Lesson 2

Computability

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Jan Reimann

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Eberly College of Science

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The Halting Problem

Uncomputable functions

Q: Does there exist $f: \mathbb{N} \to \mathbb{N}$ which is not computable?

A set $A \subseteq \mathbb{N}$ which is not decidable?

A: YES, by a counting argument

Observation: There are only countably many Turing machines.

TM : finite list of instructions

- each instruction is a 5-tuple

$$(p, \underbrace{x, y}_{\in A}, \underbrace{d}_{\in \{L, R, S\}}, q)$$

$p \in Q$, $q \in Q$

finite

# of TM's countable

vs

# of functions $f : \mathbb{N} \to \mathbb{N}$

(as the # of sets $A \subseteq \mathbb{N}$)

is **uncountable**

$f(x) = 2x$

$f(x) =$ the x-th digit of π

recursive functions

An undecidable set

- Numbering TM's : $M_0, M_1, M_2, \dots$

  code instructions

  $$I = (p, x, y, \underline{d}, q)$$

  as natural numbers

  $Q \subseteq \{0, \dots, m\}$ $\quad x, y \in \{0, 1\}$.

  d: $L = 0, R = 1, S = 2$

  $$\# \underline{I} = 3^{p+1} \cdot 5^{x+1} \cdot 7^{y+1} \cdot 11^{d+1} \cdot 13^{q+1}$$

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By uniqueness of prime decomposition, each $\underline{I}$ is assigned a unique code $\#\underline{I} \in \mathbb{N}$

TM: finite list of instructions

$$M = (\underline{I}_1 \dots \underline{I}_k)$$

$$\#M = 2^{\#\underline{I}_1} \cdot 3^{\#\underline{I}_2} \cdots P_k^{\#\underline{I}_k}$$

P_k: k-th prime

$$\# M = 2^{\# I_1} \cdot 3^{\# I_2} \cdots P_k^{\# I_k}$$

k-th prime

Gödel number of M
or index of M

$$\#: TM \longrightarrow \mathbb{N}$$

not onto
but we can modify it to make it onto

The Universal TM

Gödel numbering (GN) of TMs is <u>effective</u>.

Let $\langle x, y \rangle = \dfrac{(x+y)(x+y+1)}{2} + y$

Bijection: $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

pairing function

THM: There exists a TM U s.t. on input $\langle e, w \rangle$,

universal TM

$$U(\langle e, w \rangle) \simeq M_e(w)$$

U halts on w iff M_e halts on w.

and if it halts, outputs the same number that M_e outputs

Notation:

$M_e(w)\downarrow$

M_e halts on input w

$M_e(w)\uparrow$

M_e does not halt on w

The Halting Problem

Q: Can we effectively decide whether $M_e(w)\downarrow$ or $\uparrow$?

Does there ex. a TM M s.t.

$$M(\langle e, w\rangle) = \begin{cases} 1 & \text{if } M_e(w)\downarrow \\ 0 & \text{if } M_e(w)\uparrow \end{cases}$$

In other words, is the set

Halting problem $\longrightarrow$ $K = \{\langle e, w\rangle : M_e(w)\downarrow\}$

decidable?