Lecture 14: The Projective Hierarchy

In Lecture 12 we saw that the analytic sets are not closed under complements, which led us to the introduction of the *co-analytic* sets as a separate class.

We saw analytic sets are projections of closed sets and hence can be written as

$$x \in A \quad \Leftrightarrow \quad \exists \alpha \in \mathbb{N}^{\mathbb{N}} \, F(\alpha, x),$$

where $F \subseteq \mathbb{N}^{\mathbb{N}} \times X$ is closed. It follows that co-analytic sets can be written in the form

$$x \in A \quad \Leftrightarrow \quad \forall \alpha \in \mathbb{N}^{\mathbb{N}} \, U(\alpha, x),$$

for some open $U \subseteq \mathbb{N}^{\mathbb{N}} \times X$.

Using quantifier manipulations that allow to switch number and function quantifiers,

$$\begin{array}{lcl} \forall m \, \exists \alpha \, P(m, \alpha) & \Leftrightarrow & \exists \beta \, \forall m \, P(m, (\beta)_m) \\ \exists m \, \forall \alpha \, P(m, \alpha) & \Leftrightarrow & \forall \beta \, \exists m \, P(m, (\beta)_m), \end{array}$$

we obtain that both the analytic sets and the co-analytic sets are closed under countable unions and intersections.

We have seen (Proposition 12.2) that the analytic sets are closed under continuous images. Taking continuous images of co-analytic sets, however, leads out of the co-analytic sets.

Using continuous images (or rather, the special case of *projections*), we define the **projective hierarchy**. Recall our notation $\exists^{\mathbb{N}}$ for projection along $\mathbb{N}$, with $\forall^{\mathbb{N}}$ its dual. We denote by $\exists^{\mathbb{N}^{\mathbb{N}}}$ and $\forall^{\mathbb{N}^{\mathbb{N}}}$ projection along $\mathbb{N}^{\mathbb{N}}$ and its dual, respectively.

$$\begin{array}{rcl} \boldsymbol{\Sigma}^1_1(X) & = & \exists^{\mathbb{N}^{\mathbb{N}}} \boldsymbol{\Pi}^0_1(X) \\ \boldsymbol{\Pi}^1_n(X) & = & \neg \boldsymbol{\Sigma}^1_n(X) \\ \boldsymbol{\Sigma}^1_{n+1}(X) & = & \exists^{\mathbb{N}^{\mathbb{N}}} \boldsymbol{\Sigma}^1_1(X) \\ \boldsymbol{\Delta}^1_n(X) & = & \boldsymbol{\Sigma}^1_n(X) \cap \boldsymbol{\Pi}^1_n(X) \end{array}$$

Hence a set $P \subseteq X$ is

$$
\begin{array}{lll r}
\boldsymbol{\Sigma}^1_1 & \text{iff} & P(x) \Longleftrightarrow \exists\alpha\, F(\alpha,x) & \text{for a closed set } F \subseteq \mathbb{N}^{\mathbb{N}} \times X, \\
\boldsymbol{\Pi}^1_1 & \text{iff} & P(x) \Longleftrightarrow \forall\alpha\, F(\alpha,x) & \text{for an open set } G \subseteq \mathbb{N}^{\mathbb{N}} \times X, \\
\boldsymbol{\Sigma}^1_2 & \text{iff} & P(x) \Longleftrightarrow \exists\alpha\forall\beta\, G(\alpha,\beta,x) & \text{for an open set } G \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times X, \\
\boldsymbol{\Pi}^1_2 & \text{iff} & P(x) \Longleftrightarrow \forall\alpha\exists\beta\, F(\alpha,\beta,x) & \text{for a closed set } F \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \times X, \\
 & & \vdots &
\end{array}
$$

These characterizations clearly indicate a relation between being projective and being definable in second order arithmetic using function quantifiers. We will describe this relation in detail when we address the effective ('lightface') version of the projective hierarchy.

Examples of projective sets

Here are a few examples of projective sets that occur naturally in mathematics.

Analytic sets:

- $\{K \subseteq X : K$ compact and uncountable$\}$ is a $\boldsymbol{\Sigma}^1_1$ subset of the space $K(X)$ of compact subsets of X.
- $\{f \in \mathcal{C}[0,1] : f$ continuously differentiable on $[0,1]\}$ is a $\boldsymbol{\Sigma}^1_1$ subset of $\mathcal{C}[0,1]$.

Co-analytic sets:

- $\{f \in \mathcal{C}[0,1] : f$ differentiable on $[0,1]\}$ is a $\boldsymbol{\Pi}^1_1$ subset of $\mathcal{C}[0,1]$.
- $\{f \in \mathcal{C}[0,1] : f$ nowhere differentiable on $[0,1]\}$ is a $\boldsymbol{\Pi}^1_1$ subset of $\mathcal{C}[0,1]$.
- $\mathrm{WF} = \{\alpha \in 2^{\mathbb{N}} : \alpha$ codes a well-founded tree on $\mathbb{N}\}$ is a $\boldsymbol{\Pi}^1_1$ subset of the space Tr of trees, which can be seen as a closed subspace of $2^{\mathbb{N}^{<\mathbb{N}}}$, and hence is Polish. As we will see, the set WF is a prototypical $\boldsymbol{\Pi}^1_1$ set.

Higher levels:

- $\{f \in \mathcal{C}[0,1] : f$ satisfies the Mean Value Theorem $[0,1]\}$ is a $\boldsymbol{\Pi}^1_2$ subset of $\mathcal{C}[0,1]$.

  (Here f satisfies the Mean Value Theorem if for all $a < b \in [0,1]$ there exists c with $a < c < b$ such that $f'(c)$ exists and $f(b) - f(a) = f'(c)(b - a)$.)

The quantifier manipulations mentioned above yield the following closure properties.

Proposition 14.1:

(1) *The classes $\mathbf{\Sigma}^1_n$ are closed under continuous preimages, countable intersections and unions, and continuous images (in particular, $\exists^{\mathbb{N}^\mathbb{N}}$).*

(2) *The classes $\mathbf{\Pi}^1_n$ are closed under continuous preimages, countable intersections and unions, and co-projections $\forall^{\mathbb{N}^\mathbb{N}}$.*

(3) *The classes $\mathbf{\Delta}^1_n$ are closed under continuous preimages, complements, countable intersections and unions. (In particular, they for a σ-algebra.)*

To show that the hierarchy is proper, we need the existence of universal sets.

Proposition 14.2: *For every Polish space X, there is a $\mathbb{N}^\mathbb{N}$-universal set for $\mathbf{\Sigma}^1_n$ and for $\mathbf{\Pi}^1_n$.*

Proof. By induction on n. We have seen that there exists a $\mathbb{N}^\mathbb{N}$-universal set for $\mathbf{\Sigma}^1_1$. Now note that if $U \in \mathbf{\Sigma}^1_n(\mathbb{N}^\mathbb{N} \times X)$ is $\mathbb{N}^\mathbb{N}$-universal for $\mathbf{\Sigma}^1_n(X)$, then $\neg U$ is $\mathbb{N}^\mathbb{N}$-universal for $\mathbf{\Pi}^1_n(X)$, and if $U \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} \times X$ is $\mathbb{N}^\mathbb{N}$-universal for $\mathbf{\Pi}^1_n(\mathbb{N}^\mathbb{N} \times X)$, then

$$V = \{(\alpha, z) \colon \exists \beta \, (\alpha, \beta, z) \in U\}$$

is $\mathbb{N}^\mathbb{N}$-universal for $\mathbf{\Sigma}^1_{n+1}$. □

Corollary 14.3: *For every $n \geq 1$, $\mathbf{\Sigma}^1_n \not\subseteq \mathbf{\Pi}^1_n$ and $\mathbf{\Pi}^1_n \not\subseteq \mathbf{\Sigma}^1_n$. Morever,*

$$\mathbf{\Sigma}^1_n \subsetneq \mathbf{\Delta}^1_{n+1} \subsetneq \mathbf{\Sigma}^1_{n+1}$$
$$\mathbf{\Pi}^1_n \subsetneq \mathbf{\Delta}^1_{n+1} \subsetneq \mathbf{\Pi}^1_{n+1}$$

The proof is similar to the proofs of Theorem 10.7 and Corollary **??**.

Regularity properties of projective sets

At first sight it does not seem impossible to extend the regularity properties (LM) and (BP) to higher levels of the projective hierarchy. But we will soon see that there are metamathematical limits that prevent us from doing so.

Without explicitly mentioning it, up to now we have been working in ZF, Zermelo-Fraenkel set theory, plus a weak form of Choice ($\mathsf{AC}_\omega(\mathbb{N}^\mathbb{N})$). If we add the full Axiom of Choice (AC), we saw that the regularity properties do not extend to all sets. Solovay's model of ZF showed that the use of a strong version of Choice is necessary for this.

On the other hand, the proofs gave us no direct indication how 'complex' the non-regular sets we constructed are. In the next section we will start to study a model of ZF in which exists a $\boldsymbol{\Delta}^1_2$ set which is neither Lebesgue measurable nor does it have the Baire property. Therefore, we cannot settle in ZF the question of whether the projective sets are measurable or have the Baire property. We will have to add additional axioms.

A key feature in the construction of a non-measurable $\boldsymbol{\Delta}^1_1$ set is the use of the well-ordering principle rather than the Axiom of Choice.

Proposition 14.4: *Suppose $<_W \subseteq \mathbb{R} \times \mathbb{R}$ is a well-ordering of $\mathbb{R}$ of order-type ω_1 in Γ, then there exists a subset of $\mathbb{R}$ in Γ that is neither Lebesgue measurable nor has the Baire property.*

Lebesgue measure here refers to the product measure $\lambda \times \lambda$, which is the unique translation invariant measure defined on the Borel σ-algebra generated by the rectangles $I \times J$, where I and J are open intervals, and $(\lambda \times \lambda)(I \times J) = \lambda(I)\lambda(J)$.

Proof. Suppose $<_W$ is a well-ordering of $\mathbb{R}$ in Γ. Let $A = \{(x, y) \colon x <_W y\}$.

Since $<_W$ is of order type ω_1, for every $y \in \mathbb{R}$, the set $A_y = \{x \colon x <_W y\}$ is countable, and hence of Lebesgue measure zero.

Fubini's Theorem implies that if $A \subseteq \mathbb{R}^2$ is measurable, then

$$(\lambda \times \lambda)(A) = \int \lambda(A_y) d\lambda(y) = 0.$$

So if A is measurable, then $(\lambda \times \lambda)(A) = 0$. The complement of A is $\neg A = \{(x, y) \colon x \geq_W y\}$. As above, for any $x \in \mathbb{R}$, $\neg A_x = \{y \colon x \leq_W y\}$ is countable, and hence $\lambda(\neg A_x) = 0$ for all x. Again, by Fubini's Theorem, $(\lambda \times \lambda)(\neg A) = 0$, and thus $(\lambda \times \lambda)(\mathbb{R}) = (\lambda \times \lambda)(A \cup \neg A) = (\lambda \times \lambda)(A) + (\lambda \times \lambda)(\neg A) = 0$, a contradiction.

We can apply a similar reasoning for Baire category. The sections A_y and $\neg A_x$ are countable, and hence meager.

The following lemma provides a Baire category analogue to Fubini's Theorem.

Lemma 14.5: *Let $A \subseteq \mathbb{R}^2$ have the property of Baire. Then A is meager if and only if $A_x = \{y : (x, y) \in A\}$ is meager for all x except a meager set.*

For a proof see Kechris [1995]. □

Therefore, if the Continuum hypothesis (CH) holds in a model and we can well-order $\mathbb{R}$ (or $\mathbb{N}^{\mathbb{N}}$, $2^{\mathbb{N}}$) within a certain complexity (as a subset of $\mathbb{R}^2$), we can find a non-regular set of the same complexity. The question now becomes how (hard it is) to define a well-ordering of $\mathbb{R}$, and of course if CH holds.