Lecture 1:   Perfect Subsets of the Real Line

Descriptive set theory nowadays is understood as the study of definable subsets of Polish Spaces. Many of its problems and techniques arose out of efforts to answer basic questions about the real numbers. A prominent example is the *Continuum Hypothesis* (CH):

> If $A \subseteq \mathbb{R}$ is uncountable, does there exist a bijection between A and $\mathbb{R}$? That is, is every uncountable subset of $\mathbb{R}$ of the same cardinality as $\mathbb{R}$? [Cantor, 1890's]

Early approaches to this problem tried to show that CH holds for a number of sets with an easy topological structure. It is a standard exercise of analysis to show that every open set satisfies CH. (An open set contains an interval, which maps bijectively to $\mathbb{R}$.) For closed sets, the situation is less clear. Given a set $A \subseteq \mathbb{R}$, we call $x \in \mathbb{R}$ an *accumulation point* of A if

$$\forall \epsilon > 0 \, \exists z \in A \, [z \neq x \,\&\, z \in U_\varepsilon(x)],$$

where $U_\varepsilon(x)$ denotes the standard ε-neighborhood of x in $\mathbb{R}$[1]. Call a non-empty set $P \subseteq \mathbb{R}$ *perfect* if it is closed and every point of P is an accumulation point. In other words, a perfect set is a closed set that has no *isolated* points. It is not hard to see that for a perfect set P, every neighborhood of a point $p \in P$ contains infinitely many points.

Obviously, $\mathbb{R}$ itself is perfect, as is any closed interval in $\mathbb{R}$. There are totally disconnected perfect sets, such as the *middle-third Cantor set* in $[0, 1]$

Theorem 1.1: *A perfect subset of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$.*

Proof. Let $P \subseteq \mathbb{R}$ be perfect. We construct an injection from the set $2^{\mathbb{N}}$ of all infinite binary sequences into P. An infinite binary sequence $\xi = \xi_0 \xi_1 \xi_2 \ldots$ can be identified with a real number $\in [0, 1]$ via the mapping

$$\xi \mapsto \sum_{i \geq 0} \xi_i 2^{-i-1}.$$

Note that this mapping is onto. Hence the cardinality of P is at least as large as the cardinality of $[0, 1]$. The **Cantor-Schröder-Bernstein Theorem** (for a proof see e.g. Jech [2003]) implies that $|P| = 2^{\aleph_0}$.

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[1]There are some divergences in terminology. Some authors call an accumulation point a limit point. We reserve the latter term for any point that is the limit of a sequence of points from a given set. Hence every member of a set is a limit point of that set. In particular, isolated members of a set are limit points.

Choose $x \in P$, and let $\varepsilon_0 = 1 = 2^0$. Since P is perfect, $P \cap U_{\varepsilon_0}(x)$. Let $x_0 \neq x_1$ be two points in $P \cap U_{\varepsilon_0}(x)$, distinct from x. Let ε_1 be such that $2^{-\varepsilon_1} \leq 1/2$, $U_{\varepsilon_1}(x_0), U_{\varepsilon_1}(x_1) \subseteq U_{\varepsilon_0}(x)$, and $\overline{U_{\varepsilon_1}(x_0)} \cap \overline{U_{\varepsilon_1}(x_1)} = \emptyset$, where $\overline{U}$ denotes the closure of U.

We can iterate this procedure recursively with smaller and smaller diameters, using the fact that P is perfect. This gives rise to a so-called *Cantor scheme*, a family of open balls (U_σ). Here the index σ is a finite binary sequence, also called a *string*. The scheme has the following properties.

C1) $\mathrm{diam}(U_\sigma) \leq 2^{-|\sigma|}$, where $|\sigma|$ denotes the length of σ.

C2) If τ is a proper extension of σ, then $\overline{U_\tau} \subset U_\sigma$.

C3) If τ and σ are *incompatible* (i.e. neither extends the other), then

$$U_\tau \cap U_\sigma = \emptyset.$$
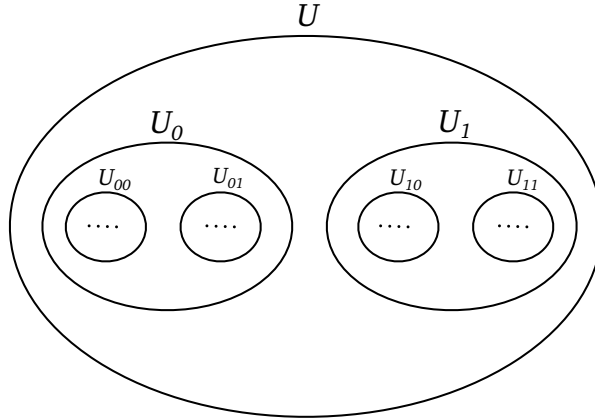
C4) The center of each U_σ, call it x_σ, is in P.

Figure 1: Cantor Scheme

Let ξ be an infinite binary sequence. Given $n \geq 0$, we denote by $\xi \!\restriction_n$ the string formed by the first n bits of ξ, i.e.

$$\xi \!\restriction_n = \xi_0 \xi_1 \dots \xi_{n-1}.$$

The finite initial segments give rise to a sequence $x_{\xi|_n}$ of centers. By (C1) and (C2), this is a Cauchy sequence. By (C4), the sequence lies in P. Since P is closed, the limit x_ξ is in P. By (C3), the mapping $\xi \mapsto x_\xi$ is well-defined and injective. $\qquad\square$

Theorem 1.2: *Every uncountable closed subset of $\mathbb{R}$ contains a perfect subset.*

Proof. Let $C \subseteq \mathbb{R}$ be uncountable and closed. We say $z \in \mathbb{R}$ is a *condensation point* of C

$$\forall \varepsilon > 0 \, [U_\varepsilon(z) \cap C \text{ uncountable}].$$

Let D be the set of all condensation points of C. Note that $D \subseteq C$, since every condensation point is clearly an accumulation point and C is closed. Furthermore, we claim that D is perfect. Clearly D is closed. Suppose $z \in D$ and $\varepsilon > 0$. Then $U_\varepsilon(z) \cap C$ is uncountable. We would like to conclude that $U_\varepsilon(z) \cap D$ is uncountable, too, since this would mean in particular that $U_\varepsilon(z) \cap D$ is infinite. The conclusion holds if $C \setminus D$ is countable. To show that $C \setminus D$ is countable, we use the fact that every open interval in $\mathbb{R}$ is the union of countably many open intervals with rational endpoints. Note that there are only countably many such intervals. If $y \in C \setminus D$, then for some $\delta > 0$, $U_\delta(y) \cap C$ is countable. y is contained in some subinterval $U_y \subseteq U_\delta(y)$ with rational endpoints. Thus, we have

$$C \setminus D \subseteq \bigcup_{y \in C \setminus D} U_y \cap C,$$

and the right hand side is a countable union of countable sets, hence countable. $\qquad\square$

We will later encounter an alternative (more constructive) proof that gives additional information about the complexity of the closed set C. For now we conclude with the fact we started out to prove.

Corollary 1.3: *Every closed subset of $\mathbb{R}$ is either countable or of the cardinality of the continuum.*