Lecture Notes on Randomness for Continuous Measures

Jan Reimann
Department of Mathematics
University of California, Berkeley

1 Introduction

Most studies on algorithmic randomness focus on reals random with respect to the uniform distribution, i.e. the $(1/2, 1/2)$-Bernoulli measure, which is measure theoretically isomorphic to Lebesgue measure on the unit interval. The theory of uniform randomness, with all its ramifications (e.g. computable or Schnorr randomness) has been well studied over the past decades and has led to an impressive theory.

Recently, a lot of attention focused on the interaction of algorithmic randomness with recursion theory: What are the computational properties of random reals? In other words, which computational properties hold effectively for almost every real? This has led to a number of interesting results, many of which will be covered in a forthcoming book by Downey and Hirschfeldt [15].

While the understanding of "holds effectively" varied in these results (depending on the underlying notion of randomness, such as computable, Schnorr, or weak randomness, or various arithmetic levels of Martin-Löf randomness, to name only a few), the meaning of "for almost every" was usually understood with respect to Lebesgue measure. One reason for this can surely be seen in the fundamental relation between uniform Martin-Löf tests and descriptive complexity in terms of (prefix-free) Kolmogorov complexity: A real is not covered by any Martin-Löf test (with respect to the uniform distribution) if and only if all of its initial segments are incompressible (up to a constant additive factor).

However, one may ask what happens if one changes the underlying measure. This question is virtually as old as the theory of randomness. Martin-Löf [45] defined randomness not only for Lebesgue measure but also for arbitrary Bernoulli distributions. Levin's contributions in the 1970's [78, 35, 36, 37] extended this to arbitrary probability measures. His framework of *semimeasures* provided an elegant uniform approach, flanked by a number of remarkable results and principles such as the existence of uniform tests, conservation of randomness, and the existence of neutral measures. This essentially defined the 'Russian' school of randomness in succession of Kolmogorov, to which Gacs, Muchnik, Shen, Uspensky, Vyugin, and many others have contributed.

Recently, partly driven by Lutz's introduction of effective fractal dimension concepts [42] and their fundamental connection with Kolmogorov complexity, interest in non-uniform randomness began to grow 'outside' the Russian school, too. It very much seems that interesting mathematics arises out of combining non-uniform randomness and logical/computational complexity in the same way it did for Lebesgue measure.

The purpose of these notes is to complement recent, far more complex endeavours of capturing research on randomness and computability (such as [15] or [52]) by focusing on randomness for non-Lebesgue measures. Of course, even this restricted plan is far too comprehensive.

2 Measures on Cantor Space

In this section we introduce the basic notions of measure on the Cantor space 2^ω. We make use of the special topological structure of 2^ω to give a unified treatment of a large class of measures, not necessarily σ-finite. We follow Rogers' approach [63] based on premeasures, which combines well with the clopen set basis of 2^ω. This way, in the general framework of randomness, we do not have to distinguish between probability measures and Hausdorff measures, for instance.

2.1 The Cantor space as a metric space

The *Cantor space* 2^ω is the set of all infinite binary sequences, also called *reals*. The mapping $x \mapsto \sum x(n)2^{-n}$ surjects 2^ω onto the unit interval $[0, 1]$. On the other hand, an element of 2^ω can be seen as the characteristic sequence of a subset of the natural numbers.

The usual metric on 2^ω is defined as follows: Given $x, y \in 2^\omega$, $x \neq y$, let $x \cap y$ be the longest common initial segment of x and y (possibly the empty string $\emptyset$). Define

$$d(x,y) = \begin{cases} 2^{-|x \cap y|} & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

Given a set $A \subseteq 2^\omega$, we define its *diameter* $d(A)$ as

$$d(A) = \sup\{d(x,y) : \ x, y \in A\}.$$

The metric d is compatible with the *product topology* on $\{0,1\}^{\mathbb{N}}$, if $\{0,1\}$ is endowed with the discrete topology.

2^ω is a compact Polish space. A countable basis is given by the *cylinder sets*

$$N_\sigma = \{x : \ x\lceil n = \sigma\},$$

where σ is a finite binary sequence. We will occasionally use the notation $N(\sigma)$ in place of N_σ to avoid multiple subscripts. $2^{<\omega}$ denotes the set of all finite binary sequences. If $\sigma, \tau \in 2^{<\omega}$, we use $\subseteq$ to denote the usual prefix partial ordering. This extends in a natural way to $2^{<\omega} \cup 2^\omega$. Thus, $x \in N_\sigma$ if and only if $\sigma \subset x$. Finally, given $U \subseteq 2^{<\omega}$, we write N_U to denote the open set induced by U, i.e. $N_U = \bigcup_{\sigma \in U} N_\sigma$.

2.2 Outer measures

A measure is a monotone, additive set function on a σ-algebra. Measures can be obtained from outer measures via restriction to a suitable family of sets. The Cantor space is a compact metric space, so we can follow the usual development of measure theory on locally compact spaces to introduce (outer) measures on 2^ω (see Halmos [21] or Rogers [63]). The following method to construct outer measures has been referred to as *Method I* [51, 63].

2.1 Definition. Let $2^{<\omega}$ be the set of all finite binary sequences. A *premeasure* is a mapping $\rho : 2^{<\omega} \to \mathbb{R}^{\geq 0}$.

If ρ is a premeasure, define the set function $\mu_\rho^* : \mathcal{P}(2^\omega) \to \mathbb{R}^{\geq 0}$ by letting

$$\mu_\rho^*(A) = \inf \left\{ \sum_{\sigma \in U} \rho(\sigma) : \ A \subseteq N_U \right\}, \tag{2.1}$$

where we set $\mu_\rho^*(\emptyset) = 0$. It can be shown that μ_ρ^* is an outer measure. An *outer measure* is a function $\nu^* : \mathcal{P}(2^\omega) \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ such that

(M1) $\nu^*(\emptyset) = 0$,

(M2) $\nu^*(A) \leq \nu^*(B)$ whenever $A \subseteq B$,

(M3) if (A_n) is a countable family of subsets of 2^ω, then

$$\nu^*\left(\bigcup_n A_n\right) \leq \sum_n \nu^*(A_n).$$

If we restrict an outer measure ν^* to sets E which satisfy

$$\nu^*(A) = \nu^*(A \cap E) + \nu^*(A \setminus E) \quad \text{for all } A \subseteq 2^\omega, \tag{2.2}$$

we obtain the *ν^*-measurable sets*. The restriction of ν^* to the measurable sets is called a *measure*, and it will be denoted by ν. It can be shown that the measurable sets form a *σ-algebra*, i.e. they are closed under countable unions, complement, and the empty set is measurable.

In the course of this article, we will always assume that a measure ν is derived from an outer measure via (2.2), and that every outer measure in turn stems from a premeasure as in (2.1). (Rogers [63] studies in great detail the relations between measures, outer measures, and premeasures.)

Of course, the nature of the outer measure μ_ρ^* obtained via (2.1) and the μ_ρ-measurable sets will depend on the premeasure ρ. In the following, we will discuss the two most important kinds of outer measures studied in randomness theory: probability measures and Hausdorff measures.

2.3 Probability measures

A *probability measure* ν is any measure that is based on a premeasure ρ which satisfies $\rho(\emptyset) = 1$ and

$$\rho(\sigma) = \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1) \tag{2.3}$$

for all finite sequences σ. The resulting measure μ_ρ preserves ρ in the sense that $\mu_\rho(N_\sigma) = \rho(N_\sigma)$ for all σ. This follows from the Caratheodory extension theorem. In the following, we will often identify probability measures with their underlying premeasure, i.e. we will write $\mu(\sigma)$ instead of $\mu(N_\sigma)$.

It is not hard to see that μ_ρ is a *Borel measure*, i.e. all Borel sets are measurable. It is also *G_δ-regular*, which means that for every measurable set A there exists a G_δ-set G such that $\mu_\rho(A) = \mu_\rho(G)$.

For $\rho(\sigma) = d(N_\sigma) = 2^{-|\sigma|}$ we obtain the *Lebesgue measure* $\mathcal{L}$ on 2^ω, which is the unique translation invariant measure on 2^ω for which $\mathcal{L}(N_\sigma) = d(N_\sigma)$.

(Generalized) Bernoulli measures correspond to product measures on the space $\{0,1\}^{\mathbb{N}}$. Suppose $\bar{p} = (p_0, p_1, p_2, \dots)$ is a sequence of real numbers such that $0 \leq p_i \leq 1$ for all i. Let $\rho_i(1) = p_i$, $\rho_i(0) = 1 - p_i$, and set

$$\rho(\sigma) = \prod_{i=0}^{|\sigma|-1} \rho_i(\sigma(i)) \tag{2.4}$$

The associated measure μ_ρ will be denoted by $\mu_{\bar{p}}$. If $p_i = p$ for all i, we call the measure $\mu_{\bar{p}} = \mu_p$ simply a *Bernoulli measure*. Note that the Bernoulli measure with $p_i = 1/2$ for all i coincides with Lebesgue measure $\mathcal{L}$.

Dirac measures are probability measures concentrated on a single point. If $x \in 2^\omega$, we define

$$\rho(\sigma) = \begin{cases} 1 & \text{if } \sigma \subset x, \\ 0 & \text{otherwise.} \end{cases}$$

For the induced outer measure we obviously have $\mu_\rho(A) = 1$ if and only if $x \in A$, and $\mu_\rho(A) = 0$ if and only if $x \notin A$. The corresponding measure is usually denoted by δ_x.

2.4 Hausdorff measures

Hausdorff measures are of fundamental importance in geometric measure theory. They share the common feature that the premeasures they stem from only depend on the diameter of an open set. Therefore, the resulting measure will be translation invariant.

Assume h is a nonnegative, nondecreasing, continuous on the right function defined on all nonnegative reals. Assume, furthermore, that $h(t) > 0$ if and only if $t > 0$. Define the premeasure ρ_h as

$$\rho_h(N_\sigma) = h(d(N_\sigma)) = h(2^{-|\sigma|}).$$

The resulting measure μ_{ρ_h} will in general not be a Borel measure. Therefore, one refines the transition from a premeasure to an outer measure, also known as *Method II* [51, 63].

Given $\delta > 0$, define the set function

$$\mathcal{H}^h_\delta(A) = \inf\left\{\sum_{\sigma\in U}\rho_h(N_\sigma) : \ A \subseteq N_U \text{ and } (\forall\sigma\in U)\, 2^{-|\sigma|} < \delta\right\}, \tag{2.5}$$

that is, we restrict the available coverings to cylinders of diameter less than δ. Now let

$$\mathcal{H}^h(A) = \lim_{\delta\to 0}\mathcal{H}^h_\delta(A).$$

Since, as δ decreases, there are fewer coverings available, $\mathcal{H}^h_\delta$ is nondecreasing, so the limit is defined, though may be infinite. It can be shown (see Rogers [63]) that the restriction of $\mathcal{H}^h$ to the measurable sets in sense of (2.2) is a Borel measure. It is called a *Hausdorff measure*. For $h(x) = x^t$, where $0 \leq t$, $\mathcal{H}^h$ is called the *t-dimensional Hausdorff measure* and is denoted by $\mathcal{H}^t$.

We will be mostly concerned with $\mathcal{H}^h$-nullsets. It is not hard to see that for any set A, $\mathcal{H}^h(A) = 0$ if and only if $\mu_{\rho_h}(A) = 0$, that is, the nullsets obtained from a premeasure via Method I and Method II coincide. Hence in the case of nullsets we can work with the less involved definition via Method I.

Due to the special nature of the metric d on 2^ω, only diameters of the form 2^{-n}, $n \in \mathbb{N}$, appear. So we can take any nondecreasing, unbounded function $h : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and set $\rho_h(N_\sigma) = 2^{-h(|\sigma|)}$. The resulting Hausdorff measure will, in slight abuse of notation, also be denoted by $\mathcal{H}^h$.

Among the numerous Hausdorff measures, the family of t-dimensional Hausdorff measures $\mathcal{H}^t$ is probably most eminent. It is not hard to see that for any set A, $\mathcal{H}^s(A) < \infty$ implies $\mathcal{H}^t(A) = 0$ for all $t > s$. Likewise, $\mathcal{H}^r(A) = \infty$ for all $r < s$. Thus there is a critical value where $\mathcal{H}^s$ 'jumps' from ∞ to 0. This value is called the *Hausdorff dimension* of A, written $\dim_{\mathrm{H}} A$. Formally,

$$\dim_{\mathrm{H}} A = \inf\{s : \ \mathcal{H}^s(A) = 0\}.$$

Hausdorff dimension is an important notion in fractal geometry, see [17].

2.5 Transformations, image measures, and semimeasures

One can obtain new measures from given measures by transforming them with respect to a sufficiently regular function. Let $f : 2^\omega \to 2^\omega$ be a function such that for every Borel set A, $f^{-1}(A)$ is Borel, too. Such functions are called *Borel* *(measurable)*. Every continuous function is Borel. If μ is a measure on 2^ω and f is Borel, then the *image measure* μ_f is defined by

$$\mu_f(A) = \mu(f^{-1}(A)).$$

It can be shown that every probability measure can be obtained from Lebesgue measure $\mathcal{L}$ by means of a measurable transformation.

2.2 Theorem (folklore, see e.g. Billingsley [6]). *If μ is a Borel probability measure on 2^ω, then there exists a measurable $f : 2^\omega \to 2^\omega$ such that $\mu = \mathcal{L}_f$.*

Proof. The proof uses a simple observation on distribution functions. For this purpose, we identify 2^ω with the unit interval. If g is the distribution function of μ, i.e.

$$g(x) = \mu([0, x]),$$

Let us define

$$f(x) = \inf\{y : \ x \leq g(y)\}.$$

g is nondecreasing and continuous on the right, so $\{y : \ x \leq g(y)\}$ is always an interval closed on the left. Therefore, $\{y : x \leq g(y)\} = [f(x), 1]$, so $f(x) \leq y$ if and only if $x \leq g(y)$, so f can be seen as an inverse to g. Clearly, f is Borel measurable. We claim that $\mathcal{L}_f = \mu$. It suffices to show that for every y, $\mathcal{L}_f([0, y]) = \mu([0, y])$. We have

$$\begin{aligned} \mathcal{L}_f([0, y]) = \mathcal{L}(f^{-1}([0, y])) &= \mathcal{L}(\{x : \ f(x) \leq y\}) \\ &= \mathcal{L}(\{x : \ x \leq g(y)\}) = \mu([0, y]). \end{aligned}$$

□

We can use the representation of functions $2^\omega \to 2^\omega$ via mappings of finite strings to obtain a finer analysis of image measures.

Let S, T be trees on $2^{<\omega}$. A mapping $\phi : S \to T$ is called *monotone* if $\sigma \subseteq \tau$ implies $\phi(\sigma) \subseteq \phi(\tau)$. Typical examples of monotone mappings are *Turing operators*.

Monotone mappings of strings induce (partial) mappings of 2^ω. Given a monotone $\phi : S \to T$, let $D(\phi) = \{x \in [S] : \ \lim_n |\phi(x\lceil n)| = \infty\}$. Then define $\widehat{\phi} : D(\phi) \to T$ by $\widehat{\phi}(x) = \bigcup_n \phi(x\lceil n)$.

It is easy to see that $\widehat{\phi}$ is continuous. On the other hand, one can show that every continuous function on 2^ω has a representation via a monotone string function.

2.3 Proposition (see [29])**.** *If $f : 2^\omega \to 2^\omega$ is continuous, then there exists a monotone mapping $\phi : 2^{<\omega} \to 2^{<\omega}$ such that $D(\phi) = 2^\omega$ and $f = \widehat{\phi}$.*

Using monotone functions, we can define a transformation of premeasures. Given a monotone ϕ and a premeasure ρ, let, for any $\tau \in 2^{<\omega}$, the set $\mathrm{Pre}(\tau)$ consist of all strings σ such that $\phi(\sigma) = \tau$, and no proper prefix of σ maps to τ. Define

$$\rho_\phi(\tau) = \sum_{\sigma \in \mathrm{Pre}(\tau)} \rho(\sigma),$$

where we let $\rho_\phi(\tau) = 0$ if $\mathrm{Pre}(\tau) = \emptyset$.

If μ is a probability measure and $\widehat{\phi}$ is total, then it is easy to see that μ_ϕ induces $\mu_{\widehat{\phi}}$. If $\widehat{\phi}$ is not total, we obtain a premeasure ρ_ϕ with the following properties.

$$\rho_\phi(\emptyset) \leq 1 \quad \text{and} \quad (\forall \sigma)\, \rho_\phi(\sigma) \geq \rho_\phi(\sigma^\frown 0) + \rho_\phi(\sigma^\frown 1).$$

Premeasures with this property have first been studied by Zvonkin and Levin [78] who introduced them as *(continuous) semimeasures*. Every semimeasure is the result of applying a monotone mapping to Lebesgue measure $\mathcal{L}$.

2.6 Effective transformations

In their seminal paper, Zvonkin and Levin [78] showed that an analysis of monotone functions (which they call *processes*) underlying continuous transformations yield a finer understanding of image measures. In particular, they showed that Theorem 2.2 holds for an "almost" continuous transformation. Furthermore, they showed that the transformation can be chosen to be at most as complex as the measures involved, in terms of their logical/computational complexity.

2.4 Definition. Let ρ be a premeasure.

(1) ρ is *computable* if there exists a computable function $g : 2^{<\omega} \times \mathbb{N} \to \mathbb{Q}$ such that for all σ, n,

$$|\rho(\sigma) - g(\sigma, n)| \leq 2^{-n}.$$

A measure μ is *computable* if it is induced by a computable premeasure.

(2) ρ is *enumerable (from below)* or simply Σ^0_1 if its *left-cut* $\{(q, \sigma) \in \mathbb{Q} \times 2^{<\omega} : q < \rho(\sigma)\}$ is recursively enumerable.

To define effective monotone mappings, interpret them as relations $R \subseteq 2^{<\omega} \times 2^{<\omega}$ such that

$$\text{if } (\sigma,\tau) \in R, \text{ then } (\forall \sigma_0 \subseteq \sigma)(\exists \tau_0)\,[\tau_0 \subseteq \tau \ \wedge\ (\sigma_0,\tau_0) \in R].$$

This way we can speak of *(partial) recursive monotone mappings*, meaning that the underlying R is recursively enumerable. Accordingly, we say that a monotone function is *recursive in some* $x \in 2^\omega$ if R is r.e. in x. Obviously, enumerable monotone mappings are precisely the mappings induced by some *Turing operator* Φ. Therefore, we will henceforth denote recursive monotone mappings simply by the name *Turing operator*.

One can show that every enumerable semimeasure is the result of applying a Turing operator to Lebesgue measure $\mathcal{L}$. For probability measures, Levin was able to show that these can be effectively generated from $\mathcal{L}$ by means of an "almost total" transformation.

2.5 Theorem (Levin, [78]). *Let $\mu : 2^{<\omega} \to [0,1]$ be a probability measure recursive in $x \in 2^\omega$*

(1) If ϕ is a computable monotone mapping such that $\mu(D(\phi)) = 1$, then μ_ϕ is a probability measure recursive in x.

(2) There exists a monotone ϕ recursive in x such that $\mu = \mathcal{L}_\phi$. ϕ can be chosen such that $\mathcal{L}(D(\phi)) = 1$, i.e. $\widehat{\phi}$ is total except on a set of $\mathcal{L}$-measure zero.

Proof. We will sketch Levin's proof of (2) for computable μ which easily relativizes. Essentially, we show that the mapping f in the proof of Theorem 2.2 can be obtained by some Turing operator ϕ. For this purpose, it is often convenient to identify 2^ω with the unit interval.

The measure μ is computable, so there exists a computable function $\gamma : 2^{<\omega} \times \mathbb{N} \to \mathbb{Q}$ such that, for all n,

$$|\mu(\sigma) - \gamma(\sigma,n)| \leq 2^{-n}.$$

We may assume that γ is approximating μ from above, so $\theta(\sigma,n) := \gamma(\sigma,n) - 2^{-n}$ is an approximation from below.

We construct ϕ as follows: Given a string $\sigma \in 2^{<\omega}$, $|\sigma| = n$, σ represents a binary interval $[a_\sigma, b_\sigma]$ of length 2^{-n} in $[0,1]$. To compute f as in the proof of

Theorem 2.2, we have to find an interval that is contained in $[c_\sigma, d_\sigma]$, where, if g is the distribution function of μ, $g(c_\sigma) = a_\sigma$, $g(d_\sigma) = b_\sigma$.

Define a set $Z = Z_\sigma$ of strings by selecting all those strings τ of length n for which

$$\sum_{\substack{|\xi|=n \\ \xi \leq \tau}} \gamma(\xi, 2n) \geq a_\sigma \quad \text{and} \sum_{\substack{|\zeta|=n \\ \zeta \leq \tau}} \theta(\zeta, 2n) \geq 1 - b_\sigma. \tag{2.6}$$

Here $\xi \leq \tau$ denotes the lexicographic ordering of strings. Let $\phi(\sigma)$ be the longest common initial segment of all strings in Z. We claim that if $x \in D(\phi)$, then $\widehat{\phi}(x) = f(x)$.

If $x \leq g(y)$, then every $x\lceil n$ is mapped to some string less or equal (lexicographically) $y\lceil n$, so $\widehat{\phi}(x) \leq y$ (as real numbers). On the other hand, since $\{y : x \leq g(y)\} = [f(x), 1]$, $z \leq f(x)$ implies $g(z) \leq x$, so an analogous argument yields $z \leq \widehat{\phi}(x)$.

It remains to show that $\mathcal{L}(D(\phi)) = 1$. There are three cases when $\widehat{\phi}$ might not be defined:

1. Suppose x is an atom of μ, i.e. $\mu(\{x\}) > 0$. This means that g has a discontinuity at x. In order to transform Lebesgue measure in to μ, $\widehat{\phi}$ must then map an interval to the single real x. Let $y < z$ be such that $f^{-1}(\{x\}) = (y, z]$. Suppose $y < x_0 < z$. From some n on the interval $[a_{x_0\lceil n} - 2^{-n}, b_{x_0\lceil n} + 2^{-n}]$ (as defined in the construction of ϕ) is contained in $[y, z]$. But then the only string to enter $Z_{x_0\lceil n}$ is $x\lceil n$, so for such x we have $\widehat{\phi}(x_0) = x$. $\widehat{\phi}$ might not be defined on y and z, but since μ is a finite measure, there can be at most countably many points x of positive μ-measure.

2. If there exists an interval $[y, z]$ such that $\mu([y, z]) = 0$, the distribution function g will remain constant on that interval, i.e. $g(x) = g(y)$ for all $y \leq x \leq z$. Thus $g(y)$ is mapped to an interval by $\widehat{\phi}$. However, as μ is finite, there can be at most countably many intervals of μ-measure zero.

3. Obviously, $\widehat{\phi}$ is also undefined if $f(x)$ is a dyadic rational number, for those numbers possess ambiguous dyadic representations, so the correspondent initial segments are always both included in Z. Thus, if $f(x) = m/2^k$, $\phi(x\lceil n)$ will map to a string of length less than k for all sufficiently large n.

$\square$

By computing approximating g, the distribution function of μ through a monotone mapping, Levin was also able to show that every probability measure can be transformed into Lebesgue measure by means of a monotone function, taking into account possible complications given by cases (1)-(3).

2.6 Theorem (Levin, [78])**.** *Let μ be a probability measure recursive in $x \in 2^\omega$. Then there exists a monotone ψ recursive in x such that $\mathcal{L} = \mu_\psi$ and such that the complement of $D(\psi)$ contains only recursive reals or reals lying in intervals of μ-measure zero.*

2.7 Transformations of Hausdorff measures

For Hausdorff measures, transformations reflecting geometric properties are particularly interesting.

Generally, a *Hölder transformation* is a mapping h between metric spaces (X, d_X) and (Y, d_Y) such that for some constants $c, \alpha > 0$,

$$d_Y(h(x), h(y)) \leq c d_X(x, y)^\alpha.$$

In the Cantor space, this implies (recall the definition of metric d)

$$|h(x) \cap h(y)| \geq \alpha |x \cap y| + \log c,$$

The last formula suggests a generalization of Hölder mappings based on string functions.

2.7 Definition. A monotone mapping $\phi : 2^{<\omega} \to 2^{<\omega}$ is *α-expansive*, $\alpha > 0$, if for all $y \in D(\phi)$,

$$\liminf_{n\to\infty} \frac{|\phi(y\lceil n)|}{n} \geq \alpha.$$

2.8 Proposition. *Let $\phi : 2^{<\omega} \to 2^{<\omega}$ be α-expansive for some $\alpha > 0$. Then for all $B \subseteq D(\phi)$, and for all $s \geq 0$,*

$$\mathcal{H}^s(B) = 0 \;\Rightarrow\; \mathcal{H}^{s/\alpha}(\widehat{\phi}(B)) = 0.$$

The case where $\alpha = 1$ is especially important. Such functions are called *Lipschitz*, and if the inverse mapping is Lipschitz, too, an easy corollary of Proposition 2.8 yields that Hausdorff dimension is *invariant under bi-Lipschitz functions*.

It is also possible to consider more general Lipschitz-like conditions and prove a related version of Proposition 2.8, see [63, Thm. 29].

3 Martin-Löf Randomness

It was Martin-Löf's fundamental idea to define randomness by choosing a *countable family* of nullsets. For any non-trivial measure, the complement of the union of these sets will have positive measure, and any point in this set will be considered *random*. There are of course many possible ways to pick a countable family of nullsets. In this regard, it is very benefiting to use the framework of recursion theory and effective descriptive set theory.

3.1 Nullsets

Before we go on to define Martin-Löf randomness formally, we note that every nullset for a measure defined via Method I (and Method II, as is easily seen) is contained in a G_δ-nullset.

3.1 Proposition. *Suppose ρ is a premeasure. Then a set $A \subseteq 2^\omega$ is μ_ρ-null if and only if there exists a set $U \subseteq \mathbb{N} \times 2^{<\omega}$ such that for all n,*

$$A \subseteq N(U_n) \quad \text{and} \quad \sum_{\sigma \in U_n} \rho(N_\sigma) \leq 2^{-n}, \tag{3.1}$$

where $U_n = \{\sigma : (n, \sigma) \in U\}$.

Of course, the G_δ-cover of A is given by $\bigcap_n U_n$. There is an alternative way of describing nullsets which turns out to be useful both in the classical and algorithmic setting (see [63, Thm. 32] and [69]).

3.2 Proposition. *Suppose ρ is a premeasure. A set $A \subseteq 2^\omega$ is μ_ρ-null if and only if there exists a set $V \subseteq 2^{<\omega}$ such that*

$$\sum_{\sigma \in V} \rho(N_\sigma) < \infty, \tag{3.2}$$

and for all $x \in A$ there exist infinitely many $\sigma \in V$ such that $x \in N_\sigma$, or equivalently, $\sigma \subseteq x$.

3.2 Martin-Löf tests and randomness

Essentially, a Martin-Löf test is an effectively presented G_δ nullset (relative to some parameter z).

3.3 Definition. Suppose $z \in 2^\omega$ is a real. A *test relative to* z, or simply a z*-test*, is a set $W \subseteq \mathbb{N} \times 2^{<\omega}$ which is recursively enumerable in z. Given a natural number $n \geq 1$, an n*-test* is a test which r.e. in $\emptyset^{(n-1)}$, the $(n-1)$st Turing jump of the empty set. A real x *passes* a test W if $x \notin \bigcap_n N(W_n)$.

Passing a test W means not being contained in the G_δ set given by W. The condition '*r.e. in* z' implies that the open sets given by the sets W_n form a uniform sequence of $\Sigma^0_1(z)$ sets, and the set $\bigcap_n N(W_n)$ is a $\Pi^0_2(z)$ subset of 2^ω. To test for randomness, we want to ensure that W actually describes a nullset.

3.4 Definition. Suppose μ is a measure on 2^ω. A test W is *correct for* μ if

$$\sum_{\sigma \in W_n} \mu(N_\sigma) \leq 2^{-n}. \tag{3.3}$$

Any test which is correct for μ will be called a *test for* μ, or μ-test.

Now we can state Martin-Löf's definition of randomness for Lebesgue measure $\mathcal{L}$.

3.5 Definition (Martin-Löf [45]). Suppose $z \in 2^\omega$. A real x is *Martin-Löf* $\mathcal{L}$*-random relative to* z, or simply $\mathcal{L}$*-*z*-random*, if x passes every z-test W for $\mathcal{L}$. Accordingly, x is $\mathcal{L}$*-*n*-random*, if it passes every n-test which for $\mathcal{L}$. Finally, x is *arithmetically* $\mathcal{L}$*-random* if it is $\mathcal{L}$-n-random for every $n \geq 1$.

Since there are only countably many Martin-Löf tests, it follows that the set of Martin-Löf random reals has Lebesgue measure 1. Martin-Löf showed that this set can be obtained as the complement of a *single* G_δ-nullset, a *universal test*.

3.6 Theorem (Martin-Löf [45]). *For every* $z \in 2^\omega$*, there exists a Martin-Löf test* U^z *such that* x *is Martin-Löf random relative to* z *if and only if* $x \notin \bigcap_n N(U^z_n)$*. Furthermore,* U^z *can be obtained uniformly in* z*.*

The existence of a universal test is of great technical value. It facilitates a lot of proofs, since one has to consider only one test instead of a whole family of them.

3.3 Randomness for arbitrary measures

Martin-Löf defined randomness not only for Lebesgue measure, but also for arbitrary computable probability measures.

The problem of extending Definition 3.5 to other measures is that the measure itself may contain non-trivial information. If one defines randomness for an arbitrary measure μ simply by considering tests which are correct for μ, this works fine as long as μ is computable. However, if the measure itself contains additional algorithmic information, this leads to possibly unacceptable phenomena.

As an example, consider any real x. Define a premeasure on 2^ω by 'perturbing' Lebesgue measure a little, so that the values $\rho(\sigma)$ remain rational and one can reconstruct x from them. If the perturbance is very small, the new measure μ_ρ will have the same nullsets as Lebesgue measure $\mathcal{L}$, and moreover it is possible to find for every $\mathcal{L}$-test a μ_ρ-test that covers the same reals, and vice versa. As a result, a real could be random with respect to a measure although it is computable from the (pre)measure.

Therefore, it seems worthwhile to incorporate the information given by the measure into the test notion. In the following, we will do this in a straightforward and most general way. This is followed by a discussion of advantages and drawbacks of this approach. Later on we will briefly address more refined concepts, which, however, due to topological reasons, have to be restricted to probability measures on 2^ω.

3.4 Representations of premeasures

To incorporate measures into the effective aspects of a randomness test we have to represent it in a form that makes it accessible for recursion theoretic methods. Essentially, this means to code a measure via an infinite binary sequence or a function $f : \mathbb{N} \to \mathbb{N}$.

The way we introduced it, an outer measure on 2^ω is completely determined by its underlying premeasure defined on the cylinder sets. It seems reasonable to represent these values via approximation by rational intervals.

3.7 Definition. Given a premeasure ρ, define its *rational representation* r_ρ by letting, for all $\sigma \in 2^{<\omega}$, $q_1, q_2 \in \mathbb{Q}$,

$$\langle \sigma, q_1, q_2 \rangle \in r_\rho \;\Leftrightarrow\; q_1 < \rho(\sigma) < q_2. \tag{3.4}$$

The real r_ρ encodes the complete information about the premeasure ρ in the sense that for each σ, the value $\rho(\sigma)$ is uniformly recursive in r_ρ. Therefore, every μ_ρ-nullset is $\Pi^0_2(r_\rho)$. This allows for a straightforward generalization of randomness tests relative to a given measure.

3.8 Definition. Suppose ρ is a premeasure on 2^ω and $z \in 2^\omega$ is a real. A real is *Martin-Löf μ_ρ-random relative to z*, or simply *μ_ρ-z-random* if it passes all $r_\rho \oplus z$-tests which are correct for μ_ρ.

Hence, a real x is random with respect to an arbitrary measure μ_ρ if and only if it passes all tests which are enumerable in the representation r_ρ of the underlying premeasure ρ.

The representation r_ρ is a very straightforward approach to represent measures. As it turns out, for probability measures this representation corresponds to a canonical representation with respect to the weak topology.

Though this approach integrates the information presented in measures into tests, it resolves the 'perturbance phenomenon' in a rather radical fashion – as the following example suggests.

Let $\bar{p} = (\frac{1}{2}+\beta_0, \frac{1}{2}+\beta_1, \frac{1}{2}+\beta_2, \dots)$ be a sequence of rational numbers such that the sequence of 'biases' (β_i) is uniformly computable and $\sum \beta_i^2 < \infty$. We will see in Section 4.1 that a real is random for $\mu_{\bar{p}}$ if and only if it is random with respect to $\mathcal{L}$.

Now consider a second sequence $\bar{p}' = (\frac{1}{2}+\gamma_0, \frac{1}{2}+\gamma_1, \frac{1}{2}+\gamma_2, \dots)$ with $0 < \gamma_i^2 < \beta_i^2$, but this time the sequence of biases (γ_i) this time is not effective, for instance it codes some Martin-Löf $\mathcal{L}$-random real y. Then, according to Definition 3.8, y is not $\mu_{\bar{p}'}$-random, although in some sense the measure $\mu_{\bar{p}'}$ is '*closer*' to $\mathcal{L}$ than $\mu_{\bar{p}}$.

For probability measures, Levin [38] has proposed alternative definitions of randomness using the topological properties of the space of probability measures on 2^ω which dwell further on this problem. However, to the authors knowledge the 'naive' representation r_ρ is the only way to incorporate the information content of measures into Martin-Löf tests in a uniform way, valid for all premeasures alike.

Finally, it should be mentioned that Martin-Löf [45] already gave a definition of randomness for for arbitrary Bernoulli measures. His approach circumvents the difficulties presented by non-computable measures. He exploited the combinatorial properties one would expect from Bernoulli random reals. This way, he was able to give a *uniform test* for all Bernoulli measures.

3.9 Theorem (Martin-Löf)***.*** *There exists a test W_B such that W_B is correct for all Bernoulli measures μ_p.*

Martin-Löf showed that a real x which passes W_B is stochastic in the sense of Von-Mises-Wald-Church (see Ambos-Spies and Kučera [1] for details on stochasticity).

Levin [35] was able to strengthen Martin-Löf's result considerably using the topological structure of the space of probability measures on 2^ω (see Section 6).

3.10 Theorem (Levin [35])***.*** *Let S be an effectively closed set of probability measures. Then there is a test W which is correct for all measures in S, and such that for every x that passes the test W, there is a measure $\mu \in S$ such that x is μ-random.*

3.5 Solovay tests

It is possible to base a definition of randomness on Proposition 3.2. This was suggested by Solovay [69]

3.11 Definition. Suppose ρ is a premeasure on 2^ω and $z \in 2^\omega$. A *Solovay z-test* is a set V r.e. in z. A Solovay test is *correct for μ_ρ* if

$$\sum_{\sigma \in V} \rho(\sigma) < \infty.$$

A real *x passes a Solovay test V* if there exist only finitely many $\sigma \in V$ such that $x \in N_\sigma$. Finally, a real is *Solovay μ_ρ-random relative to z*, or simply *Solovay μ_ρ-z-random*, if it passes all $r_\rho \oplus z$-tests which are correct for μ_ρ.

Solovay [69] observed that for Lebesgue measure, a real is Solovay random if and only if it is Martin-Löf random. This result easily extends to all probability measures.

3.12 Theorem (Solovay)***.*** *If μ is a probability measure on 2^ω, and $z \in 2^\omega$, then x is Solovay μ-z-random if and only if it is Martin-Löf μ-z-random.*

Proof. If U is a Martin-Löf test for μ_ρ, then $V = \bigcup U_n$ forms a Solovay test which is correct for μ_ρ. On the other hand, given a Solovay test V which is correct for μ, we can, by omitting finitely many elements, pass to a Solovay test V' for which $\sum_V \rho(\sigma) \leq 1$. Now define a set $U \subseteq \mathbb{N} \times 2^{<\omega}$ enumerable in V' as follows:

> Put (n, σ) into U if and only if at the stage when σ is enumerated into V',
>
> 1. no extension of σ has been enumerated into V' already, and
> 2. at least 2^{-n} predecessors of σ have been enumerated.

Then it is easy to see that $\sum_{U_n} \mu(N_\sigma) \leq 2^{-n}$, so U is a Martin-Löf test which is correct for μ. □

If μ_ρ is not a probability measure, the above construction cannot be applied to yield that Martin-Löf tests and Solovay tests are equivalent. To see this, note that if μ_ρ is a probability measure, then, if $U \subseteq V$ are open sets represented by $C_U, C_V \subseteq 2^{<\omega}$, respectively, $\sum_{C_U} \rho(\sigma) \leq \sum_{C_V} \rho(\sigma)$. This allowed us to conclude in the preceding proof that $\mu(U_n) \leq 2^{-n}$. The same reasoning is, however, not possible if, for instance, $\rho(\sigma) < \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1)$. In fact, Reimann and Stephan [62] were able to separate Martin-Löf tests and Solovay tests for a large family of premeasures.

3.13 Definition. A *geometrical premeasure* is a premeasure ρ such that $\rho(\emptyset) = 1$ and there are (computable) real numbers p, q with

(1) $1/2 \leq p < 1$ and $1 \leq q < 2$;

(2) $\forall \sigma \in 2^{<\omega}\, \forall i \in \{0,1\}\, [\rho(\sigma^\frown i) \leq p\rho(\sigma)]$;

(3) $\forall \sigma \in 2^{<\omega}\, [q\rho(\sigma) \leq \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1)]$.

We will call such ρ a (p, q)-premeasure. ρ is called an *unbounded premeasure* if it is (p, q)-premeasure for some $q > 1$. A premeasure ρ is called *length-invariant* if

$$(\forall \sigma, \tau)\, [|\sigma| = |\tau| \;\Rightarrow\; \rho(\sigma) = \rho(\tau)]$$

Note that every premeasure $\rho(\sigma) = 2^{-|\sigma| s}$ (on which the Hausdorff measure $\mathcal{H}^s$ is based), $0 < s < 1$, is an unbounded, length invariant premeasure.

3.14 Theorem (Reimann and Stephan [62])***.*** *For every computable, unbounded premeasure ρ there exists a real x which is Martin-Löf μ_ρ-random but not Solovay μ_ρ-random.*

This answered a question raised by Calude, Staiger, and Terwijn [8]. In particular, we see that for effective Hausdorff measures, Martin-Löf tests and Solovay tests do not yield the same notion of randomness.

4 Computable probability measures

Most work on algorithmic randomness beyond Lebesgue measure has been done on computable probability measures. One reason for this can certainly be seen in the fact that Martin-Löf's approach carries over to arbitrary computable probability measures without facing the problem of representing measures described above.

Computable premeasures were defined in Section 2.6. A measure μ is *computable* if it is induced by a computable premeasure.

It is easy to see that a premeasure ρ is computable if and only if its rational representation r_ρ is computable. Therefore, a real x is μ_ρ-random by Definition 3.8 if and only if it passes every test which is correct for μ_ρ. (Recall that a test was defined to be just an r.e. subset of $\mathbb{N} \times 2^{<\omega}$.)

4.1 Equivalent measures and randomness

In our discussion of how to define randomness with respect to arbitrary measures we mentioned an invariance property of randomness due to Vovk [76]. If a generalized Bernoulli measure is close enough the uniform distribution, the corresponding sets of random reals coincide. The underlying dichotomy due to Kakutani [27] has been used by Shen to separate the notions of Martin-Löf randomness and Kolmogorov-Loveland stochasticity.

4.1 Definition. Let μ, ν be two probability measures on 2^ω. μ is called *absolutely continuous* with respect to ν, written $\mu \ll \nu$, if every ν-nullset is also a μ-nullset. If two measures μ, ν are mutually absolutely continuous, we call them *equivalent* and write $\mu \sim \nu$. If on the other hand there exists a set A such that $\mu(A) = 0$ and $\nu(2^\omega \setminus A) = 0$, we call μ and ν *orthogonal*, written $\mu \perp \nu$.

The relation $\sim$ is an equivalence relation on the space of probability measures on 2^ω. For (generalized) Bernoulli measures, Kakutani [27] obtained a fundamental result concerning equivalence of measures.

4.2 Theorem (Kakutani [27])***.*** *Let $\mu_{\bar{p}}$ and $\mu_{\bar{q}}$ be two generalized Bernoulli measures with associated sequences $\bar{p} = (p_i)$ and $\bar{q} = (q_i)$, respectively, such that for some $\varepsilon > 0$, $p_i, q_i \in [\varepsilon, 1-\varepsilon]$ for all i.*

(1) If $\sum_i (p_i - q_i)^2 < \infty$, then $\mu_{\bar{p}} \sim \mu_{\bar{q}}$.

(2) If $\sum_i (p_i - q_i)^2 = \infty$, then $\mu_{\bar{p}} \perp \mu_{\bar{q}}$.

Vovk [76] showed that this dichotomy holds effectively.

4.3 Theorem (Vovk [76]). *Let $\mu_{\bar{p}}$ and $\mu_{\bar{q}}$ as in Theorem 4.2, and suppose that in addition $\mu_{\bar{p}}$ and $\mu_{\bar{q}}$ are computable.*

(1) If $\sum_i (p_i - q_i)^2 < \infty$, then a real x is $\mu_{\bar{p}}$-random if and only if it is $\mu_{\bar{q}}$-random.

(2) If $\sum_i (p_i - q_i)^2 = \infty$, then no real is random with respect to both $\mu_{\bar{p}}$ and $\mu_{\bar{q}}$.

Let $\mu_{\bar{p}}$ be a computable generalized Bernoulli measure induced by $\bar{p} = (\frac{1}{2} + \beta_0, \frac{1}{2} + \beta_1, \frac{1}{2} + \beta_2, \dots)$ with $\beta_i \in [\varepsilon, 1 - \varepsilon]$ for some $\varepsilon > 0$ and all i, and $\lim_i \beta_i = 0$. Shen [67] was able to show that if x is $\mu_{\bar{p}}$-random, then it is Kolmogorov-Loveland stochastic. (For a definition of Kolmogorov-Loveland stochasticity refer to Ambos-Spies and Kučera [1] or Muchnik, Semenov, and Uspensky [50].) However, by Theorem 4.3, if $\sum_i \beta_i^2 = \infty$, x cannot be $\mathcal{L}$-random. It follows that Martin-Löf randomness is a stricter notion than Kolmogorov-Loveland stochasticity.

One can ask whether Vovk's result holds in larger generality. Bienvenu [3] showed that if two computable probability measures have exactly the same set of random reals, then they must be equivalent. However, Bienvenu and Merkle [4] were able to show that the converse does not hold.

4.4 Theorem (Bienvenu and Merkle [4]). *There exists a computable probability measure μ and a real x such that $\mu \sim \mathcal{L}$ and x is $\mathcal{L}$-random but not μ-random. In fact, x can be chosen to be Chaitin's Ω.*

Proof idea. For measures μ, ν, and $k \in \mathbb{N} \cup \{\infty\}$, define

$$\mathcal{L}^k_{\mu/\nu} = \left\{ x \in 2^\omega : \sup_n \frac{\mu(x\lceil n)}{\nu(x\lceil n)} \right\}$$

(define $0/0 = 1$, and $c/0 = \infty$ for $c > 0$). It holds that $\mu \sim \nu$ if and only if $\mu(\mathcal{L}^\infty_{\mu/\nu}) = \nu(\mathcal{L}^\infty_{\nu/\mu}) = 0$.

Let x be a $\mathcal{L}$-random real x in Δ^0_2 such as Chaitin's Ω. Use a computable approximation to x to define a computable measure μ such that $\mathcal{L}^\infty_{\mu/\mathcal{L}} = \emptyset$ and $\mathcal{L}^\infty_{\mathcal{L}/\mu} = \{x\}$. Then $\mu \sim \mathcal{L}$, but the fact that μ along x converges much faster to 0 than $\mathcal{L}$, while on all other paths it behaves like $\mathcal{L}$, up to a multiplicative constant, can be used to define a μ-test that covers x. $\square$

4.2 Proper reals

One may ask whether a given real is random with respect to some computable probability measure. This question was first considered by Zvonkin and Levin [78]. The called reals that are random with respect to some computable probability measure *proper*. Muchnik et al. [50] used the name *natural*.

First note that a real x is trivially random with respect to a measure μ if the set $\{x\}$ does not have μ-measure 0, i.e. if x is an *atom* of μ. It is not hard to see that every atom of a computable probability measure is recursive.

4.5 Proposition (Levin, 1970)**.** *If μ is a computable probability measure and if $\mu(\{x\}) > 0$ for $x \in 2^\omega$, then x is recursive.*

Proof. Suppose $\mu(\{x\}) > c > 0$ for some computable μ and rational c. Let g be a computation function for μ, i.e. g is recursive and for all σ and n, $|g(\sigma, n) - \mu(\sigma)| \leq 2^{-n}$. Define a recursive tree T by letting $\sigma \in T$ if and only if $g(\sigma, |\sigma|) \geq c - 2^{-|\sigma|}$. By definition of T and the fact that μ is a probability measure, it holds that for sufficiently large m,

$$|\{\sigma : \ \sigma \in T \ \wedge \ |\sigma| = m\}| \leq \frac{1}{c - 2^{-m}}.$$

But this means that every infinite path through T is isolated, i.e. if x is an infinite path through T, there exists a string σ such that for all $\tau \supseteq \sigma$, $\tau \in T$ implies $\tau \subset x$. Furthermore, every isolated path through a recursive tree is recursive, and hence x is recursive. $\square$

On the other hand, if x is recursive and not a μ-atom, where μ is a computable probability measure, then one can easily use the recursiveness of x to devise a μ-test that covers x.

Concerning non-recursive reals, examples of non-proper reals can be obtained using *arithmetic Cohen forcing* (see for instance Odifreddi [54] for an introduction).

4.6 Theorem (Muchnik, 1998)**.** *If x is Cohen 1-generic, it cannot be proper.*

Proof idea. It suffices to show that for every $\sigma \in 2^{<\omega}$ and for every $n \in \mathbb{N}$, we can effectively find an extension $\tau \supseteq \sigma$ such that $\mu(N_\tau) \leq 2^{-n}$. This, however, follows easily by induction, since either $\mu(N_{\sigma^\frown 0}) \leq \mu(N_\sigma)/2$ or $\mu(N_{\sigma^\frown 1}) \leq \mu(N_\sigma)/2$, so we can use the computability of μ to search effectively for a suitable extension τ. $\square$

It is straightforward to prove the slightly more general result that any real that has a 1-generic real as a recursive subsequence cannot be proper.

The next example is probably more unexpected.

4.7 Theorem (Levin, 1970). *The halting problem $\emptyset'$ is not proper.*

Proof sketch. Let μ a computable probability measure. Given a set $S \subseteq 2^\omega$ and $n \in \mathbb{N}$, let $S_{n,i} = \{y \in S : y(n) = i\}$.

Use the recursion theorem to construct an r.e. set W_e as follows. Set $F_0 = 2^\omega$. Given F_n, let $n \in W_e$ if and only if

$$\mu(F_n \cap S_{\langle e,n\rangle,1}) < \mu(F_n \cap S_{\langle e,n\rangle,0}).$$

Let $F_{n+1} = F_n \cap S_{\langle e,n\rangle,W_e(n)}$. Hence, W_e picks its values such that the restriction of F_n to paths x which satisfy $x(\langle e,n\rangle) = W_e(n)$ has minimal measure, at most half as large as the measure of F_n.

Since $\emptyset' = \{\langle e,n\rangle : n \in W_e\}$, W_e can be used to define a Martin-Löf μ-test for $\emptyset'$. □

4.3 Computable probability measures and Turing reducibility

We saw in Section 2.6 that Turing reductions transform measures effectively. Every computable probability measure on 2^ω is the result of transforming Lebesgue measure by means of an almost everywhere defined Turing operator. On the other hand, every computable probability measure can be mapped effectively to Lebesgue measure.

Levin formulated the *principle of randomness conservation*: If a μ-random real is transformed by means of an effective continuous mapping f, then the result should be random with respect to the image measure μ_f.

The results of Section 2.6 easily yield that conservation of randomness holds for computable probability measures.

4.8 Proposition. *Let μ be a computable probability measure and ϕ a Turing operator such that $\mu(D(\phi)) = 1$. If x is μ-random, then $\widehat{\phi}(x)$ is μ_ϕ-random.*

Proof idea. If $U = \{U_n\}$ is a test for μ_ϕ, then $V = \{V_n\}$ with $V_n = \{\phi^{-1}(\sigma) : \sigma \in U_n\}$ is a μ-test. □

A finer analysis of Theorems 2.5 and 2.6 yields a much stronger result. The operator $\widehat{\phi}$ is undefined only on a set of effective $\mathcal{L}$-measure 0. Hence it is defined on every $\mathcal{L}$-random real. Likewise, the operator $\widehat{\psi}$ is undefined only on recursive reals or reals lying in intervals of μ-measure 0. We obtain the following result, which says that with regard to Turing reducibility, proper reals have the same computational power as the standard Martin-Löf, i.e. $\mathcal{L}$-random reals. It has independently been proved by Kautz [28]. (His approach is also presented in Downey and Hirschfeldt [15].)

4.9 Theorem (Levin, [78]; Kautz [28])**.** *Let μ be a computable probability measure. If x is μ-random and non-recursive, then x is Turing equivalent to some $\mathcal{L}$-random real R.*

This result can be used to obtain a number of interesting consequences. Demuth [12] observed that every real which is tt-reducible to some $\mathcal{L}$-random real is in the same Turing degree with some $\mathcal{L}$-random real.

4.10 Theorem (Demuth [12])**.** *If $x \leq_{\mathrm{tt}} R$ and R is $\mathcal{L}$-random, then there exists some $y \in 2^\omega$ such that y is $\mathcal{L}$-random and $y =_{\mathrm{T}} x$.*

Proof. If $x \leq_{\mathrm{tt}} R$ via Φ, conservation of randomness implies that x is $\mathcal{L}_\Phi$-random, where $\mathcal{L}_\Phi$ is a computable probability measure. Now apply Theorem 4.9. $\square$

It is known that below a hyperimmune-free Turing-degree, i.e. a degree $\mathbf{a}$ such that every function f recursive in $\mathbf{a}$ is majorized by some recursive function, Turing and truth-table reducibility coincide. Applying the *hyperimmune-free basis theorem* [24] to a Π^0_1 class containing only random reals, we obtain a degree below which every (non-zero) degree contains a $\mathcal{L}$-random real.

4.11 Theorem (Kautz [28])**.** *There exists a Turing degree $\mathbf{a} > \mathbf{0}$ such that any degree $\mathbf{b}$ with $\mathbf{0} < \mathbf{b} \leq \mathbf{a}$ contains a $\mathcal{L}$-random real.*

Another straightforward application concerns the halting problem $\emptyset'$. Bennett [2] investigated the notion of *logical* or *computational depth*. A main result of this investigation was that the halting problem $\emptyset'$ is not truth-table reducible to any $\mathcal{L}$-random real. (Note however that, by results of Kučera [33] and Gács [19], it is Turing reducible to some $\mathcal{L}$-random real.) This result can be derived easily from Theorem 4.7 and conservation of randomness.

4.12 Theorem (Bennett [2], see also Juedes, Lathrop, and Lutz [26])**.** *The halting problem $\emptyset'$ is not truth-table reducible to a $\mathcal{L}$-random real.*

5 Hausdorff Measures

Recently, a lot of research on randomness for non-Lebesgue measures focused on *Hausdorff measures*. One reason for this can certainly be seen in Lutz's introduction of *effective fractal dimension concepts* [41, 42]. Close connections between Kolmogorov complexity and Hausdorff dimension had been known to exist for for quite some time, e.g. through works of Ryabko [64, 65], Staiger [71, 72], or Cai and Hartmanis [7]. But Lutz's concepts brought these together with the topics and techniques that had been developed in resource-bounded measure theory and the investigation of computational properties of random reals.

We can and will not cover these new developments in full breadth, for this purpose the reader may refer to survey articles [22, 40] or the author's PhD-thesis [57]. Instead, we will focus on a few recent results which suit well in the line of this article.

5.1 Effective Hausdorff measures and Kolmogorov complexity

A lot of interesting recent research on effective dimension concepts is based on a fundamental correspondence between Hausdorff measures and Kolmogorov complexity. Although the general framework of randomness from Section 3.4 extends to arbitrary Hausdorff premeasures, investigations focused on effective Hausdorff measures.

Let $h : \mathbb{N} \to \mathbb{R}^{\geq 0}$ be a nondecreasing, unbounded function. In connection with randomness, such functions were studied by Schnorr [66], without explicit reference to Hausdorff measures. Schnorr called such functions *orders* or *order functions*. He used them to classify growth rates of martingales and give a martingale characterization of the randomness concept that is now known as *Schnorr randomness*.

If h is an order function, recall that $\mathcal{H}^h$ denotes the Hausdorff measure induced by the premeasure $2^{-h(|\sigma|)}$. Schnorr's characterization of $\mathcal{L}$-randomness via Kolmogorov complexity can be extended to $\mathcal{H}^h$-random reals. We assume the reader is familiar with the basic definitions of Kolmogorov complexity, as presented in the books by Li and Vitányi [39] and Downey and Hirschfeldt [15].

5.1 Theorem (Tadaki [74]; Reimann [57])***.*** *If h is a computable order function, a real $x \in 2^\omega$ is $\mathcal{H}^h$-random if and only if there exists a constant c such that for all n,*

$$\mathrm{K}(x \lceil n) \geq h(n) - c,$$

where K *denotes prefix-free Kolmogorov complexity.*

Proof sketch. If x is not $\mathcal{H}^h$-random, choose an r.e. test W that covers $\{x\}$ and is correct for $2^{-h(|\sigma|)}$. Define functions $m_n : 2^{<\omega} \to \mathbb{Q}$ by

$$m_n(\sigma) = \begin{cases} n2^{-h(|\sigma|)} & \text{if } \langle n, \sigma \rangle \in W, \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$m(\sigma) = \sum_{n=1}^{\infty} m_n(\sigma).$$

Obviously, all m_n and thus m are enumerable from below. Furthermore, it is not hard to see that

$$\sum_{\sigma \in 2^{<\omega}} m(\sigma) < \infty,$$

hence m is an *enumerable discrete semimeasure*. Apply the *coding theorem* (see [39]) to obtain a constant c_m such that $-\log m(\sigma) \geq \mathrm{K}(\sigma) - c_m$ for all σ. By definition of m, for every n there exists some l_n such that $m(x\lceil l_n) \geq n\,2^{-h(l_n)}$, which implies $\mathrm{K}(x\lceil l_n) - c_m \leq -\log m(x\lceil l_n) \leq h(l_n) - n$.

For the other direction, we use a result by Chaitin [10] which establishes that for any l,

$$|\{\sigma \in \{0,1\}^n : \ \mathrm{K}(\sigma) \leq n + \mathrm{K}(n) - l\}| \leq 2^{n+C-l}, \tag{5.1}$$

where C is a constant independent of n, l. (Here the natural numbers are identified with their binary representation.)

Assume that the complexity of x is not bounded from below by $h(n) - c$ for any constant c. Define

$$W_n = \{\sigma \in 2^{<\omega} : \ \mathrm{K}(\sigma) \leq h(|\sigma|) - n - C\}.$$

Then the test W covers x, since for every l there is some prefix σ of x such that $\mathrm{K}(\sigma) \leq h(|\sigma|) - l$. Furthermore, W is r.e., since K is enumerable from above. Finally, using (5.1), we have for each n,

$$\sum_{\sigma \in W_n} 2^{-h(|\sigma|)} = \sum_{k=0}^{\infty} \sum_{\substack{\sigma \in W_n \\ |\sigma| = k}} 2^{-h(|\sigma|)} = \sum_{k=0}^{\infty} 2^{-h(k)} |\{0,1\}^k \cap W_n|$$

$$\leq 2^{-n} \sum_{k=0}^{\infty} 2^{-\mathrm{K}(k)} \leq 2^{-n}.$$

□

The concept of an *effective $\mathcal{H}^h$-nullset* leads in straightforward way to *effective Hausdorff dimension*. Given $x \in 2^\omega$, let

$$\dim^1_{\mathrm{H}} x = \inf\{s \geq 0 : \ x \text{ is not } \mathcal{H}^s\text{-random}\}.$$

This was first defined by Lutz [42] via a variant of martingales (*gales*) under the name *constructive dimension*. Effective Hausdorff dimension has an elegant characterization via Kolmogorov complexity, which follows easily from Theorem 5.1.

5.2 Theorem. *For every $x \in 2^\omega$,*

$$\dim^1_{\mathrm{H}} x = \liminf_{n\to\infty} \frac{\mathrm{K}(x\lceil n)}{n}.$$

The theorem was first explicitly proved by Mayordomo [47]. However, as Staiger [70] pointed out, much of it was present in earlier work by Ryabko [64, 65], Staiger [71, 72], or Cai and Hartmanis [7]. Essentially, the characterization is a consequence of the correspondence between semimeasures and Kolmogorov complexity established by Levin [78].

Another important feature of effective dimension is the *stability property*. Although we defined effective dimension only for single reals, it is easy to use effective $\mathcal{H}^s$-nullsets (i.e. correct $\mathcal{H}^s$-tests) to define the effective Hausdorff dimension of a set A of reals, denoted by $\dim^1_{\mathrm{H}} A$.

5.3 Theorem (Lutz [42])***.*** *For every $A \subseteq 2^\omega$,*

$$\dim^1_{\mathrm{H}} A = \sup\{\dim^1_{\mathrm{H}} x : \ x \in A\}.$$

This means that, with respect to dimension, every set of reals has to contain an element of accordant complexity, measured in terms of asymptotic algorithmic complexity, as given by Theorem 5.2, where the correspondence is exact for effective dimension. This can be seen as a generalization of the fact that any set of positive Lebesgue measure contains a $\mathcal{L}$-random real.

Geometric measure theory knows a multiplicity of dimension notions besides Hausdorff dimension (see e.g. Falconer [17]). Many of these can be effectivized, most notably *packing dimension*, and be related to Kolmogorov complexity. We will not address this here, but instead refer to the aforementioned sources.

5.2 Hausdorff measures and probability measures

So far, there are few types of examples of reals which are random for some Hausdorff measure. All of them are derived from randomness for probability measures.

(1) If $0 < r < 1$ is rational, let $Z_r = \{\lfloor n/r \rfloor : n \in \mathbb{N}\}$. Given a $\mathcal{L}$-random real x, define x_r by

$$x_r(m) = \begin{cases} x(n) & \text{if } m = \lfloor n/r \rfloor, \\ 0 & \text{otherwise.} \end{cases}$$

Using Theorem 5.2, it is easy to see that $\dim_{\mathrm{H}}^1 x_r = r$. This technique can be refined to obtain sets of effective dimension s, where $0 \leq s \leq 1$ is any Δ_2^0-computable real number (see e.g. Lutz [43]), or reals which are $\mathcal{H}^h$-random, where h is a computable order function.

(2) Given a Bernoulli measure μ_p with bias $p \in \mathbb{Q} \cap (0, 1)$, the effective dimension of any set that is Martin-Löf random with respect to μ_p equals the entropy of the measure $\mathrm{H}(\mu_p) = -[p \log p + (1 - p) \log(1 - p)]$ (Lutz [41]). This is an effectivized version of a classical theorem due to Eggleston [16].

(3) Let U be a universal, prefix-free machine. Given a computable real number $0 < s \leq 1$, the binary expansion of the real number

$$\Omega^{(s)} = \sum_{\sigma \in \mathrm{dom}(U)} 2^{-|\sigma|/s}$$

has effective Hausdorff dimension s. This was shown by Tadaki [74]. For $s = 1$, we obtain Chaitin's Ω, which is $\mathcal{L}$-random [11]. The effective dimension of $\Omega^{(s)}$ is linked to the behavior of nullsets under Hoelder transformations as described in Section 2.7.

The first two examples are random with respect to a computable probability measure. However, this is not the case for every real which is $\mathcal{H}^h$-random for some order function h.

5.4 Theorem (Reimann [57]). *For every order function h there exists a real x such that x is not proper but $\mathcal{H}^h$-random.*

Proof idea. As in example (1), recursively join a real y of low complexity and a $\mathcal{L}$-random real with the appropriate density, given by h. We can choose y to be 1-generic real, which is not proper by Theorem 4.6. □

Nevertheless, every $\mathcal{H}^h$-random real *is* random with respect to some probability measure. The following result can be seen as an effective variant of *Frostman's Lemma* in geometric measure theory, which establishes a close connection between Hausdorff dimension and capacity (see e.g. [46]).

5.5 Theorem (Reimann [58])**.** *If $x \in 2^\omega$ is $\mathcal{H}^h$-random, where h is a computable order function, then there exists a probability measure μ such that x is μ-random and there exists a c such that for all σ,*

$$\mu(\sigma) \leq c 2^{-h(|\sigma|)}.$$

Proof. By Theorems of Kučera [33] and Gács [19], there exists a $\mathcal{L}$-random real y such that $x \leq_T y$ via some Turing operator Φ. To make x μ-random by conservation of randomness, we ensure that for all σ,

$$\mathcal{L}(\Phi^{-1}(\sigma)) \leq \mu(\sigma) c 2^{-h(|\sigma|)}.$$

[MORE...] □

5.3 The computational power of Hausdorff randomness

It is a question of apparently intriguing difficulty to determine the computational power of reals of non-trivial Hausdorff dimension. The examples (1) - (3) are all Turing equivalent to a $\mathcal{L}$-random real. For (1) this is obvious, for (2) this follows from Theorem 4.9. For (3), this follows from a different property of $\mathcal{H}^h$-random reals, which we will address further below.

This observation might suggest to conjecture that every real of positive effective Hausdorff dimension, or more generally, every real that is $\mathcal{H}^h$-random for some computable order function h, computes a $\mathcal{L}$-random real, or at least a real of dimension 1 (or arbitrarily close to 1).

It turns out that this is in general not the case for strong reductions, and not true with respect to Turing reducibility for every computable order function h. But it remains an open question whether such an '*extraction of randomness*' via Turing reductions is possible for higher levels of entropy, e.g. for reals of positive Hausdorff dimension (see Reimann [57] and Miller and Nies [48]).

We first address the results for strong reducibilities. Reimann and Terwijn [57] showed that a many-one reduction cannot increase the entropy of a real x random for a Bernoulli measure μ_p, p rational. It follows that every real m-reducible to x has effective dimension at most $\mathrm{H}(\mu_p)$.

However, this result does not extend to weaker reducibilities such as truth-table reducibility, since for Bernoulli-measures μ_p with $p \in (0, 1)$ the Levin-Kautz result (Theorem 4.9) holds for a total Turing reduction.

Using a different approach, Stephan [73] was able to construct an oracle relative to which there exists a wtt-lower cone of positive effective dimension at most $1/2$. A most general unrelativized result was obtained by Nies and Reimann [53].

5.6 Theorem. *For each rational r, $0 \leq r \leq 1$, there is a real $x \leq_{\mathrm{wtt}} \emptyset'$ such that $\dim^1_{\mathrm{H}} x = r$ and for all $z \leq_{\mathrm{wtt}} x$, $\dim^1_{\mathrm{H}} z \leq r$.*

Proof idea. We construct x satisfying the requirements

$$R_{\langle e,j\rangle} : z = \Psi_e(x) \;\Rightarrow\; \exists (k \geq j)\ \mathrm{K}(z\lceil k) \leq (r + 2^{-j})k + O(1)$$

where (Ψ_e) is a uniform listing of wtt reduction procedures. We can assume each Ψ_e also has a certain (non-trivial) lower bound on the use g_e, because otherwise the reduction would decrease complexity anyway.

To ensure that x has dimension r we construct it inside the Π^0_1 class

$$P = \{y : \ (\forall n \geq n_0)\ \mathrm{K}(Y\lceil n) \geq \lfloor rn \rfloor\}$$

where n_0 is chosen so that $\mathcal{L}(P) \geq 1/2$. P is given as an effective approximation through clopen sets P_s.

We approximate longer and longer initial segments σ_j of x, where σ_j is a string of length m_j, both σ_j, m_j controlled by R_j.

Define a length k_j where we intend to compress z, and let $m_j = g_e(k_j)$. Define σ_j of length m_j in a way that, if $\tau = \Psi_e^{\sigma_j}$ is defined then we compress it down to $(\alpha + 2^{-b_j})k_j$, by constructing an appropriate Martin-Löf test L.

The 'opponent's' answer could be to remove σ_j from P. (σ_j is not of high dimension.) In this case, the capital he spent for this removal exceeds what we spent for our request, so we can account our capital against his. Of course, usually σ_j is much longer than x. So we will only compress x when the measure of oracle strings computing it is large. The advantage we have in measure is reflected by the following lemma.

5.7 Lemma. *Let $C \subseteq 2^\omega$ be clopen such that $C \subseteq P_s$ and $C \cap P_t = \emptyset$ for stages $s < t$. Then*

$$\Omega_t - \Omega_s \geq (\mathcal{L}C)^r.$$

Here Ω_s is the (rational valued) approximation to Chaitin's Ω at stage s.

In the course of the construction, some R_j might have to pick a new σ_j. In this case we have to initialize all R_n of lower priority ($n > j$).

We have to make sure that this does not make us enumerate too much measure into L. Therefore, we have to assign a new length k_n to the strategies R_n. $\square$

In the course of the construction, it is essential that we know the use of the reduction related to R_j, so that we can assign proper new lengths. This is the reason why the construction does not extend to the Turing case.

However, there exists a non-extractability result for Turing reducibility.

5.8 Theorem. *There exists a computable order function h and an $\mathcal{H}^h$-random real x such that no real $y \leq_T x$ is $\mathcal{L}$-random.*

The result was independently proved by Kjos-Hanssen, Merkle, and Stephan [31] and Reimann and Slaman [59]. While Reimann and Slaman gave a direct construction, the proof by Kjos-Hanssen et al. sheds light on a fascinating connection with recursion theory.

A function $g : \mathbb{N} \to \mathbb{N}$ is *diagonally non-recursive (dnr)* if for all n, $g(n) \neq \varphi_n(n)$, where $\{\varphi_n\}_{n \in \mathbb{N}}$ is some standard effective enumeration of all partial recursive functions from $\mathbb{N}$ to $\mathbb{N}$. Dnr functions play an important role in recursion theory. It is known that computing a dnr function is equivalent to computing a *fixed-point free function*, i.e. a function f such that $\varphi_{f(e)} \neq \varphi_e$ for all e [25]. The well-known *Arslanov completeness criterion* says that an r.e. set $W \subseteq \mathbb{N}$ is Turing complete if and only if it computes a fixed-point free function.

Kjos-Hanssen et al. were able to prove the following.

5.9 Theorem (Kjos-Hanssen et al. [31])***.*** *Let $x \in 2^\omega$.*

(1) x is $\mathcal{H}^h$-random for some computable order function h if and only if it truth-table computes a dnr function.

(2) x is $\mathcal{H}^h$ random for some order function $h \leq_T x$ if and only if it Turing computes a dnr function.

Kjos-Hanssen et al. called reals which satisfy one of the equivalent conditions in (1) *complex*, those which satisfy one of the conditions in (2) *autocomplex*.

Theorem 5.9 is quite a powerful tool. For instance, together with the Arslanov completeness criterion it immediately implies that $\Omega^{(s)}$ as defined in Section 5.2, example (3), is Turing equivalent to $\emptyset'$. (Note that $\Omega^{(n)}$ is of r.e. degree.)

Furthermore, the result can be applied to prove Theorem 5.8. An intricate construction by Kumabe [34] showed the existence of a minimal degree which contains a recursively bounded dnr function g. (A function $f : \mathbb{N} \to \mathbb{N}$ is *recursively bounded* if there exists a recursive $G : \mathbb{N} \to \mathbb{N}$ such that $g(n) \leq G(n)$ for all n.) If we encode g as a real x_g (for instance, via unary representations of $g(n)$, separated by 0), the fact that g is recursively bounded implies that x_g is truth-table equivalent to g. Hence, by Theorem 5.9, x_g is complex. However, no minimal degree can contain a $\mathcal{L}$-random real, since by a theorem of Van Lambalgen [75], every recursive split of a $\mathcal{L}$-random real into two halves yields two relatively random, and hence Turing incomparable, $\mathcal{L}$-random reals.

These results are contrasted by positive results for randomness/entropy extraction if the entropy oszilllations present in a real are bounded.

Define the upper asymptotic entropy of a real x by

$$\overline{\mathrm{K}}(x) = \limsup_{n\to\infty} \frac{\mathrm{K}(x\lceil n)}{n}.$$

Note that this is a dual to the effective Hausdorff dimension of x, by Theorem 5.2. Extending earlier work by Ryabko [64, 65] and Doty [13], Bienvenu, Doty, and Stephan [5] showed the following.

5.10 Theorem. *For all $\varepsilon > 0$ and any $x \in 2^\omega$ such that $\overline{\mathrm{K}}(x) > 0$, there exists $y \equiv_{\mathrm{wtt}} x$ such that*

$$\overline{\mathrm{K}}(y) \geq 1 - \varepsilon \quad \textit{and} \quad \dim^1_{\mathrm{H}} y \geq \frac{\dim^1_{\mathrm{H}} x}{\overline{\mathrm{K}}(x)} - \varepsilon.$$

6 Arbitrary Probability Measures

In Section 3.4 we gave a definition of randomness based on the rational representation of premeasures. While the rational representation is defined for any premeasure and hence leads to a universal notion of relative Martin-Löf-style randomness, it does not reflect the topological properties of the space of probability measures on 2^ω.

In this section we will see how, by passing to a different representation of measures, one can exploit the topological structure to prove results about randomness.

It is a classic result of measure theory (see Parthasarathy [55]) that the space of probability measures $\mathcal{P}$ on 2^ω is a compact polish space. The topology is the *weak topology*, which can be metrized by the *Prokhorov metric*, for instance. There is an *effective dense subset*, given as follows: Let Q be the set of all reals of the form $\sigma^\frown 0^\omega$. Given $\bar{q} = (q_1, \dots, q_n) \in Q^{<\omega}$ and non-negative rational numbers $\alpha_1, \dots, \alpha_n$ such that $\sum \alpha_i = 1$, let

$$\delta_{\bar{q}} = \sum_{k=1}^{n} \alpha_k \delta_{q_k},$$

where δ_x denotes the *Dirac point measure* for x. Then the set of measures of the form $\delta_{\bar{q}}$ is dense in $\mathcal{P}$.

The recursive dense subset $\{\delta_{\bar{q}}\}$ and the effectiveness of the metric d between measures of the form $\delta_{\bar{q}}$ suggests that the representation reflects the topology effectively, i.e. the set of representations should be Π^0_1. However, this is not true for the set of rational representations of probability measures. Instead, we have to resort to other representations in metric spaces, such as Cauchy sequences. Using the framework of *effective descriptive set theory*, as for example presented in Moschovakis [49], one can obtain the following.

6.1 Theorem. *There is a recursive surjection*

$$\pi : 2^\omega \to \mathcal{P}$$

and a Π^0_1 subset P of 2^ω such that $\pi\lceil P$ is one-one and $\pi(P) = \mathcal{P}$.

The topological structure comes at price. No longer does every (pre)measure have a unique representation. In the case of Cauchy representations for instance, there are infinitely many for each measure. In particular, if x is a real and μ is a measure, we can find a Cauchy sequence representation r of μ such that x is recursive in μ. If we try to remedy this and pick out a Π^0_1 set on which the representation is one-one and onto, one could claim there is a certain arbitrariness in this.

Therefore, we either have to speak of *randomness with respect to a representation*, or try to define a notion a randomness which is *independent of the representation* of the measure.

The second path has first been followed by Levin [37, 38]. It has recently been extended by Gács [20] to a larger class of metric spaces on which random objects can be defined. It would go beyond the scope of this article to present this theory here, instead we refer to Gacs' excellent paper, which develops the theory in a mostly self-contained account. The interested reader may then pass on to Levin's much more succinct article [38].

The effective compactness of $\mathcal{P}$ has a number of remarkable properties in this theory. For instance, there exists a *neutral measure*, a measure relative to which every sequence is random.

Here we will follow the more *naive approach* and see that a result of similar nature holds. We single out a representation of $\mathcal{P}$ in the sense of Theorem 6.1. So in the following, when we speak of "measure", we will *at the same time refer to its unique representation in the* Π^0_1 *set* P given by Theorem 6.1.

6.1 Randomness of non-recursive reals

If x is an atom of some probability measure μ, it is trivially μ-random. Interestingly, only for the recursive reals this is the only way to become random.

6.2 Theorem (Reimann and Slaman [61])**.** *For any real x, the following are equivalent.*

(i) There exists a probability measure μ such that $\mu(\{x\}) = 0$ and x is μ-random.

(ii) x is not recursive.

Proof sketch. A fundamental result by Kučera [33] ensures that every Turing degree above $\emptyset'$ contains a $\mathcal{L}$-random real. This result relativizes. Hence one can combine it with the *Posner-Robinson Theorem* [56], which says that for every non-recursive real x there exists a z such that $x \oplus z =_{\mathrm{T}} z'$, to obtain a real R which is $\mathcal{L}$-random relative to some $z \in 2^\omega$ and which is T(z)-equivalent to x. There are Turing functionals Φ and Ψ recursive in z such that

$$\Phi(R) = x \quad \text{and} \quad \Psi(x) = R.$$

One can then use the functionals to define a Π^0_1 subset S of P, the set of representations of measures. All measures in S are consistent with the condition that it is an image measure of $\mathcal{L}$ induced by Φ, and that it is non-atomic on x. In order to apply Levin's technique of conservation of randomness, one resorts to a basis result for Π^0_1 sets regarding relative randomness. □

6.3 Theorem (Reimann and Slaman [61],Downey, Hirschfeldt, Miller, and Nies [14])**.** *Let S be $\Pi^0_1(z)$. Then, if R is $\mathcal{L}$-random relative to z, then there exists a $y \in S$ such that R is $\mathcal{L}$-random relative to $y \oplus z$.*

Theorem 6.3 is essentially a consequence of *compactness*. It seems to be quite a versatile result. For instance, it is also used in the proof of Theorem 5.5.

6.2 Randomness of for continuous measures

A natural question arising regarding Theorem 6.2 is whether the measure making a real random can be ensured to have certain regularity properties; in particular, can it be chosen *continuous*? (A probability measure is *continuous* if $\mu(\{x\}) = 0$ for all $x \in 2^\omega$.)

Reimann and Slaman [61] gave an explicit construction of a non-recursive real not random with respect to any continuous measure. Call such reals 1-*ncr*. In general, let NCR_n be the set of reals which are not n-random with respect to any continuous measure.

Kjos-Hanssen and Montalban [30] observed that any member of a countable Π^0_1 class is an element of NCR_1.

6.4 Proposition. *If $A \subseteq 2^\omega$ is Π^0_1 and countable, then no member of A can be in* NCR_1.

Proof idea. If μ is a continuous measure, then obviously $\mu(A) = 0$. One can use a recursive tree T such that $[T] = A$ to obtain a μ-test for A. □

It follows from results of Cenzer, Clote, Smith, Soare, and Wainer [9] that members of NCR_1 can be found throughout the hyperarithmetical hierarchy of Δ^1_1, whereas Kreisel [32] had shown earlier that each member of a countable Π^0_1 class is in fact hyperarithmetical.

Quite surprisingly, Δ^1_1 turned out to be the precise upper bound for NCR_1. An analysis of the proof of Theorem 6.2 shows that if x is *truth-table* equivalent to a $\mathcal{L}$-random real, then the "pull-back" procedure used to devise a measure for x yields a continuous measure. More generally, we have the following.

6.5 Theorem (Reimann and Slaman [60])**.** *Let x be a real. For any $z \in 2^\omega$, the following are equivalent.*

(i) x is random for a continuous measure recursive in z.

(ii) x is random for a continuous dyadic measure recursive in z.

(iii) There exists a functional Φ recursive in z which is an order-preserving homeomorphism of 2^ω such that $\Phi(x)$ is $\mathcal{L}$-z-random.

(iv) x is truth-table equivalent to a $\mathcal{L}$-z-random real.

Here *dyadic* measure means that the underlying premeasure is of the form $\rho(\sigma) = m/2^n$ with $m, n \in \mathbb{N}$. The theorem can be seen as an effective version of the *classical isomorphism theorem* for continuous probability measures (see for instance Kechris [29]).[1]

Woodin [77], using involved concepts from set theory, was able to prove that if $x \in 2^\omega$ is not hyperarithmetic, then there is a $z \in 2^\omega$ such that $x \oplus z \equiv_{\mathrm{tt}(z)} z'$, i.e. outside Δ^1_1 the Posner-Robinson theorem holds with truth-table equivalence. Hence we have

6.6 Theorem (Reimann and Slaman [61])**.** *If a real x is not Δ^1_1, then there exists a continuous measure μ such that x is μ-random.*

It is on the other hand an open problem whether every real in NCR_1 is a member of a countable Π^0_1 class.

One may ask how the complexity of NCR_n grows with n. There is some 'empirical' evidence that this growth is rather fast. It it, for instance, not obvious at all whether for all n, NCR_n is countable. This, however, holds true.

6.7 Theorem (Reimann and Slaman [60])**.** *For all n, NCR_n is countable.*

Proof idea. The idea is to use *Borel determinacy* to show that the complement of NCR_n contains an upper Turing cone. This follows from the fact that the complement of NCR_n contains a Turing invariant and cofinal (in the Turing degrees) Borel set. For example, we can use the set of all y that are Turing equivalent to some $z \oplus R$, where R is $\mathcal{L}$-$(n+1)$-random relative to a given z. The desired cone is given by the *Turing degree of a winning strategy* in the corresponding game (see Martin [44]).

The one can go on to show that the elements of NCR_n show up at a rather *low level of the constructible universe*. It holds that $\mathrm{NCR}_n \subseteq L_{\beta_n}$, where β_n is the least ordinal such that

$$L_{\beta_n} \models \mathsf{ZFC}^- + \text{ there exist } n \text{ many iterates of the power set of } \omega,$$

where ZFC^- is Zermelo-Fraenkel set theory without the Power Set Axiom.

[1] The theorem suggests that for continuous randomness representational issues do not really arise, since there is always a measure with a computationally minimal representation.

To show this, given $x \notin L_{\beta_n}$, construct a set G such that $L_{\beta_n}[G]$ is a model of ZFC^-_n, and for all $y \in L_{\beta_n}[G] \cap 2^\omega$, $y \leq_{\mathrm{T}} x \oplus G$. G is constructed by *Kumabe-Slaman forcing* (see [68]). The existence of G allows to conclude: If x is not in L_{β_n}, it will belong to every cone with base in $L_{\beta_n}[G]$. In particular, it will belong to the cone given by Martin's argument (relativized to G, here one has to use absoluteness), i.e. the cone avoiding NCR_n. Hence x is random relative to G for some continuous μ, an thus in particular μ-random. □

The proof of the countability of NCR_n makes essential use of Borel determinacy.

It is known from a result by Friedman [18] that the use of infinitely many iterates of the power set of ω is necessary to prove Borel determinacy. As a base for an induction on the levels of the Borel hierarchy, Friedman showed that ZFC^- does not prove the statement "All $\mathbf{\Sigma}^0_5$-games on countable trees are determined." The proof works by showing that there is a model of ZFC^- for which $\mathbf{\Sigma}^0_5$-determinacy does not hold. This model is just L_{β_0}.

Very recently, Reimann and Slaman [60] showed that for every fixed k, NCR_n is cofinal in the Turing degrees of L_{β_k}. It allowed them to infer the following result.

6.8 Theorem (Reimann and Slaman [60])**.** *For every k, the statement*

For every n, NCR_n *is countable.*

cannot be proven in

ZFC^- + *there exists k many iterates of the power set of* ω*.*

The proof uses Jensen's *master codes* [23] as witnesses for NCR_n.

This line of work indicates that questions about randomness for continuous measures formalizable in second order arithmetic (such as the one formulated in the problem above) extend far into the realm of (descriptive) set theory.

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