Homework 1 for MATH 185

Brief sketches of solutions

Problem 1

Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ be the *upper half plane*, and $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ be the *(open) unit disk*.

Show that the mapping $f : \mathbb{H} \to \mathbb{C}$ defined by

$$z \mapsto \frac{z - i}{z + i}$$

is one-one and it holds that $f(\mathbb{H}) = \mathbb{E}$ (i.e. f is a *bijection* between $\mathbb{H}$ and $\mathbb{E}$).

Solution. To show that f is injective, suppose

$$\frac{w - i}{w + i} = \frac{z - i}{z + i}.$$

This is equivalent to $2iw = 2iz$, which in turn implies $z = w$.

If $z \in \mathbb{H}$, the z is closer to i than to $-i$. Hence $|z - i| < |z + i|$, and thus

$$\left|\frac{z - i}{z + i}\right| = \frac{|z - i|}{|z + i|} < 1.$$

This proves that $f(\mathbb{H}) \subseteq \mathbb{E}$.

Finally,

$$w = \frac{z - i}{z + i} \Leftrightarrow z = i\frac{1 + w}{1 - w}.$$

If $w \in \mathbb{E}$, then $1 - w \neq 0$ and so this is well-defined. To show that for such w, $\operatorname{Im}(i(1 + w)/(1 - w)) > 0$, use the formula $\operatorname{Im}(z) = (z - \bar{z})/2i$ to infer

$$\operatorname{Im}\left(i\frac{1 + w}{1 - w}\right) = \frac{1 - |w|^2}{(1 - w)(1 - \overline{w})}.$$

If $|w| < 1$, both nominator and denominator are positive. ∎

Problem 2

Show that a quadratic equation $z^2 + pz + q = 0$, $p, q \in \mathbb{C}$ always has two solutions in $\mathbb{C}$ (counting multiplicity). What can you say about the solutions if both p and q are real numbers?

Solution. See book, page 460. ∎

Problem 3

Let $n \in \mathbb{N}$, $\zeta_n := \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} \in \mathbb{C}$. Show that for all $k \in \mathbb{N}$,

$$1 + \zeta_n^k + \zeta_n^{2k} + \cdots + \zeta_n^{(n-1)k} = \begin{cases} n, & \text{if } n \text{ divides } k \\ 0, & \text{otherwise.} \end{cases}$$

Solution. We have $\zeta_n = \exp(2\pi i/n)$. Suppose n divides k. Then, by periodicity of the exponential function, $\exp(2\pi i l k/n) = 1$ for all $l \in \mathbb{Z}$ and the sum is n. If n does not divide k, $\zeta_n^k = \exp(2\pi i k/n) \neq 1$. Furthermore,

$$\zeta_n^k(1 + \zeta_n^k + \zeta_n^{2k} + \cdots + \zeta_n^{(n-1)k}) = \zeta_n^k + \zeta_n^{2k} + \cdots + \zeta_n^{(n-1)k} + \zeta_n^{nk} = 1 + \zeta_n^k + \zeta_n^{2k} + \cdots + \zeta_n^{(n-1)k}.$$

Since $\zeta_n^k \neq 1$, it follows that $1 + \zeta_n^k + \zeta_n^{2k} + \cdots + \zeta_n^{(n-1)k} = 0$.

An alternative proof uses the geometric sum identity

$$\sum_{k=0}^{n-1} z^k = \frac{1 - z^n}{1 - z},$$

which is valid in every field, provided $z \neq 1$. ∎

Problem 4

Let U be an open subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be a continuous function. Assume there exists $a \in U$ such that $f(a) \neq 0$. Prove that there is an open ball B containing a such that $f(z) \neq 0$ for all z in B.

Solution. Assume that for every n, there is a $z_n \in U_{1/n}(a)$ such that $f(z_n) = 0$. Then $z_n \to a$, and since f is continuous, $f(z_n) \to f(a) \neq 0$. But obviously, $f(z_n) \to 0$ by choice of z_n, contradiction. Hence there is some $n > 0$ such that $f(z) \neq 0$ for all $z \in U_{1/n}(a)$. ∎