Homework 6 for MATH 104

Brief solutions to selected exercises

Problem 1

Define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ with } p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\} \text{ relatively prime}, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that g is continuous at all irrational points, and discontinuous at all rational points.

Solution. First let $x = p/q$ be rational. For every real number there exists a sequence () of irrationals converging to it, e.g. take $x_n = x + \sqrt{n}/n$. Then by definition $g(x_n) = 0$ for all n, but $g(x) = 1/q \neq 0$, so g is discontinuous at x.

Now consider an irrational x. Suppose $x_n \to x$. If all but finitely many x_n are irrational, obviously $g(x_n) = 0$ for all but finitely many n, and thus $g(x_n) \to 0 = g(x)$. Therefore, by passing to a subsequence, we may assume that all the x_n are rational, i.e. $x_n = p_n/q_n$ with q_n a positive integer. We have to show that $q_n \to \infty$. Assume for a contradiction this is not the case. Then there exists a subsequence q_{n_k} that is bounded. Using Bolzano-Weierstrass, we can pick a convergent subsequence $q_{n_{k_l}}$. Since $q_{n_{k_l}}$ consists only of positive integers, we must have $q_{n_{k_l}} = q$ for all but finitely many m and some positive integer q. Since $x_n \to x$, it must hold that $p_{n_{k_l}}/q \to x$, too. It is easy to see that the set $\{p_{n_{k_l}}/q\}$ is closed in $\mathbb{R}$, and $p_{n_{k_l}}/q \neq x$ for all l. Therefore, there exists an l such that $|p_{n_{k_l}}/q - x|$ is minimal. If we choose ε less than this minimum, the ε-neiborhood of x contains no points from $\{p_{n_{k_l}}/q\}$, contradicting $x_n \to x$. ■

Problem 2

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Let the *zero set* of f be defined as

$$Z(f) = \{x \in \mathbb{R} : \ f(x) = 0\}.$$

Show that $Z(f)$ is a closed subset of $\mathbb{R}$.

Solution. We show that $Z(f)$ contains all its limit points. Let x_n be a sequence in $Z(f)$ and assume that $x_n \to x \in \mathbb{R}$. Then by continuity of f, $f(x_n) \to f(x)$. But $f(x_n)$ is 0 for all n, so it must hold that $f(x) = 0$, and hence $x \in Z(f)$. ■

Problem 3

Call a mapping $f : \mathbb{R} \to \mathbb{R}$ *open* if for every open set $U \subseteq \mathbb{R}$, the image

$$f(U) = \{f(x) : \ x \in U\}$$

is open. Show that a continuous open mapping $f : \mathbb{R} \to \mathbb{R}$ is monotonic.

Solution. Assume for a a contradiction that f is not monotonic. Then w.l.o.g. there exist $x < y < z \in \mathbb{R}$ such that $f(x) < f(y)$ and $f(y) > f(z)$. Since the interval $[x, z]$ is compact, the function f attains a maximum on $[x, z]$ say at $t \in [x, z]$. Since $f(y) > f(x)$ and $f(y) > f(z)$, $t \neq x, z$. Therefore, f attains a maximum on the open interval (x, z). But $f(x, z)$ cannot be open, since any neighborhood of $f(t)$ contains points not in $f(x, z)$. Contradiction. ■

Problem 4

(a) Let f, g be continuous mappings from $\mathbb{R}$ into $\mathbb{R}$. Further, let D be a dense subset of $\mathbb{R}$. Show that $f(D)$ is dense in $f(\mathbb{R})$. Furthermore, show that if $f(x) = g(x)$ for all $x \in D$, then $f(z) = g(z)$ for all $z \in \mathbb{R}$ (This shows that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is uniquely determined by its values on D.)

Solution. Assume that $y_0 \in f(\mathbb{R})$. Then there exists $x_0 \in \mathbb{R}$ with $f(x_0) = y_0$. By denseness of D in $\mathbb{R}$, there exists a sequence x_n in D such that $x_n \to x_0$. Then $f(x_n)$ is a sequence in $f(D)$, and by continuity of f, $f(x_n) \to f(x_0) = y_0$. Hence $f(D)$ is dense in $f(\mathbb{R})$.

Assume now $z \in \mathbb{R}$. Since D is dense in $\mathbb{R}$, there exists a sequence x_n in D such that $x_n \to z$. By continuity of f and g, we have $f(x_n) \to f(z)$ and $g(x_n) \to g(z)$. By the assumption on f and g we have $f(x_n) = g(x_n)$ for all n. Since the limit of a sequence is unique, it must hold that $f(z) = g(z)$. ■

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that f is of the form $f(x) = c \cdot x$ for some $c \in \mathbb{R}$.

Solution. We first observe that $f(0) = 0$, since $f(1) = f(1 + 0) = f(1) + f(0)$.

Next, it follows by induction that for each $n \in \mathbb{N}$, $f(n) = f(n-1) + f(1) = \cdots = nf(1)$. Using $0 = f(0) = f(1 + (-1)) = f(1) + f(-1)$ and hence $f(-1) = -f(1)$, we can derive the identity $f(z) = zf(1)$ for all $z \in \mathbb{Z}$.

Furthermore, $f(1) = nf(\frac{1}{n})$ and thus $f(\frac{1}{n}) = \frac{1}{n}f(1)$. We use this to prove that the identity $f(q) = qf(1)$ extends to all $q \in \mathbb{Q}$.

Now observe that each function f such that $f(x+y) = f(x)+f(y)$ must be continuous. For assume that $x_n \to x$. Then $|f(x_n) - f(x)| = |f(x_n - x)| \to f(0) = 0$. We have seen that for each $c \in \mathbb{R}$, $f(x) = cx$ is continuous, and obviously $f(x) = cx$ satisfies $f(x + y) = f(x) + f(y)$. We have shown that $f(q) = qf(1)$ for all $q \in \mathbb{Q}$, and $f(x) = xf(1)$ is a continuous extension of $f(q) = qf(1)$ to $\mathbb{R}$, which by part (a) is unique. Therefore, $f(x) = xf(1)$ for all $x \in \mathbb{R}$. ■