Lesson 1

Sequence Spaces, Random Variables, And Probailistic Processes

Math 574 - Topics in Logic
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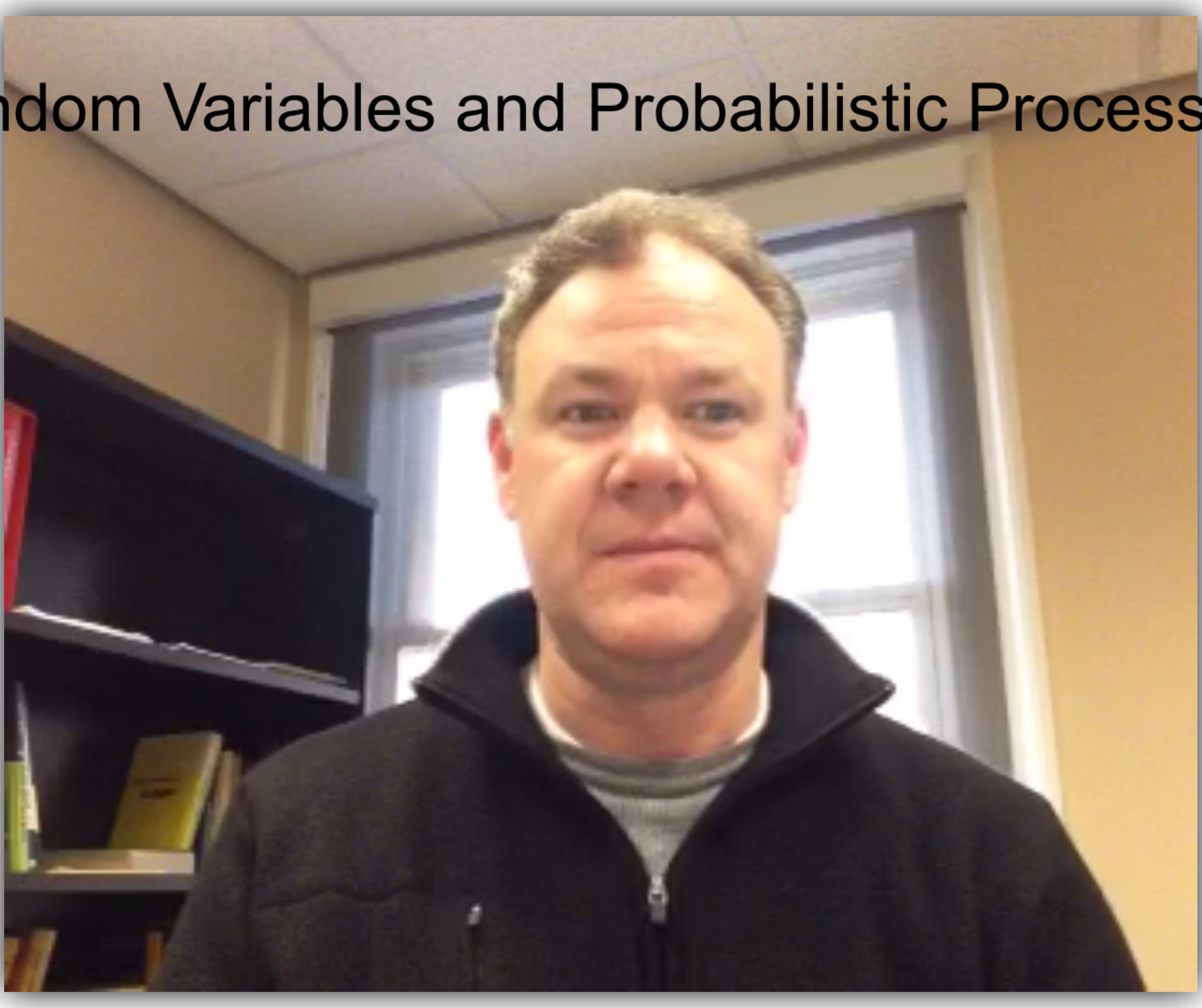
Random Variables and Probabilistic Processes



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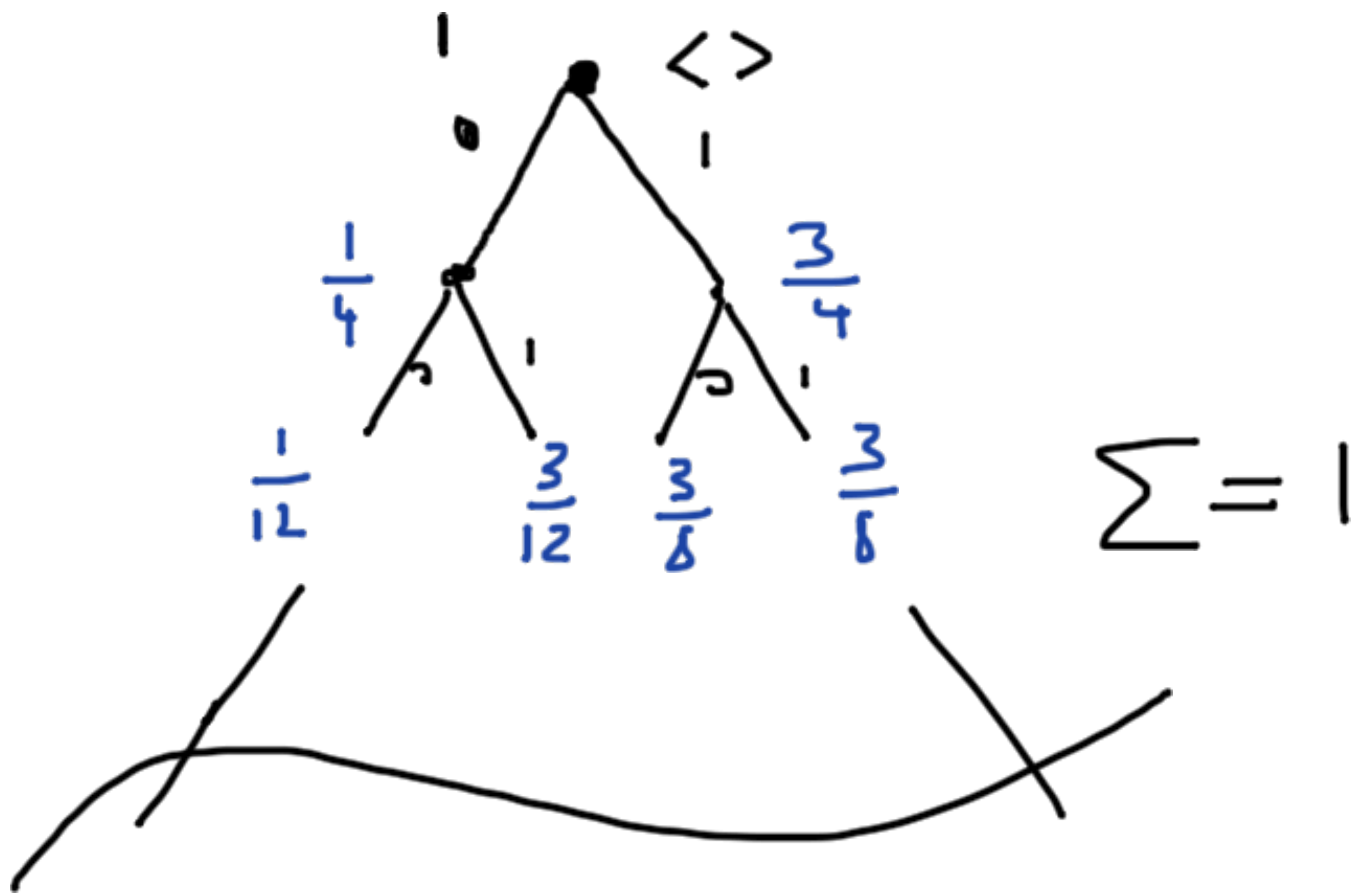
$$\mu[\sigma] = \mu[\sigma^\frown 0] + \mu[\sigma^\frown 1]$$



$$\mu[\sigma] = 2^{-|\sigma|}$$

uniform distribution.

Random Variables

$(\Omega, \mathcal{A}, \mu)$ measure space

$X : \Omega \to \mathbb{R}$ is called **measurable** if

f. all Borel $B \subseteq \mathbb{R}$, $X^{-1}(B) \in \mathcal{A}$

with respect to standard topology on $\mathbb{R}$.

A measurable function $X : \Omega \to \mathbb{R}$ is also called a **random variable**.

X induces a probability distribution on $\mathbb{R}$:

$$P(B) = \mu\left(X^{-1}(B)\right)$$

notation: $P(X \in B)$

Discrete random variables:

X takes values in a discrete set (finite, $\mathbb{N}$, or $\mathbb{Z}$)

binary random var.: $X : \Omega \to \{0,1\}$

hence induces prob. distr. on $\{0,1\}$

Stochastic process:

Sequence $X_0, X_1, X_2, \ldots$ of random variables defined over common probability space $(\Omega, \mathcal{A}, \mu)$

Discrete process: all X_i take values in the same discrete set A

The Distribution of a Process

$X_0, X_1, X_2, \ldots$ A-valued process, A discrete

notation: A-process

For any n, $X_0, X_1, \ldots, X_{n-1}$ induce a joint distribution

$$P_n(X_0 = a_0, \ldots, X_{n-1} = a_{n-1}) = \mu\left(X_0^{-1}(\{a_0\}) \cap \cdots \cap X_{n-1}^{-1}(\{a_{n-1}\})\right)$$

$(a_0, \ldots, a_{n-1} \in A)$

The sequence $(P_n : n \geq 0)$ is called the distribution of the process

Write $P_n(\underbrace{a_0 \cdots a_{n-1}}_{\text{string over } A})$ for $P_n(X_0 = a_0, \ldots, X_{n-1} = a_{n-1})$

A^n

To keep things simple, assume again $A = \{0, 1\}$.

For any string $a_0 \cdots a_{n-1}$,

$$X_0^{-1}(\{a_0\}) \cap \cdots \cap X_{n-1}^{-1}(\{a_{n-1}\}) \cap X_n^{-1}(\{0\})$$

is disjoint from

$$X_0^{-1}(\{a_0\}) \cap \cdots \cap X_{n-1}^{-1}(\{a_{n-1}\}) \cap X_n^{-1}(\{1\})$$

Since μ is a measure, we must have

$$(**) \qquad P_n(a_0 \cdots a_{n-1}) = P_{n+1}(a_0 \cdots a_{n-1} 0) + P_{n+1}(a_0 \cdots a_{n-1} 1)$$

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Hence the distributions (P_n) satisfy a consistency condition $(**)$ similar to $(*)$ in the previous lecture.

$$\mu[\overline{G}] = \mu[\overline{G}^\frown 0] + \mu[\overline{G}^\frown 1]$$

This similarity is not a coincidence, due to the

<u>Kolmogorov Extension Theorem</u>

Note: In the previous development, it does not really matter what the underlying probability space $(\Omega, \mathcal{A}, \mu)$ is.

All that matters is the distribution of the process

K. Extension Thm: Constructs an underlying space from the distribution of a process satisfying consistency condition (**)

This space is a measure on $A^{\mathbb{N}}$

THM: Suppose $(P_n)_{n \in \mathbb{N}}$ is a sequence of prob. measures, where P_n is a measure on A^n.

Suppose further that the P_n satisfy the consistency condition

$$P_n(\sigma) = \sum_{i \in A} P_{n+1}(\sigma^\frown i). \qquad (**)$$

Then there exists a unique Borel prob. measure μ on $A^{\mathbb{N}}$ s.t.

$$\mu[\sigma] = P_n(\sigma) \qquad \text{f. all } \sigma \in A^{<\mathbb{N}}$$

Sequences vs Processes

A-valued sequences

$x = x_0 x_1 x_2 \cdots$

$x_i \in A$

measure on $A^{\mathbb{N}}$:

measure on $[\sigma]$

+ consistency $(*)$

↳ algebra of finite unions of $[\sigma]$

Carathedory

A-valued rand. var.

$X_0 \, X_1 \, X_2$

joint distributions

P_n on A^n

+ consistency $(**)$

Kolmogorov

μ on $A^{\mathbb{N}}$