My PDF

Sunday 29th December, 2024

Introduction: Perfect Subsets of the Real Line

Continuum Hypothesis (Cantor, 1890s)
If $A \subseteq R$ is uncountable, then there exists a bijection between A and R. That is, is every uncountable subset of R is of the same cardinality as R.

Possible approach: show that CH holds for a number of sets with an easy topological structure.

Exercise
Show that every open set in R satisfies CH (in the sense that it either countable or can be mapped bijectively to R).

Closed sets?

Given a set $A \subseteq R$, we call $x \in R$ an **limit point** of A if

$$\forall \epsilon > 0 \, \exists z \in A \, [z \neq x \,\&\, z \in U_\varepsilon(x)], \tag{1}$$

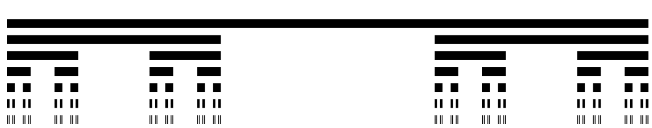
where $U_\varepsilon(x)$ denotes the standard ε-neighborhood of x in R

A non-empty set $P \subseteq R$ is **perfect** if it is closed and every point of P is an limit point.

In other words, a perfect set is a closed set that has no isolated points.

For a perfect set P, every neighborhood of a point $p \in P$ contains infinitely many points from P.

Obviously, R itself is perfect, as is any closed interval in R. There are totally disconnected perfect sets, such as the **middle-third Cantor set** in $[0, 1]$



Theorem 0.1 (Cantor, 1884). *A perfect subset of R has the same cardinality as R.*

Theorem 0.2 (Cantor-Bendixson). *Every uncountable closed subset of R contains a perfect subset.*

Corollary 0.2.1. *Every closed subset of R is either countable or of the cardinality of the continuum.*

Definition 0.1. *A family $\mathcal{F}$ of sets (of reals) has the* ***perfect set property*** *if every set in $\mathcal{F}$ is either countable or has a perfect subset.*

Question

Which families of sets have the perfect set property?