Large Cardinals

27th Mar, 2025

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1 Inaccessible cardinals

The cardinality of V_α grows rather fast relative to α. For example,

$$|V_{\omega+\alpha}| = \beth_\alpha \tag{1}$$

where the **beth function** $\beth_\alpha$ is defined as

$$\begin{aligned}\beth_0 &= \aleph_0 \\ \beth_{\alpha+1} &= 2^{\beth_\alpha} \\ \beth_\lambda &= \sup\{\beth_\alpha : \alpha < \lambda\} \quad \lambda \text{ limit}\end{aligned}$$

The existence of **inaccessible cardinals** ensures that the von-Neumann hierarchy is "long enough" for α to eventually "catch up" with the cardinality of V_α.

Recall the enumeration of all cardinals by means of the $\aleph$-**sequence**:

$$\aleph_0 = \omega, \quad \aleph_{\alpha+1} = \aleph_\alpha^+, \quad \aleph_\lambda = \sup\{\aleph_\xi : \xi < \lambda\} \text{ for limit } \lambda.$$

Here κ^+ is the least cardinal $> \kappa$. Some cardinals are limits of short sequences of cardinals – for example,

$$\aleph_\omega = \lim_n \aleph_n$$

is uncountable, but a limit of a countable sequence of smaller cardinals. Generally, cardinals who are a limit of a sequence of cardinals of length smaller than their cardinality are called **singular**. Non-singular cardinals are called **regular**:

$$\operatorname{reg}(\kappa) :\iff \forall \alpha < \kappa \, \forall f \, (f : \alpha \to \kappa \;\to\; \sup_{\xi<\alpha} f(\xi) < \kappa).$$

In other words, a regular cardinal κ cannot be reached by less then κ-many steps. The first example of a regular cardinal is $\aleph_0$.

Exercise
Show that all **successor cardinals**, i.e. cardinals of the form $\aleph_{\alpha+1}$ are regular. (Use the Axiom of Choice.)

On the other hand,

$$\aleph_\omega, \aleph_{\omega+\omega}, \aleph_{\aleph_\omega}, \aleph_{\aleph_{\aleph_\omega}}, \ldots$$

are singular and this suggests the question:

> Are there regular cardinals of the form $\aleph_\lambda$ with λ limit?

This is captured by the notion of **inaccessibility**.

Definition 1.1 (Hausdorff 1908, Tarski, Zermelo 1930)**.** *An uncountable cardinal $\kappa > \omega$ is*

- ***weakly inaccessible** if*

$$\begin{aligned}&\operatorname{reg}(\kappa) \ \wedge \ \exists\lambda(\lim(\lambda) \ \wedge \ \kappa = \aleph_\lambda)\\ &\qquad (\Leftrightarrow \operatorname{reg}(\kappa) \ \wedge \ \forall\alpha < \kappa \ \alpha^+ < \kappa)\end{aligned}$$

- ***(strongly) inaccessible** if*

$$\operatorname{reg}(\kappa) \ \wedge \ \forall\alpha < \kappa \ \ 2^\alpha < \kappa$$

Under the **Generalized Continuum Hypothesis**,

(GCH) $\forall\alpha \ \ 2^{\aleph_\alpha} = \aleph_\alpha^+$

weakly and strongly inaccessible cardinals coincide.

If $\kappa > \omega$ is inaccessible, then $\kappa = \aleph_\kappa$. Moreover, we have

Proposition 1.1. *If κ is strongly inaccessible, $|V_\kappa| = \kappa$.*

Proof. It suffices to show that $|V_\alpha| < \kappa$ for all $\alpha < \kappa$. This follows by a straightforward induction, using the fact that κ is strongly inaccessible.

□

This in turn implies we can bound the cardinality of elements of V_κ.

Proposition 1.2. *Suppose κ is strongly inaccessible and $x \subset V_\kappa$. Then*

$$x \in V_\kappa \ \Leftrightarrow \ |x| < \kappa. \tag{2}$$

Proof. ($\Rightarrow$) $x \in V_\kappa$ implies $|x| < |V_\kappa|$. Apply Proposition **??**.

($\Leftarrow$) Since $x \subseteq V_\kappa$, each $y \in x$ has rank $< \kappa$. Since $|x| < \kappa$, by regularity of κ,

$$\operatorname{rank}(x) = \sup\{\operatorname{rank}(y) + 1 \colon y \in x\} < \kappa \tag{3}$$

which implies $x \in V_\kappa$.

□

We have already seen that for limit $\alpha > \omega$, V_α is a model of all ZFC axioms except *Replacement*.

Theorem 1.1. *If κ is strongly inaccessible, then $V_\kappa \models$ ZFC.*

Proof. We verify that V_κ satisfies the axiom of *Replacement*. Suppose $x \in V_\kappa$ and $f : x \to V_\kappa$ is a function. Then $f[x] \subseteq V_\kappa$, and by Proposition 1.2, $|f[x]| \leq |x| < \kappa$. Applying the other direction of Proposition 1.2 to $f[x]$, we obtain $f[x] \in V_\kappa$, as desired.

□

Suppose an inaccessible cardinal exists, and let κ be the least inaccessible.
It is not hard to verify that

$$V_\kappa \models \text{ZFC} + \text{"there does not exist an inaccessible cardinal"}. \tag{4}$$

(You verify that being a inaccessible cardinal is absolute for V_κ.) Therefore, the existence of an inaccessible cardinal is not provable from ZFC. This fact also follows from Gödel's second incompleteness theorem.

2 Measurability

We have seen that (assuming the Axiom of Choice) there subsets of $\mathbb{R}$ that are not Lebesgue measurable. Inspecting the proof, we see that we only use the following properties of Lebesgue measure:

- σ-additivity,
- translation invariance ($\lambda(A) = \lambda(A + r)$),
- $\lambda(A) > 0$ for some A.

For spaces without an additive structures, instead of translation invariance, we can consider a **non-triviality condition**:

$$m(\{x\}) = 0 \quad \text{for all } x \tag{5}$$

The **generalized measure problem** asks whether there exists a set M together with a measure function

$$m : \mathcal{P}(M) \to [0, \infty),$$

so that the following conditions are met:

- **(M1)** $\quad m(M) = 1$
- **(M2)** $\quad \forall x \in M\ m(\{x\}) = 0$
- **(M3)** $\quad$ if $(A_i)_{i<\omega}$ is a countable sequence of disjoint sets $\subseteq M$, then

$$m\left(\bigcup_{i<\omega} A_i\right) = \sum_{i<\omega} m(A_i)$$

The structure of the set M does not play any role here, so we can replace it by a cardinal κ outright. One can also consider strengthening σ-additivity to κ-**additivity**:

> If $\gamma < \kappa$ and $(A_\xi)_{\xi<\lambda}$ is a sequence of disjoint subsets of κ, then
>
> $$m(\bigcup_{\xi<\gamma} A_\xi) = \sum_{\xi<\gamma} m(A_\xi).$$

A transfinite sum $\sum_{\xi<\gamma}$ is given as the supremum of all sums over finite subsequences:

$$\sum_{\xi<\gamma} m(A_\xi) = \sup\left\{\sum_{\xi\in F} m(A_\xi) \colon F \subseteq \gamma \text{ finite}\right\}. \tag{6}$$

Hence, ω_1-additive is the same as σ-additive.

Theorem 2.1 (Banach)**.** *If κ is the least cardinal for which a measure satisfying (M1)-(M3) exists, then any such measure on κ is already κ-additive.*

Proof. Suppose m is a measure on κ that is not κ-additive. Then, for some $\gamma < \kappa$, there exists a sequence $(A_\xi)_{\xi<\gamma}$ of disjoint subsets of κ so that

$$m(\bigcup_{\xi<\gamma} A_\xi) \neq \sum_{\xi<\gamma} m(A_\xi). \tag{7}$$

Since a measure is always σ-additive, $\gamma > \omega$ has to hold, and there can be at most countably many A_ξ with $m(A_\xi) > 0$.

We can drop those A_ξ, and by the σ-additivity of m for the remaining ξ it has to hold that $m(A_\xi) = 0$ while $m\left(\bigcup_{\xi<\gamma} A_\xi\right) = r > 0$.

By putting

$$\overline{m}(X) = \frac{m(\bigcup_{\xi \in X} A_\xi)}{r}$$

we obtain a measure on $\gamma < \kappa$, contradicting the minimality of κ.

□

2.1 Measurable cardinals

If m is a measure on κ, the **associated ideal**

$$\mathcal{I}_m = \{x \subseteq \kappa \colon m(x) = 0\} \tag{8}$$

is a σ-ideal, or, complementing the notion of ω_1-additivity, a ω_1-**complete ideal**.

Exercise
Show that $\mathcal{I}_m$ is not principal.

The corresponding filter

$$\mathcal{F}_m = \{x \subseteq \kappa \colon m(x) = 1\} \tag{9}$$

is then ω_1-complete, too.

A measure m is **two-valued** if it only assumes the values 0 and 1. In this case the corresponding filter $\mathcal{F}_m$ is an **ultrafilter** (and $\mathcal{I}_m$ is a **prime ideal**).

Conversely, if U is ω_1-complete, non-principal ultrafilter on κ, we can define a two-valued measure $m : \mathcal{P}(\kappa) \to \{0, 1\}$ on κ by letting

$$m(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Definition 2.1. *Let κ be an uncountable cardinal.*

- *κ is **real-valued measurable** if there exists a κ-additive measure on κ.*
- *κ is **measurable** if there exists a κ-additive, two-valued measure on κ, or, equivalently, if there exists a κ-complete, non-principal ultrafilter on κ.*

In the following, we will see that measurability implies inaccessibility.

Lemma 2.2. *If U is a κ-complete, non-principal ultrafilter on κ, then every $X \in U$ has cardinality κ.*

Proof. Since U is non-principal, no *singleton* set $\{x\}$ can be in U (for this would imply $\kappa \setminus \{x\} \notin U$ and therefore no subset of it would be in U either, contradicting the non-principality of U).

If $X \in U$ and $|X| < \kappa$, then X is the union of $< \kappa$ many singletons. Since $\neg U$ is a κ-complete prime ideal, this implies $X \in \neg U$, contradiction.

□

Proposition 2.1. *If κ is measurable, then it is regular.*

Proof. If κ were singular, it would be the union of $< \kappa$-many sets of cardinality $< \kappa$. Applying Lemma 2.2 leads to a contradiction.

□

Theorem 2.3. *A measurable cardinal is (strongly) inaccessible.*

Proof. By Proposition 2.1, any measurable cardinal is regular. Assume for a contradiction there exists $\gamma < \kappa$ with $2^\gamma > \kappa$. As $2^\gamma > \kappa$, there exists a set S of functions $f : \gamma \to \{0, 1\}$ with $|S| = \kappa$. Let U be a κ-complete, non-principal ultrafilter on S.

For $\alpha < \gamma, i \in \{0, 1\}$, let

$$X_{\alpha,i} = \{f \in S \colon f(\alpha) = i\} \tag{11}$$

and let $g(\alpha) = i$ if and only if $X_{\alpha,i} \in U$. Since U is an ultrafilter, g is well-defined on γ.

Since $\gamma < \kappa$ and U is κ-complete,

$$X = \bigcap_{\alpha<\gamma} X_{\alpha,g(\alpha)} \tag{12}$$

is in U. But $|X| \leq 1$, since the only function possibly in X is g. This contradicts Lemma 2.2.

□

Exercise
Show that every real-valued measurable cardinal is weakly inaccessible.

Proposition 2.2. *If κ is real-valued measurable, then κ is measurable or $\kappa \leq 2^{\aleph_0}$.*

Thus, if κ is real-valued measurable but not measurable, then the continuum $2^{\aleph_0}$ has to be very large.

3 Partition properties

Another concept of largeness is related to the existence of large **homogeneous sets** for partitions.

For given set S and $n \in \mathbb{N}$, let

$$[S]^n := \{X \subseteq S \colon |X| = n\}$$

be the set of all n-element subsets of S. For cardinals κ, λ, we define

$$\kappa \to (\lambda)^n_k$$

to mean that any partition $F : [S]^n \to \{1, \dots, k\}$ mit $|S| = \kappa$ has an F-**homogeneous subset** of cardinality λ, that is, a set H, $|H| = \lambda$, such that

$$F|_{[H]^n} \equiv \text{ constant.}$$

Ramsey's theorem (1929/39) says that for any $n, k \in \mathbb{N}$,

$$\aleph_0 \to (\aleph_0)^n_k.$$

Do there exist uncountable cardinals with similar properties?

A cardinal κ is **weakly compact** if it is uncountable and $\kappa \to (\kappa)^2_2$ holds.

> **Exercise**
> Show that for any cardinal κ, $2^\kappa \not\to (\kappa^+)^2_2$, and use this to infer that any weakly compact cardinal is inaccessible.
>
> (Thus the existence of weakly compact cardinals cannot be established in ZFC.)

Measurable cardinals have even stronger homogeneity properties. Let $[S]^{<\omega}$ be the set of all finite subsets of S. If $F : [S]^{<\omega} \to I$ is a partition, then $H \subseteq S$ is F-**homogenenous** if

$$F|_{[H]^n} \equiv \text{ constant}$$

for all $n \in \mathbb{N}$.

Theorem 3.1 (Rowbottom)**.** *Let κ be a measurable cardinal and let $F : [\kappa]^{<\omega} \to \lambda$ a partition of $[\kappa]^{<\omega}$ into $\lambda < \kappa$ pieces. Then there exists an F-homogeneous set $H \subseteq \kappa$ with $|H| = \kappa$.*

In general, any cardinal that satisfies the statement of the theorem is called **Ramsey**.

To prove Theorem **??**, we introduce **normal ultrafilters**.

Definition 3.1. *Given a sequence of sets $(A_\xi)_{\xi<\gamma}$, the* ***diagonal intersection*** *is given as*

$$\Delta_{\xi<\gamma} A_\xi = \{\alpha < \gamma \colon \alpha \in \bigcap_{\xi<\alpha} A_\xi\}. \tag{13}$$

A filter F on a cardinal κ is ***normal*** *if for any κ-sequence $(A_\xi)_{\xi<\kappa}$, $A_\xi \in F$, the diagonal intersection $\Delta_{\xi<\kappa} A_\xi$ is in F.*

Let us assume as a convention that a filter on a cardinal κ always contains the end-segments $\{\xi \colon \alpha \leq \xi < \kappa\}$.

> **Exercise**
> Show that a normal filter on κ is κ-complete.

> **Exercise**
> Show that if there is a normal filter over κ, then κ is uncountable and regular.

> **Exercise**
> Show that if κ is measurable, then there is a normal ultrafilter on κ.

Proof. (Proof of Theorem **??**)

Let U be a normal filter over κ. We show that for every n, for any $g : [\kappa]^n \to \gamma$ with $\gamma < \kappa$, there is a set $H_n \in U$ such that $g_n \restriction [H_n]^n \equiv$ const. The intersection of the H_n is again in U and satisfies the statement of the the theorem.

We proceed by induction. The case $n = 1$ follows from the κ-completeness of U. Now assume $g : [\kappa]^{n+1} \to \gamma$, with $\gamma < \kappa$.

For each $S \in [\kappa]^n$, define $g_s : \kappa \to \gamma$ by

$$g_S(\alpha) = \begin{cases} g(S \cup \{\alpha\}) & \text{if } \max S < \alpha \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

By κ-completeness of U, g_S is constant on a set $Y_S \in U$, say

$$g_S \restriction Y_S \equiv \delta_S < \gamma. \tag{15}$$

We now define a function $h : [\kappa]^n \to \gamma$ by letting

$$h(S) = \delta_S. \tag{16}$$

By induction hypothesis, h is constant on a set $Z \subseteq \kappa$ in U (and hence of size κ), say $h \restriction [Z]^n \equiv \delta < \kappa$.

For each $\alpha < \kappa$, let

$$Y_\alpha = \bigcap \{Y_S \colon \max S \leq \alpha\} \tag{17}$$

By κ-completeness, $Y_\alpha \in U$, and by normality

$$H = Z \cap \Delta_{\alpha<\kappa} Y_\alpha \in U \tag{18}$$

By Lemma 2.2, H has cardinality κ.

We claim that g is constant on $[H]^{n+1}$: Let $T \in [H]^{n+1}$. Write T as $S \cup \{\alpha\}$ with $\max S < \alpha$. Then

$$\begin{aligned} \alpha \in H \Rightarrow & \qquad \alpha \in \Delta_{\gamma<\kappa} Y_\gamma \\ \Rightarrow & \qquad \alpha \in \bigcap_{\beta<\alpha} Y_\beta \\ \Rightarrow & \qquad \alpha \in Y_{\max S} \\ \Rightarrow & \qquad \alpha \in Y_S \\ \Rightarrow & \qquad g_S(\alpha) = \delta_S \end{aligned}$$

On the other hand, $S \subseteq H$ implies $S \subseteq Z$ and hence by definition of Z, $h(S) = \delta_S = \delta$, so $g(T) = g_S(\alpha) = \delta_S = \delta$.

□