Algorithmic Independence and PA Degrees

Jan Reimann

(joint work in progress with Adam Day)

Randomness and Dynamical Systems

See algorithmic randomness as an effective complement of certain aspects of dynamical systems.

- recent progress on the dynamic stability of random reals (random reals as typical points in measure theoretic dynamical systems);
- single orbit dynamics (B. Weiss).

The Space of Probability Measures

The space $\mathcal{M}(2^{\mathbb{N}})$ of all probability measures on $2^{\mathbb{N}}$ is compact Polish.

Compatible metric:

$$d(\mu,\nu) = \sum_{n=1}^{\infty} 2^{-n} d_n(\mu,\nu)$$

$$d_n(\mu,\nu) = \frac{1}{2} \sum_{|\sigma|=n} |\mu[\sigma] - \nu[\sigma]|.$$

Countable dense subset: Basic measures

$$\nu_{\vec{\alpha},\vec{q}} = \sum \alpha_i \delta_{q_i}$$

$\sum \alpha_i = 1,\ \alpha_i \in \mathbb{Q}^{\geq 0},\ q_i$ `rational points' in $2^{\mathbb{N}}$

Representation of Probability Measures

(Nice) Cauchy sequences of basic measures yield continuous surjection

$$\rho : 2^{\mathbb{N}} \to \mathcal{M}(2^{\mathbb{N}}).$$

Surjection is effective: For any $X \in 2^{\mathbb{N}}$,

$$\rho^{-1}(\rho(X)) \text{ is } \Pi^0_1(X).$$

Randomness

A test for randomness is an effectively presented G_δ nullset (relative to a representation of a measure).

A real X is μ-Z-random if there exists a representation R_μ so that X passes all R_μ-Z-tests.

Levin developed a representation free definition. Recently, Day and Miller showed that the two approaches coincide.

Duality

We define the **randomness spectrum** of a real as

$$S_X = \{\mu \in \mathcal{M}(2^{\mathbb{N}}) \colon X \text{ is } \mu\text{-random}\}.$$

- Given a real X, what kind of randomness does X support?
- How do we find a measure that makes X random?
- Is the (logical) complexity of X reflected in its randomness spectrum?

Randomness Spectra

Some facts about the randomness spectrum.

- S_X is always non-empty (it always contains a point measure).
- If X is recursive, then S_X contains only measures that are atomic on X.
- If X is not recursive, then S_X contains a measure with $\mu\{X\} = 0$. [R. and Slaman]
- If X is not hyperarithmetical, then S_X contains a continuous measure. [R. and Slaman]
- If X is recursive in an incomplete r.e. set (in particular if X is K-trivial), then S_X does not contain a continuous measure. [BGMS]

Constructing Measures

How can we construct measures that make a real random?

compactness appears to be essential.

Example from dynamical systems:

- Let T denote the shift map on $2^{\mathbb{N}}$.

$$T(X)_i = X_{i+1}.$$

- Any limit point of the measures

$$\mu_n^X = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{T^i(X)}$$

  is shift invariant. [Krylov and Bogolyubov]

  (In ergodic theory this is called the spectrum of X.)

Constructing Measures

However, for effective randomness we also have to take into account the logical complexity of the real.

Currently two ways known to use compactness:

- transfer the randomness from a more complicated point system;
- use neutral measures.

Point Systems

Effective randomness combines the bit-by-bit aspect of dynamical systems with the complexity aspect of definability/computability.

Call a pair (X, μ) consisting of a real X and a measure μ for which X is random a point system.

Question: What is the algebraic structure of point systems?

- Joinings and disjointness have played an important role in dynamical systems (structure theorems).
- How do point systems behave under joins?

Independence

$X \oplus Y$ is random with respect to some measure. But does the spectrum of X, Y contain a product measure?

Van Lambalgen's Theorem:
For any measure μ, (X, Y) is $\mu \times \mu$-random iff X is μ-Y-random and Y is μ-X-random.

A most general version was proved by Bienvenu, Hoyrup, and Shen.

Pointwise independence: There exists a measure μ such that X is μ-Y-random and Y is μ-X-random, and $\mu\{X, Y\} = 0$.

Independence Spectrum

Similar to the randomness spectrum, we can define the **independence spectrum** of a real X as

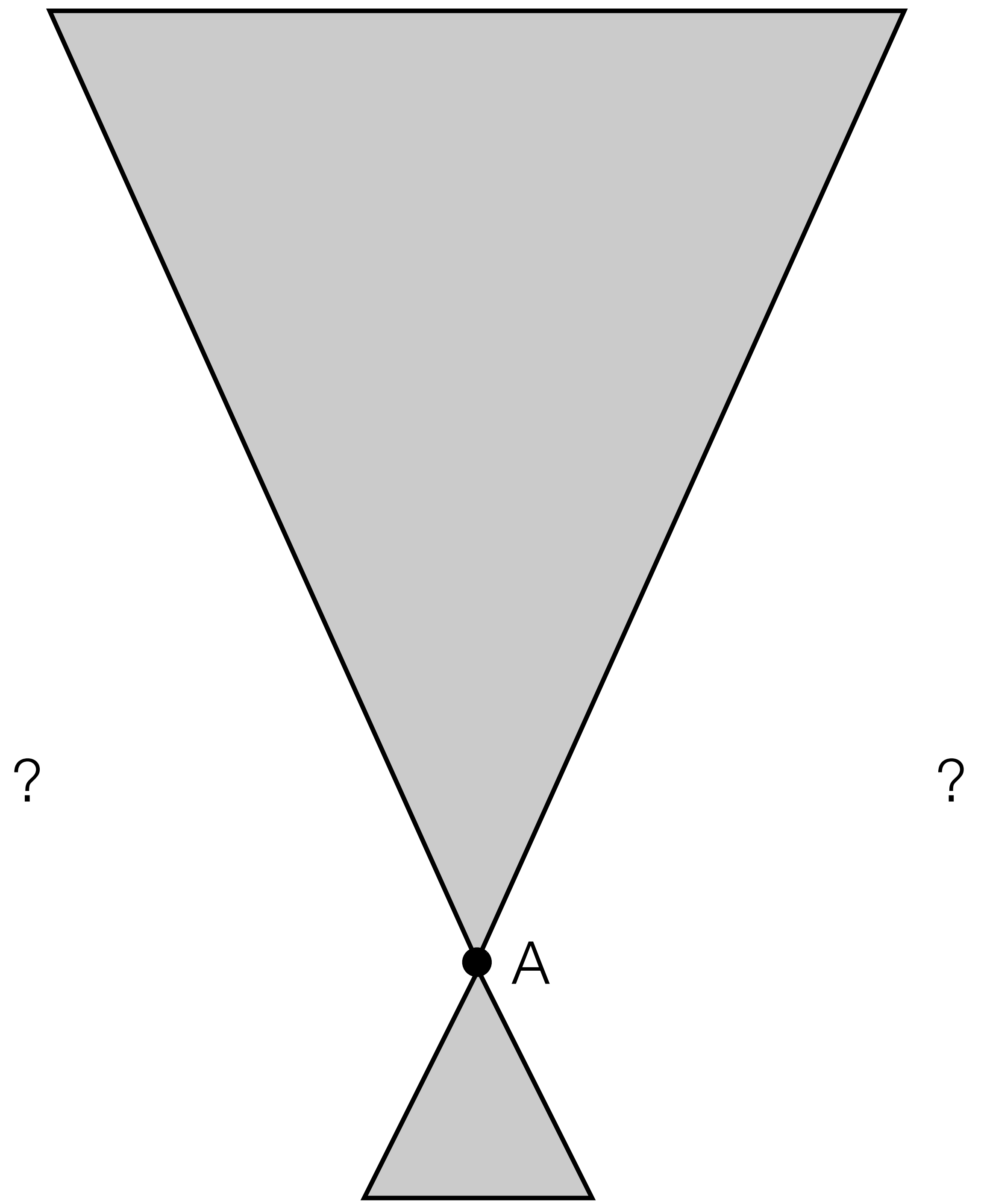
$$I_X = \{Y \in 2^{\mathbb{N}} : \exists \mu \ (X, Y) \text{ is } (\mu \times \mu)\text{-random and } \mu\{X, Y\} = 0\}.$$

The **dependence spectrum** is $D_X = 2^{\mathbb{N}} \setminus I_X$.

Basic properties.

- $X \in I_Y$ if and only if $Y \in I_X$.
- $X \in I_Y$ implies that $X \mid_T Y$.

Independence Spectrum



Independence Spectrum

The independence spectrum of a non-recursive real is large:

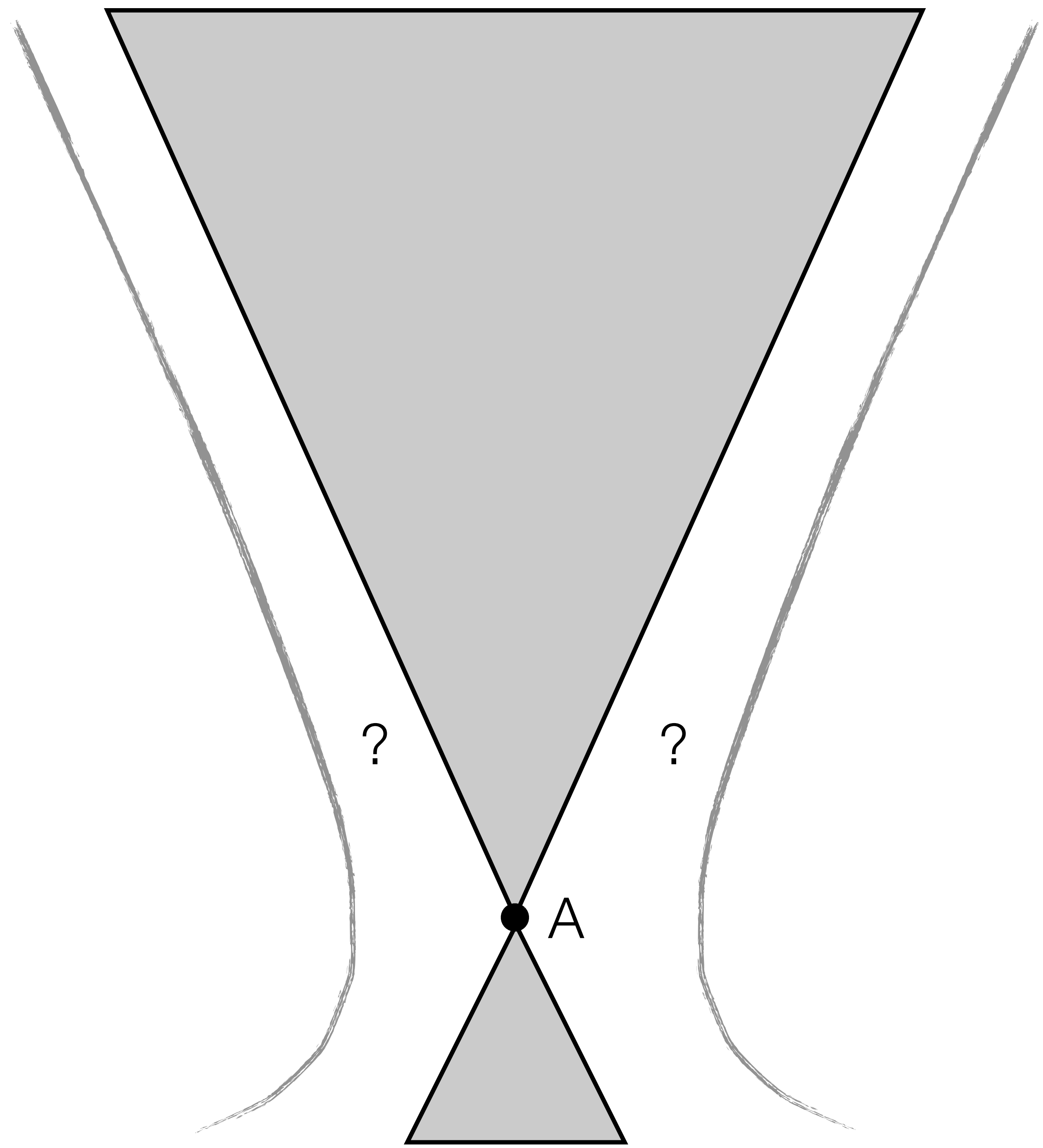
- If X is non-recursive, then I_X has measure 1 for any computable continuous probability measure.

This means I_X has effective universal measure zero.

Question: What is the size of D_X off the upper and lower cone of X?

- Does it contain a perfect subset?
- Is it countable? (D_X is a Π^1_1 set.)

Independence Spectrum



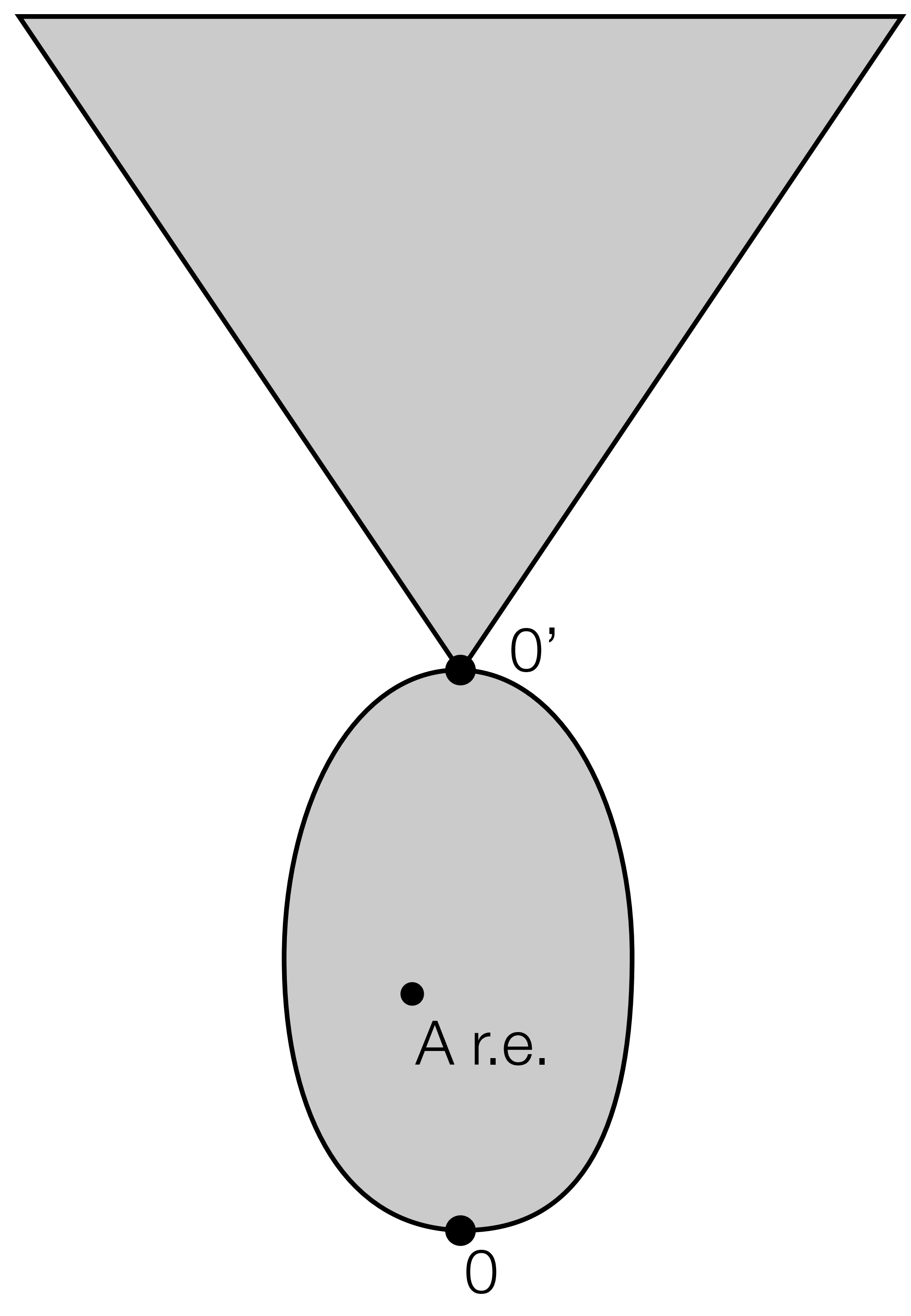
Independence Spectrum

We can rule out that the independence spectrum of a real X consists precisely of all reals Y that are Turing incomparable with X, i.e. for which $X \not\geq_T Y$ and $Y \not\geq_T X$?

Theorem:
If X is non-trivially μ-random and r.e., then then $R_\mu \oplus X \geq_T 0'$ for any representation R_μ of μ.

Corollary:
If X is r.e. and $Y \leq_T 0'$ then $Y \notin I_X$.

Independence Spectrum



Independence Spectrum

The question of how large $D_X \setminus \mathrm{Cones}(X)$ exactly is remains open.

Interesting technical aspects, e.g. Posner-Robinson style jump-inversion inside Π^0_1 classes.

In the end, it seems we are simply lacking methods to construct measures that make a given real random.

Application: PA Degrees

A real X is of PA-degree if it is Turing equivalent to a complete extension of Peano Arithmetic.

Some properties:

- PA degrees are closed upwards.
- PA degrees compute a path through any non-empty Π^0_1 class. In particular, every PA degree computes a λ-random real.
- If a λ-random set X is of PA degree, then $X \geq_T 0'$ [Stephan].

The computationally ``useful'' λ-random reals are precisely the ones above $0'$.

Neutral Measures

Levin:
There exists a measure v, called a neutral measure, such that any $X \in 2^{\mathbb{N}}$ is v-random.

Day and Miller:
Every PA degree computes a representation of a neutral measure.

R.e. Sets and PA Degrees

We can combine these results with the previous one.

Theorem: If X is r.e. and neither recursive nor T-complete then $P \oplus X \geq_T 0'$ for any set P of PA degree such that $P \not\geq_T X$.

- This extends a previous result by Kucera and Slaman.
- Direct proofs were subsequently found by Kucera and Miller.
- The result also lets us classify those incomplete r.e. sets which are bounded by an incomplete PA degree -- precisely the low ones.