# Homework 7 for MATH 104

Due: Tuesday, October 31, 9:30am in class

**Problem 1**

(a) Suppose $\sum a_n x^n$ has finite radius of convergence $R$ and that $a_n \geqslant 0$ for all $n$. Show that if the series converges at $R$, then it also converges at $-R$.

(b) Give an example of a power series whose interval of convergence is exactly $(-1, 1]$.

**Problem 2**

Show that $\sum_{n=1}^{\infty} \frac{x^n}{n^2 2^n}$ has radius of convergence 2 and that the series converges uniformly to a continuous function on $[-2, 2]$.

**Problem 3**

(a) Let $(f_n)$ be a sequence of continuous functions $f_n : S \to \mathbb{R}$, $S \subseteq \mathbb{R}$ which converges uniformly on $S$. Show that if $(x_n)$ is a sequence in $S$ such that $x_n \to x \in S$, then $\lim_n f_n(x_n) = f(x)$.

(b) Is the converse of (a) true, that is, is it true that if $f_n(x_n)$ converges to $f(x)$ whenever $(x_n)$ converges to $x$ in $S$, then $(f_n)$ converges uniformly to $f$?

**Problem 4**

Let $\mathcal{B}(\mathbb{R})$ be the set of all bounded functions $f : \mathbb{R} \to \mathbb{R}$, i.e. there exists a real number $M \geqslant 0$ such that $|f(x)| \leqslant M$ for all $x \in \mathbb{R}$.

(a) Define a function $d : \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) \to \mathbb{R}^{\geqslant 0}$ by

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}.$$

Show that $d$ defines a metric on $\mathcal{B}(\mathbb{R})$.

(b) Show that a sequence $(f_n)$ of bounded real functions $f_n : \mathbb{R} \to \mathbb{R}$ converges uniformly to a function $f : \mathbb{R} \to \mathbb{R}$ if and only if $f_n \to f$ with respect to the metric $d$. Show that the limit function $f$ is bounded, too.

(c) Show that the metric space $(\mathcal{B}(\mathbb{R}), d)$ is complete.

(d) Is $(\mathcal{B}(\mathbb{R}), d)$ compact?