Homework 7 for MATH 104

Brief solutions to selected exercises

Problem 1

(a) Suppose $\sum a_n x^n$ has finite radius of convergence R and that $a_n \geqslant 0$ for all n. Show that if the series converges at R, then it also converges at $-R$.

Solution. Obviously, since $a_n \geqslant 0$ and $R \geqslant 0$, the series $\sum_n a_n R^n$ is absolutely convergent. Furthermore, it holds that $|a_n(-R)^n| = a_n R^n$, so absolute convergence of $\sum_n a_n R^n$ implies convergence of $\sum_n a_n(-R)^n$. ■

(b) Give an example of a power series whose interval of convergence is exactly $(-1, 1]$.

Solution. $\sum_n (-1)^n x^n / n$. ■

Problem 2

Show that $\sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n}$ has radius of convergence 2 and that the series converges uniformly to a continuous function on $[-2, 2]$.

Solution. We have $\limsup_n \sqrt[n]{1/n^2 2^n} = \limsup_n 1/(\sqrt[n]{n})^2 2 = 1/2$. Therefore, $R = 2$. We now use the Weierstrass M-test: For all $|x| \leqslant 2$ it holds that

$$\left|\frac{x^n}{n^2 2^n}\right| = \frac{|x|^n}{n^2 2^n} \leqslant \frac{2^n}{n^2 2^n} = \frac{1}{n^2}.$$

Since $1/n^2 > 0$ and $\sum_n 1/n^2$ converges, $\sum_n \frac{x^n}{n^2 2^n}$ converges uniformly to a continuous function on $[-2, 2]$. ■

Problem 3

(a) Let (f_n) be a sequence of continuous functions $f_n : S \to \mathbb{R}$, $S \subseteq \mathbb{R}$ which converges uniformly on S. Show that if (x_n) is a sequence in S such that $x_n \to x \in S$, then $\lim_n f_n(x_n) = f(x)$.

Solution. Let $\varepsilon > 0$. Since $f_n \to f$ uniformly and f_n continuous for all n, we know that f is continuous. Pick $N_1 \in \mathbb{N}$ such that $|f(x_n) - f(x)| < \varepsilon/2$ whenever $n > N_1$. Furthermore, since $f_n \to f$ uniformly, we can pick N_2 such that $\sup\{|f_n(x) - f(x)| : x \in S\} < \epsilon/2$ whenever $n > N_2$ (use 4(b)). Then it holds that for all $n > \max\{N_1, N_2,$

$$|f_n(x_n) - f(x)| \leqslant |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

■

(b) Is the converse of (a) true, that is, is it true that if $f_n(x_n)$ converges to $f(x)$ whenever (x_n) converges to x in S, then (f_n) converges uniformly to f?

Solution. No, consider for example $f_n(x) = x^n$ on $S = (0, 1)$. ■

Problem 4

Let $\mathcal{B}(\mathbb{R})$ be the set of all bounded functions $f : \mathbb{R} \to \mathbb{R}$, i.e. there exists a real number $M \geqslant 0$ such that $|f(x)| \leqslant M$ for all $x \in \mathbb{R}$.

(a) Define a function $d : \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}^{\geqslant 0}$ by

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}.$$

Show that d defines a metric on $\mathcal{B}(\mathbb{R})$.

Solution. Axioms (M1) and (M2) are obvious. For (M3), let $x \in \mathbb{R}$ be such that $|f(x) - h(x)| \geqslant d(f, h) - \varepsilon$. Furthermore,

$$d(f, h) - \varepsilon \leqslant |f(x) - h(x)| \leqslant |f(x) - g(x)| + |g(x) - h(x)| \leqslant d(f, g) + d(g, h).$$

Since this holds for every ε, it follows that $d(f, h) \leqslant d(f, g) + d(g, h)$. ∎

(b) Show that a sequence (f_n) of bounded real functions $f_n : \mathbb{R} \to \mathbb{R}$ converges uniformly to a function $f : \mathbb{R} \to \mathbb{R}$ if and only if $f_n \to f$ with respect to the metric d. Show that the limit function f is bounded, too.

Solution. This is easy, since "$|f_n(x) - f(x)| \leqslant \varepsilon$ for all x" holds if and only if $\sup\{|f(x) - g(x)| : x \in \mathbb{R}\} \leqslant \varepsilon$.

The boundedness of the limit function immediately follows from the fact that f is within ε of some f_n. ∎

(c) Show that the metric space $(\mathcal{B}(\mathbb{R}), d)$ is complete.

Solution. This is Theorem 25.4 in Ross. ∎

(d) Is $(\mathcal{B}(\mathbb{R}), d)$ compact?

Solution. No, consider for example the open cover $\mathcal{U} = \{B_r(0) : r > 0\}$, where 0 denotes the function constantly 0. ∎