Homework 10 for MATH 104

Due: Tuesday, December 5, 9:30am in class

Problem 1

Suppose $f : \mathbb{R} \to \mathbb{R}$. Call x a *fixed point* of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \neq 1$ for all t, prove that f has at most one fixed point.

(b) Show that the function f defined by

$$f(t) = t + \frac{1}{1 + e^t}$$

satisfies $0 < f'(t) < 1$ for all t but has no fixed point.

(c) Show that if $|f'(t)| \leqslant M$ for all t and some $M < 1$, then f has a fixed point.

[*Hint*: Use homework 9.4.]

Problem 2

[Newton's method, 10P]
Suppose $f : [a, b]$ to$\mathbb{R}$ is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geqslant \delta > 0$ and $0 \leqslant f''(x) \leqslant M$ for all $x \in [a, b]$.

(a) Show that there exists a unique point $\xi \in (a, b)$ such that $f(\xi) = 0$.

(b) Let $x_1 \in (\xi, b)$ and define (x_n) by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Give a geometrical interpretation of this definition, by means of a tangent to the graph of f.

(c) Prove that $x_{n+1} < x_n$ and that $\lim_n x_n = \xi$.

(d) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

(e) Set $A = M/2\delta$ and deduce that

$$0 \leqslant x_{n+1} - \xi \leqslant \tfrac{1}{A}[A(x_1 - \xi)]^{2^n}$$

(f) Interpret Newton's method as a search for a fixed point of some function.

Problem 3

Let F be the Cantor set (as defined for example in Ross, p.85). Let $f : [0, 1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in F, \\ 0 & \text{if } x \notin F. \end{cases}$$

Show that f is Riemann integrable.