Homework 4 for MATH 185

Brief sketches of solutions

Problem 1

For $r = 1, 3, 5$ compute the following integral:

$$\oint_{|\xi-2|=r} \frac{\exp(\xi^2)}{(\xi^2 - 6\xi)} d\xi.$$

Solution. For $r = 1$, the function $\frac{\exp(\xi^2)}{(\xi^2-6\xi)}$ is analytic on the disk $|z-2| \leqslant 1$, so by the Cauchy integral theorem the integral is 0.

For $r = 3$, we set $f(z) = \frac{\exp(z^2)}{(z-6)}$. f is analytic on $|z-2| \leqslant 3$, so the Cauchy integral formula yields:

$$\oint_{|\xi-2|=3} \frac{\exp(\xi^2)}{(\xi^2 - 6\xi)} d\xi = \oint_{|\xi-2|=3} \frac{f(\xi)}{(\xi} d\xi = 2\pi i f(0) = \pi i/3.$$

For $r = 5$, we cannot apply this strategy any longer, since both 'critical points' $z = 0, 6$ lie inside the disk $|z-2| \leqslant 5$. We resort to a partial fraction decomposition:

$$\frac{\exp(z^2)}{z(z-6)} = \frac{-\exp(z^2)}{6z} + \frac{\exp(z^2)}{6(z-6)}.$$

Now we can apply the Cauchy integral formula:

$$\oint_{|\xi-2|=5} \frac{\exp(\xi^2)}{(\xi^2 - 6\xi)} d\xi = -\oint_{|\xi-2|=5} \frac{\exp(\xi^2)}{6\xi} d\xi + \oint_{|\xi-2|=5} \frac{\exp(\xi^2)}{6(\xi-6)} d\xi = -2\pi i e^0/6 + 2\pi i e^{6^2}/6 = \pi i(e^{36} - 1)/3.$$

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Problem 2

We will prove that the Cauchy integral formula holds in a much more general form. In particular,

$$f'(z) = \frac{1}{2\pi i} \oint_{|\xi-z_0|=r} \frac{f(\xi)}{(\xi - z)^2} d\xi,$$

for every z with $|z - z_0| < r$.

Use this to show that if $f : \mathbb{C} \to \mathbb{C}$ is analytic and $\lim_{z\to\infty} f(z)/z = 0$, then f is constant.

Solution. First observe that $|f(z)/z| \to 0$ for $|z| \to \infty$ implies $|f(z)/(z-a)| \to 0$ for $|z| \to \infty$. Let $a \in \mathbb{C}$. We claim that $f'(a) = 0$, so it follows that f is constant. Let $\varepsilon > 0$. By assumption, there exists an $R > 0$ such that $|f(z)/(z-a)| < \varepsilon$ for all z such that $|z-a| \geqslant R$. The Cauchy integral formula says that

$$f'(a) = \frac{1}{2\pi i} \oint_{|\xi-a|=R} \frac{f(\xi)}{\xi - a} d\xi.$$

We can use the standard estimate to obtain

$$|f'(a)| \leqslant \frac{1}{2\pi} \frac{\varepsilon}{R} l(|\xi - a| = R) = \frac{1}{2\pi} \frac{\varepsilon}{R} 2\pi R = \epsilon.$$

Since ε was arbitray, the result follows. ■