Lecture 20: Π^1_1 Sets of Natural Numbers

In this lecture we consider Π^1_1 sets of natural numbers. They are defined just like their counterparts in $\mathbb{N}^{\mathbb{N}}$. Using the Kleene Normal Form, a set $X \subseteq \mathbb{N}$ is Π^1_1 if there exists a bounded formula $\varphi(x, y, \beta,)$ such that

$$x \in X \quad \Leftrightarrow \quad \forall \beta \exists y \, \varphi(x, y, \beta).$$

One can show that equivalently, there exists a recursive relation $R(x, y, \beta)$ such that

$$x \in X \quad \Leftrightarrow \quad \forall \beta \exists y \, R(x, y, \beta).$$

Σ^1_1 sets are given analogously.

Recursive relations are those that are Σ^0_1 and Π^0_1 at the same time, i.e. that are Δ^0_1. There are recursive relations that are *not* definable by bounded formulas. Hence the above equivalence requires a little bit of work, for which we refer to Kanamori [2003].

On the other hand, one can show that a relation $R(x, y, \beta)$ is recursive if and only there exists an e such that for all x, y, β, $\Phi^\beta_e(x, y)\downarrow$ and

$$R(x, y, \beta) \quad \Leftrightarrow \quad \Phi^\beta_e(x, y) = 0.$$

The truth of the right hand side depends only on a finite initial segment of β (the *use principle*). This is reflected by the **Kleene T-predicate**. This is a recursive predicate T such that, for some recursive function U,

$$\Phi^\beta_e(x, y) \simeq 0 \quad \Leftrightarrow \quad \exists s \, [T(e, x, y, s, \beta \mid s) \,\&\, U(s) = 0].$$

Hence we have (using quantifier contraction) that $X \subseteq \mathbb{N}$ is Π^1_1 if and only if there exists a recursive predicate R^* such that

$$x \in X \quad \Leftrightarrow \quad \forall \beta \exists y \, R^*(x, y, \beta \mid y).$$

This allows us to derive a tree representation similar to the case of Baire space. Namely, let

$$\sigma \in S(x) \quad \Leftrightarrow \quad \forall i < |\sigma| \, \neg R^*(x, \sigma \mid i, i).$$

Then $S(x)$ is a recursive tree for each x, and

$$x \in X \quad \Leftrightarrow \quad S(x) \text{ is well-founded.}$$

Kleene's $\mathcal{O}$ and well-founded relations

The above normal form reduces deciding membership in a Π^1_1 set to deciding whether a recursive predicate is well-founded. We will now show that $\mathcal{O}$ can decide the latter question in a uniform way.

Let the e-th r.e. relation $R_e(x,y)$ be given by

$$x\,R_e\,y \quad\Leftrightarrow\quad R_e(x,y) \quad\Leftrightarrow\quad \varphi_e(x,y)\downarrow.$$

As before, the domain of R_e, $\mathrm{dom}(R_e)$ is given as

$$\mathrm{dom}(R_e) = \{x : \exists y\, R_e(x,y) \vee R_e(y,x)\}.$$

Note that $\mathrm{dom}(R_e)$ is r.e., too.

Let $\mathrm{WF}_{\mathbb{N}} = \{e : R_e \text{ is well-founded}\}$. We want to show that $\mathrm{WF}_{\mathbb{N}}$ reduces to $\mathcal{O}$. To this end we define, uniformly, an effective order-preserving mapping f from $\mathrm{dom}(R_e)$ into $\mathcal{O}$. We do this by effective transfinite recursion. Let $h(e,n)$ be a recursive function such that

$$R_{h(e,n)}(x,y) \quad\Leftrightarrow\quad R_e(x,n)\,\&\,R_e(y,n)\,\&\,R_e(x,y),$$

where $R_{h(e,n)}$ is empty if $n \notin \mathrm{dom}(R_e)$. $R_{h(e,n)}$ is the initial segment of R_e below n. Clearly the $R_{h(e,n)}$ are uniformly enumerable. Since we can enumerate R_e by enumerating the $R_{h(e,n)}$, the idea is to define f on the $R_{h(e,n)}$, and then extend f to R_e by transfinite recursion.

The images of the $R_{h(e,n)}$ will be r.e., too. Moreover, we can enumerate these images uniformly, obtaining an r.e. subset of $\mathcal{O}$. To extend our mapping f to R_e, we need an effective way to find, given an r.e. subset W of $\mathcal{O}$, an element of $\mathcal{O}$ "on top" of W.

The next lemma shows that we can do this in a uniform way.

Lemma 20.1: *There exists a recursive function g such that*

(a) *$g(e) \in O$ if and only if $W_e \subseteq \mathcal{O}$,*

(b) *If $g(e) \in \mathcal{O}$, then $|x|_{\mathcal{O}} < |g(e)|_{\mathcal{O}}$ for all $x \in W_e$.*

Proof. We use recursion along the $+_{\mathcal{O}}$ function, summing the elements of W_e. To ground the recursion, we first add 1 to W_e: Let $r(e)$ be recursive such that

$\mathrm{ran}(\varphi_{r(e)}) = W_e \cup \{1\}$ and $\varphi_{r(e)}(0) = 1$. Now define a recursive function s such that

$$\begin{aligned}\varphi_{s(e)}(0) &= \varphi_{r(e)}(0) = 1,\\ \varphi_{s(e)}(n+1) &= \varphi_{s(e)}(n) +_{\mathcal{O}} 2^{\varphi_{r(e)}(n+1)}.\end{aligned}$$

We put $g(e) = 3 \cdot 5^{s(e)}$.

We verify (a). Suppose $g(e) \in \mathcal{O}$. Then $\varphi_{s(e)}(n) \in \mathcal{O}$ for all n. It is not hard to show that $x +_{\mathcal{O}} y \in \mathcal{O}$ if and only if $x, y \in \mathcal{O}$. Therefore, $2^{\varphi_{r(e)}(n)} \in \mathcal{O}$ and hence $\varphi_{r(e)}(n) \in \mathcal{O}$ for all n. Now assume $W_e \subseteq \mathcal{O}$. It follows that for each n, $\varphi_{r(e)}(n) \in \mathcal{O}$. By the properties of $+_{\mathcal{O}}$, $\varphi_{s(e)}(n)$ for all n and $\varphi_{s(e)}(n) <_{\mathcal{O}} \varphi_{s(e)}(n+1)$. Hence $g(e) \in \mathcal{O}$.

For (b), suppose $g(e) \in \mathcal{O}$. By definition of g we have $g(e) >_{\mathcal{O}} 1$, so let $1 \neq a \in W_e$. We can choose $n > 0$ such that $\varphi_{r(e)}(n) = a$. By definition of g, $g(e) >_{\mathcal{O}} \varphi_{s(e)}(n)$ for all n. We have

$$\varphi_{s(e)}(n) = \varphi_{s(e)}(n-1) +_{\mathcal{O}} 2^a.$$

Therefore $2^a \leq_{\mathcal{O}} g(e)$ and thus $a <_{\mathcal{O}} g(e)$. $\square$

We have to deal with the possibility that $\mathrm{dom}(R_e)$ is empty, in wich case our recursion would get stuck at the very beginning and not return a value. We prevent this by dealing with this case explicitly. Let t be recursive such that

$$W_{t(b,e)} = \begin{cases} \emptyset & \text{if } R_e = \emptyset,\\ \{\varphi_b(h(e,n)) \colon n \in \mathbb{N}\} & \text{otherwise.}\end{cases}$$

Think of b as an index for f. We choose a recursive function k such that

$$\varphi_{k(b)}(e) \simeq g(t(e,b)).$$

Let c be a fixed point of k. We put

$$\begin{aligned}f(e) &= \varphi_c(e),\\ t(e) &= t(c,e).\end{aligned}$$

Then

$$W_{t(e)} = \begin{cases} \emptyset & \text{if } R_e = \emptyset,\\ \{f(h(e,n)) \colon n \in \mathbb{N}\} & \text{otherwise,}\end{cases}$$

and hence $f(e) = g(t(e))$.

Suppose R_e is well-founded. If $\mathrm{dom}(R_e) = \emptyset$, then $W_{t(e)} = \emptyset \subseteq \mathcal{O}$ and $f(e) \in O$ by the Lemma. If $R_e \neq \emptyset$, then it follows by induction that $R_{h(e,n)} \subseteq \mathcal{O}$ for all n, and by definition of f, $f(e) \in \mathcal{O}$, using Lemma 20.1.

If, on the other hand, $f(e) \in \mathcal{O}$, then, by Lemma 20.1, $|f(e)|_{\mathcal{O}} > |a|_{\mathcal{O}}$ for all $a \in W_{t(e)}$. But the elements of $W_{t(e)}$ are precisely the numbers $f(h(e,n))$. By transfinite induction on $<_{\mathcal{O}}$, this means that each $R_{h(e,n)}$ is well-founded. Hence R_e is well-founded.

Summing up, we have shown

Theorem 20.2: $\mathrm{WF}_{\mathbb{N}}$ *many-one reduces to* $\mathcal{O}$.

The proof of Theorem 20.2 also yields that $|f(e)|_{\mathcal{O}}$ bounds the rank $\rho(R_e)$ of R_e, provided R_e is well-founded. The rank of R_e in this case is simply the rank of the corresponding tree.

Corollary 20.3: *There exists a recursive function f such that if R_e is well-founded, then $\rho(R_e) \leq |f(e)|_{\mathcal{O}}$.*

Theorem 20.2 also lets us show that every recursive ordinal is constructive.

Proposition 20.4: *Every recursive ordinal is constructive.*

Proof. Suppose ξ is recursive. Let R be a recursive well-ordering of $\mathbb{N}$ of order-type ξ. Since a well-ordering is well-founded, the previous corollary yields an $x \in \mathcal{O}$ with $|x|_{\mathcal{O}} > \xi$ (namely $x = 2^{f(e)}$ for $R = R_e$). Hence ξ receives a notation and is thus constructive. $\square$

Kleene's $\mathcal{O}$ is Π^1_1-complete

We now use the previous result to show that $\mathcal{O}$ is *many-one complete* for all Π^1_1 subsets of $\mathbb{N}$. First, we establish that $\mathcal{O}$ is in fact a Π^1_1 set.

Proposition 20.5: *$\mathcal{O}$ and $<_{\mathcal{O}}$ are Π^1_1 sets.*

Proof. First note that $\mathcal{O} = \mathrm{dom}(<_{\mathcal{O}})$ and

$$x \in \mathrm{dom}(<_{\mathcal{O}}) \quad \Leftrightarrow \quad \exists y\,[x <_{\mathcal{O}} y \;\vee\; y <_{\mathcal{O}} x].$$

Since Π^1_1 sets are closed under projection along $\mathbb{N}$, $\exists^{\mathbb{N}}$, $<_{\mathcal{O}}$ being Π^1_1 implies that $\mathcal{O}$ is Π^1_1.

Consider the following predicate $A(X)$ of *reals* $X \subseteq \mathbb{N}$:

$$\begin{aligned} A(X) \quad \Leftrightarrow \quad & \forall x, y\, [\langle x, y\rangle \in X \Rightarrow \langle x, 2^y\rangle \in X] \\ & \wedge\ \forall n[\varphi_e(n)\downarrow \wedge \langle \varphi_e(n), \varphi_e(n+1)\rangle \in X] \quad \Rightarrow \quad \forall n\, \langle \varphi_e(n), 3\cdot 5^e\rangle \in X \\ & \wedge\ \forall x, y, z[(\langle x, y\rangle \in X \wedge \langle y, z\rangle \in X) \Rightarrow \langle x, z\rangle \in X]. \end{aligned}$$

Clearly $<_{\mathcal{O}}$ (as a set of coded tuples) satisfies A. In fact, every non-empty $X \subseteq \mathbb{N}$ with $A(X)$ can be seen as a set of ordinal notations. It follows by transfinite induction along the recursive ordinals that $<_{\mathcal{O}}$ is contained in any other $X \subseteq \mathbb{N}$ such that $A(X)$ and $\langle 1, 2\rangle \in X$. In other words, $<_{\mathcal{O}}$ is $\subseteq$-*minimal* among the solutions X of A that contain $\langle 1, 2\rangle$.

A is obviously Π^0_1. By the observation on the minimality of $<_{\mathcal{O}}$, we have

$$\langle x, y\rangle \in <_{\mathcal{O}} \quad \Leftrightarrow \quad \forall X\, [(\langle 1, 2\rangle \in X \wedge A(X)) \Rightarrow \langle x, y\rangle \in X].$$

The condition on the right hand side is Π^1_1. □

Theorem 20.6: *For every Π^1_1 set $X \subseteq \mathbb{N}$ there exists a recursive function f such that*

$$x \in X \quad \Leftrightarrow \quad f(x) \in \mathcal{O}.$$

Proof. By the Normal Form given at the beginning of this Lecture,

$$x \in X \quad \Leftrightarrow \quad S(x) \text{ is well-founded.}$$

The tree $S(x)$ is recursive uniformly in x, so there exists a recursive function t such that $S(x) = R_{t(x)}$, where R_e is the eth recursively enumerable binary relation on $\mathbb{N}$. If we let f be a reduction from $\mathrm{WF}_{\mathbb{N}}$ to $\mathcal{O}$. Then

$$x \in X \quad \Leftrightarrow \quad f(t(x)) \in \mathcal{O}.$$

□

It is clear from the proof that $\mathrm{WF}_{\mathbb{N}}$ is also a Π^1_1 complete set.

Corollary 20.7: $\mathcal{O}$ *is not* Σ^1_1.

Proof. Similar to showing that WF is not Σ^1_1 – exhibit a Π^1_1 subset of $\mathbb{N}$ that is not Σ^1_1. This can be done using the universality of the Kleene T-predicate. □