condensation

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1 Guided Study: The Condensation Lemma

1.1 The condensation lemma

“‘{admonition} Condensation lemma

There is a finite set T of axioms of $\mathsf{ZF} - $Power Set so that if M is a transitive set with $M \vDash T + \mathsf{VL}$, then $M = L_\lambda$ for some limit ordinal λ. “‘

Define: \$ _{VL} = \$ conjunction of the axioms in T and the axiom VL.

1.2 First application: VL implies GCH

“‘{admonition} Key Lemma

Suppose $V = L$. If κ is a cardinal and $x \subseteq \kappa$, then $x \in L_{\kappa^+}$. “‘

Guide to proof: *Verify each of following steps.*

- There exists limit $\lambda > \kappa$ such that $x \in L_\lambda$ and such that $L_\lambda \vDash \varphi_{VL}$
- Let $X = \kappa \cup \{x\}$.
  What do we know about X at this stage?
- There exists an **elementary substructure** $N \preceq L_\lambda$ such that
  $$X \subseteq N \subseteq L_\lambda \quad \text{and} \quad |N| = |X|. \tag{$*$}$$
  Use a famous theorem from logic.
- Can we apply the condensation lemma to N?
  What are the possible obstacles?
- Apply a Mostowski collapse to N. This gives us a transitive M isomorphic to N.
  Now we have a new problem. What is it? (Hint: where does x go?)
- Argue that the Mostowski isomorphism fixes x.
  Hint: What does it do with ordinals?
- Now we can apply condensation. This yields $M = L_\beta$ for some β.
  Where does this put x now?

- **Key argument**: $|\beta| = \kappa$.
  Hint: Proposition 53

- Finish: $x \in L_{\kappa^+}$

"'{admonition} $\mathsf{VL} \to \mathsf{GCH}$

If VL, then for all cardinals κ, $2^\kappa = \kappa^+$. "'

"'{admonition} Corollary

If ZF is consistent, so is $\mathsf{ZF} + \mathsf{GCH}$. "'

1.3 Second application: the complexity of constructible reals

The set of all constructible reals is defined by a Σ_1 formula over set theory:

$$\varphi(x_0) \;\equiv\; \exists y \,[y \text{ is an ordinal } \;\wedge\; x_0 \in L_y \;\wedge\; x_0 \text{ is a set of natural numbers }].$$

Can we "convert" this into a formula of second order arithmetic?

How could the condensation lemma help with this?

Key ideas:

- every constructible real shows up at a countable stage of L.
  Why?

- Hence if $\alpha \in L \cap \mathbb{N}^{\mathbb{N}}$, there exists a countable ξ such that $x \in L_\xi$.

- Then L_ξ is countable, too.
  Why?

- Hence we can hope to replace L_ξ by something like > "*there exists a real that codes a model that looks like L_ξ*"

Key ingredients:

- Condensation lemma and Mostowski collapse
  Can you think why these are important here?

This is the formula:

$$\begin{aligned} \alpha \in L \cap \mathbb{N}^{\mathbb{N}} \iff \exists \beta \exists m \,[&E_\beta \text{ is extensional and well-founded} \\ &\wedge\; (\omega, E_\beta) \vDash \phi_{VL} \;\wedge\; \pi_\beta(m) = \alpha], \end{aligned} \tag{$**$}$$

where π_β is the Isomorphism of the Mostowski collapse of E_β.

- *Why does this work?*
- *What do we still need to verfiy to make sure this is in second order arithmetic?*

Ingredient 1:

For any $n \in \mathbb{N}$, the following set is Σ^0_1:

$$\{(m,\sigma,\gamma) \in \mathbb{N} \times \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \colon m = \ulcorner \varphi \urcorner \ \wedge\ \varphi \text{ is } \Sigma_1 \ \wedge\ (\omega, E_\gamma) \vDash \varphi[\sigma]\}$$

- *You don't need to prove this fully. Just think about* ***how*** *you would prove it. In particular, how would you arithmetize the truth relation* $(\omega, E_\beta) \vDash \psi$*?*
- *Since we work with relations over* $\mathbb{N}$ *now instead of arbitrary sets, it is not that easy anymore to keep quantifiers bounded. Think of an example for this difficulty. Why can't we just convert a bounded quantifier* $\exists x \in y$ *to a bounded quantifier in arithmetic* $\exists m < n$*?*
- *But since we are only interested in the complexity of* $\vDash$ *for* Σ_1*-formulas, this helps us bound the overall complexity at* Σ^0_1

Ingredient 2:

If $\alpha \in \mathbb{N}^{\mathbb{N}}$ and E_α is well-founded and extensional, then the following set is arithmetic in α:

$$\{(m,\gamma) \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \colon \pi_\alpha(m) = \gamma\}$$

- *Again, you do not need to fully prove this, just think about* ***how*** *you would do it. In particular, how would you arithmetize* π_α*?*

Now put everything together and show

“‘{admonition} Theorem

The set $L \cap \mathbb{N}^{\mathbb{N}}$ is Σ^1_2. “‘

In a similar way we can show

“‘{admonition} Theorem

The set $\{(\alpha,\beta) \in (L \cap \mathbb{N}^{\mathbb{N}})^2 \colon \alpha <_L \beta\}$ is Σ^1_2. “‘

If VL, then the set is actually Δ^1_2, since then

$$\alpha <_L \beta \iff \alpha \neq \beta \ \wedge\ \neg(\beta <_L \alpha).$$

This has consequences for the existence of non-measurable sets. Find or recall the corresponding theorem and formulate a theorem under the hypothesis VL*.*

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