

Lecture 15: The Constructible Universe

A set X is *first-order definable* in a set Y (from parameters) if there exists a first-order formula $\varphi(x_0, x_1, \dots, x_n)$ in the language of set theory (i.e. only using the binary relation symbol \in) such that for some $a_1, \dots, a_n \in Y$,

$$X = \{y \in Y : (Y, \in) \models \varphi[y, a_1, \dots, a_n]\}.$$

Here (Y, \in) stands for the interpretation of Y as a structure of the language of set theory, i.e. Y is a set and \in is interpreted as a binary relation over Y .

The constructible universe is built as a cumulative hierarchy of sets along the ordinals. In each successor step, instead of adding all subsets of the current set, only the *definable* ones are added. Formally, L is defined as follows. Given a set Y , let

$$\mathcal{P}_{\text{DEF}}(Y) = \{X \subseteq Y : X \text{ is first order definable in } Y \text{ from parameters}\},$$

where the underlying language is the language of set theory. Now put

$$\begin{aligned} L_0 &= \emptyset \\ L_{\xi+1} &= \mathcal{P}_{\text{DEF}}(L_\xi) \\ L_\xi &= \bigcup_{\zeta < \xi} L_\zeta \quad (\xi \text{ limit ordinal}) \end{aligned}$$

Finally, let

$$L = \bigcup_{\xi \in \text{Ord}} L_\xi.$$

Basic properties of L

The first Proposition tells us that the L_ξ are set-theoretically nice structures and linearly ordered by the \subseteq -relation.

Proposition 15.1: *For each ordinal ξ :*

- (1) L_ξ is transitive.
- (2) For $\zeta < \xi$, $L_\zeta \subseteq L_\xi$.
- (3) For $\zeta < \xi$, $L_\zeta \in L_\xi$.

(4) If $\xi \geq \omega$, then $|L_{\xi+1}| = |L_\xi|$.

Proof. We show the first two statements simultaneously by induction. They are clear for $\xi = 0$ and ξ limit, so assume $\xi = \zeta + 1$. Suppose $x \in L_\zeta$. Consider the formula $\varphi(x_0) \equiv x_0 \in x$ (here x is a parameter). φ defines the set

$$x' = \{a \in L_\zeta : L_\zeta \models \varphi[a]\} = \{a \in L_\zeta : a \in x\}.$$

By induction hypothesis, L_ζ is transitive, and hence $a \in x$ implies $a \in L_\zeta$, and hence $x' = x$, so $x \in L_{\zeta+1}$. This yields $L_\zeta \subseteq L_\xi$. Now if $x \in L_\xi$, then $x \subseteq L_\zeta$, and hence $x \subseteq L_\xi$. Thus L_ξ is transitive.

For the third statement, note that the formula $\varphi(x_0) \equiv x_0 = x_0$ defines L_ζ in L_ζ , and hence $L_\zeta \in L_{\zeta+1}$. \square

Next, we show that L contains all ordinals and that ξ ‘shows up’ exactly after ξ steps.

Proposition 15.2: For any ξ ,

(1) $\xi \subseteq L_\xi$,

(2) $L_\xi \cap \text{Ord} = \xi$.

Proof. Clearly, (1) follows from (2). To show (2), one again proceeds by induction. Again, the statement is clear for 0 and limit ordinals, so assume $\xi = \zeta + 1$ and $L_\zeta \cap \text{Ord} = \zeta$. We need to show $L_{\zeta+1} \cap \text{Ord} = \zeta + 1 = \zeta \cup \{\zeta\}$. Since $L_\zeta \subseteq L_{\zeta+1}$, we have $\zeta \subseteq L_{\zeta+1} \cap \text{Ord}$. On the other hand, since $L_{\zeta+1} \subseteq \mathcal{P}(L_\zeta)$, we have $L_{\zeta+1} \cap \text{Ord} \subseteq \zeta + 1$. It thus remains to show that $\zeta \in L_{\zeta+1}$.

We need a formula φ_{Ord} that defines the ordinals (in L_ζ). Such a formula is easily found by formalizing the statement

“ x is transitive and linearly ordered by \in .”

(Note that we assume that every set is well-founded.) It then seems that we have

$$\zeta = \{a \in L_\zeta : L_\zeta \models \varphi_{\text{Ord}}[a]\}, \quad (*)$$

and hence we can conclude $\zeta \in L_{\zeta+1}$. The problem is that *being an ordinal* (i.e. satisfying φ_{Ord}) in L_ζ may not be the same as *being an ordinal in general* (in V , the universe of all sets).

The fact that $(*)$ nevertheless is true is a consequence of the *absoluteness of Δ_0 formulas for transitive sets*. We address this important concept in detail next. \square

Given a formula φ in the language of set theory and some class M , we can *relativize* φ to φ^M essentially by restricting all quantifiers occurring in φ to range over M , i.e. $\exists x \psi$ becomes $(\exists x \in M) \psi^M$, for example. We say a formula $\varphi(x_0, \dots, x_n)$ is **absolute for M** if for all $a_0, \dots, a_n \in M$

$$\varphi^M(a_0, \dots, a_n) \text{ holds} \iff \varphi(a_0, \dots, a_n) \text{ holds.}$$

Unfortunately, even simple formulas like $x \subseteq y$ can fail to be absolute. For example, let $M = \{0, a\}$, where $a = \{\{0\}\}$. Then $(a \subseteq 0)^M$ (which is defined as $\forall x \in M (x \in a \rightarrow x \in 0)$) but not $a \subseteq 0$.

However, if M is transitive, then many important formulas *are* absolute for M . A formula is Δ_0 if it contains no or only *bounded quantifiers* of the form $\forall x \in v$ or $\exists x \in v$, where x, v are set variables.

Proposition 15.3: *If M is transitive and φ is Δ_0 , then φ is absolute for M .*

Proof sketch. Clearly $x = y$ and $x \in y$ are absolute for any M . It is also not hard to see that if φ and ψ are absolute for M , then so are $\neg\varphi$ and $\varphi \wedge \psi$. Hence all quantifier free formulas are absolute.

Finally, if φ is absolute for M , so is $\psi \equiv \exists x \in y \varphi$: If $\psi^M(y, \bar{z})$ holds for $y, \bar{z} \in M$, then we have $[\exists x (x \in y \wedge \varphi(x, y, \bar{z}))]^M$, i.e., $\exists x \in M (x \in y \wedge \varphi^M(x, y, \bar{z}))$. Since $\varphi^M(x, y, \bar{z})$ if and only if $\varphi(x, y, \bar{z})$, it follows that $\exists x \in y \varphi(x, y, \bar{z})$, i.e. ψ .

Conversely, if for $y, \bar{z} \in M$, $\exists x \in y \varphi(x, y, \bar{z})$, then since M is transitive, x belongs to M , and since $\varphi(x, y, \bar{z})$ if and only if $\varphi^M(x, y, \bar{z})$, we have $\exists x \in M (x \in y \wedge \varphi^M(x, y, \bar{z}))$ and so $\psi^M(y, \bar{z})$. \square

One can show that “ x is an ordinal.” is indeed definable by a Δ_0 formula. Furthermore, using the absoluteness of Δ_0 formulas, one can also show that L is a model of ZF. More formally, this means that for every axiom σ of ZF, $\text{ZF} \vdash \sigma^L$.

Theorem 15.4: *For every axiom σ of ZF, $\text{ZF} \vdash \sigma^L$.*

L is an **inner model** of ZF, that is, L is transitive, contains all ordinals, and satisfies the axioms of ZF.

We can add to ZF the axiom that all sets are constructible, i.e.

$$\forall x \exists y (y \text{ is an ordinal} \wedge x \in L_y).$$

This axiom is usually denoted by $V = L$. We may be tempted to think that L is then trivially a model of $ZF + V = L$. But this is not at all clear, since this has to hold *relative to* L , i.e. $(V = L)^L$. This means that

$$\forall x \in L \exists y \in L (y \text{ is an ordinal} \wedge (x \in L_y)^L).$$

But it might be that $(x \in L_y)^L$ is not absolute, that is, viewed from *inside* L , not every set may be definable. To show that $(x \in L_y)^L$ is indeed absolute, one has to carefully study the notion of definability. In particular, we have to show that *definability is definable*.

The definability of L

Fix a Gödel numbering of set theoretic formulae. We can use it to formally define syntactical notions such as the satisfaction relation. More precisely, given a set X , let $\text{SAT}_X : X^{<\omega} \times \omega \rightarrow \{0, 1\}$ be the binary valued (partial) function that is defined for $((a_1, \dots, a_n), e)$ iff e is the Gödel number of a formula φ with n free variables and in this case

$$\text{SAT}_X(\vec{a}, e) = 1 \quad \text{iff} \quad X \models \varphi[\vec{a}]. \quad (15.1)$$

While Tarski's Theorem excludes the possibility that a structure X satisfying (a sufficiently large fragment of) ZFC can define its own truth predicate, one can formalize the satisfaction relation and show it works in a *relativized* environment that has sufficient closure properties. To be more precise, based on the recursive definition of the satisfaction relation one can devise a set theoretic formula φ_{SAT} aiming at describing this relation formally. This formula will “work” in any relativized environment, represented by a set Y , as long as Y satisfies some basic closure properties – it has to be transitive, closed under formation of finite sequences, and has to be able to address the Gödel numbers of formulas, i.e. it contains the natural numbers. The latter can be ensured by requiring that $V_\omega \subseteq Y$. Let us call such Y *adequate*.

Proposition 15.5: *There exists a set theoretic formula $\varphi_{\text{SAT}}(x_0, x_1)$ such that for all adequate Y , whenever $a_0, a_1 \in Y$,*

$$Y \models \varphi_{\text{SAT}}[a_0, a_1] \quad \text{iff} \quad \begin{array}{l} (1) a_0 \text{ is transitive and} \\ (2) a_1 = \text{SAT}_{a_0}. \end{array}$$

Based on φ_{SAT} , one can devise a formula φ_{DEF} with the following properties: Suppose Y is adequate. Then for all $a_0, a_1 \in Y$, if $\text{SAT}_{a_0} \in Y$ then

$$Y \models \varphi_{\text{DEF}}[a_0, a_1] \quad \text{iff} \quad \begin{aligned} &(1) \ a_0 \text{ is transitive and} \\ &(2) \ a_1 = \mathcal{P}_{\text{DEF}}(a_0). \end{aligned}$$

In other words, first-order definability is definable. With regard to absoluteness considerations, it is important to track the complexity of the formulas. It turns out φ_{DEF} is provably equivalent in ZF to both a Σ_1 and a Π_1 formula of set theory. In this case we say the predicate $a_1 = \mathcal{P}_{\text{DEF}}(a_0)$ is Δ_1 .

It is not hard to see that for limit ξ , L_ξ is adequate. One can use these closure properties of L_ξ at limit stages to show that L_ξ can definably “recover” the sequence of L_ζ ’s leading up to it.

Proposition 15.6: *Suppose ξ is a limit ordinal, $\xi > 0$. Let $G : \text{Ord} \rightarrow V$ be given by $\zeta \mapsto L_\zeta$. Then for all $\zeta < \xi$, $G|_\zeta \in L_\xi$.*

If a formula $\varphi(\bar{x}, y)$ defines a function $F(\bar{x}) = y$, then we say F is *absolute* for M if φ is. (It is not hard to show that this is independent of the particular definition of F .) One can in fact show that the function G is absolute for all transitive models of ZF (it is Δ_1).

Theorem 15.7: *L is a model of $\text{ZF} + V = L$.*

Proof. If $x \in L$, then there exists a limit ξ such that $x \in L_\xi$. Since $\text{Ord} \subseteq L$, and since G is absolute,

$$\begin{aligned} \forall x \in L \exists \xi \in L (x \in L_\xi) &\quad \Leftrightarrow \quad \forall x \in L \exists \xi, y \in L (G(\xi) = y \wedge x \in y) \\ &\quad \Leftrightarrow \quad \forall x \in L \exists \xi \in L [(x \in L_\xi)^L]. \end{aligned}$$

□

A further consequence is that L is the *smallest* transitive class model of ZF.

Theorem 15.8: *If M is any transitive proper class model of ZF, then $L = L^M \subseteq M$.*

The crucial fact used to prove Proposition 15.6 is that for transitive $X \in L_\xi$,

$$\begin{aligned} \text{SAT}_X &\text{ can be “reached” from } X \\ &\text{ within a finite number of iterations of the } \mathcal{P}_{\text{DEF}}\text{-operator.} \quad (15.2) \end{aligned}$$

From this it follows that $\text{SAT}_X \in L_\xi$, and hence L_ξ is closed under the SAT-function.

Note that the proposition is *not* an immediate consequence of the definition of L . Although we have that $L_\zeta \in L_\xi$ for all $\zeta < \xi$, it is not clear at all that in L_ξ one can define the whole ensemble of the L_ζ in first order terms. The proposition says that we can *definably* recover them in L_ξ : There is a formula $\varphi_L(x_0, x_1)$ such that for limit ξ , given $a_0, a_1 \in L_\xi$,

$$L_\xi \models \varphi_L[a_0, a_1] \quad \text{iff} \quad a_0 \text{ is an ordinal and } a_1 = L_{a_0}.$$

As a consequence, one can devise a sentence $\varphi_{V=L}$ that identifies precisely the limit levels of the constructible hierarchy: For any transitive set Y ,

$$Y \models \varphi_{V=L} \quad \text{iff} \quad Y = L_\xi \text{ for some limit ordinal } \xi.$$

This last fact has a far-reaching implication.

Theorem 15.9 (Gödel Condensation Lemma): *For every limit ordinal ζ , every elementary substructure of (L_ζ, \in) is isomorphic to an (L_η, \in) for some $\eta \leq \zeta$.*

The canonical well-ordering of L

Every well-ordering on a transitive set X can be extended to a well-ordering of $\mathcal{P}_{\text{DEF}}(X)$. Note that every element of $\mathcal{P}_{\text{DEF}}(X)$ is determined by a pair (ψ, \vec{a}) , where ψ is a set-theoretic formula, and $\vec{a} = (a_1, \dots, a_n) \in X^{<\omega}$ is a finite sequence of parameters. For each $z \in \mathcal{P}_{\text{DEF}}(X)$ there may exist more than one such pair (i.e. z can have more than one definition), but by well-ordering the pairs (ψ, \vec{a}) , we can assign each $z \in \mathcal{P}_{\text{DEF}}(X)$ its *least* definition, and subsequently order $\mathcal{P}_{\text{DEF}}(X)$ by comparing least definitions. Elements already in X will form an initial segment. Such an order on the pairs (ψ, \vec{a}) can be obtained in a definable way: First use the order on X to order $X^{<\omega}$ length-lexicographically, order the formulas through their Gödel numbers, and finally say

$$(\psi, \vec{a}) < (\varphi, \vec{b}) \quad \text{iff} \quad \psi < \varphi \text{ or } (\psi < \varphi \text{ and } \vec{a} < \vec{b}).$$

Based on this, we can order all levels of L so that the following hold:

- (1) $<_L \upharpoonright V_\omega$ is the canonical well-order on V_ω .
- (2) $<_L \upharpoonright L_{\zeta+1}$ is the order on $\mathcal{P}_{\text{DEF}}(L_\zeta)$ induced by $<_L \upharpoonright L_\zeta$.
- (3) $<_L \upharpoonright L_\xi = \bigcup_{\zeta < \xi} <_L \upharpoonright L_\zeta$ for a limit ordinal $\xi > \omega$.

It is straightforward to verify that this is indeed a well-ordering on L . But more importantly, for any limit ordinal $\xi > \omega$, $<_L \upharpoonright L_\xi$ is definable over L_ξ . To facilitate notation, we denote the restriction of $<_L$ to some L_ξ by $<_\xi$.

Proposition 15.10: *There is a Σ_1 formula $\varphi_<(x_0, x_1)$ such that for all limit ordinals $\xi > \omega$, if $a, b \in L_\xi$,*

$$L_\xi \models \varphi_<[a, b] \quad \text{iff} \quad a <_\xi b.$$

The proof of this proposition is similar to the proof that the sequence of $(L_\zeta)_{\zeta < \xi}$ is definable in L_ξ . It relies on the strong closure properties of L_ξ under the SAT-function.

Theorem 15.11: *If $V = L$ then AC holds.*

The Continuum Hypothesis in L

We show that the Generalized Continuum Hypothesis (GCH) holds if $V = L$.

Theorem 15.12 (Gödel): *If $V = L$, then for any ordinal ξ , $2^{\aleph_\xi} = \aleph_{\xi+1}$.*

Proof sketch. Suppose $A \subseteq L \cap \aleph_\xi$. Since we assume $V = L$, there exists limit $\delta > \aleph_\xi$ such that $A \in L_\delta$. Let $X = \aleph_\xi \cup \{A\}$. By choice of δ , $X \subseteq L_\delta$. The Löwenheim-Skolem Theorem (and a Mostowski collapse – see Lecture 16) yields a set M such that

- (M, \in) is a transitive, elementary substructure of (L_δ, \in) ,
- $X \subseteq M \subseteq L_\delta$,
- $|M| = |X|$.

The Condensation Lemma 15.9 yields that $M = L_\zeta$ for some $\zeta \leq \delta$.

Lemma 15.13: *For all $\xi \geq \omega$, $|L_\xi| = |\xi|$.*

Proof of Lemma. We know that $\xi \subseteq L_\xi$. Hence $|\xi| \leq |L_\xi|$. To show $|\xi| \geq |L_\xi|$, we work by induction on ξ .

If $\xi = \delta + 1$, then by Proposition 15.1 (4), $|L_\xi| = |L_\delta| = |\delta| \leq |\xi|$.

If ξ is limit, then L_ξ is a union of $|\xi|$ many sets of cardinality $\leq |\xi|$ (by inductive hypothesis), hence of cardinality $\leq |\xi|$. \square

Applying the lemma to $M = L_\zeta$, we obtain

$$|\zeta| = |L_\zeta| = |M| = |X| = \aleph_\xi < \delta.$$

Therefore, $A \subseteq L_\zeta$, $|\zeta| < \aleph_\xi$, which means that every subset of $L \cap \aleph_\xi$ appears (is constructed) at an ordinal $< \aleph_{\xi+1}$, and therefore $L \cap \mathcal{P}(\aleph_\xi) \subseteq L_{\aleph_{\xi+1}}$, and hence, by the Lemma,

$$|L \cap \mathcal{P}(\aleph_\xi)| \leq |L_{\aleph_{\xi+1}}| = \aleph_{\xi+1}.$$

□

In the previous proof we have used the Axiom of Choice in various places (Löwenheim-Skolem, proof of the lemma), but since $V = L$ implies AC, this is not a problem.