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Periodic Solutions of Differential Systems, Galerkin's Procedure and the Method of Averaging*

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In a previous paper [1] we discussed Galerkin's method in the determination and existence analysis of periodic solutions of periodic nonlinear differential systems of the form $dx/dt = X(x, t)$, therefore also strongly nonlinear. We proved there that isolated periodic solutions (see Section 1) possess Galerkin approximations of any order sufficiently high, that these converge uniformly with their first order derivatives toward the periodic solutions, and that the existence of exact isolated periodic solutions can be proved—under smoothness hypotheses—from the existence of Galerkin approximations of sufficiently high order, as they can be obtained by present day electronic computers. An existence analysis based on Galerkin approximations—even in association with very low orders of approximation and more topological in character—had been previously initiated by Cesari [2], also in view of qualitative applications.

In the present paper we discuss the Galerkin approach in comparison with the method of averaging [3] for periodic nonlinear differential systems containing a small parameter

$$\frac{dx}{dt} = \lambda X(x, t). \tag{S}$$

The main result of the present paper is an explicit formula for a bound λ_0 for the parameter λ within which it can be stated that the method of averaging is effective in affirming the existence of a periodic exact solution for all positive $\lambda \leqslant \lambda_0$.

Needless to say, our results can be easily extended to periodic systems of the more general form

$$\frac{dx}{dt} = \lambda X(x, t, \lambda)$$

with slight modifications.

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Actual numerical evaluations in association with a series of concrete examples will be discussed in another paper jointly with A. Reiter [*4*], together with the numerical results obtained by Galerkin's method and the use of high speed electronic computers.

1. Preliminaries

We shall consider first periodic differential systems

$$\frac{dx}{dt} = X(x, t), \tag{1.1}$$

where x and $X(x, t)$ are vectors of the same dimension and $X(x, t)$ is periodic in t with period 2π. For the determination and approximation of periodic solutions of period 2π, one may consider a trigonometrical polynomial

$$x_m(t) = a_0 + \sqrt{2} \sum_{n=1}^{m} (a_n \cos nt + b_n \sin nt) \tag{1.2}$$

with undetermined coefficients a_0, a_1, b_1, $\cdots$, a_m, b_m, and one may try to determine, if possible, these coefficients so that $x_m(t)$ satisfies exactly the system

$$\frac{dx_m(t)}{dt} = P_m X[x_m(t), t], \tag{1.3}$$

where P_m denotes the truncation of the Fourier series discarding the harmonic terms of order higher than m. A trigonometrical polynomial (1.2) satisfying (1.3) is called a Galerkin approximation of order m.

Equation (1.3) is evidently equivalent to the system of equations

$$\begin{aligned}
F_0^{(m)}(\alpha) &= \frac{1}{2\pi} \int_0^{2\pi} X[x_m(t), t]\, dt = 0, \\
F_n^{(m)}(\alpha) &= \frac{1}{\sqrt{2}\,\pi} \int_0^{2\pi} X[x_m(t), t] \cos nt\, dt - nb_n = 0, \\
G_n^{(m)}(\alpha) &= \frac{1}{\sqrt{2}\,\pi} \int_0^{2\pi} X[x_m(t), t] \sin nt\, dt + na_n = 0, \\
&\qquad (n = 1, 2, \cdots, m),
\end{aligned} \tag{1.4}$$

where $\alpha = (a_0, a_1, b_1, ,\cdots, a_m, b_m)$, $F_0^{(m)}$, $F_n^{(m)}$, $G_n^{(m)}$ are the functions of α defined by these relations, and $x_m(t)$ is given by (1.2).

System (1.4) is called the determining system (or equation) for the Galerkin approximation of order m.

In the paper [*1*] mentioned above we considered *isolated periodic solutions*, that is, periodic solutions of (1.1) for which the multipliers of the relative first variational system are all different from one. In particular we proved in [*1*]:

THEOREM 1. *Let us assume $X(x, t)$ and its first order partial derivatives with respect to the x-coordinates are once continuously differentiable with respect to the same x-coordinates and t in the region $D \times L$, where D is a closed bounded region of the x-space and L is the real line. If there is an isolated periodic solution $x = \hat{x}(t)$ of (1.1) lying inside D, then, for sufficiently large m_0, there exist Galerkin approximations $x = \bar{x}_m(t)$ of all orders $m \geqslant m_0$ lying in D, which converge uniformly as $m \to \infty$ to the exact solution $x = \hat{x}(t)$ together with their first order derivatives.*

For the proof we refer to Theorem I and the corollary of Theorem II of our paper [*1*].

Theorem and the corresponding numerical results which we shall discuss in [*4*] show that Galerkin's procedure is really effective in the determination of isolated periodic solutions provided the given system has the smoothness specified in the theorem.

As in paper [*1*] we use Euclidean norms for vectors and matrices, and we denote them by the symbol $\| \ \|$. In addition, for continuous periodic vector functions we shall use square norms $\| \ \|_q$ and uniform norms $\| \ \|_n$, which are defined as usual as follows:

$$\|f\|_q = \left[\frac{1}{2\pi}\int_0^{2\pi} \|f(t)\|^2 \, dt\right]^{1/2},$$

$$\|f\|_n = \max_t \|f(t)\| .$$

For any continuously differentiable periodic vector function $f(t)$ with period 2π we know that

$$\|f - P_m f\|_n \leqslant \sigma(m) \, \|\dot{f}\|_q , \tag{1.5}$$

$$\|f - P_m f\|_q \leqslant \sigma_1(m) \, \|\dot{f}\|_q , \tag{1.6}$$

where $\cdot = d/dt$ and

$$\begin{aligned} \sigma(m) &= \sqrt{2}\,[(m+1)^{-2} + (m+2)^{-2} + \cdots]^{1/2} < \sqrt{2}\, m^{-1/2}, \\ \sigma_1(m) &= (m+1)^{-1}. \end{aligned} \tag{1.7}$$

For the proofs of (1.5) and (1.6) see Lemma 2.1 of [*1*] (or (2.9) and (2.17) of [*2*]).

In the present paper we will use the following theorems and definitions of [*1*].

THEOREM 2. *Let us consider a linear periodic system*

$$\frac{dx}{dt} = A(t)\,x + \phi(t), \tag{1.8}$$

where $A(t)$ is a continuous periodic matrix with period 2π and $\phi(t)$ is a continuous periodic vector function with the same period. If the multipliers of solutions of the corresponding homogeneous system

$$\frac{dy}{dt} = A(t)\,y \tag{1.9}$$

are all different from one, then (1.8) has one and only one periodic solution with period 2π, which is given by

$$x(t) = \int_0^{2\pi} H(t, s)\,\phi(s)\,ds, \tag{1.10}$$

where $H(t, s)$ is the piecewise continuous periodic matrix defined by

$$H(t,s) = \begin{cases} \Phi(t)\,[E - \Phi(2\pi)]^{-1}\,\Phi^{-1}(s) & \text{for} \quad 0 \leqslant s < t \leqslant 2\pi, \\ \Phi(t)\,[E - \Phi(2\pi)]^{-1}\,\Phi(2\pi)\,\Phi^{-1}(s) & \text{for} \quad 0 \leqslant t \leqslant s \leqslant 2\pi, \end{cases} \tag{1.11}$$

and

$$H(t, s) = H(t + 2m\pi, s + 2n\pi) \qquad (m, n\text{: integers}).$$

Here E denotes the unit matrix and $\Phi(t)$ is the fundamental matrix solution of (1.9) such that $\Phi(0) = E$.

For the proof, see Proposition 1 of the paper [*1*].

The formula (1.10) defines a linear mapping H in the space of continuous periodic functions. This mapping is called the H-mapping corresponding to the matrix $A(t)$. Corresponding to the two norms considered above for continuous periodic vector functions we have also two norms $\| H \|_q$ and $\| H \|_n$ of the mapping H. By Schwartz's inequality,

$$\begin{aligned} \| H \|_q &\leqslant \left[\int_0^{2\pi}\int_0^{2\pi} \sum_{k,\ell} H_{k\ell}^2(t, s)\,ds\,dt\right]^{1/2}, \\ \| H \|_n &\leqslant \left[2\pi \cdot \max_t \int_0^{2\pi} \sum_{k,\ell} H_{k\ell}^2(t, s)\,ds\right]^{1/2}, \end{aligned} \tag{1.12}$$

where $H_{k\ell}(t, s)$ are the elements of the matrix $H(t, s)$.

THEOREM 3. *Let us consider a real system of differential equations*

$$\frac{dx}{dt} = X(x, t), \tag{1.13}$$

where x and $X(x, t)$ are vectors of the same dimension and $X(x, t)$ is periodic in t with period 2π and is continuously differentiable with respect to the x-coordinates in the region $D \times L$ where D is a given region of the x-space and L is the real line.

Assume that (1.13) *has a periodic approximate solution $x = \bar{x}(t)$ lying in D and that there are a continuous periodic matrix $A(t)$, a positive constant δ, and a non-negative constant $\ae < 1$ such that*

(i) *the multipliers relative to the linear homogeneous system $dy/dt = A(t)y$ are all different from one,*

(ii) $D_\delta = \{x \mid \| x - \bar{x}(t) \| \leqslant \delta$ *for some* $t \in L\} \subset D$,

(iii) $\| \psi[x, t] - A(t) \| \leqslant \ae/M$ *for all* x, t *such that* $\| x - \bar{x}(t) \| \leqslant \delta$,

(iv) $Mr/(1 - \ae) \leqslant \delta$.

Here $\psi(x, t)$ is the Jacobian matrix of $X(x, t)$ with respect to x; M is a positive constant such that $\| H \|_n \leqslant M$, where H is the H-mapping corresponding to $A(t)$; r is a non-negative constant such that

$$\| d\bar{x}(t)/dt - X[\bar{x}(t), t] \|_n \leqslant r.$$

Then the given system (1.13) *has one and only one periodic solution $x = \hat{x}(t)$ in D_δ and this is an isolated periodic solution. Further, for $x = \hat{x}(t)$, we have*

$$\| \hat{x}(t) - \bar{x}(t) \| \leqslant \frac{Mr}{1 - \ae}. \tag{1.14}$$

For the proof, see Proposition 3 of paper [*1*].

Theorem 3 gives conditions under which the existence of an exact isolated periodic solution can be inferred from the existence of a periodic approximate solution. In paper [*1*] it was shown that, if the given system has the smoothness specified in Theorem 1, the existence of an isolated periodic solution can be always inferred by means of Theorem 3 from a computed Galerkin approximation of sufficiently high order. In practical applications this is very important, because one can state the existence of an exact periodic solution from the computed results by checking the conditions of Theorem 3. For details and practical applications, see paper [*4*].

2. Galerkin Approximations for Systems (S)

For systems (S) we assume that $X(x, t)$ is periodic in t with period 2π and is twice continuously differentiable with respect to the x-coordinates in the region $D \times L$, where D is a closed bounded region of the x-space and L is the real line.

As we deduce from (1.4), the determining equation for system (S) is evidently

$$
\begin{aligned}
F_0^{(m)}(\alpha) &= \frac{\lambda}{2\pi}\int_0^{2\pi} X[x_m(t), t]\,dt = 0,\\
F_n^{(m)}(\alpha) &= \frac{\lambda}{\sqrt{2}\,\pi}\int_0^{2\pi} X[x_m(t), t]\cos nt\,dt - nb_n = 0,\\
G_n^{(m)}(\alpha) &= \frac{\lambda}{\sqrt{2}\,\pi}\int_0^{2\pi} X[x_m(t), t]\sin nt\,dt + na_n = 0\\
&\qquad (n = 1, 2, \cdots, m),
\end{aligned}
\tag{2.1}
$$

where $\alpha = (a_0\,, a_1\,, b_1\,, \cdots, a_m\,, b_m)$ and

$$
x_m(t) = a_0 + \sqrt{2}\sum_{n=1}^{m}(a_n\cos nt + b_n\sin nt). \tag{2.2}
$$

Let us suppose that $|\,\lambda\,|$ is small and that the equation

$$
\int_0^{2\pi} X(a, t)\,dt = 0 \tag{2.3}
$$

has a real solution $a = \tilde{a}_0 \in D$.

Then, we shall consider the numerical vector $\tilde{\alpha}$ defined below as an approximate solution $\alpha = \tilde{\alpha} = (\tilde{a}_0\,, \tilde{a}_1\,, \tilde{b}_1\,, \cdots, \tilde{a}_m\,, \tilde{b}_m)$ of (2.1):

$$
\begin{aligned}
\tilde{a}_n &= -\frac{\lambda}{\sqrt{2}\,n\pi}\int_0^{2\pi} X(\tilde{a}_0\,, t)\sin nt\,dt,\\
\tilde{b}_n &= \frac{\lambda}{\sqrt{2}\,n\pi}\int_0^{2\pi} X(\tilde{a}_0\,, t)\cos nt\,dt.\\
&\qquad (n = 1, 2, \cdots, m).
\end{aligned}
\tag{2.4}
$$

Let $\tilde{x}_m(t)$ be the trigonometric polynomial obtained from $x_m(t)$ replacing α by $\tilde{\alpha}$. Then we may consider $\tilde{x}_m(t)$ as an approximate Galerking approximation of order m.

Now let us consider the function

$$
f(t) = \lambda\int_0^{t} X(\tilde{a}_0\,, s)\,ds. \tag{2.5}
$$

Since

$$
\int_0^{2\pi} X(\tilde{a}_0\,, s)\,ds = 0,
$$

$f(t)$ is periodic in t with period 2π. In addition, as is readily seen by integration by parts,

$$\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(t) \cos nt \, dt = \tilde{a}_n ,$$

$$\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f(t) \sin nt \, dt = \tilde{b}_n ,$$

$$(n = 1, 2, \cdots, m).$$

Therefore, if we put

$$\tilde{x}(t) = \tilde{a}_0 - \frac{1}{2\pi} \int_0^{2\pi} f(t) \, dt + f(t), \tag{2.6}$$

then

$$\tilde{x}_m(t) = P_m \tilde{x}(t). \tag{2.7}$$

Equations (2.5) and (2.6) show that $x = \tilde{x}(t)$ is the first approximate solution of (S) in the method of averaging [3].

A proper Galerkin approximation, if it exists, is a trigonometric polynomial $\bar{x}_m(t)$ obtained from $x_m(t)$ replacing α by $\bar{\alpha}$, where $\bar{\alpha} = (\bar{a}_0, \bar{a}_1, \bar{b}_1, \cdots, \bar{a}_m, \bar{b}_m)$ is an exact solution of the determining equation (2.1).

In the present paper, the three approximate solutions $\tilde{x}(t)$, $\tilde{x}_m(t)$, and $\bar{x}_m(t)$ are all taken into consideration and compared.

3. Lemmas

By the assumptions on the function $X(x, t)$ there are positive constant K, K_1, and K_2 such that

$$\max_{D \times L} \| X(x, t) \| \leqslant K, \qquad \max_{D \times L} \| \psi(x, t) \| \leqslant K_1 ,$$

$$\left[\max_{D \times L} \sum_{k, \ell, p} \left(\frac{\partial^2 X_k(x, t)}{\partial x_\ell \partial x_p} \right)^2 \right]^{1/2} \leqslant K_2 , \tag{3.1}$$

where $\psi(x, t)$ is the Jacobian matrix of $X(x, t)$ with respect to x and $X_k(x, t)$ and x_ℓ are, respectively, the components of the vectors $X(x, t)$ and x.

In what follows, we will prove a few lemmas which are necessary for subsequent discussions.

Lemma 1. *For $\tilde{x}(t)$ and $\tilde{x}_m(t)$,*

$$\| \tilde{x}(t) - \tilde{a}_0 \| , \qquad \| \tilde{x}_m(t) - \tilde{a}_0 \| \leqslant | \lambda | \cdot \frac{\pi}{\sqrt{3}} K, \tag{3.2}$$

$$\| \tilde{x} - \tilde{x}_m \|_n \leqslant | \lambda | K \sigma(m). \tag{3.3}$$

Proof. By (2.6) and (2.5),

$$\frac{d\tilde{x}(t)}{dt} = \lambda X(\tilde{a}_0, t). \tag{3.4}$$

On the other hand, by (2.6),

$$\tilde{a}_0 = P_0\tilde{x}(t).$$

Therefore, by (1.5),

$$\| \tilde{x}(t) - \tilde{a}_0 \| \leqslant | \lambda | K\sigma(0). \tag{3.5}$$

However, as is well known,

$$1^{-2} + 2^{-2} + 3^{-2} + \cdots = \frac{\pi^2}{6},$$

and therefore, by (1.7), $\sigma(0) = \pi/\sqrt{3}$. By substituting this value into the right-hand side of (3.5), we have

$$\| \tilde{x}(t) - \tilde{a}_0 \| \leqslant | \lambda | \frac{K\pi}{\sqrt{3}}. \tag{3.6}$$

Now, by (2.7),

$$\frac{d\tilde{x}_m(t)}{dt} = P_m \frac{d\tilde{x}(t)}{dt}.$$

Therefore, by Bessel's inequality,

$$\left\| \frac{d\tilde{x}_m(t)}{dt} \right\|_q \leqslant \left\| \frac{d\tilde{x}(t)}{dt} \right\|_q = | \lambda | \cdot \| X(\tilde{a}_0, t) \|_q \leqslant | \lambda | K.$$

Then, since $\tilde{a}_0 = P_0\tilde{x}_m(t)$, by the definition of $\tilde{x}_m(t)$, in the same way as for (3.6), we have

$$\| \tilde{x}_m(t) - a_0 \| \leqslant | \lambda | \frac{K\pi}{\sqrt{3}}. \tag{3.7}$$

Inequalities (3.6) and (3.7) prove (3.2). Inequality (3.3) readily follows from (2.7) and (3.4) by (1.5).

LEMMA 2. *If* $\theta\tilde{x}(t) + (1 - \theta)\, \tilde{a}_0 \in D$, $0 \leqslant \theta \leqslant 1$, *then*

$$\left\| \frac{d\tilde{x}(t)}{dt} - \lambda X[\tilde{x}(t), t] \right\| \leqslant | \lambda |^2 \frac{KK_1\pi}{\sqrt{3}}. \tag{3.8}$$

Proof. Let us put

$$\frac{d\tilde{x}(t)}{dt} - \lambda X[\tilde{x}(t), t] = \tilde{\eta}(t).$$

Then, by (2.6) and (2.5), we have

$$\tilde{\eta}(t) = \lambda X[\tilde{a}_0, t] - \lambda X[\tilde{x}(t), t]$$
$$= -\lambda \int_0^1 \psi[\tilde{a}_0 + \theta(\tilde{x}(t) - \tilde{a}_0), t] \cdot (\tilde{x}(t) - \tilde{a}_0)\, d\theta.$$

Then, by (3.1) and (3.2), we see that

$$\| \tilde{\eta}(t) \| \leqslant | \lambda | \cdot K_1 \cdot | \lambda | \cdot \frac{K\pi}{\sqrt{3}} = | \lambda |^2 \frac{KK_1\pi}{\sqrt{3}},$$

and this proves (3.8).

LEMMA 3. *If*

$$\det \left[\int_0^{2\pi} \psi(\tilde{a}_0, t)\, dt \right] \neq 0, \tag{3.9}$$

then there are positive numbers λ_0 *and* M *such that, whenever* $0 < | \lambda | \leqslant \lambda_0$, (i) *the multipliers of the linear homogeneous system*

$$\frac{dy}{dt} = \lambda \psi(\tilde{a}_0, t)\, y \tag{3.10}$$

are all different from one, and (ii)

$$\| H_\lambda \|_n \leqslant M | \lambda |^{-1}, \tag{3.11}$$

where H_λ *is the H-mapping corresponding to* $\lambda\psi(\tilde{a}_0, t)$.

Proof. Let $\Phi(t, \lambda)$ be the fundamental matrix of (3.10) such that $\Phi(0, \lambda) = E$. Then, as it is readily seen, $\Phi(t, \lambda)$ can be expressed as follows:

$$\Phi(t, \lambda) = E + \lambda \Phi_1(t) + \lambda^2 \Phi_2(t, \lambda), \tag{3.12}$$

where

$$\Phi_1(t) = \int_0^t \psi(\tilde{a}_0, s)\, ds \tag{3.13}$$

and $\Phi_2(t, \lambda)$ is the solution of the differential equation

$$\frac{d\Phi_2}{dt} = \lambda \psi(\tilde{a}_0, t)\, \Phi_2 + \psi(\tilde{a}_0, t)\, \Phi_1(t) \tag{3.14}$$

such that

$$\Phi_2(0, \lambda) = 0. \tag{3.15}$$

By the assumption (3.9), $\det \Phi_1(2\pi) \neq 0$, and consequently $\Phi_1^{-1}(2\pi)$ exists. Let us take a small positive number λ_0 such that

$$\begin{aligned} &|\lambda| \cdot \|\Phi_1^{-1}(2\pi)\,\Phi_2(2\pi,\lambda)\| < 1, \\ &|\lambda| \cdot \|\Phi_1(t) + \lambda\Phi_2(t,\lambda)\| < 1 \end{aligned} \tag{3.16}$$

whenever $0 \leqslant |\lambda| < \lambda_0$ and $0 \leqslant t \leqslant 2\pi$. This is evidently possible.

Then, by (3.12),

$$\begin{aligned} E - \Phi(2\pi,\lambda) &= -\lambda[\Phi_1(2\pi) + \lambda\Phi_2(2\pi,\lambda)] \\ &= -\lambda\Phi_1(2\pi)\,[E + \lambda\Phi_1^{-1}(2\pi)\,\Phi_2(2\pi,\lambda)], \end{aligned} \tag{3.17}$$

and consequently, by the first relation (3.16),

$$\det\,[E - \Phi(2\pi,\lambda)] \neq 0$$

for any λ such that $0 < |\lambda| \leqslant \lambda_0$. This proves conclusion (i) of the lemma.

The matrix $H_\lambda(t,s)$ of the H-mapping H_λ corresponding to $\lambda\psi(\tilde{a}_0, t)$ is then given by Theorem 2 as follows:

$$H_\lambda(t,s) = \begin{cases} \Phi(t,\lambda)[E - \Phi(2\pi,\lambda)]^{-1}\,\Phi^{-1}(s,\lambda) \\ \qquad\qquad\qquad \text{for} \quad 0 \leqslant s < t \leqslant 2\pi, \\ \Phi(t,\lambda)\,[E - \Phi(2\pi,\lambda)]^{-1}\,\Phi(2\pi,\lambda)\,\Phi^{-1}(s,\lambda) \\ \qquad\qquad\qquad \text{for} \quad 0 \leqslant t \leqslant s \leqslant 2\pi \end{cases} \tag{3.18}$$

and

$$H_\lambda(t,s) = H_\lambda(t + 2m\pi, s + 2n\pi), \qquad m, n\text{: integers.}$$

However, for λ such that $0 < |\lambda| \leqslant \lambda_0$, from (3.17),

$$[E - \Phi(2\pi,\lambda)]^{-1} = -\frac{1}{\lambda}\,[E + \lambda\Phi_1^{-1}(2\pi)\,\Phi_2(2\pi,\lambda)]^{-1}\,\Phi_1^{-1}(2\pi) \tag{3.19}$$

and, from (3.12),

$$\Phi^{-1}(s,\lambda) = [E + \lambda\Phi_1(s) + \lambda^2\Phi_2(s,\lambda)]^{-1}, \tag{3.20}$$

and this inverse indeed exists by force of the second relation (3.16). Then substituting (3.12), (3.19), and (3.20) into (3.18), we see that $H_\lambda(t,s)$ is of the form $H_\lambda(t,s) = \lambda^{-1}K_\lambda(t,s)$, where $K_\lambda(t,s)$ is periodic in t and s with period 2π and is bounded as long as $0 \leqslant |\lambda| \leqslant \lambda_0$. This evidently implies the existence of a positive constant M for which (3.11) holds.

4. The Existence of an Exact Periodic Solution and the Errors of the Approximate Solutions

Concerning the existence of an isolated periodic solution and the errors of the approximate solutions $\tilde{x}(t)$ and $\tilde{x}_m(t)$, we have the following theorem.

Theorem 4. *Let us assume that system (S) satisfies the conditions stated in the beginning of Section 2, and suppose that equation (2.3) has a real solution $a = \tilde{a}_0$ which lies inside D and fulfills condition (3.9). Let K, K_1, and K_2 be the positive numbers satisfying (3.1) and let λ_0 and M be the positive numbers defined in Lemma 3.*

Let æ be an arbitrary fixed number such that $0 < æ < 1$ and δ be a positive nuîber such that

$$D_\delta = \{x \mid \| x - \tilde{a}_0 \| \leqslant \delta\} \subset D \qquad \textit{and} \qquad \delta \leqslant \frac{æ}{MK_2}. \tag{4.1}$$

Then for any λ such that

$$0 < |\lambda| \leqslant \min \left[\lambda_0, \frac{1}{\sqrt{3}\pi} \cdot \frac{\delta}{K}, \frac{1}{\sqrt{3}\pi} \cdot \frac{(1-æ)\,\delta}{KK_1M}\right], \tag{4.2}$$

(i) *system (S) has one and only one isolated periodic solution $x = \hat{x}(t)$ lying in D_δ;* (ii) *$\tilde{x}(t)$, $\tilde{x}_m(t)$ lie in D_δ; and* (iii)

$$\| \tilde{x} - \hat{x} \|_n \leqslant |\lambda| \cdot \frac{\pi}{\sqrt{3}} \cdot \frac{KK_1M}{1-æ}, \tag{4.3}$$

$$\| \tilde{x}_m - \hat{x} \|_n \leqslant |\lambda| \left[\frac{\pi}{\sqrt{3}} \cdot \frac{KK_1M}{1-æ} + K\sigma(m)\right]. \tag{4.4}$$

Proof. From (3.2), for λ satisfying (4.2),

$$\| \tilde{x} - \tilde{a}_0 \|_n, \qquad \| \tilde{x}_m - \tilde{a}_0 \|_n \leqslant \frac{\delta}{3}. \tag{4.5}$$

Therefore, by (4.1) it is evident that $\tilde{x}(t)$, $\tilde{x}_m(t)$ lie in D_δ. This proves conclusion (ii).

From (4.5) it is also evident that

$$\theta\tilde{x}(t) + (1-\theta)\,\tilde{a}_0 \in D_\delta \subset D, \qquad 0 \leqslant \theta \leqslant 1,$$

and therefore, by Lemma 2,

$$\frac{M}{|\lambda|} \cdot \frac{\| \tilde{\eta} \|_n}{1-æ} \leqslant |\lambda| \cdot \frac{\pi KK_1M}{(1-æ)\sqrt{3}}, \tag{4.6}$$

where

$$\tilde{\eta}(t) = \frac{d\tilde{x}(t)}{dt} - \lambda X[\tilde{x}(t), t]. \tag{4.7}$$

Then, for λ satisfying (4.2),

$$\frac{M}{|\lambda|} \cdot \frac{\| \tilde{\eta} \|_n}{1 - \varkappa} \leqslant \frac{\delta}{3}. \tag{4.8}$$

Now, for any x', x'' lying in D_δ,

$$\| \psi[x', t] - \psi[x'', t] \|^2 \leqslant \sum_{k,\ell} [\psi_{k,\ell}(x', t) - \psi_{k\ell}(x'', t)]^2,$$

where $\psi_{k\ell}(x', t)$ and $\psi_{k\ell}(x'', t)$ are, respectively, the elements of $\psi[x', t]$ and $\psi[x'', t]$. Since $x'' + \theta[x' - x''] \in D_\delta \subset D$ $(0 \leqslant \theta \leqslant 1)$, the right-hand side of the above inequality is estimated successively by means of Schwartz's inequality as follows:

$$\begin{aligned}
&\sum_{k,\ell} [\psi_{k\ell}(x', t) - \psi_{k\ell}(x'', t)]^2 \\
&= \sum_{k,\ell} \left[\int_0^1 \left\{\sum_p \frac{\partial \psi_{k\ell}}{\partial x_p}(x'' + \theta(x' - x''), t) \cdot (x_p' - x_p'')\right\} d\theta\right]^2 \\
&= \sum_{k,\ell} \left[\sum_p \left\{\int_0^1 \frac{\partial \psi_{k\ell}}{\partial x_p}\, d\theta \cdot (x_p' - x_p'')\right\}\right]^2 \\
&\leqslant \sum_{k,\ell} \left[\sum_p \left\{\int_0^1 \frac{\partial \psi_{k\ell}}{\partial x_p}\, d\theta\right\}^2 \cdot \sum_p (x_p' - x_p'')^2\right] \\
&\leqslant \sum_{k,\ell} \left[\sum_p \int_0^1 \left(\frac{\partial \psi_{k\ell}}{\partial x_p}\right)^2 d\theta\right] \cdot \| x' - x'' \|^2 \\
&= \int_0^1 \sum_{k,\ell,p} \left(\frac{\partial \psi_{k\ell}}{\partial x_p}\right)^2 d\theta \cdot \| x' - x'' \|^2 \\
&\leqslant K_2{}^2 \cdot \| x' - x'' \|^2 .
\end{aligned}$$

Hence

$$\| \psi[x', t] - \psi[x'', t] \| \leqslant K_2 \cdot \| x' - x'' \| .$$

Then, for any $x \in D_\delta$, we have

$$\begin{aligned}
\| \lambda\psi[x, t] - \lambda\psi[\tilde{a}_0, t] \| &\leqslant |\lambda| K_2 \| x - \tilde{a}_0 \| \\
&\leqslant |\lambda| K_2 \delta,
\end{aligned}$$

and consequently, by (4.1), for any $x \in D_\delta$, we have

$$\| \lambda\psi[x, t] - \lambda\psi[\tilde{a}_0, t] \| \leqslant \frac{\ae |\lambda|}{M}. \tag{4.9}$$

Let $D'_{\delta/3}$ be the set

$$D'_{\delta/3} = \left\{ x \mid \| x - \tilde{x}(t) \| \leqslant \frac{\delta}{3} \text{ for some } t \right\}. \tag{4.10}$$

Then from (4.5) it is evident that

$$D'_{\delta/3} \subset D_\delta \subset D. \tag{4.11}$$

Now expression (4.7)-(4.11) show by Lemma 3 that the condition of Theorem 3 are all fulfilled by $A(t) = \lambda\psi(\tilde{a}_0, t)$. Thus by Theorem 3 we see that system (S) has one and only one isolated periodic solution $x = \hat{x}(t)$ in $D'_{\delta/3}$ for any λ satisfying (4.2). However, as shown in the proof of Theorem 3 (see the proof of Proposition 3 in paper [*1*]), condition (4.9) implies the uniqueness of the periodic solution $\hat{x}(t)$ in D_δ. Thus, we conclude that there exists a unique isolated periodic solution in D_δ. This proves conclusion (i).

Inequality (4.3) follows from (4.6) by Theorem 3.

Inequality (4.4) follows from (4.3) and (3.3). Theorem 4 is thereby proved.

As it is seen from the proof, the exact isolated periodic solution $x = \hat{x}(t)$ lies in

$$D_{(2/3)\delta} = \{ x \mid \| x - \tilde{a}_0 \| \leqslant (2/3)\, \delta \}.$$

We should notice that, in the proof of Theorem 4, the existence theorem of implicit functions is not used at all.

COROLLARY. *For the isolated periodic solution $x = \hat{x}(t)$ stated in Theorem 4, we have*

$$\| \hat{x}(t) - \tilde{a}_0 \| \leqslant |\lambda| \cdot \frac{\pi}{\sqrt{3}} K \left(1 + \frac{K_1 M}{1 - \ae} \right). \tag{4.12}$$

The inequality (4.12) is a special case of (4.4) where $m = 0$.

From Theorem 4 we deduce now the following theorem concerned with Galerkin approximations.

THEOREM 5. *In Theorem 4 let us assume further that $X(x, t)$ and $\psi(x, t)$ are continuously differentiable with respect to t in the region $D \times L$. Then, for any λ satisfying (4.2), a Galerkin approximation $x = \bar{x}_m(t)$ of any order $m > m_0$ exists in D_δ provided m_0 is sufficiently large. For such Galerkin approximation, we have then*

$$\| \bar{x}_m - \hat{x} \|_n \leqslant \frac{M}{1 - \ae} (K_3 + |\lambda| K K_1)\, \sigma(m), \tag{4.13}$$

where $x = \hat{x}(t)$ is the isolated periodic solution stated in Theorem 4 and

$$K_3 = \max_{D \times L} \left\| \frac{\partial X(x, t)}{\partial t} \right\|.$$

Proof. By Theorem 4 system (S) has an isolated periodic solution $x = \hat{x}(t)$ in $D_{(2/3)\delta}$. Therefore, by Theorem 1, there exists a Galerkin approximation $x = \bar{x}_m(t)$ of any order $m > m_0$ lying in D_δ provided m_0 is sufficiently large. By the definition of $x = \bar{x}_m(t)$ we have

$$\frac{d\bar{x}_m(t)}{dt} = P_m[\lambda X(\bar{x}_m(t), t].$$

This can be rewritten as follows:

$$\frac{d\bar{x}_m(t)}{dt} = \lambda X[\bar{x}_m(t), t] + \eta_m(t), \tag{4.14}$$

where

$$\eta_m(t) = -(I - P_m)[\lambda X(\bar{x}_m(t), t)], \tag{4.15}$$

and I is the identity operator. Since

$$\begin{aligned} \frac{d}{dt} X[\bar{x}_m(t), t] &= \psi[\bar{x}_m(t), t] \cdot \frac{d\bar{x}_m(t)}{dt} + \frac{\partial X}{\partial t}[\bar{x}_m(t), t] \\ &= \psi[\bar{x}_m(t), t] \cdot P_m[\lambda X(\bar{x}_m(t), t)] + \frac{\partial X}{\partial t}[\bar{x}_m(t), t], \end{aligned}$$

by Bessel's inequality we have

$$\begin{aligned} \left\| \frac{d}{dt} X[\bar{x}_m(t), t] \right\|_q &\leqslant K_1 \mid \lambda \mid \cdot \| P_m X(\bar{x}_m(t), t) \|_q + K_3 \\ &\leqslant \mid \lambda \mid K_1 K + K_3 . \end{aligned}$$

Hence by (1.5) and (4.15) we have

$$\| \eta_m \|_n \leqslant \mid \lambda \mid \sigma(m)(K_3 + \mid \lambda \mid K K_1). \tag{4.16}$$

Now (4.14) can be rewritten as follows:

$$\frac{d\bar{x}_m(t)}{dt} = \lambda \psi(\tilde{a}_0, t)\, \bar{x}_m(t) + [\lambda X(\bar{x}_m(t), t) - \lambda \psi(\tilde{a}_0, t)\, \bar{x}_m(t) + \eta_m(t)].$$

By Theorem 2, therefore, $\bar{x}_m(t)$ is expressed by

$$\bar{x}_m(t) = \int_0^{2\pi} H_\lambda(t, s)\, [\lambda X(\bar{x}_m(s), s) - \lambda \psi(\tilde{a}_0, s)\, \bar{x}_m(s) + \eta_m(s)]\, ds, \tag{4.17}$$

where $H_\lambda(t, s)$ is the matrix of the H-mapping H_λ corresponding to $\lambda\psi(\tilde{a}_0, t)$. On the other hand $x = \hat{x}(t)$ is an exact solution of system (S), hence

$$\frac{d\hat{x}(t)}{dt} = \lambda X[\hat{x}(t), t].$$

This can be rewritten as follows:

$$\frac{d\hat{x}(t)}{dt} = \lambda\psi(\tilde{a}_0, t)\,\hat{x}(t) + [\lambda X(\hat{x}(t), t) - \lambda\psi(\tilde{a}_0, t)\,\hat{x}(t)].$$

Since $\hat{x}(t)$ is periodic, $\hat{x}(t)$ can be expressed again by Theorem 2 as follows:

$$\hat{x}(t) = \int_0^{2\pi} H_\lambda(t, s)\,[\lambda X(\hat{x}(s), s) - \lambda\psi(\tilde{a}_0, s)\,\hat{x}(s)]\,ds. \tag{4.18}$$

Then, subtracting (4.18) from (4.17), we have

$$\begin{aligned} \bar{x}_m(t) - \hat{x}(t) = \int_0^{2\pi} H_\lambda(t, s)\,[\{\lambda X(\bar{x}_m(s), s) - \lambda X(\hat{x}(s), s)\} \\ - \lambda\psi(\tilde{a}_0, s)\,\{\bar{x}_m(s) - \hat{x}(s)\}]\,ds + \int_0^{2\pi} H_\lambda(t, s)\,\eta_m(s)\,ds. \end{aligned} \tag{4.19}$$

Now, since both $x = \bar{x}_m(t)$ and $x = \hat{x}(t)$ lie in D_δ.

$$\begin{aligned} &\lambda X(\bar{x}_m(s), s) - \lambda X(\hat{x}(s), s) - \lambda\psi(\tilde{a}_0, s)\,\{\bar{x}_m(s) - \hat{x}(s)\} \\ &= \lambda \int_0^1 [\psi\{\hat{x}(s) + \theta(\bar{x}_m(s) - \hat{x}(s)), s\} - \psi(\tilde{a}_0, s)] \cdot [\bar{x}_m(s) - \hat{x}(s)]\,d\theta. \end{aligned}$$

By (4.9), therefore, we have

$$\|\lambda X(\bar{x}_m(s), s) - \lambda X(\hat{x}(s), s) - \lambda\psi(\tilde{a}_0, s)\,\{\bar{x}_m(s) - \hat{x}(s)\}\| \leqslant |\lambda| \frac{\alpha}{M} \|\bar{x}_m - \hat{x}\|_n.$$

Then by Lemma 3, (4.16) and (4.19) it follows that

$$\|\bar{x}_m - \hat{x}\|_n \leqslant \alpha \|\bar{x}_m - \hat{x}\|_n + \sigma(m)\,M(K_3 + |\lambda|\,KK_1).$$

From this (4.13) readily follows since $\alpha < 1$.

Remarks. In Theorem 4 we have given an explicit formula for a bound on the parameter λ within which the method of averaging is effective in affirming the existence of an exact periodic solution. This may be important in applications of the method of averaging.

However in practical problems it may be very difficult to find the value of the bound given by (4.2) for the given system. Moreover, even if one can find the value of the bound given by (4.2), it may happen that the bound is too small to check the validity of the method of averaging, since the condition (4.2) is merely a sufficient condition and is not a necessary condition. In such a case, the given system may have an isolated periodic solution even if the given value of λ exceeds the bound given by (4.2).

Such difficulties, however, can be avoided if one uses Galerkin's procedure and checks the existence of an exact periodic solution by means of Theorem 3. In such a case, of course, particular checking of the bound of the parameter is not necessary. In addition, in Galerkin's procedure, the determining equation (2.1) can be easily solved numerically by Newton's method starting from the value $\alpha = (\tilde{a}_0, 0, 0, \cdots, 0, 0)$. As stated at the end of Section 1, the Galerkin's procedure above accompanied with checking is a valid method for any system having smoothness properties as specified in Theorem 1 if an isolated periodic solution exists inside D. Consequently, by Theorem 4 the above process is always valid for systems of the form (S) having the smoothness specified in Theorem 5 in case the method of averaging is effective under the conditions of Theorem 4. It is needless to say that Galerkin's procedure accompanied with checking has broader applicability than the method of averaging.

The above discussions show that, in practical problems, Galerkin's procedure accompanied by checking will be more convenient than the method of averaging for seeking an isolated periodic solution for systems of the form (S) with a fixed value of the parameter even when this parameter is small. Numerical examples are shown in paper [*4*].

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