Measure Spaces

Measure space $(X, \mathcal{A}, \mu)$

set $\quad$ σ-algebra $\quad$ measure

σ-algebra: family $\mathcal{A}$ of subsets of X s.t.

- $\emptyset, X \in \mathcal{A}$
- closed under complements: $A \in \mathcal{A} \implies X \setminus A \in \mathcal{A}$
- —"— countable unions: $(A_i)_{i \in \mathbb{N}}, A_i \in \mathcal{A} \implies \bigcup_i A_i \in \mathcal{A}$

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Borel Sets

Important σ-algebra: **Borel sets**

smallest σ-algebra containing all open sets

We also say: σ-algebra generated by open sets

generating the Borel σ-algebra

more and more complicated sets

- Start with open sets → put into σ-algebra
- If open sets are in σ-algebra, the complements of open sets must be in there, too
  → put closed sets into σ-algebra
- By closure under unions, countable unions of closed sets must be in σ-algebra → put those in

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This process builds up the Borel sets step-by-step.

Exa: $\mathbb{R}$ with the topology generated by open intervals (a, b)

Is $\mathbb{Q}$ a Borel set?

YES: $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ ← countable union

each singleton set $\{q\}$ is closed: complement $\mathbb{R} \setminus \{q\}$ is open

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Measures

$$\mu : \mathcal{A} \longrightarrow \mathbb{R}^{\geq 0} \cup \{\infty\}$$

σ-algebra

- $\mu(\emptyset) = 0$

countably additive

- $(A_i)_{i\in\mathbb{N}}$, $A_i \in \mathcal{A}$ pairwise disjoint

$$\Longrightarrow \mu\Big(\bigcup_i A_i\Big) = \sum_i \mu(A_i)$$

Types of Measures

$(X, \mathcal{A}, \mu)$ measure space

- Borel measure : $\mathcal{A}$ is a Borel σ-algebra
- finite measure : $\mu(X) < \infty$
- probability measure : $\mu(X) = 1$
- σ-finite measure : $X = \bigcup_{i \in \mathbb{N}} X_i$, $\mu(X_i) < \infty$

Constructing Borel Measures

Caratheodory Extension Thm: To specify a measure on a σ-algebra, it suffices to specify it on a simpler set-family, an algebra.

again, meaning the smallest σ-algebra containing the ring.

The measure then extends to the $\rightarrow$ σ-algebra generated by the algebra.

extension is unique if μ is σ-finite

Algebra : Family $\mathcal{R}$ of subsets of X s.t.

closure under complements and finite unions

- $\emptyset \in \mathcal{R}$
- $A, B \in \mathcal{R} \implies A \cup B \in \mathcal{R}$
- $A \in \mathcal{R} \implies X \setminus A \in \mathcal{R}$

Goal: We want to specify Borel measures on a sequence space $A^{\mathbb{N}}$.

Can we find a ring in $A^{\mathbb{N}}$ that generates the Borel sets in $A^{\mathbb{N}}$?