Lecture 18: $\mathbf{\Sigma}^1_2$ Sets

In this lecture we extend the results of the previous lecture to Σ^1_2 sets.

Tree representations of $\mathbf{\Sigma}^1_2$ sets

Analytic sets are projections of closed sets. Closed sets are in $\mathbb{N}^{\mathbb{N}}$ are infinite paths through trees on ω.

We call a set $A \subseteq \mathbb{N}^{\mathbb{N}}$ *Y-Souslin* if A is the projection $\exists^{Y^{\mathbb{N}}}[T]$ of some $[T]$, where T is a tree on $\mathbb{N} \times Y$, i.e. $A = \exists^{Y^{\mathbb{N}}}[T] = \{\alpha \colon \exists y \in Y^{\mathbb{N}}\, (\alpha, y) \in [T]\}$.

Theorem 18.1 (Shoenfield, 1961)**:** *Every $\mathbf{\Sigma}^1_2$ set is ω_1-Souslin. In particular, if A is Σ^1_2 then there is a tree $T \in L$ on $\mathbb{N} \times \omega_1$ such that $A = \exists^{(\omega_1)^{\mathbb{N}}}[T]$.*

Proof. Assume first A is Π^1_1. There is a recursive tree T on $\mathbb{N} \times \mathbb{N}$ (and hence, in L, since 'being recursive' is definable) such that

$$\alpha \in A \quad \Leftrightarrow \quad T(\alpha) \text{ is well-founded.}$$

Hence, $\alpha \in A$ if and only if there exists an order preserving map $\pi : T(\alpha) \to \omega_1$. We recast this in terms of getting an infinite branch through a tree. Let $\{\sigma_i : i \in \mathbb{N}\}$ be a recursive enumeration of $\mathbb{N}^{<\mathbb{N}}$. We may assume for this enumeration that $|\sigma_i| \leq i$. We define a tree $\widetilde{T}$ on $\mathbb{N} \times \omega_1$ by

$$\widetilde{T} = \{(\sigma, \tau) : \ \forall i, j < |\sigma|\, [\sigma_i \supset \sigma_j \ \wedge \ (\sigma \mid |\sigma_i|, \sigma_i) \in T \ \to \ \tau(i) < \tau(j)]\}.$$

It is easy to see that $\widetilde{T}$ is in L, since it is definable from T and ω_1. Furthermore, if $\alpha \in A$, then the existence of an order-preserving map $\pi : T(\alpha) \to \omega_1$ implies that there is an infinite path (α, η) through $\widetilde{T}$. Conversely, if such a path (α, η) exists, then it is easy to see that there is an order preserving map $\pi : T(\alpha) \to \omega_1$. Hence we have

$$\alpha \in A \longleftrightarrow \exists \eta \in (\omega_1)^{\mathbb{N}}\, (\alpha, \eta) \in [\widetilde{T}] \ \longleftrightarrow \ \alpha \in \exists^{(\omega_1)^{\mathbb{N}}}[\widetilde{T}],$$

so A is of the desired form.

Now we extend the representation to Σ^1_2. If A is Σ^1_2, then there is a Π^1_1 set $B \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $A = \exists^{\mathbb{N}^{\mathbb{N}}} B$. Since $B \in \Pi^1_1$, we can employ the tree representation of Π^1_1 to obtain a tree T over $\mathbb{N} \times \mathbb{N} \times \omega_1$ such that $B = \exists^{(\omega_1)^{\mathbb{N}}}[T]$. Now we recast T as a tree T' over $\mathbb{N} \times \omega_1$ such that $\exists^{(\omega_1)^{\mathbb{N}}}[T'] = \exists^{(\omega_1)^{\mathbb{N}}} B$. This is done by using a bijection between $\mathbb{N}\times\omega_1$ and ω_1. This way we can cast the $\mathbb{N}\times\omega_1$ component of T into a single ω_1 component, and thus transform the tree T into a tree T' over $\mathbb{N}\times\omega_1$ such that $\exists^{(\omega_1)^{\mathbb{N}}}[T'] = \exists^{(\omega_1)^{\mathbb{N}}}[B]$. □

$\boldsymbol{\Sigma}^1_2$ sets as unions of Borel sets

We can use Shoenfield's tree representation to extend Corollary 17.8 to Σ^1_2 sets.

Theorem 18.2 (Sierpiński, 1925)**:** *Every Σ^1_2 set is a union of $\aleph_1$-many Borel sets.*

Sierpinski's original proof used AC. The following proof does not make use of choice.

Proof. Let $A\subseteq\mathbb{N}^{\mathbb{N}}$ be Σ^1_2. By Theorem 18.1 there exists a tree T on $\mathbb{N}\times\omega_1$ such that $A=\exists^{(\omega_1)^{\mathbb{N}}}[T]$. For any $\xi<\omega_1$ let

$$T^{\xi}=\{(\sigma,\eta)\in T:\forall i\leq|\eta|\,\eta(i)<\xi\}.$$

Since the cofinality of ω_1 is greater than ω (this can be proved without using AC), every $d:\omega\to\omega_1$ has its range included in some $\xi<\omega_1$. Thus we have

$$A=\bigcup_{\xi<\omega_1}\exists^{(\omega_1)^{\mathbb{N}}}[T^{\xi}].$$

For all $\xi<\omega_1$, the set $\exists^{(\omega_1)^{\mathbb{N}}}[T^{\xi}]$ is Σ^1_1, because the tree T^{ξ} is a tree on a product of countable sets and hence is isomorphic to a tree on $\mathbb{N}\times\mathbb{N}$. By Corollary 17.9, each Σ^1_1 set is the union of $\aleph_1$ many Borel sets, from which the result follows. □

Again, an immediate consequence of this theorem is (using the perfect set property of Borel sets):

Corollary 18.3: *Every $\boldsymbol{\Sigma}^1_2$ set has cardinality at most $\aleph_1$ or has a perfect subset and hence cardinality $2^{\aleph_0}$.*

Absoluteness of Σ^1_2 relations

Shoenfield used the tree representation of $\boldsymbol{\Sigma}^1_2$ sets to establish an important absoluteness result for Σ^1_2 sets of reals.

Suppose $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Σ^1_2. Then, by the Kleene Normal Form there exists a bounded formula $\varphi(\alpha, \beta_0, \beta_1, m)$ such that

$$\alpha \in A \quad \Longleftrightarrow \quad \exists \beta_0 \, \forall \beta_1 \, \exists m \, \varphi(\alpha, \beta_0, \beta_1, m).$$

Let M be in inner model of ZF, i.e. M is transitive and contains all ordinals. Since arithmetical formulas can be interpreted in ZF, M contains all recursive predicates over $\mathbb{N}$. In particular, since the truth of the bounded formula φ depends only on finite initial segments of α, β_0, β_1, it defines a recursive predicate $R_\varphi(\alpha, \beta_0, \beta_1, m) = R_\varphi(\sigma, \tau_0, \tau_1, m)$, which in turns defines a subset of $\mathbb{N}^4$ that is contained in M. Hence we can define the *relativization* of A to M as

$$A^M(\alpha) \quad \Longleftrightarrow \quad \exists \beta_0 \in M \, \forall \beta_1 \in M \, \exists m \, R(\alpha, \beta_0, \beta_1, m).$$

We say that A is **absolute for** M if for any $\alpha \in M$,

$$A^M(\alpha) \quad \Longleftrightarrow \quad A(\alpha).$$

Absoluteness itself can be extended and relativized in a straightforward manner to predicates analytical in some $\gamma \in \mathbb{N}^{\mathbb{N}} \cap M$.

Theorem 18.4 (Shoenfield Absoluteness)**:** *Every $\Sigma^1_2(\gamma)$ predicate and every $\Pi^1_2(\gamma)$ predicate is absolute for all inner models M of* ZFC *such that $\gamma \in M$. In particular, all Σ^1_2 and Π^1_2 relations are absolute for L.*

Proof. We show the theorem for Σ^1_2 predicates. For the relativized version, one uses the *relative constructible universe* $L[\gamma]$, see Jech [2003] or Kanamori [2003].

Let A be a Σ^1_2 relation. For simplicity, we assume that A is unary. Fix a tree representation of A as a projection of a Π^1_1 set. So, let T be a recursive tree on $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ such that

$$\alpha \in A \quad \Longleftrightarrow \quad \exists \beta \; T(\alpha, \beta) \text{ is well-founded.}$$

Note that T is in M.

Now assume $\alpha \in M$ and $\alpha \in A^M$. So there is a $\beta \in M$ such that $T(\alpha, \beta)$ is well-founded in M. This is equivalent to the fact that in M there exists an order preserving mapping $\pi : T(\alpha, \beta) \to \mathrm{Ord}^M$. Since M is an inner model and T is the same in V and M, such a mapping exists also in V. Hence $T(\alpha, \beta)$ is well-founded in V and thus $\alpha \in A$.

For the converse assume that $\alpha \in A \cap M$. Now we use the tree representation of A given by Theorem 18.1. Let $U \in L \subseteq M$ be a tree on $\mathbb{N} \times \omega_1$ such that $A = \exists^{(\omega_1)^{\mathbb{N}}} U$. This means that for any $\alpha \in \mathbb{N}^{\mathbb{N}}$,

$$\alpha \in A \quad \Longleftrightarrow \quad U(\alpha) \text{ is not well-founded.}$$

So $\alpha \in A \cap M$ implies that there exists no order preserving map $U(\alpha) \to \omega_1$. But then such a map cannot exist in M either. So, $U(\alpha)$ is a tree in M which is ill-founded in the sense of M. Thus, by Shoenfield's Representation Theorem relativized to M, $\alpha \in A^M$.

Absoluteness for Π^1_2 follows by employing the same reasoning, using that the complement is Σ^1_2. □

By analyzing the proof one sees that it actually suffices that M is a transitive $\in$-model of a certain finite collection of axioms ZF such that $\omega_1 \subseteq M$.

The result is the best possible with respect to the analytical hierarchy, since the statement

$$\exists \alpha \, [\alpha \notin L]$$

is Σ^1_3, but cannot be absolute for $M = L$.

Shoenfield's Absoluteness Theorem also holds for sentences rather than formulae, with a similar proof. This means a Σ^1_2 statement is true in L if and only if it holds in V. This has an important consequence regarding the significance of principles like CH for analysis. Many results of classical analysis are Σ^1_2 statements. The Absoluteness Theorem says that if they can be established under $\mathsf{V} = \mathsf{L}$ (and hence in a world where CH holds), they can be established in ZF alone.

Another consequence concerns the complexity of reals defined by analytical relations.

Corollary 18.5: *If $X \subseteq \omega$ is Σ^1_2, then $X \in L$. In particular, every Σ^1_2 real (and hence every Π^1_2 real) is in L.*

Proof. Let X be Σ^1_2 via some formula φ. Since $\omega \in L$, and since L is an inner model of ZF, it satisfies the axiom of separation (relativized to L) for φ. So the set $X^L = \{a \in \omega : \varphi^L(a)\}$ is in L. It is clear that the representation and absoluteness results also hold for subsets of ω. (Change the notation to include subsets of ω.) Absoluteness for φ implies that $X^L \cap L = X \cap L$, but since $X \subseteq \omega$, we have $X = X \cap L$ and $X^L \cap L = X^L$, and hence $X \in L$. □

We cannot extend this to Σ^1_2 sets of reals. In the proof of the Corollary, it is crucial that ω, the set over which we apply separation, is in L. This is not longer the case for sets of reals. For example, the set of all reals is clearly Σ^1_2, but unless $\mathsf{V} = \mathsf{L}$, it is not contained in L.