Infinite Paths Through Trees

$T \subseteq A^{<\mathbb{N}}$ tree

$x \in A^{\mathbb{N}}$ is a path through T if

f. all n, $x \upharpoonright n \in T$

first n entries of x:

$x \upharpoonright n = x_0 x_1 \dots x_{n-1}$

T^∞ = set of all infinite paths through T.

THM: $F \subseteq A^{\mathbb{N}}$ is closed iff there ex. tree $T \subseteq A^{<\mathbb{N}}$ s.t.

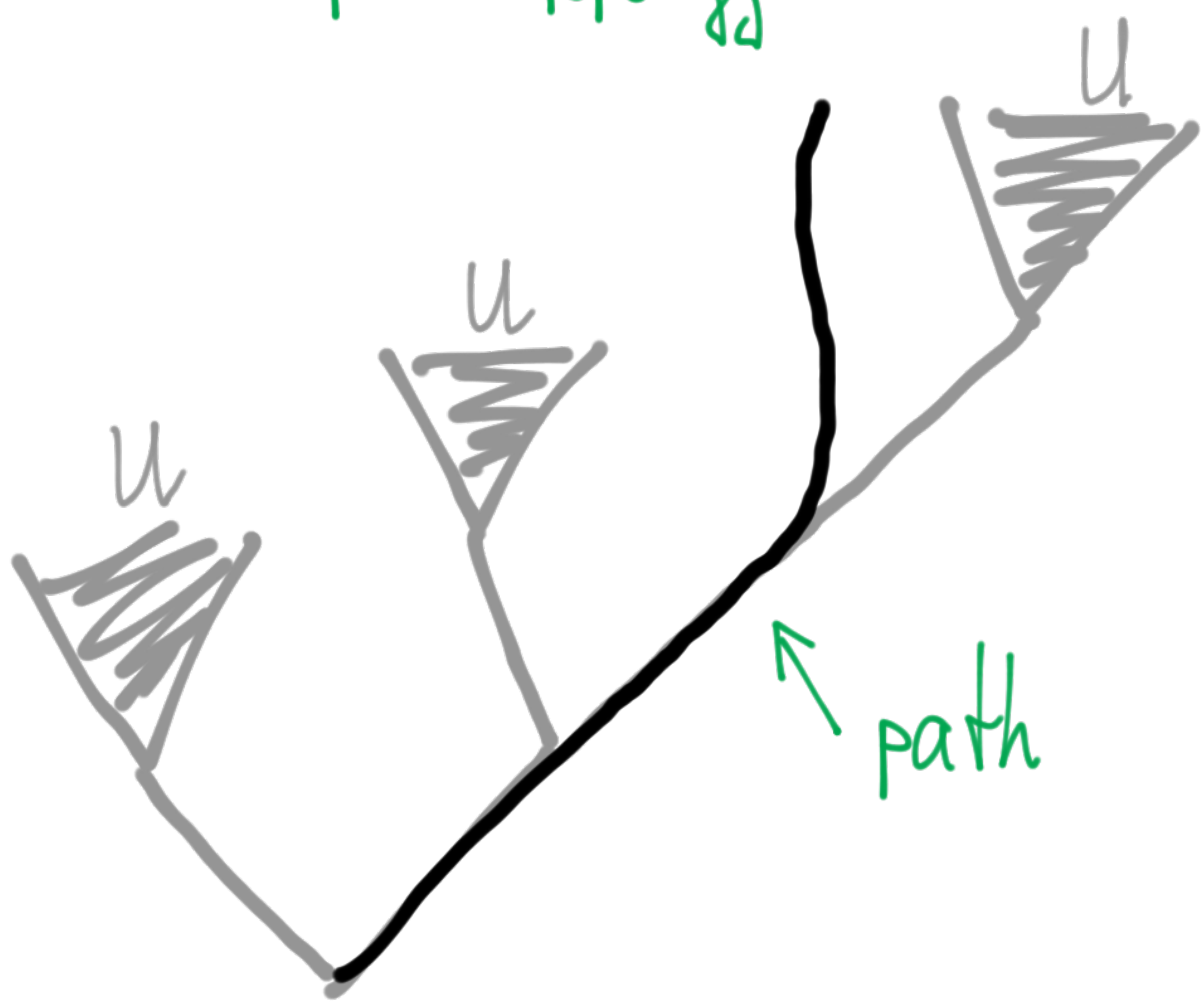
$$F = \overline{T}^{\infty}$$

Proof: $\Rightarrow$: Sup. F is closed, so $U = A^{\mathbb{N}} \setminus F$ is open.

There ex. $S \subseteq A^{\mathbb{N}}$ s.t. $U = \bigcup_{\sigma \in S} [\sigma]$

Notation: $[S]$

Recall the cylinders form a basis for the topology



So $x \in F \iff x \notin U \iff \forall \sigma \in S \quad \sigma \not\subseteq x$

Define a tree T by letting

$$\tau \in T :\iff \neg \exists \sigma \in S \quad \sigma \subseteq \tau$$

Then

$$\begin{aligned} x \in T^\infty \iff \forall n \;\; x \restriction n \in T &\iff \forall n \; \neg \exists \sigma \in S \quad \sigma \subseteq x \restriction n \\ &\iff \forall n \; \forall \sigma \in S \quad \sigma \not\subseteq x \restriction n \\ &\iff \forall \sigma \in S \quad \sigma \not\subseteq x \\ &\iff x \notin \underset{U}{\underset{\shortparallel}{[S]}} \iff x \in F \end{aligned}$$

$\Leftarrow$: Suppose $F = [T]^\infty$, T tree.

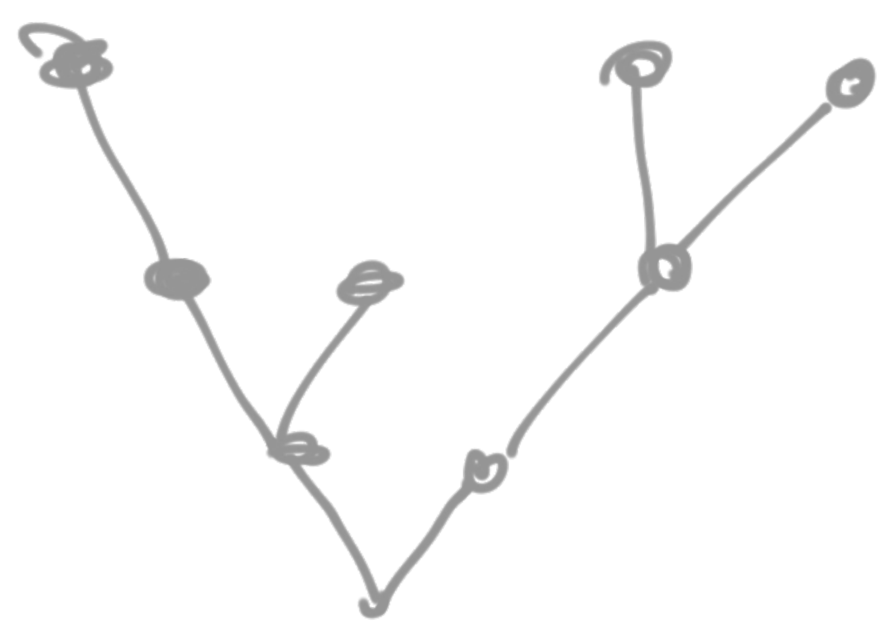
We show that $U := A^{\mathbb{N}} \setminus F$ is open

Define $S \subseteq A^{<\mathbb{N}}$ by

$$\sigma \in S \iff \sigma \notin T \ \&\ \sigma^- \in T$$

σ^- with last entry deleted

$\sigma^- = \sigma_0 \sigma_1 \cdots \sigma_{|\sigma|-2}$



T

Then: $x \in U \iff x \notin T^\infty \iff \exists n_0$ minimal s.t.

$$x\restriction_{n_0} \notin T$$

$$\iff x\restriction_{n_0-1} \in T$$

(provided $T \neq \emptyset$)

$$\iff x\restriction_{n_0} \in S$$

$$\iff x \in [S]$$

Hence we have $U = [S]$, i.e. U is a union of cylinders, hence U is open.