Randomness and Universal Machines

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Kolmogorov complexity

General idea: Amount of information needed to describe effectively a string.

Complexity H(x): $\mathbf{H(x) = \min\{|p| : U(p) = x\}}$.

Based on universal prefix-free machine U

Prefix-free: If $\mathbf{p} \prec \mathbf{q}$ and $\mathbf{U(p)}$ defined then $\mathbf{U(q)}$ undefined.

Universal: Machine which cannot be improved much. For every further prefix-free machine $\mathbf{V}$ the best program for any input can only be constantly shorter as the corres- ponding program for $\mathbf{U}$. That is, $\mathbf{U}$ satisfies the formula
$\forall \mathbf{V} \exists \mathbf{c} \forall \mathbf{p} \exists \mathbf{q}\, [\mathbf{V(p)} \text{ defined } \Rightarrow \mathbf{U(q)} = \mathbf{V(p)} \wedge |\mathbf{q}| \leq |\mathbf{p}| + \mathbf{c}]$.

Remark: There is also a variant of Kolmogorov complexity which does not request the machine to be prefix-free.

Martin-Löf random

General idea: A set $\mathbf{A}$ is Martin-Löf random iff no algorithm can derive any non-trivial information on the set $\mathbf{A}$. Many natural characterizations.

Tests: $\mathbf{A}$ is Martin-Löf random iff there is no sequence $\mathbf{S_1}, \mathbf{S_2}, \ldots$ of classes such that

- $\mathbf{A} \in \mathbf{S_n}$ for all $\mathbf{n}$;
- The Lebesgue measure of each $\mathbf{S_n}$ is at most $\mathbf{2^{-n}}$;
- The $\mathbf{S_n}$ are uniformly $\mathbf{\Sigma_1}$, that is, $\{(\mathbf{n}, \mathbf{B}) : \mathbf{B} \in \mathbf{S_n}\}$ is a $\mathbf{\Sigma_1}$-class.

Kolmogorov Complexity: $\mathbf{A}$ is Martin-Löf random iff $\exists \mathbf{c} \forall \mathbf{n}\ [\mathbf{H}(\mathbf{A}(0)\mathbf{A}(1)\ldots\mathbf{A}(\mathbf{n})) \geq \mathbf{n} - \mathbf{c}]$.

Left-r.e. sets

Definition
A set $\mathbf{A}$ is left-r.e. if $\{\sigma : \sigma \leq_{\text{lex}} \mathbf{A}\}$ is an r.e. set.

Chaitin's $\mathbf{\Omega}$ is defined by

$$\sum_{\mathbf{n} \in \mathbf{\Omega}} \mathbf{2}^{-\mathbf{n}-\mathbf{1}} = \sum_{\mathbf{p},\ \mathbf{U}(\mathbf{p})\ \text{defined}} \mathbf{2}^{-|\mathbf{p}|}$$

and is called the halting probability of the prefix-free universal machine $\mathbf{U}$.

$\mathbf{\Omega}$ is a left-r.e. Martin-Löf random set. Every left-r.e. Mar- tin-Löf random set is an $\mathbf{\Omega}$-number, that is, is the halting probability of a suitable prefix-free universal machine.

The Starting Point

Open Questions [Miller and Nies, 2005]
List of Open Questions in Algorithmic Randomness pre- sented at the Annual Meeting of the ASL in Stanford 2005.
http://www.cs.auckland.ac.nz/ ˜nies/papers/questions.pdf

Question 8.1: A set $\mathbf{A}$ relative to which $\mathbf{\Omega}$ is Martin-Löf random is called "low for $\mathbf{\Omega}$". Is every set which is low for $\mathbf{\Omega}$ either recursive or hyperimmune?

Question 8.9: Are $\mathbf{\Omega}$-numbers always truth-table equivalent?

Question 8.10: Is there a co-r.e. set $\mathbf{X}$ such that

$$\mathbf{\Omega_U[X]} = \sum_{\mathbf{p:U(p)\in X}} \mathbf{2^{-|p|}}$$

is not Martin-Löf random?

Question 8.1

A set $\mathbf{A}$ has hyperimmune Turing degree if there is a function $\mathbf{f} \leq_\mathbf{T} \mathbf{A}$ such that for all recursive functions $\mathbf{g}$ there is an $\mathbf{x}$ with $\mathbf{f}(\mathbf{x}) > \mathbf{g}(\mathbf{x})$, that is, $\mathbf{f}$ is not majorized by $\mathbf{g}$.
A set $\mathbf{A}$ has hyperimmune-free Turing degree if for every $\mathbf{f} \leq_\mathbf{T} \mathbf{A}$ there is a recursive function $\mathbf{g}$ which majorizes $\mathbf{f}$.

A set is low for $\mathbf{\Omega}$ if $\mathbf{\Omega}$ is Martin-Löf random also in the world of computations relative to $\mathbf{A}$.

There are many sets which are low for $\mathbf{\Omega}$, actually the measure of the class of these sets is $\mathbf{1}$.

Question [Miller and Nies 2005]
Does every low for $\mathbf{\Omega}$ set have hyperimmune Turing degree?

Partial Results

A recursive binary tree is a downward closed recursive subset of $\{\mathbf{0}, \mathbf{1}\}^*$.

Theorem [Jockusch and Soare 1972]
Every infinite recursive binary tree has an infinite branch of hyperimmune-free Turing degree. Every infinite recursive binary tree without infinite recursive branch has an infinite branch of hyperimmune Turing degree.

Theorem
If $\mathbf{T}$ is a recursive tree without recursive infinite branches then every infinite branch of $\mathbf{T}$ which is low for $\mathbf{\Omega}$ does also have hyperimmune Turing degree.

Theorem
If $\mathbf{A}$ is low for $\mathbf{\Omega}$, $\mathbf{A}$ is nonrecursive and the Turing degree of $\mathbf{A} \oplus \mathbf{K}$ is hyperimmune-free relative to $\mathbf{K}$ then $\mathbf{A}$ has hyperimmune Turing degree.

Question 8.9

Truth-Table Reducibility

A set $\mathbf{A}$ is truth-table reducible to $\mathbf{B}$ iff there are total recursive functions $\mathbf{f}, \mathbf{g}$ such that, for all $\mathbf{x}$,

$$\mathbf{A}(\mathbf{x}) = \mathbf{f}(\mathbf{x}, \mathbf{B}(\mathbf{0})\mathbf{B}(\mathbf{1})\ldots\mathbf{B}(\mathbf{g}(\mathbf{x}))).$$

Theorem [Calude and Nies 1998]

The halting problem $\mathbf{K}$ is not truth-table reducible to any Martin-Löf random set.

Question [Miller and Nies]

Given two universal prefix-free machines $\mathbf{U}, \mathbf{V}$, are $\mathbf{\Omega_U}$ and $\mathbf{\Omega_V}$ equivalent with respect to truth-table reducibility?

It is known that $\mathbf{\Omega_U}$ and $\mathbf{\Omega_V}$ are equivalent with respect to weak truth-table reducibility.

Truth-Table Incomparable Ω-Numbers

Theorem

There are universal machines $\mathbf{U}$ and $\mathbf{V}$ such that $\Omega_{\mathbf{U}}$ and $\Omega_{\mathbf{V}}$ are incomparable for truth-table reducibility.

Given a universal machine $\mathbf{U}$, $\Omega_{\mathbf{U}} < 1$. Now choose a recursive function $\mathbf{f}$ such that

- $\Omega_{\mathbf{U}} + 2^{-1-\mathbf{f}(0)} + 2^{-1-\mathbf{f}(1)} + \ldots < 1$;
- for all $\mathbf{n}$ there is $\mathbf{m} \notin \Omega_{\mathbf{U}}$ with $\mathbf{f}(\mathbf{n}) < \mathbf{m} < \mathbf{f}(\mathbf{n}+1)$.

Let $\mathbf{X} = \mathbf{f}(\mathbf{K})$ for the halting problem $\mathbf{K}$ and note that there is a universal machine $\mathbf{V}$ with

$$\Omega_{\mathbf{V}} = \Omega_{\mathbf{U}} + \mathbf{X}.$$

If $\Omega_{\mathbf{U}}$ would be truth-table reducible to $\Omega_{\mathbf{V}}$, so would be $\mathbf{X}$ and $\mathbf{K}$, a contradiction.

Related Questions

Theorem

For truth-table reducibility, there is an antichain of Ω-numbers.

Question

Are there universal machines $\mathbf{U}, \mathbf{V}$ such that $\Omega_{\mathbf{U}}$ is strictly truth-table below $\Omega_{\mathbf{V}}$?

For some related reducibility, any two Ω-numbers are either equivalent or incomparable.

Question 8.10

Definition [Becher and Grigorieff 2005]
Given a universal machine $\mathbf{U}$ and a set $\mathbf{X}$, let

$$\Omega_{\mathbf{U}}[\mathbf{X}] = \sum_{\mathbf{p}:\mathbf{U}(\mathbf{p})\in\mathbf{X}} 2^{-|\mathbf{p}|}$$

be the probability that $\mathbf{U}$ halts and outputs an element of $\mathbf{X}$.

A universal machine $\mathbf{U}$ is universal by adjunction if every other prefix-free machine $\mathbf{V}$ is coded into it in a direct way: there is a fixed string $\mathbf{q}$ such that $\mathbf{U}(\mathbf{qp}) = \mathbf{V}(\mathbf{p})$ for all strings $\mathbf{p}$.

Question [Miller and Nies 2005]
Given a machine $\mathbf{U}$ which is universal by adjunction and a co-r.e. set $\mathbf{X}$, is $\Omega_{\mathbf{U}}[\mathbf{X}]$ then Martin-Löf random?

Special Universal Machines

Observation [Miller and Nies 2005]
Let $\mathbf{X}$ be recursively enumerable and $\mathbf{U}$ be a universal machine. If $\mathbf{X}$ is infinite or $\mathbf{U}$ universal by adjunction then $\mathbf{\Omega_U[X]}$ is an $\mathbf{\Omega}$-number, in particular $\mathbf{\Omega_U[X]}$ is left-r.e. and Martin-Löf random.

Theorem
There is a universal machine $\mathbf{U}$ such that, for all $\mathbf{x}$, $\mathbf{\Omega_U[\{x\}] = 2^{1-H(x)}}$.

Theorem
There are a universal machine $\mathbf{U}$ and an infinite co-r.e. set $\mathbf{X}$ such that $\mathbf{\Omega_U[X]}$ is neither left-r.e. nor Martin-Löf random.

Summary

The present work deals with open questions on random- ness collected by Nies and Miller.

Question 8.1: A partial result was obtained by showing that every set which is low for Ω and infinite branch of a binary recursive tree without recursive branches has hyperimmune Turing degree. It remains open whether this second condi- tion can be removed.

Question 8.9: This question was completely solved by showing that there is an infinite antichain of Ω-numbers with respect to truth-table reducibility.

Question 8.10: A partial result was obtained by constructing a universal machine U and a co-r.e. set X such that $\Omega_U[X]$ is not Martin-Löf random. The original question asks for machines universal by adjunction what the constructed machine is not.