0: Verifications

DO NOT WRITE ANYTHING HERE. INSTEAD, SCAN YOUR ANNOTATIONS ON THE VERIFICATIONS PDF, AND SUBMIT SEPARATELY.

1: Your Average Induction

Claim: $Average(A, n) = \sum_{i=0}^{n-1} A_i/n$, where $A_i = A[i]$ for $0 \leq i \leq n$ for all $n \geq 1$.
We do this by induction.
Base Case: $Average(A, 1) = A_0/1 = A_0$ by the definition of $Average$.
Induction Hypothesis: Suppose the claim is true for some $k \geq 1$.
Induction Step:
$Average(A, k+1) = ((Average(A, k) * k) + A[k])/(k+1)$ (by the given function)
$Average(A, k) = \sum_{i=0}^{k-1} A_i/k$ (by I.H.)
Therefore,

$$\begin{aligned} Average(A, k+1) &= ((\sum_{i=0}^{k-1} A_i/k) * k + A[k])/(k+1) \\ &= ((\sum_{i=0}^{k-1} A_i) + A[k])/(k+1) \\ &= (\sum_{i=0}^{k} A_i)/(k+1) \\ &= \sum_{i=0}^{k} A_i/(k+1) \end{aligned}$$

which is exactly $Average(A, k+1)$. Since the base case and inductive step both hold, $Average(A, n) = \sum_{i=0}^{n-1} A_i/n$, where $A_i = A[i]$ for $0 \leq i \leq n$ for all $n \geq 1$.

2: No, You're Being Irrational

Claim: $\sqrt{2} + \sqrt{5}$ is irrational.
We will do this through proof by contradiction.
Lets assume that $\sqrt{2} + \sqrt{5}$ is rational. If this is true, it can be written as a fraction, a/b, where a and b are two coprime integers and $\neq 0$.

$$\sqrt{2} + \sqrt{5} = a/b \qquad \text{Definition of Rational Numbers}$$

Using Algebra to rearrange the equation, we can isolate the a and b terms.

$$\begin{aligned}
\sqrt{5} &= a/b - \sqrt{2} \\
5 &= a^2/b^2 - (2\sqrt{2})a/b + 2 \\
3 - a^2/b^2 &= -(2\sqrt{2})a/b \\
(3b)/a - a/b &= -2\sqrt{2} \\
a/(2b) - (3b)/(2a) &= \sqrt{2} \\
(a^2 - 3b^2)/(2ab) &= \sqrt{2}
\end{aligned}$$

Since we know that a and b are integers $\neq 0$, we also can state that $(a^2 - 3b^2)$ and $(2ab)$ are also integers and $(2ab) \neq 0$. If we let them be n and m respectively, the left side of the equation becomes n/m, indicating that the right side of the equation is rational. However, we know that $\sqrt{2}$ is irrational. Therefore, the initial assumption that $\sqrt{2} + \sqrt{5}$ is rational is contradicted. Thus, $\sqrt{2} + \sqrt{5}$ is irrational.

3: Prime Examples

Claim: For any prime $p > 3$, either $p \equiv_6 1$ or $p \equiv_6 5$.
We do this through proof by contradiction.
Lets assume that there exists some prime $p > 3$ such that $p \not\equiv_6 1$ and $p \not\equiv_6 5$. Lets consider all possible remainders for $(p/6) \notin \{1, 5\}$.

1. $p \equiv_6 0$: In this case, p is divisible by 6, meaning it is additionally divisible by 2 and 3 and therefore not prime.

2. $p \equiv_6 2$: In this case, p is divisible by 2. The only prime number for which this is true is when $p = 2$; however, p would no longer be > 3. Therefore, p is not prime.

3. $p \equiv_6 3$: In this case, p is divisible by 3 and not 2. However, the only prime number for which this is true is 3. Therefore, for any $p > 3$, p is not prime.

4. $p \equiv_6 4$: In this case, p is an even number and therefore not prime.

Therefore, there are no such prime values for $p > 3$ such that the claim holds. Thus, for any prime p such that $p > 3$, either $p \equiv_6 1$ or $p \equiv_6 5$.

4: Balanced Ternary

Claim: $evaluate_n$(X) is injective for all n. That is, if $evaluate_n$(X) $=$ $evaluate_n$(Y), then X $=$ Y. We do this by induction.
Base Case: By the definition of V, if $V(t_0) = 0$, $t_0 = 0$. If $V(t_0) = 1$, $t_0 = 1$. If $V(t_0) = -1$, $t_0 = T$. Thus, becaue every output only has 1 input for length $n = 1$, $evaluate_1(X)$ is injective.
Induction Hypothesis: Assume that the claim is true for some ternary number of length k.
Induction Step: Suppose $A = (a_k...a_0)$ and $B = (b_k...b_0)$ are two ternary numbers of length $k+1$ such that $evaluate_{k+1}(A) = evaluate_{k+1}(B)$. From this, we can expand.

$$\begin{aligned}
evaluate_{k+1}(A) &= evaluate_{k+1}(B) & \\
\sum_{i=0}^{k} V(a_i)3^i &= \sum_{i=0}^{k} V(b_i)3^i & \text{by Definition of V} \\
\sum_{i=0}^{k-1} V(a_i)3^i + V(a_k)3^k &= \sum_{i=0}^{k-1} V(b_i)3^i + V(b_k)3^k & \text{Algebra} \\
(\sum_{i=0}^{k-1} V(a_i)3^i + V(a_k)3^k)(mod 3^k) &= (\sum_{i=0}^{k-1} V(b_i)3^i + V(b_k)3^k)(mod 3^k) & \text{(mod } 3^k\text{) on both sides} \\
(\sum_{i=0}^{k-1} V(a_i)3^i)(mod 3^k) &= (\sum_{i=0}^{k-1} V(b_i)3^i)(mod 3^k) & \text{Definition of Modulus}
\end{aligned}$$

By the **Lemma**, both $\sum_{i=0}^{k-1} V(a_i)3^i$ and $\sum_{i=0}^{k-1} V(b_i)3^i$ are $< 3^k$ and are not affected by the $(mod 3^k)$ term. Thus, we end up with

$$\sum_{i=0}^{k-1} V(a_i)3^i = \sum_{i=0}^{k-1} V(b_i)3^i$$

Thus, $A = B$ by the Inductive Hypothesis and $evaluate_{k+1}$ is injective. Since the base case and inductive step both hold, $evaluate_n$(X) is injective for all n.

Lemma: Proving that there exists a maximum value for $\sum_{i=0}^{k-1} V(x_i)3^i$ that is $< 3^k$ for all $k \geq 1$. Note that the maximum value for $V(x_1)$ is 1, so it can be omitted from the proof.

Claim: $\sum_{i=0}^{k-1} V(x_i)3^i \leq 3^k - 1$.
We do this by induction.
Base Case: When $k = 1$, $\sum_{i=0}^{0} 3^i = 3^0 = 1$. This is $\leq 3^1 - 1 = 2$, so the base case holds.
Inductive Hypothesis: The claim is true for some $k \geq 1$.
Inductive Step: Show the claim holds for ternary numbers of length $k+1$.

$$\sum_{i=0}^{k} 3^i = 3^k + \sum_{i=0}^{k-1} 3^i \leq 3^k + 3^k - 1 \text{ by Inductive Hypothesis.}$$

Thus, the claim that $\sum_{i=0}^{k-1} V(x_i)3^i \leq 3^k - 1$ holds for all $k \geq 1$.