Homework 5 for MATH 104

Due: Tuesday, October 17, 9:30am in class

Problem 1

For $x, y \in \mathbb{R}$, define

$$d_1(x,y) = (x-y)^2 \qquad d_3(x,y) = |x^2 - y^2| \qquad d_5(x,y) = \frac{|x-y|}{1+|x-y|}$$

$$d_2(x,y) = \sqrt{|x-y|} \qquad d_4(x,y) = |x-2y|$$

Determine, for each of these, whether it is a metric or not. Justify your answer.

Problem 2

Consider the real line $\mathbb{R}$ with the standard metric $d(x,y) = |x-y|$.

(a) Prove that the set $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is compact directly from the definition (without using the Heine-Borel Theorem).

(b) Give an example of a compact subset K of $\mathbb{R}$ such that K has countably many limit points.

Problem 3

Let (X, d) be a metric space.

(a) Define the *diameter*$\delta(A)$ of a set $A \subseteq X$ as

$$\delta(A) = \sup\{d(x,y) : x, y \in A\}$$

Consider the metric space $(\mathbb{R}^n, d_2)$, where d_2 denotes the Euclidean metric. Show that for any compact subset K of $\mathbb{R}^n$, there exists $x, y \in K$ such that $\delta(K) = d(x,y)$.

(b) Define the *distance* $\delta(A, B)$ of two sets as

$$\delta(A,B) = \inf\{d(x,y) : x \in A, y \in B\}.$$

Give an example of two closed subsets of $\mathbb{R}$ (with respect to the standard metric) such that $\delta(A,B) < d(x,y)$ for all $x \in A$, $y \in B$. (As always, justify your answer.)

Problem 4

Let (X, d) be a metric space. A set $D \subset X$ is called *dense* if every point in X is a limit point of D. A metric space (X, d) is called *separable* if it contains a countable dense subset.

(a) Prove that $(\mathbb{R}^n, d_2)$ is separable. (Hint: consider the set $\mathbb{Q}^n = \{(q_1, \ldots, q_n) : q_i \in \mathbb{Q} \text{ for } 1 \leqslant i \leqslant n\}$.)

(b) Show that every compact metric space is separable.