Homework 9 for MATH 104

Due: Tuesday, November 21, 9:30am in class

Problem 1

(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}$. Show that

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x). \qquad (*)$$

(b) Find an example of a function $g : \mathbb{R} \to \mathbb{R}$ such that the limit in (*) exists for some $x \in \mathbb{R}$ but g is not even continuous at x.

Problem 2

Consider the functions

$$f(x) = \sin(\tfrac{1}{x}) \quad g(x) = x\sin(\tfrac{1}{x}) \quad h(x) = x^2\sin(\tfrac{1}{x}) \quad \text{for } x \neq 0,$$

and set $g(0) = h(0) = 0$.

(a) Show that f cannot be extended continuously to $x = 0$, i.e. show that there is no continuous function $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ such that $\widetilde{f}(x) = f(x)$ for all $x \neq 0$.

(b) Show that g is continuous but not differentiable at $x = 0$.

(c) Show that h is differentiable at $x = 0$ but h' is not continuous at $x = 0$.

Problem 3

(a) Use the mean value theorem to show that

$$\sqrt{1+x} < 1 + \frac{x}{2} \quad \text{for all } x > 0.$$

(b) Suppose f that differentiable on $\mathbb{R}$, that $1 \leqslant f'(x) \leqslant 2$ for all $x \in \mathbb{R}$ and that $f(0) = 0$. Show that $x \leqslant f(x) \leqslant 2x$ for all $x \geqslant 0$.

Problem 4

Let f be differentiable on $\mathbb{R}$ with $a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$. Select $s_0 \in \mathbb{R}$ and define $s_n = f(s_{n-1})$ for $n \geqslant 1$. Show that (s_n) converges.

[*Hint*: Prove the inequality $|s_{n+1} - s_n| \leqslant a|s_n - s_{n-1}|$.]