Homework 6 for MATH 104

Due: Tuesday, October 24, 9:30am in class

Problem 1

Define the function $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ with } p \in \mathbb{Z}, q \in \mathbb{Z} \setminus \{0\} \text{ relatively prime}, \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Show that g is continuous at all irrational points, and discontinuous at all rational points.

Problem 2

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Let the *zero set* of f be defined as

$$Z(f) = \{x \in \mathbb{R} : \ f(x) = 0\}.$$

Show that $Z(f)$ is a closed subset of $\mathbb{R}$.

Problem 3

Call a mapping $f : \mathbb{R} \to \mathbb{R}$ *open* if for every open set $U \subseteq \mathbb{R}$, the image

$$f(U) = \{f(x) : \ x \in U\}$$

is open. Show that a continuous open mapping $f : \mathbb{R} \to \mathbb{R}$ is monotonic.

Problem 4

(a) Let f, g be continuous mappings from $\mathbb{R}$ into $\mathbb{R}$. Further, let D be a dense subset of $\mathbb{R}$. Show that $f(D)$ is dense in $f(\mathbb{R})$. Furthermore, show that if $f(x) = g(x)$ for all $x \in D$, then $f(z) = g(z)$ for all $z \in \mathbb{R}$ (This shows that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is uniquely determined by its values on D.)

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that f is of the form $f(x) = c \cdot x$ for some $c \in \mathbb{R}$.