1) Let $f: R \to R$, $f(x) = \sin x$
Show that f is differentiable $\forall p \in R$

Sol^n: Given $F(x) = \sin x$

$\therefore F(p) = \sin p, \forall p \in R$

$$\therefore \lim_{x \to p} \frac{f(x) - f(p)}{x - p} = \lim_{x \to p} \frac{\sin x - \sin p}{x - p}$$

$$= \lim_{x \to p} \frac{2\cos\left(\frac{x+p}{2}\right) \cdot \sin\left(\frac{x-p}{2}\right)}{x - p}$$

$$= \lim_{x \to p} \cos\left(\frac{x+p}{2}\right) \cdot \lim_{x \to p} \frac{2\sin\left(\frac{x-p}{2}\right)}{\left(\frac{x-p}{2}\right) \times 2}$$

$$= \lim_{x \to p} \cos\left(\frac{x+p}{2}\right) \cdot \lim_{x \to p} \frac{\sin\left(\frac{x-p}{2}\right)}{\left(\frac{x-p}{2}\right)}$$

$\left(\text{here, } \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1\right)$; $$= \cos\left(\frac{p+p}{2}\right)$$

$$= \cos\left(\frac{p+p}{2}\right)$$

$\therefore F$ is differentiable at p and $f'(p) = \cos p$

2) Draw the graph of $y = 4 - 3a^2 + a^3$

Sol^n:

Given : $y = 4 - 3a^2 + 3$

i) For x-intercept put $y = 0$

$0 = 4 - 3a^2 + a^3$

$0 = (1+a)(a-2)^2$

$\therefore a = -1, \quad a = 2$

For y-intercept put $a = 0$

$y = 4 - 3(0^2) + 0^3$

$\therefore y = 4$

$\therefore$ y-intercept is 4

ii) $f'(a) = y' = -6a + 3a^2$

$= 3a(a-2)$

(a) $\therefore f'(a) = 3a(a-2) > 0$

Case (i) if $3a > 0$ and $(a-2) > 0$

i.e $a > 0$ and $a > 2$

$\Rightarrow a > 2$

$\therefore f'(a) > 0$ for $a > 2$

Case (ii) if $3a \cancel{>} 0$ and $(a-2) < 0$

i.e $a < 0$ and $a < 2$

$\Rightarrow a > 2$ for $a < 0$

$\therefore$ The function is increasing for $a > 2$ and $a < 0$

i.e $(2, \infty)$ and $(-\infty, 0)$

(b) $\therefore f'(a) = 3a(a-2) < 0$

case (i) if $3a < 0$ and $(a-2) > 0$

i.e $a < 0$ and $a > 2$

which is not possible

case (ii) if $3a > 0$ and $(a-2) < 0$

i.e $a > 0$ and $a < 2$

i.e $0 < a < 2$

i.e $(0, 2)$

$\therefore$ Then Function is decreasing in $(0, 2)$

(c) $\therefore f'(a) = 3a(a-2) = 0$

$a = 0$ or $a = 2$

(d) Increase in $(2, \infty)$ and $(-\infty, 0)$

Decreases in $(-\infty, 0)$ and $(2, \infty)$

3) Divide the number 100 into two parts such that their product is maximum

Sol^n: The given number is 100

Let a be one part of the number and b be the other part

$\therefore a + b = 100$ $\therefore b = (100 - a)$

$\therefore$ The product of two number is ab.

$\therefore f(a) = ab = a(100 - a)$

$\therefore f(a) = 100a - a^2$

$\therefore f'(a) = 100 - 2a$

$\therefore f''(a) = -2$

Now,

$f'(a) = 100 - 2a = 0$ $\therefore 100 = 2a$

$\therefore a = \frac{100}{2} = 50$ For $a = 50$,

$f''(a) = -2 < 0$

$\therefore f$ is maximum at $a = 50$

$\therefore 50 + b = 100$ i.e $b = 50$

4) Square of the sum of two parts is minimum if its divided in to 100

Soln. The given number is 100.

The given number and let a be the one part of the number and b be the other part

$\therefore a + b = 100$ $\therefore b = 100 - a$

The sum of their square is $a^2 + b^2$

$\therefore f(a) = a^2 + b^2 = a^2 + (100 - a)^2$

$= a^2 + 100 - 200a + a^2$

$f(a) = 2a^2 - 200a + 100$

$f'(a) = 4a - 200$ and $f''(a) = 4$

Now,

$f'(a) = 0$; $4a - 200 = 0$

$\therefore 4a = 200$ $\therefore a = 50$

For $a = 50$, $f''(50) = 4 > 0$

$\therefore f$ is minimum at $a = 50$

$\therefore 100 + b = 50$; $\therefore b = 50$

5) Determine the maximum and minimum values of the following $f(x) = x^2 + \frac{16}{x^2}$

Soln: Given: $f(x) = x^2 + \frac{16}{x^2}$

$$f'(x) = 2x + 16(-2x^{-3})$$
$$= 2x - \frac{32}{x^3}$$

The function has maximum or minimum if $f'(x) = 0$

i.e. $2x - \frac{32}{x^3} = 0 \Rightarrow 2x = \frac{32}{x^3}$

$2x^4 = 32$ $\therefore x^4 = 16$

$\therefore x^2 = 4$ $x = \pm 2$ are the critical point

Now, $f''(x) = 2 - 32(-3x^{-4}) = 2 + \frac{96}{x^4}$

$$f''(x) = 2 + \frac{96}{2^4} = 2 + \frac{96}{16} = 8 > 0$$

$\therefore$ By second derivative test function f has a minimum at $x = 2$ and minimum value is

$$f(2) = 2^2 + \frac{16}{2^2} = 4 + 4 = 8$$

$$f''(-2) = 2 + \frac{96}{(-2)^4} = 2 + \frac{96}{16}$$

$$= 2 + 6 = 8 > 0$$

$\therefore$ Function has a minimum at $x = -2$ and minimum value is

$f(-2) = (-2)^2 + \frac{16}{(-2)^2}$

$= 4 + \frac{16}{4}$

$= 4 + 4 = 8$

6) A rectangle has an area of 9 sq cm find its dimensions for least perimeter.

Soln:

Let x, y, A and P be the length, breadth, area and perimeter respectively of the rectangle.

$\therefore A = xy = 9$

$\therefore y = \frac{9}{x}$

$\therefore P = 2(x+y) = 2\left(x + \frac{9}{x}\right)$

Now $\frac{dP}{dx} = 2[1 + 9(-1)x^{-2}]$

$= 2 - \frac{18}{x^2}$ cm

For P is minimum $\frac{dP}{dx} = 0$

$\therefore 2 - \frac{18}{x^2} = 0$

$\therefore 2x^2 - 18 = 0$

$\therefore 2x^2 = 18$

$\boxed{\therefore x^2 = 9}$

$\therefore \boxed{x = 3}$

Now,

$$\frac{d^2p}{dx^2} = 0 - 9(-2)x^{-3} = \frac{18}{x^3}$$

$$F''(P)_{(x=3)} = \frac{18}{(3)^3} = \frac{18}{27} = \frac{2}{3} > 0$$

$\therefore$ P is minimum when $x = 3$

by using Eqⁿ (1)

$$y = \frac{9}{x} = \frac{9}{3} = 3$$

Thus, the perimeter of rectangle is the least (minimum) when length = breadth = 3 cm.

Q.2 1) Find the area between the curve $y = f(x) = x^2$ and the interval [1, 3] using $n = 4$ equal sub divisions of the interval

Solⁿ: Here f is nonnegative and continuous function on the interval [1, 3]
we divide interval [1, 3] into $n = 4$ equal sub interval with width $\Delta x = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2}$
By definition of area.

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

we can choose x_i^* as right end point.

$$x_i^* = 1 + i \frac{1}{2} = 1 + \frac{i}{2}$$

$$A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

$$= \lim_{n \to \infty} \sum_{i=1}^{4} f\left(1 + \frac{i}{2}\right) \frac{1}{2}$$

$$= \lim_{n \to \infty} \frac{1}{2} \sum_{i=1}^{4} \left(1 + \frac{i}{2}\right)^2$$

$$= \lim_{n \to \infty} \frac{1}{2} \sum_{i=1}^{4} \left(1 + \frac{2i}{2} + \frac{i^2}{2^2}\right)$$

$$= \lim_{n \to \infty} \frac{1}{2} \left[\sum_{i=1}^{4} 1 + \sum_{i=1}^{4} i + \frac{1}{4} \sum_{i=1}^{4} i^2\right]$$

$$= \lim_{n\to\infty} \frac{1}{2}\left[4 + \frac{4(4+1)}{2} + \frac{1}{4}\left(\frac{4(4+1)(2(4)+1)}{6}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{2}\left[4 + \frac{20}{2} + \frac{1}{4} \times \frac{180}{6}\right]$$

$$= \lim_{n\to\infty} \frac{1}{2}\left[4 + 10 + 7.5\right]$$

$$= \lim_{n\to\infty} \frac{1}{2}\left[21.5\right]$$

$$= \lim_{n\to\infty} \frac{1}{2} \times 21.5$$

$\therefore A = 10.75$

Simpson's Rule:

2) Use Simpson's Rule with $n=4$ to approximate the integral $\int_0^8 \sqrt{x}\, dx$

Soln:- $\therefore$ By Using Simpson's Rule,

$$\text{Area} = \int_a^b F(x)\,dx \approx \frac{\Delta x}{3}\left[y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 4y_{n-1} + y_n\right]$$

$$\approx \frac{\Delta x}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + y_6 \cdots)\right]$$

Comparing,

$$\int_a^b F(x)\,dx = \int_0^8 \sqrt{x}\cdot dx$$

we get $a=0$ and $b=8$

$y = F(x) = \sqrt{x}$

$\therefore$ For given $n = 4$, $\Delta x = \frac{8-0}{4} = 2$, $x_{i+1} = x_i + \Delta x$

$= a = 0$ and $x_4 = b = 8$

i	x_i	$y_i = \sqrt{x_i}$	multiplier	Multiplier$\times y_i$
0	$x_0 = 0$	$y_0 = \sqrt{0} = 0$	1	0
1	$x_1 = 0+2 = 2$	$y_1 = \sqrt{2}$	4	$4\sqrt{2}$
2	$x_2 = 2+2 = 4$	$y_2 = \sqrt{4} = 2$	2	4
3	$x_3 = 4+2 = 6$	$y_3 = \sqrt{6}$	4	$4\sqrt{6}$
4	$x_4 = 6+2 = 8$	$y_4 = \sqrt{8} = 2\sqrt{2}$	1	$2\sqrt{2}$

3) Find the area of the curve enclosed by the curve $y = x^2$ and $y = \sqrt{x}$

Soln:

The graph of the function $y = x^2$ and $y = \sqrt{x}$ are intersects at $x = 0$ and $x = 1$

From the fig $a = 0$, $b = 1$ bottom curve is $y = x^2$ and top curve is $y = \sqrt{x}$

$\therefore$ Area $= \int_a^b [f(x) - g(x)]\,dx = \int_a^b$ Top curve $-$ Bottom curve

$= \int_a^b [\sqrt{x} - x^2]\,dx$

$= \int_0^1 [\sqrt{x} - x^2]\,dx \quad \therefore a=0, b=1$

$= \left[\frac{2}{3}x^{3/2} - \frac{x^3}{2}\right]_0^1$

$= \frac{2}{3} - \frac{1}{3}$

$= \frac{1}{3}$ sq. units

4) Solved the differential equation $\frac{dy}{dx} = y(x+1) + (x+1)$

Soln:

$\frac{dy}{dx} = y(x+1) + (x+1)$

$\frac{dy}{dx} = (x+1)(y+1)$

$\frac{1}{(y+1)}\,dy = dx \cdot (x+1)$

$\therefore$ By Integrating both sides we get

$\int \frac{1}{y+1} \cdot dy = \int (x+1) \cdot dx$

$\therefore \log(y+1) = \frac{x^2}{2} + x + C$

5) Use Rules Method to solve initial values problem $\frac{dy}{dx} = e^{-y}, y(0) = 0, \; 0 \leq n \leq 1$ with step size $\Delta x = 0.1$

Soln: Given $\frac{dy}{dx} = y' = e^{-y}, y(0) = 0$ and step size $\Delta x = 0.1$

we compare with initial value problem $y' = \frac{dy}{dx}$

$= f(x,y), y(x_0) = y_0$

we get $f(x,y) = e^{-y}$ and $x_0 = 0$ and $y_0 = 0$

$= x_1 = x_0 + \Delta x = 0 + 0.1 = 0.1$

$x_2 = x_1 + \Delta x = 0.1 + 0.1 = 0.2$

$x_3 = x_2 + \Delta x = 0.2 + 0.1 = 0.3$

$x_4 = x_3 + \Delta x = 0.3 + 0.1 = 0.4$

$x_5 = x_4 + \Delta x = 0.4 + 0.1 = 0.5$

$x_6 = x_5 + \Delta x = 0.5 + 0.1 = 0.6$

$x_7 = x_6 + \Delta x = 0.6 + 0.1 = 0.7$

$x_8 = x_7 + \Delta x = 0.7 + 0.1 = 0.8$

$x_9 = x_8 + \Delta x = 0.8 + 0.1 = 0.9$

$x_{10} = x_9 + \Delta x = 0.9 + 0.1 = 1.0 = 1$

$\therefore$ By Rules Method Formula $y_n = y(m_n) = y_{n-1} + \Delta x$

$+ (x_{n-1}, y_{n-1})$

$\therefore$ for $x = 1, y_1 = y(x_1) = y_0 + \Delta x \, f(x_0, y_0)$

$= 0 + 0.1 \times f(0,0)$

$= 0 + 0.1 \times e^0 = 0.1 \times 1 = 0.1$

$\therefore$ For $x = 2$, $y_2 = y(x_2) = y_1 + \Delta x\, F(x_1, y_1)$

$= 0.1 + 0.1 \times F(0.1, 0.1)$

$= 0.1 + 0.1 + e^{-0.1}$

$= 0.1 + 0.1 \times 0.9048$

$= 0.1 + 0.09048$

$= 0.19048$

Next approximation use write in tabular Form

x	x_{n-1}	y_{n-1}	$F(x_{n-1}, y_{n-1})$	$\Delta x F(x_{n-1}, y_{n-1})$	$y_{n-1} + \Delta x F(x_{n-1}, y_{n-1})$
1	0	0	1	0.1	0.1
2	0.1	0.1	0.9048	0.09048	0.1904
3	0.2	0.1904	0.82655	0.082655	0.2731
4	0.3	0.2731	0.7609	0.07609	0.3492
5	0.4	0.3492	0.7052	0.07052	0.4197
6	0.5	0.4197	0.6572	0.06572	0.4854
7	0.6	0.4854	0.6154	0.06154	0.5470
8	0.7	0.5470	0.5867	0.05867	0.6048
9	0.8	0.6048	0.5461	0.05461	0.6595
10	0.9	0.6595	0.5171	0.05171	0.7112

Therefore $y(1) = 0.7112$

6) New ton's Law of cooling

Ans : . Cooling rate of hot object which is placed in cool environment is proportional to the difference in temperature between the object and the environment

. Similarly getting warm rate of cold object which is placed in to a warm environment if a hot object is placed into a cool environment is proportional to the difference in temperature between the object and the environment.

. Together, these result is Known as New ton's law of cooling

. To explain this into a Mathematical model, Suppose that $T = T(t)$ is the temperature of the object at time t and that T_e is the temperature of the environment, which is assumed to be constant.

. Since the rate of change $\frac{dT}{dt}$ is proportional to $T - T_e$,

we have,

$$\frac{dT}{dt} = K(T - T_e) \quad [\because K \text{ is proportionality constant}]$$

. Since $\frac{dT}{dt}$ is positive when $T < T_e$, and is negative when $T > T_e$ the sign of K must be negative.

. Thus if the temperature of the object is Known at some time, say $T = T_0$ at time $t = 0$, then Formula for the temperature, $T(t)$ can be obtained by solving the initial value problem.

$$\frac{dT}{dt} = K(T - T_e),\ T(0) = T_0$$

. We consider for $T > T_e$,

$$\frac{dT}{dt} = -K(T - T_e),\ T(0) = T_0$$

. Solving above equation we get

$$T(t) = T_e + (T_0 - T_e)\, e^{-K(T - T_0)}$$

1) Find the limits of following functions

i) $f(x,y) = x^2 - xy + y^2 - 2x + 3y - 2)$ at $(1,2)$

ii) $f(x,y) = \dfrac{2x+3y}{4x-3y}$ at $(1,2)$

iii) $f(x,y) = \dfrac{xy}{x^2+y^2}$

iv) $f(x,y) = \dfrac{y}{x} + \sin(xy)$ at $(1, \pi)$

Sol^n.

i) $\lim\limits_{(x,y)\to(1,2)} (x^2 - xy + y^2 - 2x + 3y - 2)$

$= [(1)^2 - (1)(2) + (2)^2 - 2(1) + 3(2) - 2]$

$= [1 - 2 + 4 - 2 + 6 - 2] = 5$

ii) $\lim\limits_{(x,y)\to(1,2)} \dfrac{2x+3y}{4x-3y}$

$= \dfrac{2(1) + 3(2)}{4(1) - 3(2)} = \dfrac{2+6}{4-6} = \dfrac{8}{-2} = -4$

iii) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$

No limit as $x=0$, $y=0$

iv) $\lim_{(x,y)\to(1,\pi)} \frac{y}{x} + \sin(xy)$

$= \frac{\pi}{1} + \sin(1\cdot\pi)$

$= \pi + \sin(1\cdot\pi)$

$= \pi + \sin(\pi)$

$= \pi + 0$

$= \pi$

2) Find all second order partial derivatives of f. Also verify whether $f_{xy} = f_{yx}$ given that $f(x,y)$ is $x^3y^4 + x^2y$

Soln:

$f(x,y) = x^3y^4 + x^2y$

$f_x = (3x^2)y^4 + (2x)y$
$= 3x^2y^4 + 2xy$

$f_y = x^3(4y^3) + x^2$
$= 4x^3y^3 + x^2$

$f_{xx} = 3(2x)y^4 + 2(1)y$
$= 6xy^4 + 2y$

$f_{yy} = 4x^3(3y^2)$
$= 12x^3y^4$

$f_{xy} = 3x^2(4y^3) + 2x(1)$
$= 12x^2y^3 + 2x$ --- (i)

$f_{yx} = 4(3x^2)y^3 + 2x$
$= 12x^2y^3 + 2x$ ---- (ii)

From eqn (i) and (ii) its verified $f_{xy} = f_{yx}$

3) Show that $f(x,y) = x^2y + xy^2$ is differentiable at $(-1,2)$ and find its linearization. then use it to approximate $f(-1.1, 2.1)$

Soln :

$f(x,y) = x^2y + xy^2$

$f(-1,2) = (-1)^2(2) + (-1)(2)^2 = -2$

$f_x = 2xy + y^2$

$f_x(-1,2) = 2(-1)(2) + (2)^2 = -4+4 = 0$

$f_y = x^2(1) + x(2y)$

$= x^2 + 2xy$

$f_y(-1,2) = (-1)^2 + 2(-1)(2) = 1-4 = -3$

Both f_x and f_y are continuous, so f is differntiab everywhere the linearization is

$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$L(x,y) = -2 + 0(x+1) + (-3)(y-2)$

$= -2 - 3y + 6$

$= -3y + 4$

The corresponding linear approximation is,

$x^2y + xy^2 = -3y + 4$

$\therefore$ It follows that $f(-1.1, 2.1) \approx -3(2.1) + 4 = -2.3$

In comparision $f(-1.1, 2.1) = (-1.1)^2(2.1) + (-1.1)(2.1)^2$

$f(-1.1, 2.1) = -2.31$

4) $z = f(x,y) = x^2 - 3xy + 2y^2$ [Chain Rule]

$x = 3\sin 2t, \quad y = 4\cos t\ 2t$

Find $\frac{dz}{dt}$

Soln:

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$$

$$= \frac{d}{dx}(x^2-3xy+2y^2)\frac{d}{dt}(3\sin 2t) + \frac{d}{dy}(x^2-3xy+2y^2)\frac{d}{dt}(4\cos 2t)$$

$$= (2xy^3 - y\sin x)\left[\frac{1}{t^2}\cdot 2t\right] + \left[x^2(3y^2)+\cos x\right]\left[4\cos 4t\right]$$

$$= \left[2(\log t^2)(\sin 4t)^3 - (\sin 4t)\sin(\log t^2)\right]\frac{1}{t} + \left[3(\log t^2)(\sin 4t)^2 + \cos(\log t^2)(4\cos 4t)\right]$$

$$= \left[6\sin 2t - 12\cos 2t\right](6\cos 2t) - 8\sin 2t\left[-9\sin 2t + 16\cos 2t\right]$$

$$= 36\sin 2t\cos 2t - 72\cos^2 2t + 72\sin^2 2t - 128\sin 2t\cos 2t$$

$$= 72\sin 2t - 72\cos^2 2t - 92\sin^2 2t\cdot\cos 2t$$

5) Final an equation for plane tangent to $2x^2+3y^2-z^2-4$ at $(1,1,-1)$

Soln:
$$F(x,y,z) = 2x^2+3y^2-z^2-4$$
$$F_x = 4x,$$
$$F_y = 6y$$
$$F_z = -2z$$
$$F_x(1,1,-1) = 4$$
$$F_y(1,1,-1) = 6$$
$$F_z(1,1,-1) = 2$$

Tangent Plane
$$F_x(x-x_0)+F_y(y-y_0)+F_z(z-z_0)=0$$
$$4(x-1)+6(y-1)+2(z+1)=0$$
$$4x-4+6y-6+2z+2=0$$
$$4x+6y+2z-8=0$$
$$\therefore\ 2x+3y+z=8$$

6) Find $\frac{dy}{dx}$ if y is defined implicity as Function of x by the equation
$$F(x,y) = x^2+xy-y^2+7x-3y-26=0$$

Soln:
$$F(x,y) = x^2+xy-y^2+7x-3y-26=0$$
$$\therefore \frac{dF}{dx} = 2x+y+7$$
$$\therefore \frac{dF}{dy} = x-2y-3$$

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$\therefore$ By implicit Formula

$$\therefore \frac{dy}{dx} = \frac{-df/dx}{df/dy}$$

$$= \frac{-(2x+y+7)}{(x-2y-3)}$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{-(2x+y+7)}{(x-2y-3)}}$$