Lecture: The Absoluteness of Constructibility

We would like to show that L is a model of $\mathsf{V} = \mathsf{L}$, or, more precisely, that L is an interpretation of $\mathsf{ZF} + \mathsf{V} = \mathsf{L}$ in ZF. We have already verified that σ^L holds (in ZF) for all axioms σ of ZF. To verify that $\mathsf{ZF} \vdash (\mathsf{V} = \mathsf{L})^L$, we need to show (in ZF) that

$$(\forall x \exists \alpha \; x \in L_\alpha)^L$$

holds. Since $L \cap \mathrm{Ord} = \mathrm{Ord}$, the former is equivalent to

$$\forall x \in L \exists \alpha \; (x \in L_\alpha)^L.$$

This could fail to hold if the definition of L "inside L" yields a different structure then the constructible hierarchy itself. Therefore, we have to analyze the function $\alpha \mapsto L_\alpha$ and show that it is *absolute* for L. We identify L with this function, i.e. we let $L : \mathrm{Ord} \to V$ be given by $L(\alpha) = L_\alpha$. We have to show that $L^L(\alpha) = L(\alpha)$ for all ordinals.

To do this, we analyze the set theoretic complexity of the definability notion.

Gödelization

We assign to every variable v_n the Gödel number (or rather the Gödel *set*)

$$\ulcorner v_n \urcorner = (1, n).$$

We also extend our language by introducing, for every set a, a new constant $\underline{a}$. This way, we can address elements of a set theoretic structure $(M, \in)$ when defining, for example, the relation $(M, \in) \models \varphi[a]$. When, for $a \in M$, the interpretation of $\underline{a}$ is to be a itself, we speak of the *canonical interpretation*. The Gödel number of a constant is

$$\ulcorner \underline{a} \urcorner = (2, a).$$

Now we can recursively assign Gödel numbers to all set theoretic formulas (in the extended language).

$$\begin{aligned}
\ulcorner x = y \urcorner &= (3, (\ulcorner x \urcorner, \ulcorner y \urcorner)) \\
\ulcorner x \in y \urcorner &= (4, (\ulcorner x \urcorner, \ulcorner y \urcorner)) \\
\ulcorner \neg\varphi \urcorner &= (5, \ulcorner \varphi \urcorner) \\
\ulcorner \varphi \wedge \psi \urcorner &= (6, (\ulcorner \varphi \urcorner, \ulcorner \psi \urcorner)) \\
\ulcorner \exists v_n \varphi \urcorner &= (7, (n, \ulcorner \varphi \urcorner))
\end{aligned}$$

Definability of syntactical notions

We can express "a is (the Gödel number of) a variable" as (recall the definition of a set theoretic pair)

$$\mathrm{Var}(a) \longleftrightarrow \exists y \in a \, \exists x \in y \; (a = (1, x) \; \wedge \; x \in \omega).$$

We want to keep track of the complexity of the definitions. All quantifiers are bounded, so Var is Δ_0, provided the expression $x \in \omega$ is Δ_0, too. A proof of the latter fact is given in Lemma 12.10 in Jech [2003], for example.

Using Var, we can go on to define set theoretic formulae saying

$$\begin{aligned} \mathrm{Fml}^n(e) \longleftrightarrow\ & e \text{ is the Gödel number of a formula } \varphi \text{ whose free variables are} \\ & \text{among } v_0, \dots, v_{n-1}, \\ \mathrm{Fml}^n_a(e) \longleftrightarrow\ & e \text{ is the Gödel number of a formula } \varphi \text{ whose free variables are} \\ & \text{among } v_0, \dots, v_{n-1}, \text{ and which contains constants } \underline{a}_i \text{ with } a_i \in a. \end{aligned}$$

The definition of Fml is not difficult, but a little tedious and has to be worked out carefully. Details can be found in Devlin [1984], Section 1.9 (but see also [Mathias, 2006]).

Informally, the definition of $\mathrm{Fml}^n(e)$ says that there exists a finite sequence of Gödel numbers of formulae and a way to put them together (a "formula tree") so that the resulting formula has Gödel number e, and the only free variable that occur are among $v_0, \dots, v_n$. One needs to resort to a suitable recursion principle to do this.

This definition of Fml is no longer Δ_0. In order to still be able to establish absoluteness results, one has to provide a careful analysis of the logical complexity of Fml.

The Levy hierarchy of set theoretic formulae

We have already discussed the notion of Δ_0 formulae. If we allow unbounded quantifiers, we obtain a hierarchy of formulae, classified according to the number of quantifier changes (similar to the arithmetical hierarchy of number theoretic formulae).

$$\begin{aligned} \varphi \text{ is a } \Sigma_1 \text{ formula} \longleftrightarrow\ & \varphi = \exists v_1 \dots \exists v_n \psi, \text{ for a } \Delta_0 \text{ formula } \psi, \\ \varphi \text{ is a } \Pi_1 \text{ formula} \longleftrightarrow\ & \varphi = \forall v_1 \dots \forall v_n \psi, \text{ for a } \Delta_0 \text{ formula } \psi, \end{aligned}$$

Continuing inductively (letting $\Sigma_0 = \Pi_0 = \Delta_0$), we put

$$\varphi \text{ is a } \Sigma_{n+1} \text{ formula} \longleftrightarrow \ \varphi = \exists v_1 \dots \exists v_n \psi, \text{ for a } \Pi_n \text{ formula } \psi,$$
$$\varphi \text{ is a } \Pi_{n+1} \text{ formula} \longleftrightarrow \ \varphi = \forall v_1 \dots \forall v_n \psi, \text{ for a } \Sigma_n \text{ formula } \psi,$$

Note that these definitions only apply to formulae in prenex normal form. However, we can extend the definition to other formulae by saying φ is Σ_n (Π_n) if it is logically equivalent to a Σ_n (Π_n) formula.

Sometimes a proof that a certain formula is Σ_n requires not only logical equivalences (such as $\exists v_1 \neg \forall v_3 \psi \ \longleftrightarrow \ \exists v_1 \exists v_2 \neg \psi$), but set theoretic axioms. For example, consider the definition of an ordinal,

$$\mathrm{Ord}(a) \longleftrightarrow a \text{ is transitive and } a \text{ is well-ordered by } \in$$

The property of being well-ordered by $\in$ is formalized as

$$\forall x \, [x \subseteq a \rightarrow \exists b \in x \forall c \in x (c \notin b)].$$

This definition is not Δ_0. However, if we assume the Axiom of Regularity, every set is well-founded with respect to $\in$, so it suffices to require that a is *linearly ordered* by $\in$. In fact, it suffices to require (exercise!) that

$$a \text{ is transitive and } \forall x, y \in a \, (x \in y \ \vee \ x = y \ \vee \ y \in x),$$

which is Δ_0.

In some cases, one can use set theoretic operations to bound quantifiers. For instance, using the definition of $\bigcup a$, we obtain that

$$\exists x \in \bigcup a \dots \longleftrightarrow \exists y \in a \exists x \in y \dots$$

is a bounded quantifier in the sense of Δ_0 formulae. Regarding other set theoretic operations, this kind of argument has to be used with caution, though. A quantifier of the form

$$\exists x \in \mathcal{P}(a)$$

cannot be regarded as bounded, since the definition of $\mathcal{P}(a)$ is not Δ_0, but Π_1.

If T is a theory (in the language of set theory) we say

$$\varphi \text{ is } \Sigma_n^T \text{ iff there exists a } \Sigma_n \text{ formula } \psi \text{ so that } T \vdash \varphi \longleftrightarrow \psi,$$
$$\varphi \text{ is } \Pi_n^T \text{ iff there exists a } \Pi_n \text{ formula } \psi \text{ so that } T \vdash \varphi \longleftrightarrow \psi,$$
$$\varphi \text{ is } \Delta_n^T \text{ iff } \varphi \text{ is } \Sigma_n^T \text{ and } \Pi_n^T.$$

Extending absoluteness

We are particularly interested in the case $n = 1$, since this allows us to extend absoluteness results beyond Δ_0 is a relatively easy manner.

Proposition 1.1: *Let M be a transitive model of T, where T is a subtheory of* ZF.

(1) *For any Σ_1^T formula φ, $\varphi^M \to \varphi$.*

(2) *For any Π_1^T formula φ, $\varphi \to \varphi^M$.*

(3) *For any Δ_1^T formula φ, $\varphi^M \longleftrightarrow \varphi$.*

We say Σ_1 formulae are *upward absolute*, whereas Π_1 are *downward absolute*.

Proof. (1) Assume φ is Σ_1^T. Suppose φ is equivalent over T to a formula $\exists v \psi(v)$, where ψ is Δ_0. Assume σ^M holds for every $\sigma \in T$. Let θ be a conjuction of finitely many sentences from T that prove $\varphi \longleftrightarrow \exists v \psi(v)$. Then, since $\vdash \theta \to (\varphi \longleftrightarrow \exists v \psi(v))$, and validities are absolute for any structure, $\theta^M \to (\varphi^M \longleftrightarrow \exists v \in M \psi^M(v))$, and hence $\varphi^M \longleftrightarrow \exists v \in M \psi^M(v)$. So if φ^M, then $\exists v \in M \psi^M(v)$ and hence $\exists v \psi^M(v)$. Since Δ_0 formulas are absolute for transitive models, we obtain $\exists v \psi(v)$. Since T is a fragment of ZF, it follows that $\mathsf{ZF} \vdash \varphi \longleftrightarrow \exists v \psi(v)$, and thus φ.

The proof for (2) is similar, and (3) follows from (1) and (2). □

Defining definability

We mentioned above that the definition of $\mathrm{Fml}^n(e)$ states the existence of a sequence of Gödel numbers of formulae and a way to put them together (a "formula tree"). This turns out to be a Σ_1 definition. However, one can bound the domain from which the sequence is drawn by a set theoretic operation A. This set theoretic operation is Δ_1^T definable for a finite fragment T of ZF. Using this operation, we can rewrite the definition of $\mathrm{Fml}^n(e)$ as "$\forall a$ *(if $u = A(e)$ then $\exists x \in u$ such that x is a sequence of Gödel numbers ...)*".

This way we can establish

Proposition 1.2: $\mathrm{Fml}^n(e)$ *and* $\mathrm{Fml}^n_a(e)$ *are* Δ_1^{ZF}.

To show $\mathrm{Fml}^n(e)$ and $\mathrm{Fml}^n_a(e)$ are Δ_1, it suffices to consider a weak fragment of ZF. *Kripke-Platek set theory* (KP) consists of the Axioms of Extensionality, Pairing, and Union, and also of the following axiom schemes:

$$\begin{aligned}
&(\Delta_0\text{-Separation}) && \exists y \forall z (z \in y \leftrightarrow x \in a \wedge \varphi(x)) \\
&(\Delta_0\text{-Replacement}) && \forall x \exists y \varphi(x,y) \to \exists z \forall x \in a \exists y \in z\, \varphi(x,y) \\
&(\text{Foundation}) && \forall x \varphi(x) \to \exists x(\varphi(x) \wedge \forall y \in x \neq \varphi(y))
\end{aligned}$$

Here, the first two schemes only apply to Δ_0 formulae φ. KP_∞ denotes the theory obtained by also adding the Axiom of Infinity.

KP_∞ can be seen as a generalized recursion theory and is strong enough to develop the recursive definitions needed to develop syntactical notions such as $\mathrm{Fml}^n(e)$. In particular, one can show that $\mathrm{Fml}^n(e)$ and $\mathrm{Fml}^n_a(e)$ are $\Delta_1^{\mathsf{KP}_\infty}$.

Now we can go on an give set-theoretic definitions of semantical notions. There exists a set theoretic formula $\mathrm{Sat}(a,e)$ which is $\Delta_1^{\mathsf{KP}_\infty}$ and expresses the following

$\mathrm{Sat}(a,e)$: e is the code of a formula $\varphi(\underline{a}_1,\dots,\underline{a}_n)$ with no free variables and φ holds in $(a,\in)$ under the canonical interpretation.

We also write $(a,\in) \models e$ instead of $\mathrm{Sat}(a,e)$. One can use Sat to formally establish the equivalence of a formula holding relativized and holding in the corresponding set theoretic structure (for *set* structures only).

Proposition 1.3: *Let $\varphi(v_0,\dots,v_{n-1})$ be a formula, let M be a set, and let $a_0,\dots,a_{n_1} \in M$. Then it holds (in KP_∞) that*

$$\varphi^M(a_0,\dots,a_{n_1}) \longleftrightarrow \mathrm{Sat}(M, \ulcorner\varphi(\underline{a}_0,\dots,\underline{a}_{n_1})\urcorner).$$

This is proved by induction over the structure of φ. The atomic case works because we require $\varphi(\underline{a}_0,\dots,\underline{a}_{n_1})$ to hold under the canonical interpretation.

The Sat predicate puts us in a position to "define" $\mathrm{Def}(M)$.

$$\mathrm{Def}(M) = \{x \subseteq M : \exists e\, (\mathrm{Fml}^1_M(e) \wedge x = \{z \in M : (M,\in) \models e(z)\})\}.$$

We have to be careful here, since "$(M,\in) \models e$" was only defined for *fixed* Gödel numbers, but here this number seems to depend on the set z. We therefore *define* $e(z)$ to be the Gödel number of the following formula: If $e = \ulcorner\varphi\urcorner$, then

$e(z)$ is the Gödel number of the formula $\varphi(\underline{z})$ that we obtain by replacing every occurrence of the (only) variable v_0 by the symbol $\underline{z}$:

$$\ulcorner\varphi(v_0)\urcorner(z) = \ulcorner\varphi(\underline{z})\urcorner.$$

(This transition is, moreover, Δ_1-definable over KP_∞.)

The absoluteness of definability

To establish the desired absoluteness, we have to check the complexity of the formula for Def.

Proposition 1.4: *The relation* $b = \mathrm{Def}(a)$ *is* $\Delta_1^{\mathsf{KP}_\infty}$.

Sketch of proof. That $\mathrm{Def}(M)$ is defined by a Σ_1 formula is not hard to see once we have established the complexity of Fml and Sat. As noted in [Jech, 2003], Lemma 13.10, graphs of functions with Δ_1 domain are Δ_1[†]. □

Having determined the complexity of $\mathrm{Def}(M)$, we can go on to show

Proposition 1.5: *The function* $\alpha \mapsto L_\alpha$ *is* $\Delta_1^{\mathsf{KP}_\infty}$.

Proof. Lemma 13.12 in [Jech, 2003] (together with the observation that graphs of Σ_1 functions with Δ_1 domain are Δ_1) reduces this task to verifying that the induction step is $\Sigma_1^{\mathsf{KP}_\infty}$.

For α a successor ordinal, this follows from 1.4. For α limit, $L_\alpha = \bigcup_{\beta<\alpha} L_\beta$ and hence $b = L_\alpha$ iff $b = \bigcup_{\beta\in b} L_\beta$. □

Putting all the pieces together, we obtain

Theorem 1.6: *L satisfies the Axiom of Constructibility,* $\mathsf{V} = \mathsf{L}$.

Furthermore, L is the smallest inner model of ZF.

Theorem 1.7: *If M is an inner model of* ZF*, then* $L \subseteq M$.

Proof. Suppose M is an inner model. Then M is in particular a model of KP_∞, and thus the function $\alpha \to L_\alpha$ is absolute for M, which means $L^M = L$ and hence $L = L^M \subseteq M$. □

[†] There is another issue here: Working in KP_∞, we cannot invoke the Power Set Axiom to claim that $\mathrm{Def}(M)$ is a set. This requires a separate argument in KP_∞ (exercise).

The Condensation Lemma

Ordinals α so that $L_\alpha \models \mathsf{KP}_\infty$ are called *admissible ordinals*. It follows from the preceding sections that for every admissible ordinal, L_α is a model of $\mathsf{V} = \mathsf{L}$. This indicates that the L_α (at certain stages) exhibit a remarkable robustness and stratification with respect to constructibility. If we refine the analysis of the absoluteness of constructibility a little bit more, we can unearth this stratification in its full glory.

Every proof leading up to Theorem 1.6 uses only finitely many sentences of the theory KP_∞. We can collect these sentences in a finite fragment T.

Theorem 1.8: *There exists a finite subtheory of KP_∞ so that $L_\alpha \models T$ for all limit ordinals α and such that the following hold.*

(1) The relations $b = \mathrm{Def}(a), b = L_\alpha, b \in L_\alpha$ are Δ_1^T.
The relation $b \in L$ is Σ_1^T.
The sentence $\mathsf{V} = \mathsf{L}$ is Π_2^T.

(2) If M is a transitive model of T, then

$$\begin{aligned} L_\alpha^M &= L_\alpha && \text{for all ordinals } \alpha \text{, and in particular:} \\ L^M &= L, && \text{if } M \text{ is a proper class,} \\ L^M &= L_\gamma, && \text{if } M \text{ is a set and } \alpha = \mathrm{Ord} \cap M. \end{aligned}$$

(3) If M is a transitive model of $T + \mathsf{V} = \mathsf{L}$, then

$$M = \begin{cases} L, & \text{if } M \text{ is a proper class} \\ L_\alpha, & \text{if } M \text{ is a set and } \alpha = \mathrm{Ord} \cap M \end{cases}$$

Proof. The preceding sections have shown that a finite fragment of KP_∞ exists so that (1) holds for any model of KP_∞. Similarly for the first statement of (2).

To establish the remaining statements, we work within KP_∞ and then argue that we needed only finitely many axioms.

First assume that M is a proper class. We first show that $\mathrm{Ord} \subseteq M$. Suppose α is an ordinal. Since M is not a set, $M \nsubseteq V_\alpha$, there exists an $x \in M$ with $\mathrm{rank}(x) \geq \alpha$. One can show that the rank-function is absolute for transitive models of KP_∞ (it is defined by recursion), thus $\mathrm{rank}(x) = \mathrm{rank}^M(x) \in M$. Since M is transitive, we have $\alpha \in M$.

Now we have, by absoluteness of $\alpha \mapsto L_\alpha$ and of Ord,

$$L^M = \bigcup_{\alpha \in M} L^M_\alpha = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha = L.$$

For the third statement of (2), let $\alpha = M \cap \mathrm{Ord}$. We make T strong enough to show that no largest ordinal exists. (Again, this can be done by including finitely many axioms from KP_∞.) Then α is a limit ordinal and hence

$$L_\alpha = \bigcup_{\beta \in M} L_\beta.$$

But by absoluteness of $\alpha \mapsto L_\alpha$

$$L^M = \bigcup_{\beta \in M} L^M_\beta = \bigcup_{\beta \in M} L_\beta$$

and thus $L^M = L_\alpha$.

To prove (3), note that if M is transitive and a model of $T + \mathsf{V} = \mathsf{L}$ (T comprising now all the sentences used to establish (1)+(2)), we have

$$(\forall x \exists \alpha (x \in L_\alpha))^M$$

which means

$$\forall x \in M \exists \alpha (x \in L^M_\alpha)$$

that is, $M = L^M$. Both cases now follow immediately from the corresponding statement in (2). □

We can rephrase (3) as follows: There exists a single sentence $\sigma_{\mathsf{V}=\mathsf{L}}$ (namely, the conjunction of all sentences in $T + \mathsf{V} = \mathsf{L}$) so that for any transitive M,

$$(M, \in) \models \sigma_{\mathsf{V}=\mathsf{L}} \quad \text{iff} \quad M = L_\alpha \text{ for some limit ordinal } \alpha.$$

Now it is easy to infer the *Gödel Condensation Lemma*, a fundamental tool in the analysis of L. The result follows directly from the preceding theorem together with the Mostowski collapse.

Theorem 1.9 (Gödel Condensation Lemma)**:** *If $(X, \in)$ is an elementary substructure of L_α, α limit, then $(X, \in)$ is isomorphic to some $(L_\beta, \in)$ with $\beta \leq \alpha$.*

The canonical well-ordering of L

Every well-ordering on a transitive set X can be extended to a well-ordering of $\mathrm{Def}(X)$. Note that every element of $\mathrm{Def}(X)$ is determined by a pair $(\psi, \vec{a})$, where ψ is a set-theoretic formula, and $\vec{a} = (a_1, \dots, a_n) \in X^{<\omega}$ is a finite sequence of parameters. For each $z \in \mathrm{Def}(X)$ there may exist more than one such pair (i.e. z can have more than one definition), but by well-ordering the pairs $(\psi, \vec{a})$, we can assign each $z \in \mathrm{Def}(X)$ its *least* definition, and subsequently order $\mathrm{Def}(X)$ by comparing least definitions. Elements already in X will form an initial segment. Such an order on the pairs $(\psi, \vec{a})$ can be obtained in a definable way: First use the order on X to order $X^{<\omega}$ length-lexicographically, order the formulas through their Gödel numbers, and finally say

$$(\psi, \vec{a}) < (\varphi, \vec{b}) \quad \text{iff} \quad \psi < \varphi \text{ or } (\psi < \varphi \text{ and } \vec{a} < \vec{b}).$$

Based on this, we can order all levels of L so that the following hold:

(1) $<_L |V_\omega$ is the canonical well-order on V_ω.

(2) $<_L |L_{\zeta+1}$ is the order on $\mathcal{P}_{\mathrm{Def}}(L_\zeta)$ induced by $<_L |L_\zeta$.

(3) $<_L |L_\zeta = \bigcup_{\zeta<\xi} <_L |L_\zeta$ for a limit ordinal $\xi > \omega$.

It is straightforward to verify that this is indeed a well-ordering on L. But more importantly, for any limit ordinal $\xi > \omega$, $<_L |L_\xi$ is definable over L_ξ. To facilitate notation, we denote the restriction of $<_L$ to some L_ξ by $<_\xi$.

Proposition 1.10: *There is a Σ_1 formula $\varphi_<(x_0, x_1)$ such that for all limit ordinals $\xi > \omega$, if $a, b \in L_\xi$,*

$$L_\xi \models \varphi_<[a, b] \quad \textit{iff} \quad a <_\xi b.$$

The proof of this proposition is similar to the proof that the sequence of $(L_\zeta)_{\zeta<\xi}$ is definable in L_ξ. It relies on the strong closure properties of L_ξ under the Sat-function.

Theorem 1.11: *If* $\mathsf{V} = \mathsf{L}$ *then* AC *holds.*

The Continuum Hypothesis in L

We can now present Gödel's proof that the Generalized Continuum Hypothesis (GCH) holds if $\mathsf{V} = \mathsf{L}$.

Theorem 1.12: *If* $\mathsf{V} = \mathsf{L}$*, then for all infinite ordinals* α*,* $\mathcal{P}(L_\alpha) \subseteq L_{\alpha^+}$.

Proof. Assume $\mathsf{V} = \mathsf{L}$ and let $A \subseteq L_\alpha$. Since we assume $\mathsf{V} = \mathsf{L}$, there exists a limit δ so that $A \in L_\delta$. Let $X = L_\alpha \cup \{A\}$. The Löwenheim-Skolem Theorem and a Mostowski collapse yield a set M such that

- $(M, \in)$ is a transitive, elementary substructure of $(L_\delta, \in)$,
- $X \subseteq M \subseteq L_\delta$,
- $|M| = |X|$.

The Condensation Lemma 1.9 yields that $M = L_\zeta$ for some $\zeta \leq \delta$. Since for all $\xi \geq \omega$, $|L_\xi| = |\xi|$, we obtain

$$|M| = |X| = |L_\alpha| = |\alpha| < \alpha^+$$

and hence $A \in L_\zeta \subseteq L_{\alpha^+}$ □

Theorem 1.13: *If* $\mathsf{V} = \mathsf{L}$ *then* GCH *holds.*

Proof. If $\mathsf{V} = \mathsf{L}$, then by the preceding theorem, for each cardinal κ,

$$\mathcal{P}(\kappa) \subseteq \mathcal{P}(L_\kappa) \subseteq L_{\kappa^+}.$$

Therefore,

$$2^\kappa \leq |L_{\kappa^+}| = \kappa^+.$$

□

In the previous proofs we have used the Axiom of Choice in various places (Löwenheim-Skolem, proof of the lemma), but since $\mathsf{V} = \mathsf{L}$ implies AC, this is not a problem.

References

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