Measures on $A^{\mathbb{N}}$

<u>Goal</u>: Find an algebra that generates the Borel σ-algebra in $A^{\mathbb{N}}$.

In the following: $\boxed{A = \{0,1\}}$

Borel sets are generated by basic open cylinders $[\sigma]$

$\Rightarrow$ suffices to find algebra containing all cylinder sets.

<u>Need</u>:

- Closure under finite unions
- Closure under complements

What is the complement of a cylinder set?

$$A^{\mathbb{N}} \setminus [\sigma] = \bigcup_{\substack{\tau \neq \sigma \\ |\tau| = |\sigma|}} [\tau]$$

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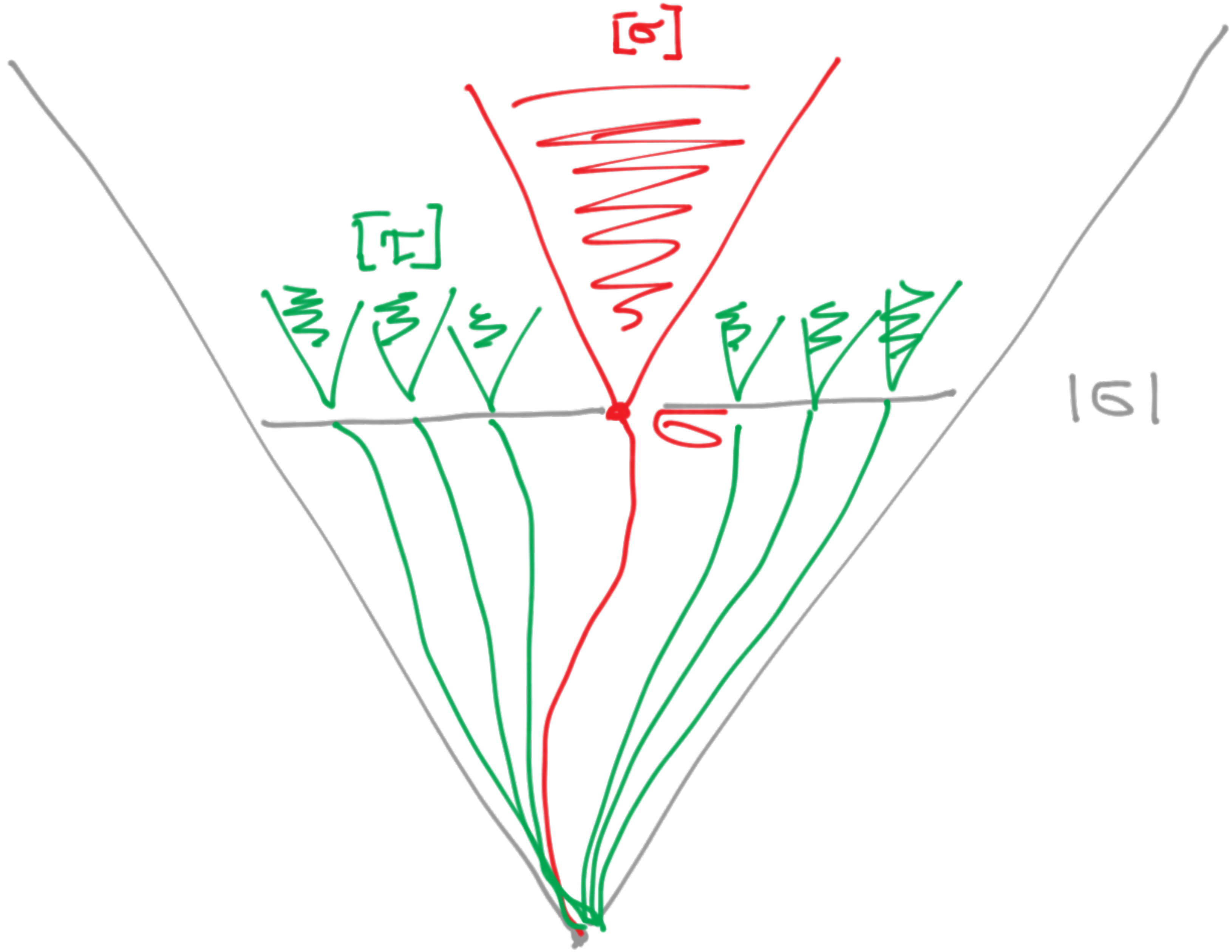

[σ]
[τ]
σ
|G|

Observation: The complement of a cylinder set is a finite union of cylinders.

$\Rightarrow$ $\mathcal{R}$ = finite unions of cylinders is an algebra.

To specify a Borel measure on $A^{\mathbb{N}}$, it suffices to specify

$$\boxed{\text{an additive set function defined on finite unions of cylinders}}$$

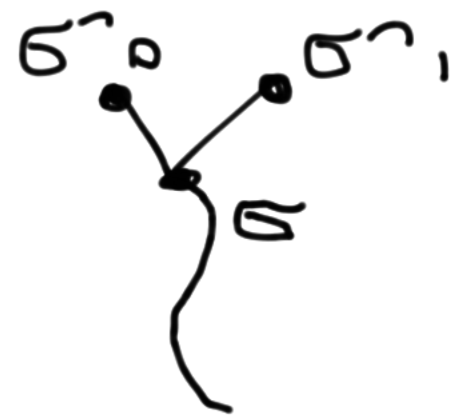
Additive set functions on cylinders

Requirement: if $[\sigma_1], \ldots, [\sigma_n]$ are pairwise disjoint,

then $\mu\left(\bigcup_i [\sigma_i]\right) = \mu[\sigma_1] + \cdots + \mu[\sigma_n]$

How do we construct such a function?

Minimal requirement: $(\ast)$ $\mu[\sigma] = \mu[\sigma^\frown 0] + \mu[\sigma^\frown 1]$



Turns out: this is actually sufficient.

Assume μ is defined for all $[\sigma]$ and satisfies $(*)$.

To extend μ to finite unions of cylinders, proceed as follows:

- Let $[\sigma_1] \cup \dots \cup [\sigma_n]$ be a finite union of cylinders

  If the $[\sigma_i]$ are pairwise disjoint, we simply put

$$\mu([\sigma_1] \cup \dots \cup [\sigma_n]) = \sum_{i=1}^{n} \mu[\sigma_i]$$

- What if the $[\sigma_i]$ are not disjoint?

  Then we cannot use def'n above, since we would count some measure multiple times.

Solution: Use the "nice" behavior of cylinders under intersections.

Observation: For cylinders $[\sigma]$, $[\tau]$ exactly one of the following holds:

(1) $[\sigma] \cap [\tau] = \emptyset$

(2) $[\sigma] \subseteq [\tau]$

(3) $[\tau] \subseteq [\sigma]$

Furthermore: (1) holds iff σ, τ incompatible

(2) iff $\sigma \sqsupseteq \tau$

(3) iff $\tau \sqsupseteq \sigma$

LEMMA: If $\{\sigma_1, \ldots, \sigma_n\}$ is a finite set of strings, there exists $\{\sigma_1^*, \ldots, \sigma_m^*\}$ s.t.

$$\bigcup_{i=1}^{n} [\sigma_i] = \bigcup_{j=1}^{m} [\sigma_j^*]$$

and for $1 \leq j < k \leq m$,

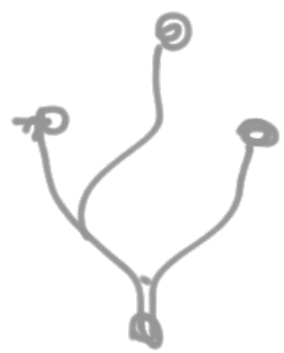
$$[\sigma_j^*] \cap [\sigma_k^*] = \emptyset$$

That is: the σ_j^* induce the same open set as the σ_i, but the $[\sigma_j^*]$ are pairwise disjoint.

Note: $[\sigma_j^*]$ p.w. disjoint means that

no σ_j^* is a prefix of another σ_k^*

Therefore such a set of strings is called prefix-free.



prefix-free

not prefix-free

Proof of Lemma (sketch):

- non-empty intersections between $[\sigma_i]$ can only arise if one σ_i extends another.
- If $\sigma_i \sqsupseteq \sigma_j$, $[\sigma_i]$ does not contribute anything to the open set, b/c it is already "covered" by $[\sigma_j]$
- Hence we can obtain $\{\sigma_1^*, \dots, \sigma_m^*\}$ by successively deleting extensions, till we remain with a prefix-free subset.

Using the Lemma, we can assign a measure to

$$[\sigma_1] \cup \dots \cup [\sigma_n]$$

as follows:

- Replace $\{\sigma_1, \dots, \sigma_n\}$ by prefix-free set of strings $\{\sigma_1^*, \dots, \sigma_m^*\}$ that generates the same open set.

- Put $\mu([\sigma_1] \cup \dots \cup [\sigma_n]) = \sum_{j=1}^{m} \mu[\sigma_j^*]$

What if there are many possible choices for $\{\sigma_1^*, \dots, \sigma_m^*\}$? Do we get the same measure?

LEMMA: If $\{\tau_1, \ldots, \tau_k\}$ and $\{u_1, \ldots, u_\ell\}$ are prefix-free sets of strings s.t.

$$\bigcup_{i=1}^{k} [\tau_i] = \bigcup_{j=1}^{\ell} [u_j]$$

and if μ is s.t. $\mu[\sigma] = \mu[\sigma^\frown 0] + \mu[\sigma^\frown 1]$, (f. all σ)

then

$$\sum_{i=1}^{k} \mu[\tau_i] = \sum_{j=1}^{\ell} \mu[u_j]$$