Lesson 3
Dynamical Systems
3-6: The Ergodic Theorem

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From Recurrence to Averages

$(X, \mathcal{B}, \mu)$ probability space, $T: X \to X$ μ-preserving.

Poincaré recurrence: For every measurable $E \subseteq X$ with $\mu(E) > 0$, almost every point in E returns to E infinitely often.

Question: Can we say something about how often a point returns on average?

x σ

- Assume $X = A^{\mathbb{N}}$, A finite, T shift mapping, $E = [\sigma]$.
- For a sequence x to return to E then means that σ occurs as a substring in x.
- The average number of returns to $[\sigma]$ by time n is then given as

$$\frac{\#\{i < n : x_i \dots x_{i+|\sigma|-1} = \sigma\}}{n}$$

$[\sigma]$ $\mu[\sigma]$

- Q: Does this average converge? To $\mu[\sigma]$?

$\longrightarrow$ Ergodic Theorem

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The Ergodic Theorem

T measure-preserving transformation on probability space $(X, \mathcal{B}, \mu)$.

- If f is a μ-integrable function on X, then the average

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x))$$

converges for μ-almost every $x \in X$.

- If we denote the limit by $f^*(x)$, then f^* is integrable ($L^1(\mu)$) and $\int f d\mu = \int f^* d\mu$. Furthermore, f^* is T-invariant, i.e. $f^* \circ T = f^*$.

- Finally, if T is ergodic, then f^* is constant μ-a.e. and hence $f^*(x) = \int f d\mu$, which means

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x)) \xrightarrow{n\to\infty} \int f d\mu.$$

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Example - Law of Large Numbers

- Let (X_n) be a binary Bernoulli process with $P(1) = p$, $P(0) = 1 - p$.
- μ_p Kolmogorov measure on $2^{\mathbb{N}}$, i.e. $\mu_p[1] = p$, $\mu_p[0] = 1 - p$. μ_p is invariant under shift map T.

i

- Let $f = \chi_{[1]}$. We have $f(T^i x) = 1$ iff $T^i x \in [1]$ iff $x_i = 1$.
- By the ergodic theorem, for μ_p-almost every $x \in 2^{\mathbb{N}}$,

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} \chi_{[1]}(T^i x) = \lim_n \frac{|\{i < n : x_i = 1\}|}{n} \longrightarrow \int \chi_{[1]} \, d\mu_p$$

$$= \lim_n \frac{x_0 + x_1 + \cdots + x_{n-1}}{n} = \mu_p[1] = p.$$

$\mu_p[1]$ //

Therefore, the ergodic theorem can be seen as an extension of the strong law of large numbers to arbitrary stationary processes.

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Proving the Ergodic Theorem (I)

It suffices to consider real-valued $f \in L^1(\mu)$. Put

$$f^*(x) = \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x) \quad \text{and} \quad f_*(x) = \liminf_n \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x).$$

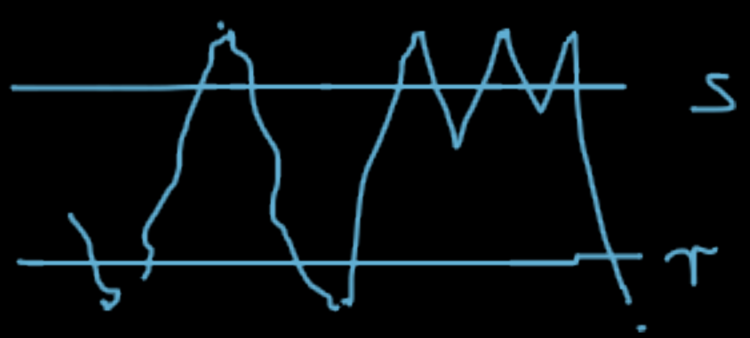
We want to show: $f^*(x) = f_*(x)$ μ-almost everywhere.

For real $r < s$, let

$$E_{r,s} = \{\underline{x \in X}\colon f_*(x) < r \text{ and } f^*(x) > s\}.$$

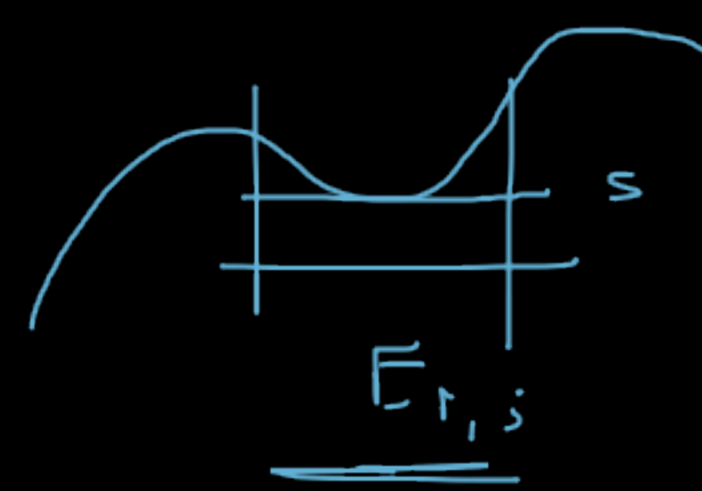
We have

$$\{x\colon f_*(x) < f^*(x)\} = \bigcup_{r<s\in\mathbb{Q}} E_{r,s}.$$



Hence it suffices to show $\mu(E_{r,s}) = 0$.

Proving the Ergodic Theorem (II)



Idea: Show that

$$\int_{E_{r,s}} f d\mu \geqslant s\mu(E_{r,s}),$$

and at the same time

$$\int_{E_{r,s}} f d\mu \leqslant r\mu(E_{r,s}).$$

If $r < s$, this forces $\mu(E_{r,s}) = 0$.

This follows from the maximal ergodic theorem.

pointwise

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Maximal Ergodic Theorem

THM: Let T be a measure-preserving transformation on a probability space $(X, \mathcal{B}, \mu)$, and $f \in L^1(\mu)$. Define

$$f^*(x) = \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x).$$

Then, for any $\lambda \in \mathbb{R}$

$$\int_{\{f^* > \lambda\}} f d\mu \geqslant \lambda \mu\{f^* > \lambda\}.$$

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A Combinatorial Approach

Suppose T is the shift on $2^{\mathbb{N}}$ and μ is shift-invariant and ergodic.

Let's try to give a more ``direct'' proof that almost surely,

$$\lim_n \frac{1}{n} \sum_{i=0}^{n-1} x_i = \mu[1].$$

Suppose the statement is false, then wlog for some $\varepsilon > 0$, the set

$$E = \{x \colon \limsup_n \frac{1}{n} \sum_{i=0}^{n-1} x_i \geqslant \mu[1] + \varepsilon\}$$

has positive measure $\mu(E) > 0$.

E is T-invariant ($TE \subseteq E$), so by ergodicity $\mu(E) = 1$.

So almost every $x \in 2^{\mathbb{N}}$ has a long stretch of ``too many'' 1's infinitely often.

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A Combinatorial Approach

On the other hand, at any finite point in time, the expected value of $\sum_{i<n} x_i/n$ must be $\mu[1]$, since

$$\mu[1] = \int \chi_{[1]}(x)\,d\mu(x) = \int (\chi_{[1]} \circ T)(x)\,d\mu(x) \qquad (*)$$

$$= \int \frac{1}{n}\sum_{i=0}^{n-1} \chi_{[1]}(T^i x)\,d\mu(x) = \int \frac{\sum_{i<n} x_i}{n}\,d\mu(x)$$

Idea: Find a long interval that contains many short intervals in which the number of 1's is too large for $(*)$ to hold.

A Combinatorial Approach

Assume $x \in E$. Since $TE \subseteq E$, for every n there exists a minimal $s(n) = s_x(n)$ for which

$$\frac{x_n + x_{n+1} + \cdots + x_{s(n)-1}}{s(n) - n} \geqslant \mu[1] + \varepsilon.$$

Only a certain fraction of x has a large $s_x(0)$: Given $\delta > 0$, there exists L such that

$$\mu(D) \leqslant \delta^2, \text{ where } D = \{x\colon s_x(0) > L\}.$$

Hence if we give up a little bit of measure, we can work with sequences that obtain an ε-excess of 1's by a fixed time L.

A Combinatorial Approach

$D = \{x : S_x(0) > L\}$

Furthermore, not many orbits will visit D very often: Let $g_N(x)$ be the average number of visits of $T^i x$ to D by time N,

$$g_N(x) = \underline{\frac{1}{N}} \sum_{i=0}^{N-1} \chi_D(T^i x).$$

$\geq \mu(D)$

T measure preserving implies $\int \underline{g_N d\mu \leqslant \delta^2}$. Then Markov's inequality yields

$$\mu(G_N) \geqslant 1 - \delta, \text{ where } G_N = \{x \colon g_N(x) \leqslant \delta\}.$$

$\leq \delta^2$

$$\mu\{x : g_N \leq \delta\} \geq 1 - \frac{\int g_N \, d\mu}{\delta}$$

$$\geq 1 - \delta$$

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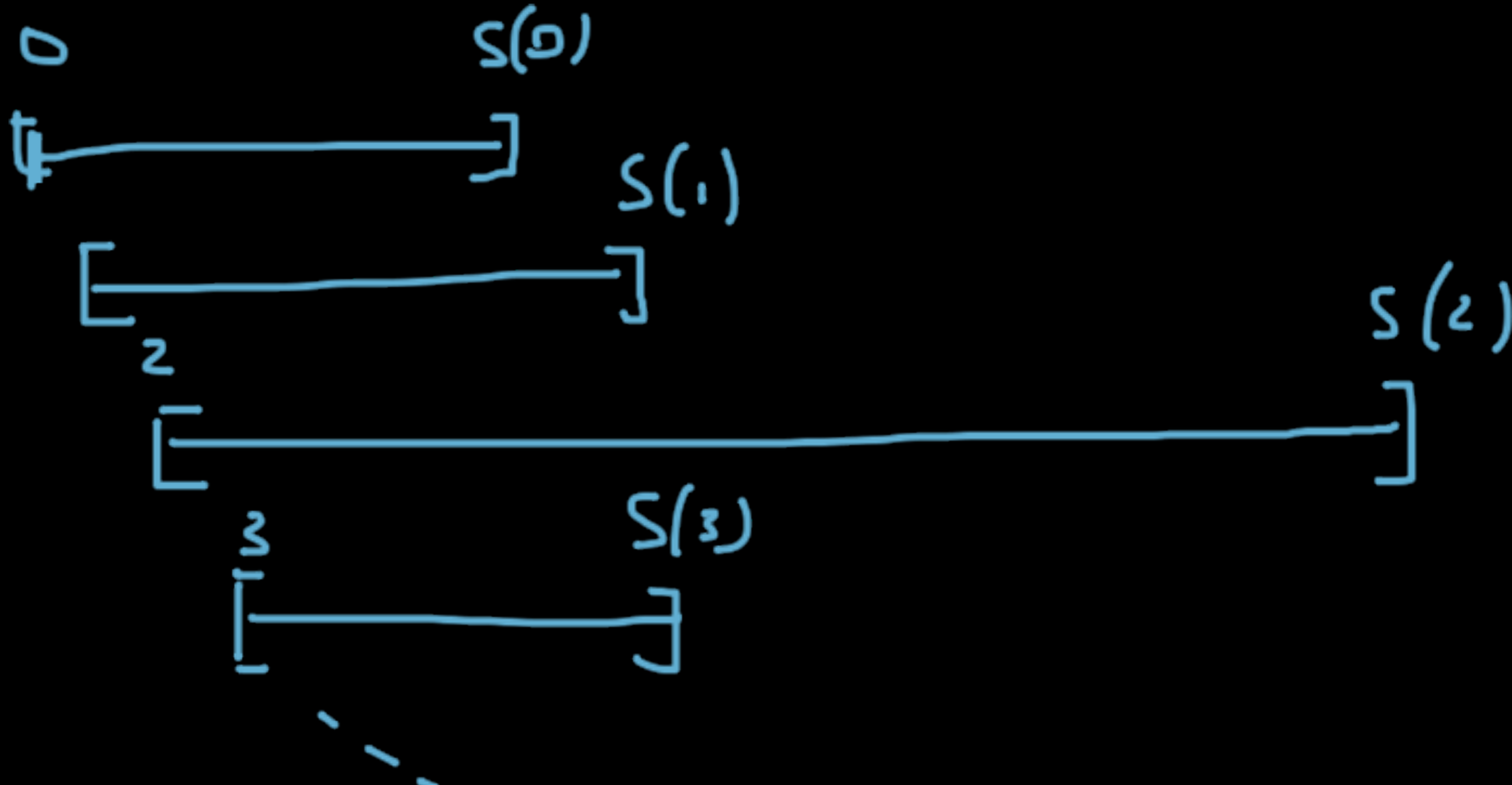
A Combinatorial Approach

For $\boldsymbol{x} \in \boldsymbol{G}_N$, the portion of long intervals with starting point in $[0, \boldsymbol{N} - 1]$ is not more than $\boldsymbol{\delta}$:

$$\frac{|\{n \in [0, N-1] \colon s(n) - n > L\}|}{N} \leqslant \delta.$$



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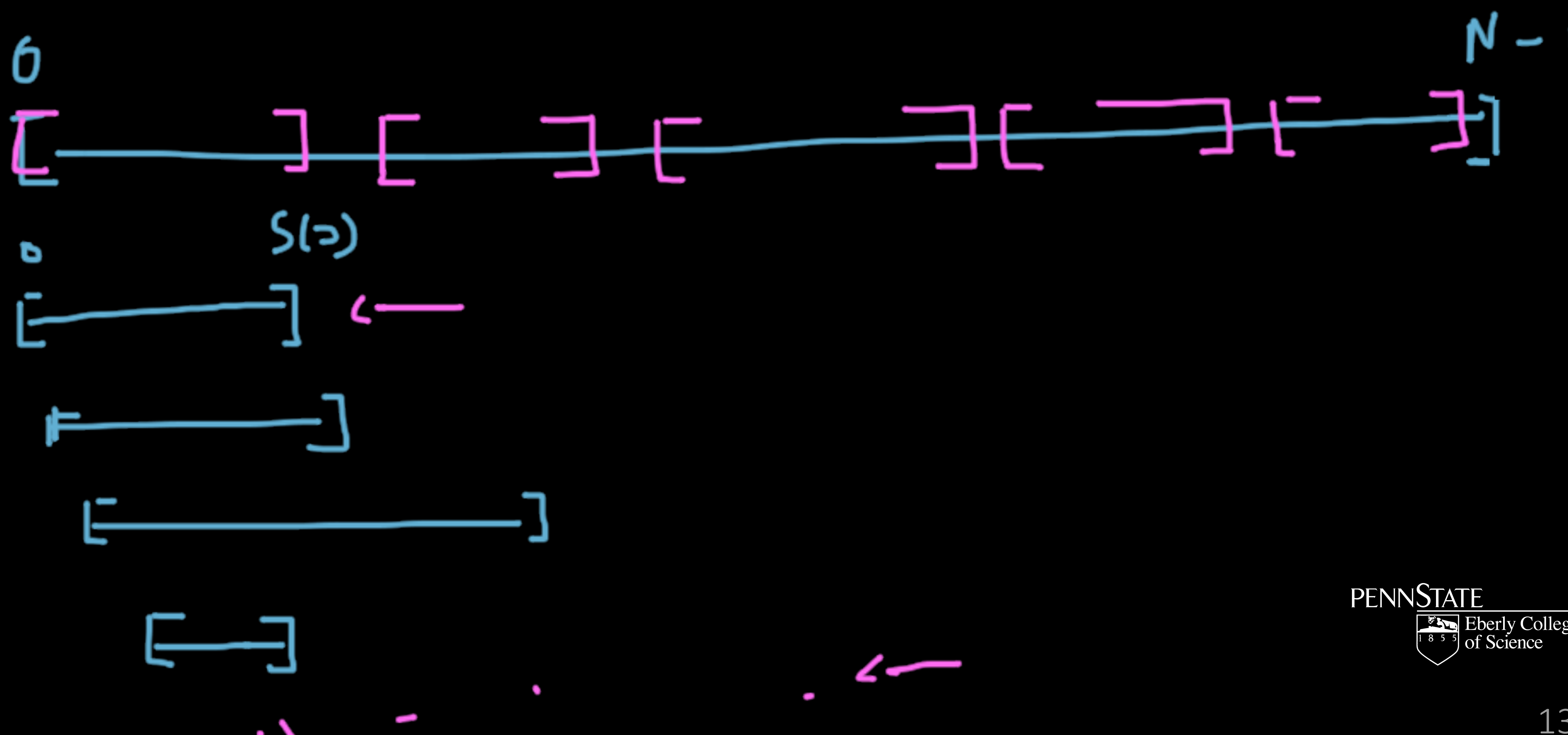
The Packing Lemma

LEMMA: Suppose $x \in G_N$. There exists a sequence $n_1 < n_2 < n_3 < \cdots < n_k$ of natural numbers such that

(1) the intervals $[n_i, s_x(n_i)]$ are disjoint,

(2) $[n_i, s_x(n_i)] \subseteq [0, N-1]$ for all $i \leqslant k$,

(3) $|\bigcup_i [n_i, s_x(n_i)]| \geqslant \underline{(1-2\delta)}N$.



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A Combinatorial Approach

The Lemma allows us to bound the number of 1's in $[0, N-1]$ from below:

$$\sum_{j=0}^{N-1} x_j \geqslant \sum_{i=1}^{k} \sum_{j=n_i}^{s(n_i)} x_j \geqslant (1-2\delta)N(\mu[1]+\varepsilon).$$

Note that this bound is independent of x (though the sequence of n_i's does depend on x), as long as $x \in G_N$. Hence we can bound the integral

$$\underline{\int \sum_{j=0}^{N-1} x_j} \geqslant (1\underline{-2\delta})\cancel{N}(\mu[1]+\varepsilon)\underbrace{\mu(G_N)}_{\geq 1-\delta}.$$

Thus,

$$\underline{\mu[1]} = \int \underline{\frac{\sum_{i<N} x_i}{N}} d\mu(x) \geqslant \underline{(1-2\delta)}\underline{(1-\delta)}\underline{(\mu[1]+\varepsilon)},$$

↯ !

which is impossible, since δ can be chosen arbitrarily small.

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[Source: Shields, *The ergodic theory of discrete sample paths*]