Counting Number of Inversions in an Array

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Introduction

- The Inversion Count of an array $\mathbf{A}$ is the count of all the pairs $(\boldsymbol{i}, \boldsymbol{j})$ such that $\boldsymbol{i} < \boldsymbol{j}$ and $\boldsymbol{A}[\boldsymbol{i}] > \boldsymbol{A}[\boldsymbol{j}]$

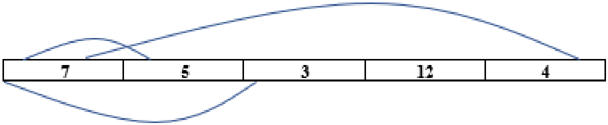


Figure: An example of inversion in array.

- Thus, Inversion Count for an array indicates – how far the array is from being sorted.

Naive Approach involves a Brute Force approach where we iterate over all the pairs of the array (which can be done through a nested loop), and if any pair (i, j) satisfies our condition, we increase our Inversion Count by one.

Divide and Conquer Approach

Suppose we divide our array A into two equal, or almost equal parts. Let's call the first one **L** and the other one **R**. Also, let's say we know the Inversion Count of both **L** and **R**. Let's call them $\boldsymbol{inv}_1$ and $\boldsymbol{inv}_2$.

Now, any inversion in $\boldsymbol{A}$ would be of type:

1. $\boldsymbol{i} \in \boldsymbol{L}$ and $\boldsymbol{j} \in \boldsymbol{R}$.
2. $\boldsymbol{i} \in \boldsymbol{L}$ and $\boldsymbol{j} \in \boldsymbol{L}$.
3. $\boldsymbol{i} \in \boldsymbol{R}$ and $\boldsymbol{j} \in \boldsymbol{R}$.

Algorithm - Divide and Conquer

For Case 1

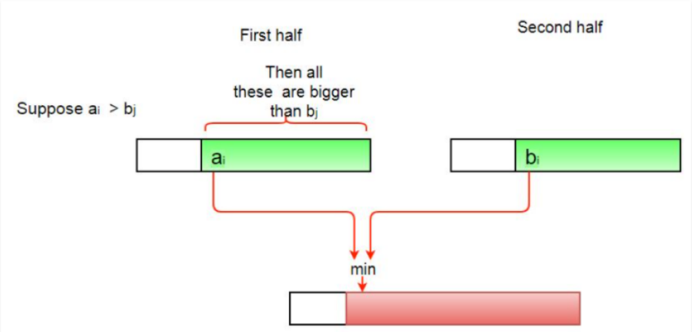
- The total inversions upon combining will be at least $\boldsymbol{inv}_1 + \boldsymbol{inv}_2$. But there may be some more inversions, as we ***merge*** **L** and **R**. So we need to find for each index $\boldsymbol{i} \in \boldsymbol{L}$, the number of indices $\boldsymbol{j} \in \boldsymbol{R}$ such that $L_i > R_j$.

- Let's say that we have sorted the arrays **L** and **R**. and their sizes are **N** and **M** respectively.

Algorithm - Divide and Conquer

Case 1(contd.)

- Say, at certain point we encounter $\boldsymbol{L}[\boldsymbol{l}] > \boldsymbol{R}[\boldsymbol{r}]$.
  Now, because **L** and **R** are sorted, we know that there are going to be **N-i** more inversions in **A**, because all elements to the right of $\boldsymbol{L}[\boldsymbol{l}]$ will also be greater than $\boldsymbol{R}[\boldsymbol{r}]$.



Source - GeeksforGeeks

Algorithm - Divide and Conquer

For Cases 2 and 3

- Now for the second and the third case: If we encounter that situation, we again divide them up into further two arrays and repeat this recursively, until we have arrived on the base case - that includes just one element. And in this case, we know the inversions will be 0, because there is just a single element in the array.

Note

Here, one more important thing is that in the first case, we require the arrays **L** and **R** to be sorted. So we need to incorporate that when we merge two arrays

Pseudo Code

mergeTwoArrays

$i, k = beg$
$j = end$
newarr

```
while i < mid AND j ≤ end do
    if arr[i] < arr[j] then
        newarr[k]=arr[i]
        increment k,i
    else
        newarr[k]=arr[j]
        invCount+= mid − i
        increment k,j
    end if
end while
while i < mid do
    temp[k] = arr[i]
    increment k,i
end while
while j < end do
    temp[k] = arr[j]
    increment k,j
end whileit = beg
while it < end do
    arr[it] = newarr[it]
    increment it
end while
return invCount
```

enhancedMergeSort

arr: The Array
beg: Start iterator
end: End iterator
$mid, invCount = 0$

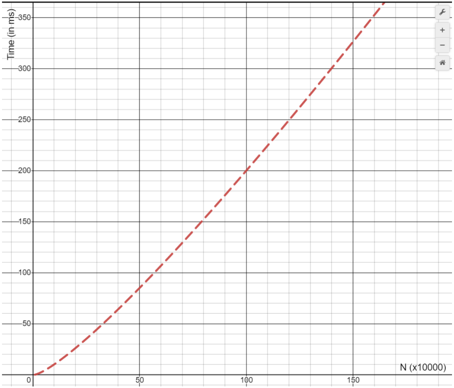
```
if end > beg then
    mid = (beg+end)/2
    invCount +=
    enhancedMergeSort(arr,left,mid)
    invCount +=
    enhancedMergeSort(arr,mid+1,right)
    invCount += mergeTwoAr-
    rays(arr,left,mid+1,right)
end if
return invCount
```

Time Complexity

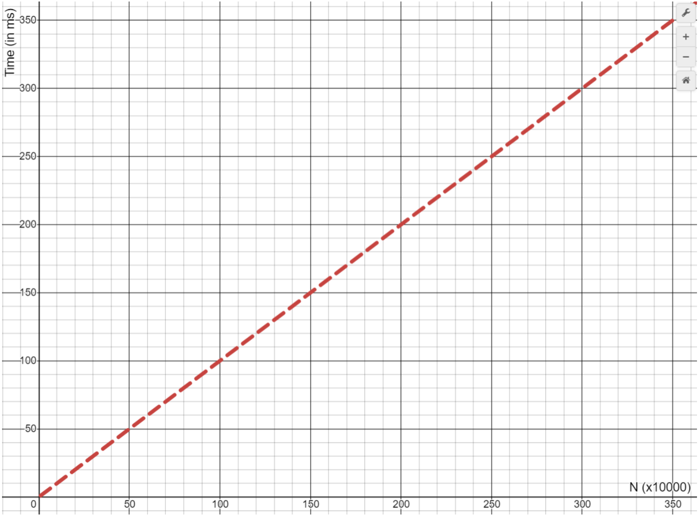
In the **Divide and Conquer** algorithm , the input array is divided into two halves each time it is processed. As such it can be expressed with following recurrence relation,

$$T(n) = 2T(n/2) + \theta(n).$$

O(N log(N))



O(N)


Time (in ms)
350
300
250
200
150
100
50
0
50
100
150
200
250
300
350
N (x10000)

References

- Introduction to Algorithms by Thomas H. Cormen, Charles E. Leiserson, Ronald Rivest, Clifford Stein
- Introduction to Design and Analysis of Algorithms by Anany Levitin
- Algorithms by Robert Sedgewick and Kevin Wayne