Lecture 6: Borel Sets as Clopen Sets

In this lecture we will learn that the Borel sets have the perfect subset property, which we already saw holds for closed subsets of Polish spaces.

The proof changes the underlying topology so that all Borel sets become clopen, and hence we can apply the Cantor-Bendixson Theorem 2.5.

We start by showing that topologically simple subspaces of Polish spaces are again Polish

Proposition 6.1: *If Y is an open or closed subset of a Polish space X, then Y is Polish, too (with respect to the subspace topology).*

Proof. Clearly subspaces of separable spaces are separable.

The statement for closed sets follows easily, since closed subsets of complete metric spaces are complete.

In case Y is open, suppose d is a compatible metric on X such that (X, d) is complete. Y may not be complete with respect to d, so we have to change the metric, but be careful not to change the induced topology.

First, replace the metric d by $\overline{d}$, given as

$$\overline{d}(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

This is again a metric, and it induces the same topology, for one can show that the identity mapping is a homeomorphism of (X, d) and $(X, \overline{d})$.

Now define

$$d_Y(x, y) = \overline{d}(x, y) + \left| \frac{1}{\overline{d}(x, X \setminus Y)} - \frac{1}{\overline{d}(y, X \setminus Y)} \right|,$$

where $\overline{d}(x, Z) = \inf\{d(x, z) \colon z \in Z\}$. With some effort, one can show that this is again a metric.

To show it is compatible with the subspace topology, assume $x_n \to z$ in (Y, d_Y). Since $d \leq d_y$, we have $d(x_n, z) \to 0$, and hence $x_n \to z$ in (Y, d).

On the other hand, if $d(x_n,z) \to 0$ in Y, it follows that $\overline{d}(x_n,z) \to 0$. Furthermore, using the triangle-inequality, we can show that the sequence

$$\frac{1}{\overline{d}(x_n,X \setminus Y)} - \frac{1}{\overline{d}(z,X \setminus Y)}$$

also goes to zero. Hence $x_n \to z$ in (Y,d_Y).

Finally, assume (x_n) is Cauchy in (Y,d_Y). Since $d \leq d_Y$, it is also Cauchy in (X,d), hence the exists $x \in X$ with $x_n \to x$. Using the triangle-inequality, one can further show that the sequence

$$\frac{1}{\overline{d}(x_n,X \setminus Y)}$$

is Cauchy in $\mathbb{R}$. Hence there exists $r \in \mathbb{R}$ such that

$$r = \lim_n \frac{1}{\overline{d}(x_n,X \setminus Y)}.$$

Now, since $\overline{d} < 1$, r cannot be 0, and hence

$$\frac{1}{\overline{d}(x_n,X \setminus Y)}$$

is bounded away from 0, which implies (triangle-inequality) that $d(x,X \setminus Y)$ is bounded away from 0, too. But this means $x \in Y$, hence (x_n) converges in (Y,d_Y). $\square$

One can strengthen this result to G_δ sets, in fact, the Polish subspaces of Polish spaces are *precisely* the G_δ subsets.

Theorem 6.2: *A subset of a Polish space is Polish (with the subspace topology) if and only if it is* $\mathbf{\Pi}^0_2$.

For a proof see Kechris [1995].

Next we show that the topology can be refined to make closed subsets clopen.

Lemma 6.3: *If X is a Polish space with topology $\mathcal{O}$, and $F \subseteq X$ is closed, then there exists a finer topology $\mathcal{O}' \supseteq \mathcal{O}$ such that $\mathcal{O}$ and $\mathcal{O}'$ give rise to the same class of Borel sets in X, and F is clopen with respect to $\mathcal{O}'$.*

Proof. By Proposition 6.1, F and $X \setminus F$ are Polish spaces with compatible metrics d_F and $d_{X \setminus F}$, respectively. Wlog $d_F, d_{X\setminus F} < 1$. We form the *disjoint union* of the *spaces* F and $X \setminus F$: This is the set $X = F \sqcup X \setminus F$ with the following topology $\mathcal{O}'$. $U \subseteq F \sqcup X \setminus F$ is in $\mathcal{O}'$ if and only if $U \cap F$ is open (in F) and $U \cap X \setminus F$ is open (in $X \setminus F$).

The disjoint union is Polish, as witnessed by the following metric.

$$d_{\sqcup}(x, y) = \begin{cases} d_F(x, y) & \text{if } x, y \in F, \\ d_{X\setminus F}(x, y) & \text{if } x, y \in X \setminus F, \\ 2 & \text{otherwise.} \end{cases}$$

It is straightforward to check that d is compatible with $\mathcal{O}'$. Furthermore, let (x_n) be Cauchy in $(X, d_{\sqcup})$. Then the x_n are completely in F or in $X \setminus F$ from some point on, and hence (x_n) converges.

Under the disjoint union topology, F is is clopen. Moreover, an open set in this topology is a disjoint union of an open set in $X \setminus F$, which also open the original topology $\mathcal{O}$, and an intersection of an open set from $\mathcal{O}$ with F. Such sets are are Borel in $(X, \mathcal{O})$, hence $(X, \mathcal{O})$ and $(X, \mathcal{O}')$ have the same Borel sets. □

Theorem 6.4: *Let X be a Polish space with topology $\mathcal{O}$, and suppose $B \subseteq X$ is Borel. Then there exists a finer topology $\mathcal{O}' \supseteq \mathcal{O}$ such that $\mathcal{O}$ and $\mathcal{O}'$ give rise to the same class of Borel sets in X, and F is clopen with respect to $\mathcal{O}'$.*

Proof. Let $\mathcal{S}$ be the family of all subsets A of X for which a finer topology exists that has the same Borel sets as $\mathcal{O}$ and in which A is open.

We will show that $\mathcal{S}$ is a σ-algebra, which by the previous Lemma contains the closed sets. Hence $\mathcal{S}$ must contain all Borel sets, and we are done.

$\mathcal{S}$ is clearly closed under complements, since the complement of a clopen set is clopen in any topology.

So assume now that $\{A_n\}$ is a countable family of sets in $\mathcal{S}$. Let $\mathcal{O}_n$ be a topology on X that makes A_n clopen and does not introduce new Borel sets.

Let $\mathcal{O}_\infty$ be the topology generated by $\bigcup_n \mathcal{O}_n$. Then $\bigcup_n A_n$ is open in $(X, \mathcal{O}_\infty)$, and we can apply Lemma 6.3. For this to work, however, we have to show that $(X, \mathcal{O}_\infty)$ is Polish and does not introduce any new Borel sets.

We know that the *product space* $\prod(X, \mathcal{O}_n)$ is Polish. Consider the mapping $\varphi : X \to \prod_n X$

$$x \mapsto (x, x, x, \dots).$$

Observe that φ is a continuous mapping between $(X, \mathcal{O}_\infty)$ and $\prod_n X$. The preimage of a basic open set $U_1 \times U_2 \times \cdots \times U_n \times X \times X \times \cdots$ under φ is just the intersection of the U_i. Furthermore, φ is clearly one-to-one, and the inverse mapping between $\varphi(X)$ and X is continuous, too.

If we can show that $\varphi(X)$ is closed in $\prod_n X$, we know it is Polish as a closed subset of a Polish space, and since $(X, \mathcal{O}_\infty)$ is homeomorphic to $\varphi(X)$, we can conclude it is Polish.

To see that $\varphi(X)$ is closed in $\prod_n X$, let $(y_1, y_2, y_3, \dots) \in \neg\varphi(X)$. Then there exist $i < j$ such that $y_i \neq y_j$. Since $(X, \mathcal{O})$ is Polish, we can pick U, V open, disjoint such that $y_i \in U$, $y_j \in V$. Since each $\mathcal{O}_n$ refines $\mathcal{O}$, U is open in $\mathcal{O}_i$, and V is open in $\mathcal{O}_j$. Therefore,

$$X_1 \times X_2 \times \cdots \times X_{i-1} \times U \times X_{i+1} \times \cdots \times X_{j_1} \times V \times X_{j+1} \times X_{j+2} \times \cdots$$

where $X_k = X$ for $k \neq i, j$, is an open neighborhood of $(y_1, y_2, y_3, \dots)$ completely contained in $\neg\varphi(X)$.

Finally, too see that the Borel sets of $(X, \mathcal{O}_\infty)$ are the same as the ones of $(X, \mathcal{O})$, for each n, let $\{U_i^{(n)}\}_{i \in \mathbb{N}}$ be a basis for $\mathcal{O}_n$. By assumption, all sets in $\mathcal{O}_n$ are Borel sets of $(X, \mathcal{O})$. The set $\{U_i^{(n)}\}_{i,n \in \mathbb{N}}$ is a subbasis for $\mathcal{O}_\infty$. This means that any open set in $(X, \mathcal{O}_\infty)$ is a countable union of finite intersections of the $U_i^{(n)}$. Since every $U_i^{(n)}$ is Borel in $(X, \mathcal{O})$, this means that any open set in $\mathcal{O}_\infty$ is Borel in $(X, \mathcal{O})$. Since the Borel sets are closed under complementation and countable unions, this in turn implies that very Borel set of $(X, \mathcal{O}_\infty)$ is already Borel in $(X, \mathcal{O})$.

□

Corollary 6.5 (Perfect subset property for Borel sets): *In a Polish space, every uncountable Borel set has a perfect subset.*

Proof. Let $(X, \mathcal{O})$ be Polish, and assume $B \subseteq X$ is Borel. We can choose a finer topology $\mathcal{O}' \supseteq \mathcal{O}$ so that B becomes clopen, but the Borel sets stay the same. B is Polish with respect to the subspace topology $\mathcal{O}'|_B$

By Theorem 2.4, there exists a continuous injection f from $2^{\mathbb{N}}$ (with respect to the standard topology) into $(B, \mathcal{O}'|_B)$. Since $2^{\mathbb{N}}$ is compact, $f(2^{\mathbb{N}})$ is closed in $(B, \mathcal{O}'|_B)$. Since $\mathcal{O}' \supseteq \mathcal{O}$, every closed set in $(B, \mathcal{O}'|_B)$ is also closed in $\mathcal{O}$. Likewise, f is continuous between $2^{\mathbb{N}}$ and $(B, \mathcal{O}|_B)$, too. Therefore, $f(2^{\mathbb{N}})$ has no isolated points with respect to $\mathcal{O}$. It follows that $f(2^{\mathbb{N}})$ is perfect with respect to $\mathcal{O}$. □