Small-dimensional Linear Programming and Convex Hulls Made Easy

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Background

Linear programming captures one of the most canonical and influential constrained optimization problems.

More precisely, it asks to maximize or minimize a linear objective under linear inequality and equality constraints.

$$\begin{aligned} \max / \min \quad & c^T x \\ \text{subject to} \quad & Ax \leq e \\ & Bx \geq f \\ & Cx = g \end{aligned}$$

Background - Geometrical Interpretation

Every constraint corresponds to a hyperplane (a halfspace, precisely).
The intersection of several halfspaces gives us a polyhedron.
We tend to optimize over this feasible area.

Background - Simplex Method

Fact

Optimal solution can always be achieved by some vertex.

Fact

If there exists any vertex with better objective value than current vertex, we always have a neighbour vertex with better value.

Firstly select any starting vertex.

Each step in Simplex Method move from one vertex to its neighbour with larger objective.

Background - Simplex Method

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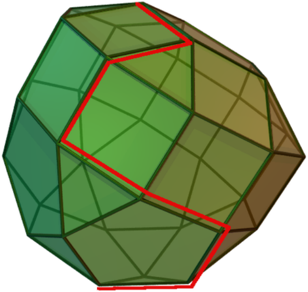


Background - Simplex Method Running Time

Add here the complexity and running time tables for three methods.

The three most famous algorithms for LP: Interior Method, Ellipsoid Method, Simplex Method.
Even though we have Interior Method and Ellipsoid Method, the most widely used algorithm is still Simplex Method due to its efficiency in practice.

Previous Work

there are some previous work showing that if d, the dimension or i.e. the number of variables in LP, is considered as a constant, then LP can be solved in time linear to m, the number of constraints. However these methods are: complicated; and running time superexponential to d: e.g. 3^{d^2}.
we also have a simple algorithm with complexity $O(d^2 m) + (\log m) O(d)^{d/2+O(1)} + O(d^4 \sqrt{m} \log m)$ but the analysis is involved.
The presented algorithm gives a simple procedure and complexity $O(d! m)$.

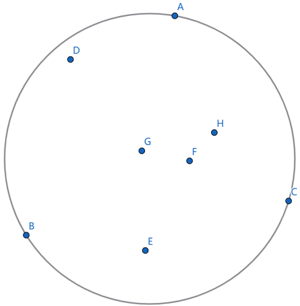
We were talking about other algorithms for LP but what do these have to do with this paper?
and we talked so much about simplex method.
But unfortunately it has nothing to do with what we are going to present.
This algorithm makes use of a very simple idea:
Suppose we delete one constraint, and somehow managed to solve the new LP. Then we check if the opt position conforms with the deleted constraint.
If it doesn't violate the constraint, then we are done. (this is obvious, since we have a larger feasible area, we certainly have better opt value)
If it violates the constraint, then the opt value lies on the hyperplane corresponding to this constraint. This is because if we consider the points along the line connecting new LP's opt point and original LP's opt point, by convexity we have that we have a better point on the hyperplane. This is also not bad, at least we decrease the dimension (we restrict the problem on a hyperplane, which has dimension $d-1$), which gives us the intuition to

Backward Analysis

Sometimes it's hard to determine the probability if thinking *forward*, while surprisingly straightforward if thinking *backward*.

Smallest Enclosing Circle

Given a set of points P in Euclidean plane, compute the smallest circle that contains them all.



Backward Analysis - Smallest Enclosing Circle

Emo Welzl gave an efficient random algorithm with surprisingly simple procedure:

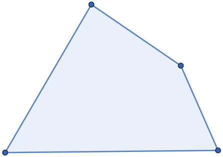
```
def Smallest_Enclosing_Circle(P, R):
    ```input : P and R are finite sets of points in the plane, |R| <= 3
    ```output: Smallest circle enclosing P with R on the boundary

    if P is empty or |R| == 3 then:
        return trivial(R)

    p = random_select(P)
    D = Smallest_Enclosing_Circle(P - {p}, R)
    if p in D then:
        return D
    else:
        return Smallest_Enclosing_Circle(P - {p}, R + {p})
```

General Problem

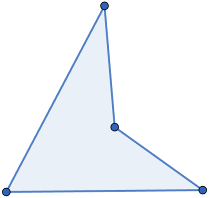
Can we find for a given k a number $g(k)$ such that: from any set containing at least $g(k)$ points it is possible to select k points forming a convex polygon[1]? Such convex polygon is denoted as convex k-gon.

Theorem

$g(k)$ exists for every $k \geq 3$ and $g(k) \leq \binom{2k-4}{k-2} + 1$.



(a) YES, convex 4-gon



(b) NO, not convex

Figure: 4-gons

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Lower Bound

$g(k) \geq 2^{k-2} + 1$.

Conjecture

$g(k) = 2^{k-2} + 1$.

[1] Erdős, Szekeres: A combinatorial problem in geometry. Compositio Math. (1935)

Improvements

Upper Bound	Author	Year
$\binom{2k-4}{k-2}+1$	Erdős, Szekeres	1934
$\binom{2k-4}{k-2}$	Chung, Graham[2]	1998
$2^{k+O(k^{2/3}\log k)}$	Suk[3]	2017
$2^{k+O(\sqrt{k\log k})}$	Holmesen et al.[4]	2020

Table: Upper Bound

Asymptotically, $\Theta(2^{k+o(k)})$ is tight.

[2] Chung, Graham: Forced convex n-gons in the plane. Discrete Comput. Geom. (1998)

[3] Suk: On the Erdős-Szekeres convex polygon problem. J. Am. Math. Soc. (2017)

[4] Holmsen, Mojarrad, Pach, Tardos: Two extensions of the Erdős-Szekeres problem. JEMS (2020)

Improvements

For small k, some results showed the conjectured bound is tight.

- When $k = 4$, we already see the proof $g(4) = 5$.
- When $k = 5$, Kalbfleisch et al. first gave a proof[5] $g(5) = 9$.
- When $k = 6$, Szekeres and Peters[6] first gave a computer-assisted proof $g(6) = 17$ using 3000 GHz Hour, while Maríc[7] improved it to 1GHz Hour using automatic formal proof.

[5] Kalbfleisch, J.D., Kalbfleisch, J.G., Stanton, R.G.: A combinatorial problem on convex n-gons. In: Proceedings of Louisiana Conference on Combinational Graph Theory Computing, Louisiana State University, Baton Rouge (1970)

[6] Szekeres, G., Peters, L.: Computer solution to the 17-point Erd "os–Szekeres problem. ANZIAM J. 48(2), 151–164 (2006)

[7] Marić, Filip. "Fast formal proof of the Erdős–Szekeres conjecture for convex polygons with at most 6 points." Journal of Automated Reasoning 62.3 (2019)

Improvements

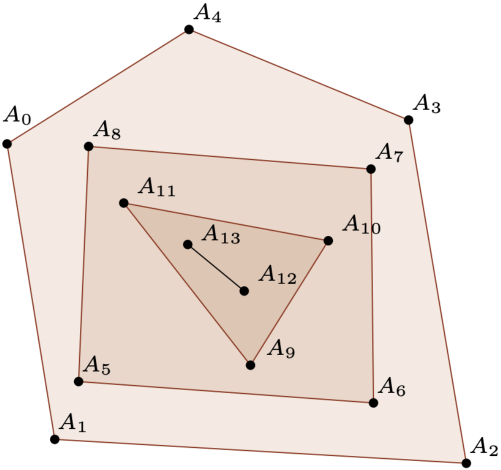
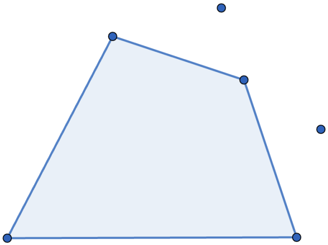


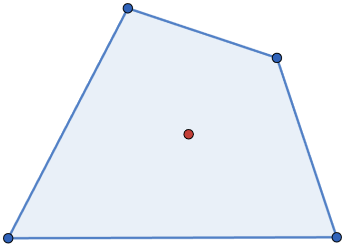
Figure: Convex hull structure: 5, 4, 3, 2. There are 73 different structures for $n = 17$.

Holes

Similar questions can be asked about k-holes, i.e. empty k-gons.



(a) YES, convex 4-hole



(b) NO, not empty

Figure: 4-holes

Holes

Similar questions can be asked about k-holes, i.e. empty k-gons.

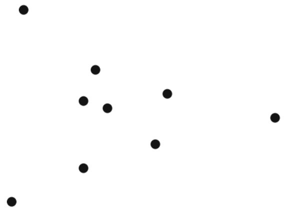
Can we find for a given k a number $h(k)$ such that: from any set containing at least $h(k)$ points it is possible to select k points forming k-hole?

Holes

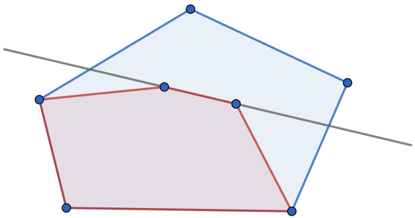
- When $k = 3$, it's trivially true that $h(3) = 3$;
- When $k = 4$, same argument for 4-gon can be applied to prove $h(4) = 5$.

Holes

- When $k = 5$, Harborth et al.[8] proved $h(5) = 10$.



(a) 9 points with no 5-hole



(b) At least one 5-gon with only one point inside

[8] Harborth, H.: Konvexe Fünfecke in ebenen Punktmengen. Elem. Math. 33, 116–118 (1978)

Holes

- When $k = 7$, Horton[9] first gave a construction for arbitrarily large n points with no 7-hole in 1983. Valtr[10] gave a much simpler inductive construction in 1992.

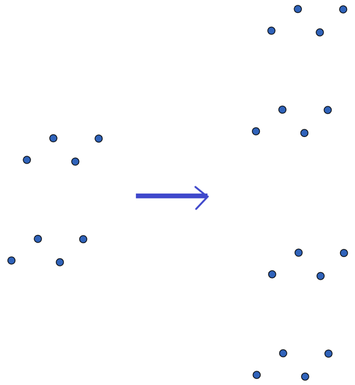


Figure: 8 / 16 points with no 7-hole

Holes

- When $k = 3, 4$, not quite interesting.
- When $k = 5$, Harborth et al. proved $h(5) = 10$.
-
- When $k = 7$, Horton first gave a construction of arbitrarily large n with no 7-hole in 1983. Valtr gave a much simpler inductive construction in 1992.

Holes

- When $k = 5$, Harborth et al. proved $h(5) = 10$.
- When $k = 6$,lower bound $h(6) \geq 30$ was given by Overmars[11] with computer assistance; and Nicolás[12] proved the finiteness by showing $h(6) \leq g(25)$ which was then improved by Gerken[13] to $h(6) \leq g(9)$.
- When $k = 7$, Horton first gave a construction of arbitrarily large n with no 7-hole in 1983. Valtr gave a much simpler inductive construction in 1992.

[11] Overmars: Finding sets of points without empty convex 6-gons. Discrete Comput. Geom.(2002)

[12] Nicolás: The empty hexagon theorem. Discrete Comput. Geom. (2007)

[13] Gerken: Empty convex hexagons in planar point sets. Discrete Comput. Geom. (2008)

Variations

What is the least number of convex k-gons / k-holes determined by any set S of n points in the plane?

What is the least/largest number of non-convex k-gons / k-holes determined by any set S of n points in the plane?

How to efficiently count the number of convex k-gons / k-holes?

Appendix: Least Number of Convex k-gons / k-holes

k	Least Number
4	$\bar{cr}(n)$
5	$\Theta(n^5)$
≤ 6	$\Theta(n^k)$

Table: least number of convex k-gons

k	Least Number
3	$n^2 - 32/7n + 22/7 \leq \cdot \leq 1.619n^2 + o(n^2)$
4	$n^2/2 - 9/4n - o(n) \leq \cdot \leq 1.9397n^2 + o(n^2)$
5	$3n/4 - o(n) \leq \cdot \leq 1.0207n^2 + o(n^2)$

Table: least number of convex k-gons

Appendix: Complexity of convex k-gon / k-hole counting

We have algorithms to with complexity $O(n^{k-2})$ and $O(kn^3)$ to count convex k-gons; $O(kn^3)$ and $O(kh_3(S))$ to count convex k-holes.

It's worth noting that the expected value of $h_3(S)$ is proven to be $\Theta(n^2)$ if points are uniformly chosen from a convex bounded body.

Appendix: Ramsey Number

Surprisingly, the finiteness of $g(k)$ can be proved with hypergraph Ramsey number.

If we color every 3-point subset $(p_i, p_j, p_k), i < j < k$ with red if the order (p_i, p_j, p_k) is clockwise order; otherwise color it with blue.

Then there exists a k-gon if and only if every 3-point subset of k points is colored in the same color.

Thus $g(k) \leq R_3(k,k)$. And it's proven that $R_3(k,k) \leq 2^{k^{k-2}\log k}$.

Appendix: High Dimension Case

Point set S on $\mathbb{R}^d$ is in general position if no $d+1$ points lie in a hyperplane; and it's convexly independent if no point lies in the convex hull of remaining points.

Similarly define $g_d(k), h_d(k)$ and we have:

- $g_d(k) \leq R_{d+2}(n, d+3)$.
- $h_d(2d+1) \leq g_d(4d+1)$, which shows the existence of $h_d(k)$ if $d+1 \leq k \leq 2d+1$.