane corresponding to the deleted halfspace as h. Let u be the optimizing direction and v stands for the optimum point for original LP problem.

If $\bar{u}$ is the projection of u on h and $\bar{\mathcal{H}} := \{H \bigcap h \mid H \in \mathcal{H}\}$, then v is exactly the optimum point for LP problem with constraints $\bar{\mathcal{H}}$ and optimizing direction $\bar{u}$.

Algorithm - Time Complexity

When $d = 1$, problem can be solved in $O(m)$.

When $m = 1$, problem can be solved in $O(d)$.

The interesting part is when we delete a constraint and the optimum point changes. Since original optimum point x has exactly d tight constraints (i.e. x is on exactly d hyperplanes) then deleting a constraint uniformly randomly ensures the probability of x' differs from x no more than d/m.

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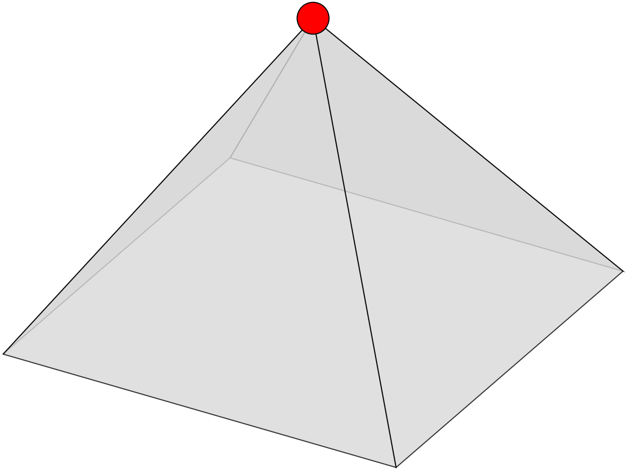


Algorithm - Time Complexity

Even if there are $k > d$ constraints are tight at x, the optimum point for original LP, at most d of them has the property that deleting it leads a better optimum.

$$
\begin{aligned}
\exists v,\ \ a_1^T v &> 0 \\
a_2^T v &\leq 0 \\
&\cdots \\
a_k^T v &\leq 0
\end{aligned}
$$

Algorithm - Time Complexity

$$T(d,m) \leq \begin{cases} O(m) & \text{if } d = 1, \\ O(d) & \text{if } m = 1, \\ T(d,m-1) + O(d) + \dfrac{d}{m}O(dm) + & \\ \dfrac{d}{m}T(d-1,m-1) & \text{otherwise.} \end{cases}$$

Expand everything and we get:

$$T(d,m) = O(\sum_{1 \leq i \leq d} \frac{i^2}{i!} d!m) = O(d!m)$$

Convex Hull

For a point set S on $\mathbb{R}^d$, assuming general position (no $d+1$ points lie in a common hyperplane), we consider construction of convex hull P of point set S.

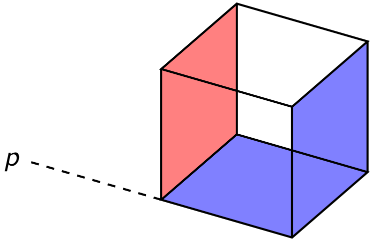
Since each facet can be uniquely determined by the consisting vertices, by using "construction of convex hull" we really mean finding all facets, represented as a d-tuple of vertices.

Terminology

Let p be some point in $\mathbb{R}^d$ in general position with respect to S.

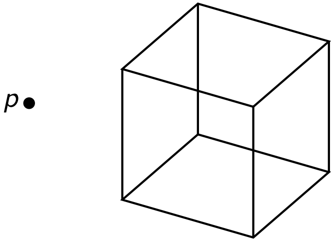
We call a facet F of P *visible* from p iff the hyperplane spanned by F seperates P and p; a facet *obscured* otherwise.

We call a face G *visible* from p iff it's only containted in facets of P that are visible from p. *Obscured* faces are defined analogously. We call G a face *horizon* iff it's contained in some visible and some obscured facet.



Visible

Obscured



Visible

Obscured

Convex Hull Sketch

Consider an incremental algorithm, from original convex hull P to construct $P' = conv(P \bigcup x)$ where x is the new point, and we have the following observations:

- no visible face of P is a face of P';
- all obsecured and horizon faces of P are faces of P';
- for each horizon face G of P, the pyramid $conv(G \bigcup x)$ is a face of P';
- this yields all faces of P'.

Convex Hull Sketch

1. Locate a facet F of P visible from x. If no such F exists then $P' = P$.
2. Determine set of facets and ridges of P visible from x and horizon ridges. Delete all visible facets and ridges.
3. For each horizon ridge G, generate new facet $conv(G \bigcup \{x\})$ (a new node for facet graph).
4. Generate new edges (i.e. ridges of P') for new node in facet graph.

Convex Hull - Time Complexity

Step 1: Locate a facet F of P visible from x. If no such F exists then $P' = P$.

This can be done by linear programming with $d + 1$ variables indicating parameters of hyperplane and each vertex gives a seperating constraint (the target hyperplane should seperate all vertices and x). The objective function is the max distance of vertices and target hyperplane.

If such hyperplane exists, then it must touch some /textitvisible vertex v. Then we iterate all facets containing v, among which we can find a seperating facet.

According to the LP algorithm, **this can be done in $O(i)$ time in i-th iteration**.

Convex Hull - Time Complexity

Step 2: Find all visible facets and ridges, then delete them.

This can be done by Depth-First-Search from the facet determined in step 1. **The time cost is proportional to number of visible facets**, which can be accounted into the creation step of these facets, amortizedly speaking.

Convex Hull - Time Complexity

Step 3: Generate new facet for each horizon ridge.

All horizon ridges can found in step 2 and adding new nodes in facet graph **cost clearly time proportional to number of new facets** N_x.

Convex Hull - Time Complexity

Step 4: Generate new edges in facet graph.

Here we need to pair up facets (i.e. nodes in facet graph) to add new edges. For each new facet, we can iterate all ridges (represented as $d-1$-tuple of vertices index). Then by Radix Sort or Trie Tree we can simply identify the corresponding old facet of each ridge.

This can be done in $O(i + N_x)$ time.

Convex Hull - Time Complexity

Put points of S in random order $p_1, \cdots, p_n$.

For i-th iteration, we compute P_i out of P_{i-1} and the running time is $O(i + N_i)$, where N_i is the number of of facets constaining p_i in P_i.

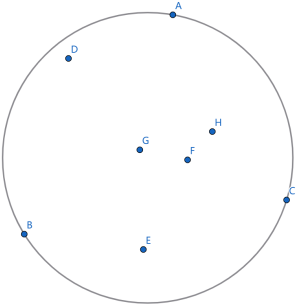
There are at most $i^{\lfloor \frac{d}{2} \rfloor}$ facets and the probability that a vertex is contained in a facet is d/i. So the expected running time for this iteration is $i^{\lfloor \frac{d}{2} \rfloor - 1}$, leading to total complexity of $n^{\lfloor \frac{d}{2} \rfloor}$.

Backward Analysis

Sometimes it's hard to determine the probability if thinking *forward*, while surprisingly straightforward if thinking *backward*.

Smallest Enclosing Circle

Given a set of points P in Euclidean plane, compute the smallest circle that contains them all.

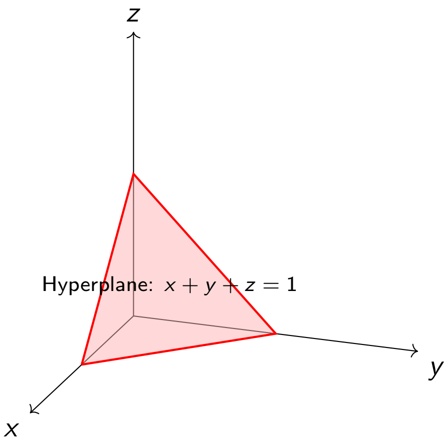


Emo Welzl gave an efficient random algorithm with surprisingly simple procedure:

```
def Smallest_Enclosing_Circle(P, R):
    ```input : P and R are finite sets of points in the plane, |R| <= 3
    ```output: Smallest circle enclosing P with R on the boundary

    if P is empty or |R| == 3 then:
        return trivial(R)

    p = random_select(P)
    D = Smallest_Enclosing_Circle(P - {p}, R)
    if p in D then:
        return D
    else:
        return Smallest_Enclosing_Circle(P - {p}, R + {p})
```


z
Hyperplane: $x + y + z = 1$
y
x

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