

## Lecture 22: Hyperarithmetical Sets

Is there an effective counterpart to Souslin's Theorem that  $\text{Borel} = \Delta_1^1$ ? Definability in second order arithmetic gives us the lightface classes  $\Sigma_n^0$  and  $\Pi_n^0$  for finite  $n$ , but what would a “lightface”  $\Sigma_\xi^0$  set be?

Instead of definability, we can also describe the lightface classes using computational properties. In Lecture 9 we saw that the  $\Sigma_n^0$  and  $\Pi_n^0$  sets of reals correspond to the Borel sets of finite level with computable codes. We could try to extend the notion of a code into the transfinite, by introducing a “limit” of codes to deal with limit ordinals. A limit code should give us a method how to effectively recover codes for the sets whose union (limit) we are taking. This is, by no coincidence, reminiscent of the concept of an ordinal notation, where the limit notation was essentially an index for the notations of the ordinals whose limit we are taking.

### The transfinite jump operation

We first illustrate the method for subsets of  $\mathbb{N}$ . Here we have the advantage that iterating definability corresponds directly to iterating the jump operator. So if we can define a transfinite extension of the Turing jump, this should give us a blueprint of how to define  $\Sigma_\xi^0, \Pi_\xi^0$  sets for infinite ordinals.

#### *The H-sets*

The template for a transfinite extension of the jump is given by

$$\begin{aligned} 0^{(n+1)} &= (0^{(n)})' \\ 0^{(\omega)} &= \{\langle n, m \rangle : m \in 0^{(n)}\}, \end{aligned}$$

that is, at limit stages we take effective unions of predecessors in the jump hierarchy. The predecessors are increasing in complexity and “lead up” to  $0^{(\omega)}$ . The general definition could read therefore something like

$$0^{(\xi)} = \{\langle n, m \rangle : n \text{ codes an ordinal } \zeta < \xi \text{ and } m \in 0^{(\zeta)}\}.$$

This clearly suggests to use ordinal notations.

**Definition 22.1:** For all  $x \in \mathcal{O}$ , we define the  $H$ -set  $H_x$  recursively as follows:

$$\begin{aligned} H_1 &= 0, \\ H_{2^x} &= (H_x)', \\ H_{3 \cdot 5^x} &= \{(y, m) : y <_{\mathcal{O}} 3 \cdot 5^x \wedge m \in H_y\}. \end{aligned}$$

The rules obviously assign a strictly ascending sequence of Turing degrees with every **path** of  $\mathcal{O}$ , i.e. every subset of  $\mathcal{O}$  that is linearly ordered by  $<_{\mathcal{O}}$  and is closed downwards under  $<_{\mathcal{O}}$ . We will see later that notations of equal  $|\cdot|_{\mathcal{O}}$ -rank define  $H$ -sets of equal Turing degree.

We start by observing that higher instances of  $H$ -sets can compute their predecessors (in a uniform way).

**Proposition 22.2:** *If  $x, y \in \mathcal{O}$ , and  $x \leq_{\mathcal{O}} y$ , then  $H_x \leq_1 H_y$  uniformly in  $x, y$  (that is, an index for the reduction from  $H_x$  to  $H_y$  can be found uniformly in  $x, y$ ).*

*Proof.* Let  $f$  be recursive such that for any  $X \subseteq \mathbb{N}$ ,  $X \leq_1 X'$  via  $f$ . Consider two cases:

$|y|_{\mathcal{O}} = |x|_{\mathcal{O}} + n$ : We can simply take  $h = f^n$ , the  $n$ -fold iterate of  $f$ .

$|y|_{\mathcal{O}} \geq |x|_{\mathcal{O}} + \omega$ : Let  $z \in \mathcal{O}, n \in \mathbb{N}$  be such that  $|z|_{\mathcal{O}}$  is limit and  $|y|_{\mathcal{O}} = |z|_{\mathcal{O}} + n$ .  
Then

$$m \in H_x \iff \langle x, m \rangle \in H_z \iff f^n(\langle x, m \rangle) \in H_y.$$

Note that  $f^n$  is one-one, that both cases can be distinguished effectively in  $x, y$ , and that  $n$  can be found effectively in  $x, y$ .  $\square$

Next we show that the jump of an  $H$ -set can compute the ordinal notations below its rank. Given  $x \in \mathcal{O}$ , let

$$\mathcal{O}_x = \{y \in \mathcal{O} : |y|_{\mathcal{O}} < |x|_{\mathcal{O}}\}.$$

**Proposition 22.3:** *For each  $x \in \mathcal{O}$ ,  $\mathcal{O}_x \leq H_{2^x}$ , uniformly in  $x$ .*

*Proof.* The proof is by effective transfinite recursion. One constructs a computable function  $f$  such that if  $x \in \mathcal{O}$ ,

$$\mathcal{O}_x = \Phi_{f(x)}^{O_{2^x}}.$$

We sketch how to do the induction step, and leave the fully formalized argument (in the manner of Lecture 19) to the reader (see [Sac90]). We consider the following cases, which can be distinguished effectively:

$x = 1$ : Then  $\mathcal{O}_x = \emptyset$ . In this case just let  $f(x)$  be an index  $c$  such that  $\Phi_c^X = \emptyset$  for any oracle  $X$ .

$x = 2^s, s = 2^t$ : Then  $\mathcal{O}_x = \mathcal{O}_s \cup \{2^y : y \in \mathcal{O}_s\}$ . By assumption,  $\mathcal{O}_s \leq_T H_{2^s} = H_x$ . It follows that  $\mathcal{O}_x$  is recursive in  $H_x$ , too. An index for the reduction can be found uniformly from an index of the reduction  $\mathcal{O}_s \leq_T H_x$ .

$x = 2^s, s = 3 \cdot 5^e$ : In this case

$$\mathcal{O}_x = \mathcal{O}_s \cup \{3 \cdot 5^e : \varphi_e \text{ total and } \forall n \varphi_e(n) <_{\mathcal{O}} \varphi_e(n+1)\}.$$

Let  $q$  be a recursive function such that  $W_{q(x)} = \{\langle y, z \rangle : y <_{\mathcal{O}} z <_{\mathcal{O}} x\}$ . Let  $X = \{3 \cdot 5^e : \varphi_e \text{ total and } \forall n \langle \varphi_e(n), \varphi_e(n+1) \rangle \in W_{q(x)}\}$ . Then  $\mathcal{O}_x = \mathcal{O}_s \cup X$ . Again by hypothesis,  $\mathcal{O}_s \leq_T H_x$ . Furthermore,  $X \leq_T 0'' = H_4$ , and thus by Proposition 22.2,  $X \leq_T H_x$ . The two reductions can be combined uniformly into a single reduction  $\mathcal{O}_x \leq_T H_x \leq_T H_{2^x}$ .

$x = 3 \cdot 5^s$ : Then  $\mathcal{O}_x = \{y : \exists n y \in \mathcal{O}_{\varphi_s(n)}\}$ . By induction hypothesis,  $\mathcal{O}_{\varphi_s(n)} \leq_T H_{2^{\varphi_s(n)}}$ , and by Proposition 22.2,  $H_{2^{\varphi_s(n)}} \leq_T H_x$  uniformly in  $n, x$ . Hence  $\mathcal{O}_x$  is r.e. in  $H_x$ , and therefore recursive in  $H_{2^x}$ . Again, all reductions are uniform. (This case is where we need the jump of  $H_x$  to compute  $\mathcal{O}_x$ .)

□

If we want to compute the set of notations of the same rank as  $x$ , we need one more jump.

**Corollary 22.4:** *For any  $x \in \mathcal{O}$ ,*

$$\mathcal{O}_{=x} = \{y \in \mathcal{O} : |y|_{\mathcal{O}} = |x|_{\mathcal{O}}\} \leq_T H_{2^{2^x}},$$

*uniformly in  $x$ .*

*Proof.* We have

$$\mathcal{O}_{=x} = \mathcal{O}_{2^x} \setminus \mathcal{O}_x.$$

Apply the previous proposition. □

We are now in a position to show that the Turing degree of an  $H$ -set is invariant under passing to a notation of equal rank.

**Theorem 22.5 (Spector):** *For any  $x, y \in \mathcal{O}$ ,*

$$|x|_{\mathcal{O}} = |y|_{\mathcal{O}} \Rightarrow H_x \equiv_T H_y,$$

*uniformly in  $x, y$ .*

*Proof.* The proof proceeds by effective transfinite recursion on the set

$$\mathcal{O}_= = \{\langle x, y \rangle : x, y \in \mathcal{O} \text{ and } |x|_{\mathcal{O}} = |y|_{\mathcal{O}}\},$$

along the well-founded relation  $\langle s, t \rangle \prec \langle x, y \rangle$  which holds if and only if  $|s|_{\mathcal{O}} < |x|_{\mathcal{O}}$ . We sketch the recursion step for the construction of a reduction  $H_x \leq_T H_y$  (the reduction  $H_y \leq_T H_x$  is obtained in a completely analogous fashion), and consider the following cases:

$\langle x, y \rangle = \langle 1, 1 \rangle$ : Choose an index  $c$  such that  $\Phi_c^\emptyset = \emptyset$ .

$\langle x, y \rangle \in \mathcal{O}_=$ ,  $\langle x, y \rangle = \langle 2^s, 2^t \rangle$ : By induction hypothesis,  $H_s \leq_T H_t$ , and thus, by the monotonicity of the jump operator,

$$H_x = H_{2^s} = (H_s)' \leq_T (H_t)' = H_{2^t} = H_y,$$

and the reduction can be found uniformly.

$\langle x, y \rangle \in \mathcal{O}_=$ ,  $\langle x, y \rangle = \langle 3 \cdot 5^s, 3 \cdot 5^t \rangle$ : We want to decide whether  $\langle z, m \rangle \in H_x$ , given an oracle for  $H_y$ . For this, we have to decide whether  $z <_{\mathcal{O}} x$  and  $m \in H_z$ .

By Proposition 19.7, the set of all  $z <_{\mathcal{O}} x$  is r.e. and hence can be decided in  $\mathcal{O}' = H_2 \leq_T H_y$ . For each such  $z$ , we have to decide whether  $m \in H_z$ . For this, enumerate all  $v <_{\mathcal{O}} y$ . Since  $|y|_{\mathcal{O}} = |x|_{\mathcal{O}}$ , we must eventually find a  $v$  such that  $|v|_{\mathcal{O}} = |z|_{\mathcal{O}}$ . The latter fact can be tested recursively in  $H_{2^{2^v}}$ , by Corollary 22.4. Since  $y$  is limit,  $H_{2^{2^v}} \leq_T H_y$ . Finally, by induction hypothesis  $H_v$  computes  $H_z$ , and by Proposition 22.2,  $H_y \geq_T H_s$ . All procedures described are uniform in  $x, y$  and an index for the uniform reduction up to  $\langle x, y \rangle$ .

□

## Hyperarithmetic = $\Delta_1^1$

A set  $X \subseteq \mathbb{N}$  is called **hyperarithmetic** if it is recursive in some  $H$ -set. We will see that the hyperarithmetic sets of natural numbers are precisely the  $\Delta_1^1$  definable sets, thereby giving an effective analog to Souslin's Theorem.

We first show that if  $X$  is hyperarithmetic, then  $X$  is  $\Delta_1^1$ . We will actually show something stronger: Uniformly in  $x \in \mathcal{O}$  we can compute a  $\Delta_1^1$ -index for  $H_x$ .

The normal form for  $\Pi_1^1$  sets discussed in Lecture 20 yields that for every  $\Pi_1^1$  set  $X$  there exists an  $e \in \mathbb{N}$  such that

$$x \in X \iff \forall \alpha \exists y [T(e, x, y, \alpha \upharpoonright_y) \wedge U(y) = 0].$$

In this case  $e$  is called a  $\Pi_1^1$ -**index** for  $X$ . A  $\Delta_1^1$ -**index** for a  $\Delta_1^1$  set  $Y$  is a number  $d = \langle e_0, e_1 \rangle$  such that  $e_0$  is a  $\Pi_1^1$ -index for  $Y$  and  $e_1$  is a  $\Pi_1^1$ -index for  $\mathbb{N} \setminus Y$  (such an  $e_1$  is also called a  $\Sigma_1^1$ -**index** for  $Y$ ).

**Theorem 22.6** (Kleene): *There exists a recursive function  $f$  such that if  $x \in \mathcal{O}$ , then  $f(x)$  is a  $\Delta_1^1$ -index for  $H_x$ .*

*Proof.* The proof proceeds as usual by effective transfinite recursion along  $<_{\mathcal{O}}$ . We skip the details and sketch how to do the recursion step.

$x = 1$ : Let  $f(x)$  be a  $\Delta_1^1$  index for the empty set.

$x = 2^s$ : By induction hypothesis, we can assume we have constructed a function  $\varphi_c$  such that  $\varphi_c(s)$  is a  $\Delta_1^1$ -index of  $H_s$ . We have to show that effectively in  $x, c$  we can find a  $\Delta_1^1$ -index for  $(H_s)'$ . Given any  $X \subseteq \mathbb{N}$ ,  $x \in X'$  if and only if  $\Phi_x^X(x) \downarrow$ , if and only if

$$\exists \sigma T(x, x, |\sigma|, \sigma) \wedge \sigma = X \upharpoonright_{|\sigma|}.$$

We can use the  $\Delta_1^1$ -index for  $X$  to express the last part of the formula in terms of a  $T$ -normal form of  $X$  and of  $\mathbb{N} \setminus X$ . We have to do this twice – for the  $\Sigma_1^1$ -index, and for the  $\Pi_1^1$ -index. Bringing the whole expression into  $T$ -normal form gives a  $\Sigma_1^1$  index and a  $\Pi_1^1$ -index for  $X'$ , respectively. (For details see [Sac90].)

$x = 3 \cdot 5^s$ : We have  $H_x = \{\langle y, m \rangle : y <_{\mathcal{O}} 3 \cdot 5^x \wedge m \in H_y\}$ . By induction hypothesis, we have constructed a function  $\varphi_c$  such that

$$H_x = \{\langle y, m \rangle : y <_{\mathcal{O}} 3 \cdot 5^x \wedge \forall \alpha \exists m [T((\varphi_c(x))_0, x, m, \alpha \upharpoonright_m) \wedge U(m) = 0]\}.$$

$y <_{\mathcal{O}} 3 \cdot 5^x$  is uniformly r.e. in  $s$ . Normalizing yields a  $\Pi_1^1$ -index for  $H_x$ . A  $\Sigma_1^1$ -index is obtained similarly.

□

**Corollary 22.7:** *If  $X \subseteq \mathbb{N}$  is hyperarithmetical, then it is  $\Delta_1^1$ .*

*Proof.* The  $\Delta_1^1$  sets are closed downward under Turing reducibility. (Exercise!) □

Finally, we show that being  $\Delta_1^1$  implies being hyperarithmetical. This is an intriguing consequence of the boundedness principle.

**Theorem 22.8** (Kleene): *If  $X \subseteq \mathbb{N}$  is  $\Delta_1^1$ , then  $X$  is hyperarithmetical.*

*Proof.* If  $X$  is  $\Delta_1^1$ , then it is many-one reducible to  $\mathcal{O}$ . Let  $g$  be recursive such that

$$x \in X \iff g(x) \in \mathcal{O}.$$

Now define  $Z \subseteq \mathbb{N}$  by

$$z \in Z \iff \exists y [y \in X \wedge z = g(y)].$$

$Z = g(X)$  is a  $\Sigma_1^1$  subset of  $\mathcal{O}$ , and by Spector's Boundedness Theorem 21.1 there exists  $b \in \mathcal{O}$  such that

$$\forall z \in Z \quad |z|_{\mathcal{O}} < |b|_{\mathcal{O}}.$$

This means

$$x \in X \iff g(x) \in \mathcal{O}_b.$$

By Proposition 22.3,  $\mathcal{O}_b$  is recursive in  $H_{2^b}$ , and hence  $X$  is hyperarithmetical.  $\square$

Let HYP be the set of hyperarithmetical sets of natural numbers.

**Corollary 22.9:** *The set HYP is a  $\Pi_1^1$  subset of  $2^{\mathbb{N}}$ .*

*Proof.* We have

$$X \in \text{HYP} \iff \exists x [x \in \mathcal{O} \wedge X \leq_T H_x].$$

Since  $\mathcal{O}$  is  $\Pi_1^1$ ,  $H_x$  has a (uniformly)  $\Delta_1^1$  definition, and Turing reducibility can be expressed via an arithmetical formula, the result follows.  $\square$