Homework 6 for **MATH 435**

Due: Monday Oct 18

Problem 1

Book, p. 190, Exercise 2.95
(Give the reasons.)

Problem 2

This exercise lets you practice the *first isomorphism theorem*, which is of fundamental importance.

(a) Show that every cyclic group is a homomorphic image of $\mathbb{Z}$. That is, given G cyclic find a homomorphism $f : \mathbb{Z} \to G$ such that $\mathrm{im}(f) = G$.

(b) Using (a) and the first isomorphism theorem, give a direct proof that every cyclic group is isomorphic to either $\mathbb{Z}$ or some $\mathbb{Z}/n\mathbb{Z}$.

Problem 3

This is an exercise for the *second isomorphism theorem*:

Book, p. 191, Exercise 2.105

Moreover, give a non-trivial concrete example of such H, K, and G. The more 'complicated' your choice of H, K, G, the better!

Problem 4

This exercise introduces the group Q of *quaternions*, which has many interesting properties and applications. It is defined in the book on page 167. Read this definition and example 2.98, and also browse a little on the internet to learn more about Q.

Then do

Book, p. 191, Exercise 2.104.

(You may also want to look at Exercise 2.86 on page 170. You are not required to solve it, but the facts stated there can help you solve this exercise.)

Problem 5

A group G is called *solvable* if there exists a sequence of subgroups

$$G = H_0 \geqslant H_1 \geqslant \cdots \geqslant H_r = \{1\},$$

such that H_{i+1} is normal in H_i for $i = 1, \ldots, r-1$, and H_i/H_{i+1} is abelian. Show that if G is a group, K is a normal subgroup of G, and both K, G/K are solvable, then G is solvable.

(*Hint*: Show as a lemma that if $f : G \to G'$ is a homomorphism and onto, and $K' \triangleleft G'$, then $K := f^{-1}(K')$ is normal in G.)

Remark: Decomposing a group into descending sequences of subgroups is an important principle, which will encounter again later.