The Axiom of Constructibility

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1 Relative consistency proofs

In this section, we are going to show that if ZF is consistent, so are ZF + AC and ZF + GCH. The usual way to do this is to exhibit a model ZF in which the additional axioms holds, too, assuming a model of ZF exists. The universe of a model is supposed to be a set, and we will work with such *set models* when we will construct a model of ZF in which CH does *not* hold.

Example 1.1. *The set of hereditarily finite sets H_ω (which is the same as V_ω) is a model of* ZF − *Infinity*+¬*Infinity. This implies that the negation of the Axiom of Infinity is consistent with* ZF − *Infinity (always provided* ZF *is consistent).*

This means, if ZF *is consistent, the Axiom of Infinity is not provable from the other axioms.*

In this section, we will work with **class models** instead, in particular, L. The satisfaction relation is not formalizable for arbitrary classes, so we have to argue syntactically.

In the previous section, we showed that L is an inner model for ZF. What the "model" part here means is simply that we can prove in ZF that every axiom of ZF holds *relative to* L, or, using the standard notation for provability,

$$\mathsf{ZF} \vdash \sigma^L \qquad \text{for all axioms } \sigma \in \mathsf{ZF}. \tag{1}$$

In this section, we will also show that

$$\mathsf{ZF} \vdash \tau^L \tag{2}$$

for $\tau =$ AC and $\tau =$ GCH. We claim that this yields

> If ZF is consistent, then ZF + τ is consistent.

For suppose ZF + τ is inconsistent. Then there exists a proof of $\theta \wedge \neg\theta$ from ZF + τ, for some formula θ. Every formal proof uses only finitely many steps, so there exists a *finitely many* $\sigma_1, \dots, \sigma_n \in$ ZF + τ such that

$$\sigma_1 \wedge \cdots \wedge \sigma_n \ \vdash \ \theta \wedge \neg\theta. \tag{3}$$

By the Deduction Theorem of first-order logic, we have

$$\vdash (\sigma_1 \wedge \cdots \wedge \sigma_n) \ \rightarrow \ (\theta \wedge \neg\theta). \tag{4}$$

This means $(\sigma_1 \wedge \cdots \wedge \sigma_n) \ \rightarrow \ (\theta \wedge \neg\theta)$ is a *validity* and derivable by purely logical arguments (not assuming any additional axioms). But any such validity will remain valid when *relativized* (recall that classes are always defined via a formula φ):

$$\vdash (\sigma_1 \wedge \cdots \wedge \sigma_n)^L \ \rightarrow \ (\theta \wedge \neg\theta)^L. \tag{5}$$

By assumption, ZF $\vdash (\sigma_1 \wedge \cdots \wedge \sigma_n)^L$, hence

$$\mathsf{ZF} \vdash (\theta \wedge \neg\theta)^L. \tag{6}$$

By the definition of relativization, the right-hand side is equivalent to $\theta^L \wedge \neg\theta^L$, which implies ZF is inconsistent - contradiction!

2 The Axiom V = L

We can add to ZF the axiom that all sets are constructible, i.e.

$$(\mathsf{V}=\mathsf{L}) \qquad \forall x \exists y\, (y \text{ is an ordinal } \wedge\ x \in L_y).$$

This axiom is usually denoted by V = L. We may be tempted to think that L is then trivially a model of ZF+V = L. But this is not at all clear, since this has to hold **relative to** L, i.e. $(\mathsf{V}=\mathsf{L})^L$.

This means that

$$\forall x \in L\, \exists y \in L\, (y \text{ is an ordinal } \wedge\ (x \in L_y)^L).$$

To verify this, we need to make sure that *inside* L, L "means the same as" L. This is, of course, an absoluteness property, and we therefore revisit the complexity of the formulas defining the constructible universe.

We have seen that the map $a \mapsto \mathcal{P}_{\mathrm{Def}}(a)$ is Σ_1. This has important implications for the map $\alpha \mapsto L_\alpha$.

Proposition 2.1. *The map $\alpha \mapsto L_\alpha$ is Δ_1.*

Proof. We first show that the mapping is Σ_1. The mapping is obtained by ordinal recursion over the functions $a \mapsto \mathcal{P}_{\mathrm{Def}}(a)$ and $a \mapsto \bigcup a$.

In general, if a function $G : \mathrm{V} \to \mathrm{V}$ is Σ_1 and $F : \mathrm{Ord} \to \mathrm{V}$ is obtained by recursion from G, i.e. $F(\alpha) = G(F \restriction \alpha)$, then F is also Σ_1. This is because

$$\begin{aligned} y = F(\alpha) \ \leftrightarrow\ \alpha \in \mathrm{Ord} \wedge \exists f\, (f \text{ function } \wedge \mathrm{dom}(f) = \alpha \\ \wedge \forall \beta < \alpha (f(\beta) = G(f \restriction \beta)) \wedge y = G(f)). \end{aligned}$$

Applying some of the various prefix transformations for Σ_1-formulas, and using that being an ordinal, being an function, being the domain of a function, etc., are all Δ_0 properties, the above formula can be shown to be Σ_1, too.

In our case, G is a function that applies either $\mathcal{P}_{\mathrm{Def}}$ or $\bigcup$ (both at most Δ_1), depending on whether the input is a function defined on a successor ordinal or a limit ordinal (or applies the identity if neither is the case). Fortunately, this case distinction is also Δ_0, and hence we obtain that $F : \alpha \mapsto L_\alpha$ is Σ_1.

Finally, as in Theorem **??**, observe that if F is a Σ_1 function with a Δ_1 domain (Ord), then F is actually Δ_1, since we have

$$F(x) \neq y \ \Leftrightarrow\ x \notin \mathrm{dom}(F)\ \vee\ \exists z (F(x) = z\ \wedge\ y \neq z) \tag{7}$$

so the complement of the graph of F is Σ_1-definable, too.

□

Corollary 2.0.1.

- *(1)* *The relations $x = L_\alpha$ and $x \in L_\alpha$ are Δ_1.*
- *(2)* *The predicate $x \in L$ is Σ_1.*
- *(3)* *The axiom V = L is Π_2.*

We can relativize the definition of L to other classes M. If M is is an inner model, then the development of L can be done *relative to* M. Since M is a ZF-model, it has to contain all the sets L_α^M (as we developed definability and proved facts about it *inside* ZF). As M is transitive, the mapping $F : \alpha \mapsto L_\alpha$ is absolute for M and we obtain, for all α,

$$L_\alpha^M = L_\alpha. \tag{8}$$

Theorem 2.1.

- *(1) If M is an inner model of* ZF*, then $L = L^M \subseteq M$.*
- *(2) L is a model of* ZF $+$ V $=$ L.

Proof. (1) follows immediately from the fact that for such M, $L_\alpha^M = L_\alpha$.

(2) We have

$$\begin{aligned}
(\mathsf{V} = \mathsf{L})^L &\leftrightarrow \forall x \in L \exists y \in L \, (y \text{ is an ordinal } \wedge \, x \in L_y)^L \\
&\leftrightarrow \forall x \in L \exists \alpha \, (x \in L_\alpha)^L && (\mathrm{Ord} \subset L \text{ and absolute}) \\
&\leftrightarrow \forall x \in L \exists \alpha \, (x \in L_\alpha) && (\text{by (1)})
\end{aligned}$$

The last statement is true since $L = \bigcup_\alpha L_\alpha$.

□

3 Constructibility and the Axiom of Choice

Every well-ordering on a transitive set X can be extended to a well-ordering of $\mathcal{P}_{\mathrm{Def}}(X)$.

Note that every element of $\mathcal{P}_{\mathrm{Def}}(X)$ is determined by a pair $(\psi, \vec{a})$, where ψ is a set-theoretic formula, and $\vec{a} = (a_1, \dots, a_n) \in X^{<\omega}$ is a finite sequence of parameters.

For each $z \in \mathcal{P}_{\mathrm{Def}}(X)$ there may exist more than one such pair (i.e. z can have more than one definition), but by well-ordering the pairs $(\psi, \vec{a})$, we can assign each $z \in \mathcal{P}_{\mathrm{Def}}(X)$ its **least** definition, and subsequently order $\mathcal{P}_{\mathrm{Def}}(X)$ by comparing least definitions. Elements already in X will form an initial segment.

Such an order on the pairs $(\psi, \vec{a})$ can be obtained in a **definable way**: First use the order on X to order $X^{<\omega}$ length-lexicographically, order the formulas by their Gödel numbers, and finally put

$$(\psi, \vec{a}) < (\varphi, \vec{b}) \quad \text{iff} \quad \psi < \varphi \text{ or } (\psi = \varphi \text{ and } \vec{a} < \vec{b}). \tag{9}$$

Based on this, we can define an order $<_L$ all levels of L so that the following hold:

- **(1)** $<_L \restriction V_\omega$ is a standard well-ordering of V_ω (as for example given by a canonical isomorphism $(V_\omega, \in) \leftrightarrow (\mathbb{N}, E)$, see Ackermann (1937))
- **(2)** $<_L \restriction L_{\alpha+1}$ is the order on $\mathcal{P}_{\mathrm{Def}}(L_\alpha)$ induced by $<_L |L_\alpha$.
- **(3)** $<_L \restriction L_\lambda = \bigcup_{\alpha<\lambda} <_L \restriction L_\alpha$ for a limit ordinal $\lambda > \omega$.

It is straightforward to verify that this is indeed a well-ordering on L. Moreover, the relation $<_L$ is Δ_1. (To verify this, we have to spell out all the details of the above definition. This is a little involved, so we skip this here and refer to Jech (2003).)

Theorem 3.1. V $=$ L *implies* AC

Since L is a model of ZF $+$ V $=$ L, we obtain

Corollary 3.1.1. *If* ZF *is consistent, then* ZF $+$ AC($=$ ZFC) *is consistent, too.*

4 Condensation and the Continuum Hypothesis

We now show that V $=$ L implies the Continuum Hypothesis. The proof works by showing that under V $=$ L, every subset of a cardinal κ will be constructed by stage κ^+. This is made possible by a "**condensation**" argument: If any subset x of κ is in L, then it must show up at some stage L_λ. κ and x generate an elementary substructure M of L_λ of cardinality κ. If we could show that this M **itself must be an** L_β, we can use the fact following fact. Essentially, it tells us that the cardinality of the L_α evolves "tamely" along the ordinals (as opposed to (V_α)).

Proposition 4.1. *For all $\alpha \geq \omega$, $|L_\alpha| = |\alpha|$.*

Proof. We know that $\alpha \subseteq L_\alpha$. Hence $|\alpha| \leq |L_\alpha|$. To show $|\alpha| \geq |L_\alpha|$, we work by induction on α.

If $\alpha = \beta + 1$, then by Proposition **??**(4), $|L_\alpha| = |L_\beta| = |\beta| \leq |\alpha|$.

If α is limit, then L_α is a union of $|\alpha|$ many sets of cardinality $\leq |\alpha|$ (by inductive hypothesis), hence of cardinality $\leq |\alpha|$.

□

But why would an elementary substructure of an L_λ have to be itself an L_β? This is where the absoluteness of the construction of L strikes yet again!

Lemma 4.1 (Condensation lemma)**.** *There is a formula $\varphi_{\mathsf{V=L}}$ so that if M is a transitive set with $M \models \varphi_{\mathsf{V=L}}$, then $M = L_\lambda$ for some limit ordinal λ.*

Proof. Let T be the axioms of ZF (including *Pairing*, *Union*, *Set Existence*) used to prove that all the theorems leading up to the fact that for all α, L_α exists and that $\alpha \mapsto L_\alpha$ is Δ_1 (and hence absolute). Any proof is finite, so we have used only finitely many (instances of) axioms of ZF to prove these facts. In particular, T is finite. Let $\varphi_{\mathsf{V=L}}$ be the sentence we obtain by taking the conjunction ($\wedge$) of all axioms in T together with the axiom $\mathsf{V} = \mathsf{L}$.

Suppose for a transitive set M, $M \models \varphi_{\mathsf{V=L}}$. Let λ be the least ordinal not in M. We must have that $\mathrm{Ord}^M = \lambda$, by the absoluteness of ordinals. Moreover, λ must be a limit ordinal since for each $\alpha \in M$, $\alpha \cup \{\alpha\}$ is in M since M satisfies *Pairing* and *Union*.

We have that

$$M \models \forall x \exists \alpha \in \mathrm{Ord}(x \in L_\alpha), \tag{10}$$

thus

$$\forall x \in M \exists \alpha < \lambda (x \in L_\alpha^M). \tag{11}$$

By absoluteness of $\alpha \mapsto L_\alpha$, we have $L_\alpha^M = L_\alpha$ and therefore

$$M \subseteq \bigcup_{\alpha \in M} L_\alpha = \bigcup_{\alpha < \lambda} L_\alpha = L_\lambda. \tag{12}$$

On the other hand, for each $\alpha < \lambda$, L_α^M exists in M (since T is strong enough to prove this), and by absoluteness

$$L_\lambda = \bigcup_{\alpha < \lambda} L_\alpha = \bigcup_{\alpha \in M} L_\alpha^M \subseteq M. \tag{13}$$

□

We now put condensation to use as described above.

Lemma 4.2. *Suppose $V = L$. If κ is a cardinal and $x \subseteq \kappa$, then $x \in L_{\kappa^+}$.*

Proof. As we assume $\mathsf{V} = \mathsf{L}$, by the Theorem **??**, there exists limit $\lambda > \kappa$ such that $x \in L_\lambda$ and such that $L_\lambda \models \varphi_{\mathsf{V=L}}$. Let $X = \kappa \cup \{x\}$. By choice of λ, $X \subseteq L_\lambda$.

By the Löwenheim-Skolem Theorem, there exists an **elementary substructure** $N \preceq L_\lambda$ such that

$$X \subseteq N \subseteq L_\lambda \quad \text{and} \quad |N| = |X|. \tag{$*$}$$

N is not necessarily transitive, but since it is well-founded we can take its **Mostowski collapse** (Theorem **??**) and obtain a **transitive** set M together with an **isomorphism** $\pi : (N, \in) \to (M, \in)$.

Since κ is contained in both M and N, and is already transitive, it is straightforward to show via induction that $\pi(\alpha) = \alpha$ for all $\alpha \in \kappa$. Since $x \subseteq \kappa$, this also yields $\pi(x) = x$. This implies in turn that $x \in M$.

As $(M, \in)$ is isomorphic to $(N, \in)$ and $N \preceq L_\lambda$, M satisfies the same sentences as $(L_\lambda, \in)$. In particular, $M \models \varphi_{\mathsf{V=L}}$. By the **condensation lemma**, $M = L_\beta$ for some β.

This implies, by Proposition **??**,

$$|\beta| = |L_\beta| = |M| = |N| = |X| = \kappa < \kappa^+ \leq \lambda. \tag{14}$$

Since $x \in L_\beta$ and $\beta < \kappa^+$, it follows that $x \in L_{\kappa^+}$, as desired.

□

Theorem 4.3 (Gödel)**.** *If* $\mathsf{V} = \mathsf{L}$*, then for all cardinals* κ*,* $2^\kappa = \kappa^+$*.*

Proof. If $\mathsf{V} = \mathsf{L}$, then by Lemma **??**, $\mathcal{P}(\kappa) \subseteq L_{\kappa^+}$. With Proposition **??**, we obtain

$$2^\kappa = |\mathcal{P}(\kappa)| \leq |L_{\kappa^+}| = \kappa^+. \tag{15}$$

□

Corollary 4.3.1. *If* ZF *is consistent, so is* ZF + AC + GCH.

References

W. Ackermann. Die Widerspruchsfreiheit der allgemeinen Mengenlehre. *Mathematische Annalen*, 114(1):305–315, 1937.

T. Jech. *Set Theory*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.