Math 497A: Introductory to Ramsey Theory

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8/26/11

1 About Ramsey-type Results

In the previous section, we saw a number of Ramsey-type results. Even though they come from different fields, graph theory, number theory, and combinatorial geometry, even the greenest mathematician can notice that these results are similar in fundamental ways. There are two phenomena which we can now discuss that overshadow most Ramsey-type results.

1. Most Ramsey-type theorems only say that at least one of the sets in the partition contains a regular substructure. However, we usually do not have knowledge as to which set it is. This leads us to the following natural question: if we know more about one of the sets in the partition, can we guarantee that it contains or lacks a regular substructure?

   An example of a positive answer to this question is Szemeredi's Theorem, proved in 1975, which is stated below. In this theorem, the extra condition known about the set is its natural density, specifically that the set in question is never too sparse in the natural numbers. Theorems which use arguments such as this are known as density results.

   Theorem 1. (Szemeredi's Theorem) If A is a subset of $\mathbb{N}$ such that

   $$\limsup_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n} > 0$$

   then A contains arbitrarily long arithmetic progressions.

   Remark: It should be noted that the inverse statement is not true; although the set of primes has a natural density of 0, it still contains aribtrarily long arithmetic progressions. This is known as the Green-Tao Theorem and is considered to be one of the most important mathematical results in the late 20^{th} century.

2. Often times, little is known about the threshold numbers obtained from Ramsey-type theorems, e.g. $R(k)$, $W(k)$, or $HJ(k)$. These values often grow very rapidly and even for small

values of k, we don't have an exact value. An exact value for $R(k)$ is unknown for $k \geq 5$; likewise, we don't know $W(k)$ for $k \geq 8$. We do however have a bound for $W(k)$; it is known that

$$\frac{2^k}{k^\varepsilon} \leq W(k) \leq 2^{2^{2^{2^{2^{k+9}}}}}$$

for all $\varepsilon > 0$.

It might seem as though the upper bound is trivial or absurd, but it was in fact a great accomplishment to obtain even that closed form for a bound. For many Ramsey-type theorems, elementary upper bounds are not known and sometimes it can be proved that no elementarily computable upper bound even exists.

2 Proof of Ramsey's Theorem on Graphs

In this section, we will provide the proof of the full Ramsey Theorem on Graphs. Let us being by restating the theorem

Theorem 2. (Ramsey's Theorem on Graphs) For every natural number k, there exists an $R(k)$ such that if $G = (V, E)$ is a complete graph on at least $R(k)$ vertices and we have a coloring

$$c : E \to \{\text{red, blue}\}$$

then G has a complete monochromatic subgraph on k vertices.

Proof. Let G be a complete graph on N vertices where N is arbitrarily large. Pick an arbitrary vertex $v_1 \in G$ and consider the $N-1$ edges which connect to it. By the pidgeonhole principle, at least $\frac{N}{2}$ of these edges must be the same color, say c_1. Now, we have partitioned all the vertices besides v_1 into two subsets: those that connect to v_1 via c_1, call this subset V_2, and the rest. Let G_2 be the induced subgraph on the vertices in V_2. Now, repeat this process by choosing a vertex $v_2 \in G_2$. Then by the pidgeonhole principle, there are at least $\frac{N}{2}$ edges connected to v_2 in G_2 which are of the same color, call this color c_2, and we can use the vertices adjacent to v_2 via these edges to form a new induced subgraph G_3.

If we continue this process, we get a sequence of vertices, colors, and subgraphs $G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots$ where the vertex v_i is connected to every vertex in G_{i+1} via an edge of the color c_i. We can again invoke the pidgeonhole principle to say that at least half of the colors c_i are equal, say c_0 and consider all the vertices v_i such that $c_i = c_0$. We now know that these vertices are all connected to each other via the same color so we simply just need N to be large enough so that we end up with at least k vertices.

$\square$

A plesant thing about this proof is that it gives us a nice approximation for $R(k)$. At each step in the proof, we were, on average, dividing our initial graph in half. Moreoever, we were trying to end with a sequence of $2k$ vertices so that we can choose k to form our monochromatic subgraph. Therefore, we needed our threshold number of vertices to start with to be approximately

$$R(k) \approx 2^{2k}.$$