Alternative proof of the equivalence (on page 180) of

$$\mathcal{N} \models \exists x_1 \forall x_2 \ldots \exists x_n \, \psi(a, \vec{c}, \vec{x})$$

and

$$\exists i_1 > i_0 \, \forall i_2 > i_1 \; \ldots \; \exists i_n > i_{n-1} \qquad \mathcal{N} \models \exists x_1 < b_{i_1} \, \forall x_2 < b_{i_2} \; \ldots \; \exists x_n < b_{i_n} \; \psi(a, \vec{c}, \vec{x}).$$

by Michael Weiss (diagonalargument.com)

First do the equivalence

$$\begin{aligned} & \mathcal{N} \models \forall x \; \exists y \; \psi(x, y, \vec{c}) \\ \Leftrightarrow \; & \forall p \, \exists q \, \mathcal{N} \models \psi(p, q, \vec{c}) \\ \Leftrightarrow \; & \forall i_1 {>} i_0 \;\; \forall p {<} b_{i_1} \;\; \exists i_2 {>} i_1 \;\; \exists q {<} b_{i_2} \;\; \mathcal{N} \models \psi(p, q, \vec{c}) \\ \Leftrightarrow \; & \forall i_1 {>} i_0 \;\; \forall p {<} b_{i_1} \;\; \exists i_2 {>} i_1 \;\; \exists q {<} b_{i_2} \;\; \mathcal{M} \models \psi(p, q, \vec{c}) \end{aligned}$$

Now that we are in $\mathcal{M}$, a model of PA, we can use the collection axioms. Define a function F by

$$F(x, \vec{u}) = \begin{cases} \mu y[\psi(x, y, \vec{u})] \text{ if } \exists y \, \psi(x, y, \vec{u}) \\ 0 \text{ otherwise} \end{cases}$$

The definition can be formalized in the language of PA (although it is not Σ_1). Using the collection axioms in $\mathcal{M}$, $\max_{0 \le z \le x} F(z, \vec{u})$ exists for all x and $\vec{u}$, and is attained at some z in the segment $[0, x]$. If x and $\vec{u}$ belong to N, then so does the maximum, since it is attained at some $z \in N$ and $\mathcal{N} \models \forall \vec{u} \forall x \, \exists y \, \mathcal{N} \models \psi(x, y, \vec{u})$. Using the cofinality of the b's in N, we conclude that F is bounded on the segment $[0, p]$ by some b_i, and we are entitled to switch the quantifiers.