Effective Capacibility

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Measures on Cantor Space

Outer measures from premeasures

- Approximate sets from outside by open sets and weigh with a general measure function.
- A premeasure is a function $\rho : 2^{<\omega} \to \mathbb{R}_0^+ \cup \{\infty\}$.
- One can obtain an outer measure μ_ρ from ρ by letting

$$\mu_\rho(X) = \inf_{C \subseteq 2^{<\omega}} \left\{ \sum_{\sigma \in C} \rho(\sigma) : \bigcup_{\sigma \in C} N_\sigma \supseteq X \right\},$$

  where N_σ is the basic open set induced by σ.
  (Set $\mu_\rho(\emptyset) = 0$.)
- The resulting $\mu = \mu_\rho$ is a countably subadditive, monotone set function, an outer measure.

Measures on Cantor Space

Types of measures

Probability measures: based on a premeasure ρ which satisfies

- $\rho(\emptyset) = 1$ and
- $\rho(\sigma) = \rho(\sigma^\frown 0) + \rho(\sigma^\frown 1)$.

For probability measures it holds that $\mu_\rho(N_\sigma) = \rho(\sigma)$.

Lebesgue measure $\mathcal{L}$: $\rho(\sigma) = 2^{-|\sigma|}$.

Hausdorff measures: based on a premeasure ρ which satisfies

- If $|\sigma| = |\tau|$, then $\rho(\sigma) = \rho(\tau)$.
- $\rho(n)$ is nonincreasing.
- $\rho(n) \to 0$ as $n \to \infty$.
- For example: $\rho(\sigma) = 2^{-|\sigma|s}$, $s \geqslant 0$.

Measures on Cantor Space

Nullsets

The way we constructed outer measures, $\mu(A) = 0$ is equivalent to the existence of a sequence $(W_n)_{n\in\omega}$, $W_n \subseteq 2^{<\omega}$, such that for all n,

$$A \subseteq \bigcup_{\sigma\in W_n} N_\sigma \quad \text{and} \quad \sum_{\sigma\in W_n} \rho(\sigma) \leqslant 2^{-n}.$$

Thus,

every nullset is contained in a G_δ nullset.

Randomness for Outer Measures

Effective G_δ sets

By requiring that the covering nullset is effectively G_δ, we obtain a notion of effective nullsets.

Definition

A μ_ρ-test relative to $z \in 2^\omega$ is a set $W \subseteq \mathbb{N} \times 2^{<\omega}$ which is c.e. in z such that

$$\sum_{\sigma \in W_n} \rho(\sigma) \leqslant 2^{-n}.$$

A real x passes a test W if $x \notin \bigcap_n N(W_n)$, where

$$W_n = \{\sigma : (n, \sigma) \in W\}.$$

Hence a real passes a test W if it is not in the G_δ-set represented by W.

Randomness for Outer Measures

Representation of measures

Definition

Given a premeasure ρ, define its rational representation r_ρ by letting, for all $\sigma \in 2^{<\omega}$, $q_1, q_2 \in \mathbb{Q}$,

$$\langle \sigma, q_1, q_2 \rangle \in r_\rho \iff q_1 < \rho(\sigma) < q_2.$$

Idea: An effective test for randomness should have access to the measure it is testing for.

- Therefore, represent it by an infinite binary sequence.
- Outer measures are determined by the underlying premeasure ρ. It seems reasonable to represent these values via approximation by rational intervals.

Randomness for Outer Measures

Tests for Arbitrary Measures

Definition

Suppose ρ is a premeasure on 2^ω. A real is μ_ρ-z-random if it passes all μ_ρ-tests relative to r_ρ.

Hence, a real x is random with respect to an outer measure μ_ρ if and only if it passes all tests which are enumerable in the representation r_ρ of the underlying premeasure ρ.

- The concept can be relativized in a natural way, by testing relative to $r_\rho \oplus z$.

Randomness for Probability Measures

The basic question

Reimann and Slaman studied the question which reals x are random for some continuous probability measure μ, i.e. $\mu(\{x\}) = 0$ for all x.

Call such reals continuously random.

- Every real $x \notin \Delta^1_1$ is continuously random.
- No member of a countable Π^0_1 class, i.e. no ranked point is. [Kjos-Hanssen and Montalban]
- However, there are reals in Δ^0_2 that are neither ranked nor continuously random.
- Furthermore, there are examples of non-continuously random reals which are not member of any “thin” recursive tree at all (Δ^0_2 and sufficiently generic).

Randomness for Probability Measures

A positive result

We will prove a positive result: If x is a member of a certain family of recursive trees, then x is continuously random.

- This family of trees will be characterized by Hausdorff measures.
- The link between Hausdorff measures and probability measures is given by capacibility.
- This relation can be made very precise, and as a by-product yields a new, useful characterization of effective dimension.

Hausdorff Dimension

Definition

Given $s \geqslant 0$, let $s(\sigma) = 2^{-|\sigma|s}$. Recall that the Hausdorff dimension of a set $A \subseteq 2^{\omega}$ is given by

$$\dim_{\mathrm{H}} A = \inf\{s : A \text{ is } \mu_s\text{-null}\}.$$

- The calculation of Hausdorff dimension is often a very difficult task, in particular, obtaining a lower bound. One of the standard tools is the mass distribution principle

Idea: If a set A supports a probability measure that is “close” to uniform, then its Hausdorff dimension is close to 1.

Hausdorff Dimension

Mass Distribution Principle

Recall: The support of a measure μ, $\mathrm{supp}(\mu)$, is the smallest closed set F such that $\mu(2^\omega \setminus F) = 0$.

$A \subseteq 2^\omega$ supports a measure μ if $\mathrm{supp}(\mu) \subseteq A$.

Mass Distribution Principle

If A supports a probability measure μ such that for all σ,

$$\mu(\sigma) \leqslant c 2^{-|\sigma| s},$$

then $\dim_{\mathrm{H}} A \geqslant s$.

Hausdorff Dimension

Frostman's Lemma

A fundamental result due to FROSTMAN (1935) asserts that the converse holds, too, as long as A is not too complex.

Frostman's Lemma

If A is analytic and $\dim_{\mathrm{H}} A > s > 0$, then there exists a probability measure μ such that $\operatorname{supp}(\mu) \subseteq A$ and for some $c > 0$,

$$(\forall \sigma)\ \mu(\sigma) \leqslant c2^{-|\sigma|s} \tag{*}$$

The theorem can be interpreted in the framework of capacity theory. Define the capacitary dimension of A to be

$$\dim_c(A) = \sup\{s :\ \exists \mu \text{ mass distr. on } A \text{ with (*)}\}.$$

Then we have for analytic sets, $\dim_c = \dim_{\mathrm{H}}$.

We prove a pointwise version of Frostman's Lemma.

Recall that an order function is a nondecreasing, unbounded function $h : \mathbb{N} \to \mathbb{N}$. h is called convex if for all n, $h(n+1) \leqslant h(n) + 1$.

We say x is h-capacitable if there exists a probability measure μ such that for all σ, $\mu(\sigma) \leqslant \gamma 2^{-h(|\sigma|)}$, and x is μ-random.

Theorem

If h is a computable, convex order function, then any 2^{-h}-random real is h-capacitable.

Topology for Probability Measures

The weak*-topology

If μ_ρ is a probability measure, the representation r_ρ can be interpreted topologically, by means of the weak*-topology of Banach spaces.

- Denote by $\mathcal{P}$ the set of all probability measures on 2^ω. For this section, we identify measures and their underlying premeasures.
- The Riesz representation theorem lets us identify measures with linear functionals on the space of continuous functions on 2^ω, by means of integration.
- The weak*-topology on $\mathcal{P}$ is the topology generated by the mappings $f \mapsto \int f \, d\mu$.

Topology for Probability Measures

A compatible metric

To generate the weak topology of $\mathcal{P}$, it suffices to consider a dense set of continuous functions on 2^ω.

- A countable dense set is given by the set of continuous functions on 2^ω that take only finitely many, rational values.
- Denote this set by $D(2^\omega) = \{f_n\}_{n\in\omega}$.

The mapping $\mu \mapsto (\int f_n \mu / \|f_n\|_\infty)_{n\in\omega}$ embeds $\mathcal{P}$ into $[-1, 1]^\omega$.

- We can pull back the product metric on $[-1, 1]^\omega$ to $\mathcal{P}$ to obtain a compatible metric

$$d(\mu, \nu) = \sum_{n=0}^{\infty} 2^{-n-1} \frac{|\int f_n d\mu - \int f_n \nu|}{\|f_n\|_\infty}.$$

Topology for Probability Measures

An effective dense subset

With the weak topology, $\mathcal{P}$ becomes a compact Polish space.

A countable dense subset of $\mathcal{P}$ is given as follows:

- Let Q be the set of all reals of the form $\sigma^\frown 0^\omega$.
- Given $\bar{q} = (q_1, \ldots, q_n) \in Q^{<\omega}$ and non-negative rational numbers $\alpha_1, \ldots, \alpha_n$ with $\sum \alpha_k = 1$, let

$$\delta_{\bar{q}} = \sum_{k=1}^{n} \alpha_k \delta_{q_k},$$

where δ_x denotes the Dirac point measure for x.

Topology for Probability Measures

Effective representations

We want to exploit the topological structure of $\mathcal{P}$ to prove results about algorithmic randomness.

- One can show that sets of the form

$$\{\mu \in \mathcal{P} : \ q_1 < \mu(\sigma) < q_2\}, \quad \sigma \in 2^{<\omega}, q_1, q_2 \in \mathbb{Q}$$

  form a subbasis of the weak topology.
- Hence, the rational representation r_μ indicates to which basic open sets μ belongs.
- However, not every real is a rational representation of some probability measure.
- Moreover, the set of all $x \in 2^\omega$ such that $x = r_\mu$ for some $\mu \in \mathcal{P}$ is not Π^0_1, so it does not effectively reflect the topological properties of $\mathcal{P}$.

Topology for Probability Measures

Effective representations

Alternative: Use the recursive dense subset $\mathcal{D} = \{\delta_{\bar{q}}\}$ and the effectiveness of the metric d between measures of the form $\delta_{\bar{q}}$ to represent measures.

Theorem

There exists a recursive sequence (r_n) and a continuous surjection

$$\pi : [T] \to \mathcal{P},$$

where $T \subset \omega^{<\omega}$ is the full (r_n)-branching tree, i.e. every node in T of length n has exactly r_n immediate successors.

Every element in $P = [T]$ represents a Cauchy sequence of measures in $\mathcal{D}$.

Effective Dimension and Continuous Randomness

Proving the effective capacibility theorem

By the Kucera-Gacs Theorem, there exists a $\mathcal{L}$-random real y such that $y \geqslant_{\mathrm{wtt}} x$ via some reduction Φ.

For every $\sigma \in 2^{<\omega}$ we define

$$\mathrm{Pre}(\sigma) = \{\tau : \ \Phi(\tau) \supseteq \sigma \ \& \ \forall \tau' \subset \tau(\Phi(\tau') \not\supseteq \sigma)\}.$$

$\mathcal{L}(\mathrm{Pre}(.))$ induces a semimeasure on $2^{<\omega}$: This is a function $\eta : 2^{<\omega} \to [0, 1]$ such that

$$\forall \sigma \, [\eta(\sigma) \geqslant \eta(\sigma^\frown 0) + \eta(\sigma^\frown 1)]. \tag{1}$$

We want to define $\mu(\sigma)$, $\sigma \in 2^{<\omega}$. We have to satisfy two requirements:

- The measure μ will dominate an image measure induced by Φ. This will ensure that any Martin-Löf random sequence is mapped by Φ to a μ-random sequence.
- The measure must respect the upper bound.

To meet these requirements, we restrict the values of μ in the following way:

$$\mathcal{L}(\Phi^{-1}(\sigma)) \leqslant \mu(\sigma) \leqslant c2^{-|\sigma|s}. \tag{+}$$

This singles out suitable completions of the semimeasure induced by Φ.

It can be shown that for some c

$$M := \{\mu : \ \mu \text{ satisfies } (+)\}$$

is a non-empty Π^0_1 subset of $\mathcal{P}$.

Note that if (V_n) were a μ-test covering x, then $\Phi^{-1}(V_n)$ would be a $\mathcal{L}$-test relative to μ covering y.

- So, what we need to show is that y is $\mathcal{L}$-random relative to μ for some $\mu \in M$.

The following result ensures the existence of such a μ.

Theorem

If $B \subseteq 2^\omega$ is nonempty and Π^0_1, then, for every y which is $\mathcal{L}$-random there is $z \in B$ such that y is $\mathcal{L}$-random relative to z.

(Downey, Hirschfeldt, Miller, and Nies; Reimann and Slaman)

Effective Capacibility and Dimension

A new characterization

As a corollary we obtain a new characterization of effective dimension.

Theorem

For any real $x \in 2^\omega$,

$$\dim_H^1 x = \sup\{s \in \mathbb{Q} : \ x \textit{ is } h\textit{-capacitable for } h(n) = sn\}.$$

Particularly with regard to effective dimension notions, several other test concepts have been suggested.

The standard structure of such tests is as follows: A notion of randomness $\mathcal{R}$ is a uniform mapping

$$\mathcal{R} : \rho \quad \mapsto \quad W \quad \mapsto \quad \bigcap_n W_n,$$

where $W \subset \mathbb{N} \times 2^{<\omega}$ is c.e. in (a representation of) ρ, and $\bigcap_n W_n$ is a μ_ρ-nullset that is $\Pi^0_2(\rho)$.

Comparisons of Dimension Notions

An application

Examples:

- Martin-Löf tests: $\rho(W_n) \leqslant 2^{-n}$
- Solovay tests: $W_1 \supseteq W_2 \supseteq \dots$, W_n contains only strings of length $\geqslant n$ and $\rho(W_n) \leqslant 1$.
- Strong tests: If $V \subseteq W_n$ is prefix-free, then $\rho(V) \leqslant 2^{-n}$.
- Vehement tests: For each n exists V_n such that $N(V_n) \supseteq N(W_n)$ and $\rho(V_n) \leqslant 2^{-n}$.

Comparisons of Dimension Notions

An application

The capacibility theorem holds for any standard notion of randomness.

Since all these notions coincide on probability measures, the capacitary characterization of effective dimension yields that they

induce the same notion of effective dimension,

that is, if $\mathcal{R}_1$ and $\mathcal{R}_2$ are two notions of randomness, then for any $x \in 2^\omega$,

$$\dim_H^{\mathcal{R}_1} x = \dim_H^{\mathcal{R}_2} x$$