Chapter 6   Counting

6.1 The Basic of Counting
1. Basic Counting Principles
   1) The Sum Rule
   2) The Product Rule
2. The Inclusion-Exclusion Principle (Subtraction Rule)
If a task can be done in either $n1$ ways or $n2$ ways, then the number of ways to do the task is $n1 + n2$ minus the number of ways to do the task that are common to the two different ways.

$$|A1 \cup A2| = |A1| + |A2| - |A1 \cap A2|.$$

3. Tree Diagrams

6.2 The Pigeonhole Principle
1.The Pigeonhole Principle : If k is a positive integer and $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

If f is a function from A to B, where A and B are finite sets with $|A| > |B|$, then there are elements a_1, a_2 in A($a_1 \neq a_2$) such that $f(a_1) = f(a_2)$

Corollary 1 A function f from a set with $k+1$ or more elements to a set with k elements is not one-to-one.

2. The Generalized Pigeonhole Principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

$f: A \to B$, If $\left\lceil \frac{|A|}{|B|} \right\rceil = i$, then there must exist elements $a_1, a_2, \cdots, a_i \in A$ such that $f(a_1) = f(a_2) = \cdots = f(a_i) = b \in B$

6.3 Permutations and Combinations
1. Permutations
   【Definition】 A ***permutation*** of a set of distinct objects is an ordered arrangement of these objects.
An ***r-permutation*** is an ordered arrangement of r elements of a set.
Notation: P(n, r)

   【**Theorem 1**】   P(n, r)=n(n-1)(n-2)...(n-r+1) $= \frac{n!}{(n-r)!}$

$B = \{b_1, b_2, ..., b_r\}$, $A = \{a_1, a_2, \cdots, a_n\}$, $f: B \to A$

1) f is an injection from B to $A \Leftrightarrow$ an r-permutation of the set A
2) the number of injections from B to $A \Leftrightarrow P(n, r)$
2. Combinations
   【Definition】 An ***r-combination*** of elements of a set is an unordered selection of r elements from the set.

Notation: $C(n, r) = \binom{n}{r}$ ←Binomial coefficient

   【**Theorem 2**】 $C(n, r) = \frac{n!}{r!(n-r)!} = n(n-1)(n-2)...\frac{n-r+1}{r!}$

   【**Corollary 1**】 ***combination corollary***   Let n and r be nonnegative integers with $r \leq n$. Then

$C\ (n,\ r)\ =C\ (n,\ n\text{-}r)$

3. Combinatorial Proofs

【**Definition**】　A *combinatorial proof* of an identity is a proof that uses one of the following methods.

- A *double counting proof* uses counting arguments to prove that both sides of an identity count the same objects, but in different ways.
- A *bijective proof* shows that there is a bijection between the sets of objects counted by the two sides of the identity.

6.4 Binomial Coefficients

【**Theorem 1**】　*The Binomial Theorem*　Let x and y be variables, and let n be a nonnegative integer. Then $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$

【**Corollary 1**】　Let n be a nonnegative integer. Then $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

【**Corollary 2**】　Let n be a positive integer. Then $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

【**Corollary 3**】　Let n be a nonnegative integer. Then $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$

【**Theorem 2**】　*PASCAL'S Identity*　Let n and k be positive integers with $k \le n$. Then $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

【**Theorem 3**】　*Vandermonde's Identity*　Let m, n and r be nonnegative integers with r not exceeding either m or n. Then $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k} \binom{n}{k}$

【**Corollary 4**】　If n is a nonnegative integer. Then $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

【**Theorem 4**】　Let n and r be nonnegative integer with $r \le n$. Then $\binom{n+1}{r+1} = \sum_{j=r}^{n} \binom{j}{r}$

6.5 Generalized Permutations and Combinations

2. Permutations with Repetition

【**Theorem 1**】　The number of r-permutations of a set of n objects with repetition allowed is n^r.

3. Permutations with Indistinguishable Objects

$A = \{ n_1 \bullet a_1, n_2 \bullet a_2, \ldots, n_k \bullet a_k \}$, where $n_1 + n_2 + \ldots + n_k = n$

【**Theorem 2**】　The number of different permutations of n objects, where there are n_1 indistinguishable objects of type1, …, and n_k indistinguishable objects of type k, is $n!/(n_1!n_2!\ldots n_k!)$

【**Theorem 3**】　The number of r- Circle permutations of a set of n objects is $P(n, r)/r$.

4. Combinations with Repetition

【**Theorem 4**】　There are $C\ (n\text{-}1+r, r)$ r-combination from a set with n elements when repetition of elements is allowed.

5. Distributing objects into boxes

1) Distinguishable objects and distinguishable boxes

【**Theorem 5**】　The number of ways to distribute n distinguishable objects into k distinguishable boxes so that n_i objects are place into box i, $i=1,2,\ldots,k$, equals $n!/(n_1!n_2!\ldots n_k!)$

2) Distinguishable objects and indistinguishable boxes

counting the ways to place n distinguishable objects into k indistinguishable boxes

$S(n, j)$: Stirling numbers of the second kind

- the number of ways to distribute n distinguishable objects into j indistinguishable boxes so that **no boxes is empty**

    (1) $S(r, 1)=S(r, r)=1$ $(r \geq 1)$

    (2) $S(r, 2)=2^{r-1}-1$

    (3) $S(r, r-1)=C(r,2)$

    (4) $S(r+1, n)=S(r,n-1)+nS(r, n)$

    (5) $S(n,j) = (\dfrac{\sum_{i=0}^{j-1}(-1)^i C_j^i (j-i)^n)}{j!}$

the number of ways to place n distinguishable objects into k indistinguishable boxes (exist empty

boxes) $\sum_{j=1}^{k} S(n,j) = \sum_{j=1}^{k} \left((\sum_{i=0}^{j-1}(-1)^i C_j^i (j-i)^n)/j! \right)$

3) Indistinguishable objects and distinguishable boxes

    -counting the ways to distribute n indistinguishable objects into k distinguishable boxes

    -Same as counting the number of n-combinations for a set with k elements when repetitions are allowed.

4) Indistinguishable objects and indistinguishable boxes

    -counting the ways to distribute indistinguishable objects into indistinguishable boxes

6.6 Generating Permutations and Combinations

1. Generating Permutations

==*The lexicographic ordering* of the set of permutations of {1, 2, …, n}== 💬

The permutation $a_1 a_2 \ldots a_n$ precedes the permutation of $b_1 b_2 \ldots b_n$, if for some k, with $1 \leq k \leq n$, $a_1 = b_1, a_2 = b_2, \ldots, a_{k-1}=b_{k-1}$, and $a_k < b_k$

●   Given permutation $a_1 a_2 \ldots a_n$, find the next larger permutation in increasing order:

    (1) Find the integers a_j, a_{j+1} with $a_j < a_{j+1}$ and $a_{j+1} > a_{j+2} > \cdots > a_n$

    (2) Put in the jth position the least integer among $a_{j+1}, a_{j+2}, \cdots, a_n$ that is greater than a_j

    (3) List in increasing order the rest of the integers $a_j, a_{j+1}, \cdots, a_n$

        124653 → 125346

2. Generating Combinations

Problem 1:

        Generate all combinations of the elements of a finite set.

Solution:

➢   A combination is just a subset. $\Rightarrow$ We need to list all subsets of the finite set.

➢   Use bit strings of length n to represent a subset of a set with n elements. $\Rightarrow$ We need to list all bit strings of length n.

➢   The 2^n bit strings can be listed in order of their increasing size as integers in their binary expansions.

Algorithm of producing all bit strings

➢   Start with the bit string 000…00, with n zeros.

➢   Then, successively find the next larger expansion until the bit string 111…11 is obtained.

The method to find the next larger binary expansion:

Locate the first position from the right that is not a 1, then changing all the 1s of the right of this position to 0s and making this first 0 a 1. **1000110011 → 1000110100**

Generate all r-combinations of the set $\{1, 2, \ldots, n\}$

The algorithm for generating the r-combination of the set $\{1, 2, \ldots, n\}$

(1) $S_1 = \{1, 2, \ldots, r\}$

(2) If $S_i = \{a_1, a_2, \cdots, a_r\}, 1 \le i \le C_n^r - 1$ has found, then the next combination can be obtained using the following rules.

First, locate the last element a_i in the sequence such that $a_i \ne n - r + i$. Then replace a_i with $a_i + 1$ and a_j with $a_i + j - i + 1$, for $j = i + 1, i + 2, \ldots, r$.