Homework 5 for
MATH 574, Topics in Logic

Due: Wednesday April 9

Problem 1

Let $T : X \to X$ be a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$. Let A be a finite set, and assume $\mathcal{P} = \{P_a : a \in A\}$ is a finite partition of X into measurable sets ($P_a \neq \emptyset$). The $(T, \mathcal{P})$-*process* is defined by putting

$$X_{\mathcal{P}}(x) = a \text{ iff } x \in P_a, \quad \text{and} \quad X_n(x) = X_{\mathcal{P}}(T^n x), \text{ for } n \geq 0.$$

Show that there exists a measurable map $F : X \to A^{\mathbb{Z}}$ such that

$$F(Tx) = T_A F(x),$$

where T_A is the shift-map on $A^{\mathbb{Z}}$, and such that the *push-forward measure* of μ under F, given as $\mu_F(B) = \mu(F^{-1}(B))$, is the Kolmogorov measure of the $(T, \mathcal{P})$-process.

Note: This shows we can ***code*** *a process as a symbolic system using partitions.*

Problem 2

Show that a measure preserving transformation $T : X \to X$ is ergodic if and only if for every measurable $f : X \to \mathbb{C}$,

$$(f \circ T)(x) = f(x) \text{ for almost every } x \text{ implies } f \text{ constant almost everywhere.}$$

Problem 3

Show that if $(\vec{p}, M)$ is a stationary Markov chain, then

$$Q = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} M^i$$

exists (i.e. the limit exists for every entry ij). Show further that Q is also stochastic and that $QM = MQ = Q$.

Problem 4

Use the preceding exercise to prove the following characterization of ergodicity for Markov chains. Let $(\vec{p}, M)$ be a stationary Markov chain. Then the following are equivalent:

(i) $(\vec{p}, M)$ is ergodic.

(ii) All rows of the limit matrix Q are identical.

(iii) 1 is a simple eigenvalue of M.