Sample Midterm 1 for MATH 104
Brief sketches of solutions

Problem 1

Show that every real number is the limit of a sequence of rational numbers.

Solution. Given $a \in \mathbb{R}$, by the density of rational numbers, there exists $q_1 \in \mathbb{Q}$ with $a < q_1 < a+1$. Apply the density property again to get rational q_2 with $a < q_2 < \min\{q_1, a+1/2\}$. Inductively, we define a nonincreasing sequence $q_1 > q_2 > q_3 > \dots$ such that for all n, $a < q_n < a + 1/n$. The sequence (q_n) is bounded and monotone, hence it converges. Since $\lim_n(a_n + 1/n) = a$, it follows that $\lim_n(q_n) = a$. ∎

Problem 2

Let (a_n) and (b_n) be two sequences. Assume that $\lim a_n = a$, $\lim b_n = b$, and that $a_n \leqslant b_n$ for all $n \in \mathbb{N}$.

Show that $a \leqslant b$.

Solution. Assume that $a > b$. Since $\lim a_n = a$, there must exist $N_a \in \mathbb{N}$ such that for all $n > N_a$, $a_n > (a+b)/2$, and there must exist $N_b \in \mathbb{N}$ such that for all $n > N_b$, $b_n < (a+b)/2$. But this would imply, in particular, that for $n > \max\{N_a, N_b\}$, $a_n > b_n$, contradicting the assumption that $a_n \leqslant b_n$ for all $n \in \mathbb{N}$. ∎

Problem 3

Let (s_n) and (t_n) be two bounded sequences. Show that

$$\limsup_n (s_n + t_n) \leqslant \limsup_n s_n + \limsup_n t_n.$$

Solution. The set $\{s_n + t_n : n > N\}$ is a subset of $\{s_n + t_m : n, m > N\}$. We know (homework 1) that $\sup\{s_n + t_m : n, m > N\} = \sup\{s_n : n > N\} + \sup\{s_n : n > N\}$. Therefore, $\sup\{s_n + t_n : n > N\} \leqslant \sup\{s_n : n > N\} + \sup\{t_n : n > N\}$. Since for any sequence (r_n), $\limsup_n r_n$ is defined as $\lim_N \sup\{r_n : n > N\}$, the result follows from Problem 2. ∎

Give an example of two bounded sequences (s_n), (t_n) such that

$$\limsup_n (s_n t_n) \neq \limsup_n s_n \limsup_n t_n.$$

Solution. Consider $s_n = (-1)^n$ and $t_n = (-1)^{n+1}$. ∎

Problem 4

Show that a countable union of countable sets is countable. More precisely, let $\{E_i : i \in \mathbb{N}\}$ be a countable family such that each E_i is countable. Show that

$$\bigcup_{i\in\mathbb{N}} E_i = \{x : \text{ exists } i \in \mathbb{N} \text{ such that } x \in E_i\}$$

is countable.

Solution. Since each E_i is countable, for all $i \in \mathbb{N}$ there exists a bijection $f_i : \mathbb{N} \to E_i$. Then $\bigcup_{i\in\mathbb{N}} E_i = \{f_i(j) : i, j \in \mathbb{N}\}$. The mapping $(i, j) \mapsto f_i(j)$ defines a surjection from $\mathbb{N}\times\mathbb{N}$ onto $\{f_i(j) : i, j \in \mathbb{N}\}$. But we already proved that $\mathbb{N}\times\mathbb{N}$ is countable, and that the surjective image of a countable set is at most countable. But since none of the E_i is finite, the union cannot be finite, hence the union is countable. ■