Small-dimensional Linear Programming and Convex Hulls Made Easy

Raimund Seidel

Teng Liu

April 3rd 2025

Background

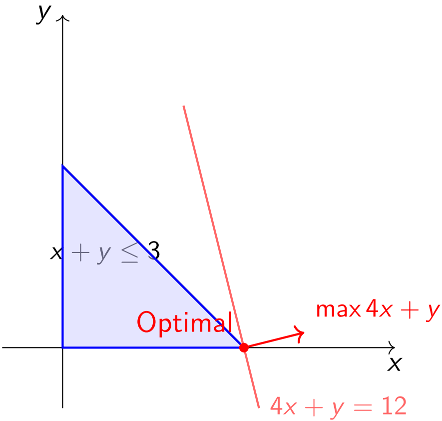
Linear programming captures one of the most canonical and influential constrained optimization problems.

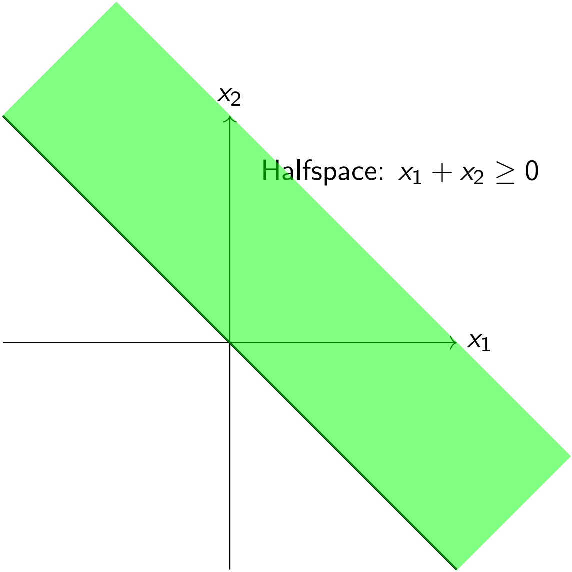
More precisely, it asks to maximize or minimize a linear objective under linear inequality and equality constraints.

$$
\begin{aligned}
\max / \min \;\; & c^T x \\
\text{subject to} \;\; & Ax \leq e \\
& Bx \geq f \\
& Cx = g
\end{aligned}
$$

Background - Geometrical Interpretation

Every constraint corresponds to a hyperplane (a halfspace, precisely). The intersection of several halfspaces gives us a polyhedron.





Hyperplane: $x_1 + x_2 = 0$

Background - Simplex Method

Fact

Optimal solution can always be achieved by some vertex.

Fact

If there exists any vertex with better objective value than current vertex, we always have a neighbour vertex with better value.

Background - Simplex Method

Fact

Optimal solution can always be achieved by some vertex.

Fact

If there exists any vertex with better objective value than current vertex, we always have a neighbour vertex with better value.

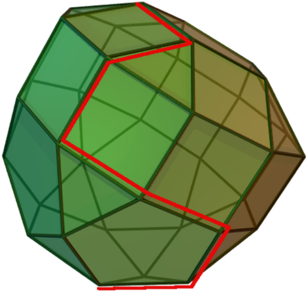
Firstly select any starting vertex.

Each step in Simplex Method move from one vertex to its neighbour with larger objective.

Background - Simplex Method

Firstly select any starting vertex.

Each step in Simplex Method move from one vertex to its neighbour with larger objective.



Background - Simplex Method Running Time

Algorithm	**Complexity**	**Speed**	**Path**
Simplex	Exponential	Fast	Edges of polyhedron
Ellipsoid	Polynomial	Slow	Ellipsoid shrinking
Interior Point	Polynomial	Fast	Through the interior

Table: Comparison of Famous LP Algorithms

Previous Work

When d, the dimension of LP (i.e. the number of variables), is considered to be constant, then LP can be solved in time linear in m, the number of constraints.

These methods[2] are complicated and running time superexponential to d: e.g. 3^{d^2}.

We also have a simple algorithm[1] with complexity $O(d^2 m) + (\log m) O(d)^{d/2+O(1)} + O(d^4 \sqrt{m} \log m)$ but the analysis is involved.

The presented algorithm gives a simple procedure and complexity $O(d!m)$.

Algorithm - Sketch

This algorithm makes use of a very simple idea:

Suppose we delete one constraint $\vec{a}x \geq b$, and somehow managed to solve the new LP. Then we check if the optimum x^* conforms with the deleted constraint.

- If it doesn't violate the constraint, i.e. $\vec{a}x^* \geq b$, then x^* is also the optimum for original LP, because deleting one constraint gives us a larger feasible area.
- If it violates the constraint, then the opt value lies on the hyperplane corresponding to this constraint. We can project the feasible area onto the corresponding hyperplane, thus reducing dimension by 1. Then solve the new LP problem with lower dimension.

Algorithm - Correctness 1

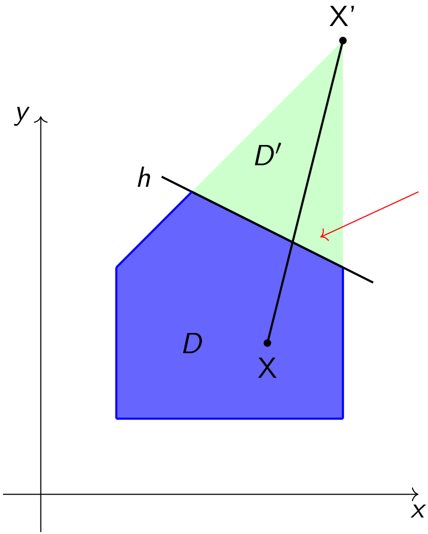
Let original feasible region be D, feasible region after deleting constraint be D', the optimum for D' be x^* . Clearly $D \subseteq D'$ and in this case $x^* \in D$.

$$u^T x^* \underset{\text{(i)}}{\leq} \max_{D} \underset{\text{(ii)}}{\leq} \max_{D'} = u^T x^*$$

(i): $x^* \in D$, by definition of max.

(ii): Optimizing on a larger area gives no worse optimal value than the original one.

Algorithm - Correctness 2



A better point lies on h

Algorithm - Further Problems

Some problems left to deal with:

- What if after deletion of bounding hyperplane the new problem become unbounded?
- When to stop the recurssion?
- What's next when we reduce the dimension?

Algorithm - Unboundedness and Recurssion

The first two problems can be solved by adding a bounding box, i.e. explicit lower and upper bounds for variables $-\alpha \leq x_i \leq \alpha$ for all d variables x_i.

Recurssion stops when all original hyperplanes are deleted and it's impossible to have unbounded case.

Algorithm - Reducing Dimension

Denote the set of m halfspaces as $\mathcal{H}$ and the hyperplane corresponding to the deleted halfspace as h. Let u be the optimizing direction and v stands for the optimum point for original LP problem.

If $\bar{u}$ is the projection of u on h and $\bar{\mathcal{H}} := \{H \bigcap h \mid H \in \mathcal{H}\}$, then v is exactly the optimum point for LP problem with constraints $\bar{\mathcal{H}}$ and optimizing direction $\bar{u}$.

Algorithm - Time Complexity

When $d = 1$, problem can be solved in $O(m)$.

When $m = 1$, problem can be solved in $O(d)$.

The interesting part is when we delete a constraint and the optimum point changes. Since original optimum point x has exactly d tight constraints (i.e. x is on exactly d hyperplanes) then deleting a constraint uniformly randomly ensures the probability of x' differs from x no more than d/m.

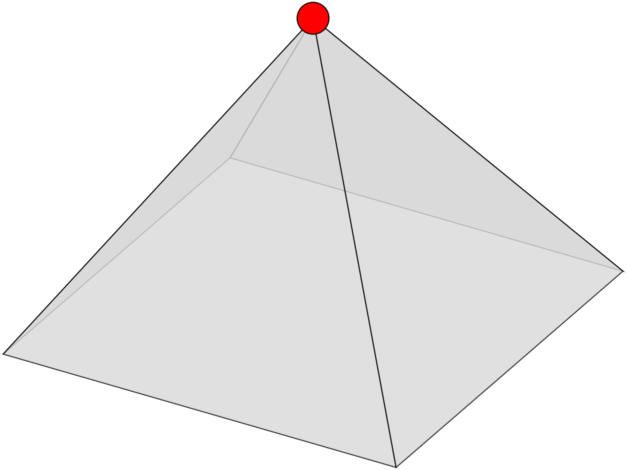
Algorithm - Time Complexity

When $d = 1$, problem can be solved in $O(m)$.

When $m = 1$, problem can be solved in $O(d)$.

The interesting part is when we delete a constraint and the optimum point changes. Since original optimum point x has exactly d tight constraints (i.e. x is on exactly d hyperplanes) then deleting a constraint uniformly randomly ensures the probability of x' differs from x no more than d/m.

Algorithm - Time Complexity



Algorithm - Time Complexity

Even if there are $k > d$ constraints are tight at x, the optimum point for original LP, at most d of them has the property that deleting it leads a better optimum.

$$\begin{aligned} \exists v,\ \ a_1^T v &> 0 \\ a_2^T v &\leq 0 \\ \cdots \\ a_k^T v &\leq 0 \end{aligned}$$

Algorithm - Time Complexity

$$T(d,m) \leq \begin{cases} O(m) & \text{if } d = 1, \\ O(d) & \text{if } m = 1, \\ T(d, m-1) + O(d) + \dfrac{d}{m} O(dm) + & \\ \dfrac{d}{m} T(d-1, m-1) & \text{otherwise.} \end{cases}$$

Expand everything and we get:

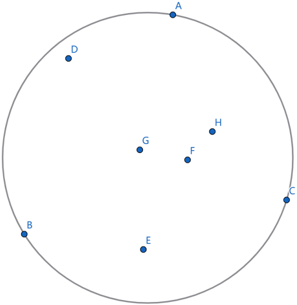
$$T(d,m) = O(\sum_{1 \leq i \leq d} \frac{i^2}{i!} d! m) = O(d!m)$$

Backward Analysis

Sometimes it's hard to determine the probability if thinking *forward*, while surprisingly straightforward if thinking *backward*.

Smallest Enclosing Circle

Given a set of points P in Euclidean plane, compute the smallest circle that contains them all.



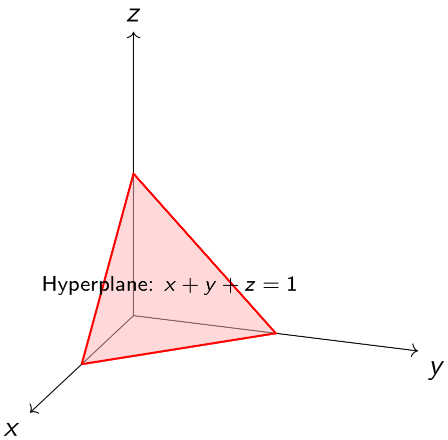
Backward Analysis - Smallest Enclosing Circle

Emo Welzl gave an efficient random algorithm with surprisingly simple procedure:

```
def Smallest_Enclosing_Circle(P, R):
    ```input : P and R are finite sets of points in the plane, |R| <= 3
    ```output: Smallest circle enclosing P with R on the boundary

    if P is empty or |R| == 3 then:
        return trivial(R)

    p = random_select(P)
    D = Smallest_Enclosing_Circle(P - {p}, R)
    if p in D then:
        return D
    else:
        return Smallest_Enclosing_Circle(P - {p}, R + {p})
```


z
y
x
Hyperplane: $x + y + z = 1$

References I

Kenneth L. Clarkson.
Las vegas algorithms for linear and integer programming when the dimension is small.
J. ACM, 42(2):488–499, March 1995.

Nimrod Megiddo.
Linear programming in linear time when the dimension is fixed.
J. ACM, 31(1):114–127, January 1984.