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Introduction: Perfect Subsets of the Real Line

Continuum Hypothesis (Cantor, 1890s)
If $A \subseteq \mathbb{R}$ is uncountable, then there exists a bijection between A and $\mathbb{R}$. That is, is every uncountable subset of $\mathbb{R}$ is of the same cardinality as $\mathbb{R}$.

Possible approach: show that CH holds for a number of sets with an easy topological structure.

Exercise
Show that every open set in $\mathbb{R}$ satisfies CH (in the sense that it either countable or can be mapped bijectively to $\mathbb{R}$).

Closed sets?

Given a set $A \subseteq \mathbb{R}$, we call $x \in \mathbb{R}$ an **limit point** of A if

$$\forall \epsilon > 0 \, \exists z \in A \, [z \neq x \,\&\, z \in U_\varepsilon(x)], \tag{1}$$

where $U_\varepsilon(x)$ denotes the standard ε-neighborhood of x in $\mathbb{R}$

A non-empty set $P \subseteq \mathbb{R}$ is **perfect** if it is closed and every point of P is an limit point.

In other words, a perfect set is a closed set that has no isolated points.

For a perfect set P, every neighborhood of a point $p \in P$ contains infinitely many points from P.

Obviously, $\mathbb{R}$ itself is perfect, as is any closed interval in $\mathbb{R}$. There are totally disconnected perfect sets, such as the **middle-third Cantor set** in $[0, 1]$

Theorem 0.1 (Cantor, 1884)**.** *A perfect subset of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$.*

Theorem 0.2 (Cantor-Bendixson)**.** *Every uncountable closed subset of $\mathbb{R}$ contains a perfect subset.*

Corollary 0.2.1. *Every closed subset of $\mathbb{R}$ is either countable or of the cardinality of the continuum.*

Definition 0.1. *A family $\mathcal{F}$ of sets (of reals) has the* ***perfect set property*** *if every set in $\mathcal{F}$ is either countable or has a perfect subset.*

Question
Which families of sets have the perfect set property?