

“A first glimpse on why MCSP is interesting”  
or something like that

...

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## 1 Introduction

In the Minimum Circuit Size Problem (MCSP), we are given a truth table of some Boolean function together with a positive integer  $s_n$  as input, and our task is to answer the question whether there exists a circuit of size at most  $s_n$  that computes the function represented by the given truth table.

Problem:	MSCP
Input:	A tuple $\langle T_n, s_n \rangle$ consisting of a truth table $T_n$ for a Boolean function of arity $n$ and an integer $s_n$
Question:	Is there a circuit $C_n$ of size at most $s_n$ computing $T_n$ ?

It is easy to see that MCSP is in **NP**. Namely, we can define a certificate as some proposed circuit  $C$  of size at most  $s_n$ , and verify whether  $C$  computes each entry of the truth table correctly in polynomial time. With that being said, a natural question arises: Is MCSP **NP**-complete?

In “Circuit Minimization Problem,” Valentine Kabanets and Jin-Yi Cai addressed the difficulty of showing MCSP to be **NP**-hard [KC00]. They showed some consequences that are unlikely to happen if there exists a polynomial-time reduction  $R$  from SAT to MCSP that is “natural,” in the sense that the size of the output depends on the size of the inputs only, and these sizes are polynomially related. Furthermore, the authors pointed out why it is challenging to prove  $\text{MCSP} \notin \mathbf{P}$  (again, by providing some consequences whose likelihood is questionable). Clearly, such proof would imply  $\mathbf{P} \neq \mathbf{NP}$  which goes beyond the currently known techniques.

In this review, we will provide some definitions required to understand the main results and key theorems of the paper in Section 2. Section 3 introduces some main consequences when  $\text{MCSP} \notin \mathbf{P}$  and MCSP is **NP**-hard under “natural” reductions. Finally, we conclude the review by providing some insightful remarks and directions for further research on this topic in Section 4.

## 2 Preliminaries

In this section, we provide the list of some definitions that, we believe, are useful for the readers.

**Definition 1.** The class Sub-Exponential:  $\text{SUBEXP} = \bigcap_{\epsilon > 0} \mathbf{DTIME}(2^{n^\epsilon})$

**Definition 2.** The class **QP** of languages decided by a TM in *quasi-polynomial* time defined by

$$\mathbf{QP} := \mathbf{DTIME}(n^{\text{polylog}(n)}) = \bigcup_{c>1} \mathbf{DTIME}(2^{\log^c n}) = \bigcup_{c>1} \mathbf{DTIME}(n^{\log^c n})$$

where we obtain the last equality since

$$2^{(\log n)^{c+1}} = \exp(\log^{c+1} n) = \exp(\log n \log^c n) = n^{\log^c n}.$$

Note that **QP** contains **P** since  $\text{polylog}(n) \in \Omega(1)$ . Also, **SUBEXP** contains **QP** since for every  $\varepsilon > 0$  and  $c > 1$  holds that  $\exp(\log^c n) \in o(\exp(n^\varepsilon))$ .

**Definition 3.** The class exponential time with linear exponential is defined as  $\mathbf{E} := \mathbf{DTIME}(2^{O(n)})$ .

Since  $\log^c n \in O(n)$  for any  $c > 0$ , we obtain that  $\mathbf{QP} \subseteq \mathbf{E}$ .

## 2.1 Natural Reductions

**Definition 4.** *Natural (Karp) Reduction:* For two problems  $A$  and  $B$  and a Karp reduction from  $A$  to  $B$ , we say the reduction  $R$  is natural if, for any instance  $I$  of  $A$ , the length of the output  $|R(I)|$ , as well as all possible output parameters  $s_n$ , depend only on the input length  $|I|$ . Furthermore,  $|I|$  and  $|R(I)|$  are polynomial related.

**Definition 5.** Assume we are given two languages  $A$  and  $B$ , where  $B$  describes the decision version of a search problem, meaning its instances are of the kind  $\langle y, s \rangle \in B$  where  $s \in \mathbb{N}$  is a numerical parameter. A many-one reduction  $R = \langle R_1, R_2 \rangle$  from  $A$  to  $B$ , whereas  $R_1$  maps to instances  $y$  of the original search problem and  $R_2$  to the parameters  $s$ , is called *natural* if for all instances  $x$  of  $A$  holds that

- there exists a  $c \in \mathbb{R}_+$  such that  $|x|^{1/c} \leq |R(x)| \leq |x|^c$ , meaning  $R_1$  does not suddenly grow or shrink, and
- $R_2$  depends only on the length of the instance  $x$ , i.e. there exists a function  $f: \mathbb{N} \rightarrow \mathbb{N}$  such that  $R_2(I) = f(|x|)$ .

For example,  $\text{SAT} \leq_p \text{3SAT}$  is “natural.” Namely, given  $\varphi$  - an instance in **SAT**, the general idea is to split some clause  $C$  in  $\varphi$  of size  $k > 3$  into a pair of two equivalent clauses  $C_1$  of size  $k - 1$  and  $C_2$  of size 3 and we repeat the process until we get the desired 3 - *CNF* formula,  $\varphi'$ . Thus, the length of  $\varphi'$  is only dependent on  $\varphi$  as we just add more clauses solely based on everything from the original formula, intuitively speaking.

Other textbook reductions that we know and love (such as  $\text{3SAT} \leq_p \text{VERTEXCOVER}$ , etc.) are “natural” in this sense. Unnatural reductions, on the other hand, are more contrived.

**Example 6.** We reduce from **Partition**<sup>1</sup> to **SubsetSum**:<sup>2</sup>

$$R(\{a_1, \dots, a_n\}) := \begin{cases} \langle \{2, 4, 8\}, 7 \rangle & \text{if } \sum_{i=1}^n a_i \equiv_2 1 \\ \langle \{a_1, \dots, a_n\}, \frac{1}{2} \sum_{i=1}^n a_i \rangle & \text{otherwise.} \end{cases}$$

<sup>1</sup>The partition problem asks, given a multiset of positive integers  $\{a_1, \dots, a_n\}$  whether there exists a partition  $(S, T)$  of  $\{1, \dots, n\}$  such that  $\sum_{i \in S} a_i = \sum_{i \in T} a_i$ .

<sup>2</sup>The partition problem asks for an instance  $\langle A, s \rangle$  of a multiset of positive integers  $A = \{a_1, \dots, a_n\}$  and an integer  $s$  whether there exists a subset  $S \subseteq \{1, \dots, n\}$  such that  $s = \sum_{i \in S} a_i$ .

Now, while this reduction is correct (if  $\sum_{i=1}^n a_i$  is congruent to 1 modulo 2, it is impossible to divide the set into two partitions of equal sum; the tuple  $\langle \{2, 4, 8\}, 7 \rangle$  is a no-instance to **SubsetSum**) and can be carried out in polynomial time. It is not natural for two reasons: First, the output size does not strictly depend on the input size as, in the case of an obvious no-instance to **Partition**, we map directly to a no-instance of **SubsetSum** of constant size. Second, the numerical parameter  $s$  is not a function of the input size only. For example, the two instances  $\{1, \dots, 1\}$  ( $n$ -times) and  $\{2^n\}$  are of equal size, but have totally different sums. With a little more care, we can, however, make this a natural reduction: Let

$$R'(\{a_1, \dots, a_n\}) := \langle \{2a_1, \dots, 2a_n, r, 2^{\sigma+2}\} \rangle \quad \text{where } \sigma := \text{size}(\{a_1, \dots, a_n\}) \\ \text{and } r := 2^{\sigma+2} - \sum_{i=1}^n a_i.$$

Now, the numerical parameter  $s$  is obviously only a function in the input size  $\sigma$ . Regarding correctness, first note that  $2^{\sigma+2} > \sum_{i=1}^n 2a_i$ . This implies that any subset with a sum of  $2^{\sigma+2}$  necessarily contains  $r$  and thus

$$2^{\sigma+2} = r + \sum_{i \in S} 2a_i \iff \frac{1}{2} \sum_{i=1}^n a_i = \sum_{i \in S} a_i$$

where  $S$  is the rest of the subset without  $r$ .

## 2.2 Pseudorandom generator

## 3 Main Results

### 3.1 MCSP and NP-completeness

To begin with, we want to emphasize that researchers have not figured out yet whether it is possible to prove the **NP**-hardness of **MCSP** or not. The difficulty of such proof was explicitly addressed through some implications for *Circuit Complexity* and **BPP**. In other words, the authors provided some consequences that are still unknown to the current state of the art if **MCSP** is **NP**-hard under the “natural” Karp reduction.

Before we move on to some key theorems of this section, let us examine some lemmas that are useful for establishing.

**Lemma 7.**  $\mathbf{QP}^{\mathbf{QP}} \subseteq \mathbf{QP}$ .

*Proof.* Let  $M$  be a TM deciding a language  $L \in \mathbf{QP}^{\mathbf{QP}}$  in quasi-polynomial time, say  $\exp(\log^c n)$ . We show that carrying out the oracle computation instead of calling the oracle does still guarantee a quasi-polynomial running time. To this end, let  $M'$  be the TM deciding the **QP**-oracle, say in time  $\exp(\log^{c'} m)$ . Now, any input to  $M'$  is at most of length  $m = \exp(\log^c n)$ . We therefore need time at most  $\exp(\log^c(\exp(\log^c n))) = \exp(\log^{cc'} n)$  for one oracle computation. At the same time, there are at most  $\exp(\log^c n)$  calls, resulting in a total running time bound of  $\exp(\log^c(\exp(\log^{cc'} n))) = \exp(\log^{c^2 c'} n)$  which is still quasi-polynomial.  $\square$

**Lemma 8.** *If  $\mathbf{NP} \subseteq \mathbf{QP}$  then  $\mathbf{PH} \subseteq \mathbf{QP}$ .*

*Proof.* Recall that we can define **PH** in terms of oracles by

$$\mathbf{PH} = \bigcup_{n>0} \underbrace{\mathbf{NP}^{\mathbf{NP}^{\dots \mathbf{NP}}}}_{n \text{ times}}.$$

We can use a straightforward induction over  $n$  to show that  $\mathbf{PH} \subseteq \mathbf{QP}$ . Note that the induction basis is just the assumption. Furthermore, by the inductive hypothesis, we have that

$$\underbrace{\mathbf{NP}^{\mathbf{NP}^{\dots \mathbf{NP}}}}_{n \text{ times}} \subseteq \mathbf{QP} \implies \underbrace{\mathbf{NP}^{\overbrace{\mathbf{NP}^{\mathbf{NP}^{\dots \mathbf{NP}}}^{n \text{ times}}}}}_{n+1 \text{ times}} \subseteq \mathbf{NP}^{\mathbf{QP}}.$$

Now,  $\mathbf{NP}^{\mathbf{QP}} \subseteq \mathbf{QP}^{\mathbf{QP}} \subseteq \mathbf{QP}$  by Lemma 7.  $\square$

**Lemma 9.**  $\mathbf{QP}^{\Sigma_k^p}$  contains a language which does not belong to  $\mathbf{P}_{/\text{poly}}$  for some  $k \in \mathbb{N}$ .

*Proof.* The proof follows a nonuniform diagonalization argument. We first define a language which will be hard to compute for any polynomial-size circuit family: Let  $L'$  be the language consisting of tuples  $\langle x, 1^{\exp(\log^3 n)} \rangle$  with  $n := |x|$  such that  $C(x) = 1$  where  $C$  is the lexicographically first circuit of size  $\exp(\log^3 n)$  which is not computed by any circuit of size  $\exp(\log^2 n)$ . The existence of such a circuit for sufficiently large  $n$  follows from a slightly more careful analysis of the nonuniform hierarchy theorem (Theorem 6.22).

We can decide membership  $\langle x, 1^{\exp(\log^3 n)} \rangle \in L'$  by a  $\Sigma_4^p$ -oracle as in Problem (1c) of Homework 7. Finally, we define our language  $L$  of superpolynomial circuit complexity as the output of a  $\mathbf{QP}^{\Sigma_4^p}$ -machine: Given an input  $x \in \{0, 1\}^n$ , query the oracle for  $L'$  with  $\langle x, 1^{\exp(\log^3 n)} \rangle$  and output its answer.

We constructed this language such that it is hard for a polynomial-size circuit to compute. To see this, assume to the contrary that there is a  $n^a$ -size circuit family. However, since  $n^a \in o(\exp(\log^2 n))$  and we particularly excluded any circuits of size less than  $\exp(\log^2 n)$ , this is a contradiction.  $\square$

**Lemma 10.** There are at most  $n^{\text{polylog}(n)}$  different circuits of size  $\log^c n$  for a constant  $c$ .

*Proof.* In a circuit with  $s \in \mathbb{N}$  many gates and inputs, each gate is connected to at most two out of  $s$  gates, and computes one of the functions  $\wedge, \vee, \neg$ . This means, there are at most  $3 \cdot s^2$  choices to construct each gate and thus  $(3s^2)^s$  choices to construct the whole circuit. Setting  $s := \log^c n$  gives us

$$(3 \log^{2c} n)^{\log^c n} = O(\exp((\log^{2c} n) \cdot (\log^c n))) = O(\exp(\log^{3c} n)) = n^{\text{polylog}(n)}$$

many ways to construct a circuit of size  $\log^c n$ .  $\square$

Now, we are ready to look at the first key theorem which is about the implication for *Circuit Complexity* if MCSP is  $\mathbf{NP}$ -hard under the *natural* reduction.

**Theorem 11** ([KC00]). If MCSP is  $\mathbf{NP}$ -hard under a natural reduction from SAT, then

1.  $\mathbf{E}$  contains a family of Boolean functions  $f_n$  not in  $\mathbf{P}_{/\text{poly}}$  (i.o.), and
2.  $\mathbf{E}$  contains a family of Boolean functions  $f_n$  of circuit complexity  $2^{\Omega(n)}$  (i.o.), unless  $\mathbf{NP} \subseteq \mathbf{SUBEXP}$  [the proof for this is yet missing]

*Proof.* We separate the prove along two cases.

- Case 1:  $\mathbf{NP} \subseteq \mathbf{QP}$

Applying Lemma 7 and Lemma 8 to this assumption yields that  $\mathbf{QP}^{\mathbf{PH}} \subseteq \mathbf{QP}^{\mathbf{QP}} \subseteq \mathbf{QP} \subseteq \mathbf{E}$ . Lemma 9 shows that  $\mathbf{E}$  also contains a language of superpolynomial circuit complexity. Hence,  $\mathbf{E} \not\subseteq \mathbf{P}_{/\text{poly}}$

- Case 2:  $\mathbf{NP} \not\subseteq \mathbf{QP}$

While it is (computationally) trivial to choose a no-instance for  $\mathbf{SAT}$  of any given size, it is not clear how to do so for  $\mathbf{MCSP}$ . The key idea for this part is, therefore, to use the natural reduction  $R$  to obtain hard instances for  $\mathbf{MCSP}$ , i.e. a family of truthtables which cannot be represented by circuits of polynomial size.

We start out by picking a quite arbitrary infinite set  $U$  of unsatisfiable CNF-formulae, say,  $U := \{\phi_n \mid n \in \mathbb{N}\}$  where

$$\phi_n(x_1, \dots, x_n) := (x_1 \wedge \bar{x}_1) \wedge (x_3 \wedge x_4 \wedge \dots \wedge x_n).$$

By construction,  $|\phi_n| \in \Theta(n)$ . Now, based on this, we want to apply  $R$  to define our hard language. For each  $k \in \mathbb{N}$ , let  $T_k$  be the truthtable in  $k$ -variables such that  $\langle T_k, s_n \rangle = R(\phi_n)$  and  $n$  is minimal. If no  $\phi_n$  maps to a truthtable in  $k$  variables, simply let  $T_k \equiv 0$ .

We know that  $R$  is natural, which in particular implies  $n^{1/c} \leq |T_k| \leq n^c$  (with an appropriate  $c \in \mathbb{N}$ ) for every  $\phi_n$  mapping to a truthtable  $T_k$ . Since  $|T_k| = 2^k$ , this is equivalent to  $\log(n^{1/c}) \leq k \leq \log(n^c)$  implying  $k = \Theta(\log n)$  or  $n = 2^{\Theta(k)}$ . Now, we proceed with defining our hard language

$$L := \{x \in \{0, 1\}^k \mid k \in \mathbb{N}, T_k(x) = 1\}$$

to obtain the following.

- (i)  $L \in \mathbf{E}$ : We show how to construct a machine to decide  $L$  in time  $2^{O(k)}$ . Given an input  $x \in \{0, 1\}^k$ , we first need to know which  $\phi_n$  maps to  $T_k$  under  $R$ .

To each candidate  $\phi_n$  for  $n \in \mathbb{N}$ , we apply  $R$  and obtain  $\langle T_{k'}, s_n \rangle = R(\phi_n)$ . We then compare whether  $k = k'$  and if so, output  $T_k(x)$ . The reduction  $R$  runs in polynomial time, say  $n^a$  for an  $a \in \mathbb{N}$ . By the above construction,  $n = 2^{\Theta(k)}$  which means we only need to check  $2^{\Theta(k)}$  candidates whereas each check runs in time at most  $n^a = (2^{\Theta(k)})^a = 2^{\Theta(k)}$ . If no candidate maps to a truthtable on  $k$  variables, we know that  $T_k \equiv 0$  by definition and reject since  $T_k(x) = 0 \neq 1$ . Overall, the decision procedure took time  $2^{O(k)}$ , proving that  $L \in \mathbf{E}$ .

- (ii)  $L \notin \mathbf{P}_{/\text{poly}}$ : We start by showing that the parameter  $s_n$  produced by  $R$  is superpolynomial in  $\log n$ . Let us thus assume to the contrary that it is bound by a polynomial  $s_n \leq \log^b n$  for any  $b \in \mathbb{N}$ . Now, this yields a simple strategy to decide  $\mathbf{SAT}$  in quasi-polynomial time: Given a CNF-formula  $\phi$  of size  $n$ , we apply  $R$  to obtain  $\langle T_{k'}, s_n \rangle := R(\phi)$  with  $s_n \leq \log^b n$ . We are going to decide the membership of this instance to  $\mathbf{MSCP}$  instead of solving the original satisfiability problem. By Lemma 10, there are at most  $n^{\text{polylog}(n)}$  circuits of size  $s_n$ . We enumerate these and check for each circuit whether it represents  $T_{k'}$ . Note that testing whether a circuit  $C$  of size  $s_n$  represents  $T_{k'}$  only requires us to do  $2^{k'}$  evaluations of  $C$ , each of which take time  $O(s_n)$ . Overall, we can decide the membership to  $\mathbf{MSCP}$ , and as such the satisfiability of  $\phi$ , in

quasi-polynomial time. However, this implies that  $\text{SAT} \in \mathbf{QP}$  contradicting our assumption that  $\mathbf{NP} \not\subseteq \mathbf{QP}$ .

We therefore established that  $s_n$  is superpolynomial in  $\log n$ . At the same time,  $R$  is a natural reduction, meaning that  $s_n$  is the same for every input of size  $n$ . In particular, we obtain that since we set  $\langle T_k, s_n \rangle = R(\phi_n)$ , the parameter  $s_n$  is superpolynomial in  $\log n = \Theta(k)$ . As  $\phi_n$  is a no-instance to  $\text{SAT}$ , we conclude that  $T_k$  cannot be represented by a polynomial-size circuit family. In other words,  $L \notin \mathbf{P}_{\text{poly}}$ .

□

Now, we will look at the implications for  $\mathbf{BPP}$  when  $\mathbf{NP}$ -hard under a natural reduction from  $\text{SAT}$ . The following two theorems on hardness-randomness trade-offs are needed to establish the one about  $\mathbf{BPP}$ .

**Theorem 12** ([IW97]). *If the class  $\mathbf{E}$  contains a family of Boolean functions  $f_n : \{0,1\}^n \rightarrow \{0,1\}$  of circuit complexity at least  $2^{\epsilon n}$  for some  $\epsilon > 0$ , (i.o.), then  $\mathbf{BPP} = \mathbf{P}$  (i.o.).*

**Theorem 13** ([BFNW93]). *If the class  $\mathbf{EXP}$  contains a family of Boolean functions of superpolynomial circuit complexity (i.o.), then  $\mathbf{BPP} \subseteq \mathbf{SUBEXP}$  (i.o.).*

**Theorem 14** ([KC00]). *If  $\text{MCSP}$  is  $\mathbf{NP}$ -hard under a natural reduction from  $\text{SAT}$ , then*

1.  $\mathbf{BPP} \subseteq \mathbf{SUBEXP}$  (i.o.), and
2.  $\mathbf{BPP} = \mathbf{P}$ , unless  $\mathbf{NP} \subseteq \mathbf{SUBEXP}$ .

Intuitively speaking, if  $\text{MCSP}$  is  $\mathbf{NP}$ -hard under a natural reduction from  $\text{SAT}$ , then a problem in  $\mathbf{BPP}$  can be efficiently solved. Finally, taking everything together, we obtain a nice corollary as follows.

**Corollary 15.** *If  $\text{MCSP}$  is  $\mathbf{NP}$ -hard under a natural reduction from  $\text{SAT}$ , then  $\mathbf{BPP} \subsetneq \mathbf{E}$*

*Proof.* Idea: From Theorem 14, we know that if  $\text{MCSP}$  is  $\mathbf{NP}$ -hard under a natural reduction from  $\text{SAT}$ , then  $\mathbf{BPP} \subseteq \mathbf{SUBEXP}$ . Thus, we can diagonalize against  $\mathbf{SUBEXP}$  with a Turing Machine  $M$  in  $\mathbf{E}$  and  $M$  should mess up when the input length is large enough. □

### 3.2 MCSP and $\mathbf{P}$

As mentioned previously, it is still unknown to us that whether we would be able to reduce  $\text{MCSP}$  to  $\text{SAT}$  using the natural Karp reduction. Namely, if such reduction exists, it yields some astonishing results (as stated in 3.1) which are unlikely to be found soon. Nonetheless, the assumption that  $\text{MCSP}$  is in  $\mathbf{P}$  gives some surprising consequences.

For starter, if  $\text{MCSP}$  can be efficiently solved, then we will be able to factor Blums integers <sup>3</sup> well on the average case. Specifically, let us examine the following theorem and its consequence.  
[... to be added]

Intuitively speaking, if  $\text{MCSP}$  is in  $\mathbf{P}$ , then our current cryptography breaks because a fast algorithm for factoring (at least for the average-case) can break any type of cryptography. It is mainly believed that factoring is hard and therefore it is very much unlikely that  $\text{MCSP}$  can be efficiently solved; however, nothing has been shown to draw an ultimate conclusion about this assumption. We will now explore some other unlikely consequences that would happen if such proof exists.

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<sup>3</sup>a Blum integer is the product of two primes, which congruent to 3 mod 4

## 4 Conclusion

[Will be added when we are done with section 3, but basically, we plan to briefly introduce some other work that tackled the open problems introduced in this paper and then conclude with some further plans of research.]

## References

- [BFNW93] L  szl   Babai, Lance Fortnow, Noam Nisan, and Avi Wigderson. Bpp has subexponential time simulations unless  $\text{exptime}$  has publishable proofs. *Computational Complexity*, 3(4):307–318, 1993.
- [IW97] Russell Impagliazzo and Avi Wigderson.  $P = \text{bpp}$  if  $e$  requires exponential circuits. pages 220–229, 01 1997.
- [KC00] Valentine Kabanets and Jin-Yi Cai. Circuit minimization problem. In *Proceedings of the Thirty-Second Annual ACM Symposium on Theory of Computing*, STOC '00, page 73–79, New York, NY, USA, 2000. Association for Computing Machinery.