# Homework 7 for
# **MATH 561**, Set Theory

Due: Thursday April 5

In the following, $M$ is always a countable transitive model of ZFC, $\mathbb{P}$ is a partial order in $M$, and $G$ is $\mathbb{P}$-generic over $M$.

### Problem 1 – Union in $M[G]$

Suppose $\tau \in M^{\mathbb{P}}$. Let

$$\pi = \{(\rho, p) \colon \exists (\sigma, q) \in \tau \, \exists r \, ((\rho, r) \in \sigma \, \wedge \, p \leq r \, \wedge \, p \leq q)\}.$$

Show that $\pi_G = \bigcup(\tau_G)$.

### Problem 2 – Images of mappings

Suppose $f \colon A \to M$ and $f \in M[G]$. Show that there is a $B \in M$ so that $f \colon A \to B$.

(*Hint:* Suppose $f = \tau_G$. Consider $B = \{b \colon \exists p \in \mathbb{P} \, (p \Vdash \check{b} \in \operatorname{ran}(\tau))\}$)

### Problem 3 – Iterated forcing

Let $\mathbb{P}$ be non-atomic. Suppose

$$M = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n \subset \ldots$$

with $M_{n+1} = M_n[G_n]$ for some $G_n$ which is $\mathbb{P}$-generic over $M_n$. Show that $\bigcup_n M_n$ does not satisfy the power set axiom.

Furthermore, show that the $G_n$ may be chosen so that there is no c.t.m. $N$ of ZFC with $\langle G_n \colon n \in \omega \rangle \in N$ and $o(N) = o(M)$. (*Hint:* $\{n \colon p \in G_n\}$ can code $o(M)$.)