Lecture 5: Borel Sets

Topologically, the Borel sets in a topological space are the σ-algebra generated by the open sets. One can build up the Borel sets from the open sets by iterating the operations of complementation and taking countable unions. This generates sets that are more and more complicated, which is refelcted in the *Borel hierarchy*. The complexity is reflected on the logical side by the number of quantifier changes needed to define the set. There is a close connection between the arithmetical and the Borel hierarchy.

Definition 5.1: Let X be a set. A **σ-algebra** $\mathcal{S}$ on X is a collection of subsets of X such that $\mathcal{S}$ is closed under complements and countable unions, that is

- if $A \in \mathcal{S}$, then $X \setminus A \in \mathcal{S}$, and
- if $(A_n)_{n\in\mathbb{N}}$ is a sequence of sets in $\mathcal{S}$, then $\bigcup_n A_n \in \mathcal{S}$,

If the enveloping space X is clear, we use $\neg A$ to denote the complement of A in X.

It is easy to derive that a σ-algebra is also closed under the following set-theoretic operations:

- *Countable intersections* – we have $\bigcap A_n = \neg \bigcup_n \neg A_n$.
- *Differences* – we have $A \setminus B = A \cap \neg B$.
- *Symmetric differences* – we have $A \triangle B = (A \cap \neg B) \cup (\neg A \cap B)$.

Definition 5.2: Let $(X, \mathcal{O})$ be a topological space. The collection of **Borel sets** in X is the smallest σ-algebra containing the open sets in $\mathcal{O}$.

One, of course, has to make sure that this collection actually exists. For this, note that the intersection of any collection of σ-algebras is again a σ-algebra, so the Borel sets are just the intersection of all σ-algebras containing $\mathcal{O}$. (Note the the full power set of X is such a σ-algebras, so we are not taking an empty intersection.)

This definition of Borel sets is rather *"external"*. It does not give us any idea what Borel sets *look like*. One can arrive at the family of Borel sets also through a construction from "within". This reveals more structure and gives rise to the *Borel hierarchy*.

The Borel hierarchy

We will restrict ourselves from now on to Polish spaces, to ensure that every closed set is a countable intersection of open sets (see exercises).

To generate the Borel sets, we start with the open sets. By closing under complements, we obtain the closed sets. We also have to close under countable unions. The open sets are already closed under this operation, but the closed sets are not. Countable unions of closed sets are classically known as F_σ sets. Their complements, i.e. countable intersections of open sets, are the G_δ *sets*. We can continue this way and form the $F_{\sigma\delta}$ sets – countable intersections of F_σ sets – the $G_{\delta\sigma}$ sets – countable unions of G_δ sets – and so on. It is obvious that the $\sigma\delta$-notation soon becomes rather impractical, and hence we replace it by something much more convenient, and much more *suggestive*, as we will see later.

Definition 5.3 (Borel sets of finite order)**:** Let X be a Polish space. We inductively define the following collection of subsets of X.

$$\begin{aligned}
&\mathbf{\Sigma}^0_1(X) = \{U : U \subseteq X \text{ open }\}\\
&\mathbf{\Pi}^0_n(X) = \{\neg A : A \in \mathbf{\Sigma}^0_n(X)\} = \neg\mathbf{\Sigma}^0_n(X)\\
&\mathbf{\Sigma}^0_{n+1}(X) = \{\bigcup_k A_k : A_k \in \mathbf{\Pi}^0_n(X)\}
\end{aligned}$$

Hence the open sets are precisely the sets in $\mathbf{\Sigma}^0_1$, the closed sets are the sets in $\mathbf{\Pi}^0_1$, the F_σ sets from the class $\mathbf{\Sigma}^0_2$ etc. If it is clear what the underlying space X is, we drop the reference to it and simply write $\mathbf{\Sigma}^0_n$ and $\mathbf{\Pi}^0_n$. Besides, we will say that a set $A \subseteq X$ *is* (or *is not*) $\mathbf{\Sigma}^0_n$ or $\mathbf{\Pi}^0_n$, respectively.

Does the collection of all $\mathbf{\Sigma}^0_n$ and $\mathbf{\Pi}^0_n$ exhaust the Borel sets of X? We will see that the answer is no. We have to extend our inductive construction into the transfinite and consider classes $\mathbf{\Sigma}^0_\xi$, where ξ is a countable infinite ordinal.

The Borel sets of finite order

We fix a Polish space X. We want to establish the basic relationships between the different classes $\mathbf{\Sigma}^0_n$ and $\mathbf{\Pi}^0_m$ for X.

It is clear that $\mathbf{\Sigma}^0_1 \nsubseteq \mathbf{\Pi}^0_1$ and $\mathbf{\Pi}^0_1 \nsubseteq \mathbf{\Sigma}^0_1$. Furthermore, it follows from the definitions that $\mathbf{\Pi}^0_n \subseteq \mathbf{\Sigma}^0_{n+1}$ and $\mathbf{\Sigma}^0_n \subseteq \mathbf{\Pi}_{n+1}$.

Lemma 5.4: *In a Polish metric space (X, d), every open set is an F_σ set.*

Proof. Let $D = \{x_1, x_2, \dots\} \subseteq X$ be a countable dense subset, and assume $U \subseteq X$ is open. For any $\varepsilon > 0$, if $\delta < \varepsilon$, then $\overline{U_\delta(x)} \subseteq U_\varepsilon(x)$ for any $x \in X$. Let $x_{i(1)}, x_{i(2)}, \dots$ and $\varepsilon_1, \varepsilon_2, \dots$ be such that

$$U = \bigcup_n U_{\varepsilon_n}(x_{i(n)}).$$

For each $n \geq 1$, let $(\delta_k^{(n)})$ be such that $\delta_k^{(n)} < \delta_{k+1}^{(n)} < \cdots < \varepsilon_i$, and $\delta_k^{(n)} \to \varepsilon_n$. Then

$$U = \bigcup_k \bigcup_n \overline{U_{\delta_k^{(n)}}(x_{i(n)})}.$$

The set on the right hand side is a countable union of closed sets. $\square$

Corollary 5.5: $\mathbf{\Sigma}_1^0 \subseteq \mathbf{\Sigma}_2^0$ *and* $\mathbf{\Pi}_1^0 \subseteq \mathbf{\Pi}_2^0$.

The second statement follows by passing to complements: If F is closed,

$$F = \neg\neg F = \neg \bigcup F_n = \bigcup \neg F_n,$$

where the F_n are closed.

There are also sets that can be both $\mathbf{\Sigma}_2^0$ and $\mathbf{\Pi}_2^0$, but neither $\mathbf{\Sigma}_1^0$ nor $\mathbf{\Pi}_1^0$. For example, consider the half-open interval $[0, 1)$.

$$[0,1) = \bigcup_n [1, 1 - 1/n] = \bigcap_m (-1/n, 1).$$

Therefore, it makes sense to define the **hybrid classes**

$$\mathbf{\Delta}_n^0 = \mathbf{\Sigma}_n^0 \cap \mathbf{\Pi}_n^0.$$

Using induction, we can extend the inclusions in a straightforward way to higher n.

Theorem 5.6 (Weak Hierarchy Theorem)**:**

$$\begin{array}{ccccccccccccc}
 & & \mathbf{\Sigma}_1^0 & & & & \mathbf{\Sigma}_2^0 & & & & \mathbf{\Sigma}_3^0 & & \\
 & \subseteq & & \subseteq & & \subseteq & & \subseteq & & \subseteq & & \subseteq & \\
\mathbf{\Delta}_1^0 & & & & \mathbf{\Delta}_2^0 & & & & \mathbf{\Delta}_3^0 & & & & \cdots \\
 & \subseteq & & \subseteq & & \subseteq & & \subseteq & & \subseteq & & \subseteq & \\
 & & \mathbf{\Pi}_1^0 & & & & \mathbf{\Pi}_2^0 & & & & \mathbf{\Pi}_3^0 & &
\end{array}$$

We also want show that the inclusions are proper. For the first two levels, this can be done by explicit counterexamples. Any countable set is in $\mathbf{\Sigma}^0_2$, since a singleton set is closed, and a countable set is a countable union of singletons. However, there are countable sets that are neither open nor closed, e.g. $\{1/n : n \geq 1\}$. The complement is consequently a $\mathbf{\Pi}^0_2$ set that is neither open nor closed. Furthermore, the rationals $\mathbb{Q}$ give an example of a $\mathbf{\Sigma}^0_2$ set that is not $\mathbf{\Pi}^0_2$. This will be shown later using the concept of *Baire category*.

It is much harder to find specific examples for the higher levels, e.g. a $\mathbf{\Sigma}^0_5$ set that is not $\mathbf{\Sigma}^0_4$. This separation will be much facilitated by the introduction of a logical/definability framework for the Borel sets. Therefore, we defer the proof for a while.

Examples of Borel sets – Continuity points of functions

Theorem 5.7 (Young)**:** *Let $f : X \to Y$ be a mapping between Polish spaces. Then*

$$C_f = \{x : f \text{ is continuous at } x\}$$

is a $\mathbf{\Pi}^0_2$ (i.e. G_δ) set.

Proof. It is not hard to see that f is continuous at a if and only if for any $\varepsilon > 0$,

$$\exists \delta > 0\, \forall x, y\ [x, y \in U_\delta(a) \Rightarrow d(f(x), f(y)) < \varepsilon]. \tag{$*$}$$

Given $\varepsilon > 0$, let

$$C_\varepsilon = \{a : (*) \text{ holds at } a \text{ for } \varepsilon\}.$$

We claim that C_ε is open. Suppose $a \in C_\varepsilon$. Choose a suitable δ that witnesses that $a \in C_\varepsilon$. We show $U_\delta(a) \subseteq C_\varepsilon$. Let $b \in U_\delta(a)$. Choose δ^* so that $U_{\delta^*}(b) \subseteq U_\delta(a)$. Then

$$x, y \in U_{\delta^*}(b) \Rightarrow x, y \in U_\delta(a) \Rightarrow d(f(x), f(y)) < \varepsilon.$$

Notice further that $\varepsilon > \varepsilon^*$ implies $C_\varepsilon \supseteq C_{\varepsilon^*}$. Hence we can represent C_f as

$$C_f = \bigcap_{n \in \mathbb{N}} C_{1/n},$$

a countable intersection of open sets. □

The function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 1 & x \text{ irrational}, \\ 0 & x = 0, \\ 1/q & x = p/q,\ p \in \mathbb{Z},\ q \in \mathbb{Z}^{>0},\ p, q \text{ relatively prime} \end{cases}$$

is a function that is continuous at every irrational, discontinuous at every rational number. As noted above, the rationals are a $\mathbf{\Sigma}^0_2$ set that is not $\mathbf{\Pi}^0_2$. Hence there cannot exist a function $g : \mathbb{R} \to \mathbb{R}$ that is discontinuous at exactly the irrationals.

We finish this lecture by showing that Young's Theorem can be reversed.

Theorem 5.8: *Given a $\mathbf{\Pi}^0_2$ subset A of a perfect Polish space X, there exists a mapping $f : X \to \mathbb{R}$ such that f is continuous at every point in A, and discontinuous at every other point, i.e. $C_f = A$.*

Proof. Fix a countable dense subset $D \subseteq X$. We first deal with the easier case that A is open. Let

$$f(x) = \begin{cases} 0 & x \in A \text{ or } x \in \neg \overline{A} \cap D, \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that f is continuous on A. Now assume $x \notin A$. If $x \notin \overline{A}$, then there exists $U_\varepsilon(x) \subseteq \neg \overline{A}$. Any $U_{\varepsilon^*}(x) \subseteq U_\varepsilon(x)$ contains points from both D and $\neg D$, so it is clear that f is not continuous at x. Finally, let $x \in \neg A \setminus A$. Then $f(x) = 1$, but points of A are arbitrarily close, where f takes value 0.

Now we extend this approach to general $\mathbf{\Pi}^0_2$ sets. Suppose

$$A = \bigcap_n G_n, \qquad G_n \text{ open.}$$

By replacing G_n with $G^*_n = G_1 \cap \cdots \cap G_n$, we can assume that

$$X = G_0 \supseteq G_1 \supseteq G_2 \supseteq G_3 \supseteq \ldots$$

The idea is to define f_n as above for each G_n and then "amalgamate" the f_n is a suitable way. Assume for each n, $f_n : X \to \mathbb{R}$ is defined as above such that $C_{f_n} = G_n$. Let (b_n) be a sequence of positive real numbers such that for all n,

$$b_n > \sum_{k>n} b_k,$$

for example, $b_n = 1/n!$. We now form the series

$$f(x) = \sum_n b_n f_n(x).$$

Since $|f_n(x)| \le 1$, $|f(x)| \le \sum_n b_n < \infty$. Furthermore, (f_n) converges uniformly to f, for

$$|f(x) - f_n(x)| \le \sum_{k>n} b_k < b_n,$$

and the last bound is independent of x and converges to 0.

It follows by uniform convergence that if each f_n is continuous at x, f is continuous on x, too. Hence f is continuous on A.

Now assume $x \notin A$. Then there exist n such that $x \in G_n \setminus G_{n+1}$. Hence

$$f_0(x) = \cdots = f_n(x) = 0.$$

Again, we distinguish two cases. First, assume $x \notin \overline{G_{n+1}}$. Then there exists $\delta > 0$ such that $U_\delta(x) \subseteq \neg G_{n+1}$. This also implies $U_\delta(x) \subseteq \neg G_k$ for any $k \ge n+1$. Besides, since G_n is open, we can chose δ sufficiently small so that $U_\delta(x) \subseteq G_n$. For $y \in \neg D \cap U_\delta(x)$ we have $f_k(y) = 1$ for all $k \ge n+1$, and hence $f(y) = \sum_{k>n} b_k f_k(y) > 0$. On the other hand, if $y \in D \cap U_\delta(x)$, then $f_k(y) = 0$ for all $k \ge n+1$, and also $f_0(y) = \cdots = f_n(y) = 0$, since $y \in G_n$, and thus $f(y) = 0$. Hence there are points arbitrarily close to x whose f-values differ by a constant lower bound, which implies f is not continuous in x.

Finally, suppose $x \in \overline{G_{n+1}}$. Then $f_{n+1}(x) = 1$ and hence $f(x) \ge b_{n+1} > 0$. On the other hand, for any $y \in G_{n+1}$, $f(y) \le \sum_{k>n+1} b_k < b_{n+1} = f(x)$. That is, there are points arbitrarily close to x whose f-value differs from $f(x)$ by a constant lower bound. Hence f is discontinuous at x in this case, too. $\square$