Homework 7 for **MATH 185**

Brief sketches of solutions

Problem 1

Let $D \subseteq \mathbb{C}$ be open, and let $M \subseteq D$. Show that the following are equivalent:

(i) M is discrete in D.

(ii) For each $p \in M$ there exists an $\varepsilon > 0$ such that $U_\varepsilon(p) \cap M = \{p\}$ and M is closed in D (i.e. there exists a closed set $A \subseteq \mathbb{C}$ such that $M = A \cap D$).

(iii) For each compact set $K \subseteq D$, the set $M \cap K$ is finite.

(iv) For each $z \in D$ there exists an $\varepsilon > 0$ such that $U_\varepsilon(z) \subseteq D$ and $M \cap U_\varepsilon(z)$ is finite.

Solution. (i) $\Rightarrow$ (ii): Let $p \in M$. Since no accumulation point of M can lie in D, p cannot be an accumulation point of M. Thus, if $\delta > 0$, $U_\delta(p) \cap M$ is finite. Let $\varepsilon > 0$ be less than the least distance from p to any of the finitely many points in $U_\delta(p) \cap M$. (If there are no other points than p, let $\varepsilon = \delta$.) Then $U_\varepsilon(p) \cap M = \{p\}$. To see that M is closed in D, recall that a set is closed in D iff for any sequence (p_n) of points in M such that $\lim p_n \in D$, $\lim p_n \in M$. Since M has no accumulation points in D, the only points in D which are sequential limits of points in M are the points in M itself, so M is closed in D.

(ii) $\Rightarrow$ (iii): Let $K \subseteq D$ be compact. Since M is closed in D, $M \cap K$ is a closed subset (in D) of a compact set and hence also compact. For any $p \in M$, let, by assumption, $\varepsilon(p)$ be such that $M \cap U_\varepsilon(p) = \{p\}$. Then $\bigcup_M U_{\varepsilon(p)} \cap D$ is an open cover of $M \cap K$. By compactness, there exists a finite subcover, and since every $U_{\varepsilon(p)}$ contains exaclty one point of $M \cap K$, $M \cap K$ is finite.

(iii) $\Rightarrow$ (iv): Let $z \in D$. Since D is open, $U_{2\varepsilon}(z) \subseteq D$ for some $\varepsilon > 0$. Then $\overline{U_\varepsilon(z)} \subseteq D$ is compact. By assumption, $\overline{U_\varepsilon(z)} \cap M$ is finite, so $U_\varepsilon(z) \cap M$ is finite, too.

(iv) $\Rightarrow$ (i): If $U_\varepsilon(z) \cap M$ is finite, this means z is not an accumulation point of M. Since such an ε exists for all $z \in D$, no $z \in D$ can be an accumulation point of M. ■

Problem 2

Let D be a domain, and let $M \subseteq D$ be discrete in D. Show that $D \setminus M$ is a domain.

Solution. We have to show that $D \setminus M$ is open and connected. By Problem 1(ii) M is closed in D, so $D \setminus M$ is open.

To show that $D \setminus M$ is connected, let $h : D \setminus M \to \mathbb{C}$ be locally constant. We extend h to a locally constant function $H : D \to \mathbb{C}$. Let $p \in M$. By Problem 1(i), there exists an $\varepsilon > 0$ such that $U_\varepsilon(p) \cap M = \{p\}$.

We claim that $U_\varepsilon(p) \setminus \{p\}$ is connected: W.l.o.g. $p = 0$. Given any two points $a, b \in U_\varepsilon(0) \setminus \{0\}$, we can connect them by a path as follows. Assume $\operatorname{Arg}(a) < \operatorname{Arg}(b)$. Go along the circle of radius $|a|$ from $\operatorname{Arg}(a)$ to $\operatorname{Arg}(b)$. Then connect to b by a straight line of length $||a| - |b||$.

Since $U_\varepsilon(p) \setminus \{p\}$ is connected, $h|_{U_\varepsilon(p)\setminus\{p\}}$ is constant, say $\equiv c_p$. Define $H(p) = c_p$. For $z \in D \setminus M$, set $H(z) = h(z)$. By construction, H is locally constant. Since D is a domain, H is constant, so h is constant, too. ■

Problem 3

Let f be the analytic function given by

$$f(z) = \sum_{n=1}^{\infty} z^{n!}.$$

Show that f cannot be analytically extended to any domain D properly containing the unit disk $\mathbb{E} = \{z : |z| < 1\}$.

Solution. Let $z = e^{2\pi i r}$, where $r = p/q$ is rational. We claim that $\sum_{n=1}^{\infty} z^{n!} = \infty$. If $n > q$, then $n!$ contains q as a factor, so

$$z^{n!} = e^{2\pi i n! p/q} = 1.$$

This means that for all but finitely many n, $z^{n!} = 1$, so the series grows unbounded.

f therefore cannot be extended to any root of unity, since at such points f would have to take the value ∞ (inside $\mathbb{E}$ f is continuous). If $U \supset \mathbb{E}$ is a open set which contains a point $|z| = 1$, then a full nbhd of this point is contained in U and with it a root of unity (infinitely many, in fact). ■

Problem 4

Determine whether there exists an analytic function $f : U_\varepsilon(0) \to \mathbb{C}$, $\varepsilon > 0$ such that for all $n \in \mathbb{N}$,

(a)

$$f(\tfrac{1}{n}) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even;} \end{cases}$$

(b)

$$f(\tfrac{1}{n}) = \begin{cases} \frac{1}{n+1} & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even;} \end{cases}$$

(c)

$$f(\tfrac{1}{n}) = \tfrac{n}{n+1}$$

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Solution. General principle following from the *identity theorem*: If (a_n) is a sequence in some domain D, $\lim a_n = a \in D$, and (b_n) is any sequence of complex numbers, then there exists *at most one* analytic function $f : D \to \mathbb{C}$ such that $f(a_n) = b_n$.

(a): There exists at most one analytic function $f : U_1(0) \to \mathbb{C}$ such that $f(1/2n) = 0$. 0 is such a function. Furthermore, there exists at most one analytic function $f : U_1(0) \to \mathbb{C}$ such that $f(1/(2n+1)) = 1/(2n+1)$. Such a function is $f(z) = z$. 0 and z do not agree in any ndhd of 0, so no such f can exist.

(b): There exists at most one analytic function $f : U_1(0) \to \mathbb{C}$ such that $f(1/(2n+1)) = 1/(2n+1)$. Such a function is $f(z) = z$. However, $f(1/1) = 1 \neq 1/2 = 1/(1+1)$, so no such f can exist.

(c): $f(z) = 1/(z+1)$ is a function with the desired properties. ■