Assignment 5

CS 311, Spring 2015
Due: June 3, 2014

Problem 1 Use a reduction to show that the language ALL_{TM} is undecidable

$$ALL_{TM} = \{\langle M\rangle \mid \text{where } M \text{ is a TM and } L(M) = \Sigma*\}$$

Lets suppose that a turing machine, R, decides ALL_{TM}. Using this assumption, lets show how to decide the acceptance problem, A_{TM}. On the input $\langle M, w\rangle$, construct M_1 as follows:

M_1 = "On input string x, simulate M on w:

1. If M accepts, *accept.*
2. If M rejects, *reject."*

Note, that $L(M_1) = \Sigma*$ if M accepts w, and will be $\emptyset$ if M rejects w. This ensures that the TM we are about to construct will only deal with the empty language or the "all" language. Nothing in the middle. Now, we can define TM S.

S = "On input $\langle M, w\rangle$, construct M_1 as shown above.

1. Run R on input $\langle M_1\rangle$. If R accepts, *accept.*
2. Otherwise, *reject."*

Here we have shown that if we assume there exists some R that can decide ALL_{TM}, we can construct a reduction that would make A_{TM} decidable. Since A_{TM} was proven undecidable (using diagonalization), there cannot exist an R that decides ALL_{TM}. Therefore, it is undecidable.

Problem 2 A *useless state* in a Turing machine is one that is never entered on any input string. Consider the problem of determining whether a Turing Machine has any useless states. Formulate this problem as a language and show that it is undecidable.

Solution We should formulate the language as follows:

$$USELESS_{TM} = \{\langle M, q\rangle \mid \text{where } M \text{ is a TM and } q \text{ is a state in } M\}$$

Suppose the turing machine R decides USELESS. R takes a Turing machine and a state and determines if that state is useless. So, give R the input $\langle M, q_{accept}\rangle$ so it can determine if the accept state of M is useless. If it is, the language is empty. ($\emptyset$) Lets define S as the turing machine that will decide E_{TM}, which was proven in the book to be undecidable. Give S the input M, and run R on $\langle M, q_{accept}\rangle$.
If R accepts, *accept.*
If R rejects, *reject.*

So, using the assumption that USELESS is decidable, we found a reduction from E_{TM} to USELESS showing that E_{TM} is decidable. But, E_{TM} was already proven to be undecidable, so our assumption is false. R does not exist.

Problem 3 If $A \leq_m B$ and B is a regular language, does this imply that A is a regular language? Why or why not?

Solution No it does not. The language 0^i1^i is not regular, but it can be mapped to this language, which is regular: 0^i1^k

make the reduction function as follows. $f(w) = 011$ if $w \in A$ or $f(w) = 10$ if $w \notin A$.

This way, $w \in A$ iff $f(w) = 011$ and $f(w) \in B$.

So B is a regular language, but A is not.
We know that f is computable because the language of A is turing decidable because it is a Context Free Language.

Problem 4 Prove that the language

$$LOOP_{TM} = \{\langle M\rangle \mid M \text{ is a TM and } M \text{ loops on all inputs}\}$$

is not recognizable.

Solution Assume LOOP is recog. Show that co-LOOP is also recog. Now show that both are decidable (theorem). Reduce A_{TM} to LOOP to show conclude undecidable, which makes LOOP unrecog.

Problem 5 Prove that the 3-SAT problem discussed in class is an element of NP by giving a verifier and a NTM decider that run in poly-time.