Math 497A: Introductory to Ramsey Theory

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1 What is Ramsey Theory?

Rather than being a single theory which focuses on a specific type of object, Ramsey Theory is instead a collection of theorems and propositions which all have a similar flavor. These "Ramsey-type" problems can come from various fields and deal with very different and unrelated ideas but their results, and at times proof methods, are all similar in spirit.

The fundamental principle of Ramsey Theory is as follows: *if we split a sufficiently large and regular object into two parts, one of these parts will inherit the regularity.* When trying to understand this principle, the important questions to ask are the following:

- What "objects" can we discuss?
- How large is "suffiently large"?
- How can we "split" an object?
- In what ways can these objects be "regular"?

Before trying to answer these answers in general, let us go to an important example which will help us understand these concepts better.

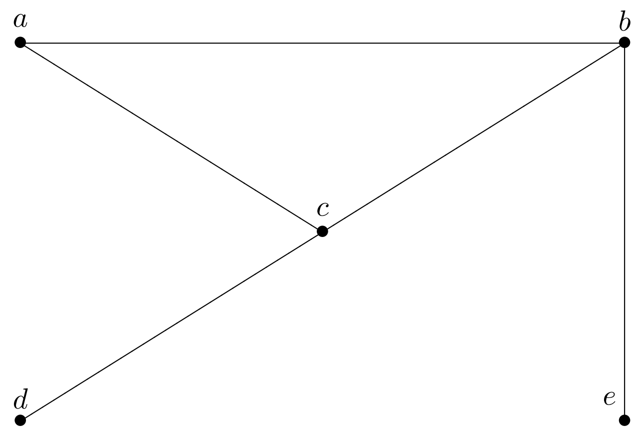
2 An Important Example in Graph Theory

The first example of Ramsey Theory that anyone is introduced to is in the field of Graph Theory. To that end, we will spend some time introducing the background and core principles of graphs.

A **graph** G is made up of two sets: a **vertex set**, V, and an **edge set**, E, where $E \subseteq [V]^2$, that is, E is a subset of the set of pairs of elements of V; we can write $G = (V, E)$. As should be expected, the elements of V are called **vertices** and the elementes of E are called **edges**. In other words, a graph is a collection of nodes placed arbitrarily with line segments connecting these nodes.

As an example, let us consider the following graph on five vertices:

$$V = \{a, b, c, d, e\}, \quad E = \{(a, b), (a, c), (b, c), (c, d), (b, e)\}.$$



When a pair of vertices is in the edge set of a graph, they are said to be **adjacent**. In the graph above, we can see that a and c are adjacent whereas d and e are not.

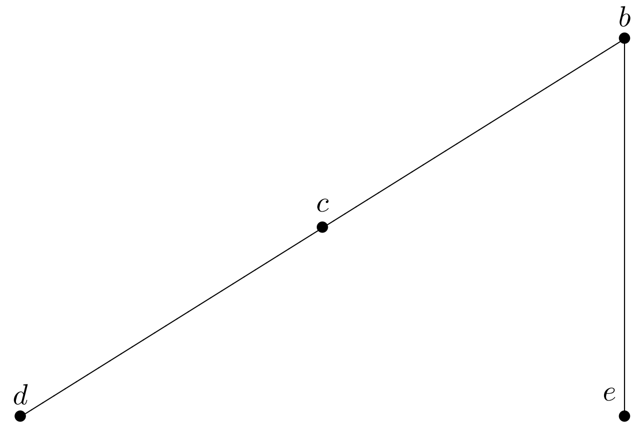
In general, graphs are allowed to have loops, i.e. edges which start and end at the same vertex, or multiple edges from one vertex to another. However, in this course, we will be discussing **simple** graphs which have no loops or multiple edges. Sometiems, we want an edge to have a direction associated to it. We can make a graph into a **directed graph** by considering the elements of E as *ordered* pairs; if the ordered pair (a, b) is in the edge set of a directed graph we would say that a is adjacent to b but b is *not* adjacent to a (unless, of course, (b, a) was also in the edge set). In the future, if we want to discuss directed graphs, we will say so explicitly, otherwise all graphs are taken to be undirected simple graphs.

A subgraph $G' = (V', E')$ of a graph $G = (V, E)$, denoted $G' \subseteq G$, is such that $V' \subseteq V$ and $E' \subseteq E$. Moreover, since G' should be a graph in its own right, we must have that $E' \subseteq [V']^2$. A subgraph is said to be **induced** if it is obtained from the original graph by removing some number of vertices, all edges containing those vertices, and nothing else.

If we consider the graph G above, then $G' = (V', E')$ where

$$V' = \{b, c, d, e\}, \quad E' = \{(b, c), (c, d), (b, e)\}$$

is an induced subgraph of G.

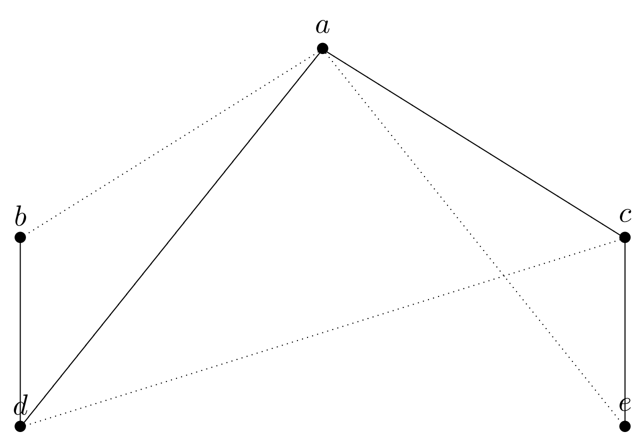


Now we can begin discussing our Ramsey-type problem. The first question we can answer is "how can we split a graph into two subgraphs?" There are many ways to break up a graph however we will be interested in edge-colorings. A **2-coloring** of a graph is simply any possible way to color some of the edges red and the rest blue. This gives us a function

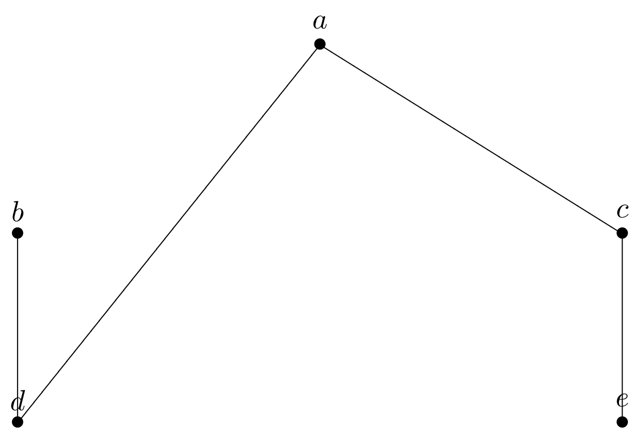
$$c : E \to \{\text{red,blue}\}$$

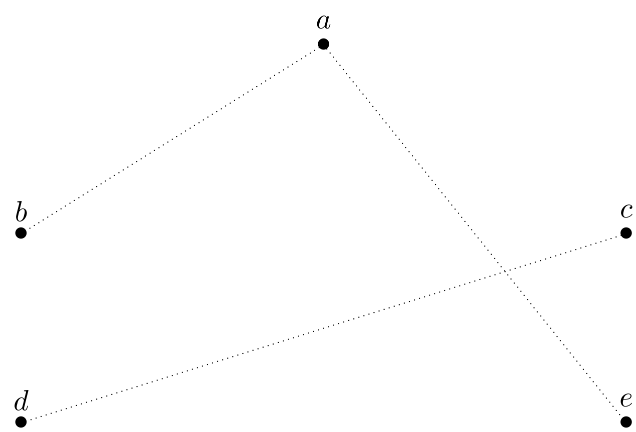
which assigns to each edge a color.

Below we can see a graph $G = (\{a, b, c, d, e\}, \{(a, b), (a, c), (a, d), (a, e), (c, d), (b, d), (c, e)\})$, with one possible 2-coloring (solid lines represent the color red while dashed lines represent blue).



From this coloring, we get two monochromatic subgraphs: the blue subgraph, G_{blue}, and the red subgraph, G_{red}.





The second question we can answer is "what kinds of graphs do we consider regular?" For this, we must define an important type of graph: a **complete** graph is one where every two vertices are adjacent. The complete graph on n vertices is denoted K^n. We can also denote a complete graph by $G = (V, [V]^2)$ since the edge set of a complete graph must contain all possible edges.

So now our Ramsey-type problem can be formulated as follows: if we split a sufficiently large complete graph via 2-coloring, will there be a complete graph (on fewer vertices) within one of the subgraphs? In 1926, Frank Ramsey answered a basic case of this rather daunting question in the proposition below.

Theorem 1. (Ramsey's Theorem on Graphs) For any 2-coloring of K^6, one of the monochromatic subgraphs must contain K^3, the complete graph on 3 vertices.

Proof. To prove the statement above, all we must do is find a monochromatic triangle. Pick any single vertex in our graph; it is adjacent to the other 5 vertices. Out of these vertices, we must have that either at least three are red or at least three are blue (otherwise, we would have only two of each color and not enough edges). Without loss of generality, we will say that there are three red edges. Now, consider the three vertices at the ends of these edges. If any of the edges connecting these three are red, then we have a red triangle. Otherwise, all three edges are blue and this forms a blue triangle. □

Remark: In the proof above, we used the **pidgeonhole principle**[1] to say that there are at least three edges of a single color coming from our initial vertex. In generality, the pidgeonhole principle says that if we have at least n objects split into k sets, then at least one set must contain $\lceil n/k \rceil$ objects.

An important thing to note about this theorem is that, a priori, we don't know whether we will end up with a red triangle or a blue triangle. We simply know that one of the two must exist.

[1] Historically, the "pidgeonholes" were not birdhouses but actually mailboxes into which letters were being placed.

Also, it is not hard to see that this result would hold if our original complete graph was on more than 6 vertices; simply pick 6 of the vertices and consider the induced subgraph on those vertices which is neccesarily K^6, then use the result.

This theorem is also commonly known as the "Party Theorem" because it can be restated in the following way: in a party of at least six people there will either be three people who mutually know each other or three people who are all strangers. This can be seen by making a complete graph where the guests are the vertices and the edges are either labelled red if two people know each other or blue otherwise. Then the existance of a monochromatic triangle gives our result.

There is an alternative formulation of this example which requires one more definition from Graph Theory. In a graph $G = (V, E)$, a subset of vertices $I \subseteq V$ is said to be **independent** if no two of the vertices are adjacent; a graph is said to be **independent** if its set of vertices is, that is, if its edge set is empty. By using this definition, Ramsey's Theorem on Graphs can now be applied to all graphs, not just those that are complete.

Theorem 2. (Ramsey's Theorem on Arbitrary Graphs) Any graph with more than 6 vertices contains a complete subgraph on 3 vertices or an independent subgraph of 3 vertices.

The proof of this is identical to the one above if we just consider edges in our graph to be colored red and those edges not in our edge set to be colored blue.

Note that the main difference between the two formulations of Ramsey's Theorem is that in the latter, the original object does not have the "regularity" property that we want the subobjects to have. This motivates a rewording of the fundamental principle stated above into a new "dual" principle: *In any sufficiently large object, we can find regular subobjects of a certain size.*

At this point, a natural question to ask is "What is so important that our complete or independent subgraphs be on 3 vertices? Can we find graphs having K^4 as a subgraph?" Ramsey's Theorem in its fullest generality partly answers this question.

Theorem 3. (Full Ramsey's Theorem on Graphs) For any $k \geq 1$, there exists some integer $R(k)$ such that any graph of at least $R(k)$ vertices contains a complete subgraph on k vertices or an independent subgraph on k vertices.

The original Ramsey's Theorem now can be written as the simple statement $R(3) = 6$.

There is also an infinite version of Ramsey's Theorem on Graphs. A (countably[2]) infinite graph is one where the size of the vertex set is (countably) infinite.

[2]Recall that a set is said to be countably infinite if there is a $1-1$ correspondence with the set and the natural numbers.

Theorem 4. (Ramsey's Theorem on Infinite Graphs) For any 2-coloring of a countably infinite complete graph, one of the monochromatic subgraphs must also contain a countably infinite complete graph.