Homework 2 for
MATH 574, Topics in Logic

Due: Wednesday Feb 12

Problem 1

Let μ be a Borel probability measure on a sequence space $A^{\mathbb{N}}$, A finite. A sequence $x \in A^{\mathbb{N}}$ is called an *atom* of μ if $\mu(\{x\}) > 0$. Show that the set of all atoms of μ is a Borel set.

Problem 2

Prove the following lemma from Lecture 1-5:

LEMMA: Suppose A is a finite alphabet. Let μ be a function defined on cylinders so that for all $\sigma \in A^{<\mathbb{N}}$,

$$\mu[\sigma] = \sum_{a \in A} \mu[\sigma^\frown a].$$

Suppose $\{\sigma_1, \dots, \sigma_n\}$ and $\{\tau_1, \dots, \tau_m\}$ are *prefix-free* sets of strings so that

$$\bigcup_{i=1}^{n} [\sigma_i] = \bigcup_{j=1}^{m} [\tau_j].$$

Then

$$\sum_{i=1}^{n} \mu[\sigma_i] = \sum_{j=1}^{m} \mu[\tau_j].$$

Problem 3

Let (P_n) be the distribution of a discrete A-valued process (X_k), and suppose the (P_n) satisfy the consistency condition

$$P_n(\sigma) = \sum_{a \in A} P_{n+1}(\sigma^\frown a).$$

The *conditional distribution on A of level k* is defined as

$$P(a|a_0 a_1 \dots a_{k-1}) = \mathrm{Prob}(X_k = a | X_0 = a_0, \dots, X_{k-1} = a_{k-1}).$$

The process (X_n) is called *independent* if for all k, a and A-strings $a_0 \dots a_{k-1}$,

$$P(a|a_0 \dots a_{k-1}) = \mathrm{Prob}(X_k = a).$$

Moreover, (X_n) is called *identically distributed* if for all $k \geq 0$, $a \in A$

$$\mathrm{Prob}(X_k = a) = \mathrm{Prob}(X_0 = a).$$

Show that a binary process (X_n) is independent and identically distributed (i.i.d.) if and only if there exists a $p \in [0,1]$ such that, if μ denotes the Kolmogorov extension of the process to $\{0,1\}^{\mathbb{N}}$,

$$\mu[\sigma] = p^N (1-p)^{|\sigma| - N},$$

where $N = \#\{i : \sigma(i) = 1\}$.