WORD REPRESENTABLE GRAPHS

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by

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CERTIFICATE

This is to certify that the work contained in this report entitled **"Word Representable Graphs"** submitted by **Ankit Tripathi** (**Roll No: 170123006**) to Department of Mathematics, Indian Institute of Technology Guwahati towards the requirement of the course **MA498 Project** has been carried out by him under my supervision.

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November 2020 Project Supervisor

ABSTRACT

Letters x and y alternate in a word w if after deleting all letters but x and y in w we get either a word $xyxy$... or a word $yxyx$...(each of these words can be of odd or even length). A graph $G = (V, E)$ is word-representable if there is a finite word w over an alphabet V such that the letters x and y alternate in w if and only if $xy \in E$. The word-representable graphs include many important graph classes, in particular, circle graphs, and comparability graphs. In this report we present the theory of word-representable graphs.

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Chapter 1

Introduction

The theory of word-representable graphs is a young but very promising research area. This theory was first introduced by Kitaev in 2004 But the first systematic study of word-representable graphs was made in paper "On Representable Graphs." by S. Kitaev and A. Pyatkin[4], where this theory had started. The report is organized as follows: In Chapter 2 the notions of word-representable graph and its representation number are introduced. In Chapter 3 the notion of permutationally representable graphs is given and its significance is justified. The non-word-representable graphs and the operations on graphs are investigated in Chapter 4 and 5 respectively. In particular, the operations of complement, edge subdivision, edge contraction, connecting two graphs by an edge, identifying two graphs by clique, Cartesian product, rooted product, and taking a line graph are considered.

A graph $G = (V, E)$ is word-representable if there exists a word w over the alphabet V such that letters x and y alternate in w if and only if $xy \in E$, that is, x and y are connected by an edge, for each $x \neq y$. Word-representable graphs are also known in the literature as representable graphs or alternation graphs.

Chapter 2

Preliminary Definitions

All objects and notions not defined in the report can be found in the text-books “Monographs in Theoretical Computer Science” by Sergey Kitaev and Vadim Lozin. theory

In this chapter first we understand the definition of Word Representable Graph and after that representation number and k-Representability of graph. Then graphs with representation number 2 ans 3 are defined.

2.1 Definiton of Word Representable Graph

Let w be a word over an alphabet and let x and y be two different letters in w. We say that x and y alternate in w if the deletion in w all letters but the copies of x and y results in either a word of type $xyxy$... or a word of type $yxyx$...(each of these words may be of even or odd length). For example, in the word 2311251324132 the letters 2 and 3 alternate as well as the letters 4 and 5, while every other pair of letters do not alternate.

Definition 2.1.1. A graph $G = (V, E)$ is word-representable if there exists a word w containing each letter of the alphabet V such that letter x and y, $x \neq y$, alternate in w if and only if $xy \in E$. In this case we also say that the word w represents G, and call w a word-representant.

2.2 k-Representability and Representation Number of a Graph

A word w is called k-uniform if every letter appears exactly k times in w. For example, the word 243321442311 is 3-uniform and every permutation is 1-uniform.

Definition 2.2.1. A graph G is called k-word-representable (or, shortly, k-representable) if there is some k-uniform word w representing G. In this case we say that w k-represents G

Theorem 2.2.2. *A graph is word-representable if and only if it is k-representable for some k.*

Proof. It is clear that k-representability implies word-representability. Let us prove the converse. Suppose that the word w_0 represents a graph G, and assume that each letter enters w_0 at most k times. Let A denote the set of all letters met exactly k times. If $A \neq V$ then consider the word $w^{'}$ obtained from w_0 by deletion of all letters from A. Let $p(w^{'})$ be its initial permutation (i.e., the permutation obtained by deletion in $w^{'}$ of all but the first copies of each letter). We can observe that the word $w_0 = p(w^{'})w_0$ also represents the graph G. If it is not k-uniform then repeat the same procedure until some k-uniform word representing G would be obtained. Theorem 2.1.2 is proved. $\square$

For example, if $k = 3$ and $w_0 = 3412132154$ then $w_1 = 34253412132154$ and $w_3 = 534253412132154$ represent the same graph and, moreover, the last word is 3-uniform. Using the same arguments as in Theorem 2.1.2, we can prove that each word-representable graph has infinitely many word-representants:

Remark 2.2.3. Let $w = uv$ be a k-uniform word representing G where u and v are some words over an alphabet V. Then the word $w^{'} = vu$ represents G too.

Definition 2.2.4. A representation number of G is the minimum k for which G is k-representable. For non-word-representable graphs (whose existence is shown below) we may assume $k = \infty$. Denote the representation number of G by $R(G)$ and put $R_k = \{G \mid R(G) = k\}$.

Evidently, $R_1 = \{G \mid \text{G is a complete graph}\}$. In the next subsection the description of R_2 is given.

2.3 Graphs with Representation Number 2

First we present the four classes of graphs with representation number 2; namely, edgeless graphs, forests and trees, cycles and ladders. Then we show that the graphs with representation number 2 are circle graphs.

2.3.1 Edgeless Graphs

Since the graph E_n is a graph on n vertices, $n \geq 2$, is not complete, $R(E_n) \geq 2$. On the other hand, E_n is representable as the concatenation of two permutations and so $R(E_n) = 2$.

2.3.2 Trees and Forests

We show by induction that every tree T is 2-representable. Hence, if T has at least three vertices then $R(T) = 2$. If a tree has one or two vertices then it can be represented by the word 11 or 1212 respectively. Let every tree on $n-1$ vertices be 2-representable for $n \geq 3$. Consider some tree T on n vertices, and let x be a leaf of T adjacent to a vertex y. Removing x yields the tree $T^{'}$ that can be represented by a 2-uniform word of type $w_1yw_2yw_3$, where the subwords w_1 , w_2 , and w_3 do not contain the letter y (note that some of these subwords could be empty). Then the word $w_1yw_2xyxw_3$ (as well as $w_1xyxw_2yw_3$) 2-represents the initial tree T.

If some forest F includes several components (trees) then a concatenation of 2-uniform words representing the trees, clearly, 2-represents F . So, $R(F) = 2$ for every forest F with at least two trees.

2.3.3 Cycles

For 2-representation of a cycle C_n (cycle C_n is the graph on n vertices) first find a representation of a path P_n on n vertices using the above mentioned way of 2-representation of trees, then shift the so-obtained word by one position to the right (by Remark 2.1.3, this word still represents P_n), and swap

n	2-representation of a ladder L_n
1	$11'11'$
2	$1'212'21'2'1$
3	$12'1'323'32'3'121'$
4	$1'213'2'434'43'4'231'2'1$

Table 2.1: 2-representation of a ladder L_n for $n = 1, 2, 3, 4$

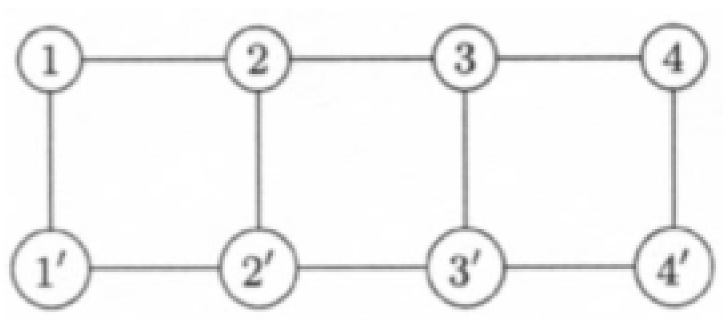

Figure 2.1: Ladder L_4

first two letters. Let us demonstrate this for the cycle C_6. Using the induction from the previous example, construct a 2-representation of the path P_6: $1212 \to 121323 \to 12132434 \to 1213243545 \to 121324354656$.

Shifting one position to the right results in the word 612132435465. Swapping the first two letters yields 162132435465 that 2-represents C_6.

2.3.4 Ladder

A ladder L_n is a graph on $2n$ vertices with $3n-2$ edges that is obtained from two copies of the path P_n by adding the edges connecting the copies of the same vertices. Figure 2.1 provides an example of a ladder for $n = 4$. It is proved in [2] by induction that $R(L_n) = 2$ for $n \geq 2$. Table 2.1 gives the 2-representations of L_n for $n = 1, 2, 3, 4$.

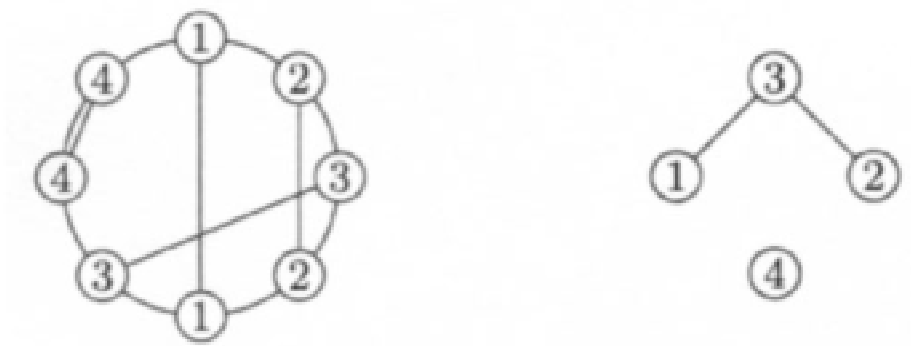

Figure 2.2: A family of chords(On left) and a circle graph induced by it(On right).

2.3.5 Circle Graphs

A graph G is called a circle graph if its vertices correspond to chords of some circle and two vertices are adjacent if and only if the corresponding chords intersect (as curves on the plane). In Figure 2.2 an example is shown of such graph and a corresponding family of chords.

The next completes the characterization of the class R_2:

Theorem 2.3.1. $R_2 = \{G \mid G$ *is a noncomplete circle graph*$\}$.

Proof. Let G be a circle graph. Consider the corresponding family of chords. We may assume that all endpoints of the chords are distinct. Starting from an arbitrary endpoint of some chord, traverse the circle clockwise recording the chords labels in the order they are met. We obtain some 2-uniform word w in which two letters x and y alternate if and only if the corresponding chords (labeled by x and y) intersect; i.e., the vertices x and y are adjacent in G. For instance, for the graph in Fig. 2 we obtain the word 13441232 that 2-represents this graph.

Conversely, let G be 2-represented by some word w. Choose on some circle $2n$ points, label them in cyclic order by the letters of w, and connect the identically labeled points by the chords. And we can observe that this is the family of chords induces G.
So, G is a circle graph if and only if $G \in R_2$, except for the complete graphs that lie in R_1 . This completes the proof.

$\square$

Note, Theorem 2.2.1 can be used for verifying whether a given graph is a circle graph or not.

2.4 Graphs with Representation Number 3

In contrast to the graphs with representation number 2, no complete characterization of 3-representable graphs is known now. However, we present below some interesting results about this class.

2.4.1 Petersen Graph

In 2010, Konovalov and Linton not only showed (by computer) that the Petersen graph (Fig. 3) is not 2-representable, but also found two different 3-representations for it:

- 138729607493541283076850194562,
- 134058679027341283506819726495.

Theorem 2.4.1. *The Petersen graph is not 2-representable.*

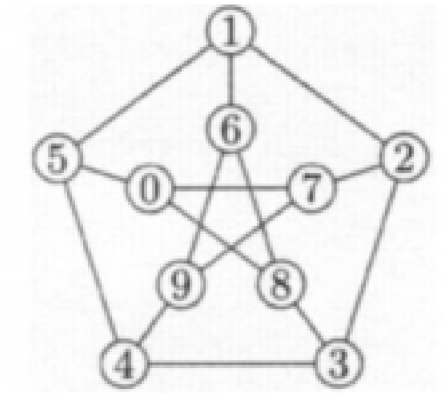

Figure 2.3: Petersen graph

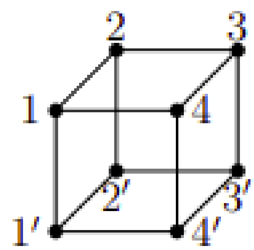

Figure 2.4: Prism Pr_4

The theorem given above is from the paper "Graphs Capturing Alternations in Words" by M. Halldórsson, S. Kitaev, and A. Pyatkin.

2.4.2 Prism

A prism Pr_n is a graph consisting of two copies of the cycle C_n with the vertices $1, 2, ..., n$ and $1^{'}, 2^{'}, ..., n'$ (here $n \geq 3$ and the vertices are listed in some cyclic order), connected by the edges of type $ii^{'}$ for $i = 1, ..., n$. For instance, three-dimensional cube is a prism Pr_4 The theorem given below is from the paper "On Representable Graphs" by S. Kitaev and A. Pyatkin.

Theorem 2.4.2. [4] *Every prism Pr_n is 3-representable.*

The above theorem is from the paper "On Representable Graphs." by S. Kitaev and A. Pyatkin.

Chapter 3

Permutationally Representable Graphs

In this chapter first, we are going to discuss about Transitive orientation and Comparability Graph then after that Permutationally Representable Graph and some theorems connecting Comparability Graph, Permutationally Representable Graph and Word Representable Graph.

3.1 Comparability Graph

An orientation is called transitive if the existence of arcs $u \to v$ and $v \to z$ implies the existence of the arc $u \to z$. An undirected graph is a comparability graph if it has transitive orientation. Example a noncomparability graph is the cycle C_5.

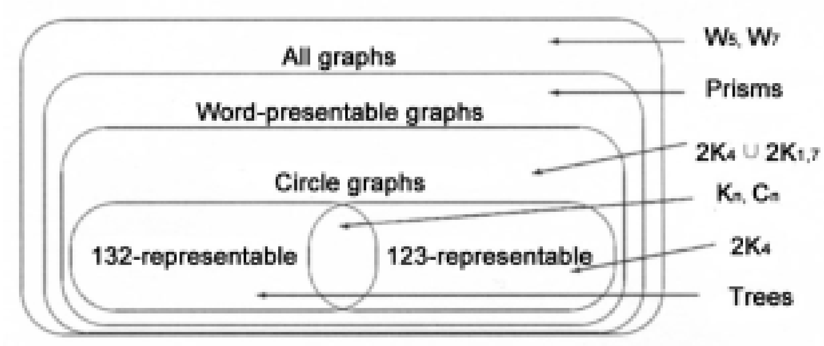

Figure 3.1: Relationship between Graph classes

3.2 Permutationally Representable Graph

Definition 3.2.1. A graph $G = (V, E)$ is permutationally representable if G can be represented by a word of type $p_1p_2...p_k$, where each p_i is a permutation over V. If there are k permutations in such representation then G is permutationally k-representable

For example, the cycle C_4 is permutationally 2-representable as the word 13243142 shows.
The theorem given below is from the paper "Word Problem of the Perkins Semigroup via Directed Acyclic Graphs." by S. Kitaev and S. Seif.

Theorem 3.2.2. *A graph is permutationally representable if and only if it is a comparability graph.*

The theorem given below is from the paper "On Graphs with Representation Number 3." by S. Kitaev.

Theorem 3.2.3. [2]*Let n be the number of vertices in a graph G and $x \in V(G)$ be a vertex of degree $n-1$ (called a dominant or all-adjacent* *vertex). Let $H = G - \{x\}$ be the graph obtained from G by removing x and all edges incident to it. Then G is word-representable if and only if H is permutationally representable.*

Chapter 4

Nonrepresentable Graphs

In this chapter first we going to see the definition of Wheel Graph and understand the term All-Adjacent Vertex after that Nonrepresentable Graphs with an All-Adjacent Vertex and then Nonrepresentable Line Graphs.

4.1 Wheel Graph

Definition 4.1.1. The wheel graph W_n is the graph on n + 1 vertices obtained from the cycle graph C_n by adding an all-adjacent vertex. all-adjecent vertex is a vertex of degree n in a graph on $n + 1$ vertices.

The graph W_5 is presented in Figure 4.1

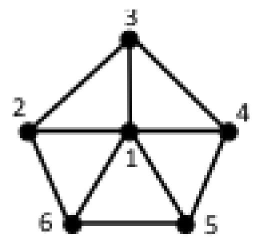

Figure 4.1: Graph W_5

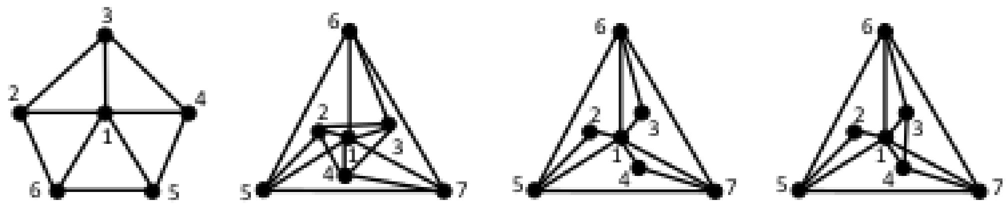

Figure 4.2: Examples of smallest non-word-representable graphs

4.2 Nonrepresentable Graphs with an All-Adjacent Vertex

Theorems 3.2.2 and 3.2.3 give us a method appearing in [4] to construct nonrepresentable graphs. We can take a non-comparability graph and add an all-adjacent vertex to it. Several examples of small non-word-representable graphs obtained using this method can be found in Figure 4.2. In particular, the smallest non-word-representable graph is the wheel W_5.

To show that the examples in Figure 4.2 are correct, we need to show that the neighbourhoods of the all-adjacent vertices in the graphs presented in Figure 4.3 are non-comparability graphs. To this end, we show that the graphs in Figure 4.3 do not accept transitive orientations, where by $i \to j$ we denote orientation of an edge ij from i to j

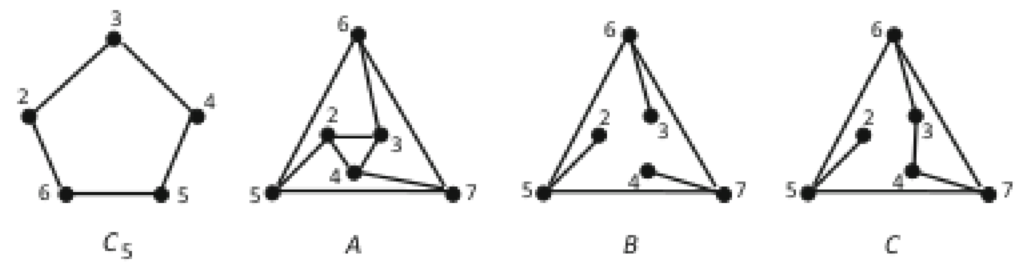

Figure 4.3: Neighbourhoods of all-adjacent vertices in the respective graphs in Figure 4.2

• For the 5-cycle C_5, let the orientation of the edge 56 be $6 \to 5$. To be a transitive orientation, the edge 45 must be oriented as $4 \to 5$ because there is no edge between the vertices 6 and 4. Because there is no edge between 3 and 5, to have a transitive orientation, we must orient 34 as $4 \to 3$, and because there is no edge between 2 and 4, to have a transitive orientation, we must orient 23 as $2 \to 3$. However, there is no way to orient the edge 26 so that the obtained orientation is transitive, and thus C_5 is not a comparability graph.

• Using arguments similar to those used for C_5 , we see that if in the graph A, we have $5 \to 7$, then we also have $4 \to 7$ leading to $4 \to 3$ and $4 \to 2$. But then we must have $5 \to 2$ and $6 \to 3$ leaving us no chance to orient the edge 23 so that the obtained orientation is transitive. Thus, A in Figure 4.3 is not a comparability graph.

• For the graph B in Figure 4.3, if, we have $5 \to 7$, then we also have $5 \to 2$ and $4 \to 7$. Further, $5 \to 2$ implies $5 \to 6$, and $4 \to 7$ implies $6 \to 7$.

But now, there is no way to orient properly the edge 36, so that B is not a comparability graph.

• Exactly the same arguments as those for the graph B show that the graph C in Figure 4.3 is not a comparability graph.

Theorem 4.2.1. [4] *For any $n \geq 2$, the wheel W_{2n+1} is non-word-representable.*

Theorem mentioned above is from the paper "On Representable Graphs" by S. Kitaev and A. Pyatkin,

4.3 Nonrepresentable Line Graphs

A line graph for a graph $G = (V, E)$ is the graph with the vertex set E, where two vertices are adjacent if and only if the corresponding edges in G have a joint endpoint; such graph is denoted by $L(G)$. The line graphs can be helpful for constructing nonrepresentable graphs. Example Figure 4.4 and 4.5 shows a graph G and its Line graph L(G).

Theorem 4.3.1. *If $n \geq 4$ then $L(W_n)$ of every wheel W_n is nonrepresentable.*

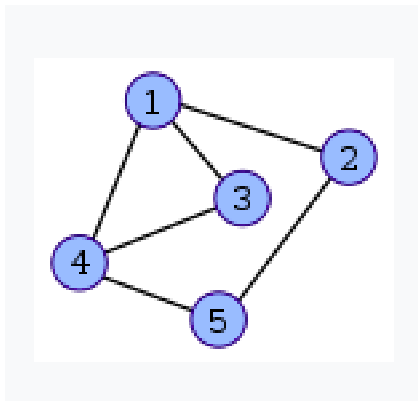

Figure 4.4: Graph G.

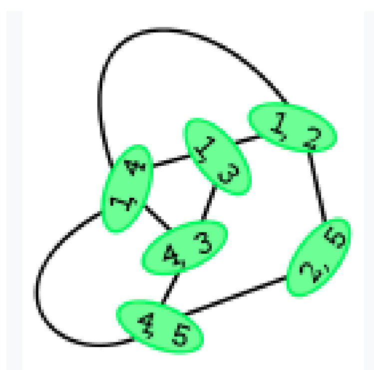

Figure 4.5: Line graph L(G).

Chapter 5

Operations On Graph

In this section we consider several basic operations on graphs, namely, taking the complement, edge subdivision, edge contraction, connecting two graphs by an edge, gluing two graphs in a clique, Cartesian product, rooted product, and taking line graph.

5.1 Complement of graph

In graph theory, the complement or inverse of a graph G is a graph H on the same vertices such that two distinct vertices of H are adjacent if and only if they are not adjacent in G. That is, to generate the complement of a graph, one fills in all the missing edges required to form a complete graph, and removes all the edges that were previously there.
Example Figure 5.1 contains Petersen graph and its complement graph.
Starting with a word-representable graph and taking its complement, we may either obtain a word-representable graph or not.

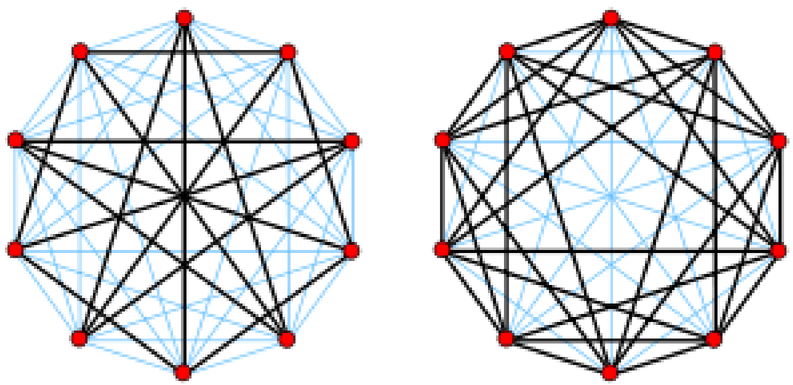

Figure 5.1: The Petersen graph(on the left) and its complement graph(on the right).

5.2 Edge Contraction

The edge contraction operation occurs relative to a particular edge, e. The edge e is removed and its two incident vertices, u and v, are merged into a new vertex w, where the edges incident to w each correspond to an edge incident to either u or v. More generally, the operation may be performed on a set of edges by contracting each edge (in any order).

Example in Figure 5.2 shown Contraction of an edge.

Edge contraction does not preserve the property of being a word-representable graph.

5.3 Connecting Two Graphs

The operations of connecting two graphs, G_1 and G_2 ,can be performed in two ways first by an edge and second by gluing two graphs at a vertex. These operations are presented schematically in Figure 5.3. If both G_1 and G_2 are

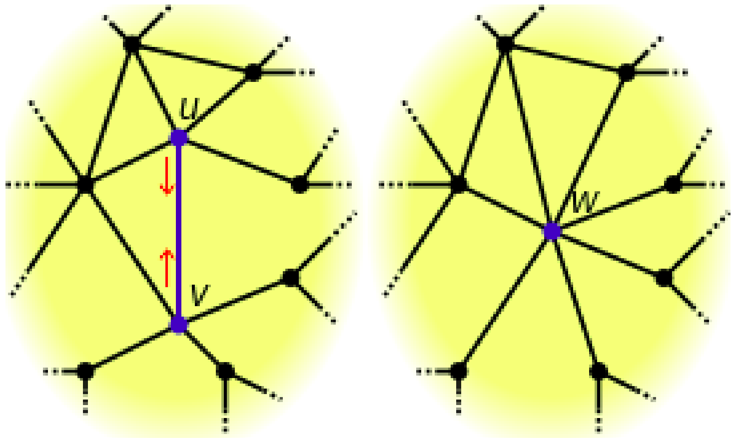

Figure 5.2: Contracting an edge

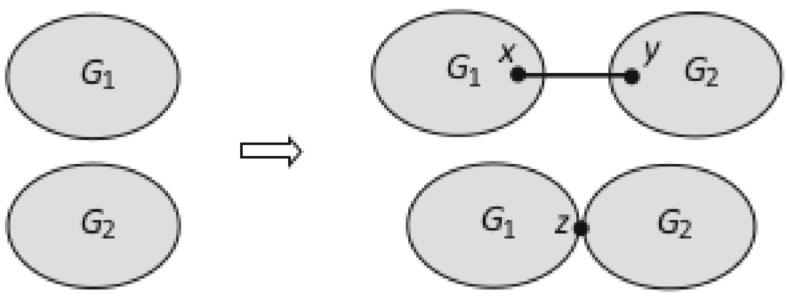

Figure 5.3: Connecting graphs by an edge and gluing graphs at a vertex

word-representable then the resulting graphs will be word-representable too, while if at least one of G_1 or G_2 is nonrepresentable then the resulting graphs will be nonrepresentable.

5.4 Cartesian Product of two Graphs

A Cartesian product $G \Box H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph with the vertex set $V(G) \times V(H)$ where vertices $(u, u^{'})$ and $(v, v^{'})$

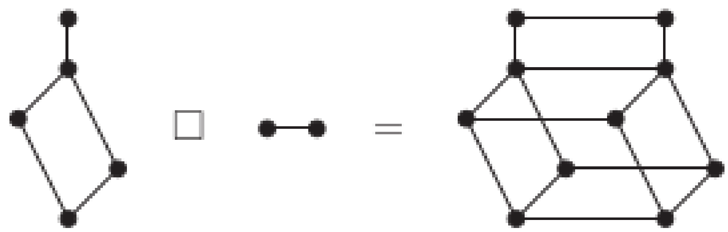

Figure 5.4: Cartesian product of two graphs

are adjacent if and only if either $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G.
Example Figure 5.4 shows Cartesian product of two graphs.

Theorem 5.4.1. *Let G and H be word-representable graphs. Then their Cartesian product $G \square H$ is word-representable.*

5.5 Rooted Product of Graphs

Given two graphs G and H, assume that one vertex of the graph H is chosen as a root. Then the rooted product of G and H is the graph $G \circ H$ defined as follows: Consider $\mid V(G) \mid$ copies of the graph H and identify each vertex v_i of the graph G with the root of i^{th} copy of the graph H. Example Figure 5.5 shows rooted product of two graphs.

Theorem 5.5.1. *Let G and H be two word-representable graphs. Then the rooted product $G \circ H$ is also word-representable.*

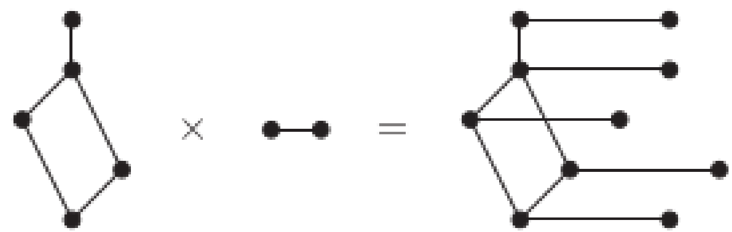

Figure 5.5: Rooted product of two graphs

5.6 Taking Line Graphs

This operation was already studied in Section 4.3. Based on the obtained results, this operation can turn a word-representable graph into either a word-representable graph or nonrepresentable graph. The examples of nonrepresentable graphs whose the line graphs are nonrepresentable are also known. However, it remains an open problem if a line graph of a nonrepresentable graph can be word- representable

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