Homework 3 for **MATH 574**

Due: Wednesday February 16

Problem 1

Show that the following sets are Borel and try to find a best possible upper bound on the level in the Borel hierarchy.

(a) $\{\alpha \in \mathbb{N}^{\mathbb{N}} \colon \ \alpha \text{ is a permutation}\}$

(b) $\{\alpha \in 2^{\mathbb{N}} \colon \ \lim_n (\alpha_0 + \cdots + \alpha_{n-1})/n \text{ exists and is rational}\}$

Problem 2

Let X, Y be topological spaces. A mapping $f : X \to Y$ is **Borel** if the preimage of any Borel set is Borel.

(a) Suppose $f : [0,1] \to \mathbb{R}$ and there exists $F : [0,1] \to \mathbb{R}$ such that F is differentiable and $F' = f$. Show that f is Borel.

(b) Suppose $f : X \to \mathbb{R}$ is *lower semicontinuous*, i.e. for all $a \in \mathbb{R}$, $\{x \colon f(x) > a\}$ is open. Show that f is Borel.

Problem 3

Show that every filter on a set X can be extended to an *ultrafilter* on X.
(This requires the Axiom of Choice.)

Problem 4

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$, and let (a_n) be a bounded sequence of real numbers. Show that there exists a real number a such that

$$\forall \varepsilon > 0 \, \{n \colon |a_n - a| < \varepsilon\} \in \mathcal{U}.$$

You can think of a as the $\mathcal{U}$-*limit* of (a_n).

Problem 5

A *metric outer measure* is an outer measure μ^* on a metric space X such that

$$\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$$

whenever A and B are *positively separated*: $\inf\{d(x,y) \colon x \in A, y \in B\} > 0$.

(a) Show that if μ^* is a metric outer measure, then μ^* is stable under upper limits: If $A_1 \subseteq A_2 \subseteq \dots$ and $A = \bigcup A_n$ is such that A_n and $A \setminus A_{n+1}$ are positively separated for all n, then $\mu^*(A) = \lim_n \mu^*(A_n)$.

(b) Use (a) to show that for a metric outer measure, all Borel sets are measurable.