Preferred Language: English

Sign Up Features Support Story Advanced

Blogs about: 254a Ergodic Theory

Featured Blog

254A, Lecture 17: A Ratner-type theorem for SL_2(R) orbits

In this final lecture, we establish a Ratner-type theorem for actions of the special linear group on homogeneous spaces. More precisely, we show: Theorem 1. Let G be a Lie group, let be a discrete sub … **more »**

What's new

254A, Lecture 16: A Ratner-type theorem for nilmanifolds

Terence Tao wrote 1 week ago: The last two lectures of this course will be on Ratner's theorems on equidistribution of orbits on homogeneous spaces. Due … more »

Tags: math.DS, math.GR

254A, Lecture 15: The Furstenberg-Zimmer structure theorem and the Furstenberg recurrence theorem — 2 comments

Terence Tao wrote 1 week ago: In this lecture – the final one on general measure-preserving dynamics – we put together the results from the past … more »

Tags: math.DS

254A, Lecture 14: Weakly mixing extensions — 6 comments

Terence Tao wrote 2 weeks ago: Having studied compact extensions in the previous lecture, we now consider the opposite type of extension, namely that of a … more »

Tags: math.DS, math.OA

254A, Lecture 13: Compact extensions — 3 comments

Terence Tao wrote 2 weeks ago: In Lecture 11, we studied compact measure-preserving systems – those systems in which every function was almost periodic, which meant … more »

Tags: math.DS, math.OA

Related Tags

All »

ergodicity
Dichotomy
nilpotent groups
nilmanifolds
Ratner's theorem
math.GR
Structure
math.DS

Follow this tag via RSS

Find other items tagged with "254a-ergodic-theory":

Technorati
Del.icio.us
IceRocket

254A, Lecture 12: Weakly mixing systems — 4 comments

Terence Tao wrote 3 weeks ago: In the previous lecture, we studied the recurrence properties of compact systems, which are systems in which all measurable functions … more »

Tags: math.DS, math.FA

254A, Lecture 11: Compact systems — 9 comments

Terence Tao wrote 1 month ago: The primary objective of this lecture and the next few will be to give a proof of the Furstenberg recurrence … more »

Tags: math.CA, math.DS

254A, Lecture 10: The Furstenberg correspondence principle — 4 comments

Terence Tao wrote 1 month ago: In this lecture, we describe the simple but fundamental Furstenberg correspondence principle which connects the “soft analysis” subject of ergodic … more »

Tags: math.CO, math.DS

254A, Lecture 9: Ergodicity — 9 comments

Terence Tao wrote 1 month ago: We continue our study of basic ergodic theorems, establishing the maximal and pointwise ergodic theorems of Birkhoff. Using these theorems, … more »

Tags: math.DS

254A, Lecture 8: The mean ergodic theorem — 16 comments

Terence Tao wrote 1 month ago: We now begin our study of measure-preserving systems , i.e. a probability space together with a probability space isomorphism (thus … more »

Tags: math.DS

254A, Lecture 7: Structural theory of topological dynamical systems — 6 comments

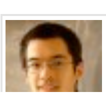
Terence Tao wrote 1 month ago: In our final lecture on topological dynamics, we discuss a remarkable theorem of Furstenberg that classifies a major type of … more »

Tags: math.DS

254A, Lecture 6: Isometric systems and isometric extensions — 14 comments

Terence Tao wrote 1 month ago: In this lecture, we move away from recurrence, and instead focus on the structure of topological dynamical systems. One remarkable … more »

Tags: math.AT, math.DS, math.GR, math.MG



254A, Lecture 5: Other topological recurrence results — 12 comments

Terence Tao wrote 1 month ago: In this lecture, we use topological dynamics methods to prove some other Ramsey-type theorems, and more specifically the polynomial van … more »

Tags: math.CO, math.DS

254A, Lecture 4: Multiple recurrence — 17 comments

Terence Tao wrote 2 months ago: In the previous lecture, we established single recurrence properties for both open sets and for sequences inside a topological dynamical … more »

Tags: math.CO, math.DS

254A, Lecture 3: Minimal dynamical systems, recurrence, and the Stone-Čech compactification — 35 comments

Terence Tao wrote 2 months ago: We now begin the study of recurrence in topological dynamical systems - how often a non-empty open set U in … more »

Tags: math.DS, math.GN, math.LO

254A, Lecture 2: Three categories of dynamical systems — 21 comments

Terence Tao wrote 2 months ago: Before we begin or study of dynamical systems, topological dynamical systems, and measure-preserving systems (as defined in the previous lecture), … more »

Tags: math.CT, math.DS

254A, Lecture 1: Overview — 18 comments

Terence Tao wrote 2 months ago: In this lecture, I define the basic notion of a dynamical system (as well as the more structured notions of … more »

Tags: math.DS

254A: Topics in Ergodic Theory — 4 comments

Terence Tao wrote 3 months ago: Next quarter, starting on Wednesday January 9, I will be teaching a graduate course entitled "Topics in Ergodic Theory". As … more »

Tags: math.DS

254A, Lecture 1: Overview

8 January, 2008 in 254A - ergodic theory, math.DS

Tags: course overview, ergodic theory, Ratner's theorem, recurrence theorem

In this lecture, I define the basic notion of a *dynamical system* (as well as the more structured notions of a *topological dynamical system* and a *measure-preserving system*), and describe the main topics we will cover in this course.

We'll begin abstractly. Suppose that X is a non-empty set (whose elements will be referred to as *points*), and $T: X \to X$ is a transformation. Later on we shall put some structures on X (such as a topology, a σ-algebra, or a probability measure), and some assumptions on T, but let us work in total generality for now. (Indeed, a guiding philosophy in the first half of the course will be to try to study dynamical systems in as maximal generality as possible; later on, though, when we turn to more algebraic dynamical systems such as nilsystems, we shall exploit the specific structure of such systems more thoroughly.)

One can think of X as a state space for some system, and T as the evolution of some discrete deterministic (autonomous) dynamics on X: if x is a point in X, denoting the current state of a system, then Tx can be interpreted as the state of the same system after one unit of time has elapsed. (In particular, evolution equations which are well-posed can be viewed as a continuous dynamical system.) More geometrically, one can think of T as some sort of shift operation (e.g. a rotation) on the space X.

Given X and T, we can define the iterates $T^n: X \to X$ for every non-negative integer n; if T is also invertible, then we can also define T^n for negative integer n as well. In the language of representation theory, T induces a representation of either the additive semigroup $\mathbb{Z}^+$ or the additive group $\mathbb{Z}$. (From the dynamical perspective, this representation is the mathematical manifestation of *time*.) More generally, one can consider representations of other groups, such as the real line $\mathbb{R}$ (corresponding the dynamics $t \mapsto T^t$ of a continuous time evolution) or a lattice $\mathbb{Z}^d$ (which corresponds to the dynamics of d commuting shift operators $T_1, \ldots, T_d: X \to X$), or of many other semigroups or groups (not necessarily commutative). However, for simplicity we shall mostly restrict our attention to $\mathbb{Z}$-actions in this course, though many of the results here can be generalised to other actions (under suitable hypotheses on the underlying semigroup or group, of course).

Henceforth we assume T to be invertible, in which case we refer to the pair (X, T) as a *cyclic dynamical system*, or *dynamical system* for short. Here are some simple examples of such systems:

1. **Finite systems**. Here, X is a finite set, and $T: X \to X$ is a permutation on X.
2. **Group actions**. Let G be a group, and let X be a homogeneous space for G, i.e. a non-empty space with a transitive G-action; thus X is isomorphic to G/Γ, where $\Gamma := \mathrm{Stab}(x)$ is the stabiliser of one of the points x in X. Then every group element $g \in G$ defines a dynamical system (X, T_g) defined by $T_g x := gx$.
3. **Circle rotations**. As a special case of Example 2 (or Example 1), every real number $\alpha \in \mathbb{R}$ induces a dynamical system $(\mathbb{R}/\mathbb{Z}, T_\alpha)$ given by the rotation $T_\alpha x := x + \alpha$. This is the prototypical

example of a very *structured* system, with plenty of algebraic structure (e.g. the shift map T_α is an isometry on the circle, thus two points always stay the same distance apart under shifts).

4. **Cyclic groups**. Another special case of Example 2 is the cyclic group $\mathbb{Z}/N\mathbb{Z}$ with shift $x \mapsto x+1$; this is the prototypical example of a finite dynamical system.
5. **Bernoulli systems**. Every non-empty set Ω induces a dynamical system $(\Omega^{\mathbb{Z}}, T)$, where T is the left shift $T(x_n)_{n\in\mathbb{Z}} := (x_{n+1})_{n\in\mathbb{Z}}$. This is the prototypical example of a very *pseudorandom* system, with plenty of mixing (e.g. the shift map tends to move a pair of two points randomly around the space).
6. **Boolean Bernoulli system**. This is isomorphic to a special case of Example 5, in which $X = 2^{\mathbb{Z}} := \{A : A \subset \mathbb{Z}\} \equiv \{0,1\}^{\mathbb{Z}}$ is the power set of the integers, and $TA := A - 1 := \{a-1 : a \in A\}$ is the left shift. (Here we endow $\{0,1\}$ with the discrete topology.)
7. **Baker's map**. Here, $X := [0,1)^2$, and $T(x,y) := (\{2x\}, \frac{y+\lfloor 2x \rfloor}{2})$, where $\lfloor x \rfloor$ is the greatest integer function, and $\{x\} := x - \lfloor x \rfloor$ is the fractional part. This is isomorphic to Example 6, as can be seen by inspecting the effect of T on the binary expansions of x and y.

The map T^n can be interpreted as an isomorphism in several different categories:

1. as a set isomorphism (i.e. a bijection) $T^n : X \to X$ from points $x \in X$ to points $T^n x \in X$;
2. as a Boolean algebra isomorphism $T^n : 2^X \to 2^X$ from sets $E \subset X$ to sets $T^n E := \{T^n x : x \in E\}$; or
3. as an algebra isomorphism $T^n : \mathbb{R}^X \to \mathbb{R}^X$) from real-valued functions $f : X \to \mathbb{R}$ to real-valued functions $T^n f : X \to \mathbb{R}$, defined by
$$T^n f(x) := f(T^{-n} x); \quad (1)$$
4. as an algebra isomorphism $T^n : \mathbb{C}^X \to \mathbb{C}^X$ of complex valued functions, defined exactly as in 3.

We will abuse notation and use the same symbol T^n to refer to all of the above isomorphisms; the specific meaning of T^n should be clear from context in all cases. Our sign conventions here are chosen so that we have the pleasant identities

$$T^n\{x\} = \{T^n x\}; \quad T^n 1_E = 1_{T^n E} \quad (2)$$

for all points x and sets E, where of course 1_E is the indicator function of E.

One of the main topics of study in dynamical systems is the asymptotic behaviour of T^n as $n \to \infty$. We can pose this question in any of the above categories, thus

1. For a given point $x \in X$, what is the behaviour of $T^n x$ as $n \to \infty$?
2. For a given set $E \subset X$, what is the behaviour of $T^n E$ as $n \to \infty$?
3. For a given real or complex-valued function $f : X \to \mathbb{R}, \mathbb{C}$, what is the behaviour of $T^n f$ as $n \to \infty$?

These are of course very general and vague questions, but we will formalise them in many different ways later in the course. (For instance, one can distinguish between worst-case, average-case, and best-case behaviour in x, E, f, or n.) The answer to these questions also depends very much on the dynamical system; thus a major focus of study in this subject is to seek classifications of dynamical systems which allow one to answer the above questions satisfactorily. (In particular, ergodic theory is a framework in which our understanding of the dichotomy between structure and randomness is at is most developed.)

One can also ask for more *quantitative* versions of the above asymptotic questions, in which n ranges in a finite interval (e.g. $[N] := \{1, \ldots, N\}$ for some large integer N), as opposed to going off to infinity, and one wishes to estimate various numerical measurements of $T^n x$, $T^n E$, or $T^n f$ in this range.

In this very general setting, in which X is an unstructured set, and T is an arbitrary bijection, there is not much of interest one can say with regards to these questions. However, one obtains a surprisingly rich and powerful theory when one adds a little bit more structure to X and T (thus changing categories once more). In particular, we will study the following two structured versions of a dynamical system:

1. *Topological dynamical systems* $(X, T) = (X, \mathcal{F}, T)$, in which $X = (X, \mathcal{F})$ is a compact metrisable (and thus Hausdorff) topological space, and T is a topological isomorphism (i.e. a homeomorphism); and
2. *Measure-preserving systems* $(X, T) = (X, \mathcal{X}, \mu, T)$, in which $X = (X, \mathcal{X}, \mu)$ is a probability space, and T is a probability space isomorphism, i.e. T and T^{-1} are both measurable, and $\mu(TE) = \mu(E)$ for all measurable $E \in \mathcal{X}$. [In this course we shall tilt towards a measure-theoretic perspective rather than a probabilistic one, thus it might be better to think of μ of as a normalised finite measure rather than as a probability measure. On the other hand, we will rely crucially on the probabilistic notions of *conditional expectation* and *conditional independence* later in this course.] For technical reasons we also require the measurable space $(X, \mathcal{X})$ to be *separable* (i.e. $\mathcal{X}$ is countably generated).

Remark 1. By Urysohn's metrisation theorem, a compact space is metrisable if and only if it is Hausdorff and second countable, thus providing a purely topological characterisation of a topological dynamical system. $\diamond$

[It is common to add a bit more structure to each of these systems, for instance endowing a topological dynamical system with a metric, or endowing a measure preserving system with the structure of a standard Borel space; we will see examples of this in later lectures.] The study of topological dynamical systems and measure-preserving systems is known as *topological dynamics* and *ergodic theory* respectively. The two subjects are closely analogous at a heuristic level, and also have some more rigorous connections between them, so we shall pursue them in a somewhat parallel fashion in this course.

Observe that we assume compactness in 1. and finite measure in 2.; these "boundedness" assumptions ensure that the dynamics somewhat resembles the (overly simple) case of a finite dynamical system. Dynamics on non-compact topological spaces or infinite measure spaces ~~can be quite nasty, and there does not appear to be a useful general theory in these cases~~ is a more complicated topic; see for instance this book of Aaronson. [Updated, Jan 9; thanks to Tamar Ziegler for the link.]

Note that the action of the isomorphism T^n on sets E and functions f will be compatible with the topological or measure-theoretic structure:

1. If $(X, T) = (X, \mathcal{F}, T)$ is a topological dynamical system, then $T^n : \mathcal{F} \to \mathcal{F}$ is a topological isomorphism on open sets, and $T^n : C(X) \to C(X)$ is also a C^*-algebra isomorphism on the space

$C(X)$ of real -valued (or complex-valued) continuous functions on X.
2. If $(X, T) = (X, \mathcal{X}, \mu, T)$ is a measure-preserving system, then $T^n : \mathcal{X} \to \mathcal{X}$ is a σ-algebra isomorphism on measurable sets, and $T^n : L^p(\mathcal{X}, \mu) \to L^p(\mathcal{X}, \mu)$ is a Banach space isomorphism on p^{th}-power integrable functions for $1 \leq p \leq \infty$. (For $p = \infty$, T^n is a von Neumann algebra isomorphism, whilst for $p = 2$, T^n is a Hilbert space isomorphism (i.e. a unitary transformation).)

We can thus see that tools from the analysis of Banach spaces, von Neumann algebras, and Hilbert spaces may have some relevance to ergodic theory; for instance, the spectral theorem for unitary operators is quite useful.

In the first half of this course, we will study topological dynamical systems and measure-preserving systems in great generality (with few assumptions on the structure of such systems), and then specialise to specific systems as appropriate. This somewhat abstract approach is broadly analogous to the combinatorial (as opposed to algebraic or arithmetic) approach to additive number theory. For instance, we will shortly be able to establish the following general result in topological dynamics:

Birkhoff recurrence theorem. Let (X,T) be a topological dynamical system. Then there exists a point $x \in X$ which is *recurrent* in the sense that there exists a sequence $n_j \to \infty$ such that $T^{n_j} x \to x$ as $j \to \infty$.

As a corollary, we will be able to obtain the more concrete result:

Weyl recurrence theorem. Let $P : \mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a polynomial (modulo 1). Then there exists a sequence $n_j \to \infty$ such that $P(n_j) \to P(0)$.

This is already a somewhat non-trivial theorem; consider for instance the case $P(n) := \sqrt{2}n^2 \mod 1$.

In a similar spirit, we will be able to prove the general topological dynamical result

Topological van der Waerden theorem. Let $(U_\alpha)_{\alpha \in A}$ be an open cover of a topological dynamical system (X,T), and let $k \geq 1$ be an integer. Then there exists an open set U in this cover and a shift $n \geq 1$ such that $U \cap T^n U \cap \ldots \cap T^{(k-1)n} U \neq \emptyset$. (Equivalently, there exists U, n, and a point x such that $x, T^n x, \ldots, T^{(k-1)n} x \in U$.)

and conclude an (equivalent) combinatorial result

van der Waerden theorem. Let $\mathbb{N} = U_1 \cup \ldots \cup U_m$ be a finite colouring of the integers. Then one of the colour classes U_j contains arbitrarily long arithmetic progressions.

More generally, topological dynamics is an excellent tool for establishing colouring theorems of Ramsey type.

Analogously, we will be able to show the following general ergodic theory result

Furstenberg multiple recurrence theorem. Let (X, T) be a measure-preserving system, let $E \in \mathcal{X}$ be a set of positive measure, and let $k \geq 1$. Then there exists $n \geq 1$ such that $E \cap T^n E \cap \ldots \cap T^{(k-1)n} E \neq \emptyset$ (or equivalently, there exists $x \in X$ and $n \geq 1$ such that $x, T^n x, \ldots, T^{(k-1)n} \in E$).

Similarly, if $f : X \to \mathbb{R}^+$ is a bounded measurable non-negative function which is not almost everywhere zero, and $k \geq 1$, then

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=1}^N \int_X f T^n f \ldots T^{(k-1)n} f > 0. \quad (3)$$

and deduce an equivalent (and highly non-trivial) combinatorial analogue:

Szemerédi's theorem. Let $E \subset \mathbb{Z}$ be a set of positive upper density, thus $\limsup_{N \to \infty} \frac{|E \cap [-N,N]|}{2N+1} > 0$. Then E contains arbitrarily long arithmetic progressions.

More generally, ergodic theory methods are extremely powerful in deriving "density Ramsey theorems". Indeed, there are several theorems of this type which currently have no known non-ergodic theory proof. [From general techniques in proof theory, one could, in principle, take an ergodic theory proof and mechanically convert it into what would technically be a non-ergodic proof, for instance avoiding the use of infinitary objects, but this is not really in the spirit of what most mathematicians would call a genuinely new proof.]

The first half of this course will be devoted to results of the above type, which apply to general topological dynamical systems or general measure-preserving systems. One important insight that will emerge from analysis of the latter is that in many cases, a large portion of the measure-preserving system is irrelevant for the purposes of understanding long-time average behaviour; instead, there will be a smaller system, known as a *characteristic factor* for the system, which completely controls these asymptotic averages. A deep and powerful fact is that in many situations, this characteristic factor is extremely structured algebraically, even if the original system has no obvious algebraic structure whatsoever. Because of this, it becomes important to study algebraic dynamical systems, such as the group actions on homogeneous spaces described earlier, as it allows one to obtain more precise results. (For instance, this algebraic structure was used to show that the limit in (3) actually converges, a result which does not seem accessible purely through the techniques used to prove the Furstenberg recurrence theorem.) This study will be the focus of the second half of the course, particularly in the important case of *nilsystems* - group actions arising from a nilpotent Lie group with discrete stabiliser. One of the key results here is Ratner's theorem, which describes the distribution of orbits $\{T^n x : n \in \mathbb{Z}\}$ in nilsystems, and also in a more general class of group actions on homogeneous spaces. It is unlikely that we will end up proving Ratner's theorem in full generality, but we will try to cover a few special cases of this theorem.

In closing, I should mention that the topics I intend to cover in this course are only a small fraction of the vast area of ergodic theory and dynamical systems; for instance, there are parts of this field connected with complex analysis and fractals, ODE, probability and information theory, harmonic analysis, group theory, operator algebras, or mathematical physics which I will say absolutely nothing about here.

[*Update*, Jan 28: Definition of measure-preserving system modified to add the separability condition.]

18 comments

Comments feed for this article

9 January, 2008 at 12:11 am

Tamar Ziegler

Hi Terry,

Just a comment regarding infinite measure preserving systems. There
is a rather developed general theory in this case as well. There is an overview
paper on Jon Aaronson's web page (he has also written a book on the subject).

Best,
Tammy

9 January, 2008 at 12:38 am

Anonymous

Dear Terry,

What text book would you recommend for this topic? Thanks.

–Anon

9 January, 2008 at 8:18 am

Anonymous 2

Dear Prof. Tao, are you going to have the notes available in a more "printable" format such as pdf? I hope so.

Thank you.

9 January, 2008 at 10:20 am

Terence Tao

254A, Lecture 2: Three categories of dynamical systems

10 January, 2008 in 254A - ergodic theory, math.CT, math.DS

Tags: minimal system, Morse sequence, topological dynamics, Zorn's lemma

Before we begin or study of dynamical systems, topological dynamical systems, and measure-preserving systems (as defined in the previous lecture), it is convenient to give these three classes the structure of a category. One of the basic insights of category theory is that a mathematical objects in a given class (such as dynamical systems) are best studied not in isolation, but in relation to each other, via morphisms. Furthermore, many other basic concepts pertaining to these objects (e.g. subobjects, factors, direct sums, irreducibility, etc.) can be defined in terms of these morphisms. One advantage of taking this perspective here is that it provides a unified way of defining these concepts for the three different categories of dynamical systems, topological dynamical systems, and measure-preserving systems that we will study in this course, thus sparing us the need to give any of our definitions (except for our first one below) in triplicate.

Informally, a morphism between two objects in a class is any map which respects all the structures of that class. For the three categories we are interested in, the formal definition is as follows.

Definition 1. (Morphisms)

1. A *morphism* $\phi: (X,T) \to (Y,S)$ between two dynamical systems is a map $\phi: X \to Y$ which intertwines T and S in the sense that $S \circ \phi = \phi \circ T$.
2. A *morphism* $\phi: (X,\mathcal{F},T) \to (Y,\mathcal{G},S)$ between two topological dynamical systems is a morphism $\phi: (X,T) \to (Y,S)$ of dynamical systems which is also continuous, thus $\phi^{-1}(U) \in \mathcal{F}$ for all $U \in \mathcal{G}$.
3. A *morphism* $\phi: (X,\mathcal{X},\mu,T) \to (Y,\mathcal{Y},\nu,S)$ between two measure-preserving systems is a morphism $\phi: (X,T) \to (Y,S)$ of dynamical systems which is also measurable (thus $\phi^{-1}(E) \in \mathcal{X}$ for all $E \in \mathcal{Y}$) and measure-preserving (thus $\mu(\phi^{-1}(E)) = \nu(E)$ for all $E \in \mathcal{Y}$). Equivalently, $\nu = \phi_*(\mu)$ is the push-forward of μ by ϕ.

When it is clear what category we are working in, and what the shifts are, we shall often refer to a system by its underlying space, thus for instance a morphism $\phi: (X,\mathcal{X},\mu,T) \to (Y,\mathcal{Y},\nu,S)$ might be abbreviated as $\phi: X \to Y$.

If a morphism $\phi: X \to Y$ has an inverse $\phi^{-1}: Y \to X$ which is also a morphism, we say that ϕ is an isomorphism, and that X and Y are *isomorphic* or *conjugate*.

It is easy to see that morphisms obey the axioms of a (concrete) category, thus the identity map $\mathrm{id}_X : X \to X$ on a system is always a morphism, and the composition $\psi \circ \phi : X \to Z$ of two morphisms $\phi : X \to Y$ and $\psi : Y \to Z$ is again a morphism.

Let's give some simple examples of morphisms.

- **Example 1.** If (X,T) is a dynamical system, a topological dynamical system, or a measure-preserving dynamical system, then $T^n : X \to X$ is an isomorphism for any integer n. (Indeed, one can view the map $X \mapsto T^n$ as a ~~covariant functor~~ natural transformation from the identity functor on the category of dynamical systems (or topological dynamical systems, etc.) to itself, although we will not take this perspective here.)
- **Example 2** (Subsystems). Let (X,T) be a dynamical system, and let E be a subset of X which is T-invariant in the sense that $T^n E = E$ for all n. Then the restriction of $(E, T\mid_E)$ of (X,T) to E is itself a dynamical system, and the inclusion map $\iota : E \to X$ is a morphism. In the category of topological dynamical systems $(X, \mathcal{F}, T)$, we have the same assertion so long as E is *closed* (hence compact, since X is compact). In the category of measure-preserving systems $(X, \mathcal{X}, \mu, T)$, we have the same assertion so long as E has full measure (thus $E \in \mathcal{X}$ and $\mu(E) = 1$). We thus see that subsystems are not very common in measure-preserving systems and will in fact play very little role there; however, subsystems (and specifically, *minimal* subsystems) will play a fundamental role in topological dynamics.
- **Example 3** (Skew shift). Let $\alpha \in \mathbb{R}$ be a fixed real number. Let (X, T) be the dynamical system $X := (\mathbb{R}/\mathbb{Z})^2, T : (x_1, x_2) \mapsto (x_1 + \alpha, x_2 + x_1)$, let (Y, S) be the dynamical system $Y := \mathbb{R}/\mathbb{Z}, S : y \mapsto y + \alpha$, and let $\pi : X \to Y$ be the projection map $\pi : (x_1, x_2) \to x_1$. Then π is a morphism. If one converts X and Y into either a topological dynamical system or a measure-preserving system in the obvious manner, then π remains a morphism. Observe that π foliates the big space X "upstairs" into "vertical" fibres $\pi^{-1}(\{y\}), y \in Y$ indexed by the small "horizontal" space "downstairs"; the shift S on the factor space Y downstairs determines how the fibres move (the shift T upstairs sends each vertical fibre $\pi^{-1}(\{y\})$ to another vertical fibre $\pi^{-1}(\{Sy\})$, but does not govern the dynamics *within* each fibre. More generally, any *factor map* (i.e. a surjective morphism) exhibits this type of behaviour. (Another example of a factor map is the map $\pi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/M\mathbb{Z}$ between two cyclic groups (with the standard shift $x \mapsto x + 1$) given by $\pi : x \mapsto x \bmod M$. This is a well-defined factor map when M is a factor of N, which may help explain the terminology. If we wanted to adhere strictly to the category theoretic philosophy, we should use epimorphisms rather than surjections, but we will not require this subtle distinction here.)
- **Example 4** (Universal pointed dynamical system). Let $\mathbb{Z} = (\mathbb{Z}, +1)$ be the dynamical system given by the integers with the standard shift $n \mapsto n + 1$. Then given any other dynamical system (X,T) with a distinguished point $x \in X$, the orbit map $\phi : n \mapsto T^n x$ is a morphism from $\mathbb{Z}$ to X. This allows us to lift most questions about dynamical systems (with a distinguished point x) to those for a single "universal" dynamical system, namely the integers (with distinguished point 0). One cannot pull off the same trick directly with topological dynamical systems or measure-preserving systems, because $\mathbb{Z}$ is non-compact and does not admit a shift-invariant probability measure. As we shall see later, the former difficulty can be resolved by passing to a universal compactification of the integers, namely the Stone–Čech compactification $\beta\mathbb{Z}$ (or equivalently, the space of ultrafilters on the integers), though with the important caveat that this compactification is not metrisable. To

resolve the second difficulty (with the assistance of a distinguished set rather than a distinguished point), see the next example.

- **Example 5** (Universal dynamical system with distinguished set). Recall the boolean Bernoulli system $(2^{\mathbb{Z}}, U)$ (Example 6 from the previous lecture). Given any other dynamical system (X, T) with a distinguished set $A \subset X$, the *recurrence map* $\phi : X \to 2^{\mathbb{Z}}$ defined by $\phi(x) := \{n \in \mathbb{Z} : T^n x \in A\}$ is a morphism. Observe that $A = \phi^{-1}(B)$, where B is the cylinder set $B := \{E \in 2^{\mathbb{Z}} : 0 \in E\}$. Thus we can push forward an arbitrary dynamical system (X, T, A) with distinguished set to a universal dynamical system $(2^{\mathbb{Z}}, U, B)$. Actually one can restrict $(2^{\mathbb{Z}}, U, B)$ to the subsystem $(\phi(X), U\downharpoonright_{\phi(X)}, B \cap \phi(X))$, which is easily seen to be shift-invariant. In the category of topological dynamical systems, the above assertions still hold (giving $2^{\mathbb{Z}}$ the product topology), so long as A is clopen. In the category of measure-preserving systems $(X, \mathcal{X}, \mu, T)$, the above assertions hold as long as A is measurable, $2^{\mathbb{Z}}$ is given the product σ-algebra, and the push-forward measure $\phi_*(\mu)$.

Now we begin our analysis of dynamical systems. When studying other mathematical objects (e.g. groups or representations), often one of the first steps in the theory is to decompose general objects into "irreducible" ones, and then hope to classify the latter. Let's see how this works for dynamical systems (X,T) and topological dynamical systems $(X, \mathcal{F}, T)$. (For measure-preserving systems, the analogous decomposition will be the *ergodic decomposition*, which we will discuss later in this course.)

Define a *minimal* dynamical system to be a system (X,T) which has no proper subsystems (Y,S). Similarly define a minimal topological dynamical system to be a system $(X, \mathcal{F}, T)$ with no proper subsystems $(Y, \mathcal{G}, S)$. [One could make the same definition for measure-preserving systems, but it tends to be a bit vacuous - given any measure preserving system that contains points of measure zero, one can make it trivially smaller by removing the orbit $T^{\mathbb{Z}}x := \{T^n x : n \in \mathbb{Z}\}$ of any point x of measure zero. One could place a topology on the space X and demand that it be compact, in which case minimality just means that the probability measure μ has full support.]

For a dynamical system, it is not hard to see that for any $x \in X$, the orbit $Y = T^{\mathbb{Z}}x = \{T^n x : n \in \mathbb{Z}\}$ is a minimal system, and conversely that all minimal systems arise in this manner; in particular, every point is contained in a minimal orbit. It is also easy to see that any two minimal systems (i.e. orbits) are either disjoint or coincident. Thus every dynamical system can be uniquely decomposed into the disjoint union of minimal systems. Also, every orbit $T^{\mathbb{Z}}x$ is isomorphic to $\mathbb{Z}/\mathrm{Stab}(x)$, where $\mathrm{Stab}(x) := \{n \in \mathbb{Z} : T^n x = x\}$ is the stabiliser group of x. Since we know what all the subgroups of $\mathbb{Z}$, we conclude that every minimal system is either equivalent to a cyclic group shift $(\mathbb{Z}/N\mathbb{Z}, x \mapsto x+1)$ for some $N \geq 1$, or to the integer shift $(\mathbb{Z}, x \mapsto x+1)$. Thus we have completely classified all dynamical systems up to isomorphism as the arbitrary union of these minimal examples. [In the case of finite dynamical systems, the integer shift does not appear, and we have recovered the classical fact that every permutation is uniquely decomposable as the product of disjoint cycles.]

For topological dynamical systems, it is still true that any two minimal systems are either disjoint or coincident (why?), but the situation nevertheless is more complicated. First of all, orbits need not be closed (consider for instance the circle shift $(\mathbb{R}/\mathbb{Z}, x \mapsto x + \alpha)$ with α irrational). If one considers the *orbit closure* $\overline{T^{\mathbb{Z}}x}$ of a point x, then this is now a subsystem (why?) , and every minimal system is the orbit closure of any of its elements (why?), but in the converse direction, not all orbit closures are minimal. Consider for instance the boolean Bernoulli system $(2^{\mathbb{Z}}, A \mapsto A-1)$ with $x = \mathbb{N} := \{0, 1, 2, \ldots\} \in 2^{\mathbb{Z}}$ being the natural numbers. Then the orbit $T^{\mathbb{Z}}x$ of x consists of all the half-lines $\{a, a+1, \ldots, \} \in 2^{\mathbb{Z}}$ for $a \in \mathbb{Z}$, but it is not closed; it has the point $\mathbb{Z} \in 2^{\mathbb{Z}}$ and the point $\emptyset \in 2^{\mathbb{Z}}$ as limit points (recall that $2^{\mathbb{Z}}$ is given the product (i.e. pointwise) topology). Each of these points is an invariant point of T and thus forms its own orbit closure, which is obviously minimal. [In particular, this shows that x itself is not contained in any minimal system - why?]

Thus we see that finite dynamical systems do not quite form a perfect model for topological dynamical systems. A slightly better (but still imperfect) model would be that of *non-invertible* finite dynamical systems (X, T), in which $T : X \to X$ is now just a function rather than a permutation. Then we can still verify that all minimal orbits are given by disjoint cycles, but they no longer necessarily occupy all of X; it is quite possible for the orbit $T^{\mathbb{N}}x = \{T^n x : n \in \mathbb{N}\}$ of a point x to start outside of any of the minimal cycles, although it will eventually be absorbed in one of them.

In the above examples, the limit points of an orbit formed their own minimal orbits. In some cases, one has to pass to limits multiple times before one reaches a minimal orbit (cf. the "Glaeser refinements" in this lecture of Charlie Fefferman). For instance, consider the boolean Bernoulli system again, but now consider the point

$$y := \bigcup_{n=0}^{\infty} [4^n, 2 \times 4^n] = [1,2] \cup [4,8] \cup [16,32] \cup \ldots \in 2^{\mathbb{Z}}$$

where we use the notation $[N, M] := \{n \in \mathbb{Z} : N \leq n \leq M\}$. Observe that the point x defined earlier is not in the orbit $T^{\mathbb{N}}y$, but lies in the orbit closure, as it is the limit of $T^{4^n}y$. On the other hand, the orbit closure of x does not contain y. So the orbit closure of x is a subsystem of that of y, and then inside the former system one has the minimal systems $\{\mathbb{Z}\}$ and $\{\emptyset\}$. It is not hard to iterate this type of example and see that we can have quite intricate hierarchies of systems.

Exercise 1. Construct a topological dynamical system $(X, \mathcal{F}, T)$ and a sequence of orbit closures $\overline{T^{\mathbb{Z}}x_n}$ in X which form a proper nested sequence, thus

$$\overline{T^{\mathbb{Z}}x_1} \supsetneq \overline{T^{\mathbb{Z}}x_2} \supsetneq \overline{T^{\mathbb{Z}}x_3} \supsetneq \ldots$$

[*Hint*: Take a countable family of nested Bernoulli systems, and find a way to represent each one as a orbit closure.] $\diamond$

Despite this apparent complexity, we can always terminate such hierarchies of subsystems at a minimal system:

Lemma 1. Every topological dynamical system $(X, \mathcal{F}, T)$ contains a minimal dynamical system.

Proof. Observe that the intersection of any chain of subsystems of X is again a subsystem (here we use the finite intersection property of compact sets to guarantee that the intersection is non-empty, and we also use the fact that the arbitrary intersection of closed or T-invariant sets is again closed or T-invariant). The claim then follows from Zorn's lemma. [We will always assume the axiom of choice throughout this course.] $\square$

Exercise 2. Every compact metrisable space is second countable and thus has a countable base. Suppose we are given an explicit enumeration $V_1, V_2, \ldots$ of such a base. Then find a proof of Lemma 1 which avoids the axiom of choice. $\diamond$

It would be nice if we could use Lemma 1 to decompose topological dynamical systems into the union of minimal subsystems, as we did in the case of non-topological dynamical systems. Unfortunately this does not work so well; the problem is that the complement of a minimal system is an open set rather than a closed set, and so we cannot cleanly separate a minimal system from its complement. (In any case, the preceding examples already show that there can be some points in a system that are not contained in any minimal subsystem. Also, in contrast with non-invertible non-topological dynamical systems, our examples also show that a closed orbit can contain multiple minimal subsystems, so we cannot reduce to some sort of "nilpotent" system that has only one minimal system.)
We will study minimal dynamical systems in detail in the next few lectures. I'll close now with some examples of minimal systems.

Example 6 (Cyclic group shift). The cyclic group shift $(\mathbb{Z}/N\mathbb{Z}, x \mapsto x+1)$, where N is a positive integer, is a minimal system, and these are the only discrete minimal topological dynamical systems. More generally, if x is a periodic point of a topological dynamical system (thus $T^N x = x$ for some $N \geq 1$), then the closed orbit of x is isomorphic to a cyclic group shift and is thus minimal.

Example 7 (Torus shift). Consider a torus shift $((\mathbb{R}/\mathbb{Z})^d, x \mapsto x + \alpha)$, where $\alpha \in \mathbb{R}^d$ is a fixed vector. It turns out that this system is minimal if and only if α is *totally irrational*, which means that $n \cdot \alpha$ is not an integer for any non-zero $n \in \mathbb{Z}^d$. (The "if" part is slightly non-trivial, requiring Weyl's equidistribution theorem; but the "only if" part is easy, and is left as an exercise.)

Example 8 (Morse sequence): Let $A = \{a, b\}$ be a two-letter alphabet, and consider the Bernoulli system $(A^{\mathbb{Z}}, T)$ formed from doubly infinite words

$$\ldots x_{-2}x_{-1}.x_0x_1x_2\ldots$$

in A with the left-shift. Now define the sequence of finite words

$$w_1 := a.b,\ w_2 := abba.baab,\ w_3 := abbabaabbaababba.baababbaabbabaab, \text{ etc.}$$

by the recursive formula

$$w_1 := a.b; \quad w_{i+1} := f(w_i)$$

where $f(w)$ denotes the word formed from w by replacing each occurrence of a and b by abba and baab respectively. These words w_i converge pointwise to an infinite word

$$w = \ldots abbabaababbabaabbaababba.baababbaabbabaababbabaab\ldots.$$

Exercise 3. Show that w is not a periodic element of $A^{\mathbb{Z}}$, but that the orbit $\overline{T^{\mathbb{Z}}w}$ is both closed and minimal. [Hint: find large subwords of w which appear syndetically, which means that the gaps between each appearance are bounded. In fact, all subwords of w appear syndetically. One can also work with a more explicit description of w involving the number of non-zero digits in the binary expansion of the index.] This set is an example of a *substitution minimal set*. $\diamond$

Exercise 4. Let $(X, \mathcal{F}, T)$ and $(Y, \mathcal{G}, S)$ be topological dynamical systems. Define the *product* of these systems to be $(X \times Y, \mathcal{F} \times \mathcal{G}, T \times S)$, where $X \times Y$ is the Cartesian product, $\mathcal{F} \times \mathcal{G}$ is the product topology, and $T \times S$ is the map $(x, y) \mapsto (Tx, Sy)$. Note that there are obvious projection morphisms from this product system to the two original systems. Show that this product system is indeed a product in the sense of category theory. Establish analogous claims in the categories of dynamical systems and measure-preserving systems. $\diamond$

Exercise 5. Let $(X, \mathcal{F}, T)$ and $(Y, \mathcal{G}, S)$ be topological dynamical systems. Define the *disjoint union* of these systems to be $(X \uplus Y, \mathcal{F} \uplus \mathcal{G}, T \uplus S)$ where $(X \uplus Y, \mathcal{F} \uplus \mathcal{G})$ is the disjoint union of $(X, \mathcal{F})$ and $(Y, \mathcal{G})$, and $T \uplus S$ is the map which agrees with T on X and agrees with S on Y. Note that there are obvious embedding morphisms from the original two systems into the disjoint union. Show that the disjoint union is a coproduct in the sense of category theory. Are analogous claims true for the categories of dynamical systems and measure-preserving systems? $\diamond$

[*Update*, Jan 11: several corrections.]

[*Update*, Jan 14: A required to be clopen in the topological version of Example 4.]

[*Update*, Jan 17: Slight change to Example 7.]

[Update, Jan 20: More exercises added.]

21 comments

Comments feed for this article

11 January, 2008 at 1:01 pm

Hao

Dear Prof. Tao,

Is the covariant functor in example 1 given by:
sends (X,T) to (X,T^n)
and the morphism (a map that interwines T and S) to a map that interwines T^n and S^n?

I am not sure about what X->T^n means in the example.

11 January, 2008 at 1:35 pm

Andy P.

254A, Lecture 3: Minimal dynamical systems, recurrence, and the Stone-Čech compactification

13 January, 2008 in 254A - ergodic theory, math.DS, math.GN, math.LO

Tags: almost periodicity, lamplighter group, recurrence, Stone-Cech compactification, syndetic sets, ultrafilter

We now begin the study of *recurrence* in topological dynamical systems $(X, \mathcal{F}, T)$ - how often a non-empty open set U in X returns to intersect itself, or how often a point x in X returns to be close to itself. Not every set or point needs to return to itself; consider for instance what happens to the shift $x \mapsto x+1$ on the compactified integers $\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$. Nevertheless, we can always show that at least one set (from any open cover) returns to itself:

Theorem 1. (Simple recurrence in open covers) Let $(X, \mathcal{F}, T)$ be a topological dynamical system, and let $(U_\alpha)_{\alpha \in A}$ be an open cover of X. Then there exists an open set U_α in this cover such that $U_\alpha \cap T^n U_\alpha \neq \emptyset$ for infinitely many n.

Proof. By compactness of X, we can refine the open cover to a finite subcover. Now consider an orbit $T^{\mathbb{Z}}x = \{T^n x : x \in \mathbb{Z}\}$ of some arbitrarily chosen point $x \in X$. By the infinite pigeonhole principle, one of the sets U_α must contain an infinite number of the points $T^n x$ counting multiplicity; in other words, the recurrence set $S := \{n : T^n x \in U_\alpha\}$ is infinite. Letting n_0 be an arbitrary element of S, we thus conclude that $U_\alpha \cap T^{n_0 - n} U_\alpha$ contains $T^{n_0} x$ for every $n \in S$, and the claim follows. $\Box$

Exercise 1. Conversely, use Theorem 1 to deduce the infinite pigeonhole principle (i.e. that whenever $\mathbb{Z}$ is coloured into finitely many colours, one of the colour classes is infinite). *Hint*: look at the orbit closure of c inside $A^{\mathbb{Z}}$, where A is the set of colours and $c : \mathbb{Z} \to A$ is the colouring function.) $\diamond$

Now we turn from recurrence of sets to recurrence of individual points, which is a somewhat more difficult, and highlights the role of minimal dynamical systems (as introduced in the previous lecture) in the theory. We will approach the subject from two (largely equivalent) approaches, the first one being the more traditional "epsilon and delta" approach, and the second using the Stone-Čech compactification $\beta\mathbb{Z}$ of the integers (i.e. ultrafilters).

Before we begin, it will be notationally convenient to place a metric d on our compact metrisable space X [though, as an exercise, the reader is encouraged to recast all the material here in a manner which does not explicitly mention a metric]. There are of course infinitely many metrics that one could place here, but they are all coarsely equivalent in the following sense: if d, d' are two metrics on X, then for every $\delta > 0$ there exists an $\varepsilon > 0$ such that $d'(x,y) < \delta$ whenever $d(x,y) < \varepsilon$, and similarly with the role d and d' reversed. This claim follows from the standard fact that continuous functions between compact metric spaces are uniformly continuous. Because of this equivalence, it will not actually matter for any of our results what metric we place on our spaces. For instance, we could endow a Bernoulli system $A^{\mathbb{Z}}$, where A is itself a compact metrisable space (and thus $A^{\mathbb{Z}}$ is compact by Tychonoff's theorem), with the metric

$$d((a_n)_{n\in\mathbb{Z}},(b_n)_{n\in\mathbb{Z}}) := \sum_{n\in\mathbb{Z}} 2^{-|n|} d_A(a_n,b_n) \quad (1)$$

where d_A is some arbitrarily selected metric on A. Note that this metric is not shift-invariant.

Exercise 2. Show that if A contains at least two points, then the Bernoulli system $A^{\mathbb{Z}}$ (with the standard shift) cannot be endowed with a shift-invariant metric. (*Hint*: find two distinct points which converge to each other under the shift map.) $\diamond$

Fix a metric d. For each n, the shift $T^n : X \to X$ is continuous, and hence uniformly continuous since X is compact, thus for every $\delta > 0$ there exists $\varepsilon > 0$ depending on δ and n such that $d(T^n x, T^n y) < \delta$ whenever $d(x,y) < \varepsilon$. However, we caution that the T^n need not be uniformly equicontinuous; the quantity ε appearing above can certainly depend on n. Indeed, they need not even be equicontinuous. For instance, this will be the case for the Bernoulli shift with the metric (1) (why?), and more generally for any system that exhibits "mixing" or other chaotic behaviour. At the other extreme, in the case of *isometric* systems - systems in which T preserves the metric d - the shifts T^n are all isometries, and thus are clearly uniformly equicontinuous.

We can now classify points x in X based on the dynamics of the orbit $T^{\mathbb{Z}}x := \{T^n x : n \in \mathbb{Z}\}$:

1. x is *invariant* if $Tx = x$.
2. x is *periodic* if $T^n x = x$ for some non-zero n.
3. x is *almost periodic* if for every $\varepsilon > 0$, the set $\{n \in \mathbb{Z} : d(T^n x, x) < \varepsilon\}$ is syndetic (i.e. it has bounded gaps);
4. x is *recurrent* if for every $\varepsilon > 0$, the set $\{n \in \mathbb{Z} : d(T^n x, x) < \varepsilon\}$ is infinite. Equivalently, there exists a sequence n_j of integers with $|n_j| \to \infty$ such that $\lim_{j\to\infty} T^{n_j} x = x$.

It is clear that every invariant point is periodic, that every periodic point is almost periodic, and every almost periodic point is recurrent. These inclusions are all strict. For instance, in the circle shift system $(\mathbb{R}/\mathbb{Z}, x \mapsto x+\alpha)$ with $\alpha \in \mathbb{R}$ irrational, it turns out that every point is almost periodic, but no point is periodic.

Exercise 3. In the boolean Bernoulli system $(2^{\mathbb{Z}}, A \mapsto A-1)$, show that the discrete Cantor set

$$x := \bigcup_{N=1}^{\infty} \{\sum_{n=0}^{N} \epsilon_n 10^n : \epsilon_n \in \{-1,+1\}\} \quad (2)$$

is recurrent but not almost periodic. $\diamond$

In a general topological dynamical system, it is quite possible to have points which are non-recurrent (as the example of the compactified integer shift already shows). But if we restrict to a *minimal* dynamical system, things get much better:

Lemma 1. If $(X, \mathcal{F}, T)$ is a minimal topological dynamical system, then every element of X is almost periodic (and hence recurrent).

Proof. Suppose for contradiction that we can find a point x of X which is not almost periodic. This means that we can find $\varepsilon > 0$ such that the set $\{n : d(T^n x, x) < \varepsilon\}$ is not syndetic. Thus, for any $m > 0$, we can find an n_m such that $d(T^n x, x) \geq \varepsilon$ for all $n \in [n_m - m, n_m + m]$ (say).

Since X is compact, the sequence $T^{n_m} x$ must have at least one limit point y. But then one verifies (using the continuity of the shift operators) that

$$d(T^h y, x) = \lim_{m \to \infty} d(T^{n_m + h} x, x) \geq \varepsilon \quad (3)$$

for all h. But this means that the orbit closure $\overline{T^{\mathbb{Z}} y}$ of y does not contain x, contradicting the minimality of X. The claim follows. $\Box$

Exercise 4. If x is a point in a topological dynamical system, show that x is almost periodic if and only if it lies in a minimal system. Because of this, almost periodic points are sometimes referred to as *minimal* points. $\diamond$

Combining Lemma 1 with Lemma 1 of the previous lecture, we immediately obtain the

Birkhoff recurrence theorem. Every topological dynamical system contains at least one point x which is almost periodic (and hence recurrent).

Note that this is stronger than Theorem 1, as can be seen by considering the element U_α of the open cover which contains the almost periodic point. Indeed, we now have obtained a stronger conclusion, namely that the set of return times $\{n : T^n U_\alpha \cap U_\alpha \neq \emptyset\}$ is not only infinite, it is syndetic.

Exercise 5. State and prove a version of the Birkhoff recurrence theorem in which the map $T: X \to X$ is continuous but not assumed to be invertible. (Of course, all references to $\mathbb{Z}$ now need to be replaced with $\mathbb{N}$.) $\diamond$

The Birkhoff recurrence theorem does not seem particularly strong, as it only guarantees existence of a single recurrent (or almost periodic point). For general systems, this is inevitable, because it can happen that the majority of the points are non-recurrent (look at the compactified integer shift system, for instance). However, suppose the system is a group quotient $(G/\Gamma, x \mapsto gx)$. To make this a topological dynamical system, we need G to be a topological group, and Γ to be a cocompact subgroup of G (such groups are also sometimes referred to as *uniform* subgroups). Then we see that the system is a homogeneous space: given any two points $x, y \in G/\Gamma$, there exists a group element $h \in G$ such that hx=y. Thus we expect any two points in G/Γ to behave similarly to each other. Unfortunately, this does not quite work in general, because the action of h need not preserve the shift $x \mapsto gx$, as there is no reason that h commutes with g. But suppose that g is a central element of G (which is for instance the case if G is abelian). Then the action of h is now an isomorphism on the dynamical system $(G/\Gamma, x \mapsto gx)$. In particular, if hx = y, we see that x is almost periodic (or recurrent) if and only if y is. We thus conclude

Theorem 2. (Kronecker type approximation theorem) Let $(G/\Gamma, x \mapsto gx)$ be a topological group quotient dynamical system such that g lies in the centre Z(G) of G. Then *every* point in this system is almost periodic (and hence recurrent).

Applying this theorem to the torus shift $((\mathbb{R}/\mathbb{Z})^d, x \mapsto x + \alpha)$, where $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ is a vector, we thus obtain that for any $\varepsilon > 0$, the set

$$\{n \in \mathbb{Z} : \mathrm{dist}(n\alpha, \mathbb{Z}^d) < \varepsilon\} \quad (4)$$

is syndetic (and in particular, infinite). This should be compared with the classical Kronecker approximation theorem.

It is natural to ask what happens when g is not central. If G is a Lie group and the action of g on the Lie algebra $\mathfrak{g}$ is unipotent rather than trivial, then Theorem 2 still holds; this follows from Ratner's theorem, of which we will discuss much later in this course. But the claim is not true for all group quotients. Consider for instance the Bernoulli shift system $(X, T) = ((\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}, T)$, which is isomorphic to the boolean Bernoulli shift system. As the previous examples have already shown, this system contains both recurrent and non-recurrent elements. On the other hand, it is intuitive that this system has a lot of symmetry, and indeed we can view it as a group quotient $(G/\Gamma, x \mapsto gx)$. Specifically, G is the lamplighter group $G = \mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$. To describe this group, we observe that the group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}}$ acts on X by addition, whilst the group $\mathbb{Z}$ acts on X via the shift map T. The lamplighter group $G := (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}} \times \mathbb{Z}$ then acts by both addition and shift:

$$(a, n) : x \mapsto T^n x + a \text{ for all } (a, n) \in G. \quad (5)$$

In order for this to be a group action, we endow G with the multiplication law

$$(a, n)(b, m) := (a + T^n b, n + m); \quad (6)$$

one easily verifies that this really does make G into a group; if we give G the product topology, it is a topological group. G clearly acts transitively on the compact space X, and so $X \equiv G/\Gamma$ for some cocompact subgroup Γ (which turns out to be isomorphic to $\mathbb{Z}$ - why?). By construction, the shift map T can be expressed using the group element $(0, 1) \in G$, and so we have turned the Bernoulli system into a group quotient. Since this system contains non-recurrent points (e.g. the indicator function of the natural numbers) we see that Theorem 2 does not hold for arbitrary group quotients.

– The ultrafilter approach –

Now we turn to a different approach to topological recurrence, which relies on compactifying the underlying group $\mathbb{Z}$ that acts on topological dynamical systems. By doing so, all the epsilon management issues go away, and the subject becomes very algebraic in nature. On the other hand, some subtleties arise also; for instance, the compactified object $\beta\mathbb{Z}$ is not a group, but merely a left-continuous semigroup.

This approach is based on ultrafilters or (equivalently) via the Stone-Čech compactification. Let us recall how this compactification works:

Theorem 3. (Stone-Čech compactification) Every locally compact Hausdorff (LCH) space X can be embedded in a compact Hausdorff space βX in which X is an open dense set. (In particular, if X is already compact, then $\beta X = X$.) Furthermore, any continuous function $f : X \to Y$ between LCH spaces extends uniquely to a continuous function $\beta f : \beta X \to \beta Y$.

Proof. (Sketch) This proof uses the intuition that βX should be the "finest" compactification of X. Recall that a compactification of a LCH space X is any compact Hausdorff space containing X as an open dense set. We say that one compactification Y of X is *finer* than another Z if there is a surjective continuous map from Y to Z that is the identity on X. (Note that as X is dense in Y, and Z is Hausdorff, this surjection is unique.) For instance, the two-point compactification $\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$ of the integers is finer than the one-point compactification $\mathbb{Z} \cup \{\infty\}$. This is clearly a partial ordering; also, the inverse limit of any chain of compactifications can be verified (by Tychonoff's theorem) to still be a compactification. Hence, by Zorn's lemma (modulo a technical step in which one shows that the moduli space of compactifications of X is a set rather than a class), there is a maximal compactification βX. To verify the extension property for continuous functions $f : X \to Y$, note (by replacing Y with βY if necessary) that we may take Y to be compact. Let Z be the closure of the graph $X' := \{(x, f(x)) : x \in X\}$ in $(\beta X) \times Y$. X' is clearly homeomorphic to X, and so Z is a compactification of X. Also, there is an obvious surjective continuous map from Z to βX; thus by maximality, this map must be a homeomorphism, thus Z is the graph of a continuous function $\beta f : \beta X \to \beta Y$, and the claim follows (the uniqueness of βf is easily established). $\Box$

Exercise 6. Let X be discrete (and thus clearly LCH), and let βX be the Stone-Čech compactification. For any $p \in \beta X$, let $[p] \in 2^{2^X}$ be the collection of all sets $A \subset X$ such that $\beta 1_A(p) = 1$. Show that [p] is an ultrafilter, or in other words that it obeys the following four properties:

1. $\emptyset \not\in [p]$.
2. If $U \in [p]$ and $V \in 2^X$ are such that $U \subset V$, then $V \in [p]$.
3. If $U, V \in [p]$, then $U \cap V \in [p]$.
4. If $U, V \in 2^X$ are such that $U \cup V = X$, then at least one of U and V lie in [p].

Furthermore, show that the map $p \mapsto [p]$ is a homeomorphism between βX and the space of ultrafilters, which we endow with the topology induced from the product topology on $2^{2^X} \equiv \{0, 1\}^{2^X}$, where we give $\{0, 1\}$ the discrete topology (one can place some other topologies here also). Thus we see that in the discrete case, we can represent the Stone-Čech compactification explicitly via ultrafilters. $\diamond$

It is easy to see that $\beta(g \circ f) = (\beta g) \circ (\beta f)$ whenever $f : X \to Y$ and $g : Y \to Z$ are continuous maps between LCH spaces. In the language of category theory, we thus see that β is a covariant functor from the category of LCH spaces to the category of compact Hausdorff spaces. (The above theorem does not explicitly define βX, but it is not hard to see that this compactification is unique up to homeomorphism, so the exact form of βX is somewhat moot. However, it is possible to create an ultrafilter-based description of βX for general LCH spaces X, though we will not do so here.)

Exercise 6′. Let X and Y be two LCH spaces. Show that the disjoint union $(\beta X) \uplus (\beta Y)$ of βX and βY

is isomorphic to $\beta(X \uplus Y)$. (Indeed, this isomorphism is a natural isomorphism.) In the language of category theory, this means that β preserves coproducts. (Unfortunately, β does not preserve products, which leads to various subtleties, such as the non-commutativity of the compactification of commutative groups.) $\diamond$

Note that if $f : X \to Y$ is continuous, then $\beta f : \beta X \to \beta Y$ is continuous also; since X is dense in βX, we conclude that

$$\beta f(p) = \lim_{x \to p} f(x) \quad (7)$$

for all $p \in \beta X$, where x is constrained to lie in X. In particular, the limit on the right exists for any continuous $f : X \to Y$, and thus if X is discrete, it exists for any (!) function $f : X \to Y$. Each p can then be viewed as a recipe for taking limits of arbitrary functions in a consistent fashion (although different p's can give different limits, of course). It is this ability to take limits without needing to check for convergence and without running into contradictions that makes the Stone-Čech compactification a useful tool here. (See also my post on ultrafilters for further discussion.)

The integers $\mathbb{Z}$ are discrete, and thus are clearly LCH. Thus we may form the compactification $\beta\mathbb{Z}$. The addition operation $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ can then be extended to $\beta\mathbb{Z}$ by the plausible-looking formula

$$p + q := \lim_{n \to p} \lim_{m \to q} n + m \quad (8)$$

for all $p, q \in \beta\mathbb{Z}$, where n, m range in the integers $\mathbb{Z}$. Note that the double limit is guaranteed to exist by (7). Equivalently, we have

$$\lim_{l \to p+q} f(l) = \lim_{n \to p} \lim_{m \to q} f(n+m) \quad (8')$$

for all functions $f : \mathbb{Z} \to X$ into an LCH space X; one can derive (8′) from (8) by applying $\beta f : \beta\mathbb{Z} \to \beta X$ to both sides of (8) and using (7) and the continuity of βf repeatedly.

This addition operation clearly extends that of $\mathbb{Z}$ and is associative, thus we have turned $\beta\mathbb{Z}$ into a semigroup. We caution however that this semigroup is not commutative, due to the usual difficulty that double limits in (8) cannot be exchanged. (We will prove non-commutativity shortly.) For similar reasons, $\beta\mathbb{Z}$ is not a group; the obvious attempt to define a negation operation $-p := \lim_{n \to p} -n$ is well-defined, but does not actually invert addition. The operation $(p, q) \mapsto p + q$ is continuous in p for fixed q (why?), but is not necessarily continuous in q for fixed p - again, due to the exchange of limits problem. Thus $\beta\mathbb{Z}$ is merely a left-continuous semigroup. If however p is an integer, then the first limit in (8) disappears, and one easily shows that $q \mapsto q + p$ is continuous in this case (and for similar reasons one also recovers commutativity, $q + p = p + q$).

Exercise 7. Let us endow the two-point compactification $\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$ with the semigroup structure + in which $x + (+\infty) = +\infty$ and $x + (-\infty) = -\infty$ for all $x \in \{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$ (compare with (8)). Show that there is a unique continuous map $\pi : \beta\mathbb{Z} \to \mathbb{Z} \cup \{-\infty\} \cup \{+\infty\}$ which is the identity on $\mathbb{Z}$, and that this map is a surjective semigroup homomorphism. Using this homomorphism, conclude:

1. $\beta\mathbb{Z}$ is not commutative. Furthermore, show that the centre

$Z(\beta\mathbb{Z}) := \{p \in \beta\mathbb{Z} : p + q = q + p \text{ for all } q \in \beta\mathbb{Z}\}$ is exactly equal to $\mathbb{Z}$.
2. Show that if $p, q \in \beta\mathbb{Z}$ are such that $p + q \in \mathbb{Z}$, then $p, q \in \mathbb{Z}$. ("Once you go to infinity, you can never return.") Conclude in particular that $\beta\mathbb{Z}$ is not a group. (Note that this conclusion could already be obtained using the coarser one-point compactification $\mathbb{Z} \cup \{\infty\}$ of the integers.) $\diamond$

Remark 1. More generally, we can take any LCH left-continuous semigroup S and compactify it to obtain a compact Hausdorff left-continuous semigroup βS. Observe that if $f : S \to S'$ is a homomorphism between two LCH left-continuous semigroups, then $\beta f : \beta S \to \beta S'$ is also a homomorphism. Thus, from the viewpoint of category theory, β can be viewed as a covariant functor from the category of LCH left-continuous semigroups to the category of CH left-continuous semigroups. We will see this functorial property being used a little later in this course. $\diamond$

The left-continuous non-commutative semigroup structure of $\beta\mathbb{Z}$ may appear to be terribly weak when compared against the jointly continuous commutative group structure of $\mathbb{Z}$, but $\beta\mathbb{Z}$ has a decisive trump card over $\mathbb{Z}$: it is *compact*. We will see the power of compactness a little later in this lecture.

A topological dynamical system $(X, \mathcal{F}, T)$ yields an action $n \mapsto T^n$ of the integers $\mathbb{Z}$. But we can automatically extend this action to an action $p \mapsto T^p$ of the compactified integers $\beta\mathbb{Z}$ by the formula

$$T^p x := \lim_{n \to p} T^n x. \quad (9)$$

(Note that X is already compact, so that the limit in (9) stays in X.) One easily checks from (8′) that this is indeed an action of $\beta\mathbb{Z}$ (thus $T^p T^q = T^{p+q}$ for all $p, q \in \mathbb{Z}$). The map $T^p x$ is continuous in p by construction; however we caution that it is no longer continuous in x (it's the exchange-of-limits problem once more!). Indeed, the map $T^p : X \to X$ can be quite nasty from an analytic viewpoint; for instance, it is possible for this map to not be Borel measurable. (This is the price one pays for introducing beasts generated from the axiom of choice into one's mathematical ecosystem.) But as we shall see, the *algebraic* properties of T^p are very good, and suffice for applications to recurrence, because once one has compactified the underlying semigroup $\beta\mathbb{Z}$, the need for point-set topology (and for all the epsilons that come with it) mostly disappears. For instance, we can now replace orbit closures by orbits:

Lemma 2. Let $(X, \mathcal{F}, T)$ be a topological dynamical system, and let $x \in X$. Then

$$\overline{T^{\mathbb{Z}}(x)} = T^{\beta\mathbb{Z}} x := \{T^p x : p \in \beta\mathbb{Z}\}.$$

Proof: Since $\beta\mathbb{Z}$ is compact, $T^{\beta\mathbb{Z}} x$ is compact also. Since $\mathbb{Z}$ is dense in $\beta\mathbb{Z}$, $T^{\mathbb{Z}} x$ is dense in $T^{\beta\mathbb{Z}} x$. The claim follows. $\square$

From (9) we see that T^p is some sort of "limiting shift" operation. To get some intuition, let us consider the compactified integer shift $(\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}, x \mapsto x + 1)$, and look at the orbit of the point 0. If one only shifts by integers $n \in \mathbb{Z}$, then $T^n 0$ can range across the region $\mathbb{Z}$ in the system but cannot reach $-\infty$ or $+\infty$. But now let $p \in \beta\mathbb{Z} \backslash \mathbb{Z}$ be any limit point of the positive integers $\mathbb{Z}^+$ (note that at least one such limit point must exist, since $\mathbb{Z}^+$ is not compact. Indeed, in the language of Exercise 7, the set of all such limit points is $\pi^{-1}(+\infty)$.) Then from (9) we see that $T^p 0 = +\infty$. Similarly, if $q \in \beta\mathbb{Z}\backslash\mathbb{Z}$ is a limit point of the negative integers $\mathbb{Z}^-$ then $T^q 0 = -\infty$. Now, since $+\infty$ invariant, we have $T^q(+\infty) = +\infty$ by (9) again, and thus $T^q T^p 0 = +\infty$, while $T^p T^q 0 = -\infty$. In particular, we see that $p + q \neq q + p$, demonstrating non-commutativity in $\beta\mathbb{Z}$ (again, compare with Exercise 7). Informally, the problem here is that in (8), n+m will go to $+\infty$ if we let m go to $+\infty$ first and then $n \to -\infty$ next, but if we take $n \to -\infty$ first and then $m \to +\infty$ next, n+m instead goes to $-\infty$.

Exercise 8. Let $A \subset \mathbb{Z}$ be a set of integers.

1. Show that βA can be canonically identified with the closure of A in $\beta\mathbb{Z}$, in which case βA becomes a clopen subset of $\beta\mathbb{Z}$.
2. Show that A is infinite if and only if $\beta A \not\subset \mathbb{Z}$.
3. Show that A is syndetic if and only if $\beta A \cap (\beta\mathbb{Z} + p) \neq \emptyset$ for every $p \in \beta\mathbb{Z}$. (Since βA is clopen, this condition is also equivalent to requiring $\beta A \cap (\mathbb{Z} + p) \neq \emptyset$ for every $p \in \beta\mathbb{Z}$.)
4. A set of integers A is said to be *thick* if it contains arbitrarily long intervals [a_n,a_n+n]; thus syndetic and thick sets always intersect each other. Show that A is thick if and only if there exists $p \in \beta\mathbb{Z}$ such that $\beta\mathbb{Z} + p \subset \beta A$. (Again, this condition is equivalent to requiring $\mathbb{Z} + p \subset \beta A$ for some p.) $\diamond$

Recall that a system is *minimal* if and only if it is the orbit closure of every point in that system. We thus have a purely algebraic description of minimality:

Corollary 1. Let $(X, \mathcal{F}, T)$ be a topological dynamical system. Then X is minimal if and only if the action of $\beta\mathbb{Z}$ is transitive; thus for every $x, y \in \mathbb{Z}$ there exists $p \in \beta\mathbb{Z}$ such that $T^p x = y$.

One also has purely algebraic descriptions of almost periodicity and recurrence:

Exercise 9. Let $(X, \mathcal{F}, T)$ be a topological dynamical system, and let x be a point in X.

1. Show that x is almost periodic if and and only if for every $p \in \beta\mathbb{Z}$ there exists $q \in \beta\mathbb{Z}$ such that $T^q T^p x = x$. (In particular, Lemma 1 is now an immediate consequence of Corollary 1.)
2. Show that x is recurrent if and only if there exists $p \in \beta\mathbb{Z}\backslash\mathbb{Z}$ such that $T^p x = x$. $\diamond$

Note that $\beta\mathbb{Z}$ acts on itself $\beta\mathbb{Z}$ by addition, $p : q \mapsto p + q$, with the action being continuous when p is an integer. Thus one can view $\beta\mathbb{Z}$ itself as a topological dynamical system, except with the caveat that $\beta\mathbb{Z}$ is not metrisable or even first countable (see Exercise 12). Nevertheless, it is still useful to think of $\beta\mathbb{Z}$ as behaving like a topological dynamical system. For instance:

Definition 1. An element $p \in \beta\mathbb{Z}$ is said to be *minimal* or *almost periodic* if for every $q \in \beta\mathbb{Z}$ there exists $r \in \beta\mathbb{Z}$ such that $r + q + p = p$.

Equivalently, p is minimal if $\beta\mathbb{Z} + p$ is a minimal left-ideal of $\beta\mathbb{Z}$, which explains the terminology.

Exercise 10. Show that for every $p \in \beta\mathbb{Z}$ there exists $q \in \beta\mathbb{Z}$ such that $q + p$ is minimal. (Hint: adapt the proof of Lemma 1 from the previous lecture.) Also, show that if p is minimal, then q+p and p+q are also minimal for any $q \in \beta\mathbb{Z}$. This shows that minimal elements of $\beta\mathbb{Z}$ exist in abundance. However, observe from Exercise 6 that no integer can be minimal. ◇

Exercise 11. Show that if $p \in \beta\mathbb{Z}$ is minimal, and x is a point in a topological dynamical system $(X, \mathcal{F}, T)$, then $T^p x$ is almost periodic. Conversely, show that x is almost periodic if and only if $x = T^p x$ for some minimal p. This gives an alternate (and more "algebraic") proof of the Birkhoff recurrence theorem. ◇

Exercise 12. Show that no element of $\beta\mathbb{Z}\backslash\mathbb{Z}$ can be written as a limit of a sequence in $\mathbb{Z}$. (*Hint*: if a sequence $n_j \in \mathbb{Z}$ converged to a limit $p \in \beta\mathbb{Z}$, one must have $\beta f(p) = \lim_{j\to\infty} f(n_j)$ for all functions $f : \mathbb{Z} \to K$ mapping into a compact Hausdorff space K.) Conclude in particular that $\beta\mathbb{Z}$ is not metrisable, first countable, or sequentially compact. ◇

[*Update*, Jan 14: various corrections and reorganisation.]

[*Update*, Jan 30: Some exercises added or expanded.]

[*Update*, Feb 4: Hint for Exercise 12 added.]

35 comments

Comments feed for this article

13 January, 2008 at 7:11 pm

Richard

Just a comment on terminology. A "thick" subset of the integers (or the generalization of the concept to any abstract group) has also been referred to as a "replete" or a "fat" subset. I think that usage of the word replete, which I prefer, preceded the usage of the others. I just thought I would comment on this so that people do not get confused reviewing the literature in this area.

13 January, 2008 at 11:38 pm

R.A.

Probably in formula (1) we should have the absolute value in the exponent

14 January, 2008 at 6:05 am

Eric

254A, Lecture 4: Multiple recurrence

15 January, 2008 in 254A - ergodic theory, math.CO, math.DS

Tags: multiple recurrence, van der Waerden's theorem
In the previous lecture, we established single recurrence properties for both open sets and for sequences inside a topological dynamical system $(X, \mathcal{F}, T)$. In this lecture, we generalise these results to multiple recurrence. More precisely, we shall show

Theorem 1. (Multiple recurrence in open covers) Let $(X, \mathcal{F}, T)$ be a topological dynamical system, and let $(U_\alpha)_{\alpha \in A}$ be an open cover of X. Then there exists U_α such that for every $k \geq 1$, we have $U_\alpha \cap T^{-r} U_\alpha \cap \ldots \cap T^{-(k-1)r} U_\alpha \neq \emptyset$ for infinitely many r.

Note that this theorem includes Theorem 1 from the previous lecture as the special case $k = 2$. This theorem is also equivalent to the following well-known combinatorial result:

Theorem 2. (van der Waerden's theorem) Suppose the integers $\mathbb{Z}$ are finitely coloured. Then one of the colour classes contains arbitrarily long arithmetic progressions.

Exercise 1. Show that Theorem 1 and Theorem 2 are equivalent. $\diamond$

Exercise 2. Show that Theorem 2 fails if "arbitrarily long" is replaced by "infinitely long". Deduce that a similar strengthening of Theorem 1 also fails. $\diamond$

Exercise 3. Use Theorem 2 to deduce a finitary version: given any positive integers m and k, there exists an integer N such that whenever $\{1, \ldots, N\}$ is coloured into m colour classes, one of the colour classes contains an arithmetic progression of length k. (Hint: use a "compactness and contradiction" argument, as in my article on hard and soft analysis.) $\diamond$

We also have a stronger version of Theorem 1:

Theorem 3. (Multiple Birkhoff recurrence theorem) Let $(X, \mathcal{F}, T)$ be a topological dynamical system. Then for any $k \geq 1$ there exists a point $x \in X$ and a sequence $r_j \to \infty$ of integers such that $T^{ir_j} x \to x$ as $j \to \infty$ for all $0 \leq i \leq k-1$.

These results already have some application to equidistribution of explicit sequences. Here is a simple example (which is also a consequence of Weyl's equidistribution theorem):

Corollary 1. Let α be a real number. Then there exists a sequence $r_j \to \infty$ of integers such that $\mathrm{dist}(r_j^2 \alpha, \mathbb{Z}) \to 0$ as $j \to \infty$.

Proof. Consider the skew shift system $X = (\mathbb{R}/\mathbb{Z})^2$ with $T(x,y) := (x+\alpha, y+x)$. By Theorem 3, there exists $(x,y) \in X$ and a sequence $n_j \to \infty$ such that $T^{r_j}(x,y)$ and $T^{2r_j}(x,y)$ both convege to (x,y). If we then use the easily verified identity

$$(x,y) - 2T^{r_j}(x,y) + T^{2r_j}(x,y) = (0, r_j^2 \alpha) \quad (1)$$

we obtain the claim. □

Exercise 4. Use Theorem 1 or Theorem 2 in place of Theorem 3 to give an alternate derivation of Corollary 1. ◇

As in the previous lecture, we will give both a traditional topological proof and an ultrafilter-based proof of Theorem 1 and Theorem 3; the reader is invited to see how the various proofs are ultimately equivalent to each other.

— Topological proof of van der Waerden –

We begin by giving a topological proof of Theorem 1, due to Furstenberg and Weiss, which is secretly a translation of van der Waerden's original "colour focusing" combinatorial proof of Theorem 2 into the dynamical setting.

To prove Theorem 1, it suffices to show the following slightly weaker statement:

Theorem 4. Let $(X, \mathcal{F}, T)$ be a topological dynamical system, and let $(U_\alpha)_{\alpha \in A}$ be an open cover of X. Then for every $k \geq 1$ there exists an open set U_α which contains an arithmetic progression $x, T^r x, T^{2r} x, \ldots, T^{(k-1)r} x$ for some $x \in X$ and $r > 0$.

To see how Theorem 4 implies Theorem 1, first observe from compactness that we can take the open cover to be a finite cover. Then by the infinite pigeonhole principle, it suffices to establish Theorem 1 for each $k \geq 1$ separately. For each such k, Theorem 4 gives a single arithmetic progression $x, T^r x, \ldots, T^{(k-1)r} x$ inside one of the U_α. By replacing the system (X, T) with the product system $(X \times \mathbb{Z}/N\mathbb{Z}, (x,m) \mapsto (Tx, m+1))$ for some large N and replacing the open cover $(U_\alpha)_{\alpha \in A}$ of X with the open cover $(U_\alpha \times \{m\})_{\alpha \in A, m \in \mathbb{Z}/N\mathbb{Z}}$ of $X \times \mathbb{Z}/N\mathbb{Z}$, one can make the spacing r in the arithmetic progression larger than any specified integer N. Thus by another application of the infinite pigeonhole principle, one of the U_α contains arithmetic progressions with arbitrarily large step r, and the claim follows.

Now we need to prove Theorem 4. By Lemma 1 of Lecture 2 to establish this theorem for minimal dynamical systems. We will need to note that for minimal systems, Theorem 4 automatically implies the following stronger-looking statement:

Theorem 5. Let $(X, \mathcal{F}, T)$ be a minimal topological dynamical system, let U be a non-empty open set in X, and let $k \geq 1$. Then U contains an arithmetic progression $x, T^r x, \ldots T^{(k-1)r} x$ for some $x \in X$ and $r \geq 1$.

Indeed, the deduction of Theorem 5 from Theorem 4 is immediate from the following useful fact (cf. Lemma 1 from Lecture 3):

Lemma 1. Let $(X, \mathcal{F}, T)$ be a minimal topological dynamical system, and let U be a non-empty open set in X. Then X can be covered by a finite number of translates $T^n U$ of U.

Proof. The set $X \backslash \bigcup_{n \in \mathbb{Z}} T^n U$ is a proper closed invariant subset of X, which must therefore be empty since X is minimal. The claim then follows from the compactness of X. $\Box$.

(Of course, the claim is highly false for non-minimal systems; consider for instance the case when T is the identity. More generally, if X is non-minimal, consider an open set U which is the complement of a proper subsystem of X.)

Now we need to prove Theorem 4. We do this by induction on k. The case k=1 is trivial, so suppose $k \geq 2$ and the claim has already been shown for k-1. By the above discussion, we see that Theorem 5 is also true for k-1.

Now fix a minimal system $(X, \mathcal{F}, T)$ and an open cover $(U_\alpha)_{\alpha \in A}$, which we can take to be finite. We need to show that one of the U_α contains an arithmetic progression $x, T^r x, \ldots, T^{(k-1)r} x$ of length k.

To do this, we first need an auxiliary construction.

Lemma 2. (Construction of colour focusing sequence) Let the notation and assumptions be as above. Then for any $J \geq 0$ there exists a sequence $x_0, \ldots, x_J$ of points in X, a sequence $U_{\alpha_0}, \ldots, U_{\alpha_J}$ of sets in the open cover (not necessarily distinct), and a sequence $r_1, \ldots, r_J$ of positive integers such that $T^{i(r_{a+1}+\ldots+r_b)} x_b \in U_{\alpha_a}$ for all $0 \leq a \leq b \leq J$ and $1 \leq i \leq k-1$.

Proof. We induct on J. The case J=0 is trivial. Now suppose inductively that $J \geq 1$, and that we have already constructed $x_0, \ldots, x_{J-1}$, $U_{\alpha_0}, \ldots, U_{\alpha_{J-1}}$, and $r_1, \ldots, r_{J-1}$ with the required properties. Now let V be a suitably small neighbourhood of x_{J-1} (depending on all the above data) to be chosen later. By Theorem 5 for k-1, V contains an arithmetic progression $y, T^{r_J} y, \ldots, T^{(k-2)r_J} y$ of length k-1. If one sets $x_J := T^{-r_J} y$, and lets U_{α_J} be an arbitrary set in the open cover containing x_J, then we observe that

$$T^{i(r_{a+1}+\ldots+r_J)} x_J = T^{i(r_{a+1}+\ldots+r_{J-1})} (T^{(i-1)r_J} y) \in T^{i(r_{a+1}+\ldots+r_{J-1})}(V) \quad (2)$$

for all $0 \leq a < J$ and $1 \leq i \leq k-1$. If V is a sufficiently small neighbourhood of x_{J-1}, we thus see (from the continuity of the $T^{i(r_{a+1}+\ldots+r_{J-1})}$ that we verify all the required properties needed to close the induction. □

We apply the above lemma with J equal to the number of sets in the open cover. By the pigeonhole principle, we can thus find $0 \leq a < b \leq J$ such that $U_{\alpha_a} = U_{\alpha_b}$. If we then set $x := x_b$ and $r := r_{a+1} + \ldots + r_b$ we obtain Theorem 4 as required. □

It is instructive to compare the k=2 case of the above arguments with the proof of Theorem 1 from the previous lecture. (For a comparison of this type of proof with the more classical combinatorial proof, see my Montreal lecture notes.)

– Ultrafilter proof of van der Waerden –

We now give a translation of the above proof into the language of ultrafilters (or more precisely, the language of Stone-Čech compactifications). This language may look a little strange, but it will be convenient when we study more general colouring theorems in the next lecture. As before, we will prove Theorem 4 instead of Theorem 1 (thus we only need to find one progression, rather than infinitely many). The key proposition is

> **Proposition 1**. (Ultrafilter version of van der Waerden) Let p be a minimal element of $\beta\mathbb{Z}$. Then for any $k \geq 1$ there exists $q \in \beta(\mathbb{Z} \times \mathbb{N})$ such that
> $$\lim_{(n,r)\to q} n + ir + p = p \hbox{ for all } 0 \leq i \leq k-1. \quad (3)$$

Suppose for the moment that this proposition is true. Applying it with some minimal element p of $\beta\mathbb{Z}$ (which must exist, thanks to Exercise 10 of the previous lecture), we obtain $q \in \beta(\mathbb{Z} \times \mathbb{N})$ obeying (3). If we let $x := T^p y$ for some arbitrary $y \in X$, we thus obtain

$$\lim_{(n,r)\to q} T^{n+ir} x = x \hbox{ for all } 0 \leq i \leq k-1. \quad (4)$$

If we let U_α be an element of the open cover that contains x, we thus see that $T^{n+ir}x \in U_\alpha$ for all $0 \leq i \leq k-1$ and all $(n,r) \in \mathbb{Z} \times \mathbb{N}$ which lie in a sufficiently small neighbourhood of q. Since a LCH space is always dense in its Stone-Čech compactification, the space of all (n,r) with this property is non-empty, and Theorem 4 follows.

Proof of Proposition 1. We induct on k. The case k=1 is trivial (one could take e.g. $q = (0, 1)$, so suppose $k > 1$ and that the claim has already been proven for k-1. Then we can find $q' \in \beta(\mathbb{Z} \times \mathbb{N})$ such that

$$\lim_{(n,r)\to q'} n + ir + p = p \quad (5)$$

for all $0 \leq i \leq k-2$.

Now consider the expression

$$p_{i,a,b} := \lim_{(n_1,r_1)\to q'} \ldots \lim_{(n_b,r_b)\to q'} i(r_{a+1} + \ldots + r_b) + m_b + p \quad (6)$$

for any $1 \leq a \leq b$ and $1 \leq i \leq k-1$, where

$$m_b := \sum_{i=1}^b n_i - r_i. \quad (7)$$

Applying (5) to the (n_b, r_b) limit in (6), we obtain the recursion $p_{i,a,b} = p_{i,a,b-1}$ for all $b > a$. Iterating this, we conclude that

$$p_{i,a,b} = p_{i,a,a} = p_{0,a,a} \quad (8)$$

for all $1 \leq i \leq k-1$. For i=0, (8) need not hold, but instead we have the easily verified identity

$$p_{0,a,b} = p_{0,b,b}. \quad (8')$$

Now let $p_* \in \beta\mathbb{Z}\backslash\mathbb{Z}$ be arbitrary (one could pick $p_* := p$, for instance) and define $p' := \lim_{a \to p_*} p_{0,a,a} = \lim_{b \to p_*} p_{0,b,b}$. Observe from (6) that all the $p_{i,a,b}$ lie in the closed set $\beta\mathbb{Z} + p$, and so p' does also. Since p is minimal, there must exist $p'' \in \beta\mathbb{Z}$ such that p = p" + p'. Expanding this out using (8) or (8′), we conclude that

$$\lim_{h \to p''} \lim_{a \to p_*} \lim_{b \to p_*} h + p_{i,a,b} = p \quad (9)$$

for all $0 \leq i \leq k-1$. Applying (6), we conclude

$$\lim_{h \to p''} \lim_{a \to p_*} \lim_{b \to p_*} \lim_{(n_1,r_1) \to q'} \ldots \lim_{(n_b,r_b) \to q'} n + ir + p = p \quad (10)$$

where $n := h + m_b$ and $r := r_{a+1} + \ldots + r_b$. Now, define $q \in \beta(\mathbb{Z} \times \mathbb{N})$ to be the limit

$$q := \lim_{h \to p''} \lim_{a \to p_*} \lim_{b \to p_*} \lim_{(n_1,r_1) \to q'} \ldots \lim_{(n_b,r_b) \to q'} (n, r) \quad (11)$$

then we obtain Proposition 1 as desired. □

Exercise 5. Strengthen Proposition 1 by adding the additional conclusion $\lim_{(n,r) \to q} r \not\in \mathbb{N}$. Using this stronger version, deduce Theorem 1 directly without using the trick of multiplying X with a cyclic shift system that was used to deduce Theorem 1 from Theorem 4. ◇

Theorem 1 can be generalised to multiple commuting shifts:

Theorem 6. (Multiple recurrence in open covers) Let $(X, \mathcal{F})$ be a compact topological space, and let $T_1, \ldots, T_k : X \to X$ be commuting homeomorphisms. Let $(U_\alpha)_{\alpha \in A}$ be an open cover of X. Then there exists U_α such that $T_1^{-r} U_\alpha \cap \ldots \cap T_k^{-r} U_\alpha \neq \emptyset$ for infinitely many r.

Exercise 6. By adapting one of the above arguments, prove Theorem 6. ◇

Exercise 7. Use Theorem 6 to establish the following the *multidimensional van der Waerden theorem* (due to Gallai): if a lattice $\mathbb{Z}^d$ is finitely coloured, and $v_1, \ldots, v_d \in \mathbb{Z}^d$, then one of the colour classes contains a pattern of the form $n + rv_1, \ldots, n + rv_d$ for some $n \in \mathbb{Z}^d$ and some non-zero r. ◇

Exercise 8. Show that Theorem 6 can fail, even for $k = 3$ and $T_1 = \mathrm{id}$, if the shift maps T_j are not assumed to commute. (*Hint*: First show that in the free group F_2 on two generators a, b, and any word $w \in F_2$ and non-zero integer r, the three words $w, a^n w, b^n w$ cannot all begin with the same generator after reduction. This can be used to disprove a non-commutative multidimensional van der Waerden theorem, which can turn be used to disprove a non-commutative version of Theorem 6.) $\diamond$

– Proof of multiple Birkhoff –

We now use van der Waerden's theorem and an additional Baire category argument to deduce Theorem 3 from Theorem 1. The key new ingredient is

Lemma 3. (Semicontinuous functions are usually continuous) Let (X, d) be a metric space, and let $F : X \to \mathbb{R}$ be semicontinuous. Then the set of points x where F is discontinuous is a set of the first category (i.e. a countable union of nowhere dense sets). In particular, by the Baire category theorem, if X is complete and non-empty, then F is continuous at at least one point.

Proof. Without loss of generality we can take F to be upper semicontinuous. Suppose F is discontinuous at some point x. Then, by upper continuity, there exists a rational number q such that

$$\liminf_{y \to x} F(y) < q \leq F(x). \quad (3)$$

In other words, x lies in the boundary of the closed set $\{x : F(x) \geq q\}$. But boundaries of closed sets are always nowhere dense, and the claim follows. $\square$

Now we prove Theorem 3. Without loss of generality we can take X to be minimal. Let us place a metric d on the space X. Define the function $F : X \to \mathbb{R}^+$ by the formula

$$F(x) := \inf_{n \geq 1} \sup_{1 \leq i \leq k-1} d(T^{in} x, x). \quad (4)$$

It will suffice to show that F(x)=0 for at least one x (notice that if the infimum is actually attained at zero for some n, then x is a periodic point and the claim is obvious). Suppose for contradiction that F is always positive. Observe that F is upper semicontinuous, and so by Lemma 3 there exists a point of continuity of F. In particular there exists a non-empty open set U such that F is bounded away from zero.

By uniform continuity of T^n, we see that if F is bounded away from zero on U, it is also bounded away from zero on $T^n V$ for any n (though the bound from below depends on n). Applying Lemma 1, we conclude that F is bounded away from zero on all of X, thus there exists $\varepsilon > 0$ such that $F(x) > \varepsilon$ for all $x \in X$. But this contradicts Theorem 1 (or Theorem 4), using the balls of radius $\varepsilon/2$ as the open cover. This contradiction completes the proof of Theorem 3.

Exercise 9. Generalise Theorem 3 to the case in which T is merely assumed to be continuous, rather than be a homeomorphism. (Hint: let $\tilde{X} \subset X^{\mathbb{Z}}$ denote the space of all sequences $(x_n)_{n \in \mathbb{Z}}$ with $x_{n+1} = Tx_n$ for all n, with the topology induced from the product space $X^{\mathbb{Z}}$. Use a limiting argument to show that $\tilde{X}$ is non-empty. Then turn $\tilde{X}$ into a topological dynamical system and apply Theorem 3.) $\diamond$

Exercise 10. Generalise Theorem 3 to multiple commuting shifts (analogously to how Theorem 6 generalises Theorem 1). ◇

Exercise 11. Combine Exercises 9 and 10 by obtaining a generalisation of Theorem 3 to multiple non-invertible commuting shifts. ◇

Exercise 12. Let $(X, \mathcal{F}, T)$ be a minimal topological dynamical system, and let $k \geq 1$. Call a point x in X *k-fold recurrent* if there exists a sequence $n_j \to \infty$ such that $T^{in_j}x \to x$ for all $0 \leq i \leq k-1$. Show that the set of k-fold recurrent points in X is residual (i.e. the complement is of the first category). In particular, the set of k-fold recurrent points is dense. ◇

Exercise 13. In the boolean Bernoulli system $(2^{\mathbb{Z}}, A \mapsto A+1)$, show that the set A consisting of all non-zero integers which are divisible by 2 an even number of times is almost periodic. Conclude that there exists a minimal topological dynamical system $(X, \mathcal{F}, T)$ such that not every point in X is 3-fold recurrent (in the sense of the previous exercise). (Compare this with the arguments in the previous lecture, which imply that every point in X is 2-fold recurrent.) ◇

Exercise 14. Suppose that a sequence of continuous functions $f_n : X \to \mathbb{R}$ on a metric space converges pointwise everywhere to another function $f : X \to \mathbb{R}$. Show that f is continuous on a residual set. ◇

Exercise 15. Let $(X, \mathcal{F}, T)$ be a minimal topological dynamical system, and let $f : X \to \mathbb{R}$ be a function which is T-invariant, thus Tf = f. Show that if f is continuous at even one point x_0, then it has to be constant. (Hint: x_0 is in the orbit closure of every point in X.) ◇

[*Update*, Jan 15: bad link fixed.]

[*Update*, Jan 21: Additional exercise added.]

[*Update*, Jan 26: Another additional exercise added.]

[Update, Mar 4: Slight correction to proof of Proposition 1.]

17 comments

Comments feed for this article

15 January, 2008 at 7:23 pm

Anonymous

The first link (to the previous lecture) is broken.

15 January, 2008 at 7:44 pm

Terence Tao

254A, Lecture 5: Other topological recurrence results

21 January, 2008 in 254A - ergodic theory, math.CO, math.DS

Tags: Hales-Jewett theorem, Hindman's theorem, hypergraphs, idempotents, polynomial van der Waerden theorem, Ramsey theory, Ramsey's theorem

In this lecture, we use topological dynamics methods to prove some other Ramsey-type theorems, and more specifically the polynomial van der Waerden theorem, the hypergraph Ramsey theorem, Hindman's theorem, and the Hales-Jewett theorem. In proving these statements, I have decided to focus on the ultrafilter-based proofs, rather than the combinatorial or topological proofs, though of course these styles of proof are also available for each of the above theorems.

— The polynomial van der Waerden theorem –

We first prove a significant generalisation of van der Waerden's theorem (Theorem 2 from the previous lecture):

Theorem 1 (Polynomial van der Waerden theorem). Let $(P_1, \ldots, P_k)$ be a tuple of be integer-valued polynomials $P_1, \ldots, P_k : \mathbb{Z} \to \mathbb{Z}$) (or *tuple* for short) with $P_1(0) = \ldots = P_k(0)$. Then whenever the integers are finitely coloured, one of the colour classes will contain a pattern of the form $n + P_1(r), \ldots, n + P_k(r)$ for some $n \in \mathbb{Z}$ and $r \in \mathbb{N}$.

This result is due to Bergelson and Leibman, who proved it using "epsilon and delta" topological dynamical methods. A combinatorial proof was only obtained rather recently by Walters. In these notes, I will translate the Bergelson-Leibman argument to the ultrafilter setting.

Note that the case $P_j(r) := (j-1)r$ recovers the ordinary van der Waerden theorem. But the result is significantly stronger; it implies for instance that one of the colour classes contains arbitrarily many shifted geometric progressions $n + r, n + r^2, \ldots, n + r^k$, which does not obviously follow from the van der Waerden theorem. The result here only claims a single monochromatic pattern $n + P_1(r), \ldots, n + P_k(r)$, but it is not hard to amplify this theorem to show that at least one colour class contains infinitely many such patterns.

Remark 1. The theorem can fail if the hypothesis $P_1(0) = \ldots = P_k(0)$ is dropped; consider for instance the case $k = 2$, $P_1(r) = 0$, $P_2(r) = 2r + 1$, and with the integers partitioned (or coloured) into the odd and even integers. More generally, the theorem fails whenever there exists a modulus N such that the polynomials $P_1, \ldots, P_k$ are never simultaneously equal modulo N. This turns out to be the only obstruction; this is a somewhat difficult recent result of Bergelson, Leibman, and Lesigne. $\diamond$

Exercise 1. Show that the polynomial $P(r) := (r^2 - 2)(r^2 - 3)(r^2 - 6)(r^2 - 7)(r^3 - 3)$ has a root modulo N for every positive integer N, but has no root in the integers. Thus we see that the Bergelson-

Leibman-Lesigne result is stronger than the polynomial van der Waerden theorem; it does not seem possible to directly use the latter to conclude that in every finite colouring of the integers, one of the classes contains the pattern $n, n+P(r)$. $\diamond$

Here are the topological dynamics and ultrafilter versions of the above theorem.

Theorem 2 (Polynomial van der Waerden theorem, topological dynamics version). Let $(P_1, \ldots, P_k)$ be a tuple with $P_1(0) = \ldots = P_k(0)$. Let $(U_\alpha)_{\alpha \in A}$ be an open cover of a topological dynamical system $(X, \mathcal{F}, T)$. Then there exists a set U_α in this cover such that $T^{P_1(r)}U \cap \ldots \cap T^{P_k(r)}U \neq \emptyset$ for at least one $r > 0$.

Theorem 3 (Polynomial van der Waerden theorem, ultrafilter version). Let $(P_1, \ldots, P_k)$ be a tuple with $P_1(0) = \ldots = P_k(0)$, and let $p \in \beta\mathbb{Z}$ be a minimal ultrafilter. Then there exists $q \in \beta(\mathbb{Z} \times \mathbb{N})$ such that

$$\lim_{(n,r) \to q} n + P_i(r) + p = p \text{ for all } 1 \leq i \leq k. \quad (1)$$

Exercise 2. Show that Theorem 1 and Theorem 2 are equivalent, and that Theorem 3 implies Theorem 2 (or Theorem 1). [For the converse implication, see Exercise 21.] $\diamond$

As in the previous lecture, we shall prove this proposition by induction. However, the induction will be more complicated than just inducting on the number k of polynomials involved, or on the degree of these polynomials, but will instead involve a more complicated measure of the "complexity" of the polynomials being measured. Let us say that a tuple $(P_1, \ldots, P_k)$ *obeys the vdW property* if the conclusion of Theorem 3 is true for this tuple. Thus for instance, from Proposition 1 from the previous lecture we know that any tuple of *linear* polynomials which vanish at the origin will obey the vdW property.

Our goal is to show that every tuple of polynomials which simultaneously vanish at the origin has the vdW property. The strategy will be to reduce from any given tuple to a collection of "simpler" tuples. We first begin with an easy observation, that one can always shift one of the polynomials to be zero:

Lemma 1. (Translation invariance) Let Q be any integer-valued polynomial. Then a tuple $(P_1, \ldots, P_k)$ obeys the vdW property if and only if $(P_1 - Q, \ldots, P_k - Q)$ has the vdW property.

Proof. Let $p \in \beta\mathbb{Z}$ be minimal. If $(P_1 - Q, \ldots, P_k - Q)$ has the vdW property, then we can find $q \in \beta(\mathbb{Z} \times \mathbb{N})$ such that

$$\lim_{(n,r) \to q} n + P_i(r) - Q(r) + p = p \text{ for all } 1 \leq i \leq k. \quad (2)$$

If we then define $q' := \lim_{(n,r)\to q} n - Q(r) \in \beta(\mathbb{Z}\times\mathbb{N})$ one easily verifies that (1) holds (with q replaced by q'), and the claim holds. The converse implication is similar. $\square$

Now we come to the key inductive step.

Lemma 2 (Inductive step). Let $(P_0, P_1, \ldots, P_k)$ be a tuple with $P_0 = 0$, and let Q be another integer-valued polynomial. Suppose that for every finite set of integers $h_1, \ldots, h_m$, the tuple $(P_i(\cdot + h_j) - P_i(h_j) - Q(\cdot))_{1\leq i\leq k; 1\leq j\leq m}$ has the vdW property. Then $(0, P_1, \ldots, P_k)$ also has the vdW property.

Proof. This will be a reprise of the proof of Proposition 1 from the previous lecture. Given any finite number of pairs $(n_1, r_1), \ldots, (n_{b-1}, r_{b-1}) \in \mathbb{Z}\times\mathbb{N}$ with $b \geq 1$, we see from hypothesis that there exists $q_b \in \beta(\mathbb{Z}\times\mathbb{N})$ (depending on these pairs) such that

$$\lim_{(n_b,r_b)\to q_b} n_b + P_i(r_{a+1} + \ldots + r_b) - P_i(r_{a+1} + \ldots + r_{b-1}) - Q(r_b) + p = p \quad (3)$$

for all $0 \leq a < b$.

Now, for every $0 \leq a \leq b$ and $0 \leq i \leq k$, consider the expression $p_{a,b,i} \in p + \beta\mathbb{Z}$ defined by

$$p_{a,b,i} := \lim_{(n_1,r_1)\to q_1} \ldots \lim_{(n_1,r_1)\to q_b} P_i(r_{a+1} + \ldots + r_b) + m_b + p, \quad (4)$$

where $q_1, \ldots, q_b$ are defined recursively as above and

$$m_b := \sum_{i=1}^b n_i - Q(r_i) \quad (5).$$

From (3) we see that

$$p_{a,b,i} = p_{a,b-1,i} \quad (6)$$

for all $0 \leq a < b$ and $1 \leq i \leq k$, and thus

$$p_{a,b,i} = p_{a,a,i} = p_{a,a,0} \quad (7)$$

in this case. For i=0, we have the slightly different identity

$$p_{a,b,0} = p_{b,b,0}. \quad (7')$$

We let $p_* \in \beta\mathbb{Z}/\mathbb{Z}$ be arbitrary, and set $p' := \lim_{a\to p_*} p_{a,a,0} = \lim_{b\to p_*} p_{b,b,0} \in p + \beta\mathbb{Z}$. By the minimality of p, we can find $p'' \in \beta\mathbb{Z}$ such that $p'' + p' = p$. We thus have

$$\lim_{h\to p''} \lim_{a\to p_*} \lim_{b\to p_*} h + p_{a,b,i} = p \quad (8)$$

for all $0 \leq i \leq k$. If one then sets

$$q := \lim_{h \to p''} \lim_{a \to p_*} \lim_{b \to p_*} \lim_{(n_1, r_1) \to q_1} \ldots \lim_{(n_1, r_1) \to q_b} (n, r) \quad (9)$$

where $n := h + m_b$ and $r := r_{a+1} + \ldots + r_b$, one easily verifies (1) as required. $\Box$

Let's see how this lemma is used in practice. Suppose we wish to show that the tuple $(0, r^2)$ has the vdW property (where we use r to denote the independent variable). Applying Lemma 2 with $Q(r) := r^2$, we reduce to showing that the tuples $((r+h_1)^2 - h_1^2 - r^2, \ldots, (r+h_m)^2 - h_m^2 - r_m^2)$ have the vdW property for all finite collections $h_1, \ldots, h_m$ of integers. But observe that all the polynomials in these tuples are linear polynomials that vanish at the origin. By the ordinary van der Waerden theorem, these tuples all have the vdW property, and so $(0, r^2)$ has the vdW property also.

A similar argument shows that the tuple $(0, r^2 + P_1(r), \ldots, r^2 + P_k(r))$ has the vdW property whenever $P_1, \ldots, P_k$ are linear polynomials that vanish at the origin. Applying Lemma 1, we see that $(Q_1(r), r^2 + P_1(r), \ldots, r^2 + P_k(r))$ obeys the vdW property when Q_1 is also linear and vanishing at the origin.

Now, let us consider a tuple $(Q_1(r), Q_2(r), r^2 + P_1(r), \ldots, r^2 + P_k(r))$ where Q_2 is also a linear polynomial that vanishes at the origin. The vdW property for this tuple follows from the previously established vdW properties by first applying Lemma 1 to reduce to the case $Q_1 = 0$, and then applying Lemma 2 with $Q = Q_2$. Continuing in this fashion, we see that a tuple $(Q_1(r), \ldots, Q_l(r), r^2 + P_1(r), \ldots, r^2 + P_k(r))$ will also obey the vdW property for any linear $Q_1, \ldots, Q_l, P_1, \ldots, P_k$ that vanish at the origin, for any k and l.

Now the vdW property for the tuple $(0, r^2, 2r^2)$ follows from the previously established cases and Lemma 2 with $Q(r) = r^2$.

Remark 2. It is possible to continue this inductive procedure (known as *PET induction;* the PET stands, variously, for "polynomial ergodic theorem" or "polynomial exhaustion theorem"); this is carried out in Exercise 3 below. $\diamond$

Exercise 3. Define the *top order monomial* of a non-zero polynomial $P(r) = a_d r^d + \ldots + a_0$ with $a_d \neq 0$ to be $a_d r^d$. Define the top order monomials of a tuple $(0, P_1, \ldots, P_k)$ to be the set of top order monomials of the $P_1, \ldots, P_k$, not counting multiplicity; for instance, the top order monomials of $(0, r^2, r^2 + r, 2r^2, 2r^2 + r)$ are $\{r^2, 2r^2\}$. Define the *weight vector* of a tuple $(P_1, \ldots, P_k)$ relative to one of its members P_i to be the infinite vector $(w_1, w_2, \ldots) \in \mathbb{Z}_{\geq 0}^{\mathbb{N}}$, where each w_d denotes the number of monomials of degree d in the top order monomials of $(P_1 - P_i, \ldots, P_k - P_i)$. Thus for instance, the tuple $(0, r^2, r^2 + r, 2r^2, 2r^2 + r)$ has weight vector $(0, 2, 0, \ldots)$ with respect to 0, but weight vector $(1, 2, 0, \ldots)$ with respect to (say) r^2. Let us say that one weight vector $(w_1, w_2, \ldots)$ is larger than another $(w'_1, w'_2, \ldots)$ if there exists $d \geq 1$ such that $w_d > w'_d$ and $w_i = w'_i$ for all $i > d$.

1. Show that the space of all weight vectors is a well-ordered set.
2. Show that if $(0, P_1, \ldots, P_k)$ is a tuple with $k \geq 1$ and P_1 nonlinear, and $h_1, \ldots, h_m$ are integers with $m \geq 1$, then the weight vector of $(P_i(\cdot + h_j) - P_i(h_j))_{1 \leq i \leq k; 1 \leq j \leq m}$ with respect to $P_1(\cdot + h_1)$ is strictly smaller than the weight vector of $(0, P_1, \ldots, P_k)$ with respect to P_1.

3. Using 1., 2., Lemma 1, and Lemma 2, deduce Theorem 3. ◇

Exercise 4. Find a direct proof of Theorem 2 analogous to the "epsilon and delta" proof of Theorem 4 from the previous lecture. (You can look up the paper of Bergelson and Leibman if you're stuck.) ◇

Exercise 5. Let $P_1, \ldots, P_k : \mathbb{Z} \to \mathbb{Z}^d$ be vector-valued polynomials (thus each of the d components of each of the P_i is a polynomial) which all vanish at the origin. Show that if $\mathbb{Z}^d$ is finitely coloured, then one of the colour classes contains a pattern of the form $n + P_1(r), \ldots, n + P_k(r)$ for some $n \in \mathbb{Z}^d$ and $r \in \mathbb{N}$. ◇

Exercise 6. Show that for any polynomial sequence $P : \mathbb{Z} \to (\mathbb{R}/\mathbb{Z})^d$ taking values in a torus, there exists integers $n_j \to \infty$ such that $P(n_j)$ converges to $P(0)$. (One can also tweak the argument to make the n_j converge to positive infinity, by the "doubling up" trick of replacing P(n) with (P(n),P(-n)).) On the other hand, show that this claim can fail with exponential sequences such as $P(n) := 10^n \alpha \mod 1 \in \mathbb{R}/\mathbb{Z}$ for certain values of α. Thus we see that polynomials have better recurrence properties than exponentials. ◇

– Ramsey's theorem –

Given any finite palette K of colours, a vertex set V, and an integer $k \geq 1$, define a *K-coloured hypergraph* G = (V,E) of order k on V to be a function $E : \binom{V}{k} \to K$, where $\binom{V}{k} := \{e \subset V : |e| = k\}$ denotes the k-element subsets of V. Thus for instance a hypergraph of order 1 is a vertex colouring, a hypergraph of order 2 is an edge-coloured complete graph, and so forth. We say that a hypergraph G is *monochromatic* if the edge colouring function E is constant. If W is a subset of V, we refer to the hypergraph $G\downharpoonright_W := (W, E\downharpoonright_{\binom{W}{k}})$ as a *subhypergraph* of G.

We will now prove the following result:

Theorem 4 (Hypergraph Ramsey theorem). Let K be a finite set, let $k \geq 1$, and let G = (V,E) be a K-coloured hypergraph of order k on a countably infinite vertex set V. Then G contains arbitrarily large finite monochromatic subhypergraphs.

Remark 3. There is a stronger statement known, namely that G contains an infinitely large monochromatic subhypergraph, but we will not prove this statement, known as the *infinite hypergraph Ramsey theorem*. In the case k=1, these statements are the pigeonhole principle and infinite pigeonhole principle respectively, and are compared in my article on hard and soft analysis. ◇

Exercise 7. Show that Theorem 4 implies a finitary analogue: given any finite K and positive integers k, m, there exists N such that every K-coloured hyeprgraph of order k on $\{1, \ldots, N\}$ contains a monochromatic subhypergraph on m vertices. (*Hint*: as in Exercise 3 from the previous lecture [note - exercises have been renumbered recently], one should use a compactness and contradiction argument as in my article on hard and soft analysis.) ◇

It is not immediately obvious, but Theorem 4 is a statement about a topological dynamical system, albeit one in which the underlying group is not the integers $\mathbb{Z}$, but rather the symmetric group $\mathrm{Sym}_0(V)$, defined as the group of bijections from V to itself which are the identity outside of a finite set. More precisely, we have

Theorem 5. (Hypergraph Ramsey theorem, topological dynamics version) Let V be a countably infinite set, and let W be a finite subset of V, thus $\mathrm{Sym}_0(W) \times \mathrm{Sym}_0(V \backslash W)$ is a subgroup of $\mathrm{Sym}_0(V)$. Let $(X, \mathcal{F}, T)$ be a $\mathrm{Sym}_0(V)$-topological dynamical system, thus $(X, \mathcal{F})$ is compact metrisable and $T : \sigma \mapsto T^\sigma$ is an action of $\mathrm{Sym}_0(V)$ on X via homeomorphisms. Let $(U_\alpha)_{\alpha \in A}$ be an open cover of X, such that each U_α is $\mathrm{Sym}_0(W) \times \mathrm{Sym}_0(V \backslash W)$. Then there exists an element U_α of this cover such that for every finite set $\Gamma \subset \mathrm{Sym}_0(V)$ there exists a group element $\sigma \in \mathrm{Sym}_0(V)$ such that $\bigcap_{\gamma \in \Gamma} T^{\gamma\sigma}(U_\alpha) \neq \emptyset$.

This claim should be compared with Theorem 2 of this lecture, or Theorem 1 of the previous lecture.

Exercise 8. Show that Theorem 4 and Theorem 5 are equivalent. (*Hint*: At some point, you will need the use the fact that the quotient space $\mathrm{Sym}_0(V)/\mathrm{Sym}_0(W) \times \mathrm{Sym}_0(V \backslash W)$ is isomorphic to $\binom{V}{|W|}$.) $\diamond$

As before, though, we shall only illustrate the ultrafilter approach to Ramsey's theorem, leaving the other approaches to exercises. Here, we will not work on the compactified integers $\beta\mathbb{Z}$, but rather on the compactified permutations $\beta\mathrm{Sym}_0(V)$. (We will view $\mathrm{Sym}_0(V)$ here as a discrete group; one could also give this group the topology inherited from the product topology on V^V, leading to a slightly coarser (and thus less powerful) compactification, though one which is still sufficient for the arguments here.) This is a semigroup with the usual multiplication law

$$pq := \lim_{\sigma \to p} \lim_{\rho \to q} \sigma\rho. \quad (10)$$

Let us say that $p \in \beta\mathrm{Sym}_0(V)$ is *minimal* if $\beta\mathrm{Sym}_0(V)p$ is a minimal left-ideal of $\beta\mathrm{Sym}_0(V)$. One can show (by repeating Exercise 10 from Lecture 3) that every left ideal $\beta\mathrm{Sym}_0(V)p$ contains at least one minimal element; in particular, minimal elements exist.

Note that if W is a k-element subset of V, then there is an image map $\pi_W : \mathrm{Sym}_0(V) \to \binom{V}{k}$ which maps a permutation σ to its inverse image $\sigma^{-1}(W)$ of W. We can compactify this to a map $\beta\pi_W : \beta\mathrm{Sym}_0(V) \to \beta\binom{V}{k}$. (Caution: $\beta\binom{V}{k}$ is *not* the same thing as $\binom{\beta V}{k}$, for instance the latter is not even compact.) We can now formulate the ultrafilter version of Ramsey's theorem:

Theorem 6. (Hypergraph Ramsey theorem, ultrafilter version) Let V be countably infinite, and let $p \in \beta\mathrm{Sym}_0(V)$ be minimal. Then for every finite set W, $\beta\pi_W$ is constant on $\beta\mathrm{Sym}_0(V)p$, thus $\beta\pi_W(qp) = \beta\pi_W(p)$ for all $q \in \beta\mathrm{Sym}_0(V)$.

This result should be compared with Proposition 1 from the previous lecture (or Theorem 3 from this lecture).

Exercise 9. Show that Theorem 6 implies both Theorem 4 and Theorem 5. $\diamond$

Proof of Theorem 6. By relabeling we may assume $V = \{1, 2, 3, \ldots\}$ and $W = \{1, \ldots, k\}$ for some k.

Given any integers $1 \leq a < i_1 < i_2 < \ldots < i_a$, let $\sigma_{i_1,\ldots,i_a} \in \mathrm{Sym}_0(V)$ denote the permutation that swaps j with i_j for all $1 \leq j \leq a$, but leaves all other integers unchanged. We select some non-principal ultrafilter $p_* := \beta V \backslash V$ and define the sequence $p_1, p_2, \ldots \in \beta\mathrm{Sym}_0(V)$ by the formula

$$p_a := \lim_{i_1 \to p_*} \ldots \lim_{i_a \to p_*} \sigma_{i_1,\ldots,i_a} p. \quad (11)$$

(Note that the condition $a < i_1 < \ldots < i_a$ will be asymptotically true thanks to the choice of limits here.)

Let $a \geq k$, and let $\alpha \in \mathrm{Sym}_0(V)$ be the a permutation which is the identity outside of $\{1, \ldots, a\}$. Then we have the identity

$$\pi_W(\alpha \sigma_{i_1,\ldots,i_a} \rho) = \pi_W(\sigma_{i_{j_1},\ldots,i_{j_k}} \rho) \quad (12)$$

for every $\rho \in \mathrm{Sym}_0(V)$, where $j_1 < \ldots < j_k$ are the elements of $\alpha^{-1}(\{1, \ldots, k\})$ in order. Taking limits as $\rho \to p$, and then inserting the resulting formula into (11), we conclude (after discarding the trivial limits and relabeling the rest) that

$$\beta\pi_W(\alpha p_a) = \lim_{i_1 \to p_*} \ldots \lim_{i_k \to p_*} \beta\pi_W(\sigma_{i_1,\ldots,i_k} p) \quad (13)$$

and in particular that $\beta\pi_W(\alpha p_a)$ is independent of α (if α is the identity outside of $\{1, \ldots, j\}$. Now let $p' := \lim_{a \to p_*} p_a$, then we have $\beta\pi_W(\alpha p')$ independent of p' for all $\alpha \in \mathrm{Sym}_0(V)$. Taking limits we conclude that $\beta\pi_W$ is constant on $(\beta\mathrm{Sym}_0(V))p'$. But from construction we see that p' lies in the closed minimal ideal $(\beta\mathrm{Sym}_0(V))p$, thus $(\beta\mathrm{Sym}_0(V))p' = (\beta\mathrm{Sym}_0(V))p$. The claim follows. $\square$

Exercise 10. Establish Theorem 4 directly by a combinatorial argument without recourse to topological dynamics or ultrafilters. (If you are stuck, I recommend reading the classic text of Graham, Rothschild, and Spencer.) $\diamond$

Exercise 11. Establish Theorem 5 directly by a topological dynamics argument, using combinatorial arguments for the k=1 case but then proceeding by induction afterwards (as in the proof of Theorem 4 from the previous lecture). $\diamond$

Remark 4. More generally, one can interpret the theory of graphs and hypergraphs on a vertex set V through the lens of dynamics of $\mathrm{Sym}_0(V)$ actions; I learned this perspective from Balazs Szegedy, and it underlies my paper on the hypergraph correspondence principle, as well as my more recent paper with Tim Austin on hypergraph property testing. $\diamond$

– Idempotent ultrafilters and Hindman's theorem –

Thus far, we have been using ultrafilter technology rather lightly, and indeed all of the arguments so far can be converted relatively easily to the topological dynamics formalism, or even a purely combinatorial formalism, with only a moderate amount of effort. But now we will exploit some deeper properties of ultrafilters, which are more difficult to replicate in other settings. In particular, we introduce the notion of an idempotent ultrafilter.

Definition 1. Let $(S, \cdot)$ be a discrete semigroup, and let βS be given the usual semigroup operation $\cdot$. An element $p \in \beta S$ is *idempotent* if $p \cdot p = p$. (We of course define idempotence analogously if the group operation on S is denoted by + instead of $\cdot$.)

Of course, 0 is idempotent, but the remarkable fact is that many other idempotents exist as well. The key tool for creating this is

Lemma 3 (Ellis-Nakamura lemma). Let S be a discrete semigroup, and let K be a compact non-empty sub-semigroup of βS. Then K contains at least one idempotent.

Proof: A simple application of Zorn's lemma shows that K contains a compact non-empty sub-semigroup K' which is minimal with respect to set inclusion. We claim that every element of K' is idempotent. To see this, let p be an arbitrary element of K'. Then observe that K'p is a compact non-empty sub-semigroup of K' and must therefore be equal to K'; in particular, $p \in K'p$. (Note that semigroups need not contain an identity.) In particular, the stabiliser $K'' := \{q \in K' : qp = p\}$ is non-empty. But one easily observes that this stabiliser is also a compact sub-semigroup of K', and so K''=K'. In particular, p must stabilise itself, i.e. it is idempotent. $\square$

Remark 5. *A posteriori*, this results shows that the minimal non-empty sub-semigroups K' are in fact just the singleton sets consisting of idempotents. Of course, one cannot see this without first deriving all of Lemma 3. $\diamond$

Idempotence turns out to be particularly powerful when combined with minimality, and to this end we observe the following corollary of the above lemma:

Corollary 4. Let S be a discrete semigroup. For every $p \in \beta S$, there exists $q \in (\beta S)p$ which is both minimal and idempotent.

Proof. By Exercise 10 from Lecture 3, there exists $r \in (\beta S)p$ which is minimal. It is then easy to see that every element of $(\beta S)r$ is minimal. Since $(\beta S)r \subset (\beta S)p$ is a compact non-empty sub-semigroup of βS, the claim now follows from Lemma 3. $\diamond$

Remark 6. Somewhat amusingly, minimal idempotent ultrafilters require *three* distinct applications of Zorn's lemma to construct: one to define the compactified space βS, one to locate a minimal left-ideal, and one to locate an idempotent inside that ideal! It seems particularly challenging therefore to define civilised substitutes for this tool which do not explicitly use the axiom of choice. $\diamond$

What can we do with minimal idempotent ultrafilters? One particularly striking example is *Hindman's theorem*. Given any set A of positive integers, define FS(A) to be the set of all finite sums $\sum_{n \in B} n$ from A, where B ranges over all finite non-empty subsets of A. (For instance, if $A = \{1, 2, 4, \dots\}$ are the powers of 2, then $FS(A) = \mathbb{N}$.)

Theorem 7 (Hindman's theorem) Suppose that the natural numbers $\mathbb{N}$ are finitely coloured. Then one of the colour classes contains a set of the form FS(A) for some infinite set A.

Remark 7. Theorem 7 implies *Folkman's theorem*, which has the same hypothesis but concludes that one of the colour classes contains sets of the form FS(A) for arbitrarily large but finite sets A. It does not seem possible to easily deduce Hindman's theorem from Folkman's theorem. $\diamond$

Exercise 12. Folkman's theorem in turn implies *Schur's theorem*, which asserts that if the natural numbers are finitely coloured, one of the colour classes contains a set of the form $FS(\{x, y\}) = \{x, y, x+y\}$ for some x, y (compare with the k=3 case of van der Waerden's theorem). Using the Cayley graph construction, deduce Schur's theorem from Ramsey's theorem (the k=2 case of Theorem 4). Thus we see that there are some connections between the various Ramsey-type theorems discussed here. $\diamond$

Proof of Theorem 7. By Corollary 4, we can find a minimal idempotent element p in $\beta\mathbb{N}$; note that as no element of $\mathbb{N}$ is minimal (cf. Exercise 7 from Lecture 3), we know that $p \not\in \mathbb{N}$. Let $c : \mathbb{N} \to \{1, \ldots, m\}$ denote the given colouring function, then $\beta c(p)$ is a colour in $\{1, \ldots, m\}$. Since

$$\lim_{n \to p} \beta c(n) = \beta c(p) \quad (14)$$

and

$$\lim_{n \to p} \beta c(n+p) = \beta c(p+p) = \beta c(p) \quad (15)$$

we may find a positive integer n_1 such that $\beta c(n_1) = \beta c(n_1 + p) = \beta c(p)$. Now from (14), (15) and the similar calculations

$$\lim_{n \to p} \beta c(n_1 + n) = \beta c(n_1 + p) = \beta c(p) \quad (16)$$

and

$$\lim_{n \to p} \beta c(n_1 + n + p) = \beta c(n_1 + p + p) = \beta c(n_1 + p) = \beta c(p) \quad (17)$$

we can find an integer $n_2 > n_1$ such that $\beta c(n_2) = \beta c(n_2 + p) = \beta c(n_2 + n_1) = \beta c(n_2 + n_1 + p) = \beta c(p)$, thus $\beta c(m) = \beta c(m+p) = \beta c(p)$ for all $m \in FS(\{n_1, n_2\})$. Continuing inductively in this fashion, one can find $n_1 < n_2 < n_3 < \ldots$ such that $\beta c(m) = \beta c(m+p) = \beta c(p)$ for all $m \in FS(\{n_1, \ldots, n_k\})$ and all k. If we set $A := \{n_1, n_2, \ldots\}$, the claim follows. $\square$

Remark 8. Purely combinatorial (and quite succinct) proofs of Hindman's theorem exist - see for instance the one in the text by Graham, Rothschild, and Spencer - but they generally rely on some *ad hoc* trickery. Here, the trickery has been encapsulated into the existence of minimal idempotent ultrafilters, which can be reused in other contexts (for instance, we will use it to prove the Hales-Jewett theorem below). $\diamond$

Exercise 13. Define an IP-set to be a set of positive integers which contains a subset of the form FS(A) for some infinite A. Show that if an IP-set S is finitely coloured, then one of its colour classes is also an

IP-set. (*Hint*: S contains FS(A) for some infinite $A = \{a_1, a_2, a_3, \ldots\}$. Show that the set $\bigcap_{n=1}^\infty \beta FS(\{a_n, a_{n+1}, \ldots\})$ is a compact non-empty semigroup and thus contains a minimal idempotent ultrafilter p. Use this p to repeat the proof of Theorem 7.) $\diamond$

– The Hales-Jewett theorem –

Given a finite alphabet A, let $A^{<\omega}$ be the free semigroup generated by A, i.e. the set of all finite non-empty words using the alphabet A, with concatenation as the group operation. (E.g. if $A = \{a, b, c\}$, then $A^{<\omega}$ contains words such as abc and cbb, with $abc \cdot cbb = abccbb$.) If we add another letter * to A (the "wildcard" letter), we create a larger semigroup $(A \cup \{*\})^{<\omega}$ (e.g. containing words such as ab**c*). We of course assume that * was not already present in A. Given any letter $x \in A$, we have a semigroup homomorphism $\pi_x : (A \cup \{*\})^{<\omega} \to A^{<\omega}$ which substitutes every occurrence of the wildcard * with x and leaves all other letters unchanged. (For instance, $\pi_a(ab**c*) = abaaca$.) Define a combinatorial line in $A^{<\omega}$ to be any set of the form a $\{\pi_x(v) : x \in A\}$ for some $v \in (A \cup \{*\})^{<\omega} \backslash A^{<\omega}$. For instance, if $A = \{a, b, c\}$, then $\{abaaca, abbbcb, abcccc\}$ is a combinatorial line, generated by the word $v = ab**c*$.

We shall prove the following fundamental theorem.

Theorem 8 (Hales-Jewett theorem). Let A be a finite alphabet. If $A^{<\omega}$ is finitely coloured, then one of the colour classes contains a combinatorial line.

Exercise 14. Show that the Hales-Jewett theorem has the following equivalent formulation: for every finite alphabet A and any $m \geq 1$ there exists N such that if A^N is partitioned into m classes, then one of the classes contains a combinatorial line. $\diamond$

Exercise 15. Assume the Hales-Jewett theorem. In this exercise we compare the strength of this theorem against other Ramsey-type theorems.

1. Deduce van der Waerden's theorem (Theorem 2 from the previous lecture.) *Hint*: the base k representation of the non-negative natural numbers provides a map from $\{0, \ldots, k-1\}^{<\omega}$ to $\mathbb{Z}_{\geq 0}$.
2. Deduce the multidimensional van der Waerden's theorem of Gallai (Exercise 7 from the previous lecture.)
3. Deduce the *syndetic van der Waerden theorem* of Furstenberg: if the integers are finitely coloured and k is a positive integer, then there are infinitely many monochromatic arithmetic progressions $n, n+r, \ldots, n+(k-1)r$ of length k, and furthermore the set of all the step sizes r which appear in such progressions is syndetic. (Hint: argue by contradiction, assuming that the set of all step sizes has arbitrarily long gaps, and use the Hales-Jewett theorem in a manner adapted to these gaps.) For an additional challenge, show that there exists a *single* colour class whose progressions of length k have spacings in a syndetic set for every k.
4. Deduce the *IP-van der Waerden theorem*: If the integers are finitely coloured, k is a positive integer, and S is an IP-set (see Exercise 13), show that there are infinitely many monochromatic arithmetic progressions whose step size lies in S. (For an additional challenge, show that one of the classes has the property that for every k, the spacings of the k-term progressions in that class forms an IP^**-set*, i.e. it has non-empty intersection with every IP-set. There is an even stronger topological dynamics version of this statement, due to Furstenberg and Weiss, which I will not

describe here.)

5. For any $d \geq 1$, define a *d-dimensional combinatorial subspace* of $A^{<\omega}$ to be a set of the form $\{\pi_{x_1,\ldots,x_d}(v) : x_1, \ldots, x_d \in A\}$, where $v \in (A \cup \{*_1, \ldots, *_d\})^{<\omega}$ is a word containing at least one copy of each of the d wildcards $_1, \ldots, *_d$, and $\pi_{x_1,\ldots,x_d} : (A \cup \{*_1, \ldots, *_d\})^{<\omega} \to A^{<\omega}$ is the homomorphism that substitutes each wildcard $_j$ with x_j. Show that if $A^{<\omega}$ is finitely coloured, then one of the colour classes contains arbitrarily high-dimensional combinatorial subspaces.
6. Let F be a finite field. If the vector space $\lim_{n\to\infty} F^n$ (the inverse limit of the finite vector spaces F^n) is finitely coloured, show that one of the colour classes contains arbitrarily high-dimensional affine subspaces over F. (This *geometric Ramsey theorem* is due to Graham, Leeb, and Rothschild.)

We now give an ultrafilter-based proof of the Hales-Jewett theorem due to Blass. As usual, the first step is to obtain a statement involving ultrafilters rather than colourings:

Proposition 1. (Hales-Jewett theorem, ultrafilter version) Let A be a finite alphabet, and let p be a minimal idempotent element of the semigroup $\beta(A^{<\omega})$. Then there exists $q \in \beta(A \cup \{*\})^{<\omega})\backslash\beta(A^{<\omega})$ such that $\beta\pi_x(q) = p$ for all $x \in A$.

Exercise 16. Deduce Theorem 8 from Proposition 1. $\diamond$

To prove Proposition 1, we need a variant of Corollary 4. If $(S, \cdot)$ is a discrete semigroup and p and q are two idempotents in βS, let us write $p \prec q$ if we have $pq = qp = p$.

Exercise 17. Show that $\prec$ is a partial ordering on the idempotents of βS, and that an idempotent is minimal in βS if and only if it is minimal with respect to $\prec$. $\diamond$

Lemma 4. Let S be a discrete semigroup, and let p be an idempotent in βS. Then there exists a minimal idempotent q in βS such that $q \prec p$.

Proof of Lemma 4. By Exercise 10 from Lecture 3 (generalised to arbitrary discrete semigroups S), $(\beta S)p$ contains a minimal left-ideal $(\beta S)r$. By Lemma 3, $(\beta S)r$ contains an idempotent s. Since $s \in (\beta S)p$ and p is idempotent, we conclude sp=s. If we then set $q := ps$, we easily check that q is idempotent, that $q \prec p$, and (since q lies in the minimal left-ideal $(\beta S)r$) it is minimal. The claim follows. $\Box$

Proof of Proposition 1. Since p is an idempotent element of $\beta(A^{<\omega})$, it is also an idempotent element of $\beta(A \cup \{*\})^{<\omega}$. It need not be minimal in that semigroup, though. However, by Lemma 4, we can find a minimal idempotent q in $\beta(A \cup \{*\})^{<\omega}$ such that $q \prec p$.

Now let $x \in A$. Since $\pi_x : (A \cup \{*\})^{<\omega} \to A^{<\omega}$ is a homomorphism, $\beta\pi_x : \beta(A \cup \{*\})^{<\omega} \to \beta A^{<\omega}$ is also a homomorphism (why?). Since q is idempotent and $q \prec p$ (note that these are both purely *algebraic* statements), we conclude that $\beta\pi_x(q)$ is idempotent and $\beta\pi_x(q) \prec \beta\pi_x(p)$. But $\beta\pi_x(p) = p$ is minimal in $\beta A^{<\omega}$, hence by Exercise 17, we have $\beta\pi_x(q) = p$. The claim follows. $\Box$

Exercise 18. Adapt the above proof to give an alternate proof of the ultrafilter version of van der Waerden's theorem (Proposition 1 from the previous lecture) which relies on idempotence rather than on induction on k. (If you are stuck, read the proof of Proposition 1.55 of Glasner's book.) $\diamond$

Remark 9. Several of the above Ramsey-type theorems can be unified. For instance, the polynomial van der Waerden theorem and the Hales-Jewett theorem have been unified into the polynomial Hales-Jewett theorem of Bergelson and Leibman (see also the more recent paper of Walters on this topic). This type of Ramsey theory is still an active subject, and we do not yet have a comprehensive and systematic theory (or a "universal" Ramsey theorem) that encompasses all known examples. $\diamond$

Exercise 19. Let X be an at most countable set (with the discrete topology), and let $\mathcal{F}$ be a family of subsets of X. Show that the following two statements are equivalent:

1. Whenever X is finitely coloured, one of the colour classes contains a subset in $\mathcal{F}$.
2. There exists $p \in \beta X$ such that every neighbourhood of p contains a subset in $\mathcal{F}$. $\diamond$

Exercise 20. Let X, Y be at most countable sets with the discrete topology, and let $f_1, \ldots, f_k : Y \to X$ be a finite collection of functions. Show that the following two statements are equivalent:

1. Whenever X is finitely coloured, one of the colour classes contains a set $\{ f_1(y), \ldots, f_k(y) \}$ for some $y \in Y$.
2. There exists $q \in \beta Y$ such that $\beta f_1(q) = \ldots = \beta f_k(q)$.

(*Hint*: look at the closure of $\{ (f_1(y), \ldots, f_k(y)) : y \in Y \}$ in $(\beta X)^k$.) $\diamond$ **Exercise 21.** Using the previous exercise, deduce Theorem 3 from Theorem 1, and deduce Theorem 6 from Theorem 4. $\diamond$

[*Update*, Jan 24: extra factors added to the polynomial in Exercise 1 to sidestep issues at 2 and 3.]

[*Update*, Jan 30: remarks added and renumbered.]

[*Update*, Feb 1: More exercises added.]

12 comments

Comments feed for this article

24 January, 2008 at 1:30 pm

Yury

Dear Prof. Tao,

the polynomial from Exercise 1 has no roots in case N=8.

24 January, 2008 at 2:29 pm

Terence Tao

254A, Lecture 6: Isometric systems and isometric extensions

24 January, 2008 in 254A - ergodic theory, math.AT, math.DS, math.GR, math.MG

Tags: Isometric extensions, Kronecker factor, polynomial sequences, Ratner's theorem, structure

In this lecture, we move away from recurrence, and instead focus on the *structure* of topological dynamical systems. One remarkable feature of this subject is that starting from fairly "soft" notions of structure, such as topological structure, one can extract much more "hard" or "rigid" notions of structure, such as *geometric* or *algebraic* structure. The key concept needed to capture this structure is that of an *isometric system*, or more generally an *isometric extension*, which we shall discuss in this lecture. As an application of this theory we characterise the distribution of polynomial sequences in torii (a baby case of a variant of Ratner's theorem due to (Leon) Green, which we will cover later in this course).

– Isometric systems –

We begin with a key definition.

Definition 1 (Equicontinuous and isometric systems). Let $(X, \mathcal{F}, T)$ be a topological dynamical system.

1. We say that the system is *isometric* if there exists a metric d on X such that the shift maps $T^n : X \to X$ are all isometries, thus $d(T^n x, T^n y) = d(x, y)$ for all n and all x,y. (Of course, once T is an isometry, all powers T^n are automatically isometries also, so it suffices to check the n=1 case.)
2. We say that the system is *equicontinuous* if there exists a metric d on X such that the shift maps $T^n : X \to X$ form a uniformly equicontinuous family, thus for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(T^n x, T^n y) \leq \varepsilon$ whenever n, x, y are such that $d(x, y) \leq \delta$. (As X is compact, equicontinuity and uniform equicontinuity are equivalent concepts.)

Example 1. The circle shift $x \mapsto x + \alpha$ on $\mathbb{R}/\mathbb{Z}$ is both isometric and equicontinuous. On the other hand, the Bernoulli shift on $\{0, 1\}^{\mathbb{Z}}$ is neither isometric nor equicontinuous (why?). $\diamond$

Example 2. Any finite dynamical system is both isometric and equicontinuous (as one can see by using the discrete metric). $\diamond$

Since all metrics are essentially equivalent, we see that the choice of metric is not actually important when checking equicontinuity, but it seems to be more important when checking for isometry. Nevertheless, there is actually no distinction between the two properties:

Exercise 1. Show that a topological dynamical system is isometric if and only if it is equicontinuous.

(Hint: one direction is obvious. For the other, if T^n is a uniformly equicontinuous family with respect to a metric d, consider the modified metric $\tilde{d}(x,y) := \sup_n d(T^n x, T^n y)$.) ◇

Remark 1. From this exercise we see that we can upgrade *topological* structure (equicontinuity) to *geometric* structure (isometry). The motif of studying topology through geometry pervades modern topology; witness for instance Perelman's proof of the Poincaré conjecture. ◇

Exercise 2. (Ultrafilter characterisation of equicontinuity) Let $(X, \mathcal{F}, T)$ be a topological dynamical system. Show that X is equicontinuous if and only if the maps $T^p : X \to X$ are homeomorphisms for every $p \in \beta\mathbb{Z}$. ◇

Now we upgrade the geometric structure of isometry to the *algebraic* structure of being a compact abelian group action.

Definition 2. (Kronecker system) A topological dynamical system $(X, \mathcal{F}, T)$ is said to be a *Kronecker system* if it is isomorphic to a system of the form $(K, \mathcal{K}, S)$, where $(K, +, \mathcal{K})$ is a compact abelian metrisable topological group, and $S : x \mapsto x + \alpha$ is a group rotation for some $\alpha \in S$.

Example 3. The circle rotation system is a Kronecker system, as is the standard shift $x \mapsto x+1$ on a cyclic group $\mathbb{Z}/N\mathbb{Z}$. Any product of Kronecker systems is again a Kronecker system. ◇

Let us first observe that a Kronecker system is equicontinuous (and hence isometric). Indeed, the compactness of the topological group K (and the joint continuity of the addition law $+ : K \times K \to K$) easily ensures that the group rotations $g : x \mapsto x + g$ are uniformly equicontinuous as $g \in K$ varies. Since the shifts $T^n : x \mapsto x + n\alpha$ are all group rotations, the claim follows.

On the other hand, not every equicontinuous or isometric system is Kronecker. Consider for instance a finite dynamical system which is the disjoint union of two cyclic shifts of distinct order; it is not hard to see that this is not a Kronecker system. Nevertheless, it clearly contains Kronecker systems within it. Indeed, we have

Proposition 1. Every *minimal* equicontinuous (or isometric) system $(X, \mathcal{F}, T)$ is a Kronecker system, i.e. isomorphic to an abelian group rotation $(K, \mathcal{K}, x \mapsto x + \alpha)$. Furthermore, the orbit $\{n\alpha : n \in \mathbb{Z}\}$ is dense in K.

Proof. By Exercise 1, we may assume that the system is isometric, thus we can find a metric d such that all the shift maps T^n are isometries. We view the T^n as lying inside the space $C(X \to X)$ of continuous maps from X to itself, endowed with the uniform topology. Let $G \subset C(X \to X)$ be the closure of the maps $\{T^n : n \in \mathbb{Z}\}$. One easily verifies that G is a closed metrisable topological group of isometries in $C(X \to X)$; from the Arzelà-Ascoli theorem we see that G is compact. Also, since T^n and T^m commute for every n and m, we see upon taking limits that G is abelian.

Now let $x \in X$ be an arbitrary point. Then we see that the image $\{f(x) : f \in G\}$ of G under the evaluation map $f \mapsto f(x)$ is a compact non-empty invariant subset of X, and thus equal to all of X by minimality. If we then define the stabiliser $\Gamma := \{f \in G : f(x) = x\}$, we see that Γ is a closed (hence compact) subgroup of the abelian group G. Since $X = \{f(x) : f \in G\}$, we thus see that there is a continuous bijection $f\Gamma \mapsto f(x)$ from the quotient group $K := G/\Gamma$ (with the quotient topology) to X. Since both spaces here are compact Hausdorff, this map is a homeomorphism. This map is thus an isomorphism of topological dynamical systems between the Kronecker system K (with the group rotation given by $\alpha := T \mod \Gamma \in G/\Gamma$) and X. Since K is a compact metrisable (thanks to Hausdorff distance) topological group, the claim follows (relabeling the group operation as +). Note that the density of $\{n\alpha : n \in \mathbb{Z}\}$ in K is clear from construction. □

Remark 2. Once one knows that X is homeomorphic to a Kronecker system with $\{n\alpha : n \in \mathbb{Z}\}$ dense, one can *a posteriori* return to the proof and conclude that the stabiliser Γ is trivial. But I do not see a way to establish that fact directly. In any case, when we move to isometric extensions below, the analogue of the stabiliser Γ can certainly be non-trivial. ◇

To get from minimal isometric systems to non-minimal isometric systems, we have

Proposition 2. Any isometric system $(X, \mathcal{F}, T)$ can be partitioned as the union of disjoint minimal isometric systems.

Proof. Since minimal systems are automatically disjoint, it suffices to show that every point $x \in X$ is contained in a minimal dynamical system, or equivalently that the orbit closure $\overline{T^{\mathbb{Z}}x}$ is minimal. If this is not the case, then there exists $y \in \overline{T^{\mathbb{Z}}x}$ such that x does not lie in the orbit closure of y. But by definition of orbit closure, we can find a sequence n_j such that $T^{n_j}x$ converges to y. By the isometry property, this implies that $T^{-n_j}y$ converges to x, and so x is indeed in the orbit closure of y, a contradiction. □

Thus every equicontinuous or isometric system can be expressed as a union of disjoint Kronecker systems.

We can use the algebraic structure of isometric systems to obtain much quicker (and slightly stronger) proofs of various recurrence theorems. For instance, we can give a short proof of (a slight strengthening of) the multiple Birkhoff recurrence theorem (Theorem 3 from Lecture 4) as follows:

Proposition 3. (Multiple Birkhoff for isometric systems) Let $(X, \mathcal{F}, T)$ be an isometric system. Then for every $x \in X$ there exists a sequence $n_j \to \infty$ such that $T^{kn_j}x \to x$ for every integer k.

Proof. By Proposition 2 followed by Proposition 1, it suffices to check this for Kronecker systems $(K, \mathcal{K}, x \mapsto x + \alpha)$ in which $\{n\alpha : n \in \mathbb{Z}\}$ is dense in K. But then we can find a sequence n_j such that $n_j\alpha \to 0$ in K, and thus (since K is a topological group) $kn_j\alpha \to 0$ in K for all k. The claim follows. □

The above argument illustrates one of the reasons why it is desirable to have an algebraic structural theory of various types of dynamical systems; it makes it much easier to answer many interesting questions regarding such systems, such as those involving recurrence.

— The Kronecker factor –

We have seen isometric systems are basically Kronecker systems (or unions thereof). Of course, not all systems are isometric. However, it turns out that every system contains a maximal isometric *factor*. Recall that a factor of a topological dynamical system $(X, \mathcal{F}, T)$ is a surjective morphism $\pi : X \to Y$ from X to another topological dynamical system $(Y, \mathcal{G}, S)$. (We shall sometimes abuse notation and refer to $\pi : X \to Y$ as the factor, when it is really the quadruplet $(\pi, Y, \mathcal{G}, S)$.) We say that one factor $\pi : X \to Y$ *refines* or is *finer than* another factor $\pi' : X \to Y'$ if we can factorise $\pi' = f \circ \pi$ for some continuous map $f : Y \to Y'$. (Note from surjectivity that this map, if it exists, is unique.) We say that two factors are *equivalent* if they refine each other. Observe that modulo equivalence, refinement is a partial ordering on factors.

Example 4. The identity factor $\mathrm{id} : X \to X$ is finer than any other factor of X, which in turn is finer than the trivial factor $\mathrm{pt} : X \to \mathrm{pt}$ that maps to a point. $\diamond$

Exercise 3. Show that any factor of a minimal topological dynamical system is again minimal. $\diamond$

We note two useful operations on factors. Firstly, given two factors $\pi : X \to Y = Y$ and $\pi' : X \to Y'$, one can define their join $\pi \vee \pi' : X \to Y \vee Y'$, where $Y \vee Y' := \{(\pi(x), \pi'(x)) : x \in X\} \subset Y \times Y'$ is the compact subspace of the product system $Y \times Y'$, and $\pi \vee \pi' : X \to Y \vee Y'$ is the surjective morphism $\pi \vee \pi' : x \mapsto (\pi(x), \pi'(x))$. One can verify that $\pi \vee \pi'$ is the least common refinement of π and π', hence the name.

Secondly, given a chain $(\pi_\alpha)_{\alpha \in A}$ of factors $\pi_\alpha : X \to Y_\alpha$ (thus π_α refines π_β for all $\alpha > \beta$), one can form their inverse limit $\pi = \lim_{\leftarrow} (\pi_\alpha)_{\alpha \in A} : X \to Y = \lim_{\leftarrow} (Y_\alpha)_{\alpha \in A}$ by first letting $f_{\alpha\beta} : Y_\alpha \to Y_\beta$ be the factoring maps for all $\alpha > \beta$, observing that $f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma}$ for all $\alpha > \beta > \gamma$, and then defining $Y \subset \prod_\alpha Y_\alpha$ to be the compact subspace of the product system $\prod_\alpha Y_\alpha$ defined as

$$Y := \{(y_\alpha)_{\alpha \in A} : f_{\alpha\beta}(y_\alpha) = y_\beta \text{ whenever } \alpha > \beta\} \quad (1)$$

and then setting $\pi : x \mapsto (\pi_\alpha(x))_{\alpha \in A}$. One easily verifies that π is indeed a factor of X, and it is the least upper bound of the π_α.

Now we observe that these operations interact well with the isometry property:

Exercise 4. Let $\pi : X \to Y$ and $\pi' : X \to Y'$ be two factors such that Y and Y' are both isometric. Then $\pi \vee \pi' : X \to Y \vee Y'$ is also isometric. $\diamond$

Lemma 1. Let $(\pi_\alpha)_{\alpha \in A}$ be a totally ordered set of factors $\pi_\alpha : X \to Y_\alpha$ with $Y_\alpha = (Y_\alpha, \mathcal{G}_\alpha, S_\alpha)$ isometric. Then the inverse limit $\pi : X \to Y$ of the π_α is such that Y is also isometric.

Proof. Observe that we have factor maps $f_\alpha : Y \to Y_\alpha$ which are surjective morphisms, which themselves factor as $f_\beta = f_{\alpha\beta} \circ f_\alpha$ for $\alpha > \beta$ and some surjective morphisms $f_{\alpha\beta} : Y_\alpha \to Y_\beta$. Let us fix some metric d on Y. For each $\alpha \in A$, consider the compact subset $\Delta_\alpha := \{(y, y') \in Y \times Y : f_\alpha(y) = f_\alpha(y')\}$ of $Y \times Y$. These sets decrease as α increases, and their intersection is the diagonal $\{(y, y) : y \in Y\}$ (why?). Applying the finite intersection property in the compact sets $\{(y, y') \cap \Delta_\alpha : d(y, y') \geq \varepsilon\}$, we conclude that for every $\varepsilon > 0$ there exists α such that $d(y, y') < \varepsilon$ whenever $f_\alpha(y) = f_\alpha(y')$.

Now suppose for contradiction that Y is not isometric, and hence not uniformly equicontinuous. Then there exists a sequences $y_j, y'_j \in Y$ with $d(y_j, y'_j) \to 0$, an $\varepsilon > 0$, and a sequence n_j of integers such that $d(S^{n_j} y_j, S^{n_j} y'_j) > \varepsilon$. By compactness we may assume that y_j, y'_j both converge to the same point. But by the preceding discussion, we can find $\alpha \in A$ such that $d(y, y') < \varepsilon/4$ whenever $f_\alpha(y) = f_\alpha(y')$. In other words, for any z in Y_α, the fibre $f_\alpha^{-1}(\{z\})$ has diameter at most $\varepsilon/2$.

Now let $z_j := f_\alpha(y_j)$ and $z'_j := f_\alpha(y'_j)$. Then z_j and z'_j converge to the same point z in Y_α, and so by equicontinuity of Y_α, $d(S_\alpha^{n_j} z_j, S_\alpha^{n_j} z'_j)$ goes to zero. By compactness and passing to a subsequence we can assume that $S_\alpha^{n_j} z_j$ and $S_\alpha^{n_j} z'_j$ both converge to some point z_* in Y_α. On the other hand, from the preceding discussion and the triangle inequality, we see that the fibres $f_\alpha^{-1}(\{S_\alpha^{n_j} z_j\})$ and $f_\alpha^{-1}(\{S_\alpha^{n_j} z'_j\})$ are separated by a distance at least $\varepsilon/2$ in Y. On the other hand, the distance between $f_\alpha^{-1}(\{S_\alpha^{n_j} z_j\})$ and $f_\alpha^{-1}(\{z_*\})$ must go to zero as $j \to \infty$ (as a simple sequential compactness argument shows), and similarly the distance between $f_\alpha^{-1}(\{S_\alpha^{n_j} z'_j\})$ and $f_\alpha^{-1}(\{z_*\})$ goes to zero. Since $f_\alpha^{-1}(\{z_*\})$ has diameter at most $\varepsilon/4$, we obtain a contradiction. The claim follows. $\Box$

Combining Exercise 2 and Lemma 1 with Zorn's lemma (and noting that with the trivial factor $\mathrm{pt} : X \to \mathrm{pt}$, the image pt is clearly isometric) we obtain

> **Corollary 1**. (Existence of maximal isometric factor) For every topological dynamical system $(X, \mathcal{F}, T)$ there is a factor $\pi : X \to Y$ with Y isometric, and which is maximal with respect to refinement among all such factors with this property. This factor is unique up to equivalence.

By Proposition 1 and Exercise 3, the maximal isometric factor of a minimal system is a Kronecker system, and we refer to it as the *Kronecker factor* of that minimal system X.

Exercise 5. (Explicit description of Kronecker factor) Let $(X, \mathcal{F}, T)$ be a minimal topological dynamical system, and let $Q \subset X \times X$ be the set

$$Q := \bigcap_V \overline{(T \times T)^{\mathbb{Z}}(V)} \quad (2)$$

where V ranges over all open neighbourhoods of the diagonal $\{(x, x) : x \in X\}$ of $X \times X$, and $T \times T : (x, y) \mapsto (Tx, Ty)$ is the product shift. Let $\sim$ be the finest equivalence relation on X such that the set $R_\sim := \{(x, y) \in X \times X : x \sim y\}$ is closed and contains Q. (The existence and uniqueness of $\sim$ can be established by intersecting $R_\sim$ over all candidates $\sim$ together.) Show that the projection map $\pi : X \to X/\sim$ to the equivalence classes of $\sim$ (with the quotient topology) is (up to isomorphism) the

Kronecker factor of X. $\diamond$

The Kronecker factor is also closely related to the concept of an eigenfunction. We say that a continuous function $f : X \to \mathbb{C}$ is an *eigenfunction* of a topological dynamical system $(X, \mathcal{F}, T)$ if it is not identically zero and we have $Tf = \lambda f$ for some $\lambda \in \mathbb{C}$, which we refer to as an *eigenvalue* for T.

Exercise 6. Let $(X, \mathcal{F}, T)$ be a minimal topological dynamical system.

1. Show that if λ is an eigenvalue for T, then λ lies in the unit circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$, and furthermore there exists a unimodular eigenfunction $g : X \to S^1$ with this eigenvalue. (*Hint*: the zero set of an eigenfunction is a closed shift-invariant subset of X.)
2. Show that for every eigenvalue λ, the eigenspace $\{f \in C(X) : Tf = \lambda f\}$ is one-dimensional, i.e. all eigenvalues have geometric multiplicity 1. (*Hint*: first establish this in the case $\lambda = 1$.)
3. If $g : X \to S^1$ is a unimodular eigenfunction with non-trivial eigenvalue $\lambda \neq 1$, show that $g : X \to g(X)$ is an isometric factor of X, where $g(X) \subset S^1$ is given the shift $z \mapsto \overline{\lambda} z$. Conclude in particular that $g = c\chi \circ \pi$, where $\pi : X \to K$ is the Kronecker factor, $\chi : K \to S^1$ is a character of K, and c is a constant. Conversely, show that all functions of the form $c\chi \circ \pi$ are eigenfunctions. (From this, it is possible to reconstruct the Kronecker factor canonically from the eigenfunctions of X; we leave the details to the reader.)

We will see eigenfunctions (and various generalisations of the eigenfunction concept) playing a decisive role in the structure theory of measure-preserving systems, which we will get to in a few lectures.

– Isometric extensions –

To cover more general systems than just the isometric systems, we need the more flexible concept of an *isometric extension*.

Definition 3 (Extensions). If $\pi : X \to Y = (Y, \mathcal{G}, S)$ is a factor of $(X, \mathcal{F}, T)$, we say that $(X, \mathcal{F}, T)$ is an *extension* of $(Y, \mathcal{G}, S)$, and refer to $\pi : X \to Y$ as the *projection map* or *factor map*. We refer to the (compact) spaces $\pi^{-1}(\{y\})$ for $y \in Y$ as the *fibres* of this extension.

Example 5. The skew shift is an extension of the circle shift, with the fibres being the "vertical" circles. All systems are extensions of a point, and (somewhat trivially) are also extensions of themselves. $\diamond$

Definition 4 (Isometric extensions). Let $(X, \mathcal{F}, T)$ be an extension of a topological dynamical system $(Y, \mathcal{G}, S)$ with projection map $\pi : X \to Y$. We say that this extension is *isometric* if there exists a metric $d_y : \pi^{-1}(\{y\}) \times \pi^{-1}(\{y\}) \to \mathbb{R}^+$ on each fiber $\pi^{-1}(\{y\})$ with the following properties:

1. (Isometry) For every $y \in Y$ and $x, x' \in \pi^{-1}(\{y\})$, we have $d_{Sy}(Tx, Tx') = d_y(x, x')$.
2. (Continuity) The function $d : \bigcup_{y \in Y} \pi^{-1}(\{y\}) \times \pi^{-1}(\{y\}) \to \mathbb{R}^+$ formed by gluing together all the d_y is continuous (where we view the domain as a compact subspace

$\{(x, x') \in X \times X : \pi(x) = \pi(x')\}$ of $X \times X$).
3. (Isometry, again) For any $y, y' \in Y$, the metric spaces $(\pi^{-1}(\{y\}), d_y)$ and $(\pi^{-1}(\{y'\}), d_{y'})$ are isometric.

Example 6. The skew shift is an isometric extension of the circle shift, where we give each fibre the standard metric. ◇

Example 7. A topological dynamical system is an isometric extension of a point if and only if it is isometric. ◇

Exercise 7. If X is minimal, show that properties 1 and 2 in the above definition automatically imply property 3. Furthermore, in this case show that the isometry group $\mathrm{Isom}(\pi^{-1}(\{y\}))$ of any fibre acts transitively on that fibre. Show however that property 3 can fail even when properties 1 and 2 hold if X is not assumed to be minimal. ◇

Exercise 7′. (Topological characterisation of isometric extensions) Let $(X, \mathcal{F}, T)$ be an extension of a topological dynamical system $(Y, \mathcal{G}, S)$ with factor map $\pi : X \to Y$, and let d be a metric on X. Show that X is an isometric extension if and only if the shift maps T^n are uniformly equicontinuous relative to π in the sense that for every $\varepsilon > 0$ there exists $\delta > 0$ such that every $x, y \in X$ with $\pi(x) = \pi(y)$ and $d(x, y) < \delta$, we have $d(T^n x, T^n y) < \varepsilon$ for all n. ◇

An important subclass of isometric extensions are the *group extensions*. Recall that an *automorphism* of a topological dynamical system is an isomorphism of that system to itself, i.e. a homeomorphism that commutes with the shift.

Definition 5. (Group extensions) Let $(X, \mathcal{F}, T)$ be a topological dynamical system. Suppose that we have a compact group G of automorphisms of X (where we endow G with the uniform topology). Then the quotient space $Y := G \backslash X = \{Gx : x \in X\}$ is also a compact metrisable space, and one easily sees that the projection map $\pi : X \mapsto Y$ is a factor map. We refer to X as a *group extension* of Y (or of any other system isomorphic to Y). We refer to G as the *structure group* of the extension. We say that the group extension is an *abelian group extension* if G is abelian.

Example 8. (Cocycle extensions) If G is a compact topological metrisable group, $(Y, \mathcal{G}, S)$ is a topological dynamical system, and a continuous map $\sigma : Y \to G$, then we define the *cocycle extension* $X = Y \times_\sigma G$ to be the product space $Y \times G$ with the shift $T : (y, \zeta) \mapsto (Sy, \sigma(y)\zeta)$, and with the factor map $\pi : (y, \zeta) \mapsto y$. One easily verifies that X is a group extension of Y with structure group G. The converse is not quite true for topological reasons; not every G-bundle can be globally trivialised, although one can still describe general group extensions by patching together cocycle extensions on local trivalisations. ◇

Example 9. The skew shift is a cocycle extension (and hence group extension) $Y \times_\sigma (\mathbb{R}/\mathbb{Z})$ of the circle shift Y, with $\sigma(y) := y$ being the identity map. Any Kronecker system is an abelian group extension of a point. ◇

Exercise 8. Show that every group extension is an isometric extension. (Hint: the group G acts equicontinuously on itself, and thus isometrically on itself by choosing the right metric, as in Exercise 1.) ◇

Exercise 9. Let $(Y, \mathcal{G}, S)$ be a topological dynamical system, and G a compact topological metrisable group. We say that two cocycles $\sigma, \sigma' : Y \to G$ are *cohomologous* if we have $\sigma'(y) = \phi(Sy)\sigma(y)\phi(y)^{-1}$ for some continuous map $\phi : Y \to G$. Show that if σ, σ' are cohomologous, then the cocycle extensions $Y \times_\sigma G$ and $Y \times_{\sigma'} G$ are isomorphic. Understanding exactly which cocycles are cohomologous to each other is a major topic of study in dynamical systems (though not one which we will pursue here). ◇

In view of Proposition 1 and Exercise 14, it is reasonable to ask whether every minimal isometric extension is a group extension. The answer is no (though actually constructing a counterexample is a little tricky). The reason is that we can form intermediate systems between a system Y=G\X and a group extension X of that system by quotienting out a subgroup. Indeed, if H is a closed subgroup of the structure group G, then H\X is a factor of X and an isometric extension of G\X, but need not be a group extension of G\X (basically because G/H need not be a group). But this is the only obstruction to obtaining an analogue of Proposition 1:

Lemma 2. Suppose that X is an isometric extension of another topological dynamical system Y with projection map $\pi : X \to Y$. Suppose also that X is minimal. Then there exists a group extension Z of Y with structure group G (thus $Y \equiv G\backslash Z$) and a closed subgroup H of G such that X is isomorphic to H\Z, and π is (after applying the isomorphisms) the projection map from H\Z to G\Z; thus we have the commutative diagram

$$\begin{array}{ccc} Z & \to & X = H\backslash Z \\ & \searrow & \downarrow \\ & & Y = G\backslash Z \end{array} . \quad (4)$$

Proof. For each $y \in Y$, let V_y be the metric space $\pi^{-1}(\{y\})$ with the metric d_y given by Definition 5. Thus for any integer n and any $y \in Y$, T^n is an isometry from V_y to $V_{S^n y}$; taking limits, we see for any $p \in \beta\mathbb{Z}$ that T^p is an isometry from V_y to $V_{S^p y}$. Also, the T^p clearly commute with the shift T.

Fix a point $y_0 \in Y$, and set $G := \mathrm{Isom}(Y_0)$.

Let W be the space of all pairs (y, f) where $y \in Y$ and f is an isometry from V_{y_0} to V_y. This is a compact metrisable space with a shift $U : (y, f) \mapsto (Sy, T \circ f)$ and an action $g : (y, f) \mapsto (y, f \circ g^{-1})$ of G that commutes with U. We let Z be the orbit closure in W of the G-orbit $\{y_0\} \times G$ under the shift U. If we fix a point $x_0 \in V_{y_0}$, then Z projects onto X by the map $f \mapsto f(y_0)$, and onto Y by the map $(y, f) \mapsto y$; these maps of course commute with the projection $\pi : x \mapsto \pi(x)$ from X to Y. Because X is minimal (and thus equal to all of its orbit closures), one sees that all of these projections are surjective morphisms, thus Z extends both Y and X. Also, one verifies that Z is a group extension over Y with structure group G, and a group extension over X with structure group given by the stabiliser $H := \{g \in G : gx_0 = x_0\}$. The claim follows. □

Exercise 10. Show that if an minimal extension $\pi : X \to Y$ is finite, then it is automatically an abelian group extension. (*Hint*: recall from Lecture 2 that minimal finite systems are equivalent to shifts on a cyclic group.) $\diamond$

An important feature of isometric or group extensions is that they tend to preserve recurrence properties of the system. We will see this phenomenon prominently when we turn to the ergodic theory analogue of isometric extensions, but for now let us give a simple illustrative result in this direction:

> **Proposition 4.** Let $(X, \mathcal{F}, T)$ be an isometric extension of $(Y, \mathcal{G}, S)$ with factor map $\pi : X \to Y$, and let y be a recurrent point of Y (see Lecture 3 for a definition). Then every point x in the fibre $\pi^{-1}(\{y\})$ is a recurrent point in X.

Proof. It will be convenient to use ultrafilters. In view of Lemma 2 it suffices to prove the claim for isometric extensions (note that recurrence is preserved under morphisms). Since y is recurrent, there exists $p \in \beta\mathbb{Z}\backslash\mathbb{Z}$ such that $S^p y = y$ (see Exercise 9 from Lecture 3). Thus $\pi(T^p x) = \pi(x)$. Since Y = G\X, this implies that $T^p x = gx$ for some $g \in G$. We can iterate this (recalling that G commutes with T) to conclude that $T^{np} x = g^n x$ for all positive integers n. But by considering the action of g on G, we know (from the Birkhoff recurrence theorem from Lecture 3) that we have $g^{n_j} h \to h$ for some $h \in G$ and $n_j \to +\infty$; canceling the h, and then applying to x, we conclude that $g^{n_j} x \to x$, and thus $T^{n_j p} x \to x$. If we write $q := \lim_{j \to r} n_j p$ for some $r \in \beta\mathbb{N}\backslash\mathbb{N}$, we conclude that $T^q x = x$ and so x is recurrent as desired. $\square$

– Application: distribution of polynomial sequences in torii –

Now we apply the above theory to the following specific problem:

> **Problem 1.** Let $P : \mathbb{Z} \to (\mathbb{R}/\mathbb{Z})^d$ be a polynomial sequence in a d-dimensional torus, thus $P(n) = \sum_{j=0}^k c_j n^j$ for some $c_0, \ldots, c_k \in (\mathbb{R}/\mathbb{Z})^d$. Compute the orbit closure $\overline{P(\mathbb{Z})} = \overline{\{P(n) : n \in \mathbb{Z}\}}$.

(We will be vague here about what "compute" means.)

Example 10. Is the orbit $\{(\sqrt{2}n \bmod 1, \sqrt{3}n^2 \bmod 1) : n \in \mathbb{Z}\}$ dense in the two-dimensional torus $(\mathbb{R}/\mathbb{Z})^2$? $\diamond$

The answer should of course depend on the polynomial P; for instance if P is constant then the orbit closure is clearly a point. Similarly, if the polynomial P has a constraint of the form $m \cdot P = c$ for some non-zero $m \in \mathbb{Z}^d$ and $c \in \mathbb{R}/\mathbb{Z}$, then the orbit closure is clearly going to be contained inside the proper subset $\{x \in (\mathbb{R}/\mathbb{Z})^d : m \cdot x = c\}$ of the torus. For instance, $\{(\sqrt{2}n^2 \bmod 1, 2\sqrt{2}n^2 \bmod 1) : n \in \mathbb{Z}\}$ is clearly not dense in the two-dimensional torus, as it is contained in the closed one-dimensional subtorus $\{(x, 2x) : x \in \mathbb{R}/\mathbb{Z}\}$.

In the above example, it is clear that the problem of computing the orbit closure of

$(\sqrt{2}n^2 \mod 1, 2\sqrt{2}n^2 \mod 1)$ reduces to computing the orbit closure of $(\sqrt{2}n^2 \mod 1)$. More generally, if a polynomial $P: \mathbb{Z} \to (\mathbb{R}/\mathbb{Z})^d$ obeys a constraint $m \cdot P = c$ for some non-zero *irreducible* $m \in \mathbb{Z}^d$ (i.e. m does not factor as m = q m' for some $q > 1$ and $m' \in \mathbb{Z}^d$, or equivalently that the greatest common divisor of the coefficients of m is 1), then some elementary number theory shows that the set $\{x \in (\mathbb{R}/\mathbb{Z})^d : m \cdot x = c\}$ is isomorphic (after an invertible affine transformation with integer coefficients on the torus) to the standard subtorus $(\mathbb{R}/\mathbb{Z})^{d-1}$).

Exercise 11. Prove the above claim. (Hint: the Euclidean algorithm may come in handy.) $\diamond$

Because of this, we see that whenever we have a constraint of the form $m \cdot P = c$ with m irreducible, we can reduce Problem 1 to an instance of Problem 1 with one lower dimension. What about if m is not irreducible? A typical example of this would be when $P(n) := (\sqrt{2}n^2, 2\sqrt{2}n^2 + \frac{1}{2})$ (I'm going to drop the "mod 1" terms to remove clutter). Here, we have the constraint $(-4, 2) \cdot P(n) = 0$, which constrains P to the union of two one-dimensional torii, rather than a single one-dimensional torus. But we can eliminate this multiplicity by the trick of working with the odd and even components $\{P(2n+1) : n \in \mathbb{Z}\}$ and $\{P(2n) : n \in \mathbb{Z}\}$ respectively. One observes that each component obeys an irreducible constraint, namely $(-2, 1) \cdot P(2n) = 0$ and $(-2, 1) \cdot P(2n+1) = \frac{1}{2}$ respectively, and so by the preceding discussion, the problem of computing the orbit closures for each of these components reduces to that of computing an orbit closure in a torus of one lower dimension.

Exercise 12. More generally, show that whenever P obeys a constraint $m \cdot P(n) = c$ with m not necessarily irreducible, then there exists an integer $q \geq 1$ such that the orbits $\{P(qn + r) : n \in \mathbb{Z}\}$ obey a constraint $m' \cdot P(qn + r) = c_r$ with m' irreducible. $\diamond$

From Exercises 11 and 12, we see that every time we have a constraint of the form $m \cdot P(n) = c$ for some non-zero m, we can reduce Problem 1 to one or more copies of Problem 1 in one lower dimension. So, without loss of generality (and by inducting on dimension) we may assume that no such constraint exists. (We will see this "induction on dimension" type of argument much later in this course, when we study Ratner-type theorems in more detail.)

Now that all the "obvious" restrictions on the orbit have been removed, one might now expect P(n) to be uniformly distributed throughout the torus. Happily, this is indeed the case (at least at the topological level):

> **Theorem 1.** (Equidistribution theorem) Let $P: \mathbb{Z} \to (\mathbb{R}/\mathbb{Z})^d$ be a polynomial sequence which does not obey any constraint of the form $m \cdot P(n) = c$ with $m \in \mathbb{Z}^d$ non-zero. Then the orbit $P(\mathbb{Z})$ is dense in $(\mathbb{R}/\mathbb{Z})^d$ (i.e. the orbit closure is the whole torus).

Remark 3. The recurrence theorems we have already encountered (e.g. Corollary 1 from Lecture 4, or Theorem 1 from Lecture 5) do not seem to directly establish this result, instead giving the weaker result that every element in $P(\mathbb{Z})$ is a limit point. $\diamond$

Exercise 13. Assuming Theorem 1, show that the answer to Problem 1 is always "a finite union of subtorii", regardless of what the coefficients of P are. $\diamond$

Theorem 1 can be proven using Weyl's theory of equidistribution, which is based on bounds on exponential sums; but we shall instead use a topological dynamics argument based on some ideas of Furstenberg. Amusingly, this argument will use some *global* topology (specifically, winding numbers) and not just *local* (point-set) topology.

To begin proving this theorem, let us first consider the linear one-dimensional case, in which one considers the orbit closure of $\{n\alpha+\beta : n \in \mathbb{Z}\}$ for some $\alpha, \beta \in \mathbb{R}/\mathbb{Z}$. The constant term β only affects this closure by a translation and we can ignore it. One then easily checks that the orbit closure $\overline{\{n\alpha : n \in \mathbb{Z}\}}$ is a closed subgroup of $\mathbb{R}/\mathbb{Z}$. Fortunately, we have a classification of these objects:

> **Lemma 3.** Let H be a closed subgroup of $\mathbb{R}/\mathbb{Z}$. Then either $H = \mathbb{R}/\mathbb{Z}$, or H is a cyclic group of the form $H = \{x \in \mathbb{R}/\mathbb{Z} : Nx = 0\}$ for some $N \geq 1$.

Proof. If H is not all of $\mathbb{R}/\mathbb{Z}$, then its complement, being a non-empty open set, is the union of disjoint open intervals. Let x be the boundary of one of these intervals, then x lies in the closed set H. Translating the group H by x, we conclude that 0 is also the boundary of one of these intervals. Since H = -H, we thus see that 0 is an isolated point in H. If we then let y be the closest non-zero element of H to the origin (the case when H={0} can of course be checked separately), we check (using the Euclidean algorithm) that y generates H, and the claim easily follows. $\Box$

Exercise 14. Using the above lemma, prove Theorem 1 in the case when d=1 and P is linear. $\diamond$

Exercise 15. Obtain another proof of Lemma 3 using Fourier analysis and the fact that the only non-trivial subgroups of $\mathbb{Z}$ (the Pontryagin dual of $\mathbb{R}/\mathbb{Z}$ are the groups $N \cdot \mathbb{Z}$ for $N \geq 1$. $\diamond$

Now we consider the linear case in higher dimensions. The key lemma is

> **Lemma 4.** Let H be a closed subgroup of $(\mathbb{R}/\mathbb{Z})^d$ for some $d \geq 1$ such that $\pi(H) = (\mathbb{R}/\mathbb{Z})^{d-1}$, where $\pi : (\mathbb{R}/\mathbb{Z})^d \to (\mathbb{R}/\mathbb{Z})^{d-1}$ is the canonical projection. Then either $H = (\mathbb{R}/\mathbb{Z})^d$ or $H = \{x \in (\mathbb{R}/\mathbb{Z})^d : m \cdot x = 0\}$ for some $m \in \mathbb{Z}^d$ with final coefficient non-zero.

Proof. The fibre $H \cap \pi^{-1}(\{0\})$ is isomorphic to a closed subgroup of $\mathbb{R}/\mathbb{Z}$, so we can apply Lemma 3. If this subgroup is full, then it is not hard to see that $H = (\mathbb{R}/\mathbb{Z})^d$, so suppose instead that $H \cap \pi^{-1}(\{0\})$ is isomorphic to the cyclic group of order N. We then apply the homomorphism $f_N : (x_1, \ldots, x_d) \to (x_1, \ldots, x_{d-1}, Nx_d)$, and observe that $H_N := f_N(H)$ is a closed subgroup of $(\mathbb{R}/\mathbb{Z})^d$ whose fibres are a point, i.e. H_N is a graph $\{(x, \phi(x)) : x \in (\mathbb{R}/\mathbb{Z})^{d-1}\}$ for some $\phi : (\mathbb{R}/\mathbb{Z})^{d-1} \to \mathbb{R}/\mathbb{Z}$. Observe that the projection map $(x, \phi(x)) \mapsto x$ is a continuous bijection from the compact Hausdorff space H_N to the compact Hausdorff space $(\mathbb{R}/\mathbb{Z})^{d-1}$, and is thus a homeomorphism; in particular, ϕ is continuous. Also, since H_N is a group, ϕ must be a homomorphism. It is then a standard exercise to conclude that ϕ is linear, and therefore takes the form $(x_1, \ldots, x_{d-1}) \mapsto m_1 x_1 + \ldots + m_{d-1} x_{d-1}$ for some integers $m_1, \ldots, m_{d-1}$. The claim then follows by some routine algebra. $\Box$

Exercise 16. Using the above lemma, prove Theorem 1 in the case when d is arbitrary and P is linear. ◇

We now turn to the polynomial case. The basic idea is to re-express P(n) in terms of the orbit $T^n x$ of some topological dynamical system on a torus. We have already seen this happen with the skew shift $((\mathbb{R}/\mathbb{Z})^2, (x,y) \mapsto (x+\alpha, y+x))$, where the orbits $T^n x$ exhibit quadratic behaviour in n. More generally, an iterated skew shift such as

$$(\mathbb{R}/\mathbb{Z})^d, (x_1, \ldots, x_d) \mapsto (x_1+\alpha, x_2+x_1, \ldots, x_d+x_{d-1}) \quad (5)$$

generates orbits $T^n x$ whose final coefficient contains degree d terms such as $\frac{n(n-1)\ldots(n-d+1)}{d!}\alpha$. What we would like to do is find criteria under which we could demonstrate that systems such as (5) are *minimal*; this would mean that every orbit closure in that system is dense, which would clearly be relevant for proving results such as Theorem 1.

To do this, we will exploit the fact that systems such as (5) can be built as towers of isometric extensions; for instance, the system (5) is an isometric extension over the same system (5) associated to d-1 (which, in the case d=1, is simply a point). Now, isometric extensions don't always preserve minimality; for instance, if one takes a trivial cocycle extension $Y \times_0 G$ then the system is certainly non-minimal, as every horizontal slice $Y \times \{g\}$ of that system is a subsystem. More generally, any cocycle extension which is cohomologous to the trivial cocycle (see Exercise 9) will not be minimal. However, it turns out that if one has a topological obstruction to triviality, then minimality is preserved. We will formulate this fact using the machinery of *winding numbers*. Recall that every continuous map $f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ has a winding number $[f] \in \mathbb{Z}$, which can be defined as the unique integer such that f is homotopic to the linear map $x \mapsto [f]x$. Note that the map $f \mapsto [f]$ is linear, and also that $[f]$ is unchanged if one continuously deforms f.

We now give a variant of a lemma of Furstenberg.

Lemma 5. Let $(Y, \mathcal{G}, S)$ be a minimal topological dynamical system. Let $\sigma: Y \to (\mathbb{R}/\mathbb{Z})^d$ be a cocycle such that for every non-zero $m \in \mathbb{Z}^d$ there exists a loop $\gamma: \mathbb{R}/\mathbb{Z} \to Y$ such that $S \circ \gamma$ is homotopic to γ and $[m \cdot \sigma \circ \gamma] \neq 0$. Then $Y \times_\sigma \mathbb{R}/\mathbb{Z}$ is also minimal.

Proof. We induct on d. The case d=0 is trivial, so suppose $d \geq 1$ and the claim has already been proven for d-1. Suppose for contradiction that $Y \times_\sigma (\mathbb{R}/\mathbb{Z})^d$ contains a proper minimal subsystem Z. Then $\pi(Z)$ is a subsystem of Y, and must therefore equal all of Y, by minimality of Y. Now we use the action of $(\mathbb{R}/\mathbb{Z})^d$ on $Y \times_\sigma (\mathbb{R}/\mathbb{Z})^d$, which commutes with the shift $T: (y, \zeta) \mapsto (Sy, \sigma(y)+\zeta)$. For every $\theta \in (\mathbb{R}/\mathbb{Z})^d$, we see that $\theta + Z$ is also a minimal subsystem, and so is either equal to Z or disjoint from Z. If we let $H := \{\theta \in (\mathbb{R}/\mathbb{Z})^d : \theta + Z = Z\}$, we conclude that H is a closed subgroup of $(\mathbb{R}/\mathbb{Z})^d$.

We now claim that the projection of H to $(\mathbb{R}/\mathbb{Z})^{d-1}$ must be all of $(\mathbb{R}/\mathbb{Z})^{d-1}$. For if this were not the case, we could project Z down to $Y \times_{\sigma'} (\mathbb{R}/\mathbb{Z})^{d-1}$, where $\sigma': Y \to (\mathbb{R}/\mathbb{Z})^{d-1}$ is the projection of σ, and obtain a proper subsystem of that extension. But by induction hypothesis we see that $Y \times_{\sigma'} (\mathbb{R}/\mathbb{Z})^{d-1}$ is minimal, a contradiction, thus proving the claim.

We can now apply Lemma 4. If H is all of $(\mathbb{R}/\mathbb{Z})^d$ then Z is all of $Y \times_\sigma (\mathbb{R}/\mathbb{Z})^d$, a contradiction. Thus we have $H = \{\zeta \in (\mathbb{R}/\mathbb{Z})^d : m \cdot \zeta = 0\}$ for some non-zero $m \in \mathbb{Z}^d$, and thus Z must take the form

$Z = \{(y, \zeta) \in Y \times_\sigma (\mathbb{R}/\mathbb{Z})^d : m \cdot \zeta = \phi(y)\}$ (6)

for some $\phi : Y \to \mathbb{R}/\mathbb{Z}$. Arguing as in the proof of Lemma 4 we can show that Y is homeomorphic to the image of Z under the map $(y, \zeta) \mapsto (y, m \cdot \zeta)$ and so ϕ must be continuous. Since Z is shift-invariant, we must have the equation

$$\phi(Sy) = \phi(y) + m \cdot \sigma(y). \quad (7)$$

We apply this for y in the loop γ associated to m by hypothesis, and take degrees to conclude

$$[\phi \circ S \circ \gamma] = [\phi \circ \gamma] + [m \cdot \sigma \circ \gamma]. \quad (8)$$

But as $S \circ \gamma$ is homotopic to γ, we have $[\phi \circ S \circ \gamma] = [\phi \circ \gamma]$ and thus $[m \cdot \sigma \circ \gamma] = 0$, contradicting the hypothesis. $\square$

Exercise 17. Using the above lemma and an induction on d, show that the system (5) is minimal whenever α is irrational. [The key, of course, is to make a good choice for the loop γ that makes all computations easy.] $\diamond$

Exercise 18. More generally, show that the product of any finite number of systems of the form (5) remains minimal, as long as the numbers α that generate each factor system are linearly independent with respect to each other and to 1 over the rationals $\mathbb{Q}$. $\diamond$

It is now possible to deduce Theorem 1 from Exercise 18 and a little bit of linear algebra. We sketch the ideas as follows. Firstly we take all the non-constant coefficients that appear in P and look at the space they span, together with 1, over the rationals $\mathbb{Q}$. This is a finite-dimensional space, and so has a basis containing 1 which is linearly independent over $\mathbb{Q}$. The non-constant coefficients of P are rational linear combinations of elements of this basis; by dividing the basis elements by some suitable integer (and using the trick of passing from P(n) to P(qn+r) if necessary) we can ensure that the coefficients of P are in fact integer linear combinations of basis elements. This allows us to write P as an affine-linear combination (with integer coefficients) of the coefficients of an orbit in the type of product system considered in Exercise 18. If this affine transformation has full rank, then we are done; otherwise, the affine transformation maps to some subspace of the torus of the form $\{x : m \cdot x = c\}$, contradicting the hypothesis on P. Theorem 1 follows.

[*Update*, Feb 9: Proof of Lemma 2 corrected.]

[*Update*, Feb 11: Exercise 6 corrected.]

[*Update*, Feb 12: Proof of Lemma 1 corrected.]

[*Update*, Feb 18: Exercise 6.3 corrected.]

[*Update*, Feb 19: Exercise 6.3 corrected again.]

254A, Lecture 7: Structural theory of topological dynamical systems

28 January, 2008 in 254A - ergodic theory, math.DS

Tags: distal systems, Furstenberg structure theorem, structure, weak mixing
In our final lecture on topological dynamics, we discuss a remarkable theorem of Furstenberg that classifies a major type of topological dynamical system - *distal* systems - in terms of highly structured (from an algebraic point of view) systems, namely towers of isometric extensions. This theorem is also a model for an important analogous result in ergodic theory, the *Furstenberg-Zimmer structure theorem*, which we will turn to in a few lectures. We will not be able to prove Furstenberg's structure theorem for distal systems here in full, but we hope to illustrate some of the key points and ideas.

– Distal systems –

Furstenberg's theorem concerns a significant generalisation of the equicontinuous (or isometric) systems, namely the *distal* systems.

Definition 1. (Distal systems) Let $(X, \mathcal{F}, T)$ be a topological dynamical system, and let d be an arbitrarily metric on X (it is not important which one one picks here). We say that two points x, y in X are *proximal* if we have $\liminf_{n\to\infty} d(T^n x, T^n y) = 0$. We say that X is distal if no two distinct points $x \neq y$ in X are proximal, or equivalently if for every distinct x, y there exists $\varepsilon > 0$ such that $d(T^n x, T^n y) \geq \varepsilon$ for all n.

It is obvious that every isometric or equicontinuous system is distal, but the converse is not true, as the following example shows:

Example 1. If $\alpha \in \mathbb{R}$, then the skew shift $((\mathbb{R}/\mathbb{Z})^2, (x, y) \mapsto (x+\alpha, y+x))$ turns out to be not equicontinuous; indeed, if we start with a pair of nearby points $(0,0), (0, 1/2n)$ for some large n and apply T^n, one ends up with $(n\alpha, \frac{n(n-1)}{2}\alpha)$ and $(\alpha, \frac{n(n-1)}{2}\alpha + \frac{1}{2})$, thus demonstrating failure of equicontinuity. On the other hand, the system is still distal: given any pair of distinct points $(x, y), (x', y')$, either $x \neq x'$ (in which case the horizontal separation between $T^n(x, y)$ and $T^n(x', y')$ is bounded from below) or $x = x'$ (in which case the vertical separation is bounded from below). $\diamond$

Exercise 1. Show that any non-trivial Bernoulli system $\Omega^{\mathbb{Z}}$ is not distal. $\diamond$

Distal systems interact nicely with the action $p \mapsto T^p$ of the compactified integers $\beta\mathbb{Z}$:

Exercise 2. Let $(X, \mathcal{F}, T)$ be a topological dynamical system.

1. Show that two points x, y in X are proximal if and only if $T^p x = T^p y$ for some $p \in \beta\mathbb{Z}$.
2. Show that X is distal if and only if all the maps T^p for $p \in \beta\mathbb{Z}$ are injective.
3. If X is distal, show that $T^p = \mathrm{id}$ whenever $p \in \beta\mathbb{Z}$ is idempotent. (Hint: use part 2.)
4. If X is distal, show that the set of transformations $G := \{T^p : p \in \beta\mathbb{Z}\}$ on X forms a group, known as the *Ellis group* of X. (Hint: use part 3, together with Lemma 3 from Lecture 5.) Show that G is a compact subset of X^X (with the product topology), and that G acts transitively on X if and only if X is minimal. ◇

Exercise 3. Show that an inverse limit of a totally ordered set $(Y_\alpha)_{\alpha \in A}$ of distal factors is still distal. (This turns out to be slightly easier than Lemma 1 from the previous lecture.) ◇

Exercise 4. Show that every topological dynamical system has a maximal distal factor. (Hint: repeat the proof of Corollary 1 from the previous lecture.) ◇

Exercise 5. Show that any distal system can be partitioned into disjoint minimal distal systems. (One can of course adapt the proof of Proposition 2 from the previous lecture to do this; but there is a slicker way to do it by exploiting the Ellis group.) ◇

Note that the skew shift system, while not isometric, does have a non-trivial isometric factor, namely the circle shift $(\mathbb{R}/\mathbb{Z}, x \mapsto x + \alpha)$ with the projection map $\pi : (x, y) \mapsto x$. It turns out that this phenomenon is general:

Theorem 1 (Baby Furstenberg structure theorem). Let $(X, \mathcal{F}, T)$ be minimal, distal and non-trivial (i.e. not a point). Then X has a non-trivial isometric factor $\pi : X \to Y$.

This result - a toy case of Furstenberg' s full structure theorem - is already rather difficult to establish. We will not give Furstenberg's original proof here (though see Exercise 13 below), but will at least sketch how the factor $\pi : X \to Y$ is constructed. A key object in the construction is the symmetric function $F : X \times X \to \mathbb{R}^+$ defined by the formula

$$F(x, y) := \inf_{n \in \mathbb{Z}} d(T^n x, T^n y). \quad (3)$$

Example 2. We again consider the skew shift $((\mathbb{R}/\mathbb{Z})^2, (x, y) \mapsto (x + \alpha, y + x))$ with α irrational. For sake of concreteness let us choose the taxicab metric $d((x, y), (x', y')) := \|x - x'\|_{\mathbb{R}/\mathbb{Z}} + \|y - y'\|_{\mathbb{R}/\mathbb{Z}}$, where $\|x\|_{\mathbb{R}/\mathbb{Z}}$ is the distance from x to the integers. Then one can check that $F((x, y), (x', y'))$ is equal to $\|x - x'\|_{\mathbb{R}/\mathbb{Z}}$ when $x - x'$ is irrational, and equal to $\|x - x'\|_{\mathbb{R}/\mathbb{Z}} + \frac{1}{q}\|q(y - y')\|_{\mathbb{R}/\mathbb{Z}}$ when $x - x'$ is rational, where q is the least positive integer such that $q(x - x')$ is an integer. Thus F is highly discontinuous, but it is at least upper semi-continuous in each of its two variables. (Actually, the upper semi-continuity of F holds for arbitrary topological dynamical systems, since F is the infimum of continuous functions.) ◇

Exercise 6. Let G be the Ellis group of a minimal distal system X.

1. For any $x, y \in X$, show that $F(x, y) = \inf_{g \in G} d(gx, gy)$. In particular, $F(gx, gy) = F(x, y)$ for all $g \in G$.

2. For any $x, y \in X$, show that the set $\{(gx, gy) : g \in G\}$ is a minimal subsystem of $X \times X$ (with the product shift $(x, y) \mapsto (Tx, Ty)$. Conclude in particular that if $F(x, y) < a$, then the set $\{n \in \mathbb{Z} : d(T^n x, T^n y) < a)\}$ is syndetic.
3. If $x, y \in X$ and $a > 0$ is such that $F(x, y) < a$, show that there exists ε such that $F(x, z) < a$ whenever $F(y, z) < \varepsilon$.
4. Let $X_F = (X, \mathcal{F}_F)$ be the space X whose topology is generated by the basic open sets $U_{a,x} := \{y \in X : F(x, y) < a\}$. (That this is a base follows from 3.) Equivalently, X_F is equipped with the weakest topology on which F is upper semi-continuous in each variable. Show that X_F is a weaker topological space than X (i.e. the identity map from X to X_F is continuous); in particular, X_F is compact. Also show that all the maps in G are homeomorphisms on X_F. $\diamond$

If X_F were Hausdorff, then the system $(X_F, \mathcal{F}_F, T)$ would be equicontinuous, by Exercise 2 from the previous lecture. Unfortunately, X_F is not Hausdorff in general. However, it turns out that we can "quotient out" the non-Hausdorff nature of X_F. Define the equivalence relation $\sim$ on X_F by declaring $x \sim y$ if we have $F(x, z) = F(y, z)$ for all z outside of a set of the first category in X. This is clearly an equivalence relation, and so we can create the quotient space $Y := X_F / \sim$; since X embeds into X_F we thus have a factor map $\pi : X \to Y$. It is a deep fact (which we will not prove here) that this quotient space is non-trivial and Hausdorff, and that $\sim$ is preserved by the shift T and even by the Ellis group G (thus if $x \sim y$ and $g \in G$ then $gx \sim gy$. Because of this, G continues to act on Y homeomorphically, and so by Exercise 2 from the previous lecture, $\pi : X \to Y$ is a non-trivial isometric factor of X as desired.

Exercise 7. Show that in the case of the skew shift (Example 2), this construction recovers the factor that was discussed just before Theorem 1. (The trickiness of this exercise should already give you some idea of the difficulty level of Theorem 1.) $\diamond$

– The Furstenberg structure theorem for distal systems –

We have already noted that isometric systems are distal systems. More generally, we have

Exercise 8. Show that an isometric extension of a distal system is still distal. (Hint: Example 1 is a good model case.) $\diamond$

Thus, for instance, the iterated skew shifts that appear in (5) from the previous lecture are distal. Also, recall from Exercise 7 that the inverse limit of distal systems is again distal. It turns out that these are the *only* ways to generate distal systems, in the following sense:

Theorem 2. (Furstenberg's structure theorem for distal systems) Let $(X, \mathcal{F}, T)$ be a distal system. Then there exists an ordinal α and a factor Y_β for every $\beta \leq \alpha$ with the following properties:

1. $Y_\emptyset$ is a point.
2. For every successor ordinal $\beta + 1 \leq \alpha$, $Y_{\beta+1}$ is an isometric extension of Y_β.
3. For every limit ordinal $\beta \leq \alpha$, Y_β is an inverse limit of the Y_γ for $\gamma < \beta$.
4. Y_α is equal to X.

The collection of factors $(Y_\beta)_{\beta\leq\alpha}$ is sometimes known as a "Furstenberg tower".

Theorem 2 follows by applying Zorn's lemma with the following key proposition:

Proposition 1. (Key inductive step) Let $(X, \mathcal{F}, T)$ be a distal system, and let Y be a proper factor of X (i.e. the factor map is not an isomorphism). Then there exists another factor Z of X which is a proper isometric extension of Y.

Note that Theorem 1 is the special case of Proposition 1 when Y is a point. Indeed, Proposition 1 is proven in the same way as Theorem 1, but with several additional technicalities which I will not discuss here; see the original paper of Furstenberg for details.

Exercise 9. Deduce Theorem 2 from Proposition 1 and Zorn's lemma. $\diamond$
Remark 1. It is known that in Theorem 2, one can take the ordinal α to be countable, and conversely that for every countable ordinal α, there exists a system whose smallest Furstenberg tower has height α. $\diamond$

Remark 2. Several generalisations and extensions of Furstenberg's structure theorem are known, but they are somewhat technical to state and will not be detailed here; see this survey of Glasner for a discussion. $\diamond$

– Weak mixing and isometric factors –

We have seen that distal systems always contain non-trivial isometric factors. What about more general systems? It turns out that there is in fact a nice dichotomy between systems with non-trivial isometric factors, and those without.

Definition 2. (Topological transitivity) A topological dynamical system $(X, \mathcal{F}, T)$ is *topologically transitive* if, for every pair U, V of non-empty open sets, there exists an integer n such that $T^n U \cap V \neq \emptyset$.

Exercise 10. Show that a topological dynamical system is topologically transitive if and only if it is equal to the orbit closure of one of its points. (Compare this with minimal systems, which is the orbit closure of *any* of its points. Thus minimality is stronger than topological transitivity; for instance, the compactified integers $\{-\infty\} \cup \mathbb{Z} \cup \{+\infty\}$ with the usual shift is topologically transitive but not minimal.) $\diamond$

Exercise 11. Show that any factor of a topologically transitive system is again topologically transitive. $\diamond$

Definition 3. (Topological weak mixing) A topological dynamical system $(X, \mathcal{F}, T)$ is *topologically weakly mixing* if the product system $X \times X$ is topologically transitive.

Exercise 12. A system is said to be *topologically mixing* if for every pair U, V of non-empty open sets, one has $T^n U \cap V \neq \emptyset$ for all sufficiently large n. Show that topological mixing implies topological weak mixing. (The converse is false, but actually constructing a counterexample is somewhat tricky.) $\diamond$

Example 3. No circle shift $(\mathbb{R}/\mathbb{Z}, x \mapsto x + \alpha)$ is topologically weak mixing (or topologically mixing), even though such shifts are minimal (and hence transitive) when α is irrational. On the other hand, any Bernoulli shift is easily seen to be topologically mixing (and hence topologically weak mixing). $\diamond$

We have the following dichotomy, first proven by Keynes and Robertson (using ideas from the above-mentioned paper of Furstenberg):

Theorem 3. (Dichotomy between structure and randomness) Let $(X, \mathcal{F}, T)$ be a minimal topological dynamical system. Then exactly one of the following statements is true:

1. (Structure) X has a non-trivial isometric factor.
2. (Randomness) X is topologically weakly mixing.

Remark 3. Combining this with Exercise 6 from the previous lecture, we obtain an equivalent formulation of this theorem: a minimal system is topologically weakly mixing if and only if it has no non-trivial eigenfunctions. $\diamond$

Proof. We first prove the easy direction: that if X has a non-trivial isometric factor, then it is not topologically weakly mixing. In view of Exercise 11, it suffices to prove this when X itself is isometric. Let x, x' be two distinct points of Y, let r denote the distance between x and x' with respect to the metric that makes X isometric, and let B and B' be the open balls of radius r/10 centred at x and x' respectively. As X is isometric, we see for any integer n that $T^n B$ cannot intersect both B and B', or equivalently that $(T \times T)^n (B \times B)$ cannot intersect $B \times B'$. Thus X is not topologically transitive as desired.

Now we prove the difficult direction: if X is not topologically weakly mixing, then it has a non-trivial isometric factor. For this we use an argument of Blanchard, Host, and Maass, based on earlier work of McMahon. By Definition 3, there exist open non-empty sets U, V in $X \times X$ such that $(T \times T)^n U \cap V = \emptyset$ for all n. If we thus set $K := \overline{\bigcup_n (T \times T)^n U}$, we see that K is a compact proper $T \times T$-invariant subset of $X \times X$ with non-empty interior. On the other hand, the projection of K to either factor of $X \times X$ is a non-empty compact invariant subset of X and thus must be all of X.

We need to somehow use K to build an isometric factor of X. For this, we shall move from the topological dynamics setting to that of the ergodic theory setting. By Corollary 1 in the appendix, X admits an invariant Borel measure μ. The support of μ is a non-empty closed invariant subset of X, and is thus equal to all of X by minimality.

The space $L^1(X, \mu)$ is a metric space, with an isometric shift map $Tf := f \circ T^{-1}$. We define the map $\pi : X \to L^1(X, \mu)$ by the formula

$$\pi(x) : y \mapsto 1_K(x, y) \quad (1)$$

for all $x \in X$, where 1_K is the indicator function of K. Because K has non-empty interior and non-empty exterior, and because μ has full support, it is not hard to show that π is non-constant. By the T-invariance of W, it also preserves the shift T. So if we can show that π is continuous, we see that $\pi(X)$ will be a non-trivial isometric factor of X and we will be done.

Let us first consider the scalar function $f(x) := \int_X 1_K(x,y)\ d\mu(y)$. From the dominated convergence theorem and the fact that K is closed, we see that f is upper semi-continuous, and continuous at at least one point, thanks to Lemma 3 from Lecture 4. On the other hand, since K is $T \times T$-invariant and μ is T-invariant, we see that f is T-invariant. Applying Exercise 15 from Lemma 4 we see that f is constant. On the other hand, as K is closed we have $\limsup_{x \to x_0} 1_K(x,y) \leq 1_K(x_0,y)$ for any $x_0 \in X$, and so by dominated convergence again we see that $1_K(x,\cdot)$ converges in L^1 to zero outside of the support of $1_K(x_0,\cdot)$. Combining this with the constancy of f we conclude that $1_K(x,\cdot)$ converges to $1_K(x_0,\cdot)$ in L^1 on all of X, and thus π is continuous as required. $\square$

Remark 4. Note how the measure-theoretic structure was used to obtain metric structure, by passing from the measure space (X,μ) to the metric space $L^1(X,\mu)$. This again shows that one can sometimes upgrade weak notions of structure (such as topological or measure-theoretic structure) to strong notions (such as geometric or algebraic structure). $\diamond$

Exercise 13. Use Theorem 3 to prove Theorem 1. (Hint: use Exercise 10.) $\diamond$

Remark 5. It would be very convenient if one had a relative version of Theorem 3, namely that if X is an extension of Y, then X is either relatively topologically weakly mixing with respect to Y (which means that the relative product $X \times_Y X := \{(x,x') \in X \times X: \pi(x) = \pi(x')\}$ is topologically transitive), or else X has a factor Z which is a non-trivial isometric extension of Y; among other things, this would have given a new proof of Theorem 2, and in fact establish a somewhat stronger structural theorem. Unfortunately, this relative version fails; a counterexample (based on the Morse sequence) can be found in Exercise 1.19.3 of Glasner's book. Nevertheless, the analogue of this claim does hold true in the measure-theoretic setting, as we shall see in a few lectures. $\diamond$

– Appendix: sequential compactness of Borel probability measures –

We now recall some standard facts from measure theory about Borel probability measures on a compact metrisable space X. Recall that a sequence of such measures μ_n converges in the vague topology to another μ if we have $\int_X f\ d\mu_n \to \int_X f\ d\mu$ for all $f \in C(X)$.

Lemma 1. (Vague sequential compactness) The space $\Pr(X)$ of Borel probability measures on X is sequentially compact in the vague topology.

Proof. From the Stone-Weierstrass theorem we know that C(X) is separable. The claim then follows from Riesz representation theorem and the usual Arzelà-Ascoli diagonalisation argument. $\square$

Corollary 1. (Krylov-Bogolubov theorem) Let $(X,\mathcal{F},T)$ be a topological dynamical system. Then there exists a T-invariant probability measure μ on X.

Proof. Pick any point $x_0 \in X$ and consider the finite probability measures

$$\mu_N := \frac{1}{N} \sum_{n=1}^N \delta_{T^n x_0} \quad (1)$$

where δ_x is the Dirac mass at x. By Lemma 1, some subsequence μ_{N_j} converges in the vague topology to another Borel probability measure μ. Since we have

$$\int Tf\ d\mu_N = \int f\ d\mu_N + O_f(1/N) \quad (2)$$

for all bounded continuous f, we conclude on taking vague limits and using the Riesz representation theorem that μ is T-invariant as required. □

Remark 6. Note that Corollary 1, like many other results obtained via compactness methods, guarantees existence of an invariant measure but not uniqueness (this latter property is known as *unique ergodicity*). Even for minimal systems, it is possible for uniqueness to fail, although actually constructing an example is tricky (see for instance this paper of Furstenberg). However, as already observed in the proof of Theorem 3, any invariant measure on a minimal topological dynamical system must be *full* (i.e. its support must be the whole space). ◇

Exercise 14. Show that any topological dynamical system which is uniquely ergodic is necessarily minimal. ◇

6 comments

Comments feed for this article

28 January, 2008 at 10:31 pm

Made Eka

I use dynamical system to solve my problem in neurology. But I dont understant this lecture. Unlucky me n glad to know you.

Sucsess,
Made Eka

5 February, 2008 at 11:49 am

254A, Lecture 9: Ergodicity « What's new

[…] if and only if the only T-invariant Borel probability measure on T is . (Hint: use Lemma 1 from Lecture 7.) Because of this fact, one can sensibly define what it means for a topological dynamical system […]

10 February, 2008 at 5:20 pm

254A, Lecture 10: The Furstenberg correspondence principle « What's new

254A, Lecture 8: The mean ergodic theorem

30 January, 2008 in 254A - ergodic theory, math.DS

Tags: conditional expectation, dual functions, Hilbert spaces, mean ergodic theorem, spectral theorem, von Neumann ergodic theorem

We now begin our study of *measure-preserving systems* $(X, \mathcal{X}, \mu, T)$, i.e. a probability space $(X, \mathcal{X}, \mu)$ together with a probability space isomorphism $T : (X, \mathcal{X}, \mu) \to (X, \mathcal{X}, \mu)$ (thus $T : X \to X$ is invertible, with T and T^{-1} both being measurable, and $\mu(T^n E) = \mu(E)$ for all $E \in \mathcal{X}$ and all n). For various technical reasons it is convenient to restrict to the case when the σ-algebra $\mathcal{X}$ is separable, i.e. countably generated. One reason for this is as follows:

Exercise 1. Let $(X, \mathcal{X}, \mu)$ be a probability space with $\mathcal{X}$ separable. Then the Banach spaces $L^p(X, \mathcal{X}, \mu)$ are separable (i.e. have a countable dense subset) for every $1 \leq p < \infty$; in particular, the Hilbert space $L^2(X, \mathcal{X}, \mu)$ is separable. Show that the claim can fail for $p = \infty$. (We allow the L^p spaces to be either real or complex valued, unless otherwise specified.) $\diamond$

Remark 1. In practice, the requirement that $\mathcal{X}$ be separable is not particularly onerous. For instance, if one is studying the recurrence properties of a function $f : X \to \mathbb{R}$ on a non-separable measure-preserving system $(X, \mathcal{X}, \mu, T)$, one can restrict $\mathcal{X}$ to the separable sub-σ-algebra $\mathcal{X}'$ generated by the level sets $\{x \in X : T^n f(x) > q\}$ for integer n and rational q, thus passing to a separable measure-preserving system $(X, \mathcal{X}', \mu, T)$ on which f is still measurable. Thus we see that in many cases of interest, we can immediately reduce to the separable case. (In particular, for many of the theorems in this course, the hypothesis of separability can be dropped, though we won't bother to specify for which ones this is the case.) $\diamond$

We are interested in the recurrence properties of sets $E \in \mathcal{X}$ or functions $f \in L^p(X, \mathcal{X}, \mu)$. The simplest such recurrence theorem is

> **Theorem 1.** (Poincaré recurrence theorem) Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let $E \in \mathcal{X}$ be a set of positive measure. Then $\limsup_{n \to +\infty} \mu(E \cap T^n E) \geq \mu(E)^2$. In particular, $E \cap T^n E$ has positive measure (and is thus non-empty) for infinitely many n.

(Compare with Theorem 1 of Lecture 3.)

Proof. For any integer $N > 1$, observe that $\int_X \sum_{n=1}^N 1_{T^n E}\ d\mu = N\mu(E)$, and thus by Cauchy-Schwarz

$$\int_X (\sum_{n=1}^N 1_{T^n E})^2\ d\mu \geq N^2 \mu(E)^2. \quad (1)$$

The left-hand side of (1) can be rearranged as

$$\sum_{n=1}^N \sum_{m=1}^N \mu(T^n E \cap T^m E). \quad (2)$$

On the other hand, $\mu(T^n E \cap T^m E) = \mu(E \cap T^{m-n} E)$. From this one easily obtains the asymptotic

$$(2) \leq (\limsup_{n\to\infty} \mu(E \cap T^n E) + o(1)) N^2, \quad (3)$$

where o(1) denotes an expression which goes to zero as N goes to infinity. Combining (1), (2), (3) and taking limits as $N \to +\infty$ we obtain

$$\limsup_{n\to\infty} \mu(E \cap T^n E) \geq \mu(E)^2. \quad (4)$$

By shift-invariance we have $\mu(E \cap T^{-n} E) = \mu(E \cap T^n E)$, and the claim follows. □

Remark 2. In classical physics, the evolution of a physical system in a compact phase space is given by a (continuous-time) measure-preserving system (this is Hamilton's equations of motion combined with Liouville's theorem). The Poincaré recurrence theorem then has the following unintuitive consequence: every collection E of states of positive measure, no matter how small, must eventually return to overlap itself given sufficient time. For instance, if one were to burn a piece of paper in a closed system, then there exist arbitrarily small perturbations of the initial conditions such that, if one waits long enough, the piece of paper will eventually reassemble (modulo arbitrarily small error)! This seems to contradict the second law of thermodynamics, but the reason for the discrepancy is because the time required for the recurrence theorem to take effect is inversely proportional to the measure of the set E, which in physical situations is exponentially small in the number of degrees of freedom (which is already typically quite large, e.g. of the order of the Avogadro constant). This gives more than enough opportunity for Maxwell's demon to come into play to reverse the increase of entropy. (This can be viewed as a manifestation of the curse of dimensionality.) The more sophisticated recurrence theorems we will see later have much poorer quantitative bounds still, so much so that they basically have no direct significance for any physical dynamical system with many relevant degrees of freedom. ◇

Exercise 2. Prove the following generalisation of the Poincaré recurrence theorem: if $(X, \mathcal{X}, \mu, T)$ is a measure-preserving system and $f \in L^1(X, \mathcal{X}, \mu)$ is non-negative, then $\limsup_{n\to+\infty} \int_X f T^n f \geq (\int_X f \, d\mu)^2$. ◇

Exercise 3. Give examples to show that the quantity $\mu(X)^2$ in the conclusion of Theorem 1 cannot be replaced by any smaller quantity in general, regardless of the actual value of $\mu(X)$. (Hint: use a Bernoulli system example.) ◇

Exercise 4. Using the pigeonhole principle instead of the Cauchy-Schwarz inequality (and in particular, the statement that if $\mu(E_1) + \ldots + \mu(E_n) > 1$, then the sets $E_1, \ldots, E_n$ cannot all be disjoint), prove the weaker statement that for any set E of positive measure in a measure-preserving system, the set $E \cap T^n E$ is non-empty for infinitely many n. (This exercise illustrates the general point that the Cauchy-Schwarz inequality can be viewed as a quantitative strengthening of the pigeonhole principle.) ◇

For this lecture and the next we shall study several variants of the Poincaré recurrence theorem. We begin by looking at the mean ergodic theorem, which studies the limiting behaviour of the ergodic averages $\frac{1}{N}\sum_{n=1}^N T^n f$ in various L^p spaces, and in particular in L^2.

— Hilbert space formulation –

We begin with the Hilbert space formulation of the mean ergodic theorem, due to von Neumann.

Theorem 2. (Von Neumann ergodic theorem) Let $U : H \to H$ be a unitary operator on a separable Hilbert space H. Then for every $v \in H$ we have

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{n=0}^{N-1} U^n v = \pi(v), \quad (5)$$

where $\pi : H \to H^U$ is the orthogonal projection from H to the closed subspace and let $H^U := \{v \in H : Uv = v\}$ consisting of the U-invariant vectors.

Proof. We give the slick (but not particularly illuminating) proof of von Neumann. It is clear that (5) holds if v is already invariant (i.e. $v \in H^U$). Next, let W denote the (possibly non-closed) space $W := \{Uw - w : w \in H\}$. If Uw-w lies in W and v lies in H^U, then by unitarity

$$\langle Uw - w, v\rangle = \langle w, U^{-1}v\rangle - \langle w, v\rangle = \langle w, v\rangle - \langle w, v\rangle = 0 \quad (6)$$

and thus W is orthogonal to H^U. In particular $\pi(Uw - w) = 0$. From the telescoping identity

$$\frac{1}{N}\sum_{n=0}^{N-1} U^n(Uw - w) = \frac{1}{N}(U^N w - w) \quad (7)$$

we conclude that (5) also holds if $v \in W$; by linearity we conclude that (5) holds for all v in $H^U + \overline{W}$. A standard limiting argument (using the fact that the linear transformations $v \mapsto \pi(v)$ and $v \mapsto \frac{1}{N}\sum_{n=0}^{N-1} U^n v$ are bounded on H, uniformly in n) then shows that (5) holds for v in the closure $\overline{H^U + W}$.

To conclude, it suffices to show that the closed space $\overline{H^U + W}$ is all of H. Suppose for contradiction that this is not the case. Then there exists a non-zero vector w which is orthogonal to all of $\overline{H^U + W}$. In particular, w is orthogonal to Uw - w. Applying the easily verified identity $\|Uw - w\|^2 = -\langle Uw - w, w\rangle$ (related to the parallelogram law) we conclude that Uw=w, thus w lies in H^U. This implies that w is orthogonal to itself and is thus zero, a contradiction. $\Box$

On a measure-preserving system $(X, \mathcal{X}, \mu, T)$, the shift map $f \mapsto Tf$ is a unitary transformation on the separable Hilbert space $L^2(X, \mathcal{X}, \mu)$. We conclude

Corollary 1. (mean ergodic theorem) Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let $f \in L^2(X, \mathcal{X}, \mu)$. Then we have $\frac{1}{N}\sum_{n=1}^{N} T^n f$ converges in $L^2(X, \mathcal{X}, \mu)$ norm to $\pi(f)$, where $\pi(f) : L^2(X, \mathcal{X}, \mu) \to L^2(X, \mathcal{X}, \mu)^T$ is the orthogonal projection to the space $\{f \in L^2(X, \mathcal{X}, \mu) : Tf = f\}$ consists of the shift-invariant functions in $L^2(X, \mathcal{X}, \mu)$.

Example 4. (Finite case) Suppose that $(X, \mathcal{X}, \mu, T)$ is a finite measure-preserving system, with $\mathcal{X}$ discrete and μ the uniform probability measure. Then T is a permutation on X and thus decomposes as the direct sum of disjoint cycles (possibly including trivial cycles of length 1). Then the shift-invariant functions are precisely those functions which are constant on each of these cycles, and the map $f \mapsto \pi(f)$ replaces a function $f : X \to \mathbb{C}$ with its average value on each of these cycles. It is then an instructive exercise to verify the mean ergodic theorem by hand in this case. $\diamond$

Exercise 5. With the notation and assumptions of Corollary 1, show that the limit $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \int_X T^n f \overline{f}\ d\mu$ exists, is real, and is greater than or equal to $(\int_X f)^2$. (Hint: the constant function 1 lies in $L^2(X, \mathcal{X}, \mu)^T$.) Note that this is stronger than the conclusion of Exercise 2. $\diamond$

Let us now give some other proofs of the von Neumann ergodic theorem. We first give a proof using the spectral theorem for unitary operators. This theorem asserts (among other things) that a unitary operator $U : H \to H$ can be expressed in the form $U = \int_{S^1} \lambda\ d\mu(\lambda)$, where $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle and μ is a projection-valued Borel measure on the circle. More generally, we have

$$U^n = \int_{S^1} \lambda^n \ d\mu(\lambda) \quad (8)$$

and so for any vector v in H and any positive integer N

$$\frac{1}{N}\sum_{n=0}^{N-1} U^n v = \int_{S^1} \frac{1}{N}\sum_{n=0}^{N-1} \lambda^n \ d\mu(\lambda) v. \quad (9)$$

We separate off the $\lambda = 1$ portion of this integral. For $\lambda \neq 1$, we have the geometric series formula

$$\frac{1}{N}\sum_{n=0}^{N-1} \lambda^n = \frac{1}{N}\frac{\lambda^N - 1}{\lambda - 1} \quad (10)$$

(compare with (7)), thus we can rewrite (9) as

$$\mu(\{1\})v + \int_{S^1\backslash\{1\}} \frac{1}{N}\frac{\lambda^N - 1}{\lambda - 1} \ d\mu(\lambda) v. \quad (11)$$

Now observe (using (10)) that $\frac{1}{N}\frac{\lambda^N - \lambda}{\lambda - 1}$ is bounded in magnitude by 1 and converges to zero as $N \to \infty$ for any fixed $\lambda \neq 1$. Applying the dominated convergence theorem (which requires a little bit of justification in this vector-valued case), we see that the second term in (11) goes to zero as $N \to \infty$. So we see that (9) converges to $\mu(\{1\})v$. But $\mu(\{1\})$ is just the orthogonal projection to the eigenspace of U with eigenvalue 1, i.e. the space H^U, thus recovering the von Neumann ergodic theorem. (It is instructive to use spectral theory to interpret von Neumann's proof of this theorem and see how it relates to the argument just given.)

Remark 3. The above argument in fact shows that the rate of convergence in the von Neumann ergodic theorem is controlled by the spectral gap of U - i.e. how well-separated the trivial component $\{1\}$ of the spectrum is from the rest of the spectrum. This is one of the reasons why results on spectral gaps of various operators are highly prized. $\diamond$

We now give another proof of Theorem 2, based on the *energy decrement method*; this proof is significantly lengthier, but is particularly well suited for conversion to finitary quantitative settings. For any positive integer N, define the averaging operators $A_N := \frac{1}{N}\sum_{n=0}^{N-1} U^n$; by the triangle inequality we see that $\|A_N v\| \leq \|v\|$ for all v. Now we observe

Lemma 1. (Lack of uniformity implies energy decrement) Suppose $\|A_N v\| \geq \varepsilon$. Then $\|v - A_N^* A_N v\|^2 \leq \|v\|^2 - \varepsilon^2$.

Proof. This follows from the identity

$$\|v - A_N^* A_N v\|^2 = \|v\|^2 - 2\|A_N v\|^2 + \|A_N^* A_N v\|^2 \quad (12)$$

and the fact that A_N^* has operator norm at most 1. $\square$

We now iterate this to obtain

Proposition 1. (Koopman-von Neumann type theorem) Let v be a unit vector, let $\varepsilon > 0$, and let $1 < N_1 < N_2 < \ldots < N_J$ be a sequence of integers with $J > 1/\varepsilon^2 + 2$. Then there exists $1 \leq j < J$ and a decomposition $v = s + r$ where $\|Us - s\| = O(J\frac{1}{N_{j+1}})$ and $\|A_N r\| \leq \varepsilon$ for all $N \geq N_j$.

(The letters s, r stand for "structured" and "random" (or "residual") respectively. For more on decompositions into structured and random components, see my FOCS lecture notes.)

Proof. We perform the following algorithm:

1. Initialise j := J-1, s := 0, and r := v.
2. If $\|A_N r\| \leq \varepsilon$ for all $N \geq N_j$ then STOP. If instead $\|A_N r\| > \varepsilon$ for some $N \geq N_j$, observe from Lemma 1 that $\|r - A_N^* A_N r\|^2 \leq \|r\|^2 - \varepsilon^2$.
3. Replace r with $r - A_N^* A_N r$, replace s with $s + A_N^* A_N r$, and replace j with j-1. Then return to Step 2.

Observe that this procedure must terminate in at most $1/\varepsilon^2$ steps (since the energy $\|r\|^2$ starts at 1, drops by at least ε^2 at each stage, and cannot go below zero). In particular, j stays positive. Observe also that r always has norm at most 1, and thus $\|(U-I)A_N^* A_N r\| = O(1/N)$ at any given stage of the algorithm. From this and the triangle inequality one easily verifies the required claims. □

Corollary 2 (partial von Neumann ergodic theorem). For any vector v, the averages $A_N v$ form a Cauchy sequence in H.

Proof. Without loss of generality we can take v to be a unit vector. Suppose for contradiction that $A_N v$ was not Cauchy. Then one could find $\varepsilon > 0$ and $1 < N_1 < M_1 < N_2 < M_2 < \ldots$ such that $\|A_{N_j} v - A_{M_j} v\| \geq 5\varepsilon$ (say) for all j. By sparsifying the sequence if necessary we can assume that N_{j+1} is large compared to N_j, M_j and ε. Now we apply Proposition 1 to find $j = O_\varepsilon(1)$ and a decomposition v = s + r such that $\|Us - s\| = O_\varepsilon(1/N_{j+1})$ and $\|A_{N_j} r\|, \|A_{M_j} r\| \leq \varepsilon$. If N_{j+1} is large enough depending on N_j, M_j, ε, we thus have $\|A_{N_j} s - s\|, \|A_{M_j} s - s\| \leq \varepsilon$, and thus by the triangle inequality, $\|A_{N_j} v - A_{M_j} v\| \leq 4\varepsilon$, a contradiction. ◇

Remark 4. This result looks weaker than Theorem 2, but the argument is much more robust; for instance, one can modify it to establish convergence of multiple averages such as $\frac{1}{N}\sum_{n=1}^N T_1^n f_1 T_2^n f_2 T_3^n f_3$ in L^p norms for commuting shifts T_1, T_2, T_3, which does not seem possible using the other arguments given here; see this paper of mine for details. Further quantitative analysis of the mean ergodic theorem can be found in this paper of Avigad, Gerhardy, and Towsner. ◇

Corollary 2 can be used to recover Theorem 2 in its full strength, by combining it with a weak form of Theorem 2:

Proposition 2 (Weak von Neumann ergodic theorem) The conclusion (5) of Theorem 2 holds in the weak topology.

Proof. The averages $A_N v$ lie in a bounded subset of the separable Hilbert space H, and are thus precompact in the weak topology by the sequential Banach-Alaoglu theorem. Thus, if (5) fails, then there exists a subsequence $A_{N_j} v$ which converges in the weak topology to some limit w other than $\pi(w)$. By telescoping series we see that $\|UA_{N_j} v - A_{N_j} v\| \leq 2\|v\|/N_j$, and so on taking limits we see that $\|Uw - w\| = 0$, i.e. $w \in H^U$. On the other hand, if y is any vector in H^U, then $A_{N_j}^* y = y$, and thus on taking inner products with v we obtain $\langle y, A_{N_j} v\rangle = \langle y, v\rangle$. Taking limits we obtain $\langle y, w\rangle = \langle y, v\rangle$, i.e. v-w is orthogonal to H^U. These facts imply that $w = \pi(v)$, giving the desired contradiction. □

– Conditional expectation –

We now turn away from the abstract Hilbert approach to the ergodic theorem (which is excellent for proving the mean ergodic theorem, but not flexible enough to handle more general ergodic theorems) and turn to a more measure-theoretic dynamics approach, based on manipulating the four components $X, \mathcal{X}, \mu, T$ of the underlying system separately, rather than working with the single object $L^2(X, \mathcal{X}, \mu)$ (with the unitary shift T). In particular it is useful to replace the σ-algebra $\mathcal{X}$ by a sub-σ-algebra $\mathcal{X}' \subset \mathcal{X}$, thus reducing the number of measurable functions. This creates an isometric embedding of Hilbert spaces

$$L^2(X, \mathcal{X}', \mu) \subset L^2(X, \mathcal{X}, \mu) \quad (13)$$

and so the former space is a closed subspace of the latter. In particular, we have an orthogonal projection $\mathbb{E}(\cdot|\mathcal{X}') : L^2(X, \mathcal{X}, \mu) \to L^2(X, \mathcal{X}', \mu)$, which can be viewed as the adjoint of the inclusion (13). In other words, for any $f \in L^2(X, \mathcal{X}, \mu)$, $\mathbb{E}(f|\mathcal{X}')$ is the unique element of $L^2(X, \mathcal{X}', \mu)$ such that

$$\int_X \mathbb{E}(f|\mathcal{X}')\overline{g}\ d\mu = \int_X f\overline{g}\ d\mu \quad (14)$$

for all $g \in L^2(X, \mathcal{X}', \mu)$. (A reminder: when dealing with L^p spaces, we identify any two functions which agree μ-almost everywhere. Thus, technically speaking, elements of L^p spaces are not actually functions, but rather equivalence classes of functions.)

Example 5. (Finite case) Let X be a finite set, thus $\mathcal{X}$ can be viewed as a partition of X, and $\mathcal{X}' \subset \mathcal{X}$ is a coarser partition of X. To avoid degeneracies, assume that every point in X has positive measure with respect to μ. Then an element f of $L^2(X, \mathcal{X}, \mu)$ is just a function $f : X \to \mathbb{C}$ which is constant on each atom of $\mathcal{X}$. Similarly for $L^2(X, \mathcal{X}', \mu)$. The conditional expectation $\mathbb{E}(f|\mathcal{X}')$ is then the function whose value on each atom A of $\mathcal{X}'$ is equal to the average value $\frac{1}{\mu(A)}\int_A f\ d\mu$ on that atom. (What needs to be changed here if some points have zero measure?) $\diamond$

We leave the following standard properties of conditional expectation as an exercise.

Exercise 6. Let $(X, \mathcal{X}, \mu)$ be a probability space, and let $\mathcal{X}'$ be a sub-σ-algebra. Let $f \in L^2(X, \mathcal{X}, \mu)$.

1. The operator $f \mapsto \mathbb{E}(f|\mathcal{X}')$ is a bounded self-adjoint projection on $L^2(X, \mathcal{X}, \mu)$. It maps real functions to real functions, it preserves constant functions (and more generally preserves $\mathcal{X}'$-valued functions), and commutes with complex conjugation.
2. If f is non-negative, then $\mathbb{E}(f|\mathcal{X}')$ is non-negative (up to sets of measure zero, of course). More generally, we have a *comparison principle*: if f, g are real-valued and $f \leq g$ pointwise a. e., then $\mathbb{E}(f|\mathcal{X}') \leq \mathbb{E}(g|\mathcal{X}')$ a.e. Similarly, we have the *triangle inequality* $|\mathbb{E}(f|\mathcal{X}')| \leq \mathbb{E}(|f||\mathcal{X}')$ a.e..
3. (Module property) If $g \in L^\infty(X, \mathcal{X}', \mu)$, then $\mathbb{E}(fg|\mathcal{X}') = \mathbb{E}(f|\mathcal{X}')g$ a.e..
4. (Contraction) If $f \in L^2(X, \mathcal{X}, \mu) \cap L^p(X, \mathcal{X}, \mu)$ for some $1 \leq p \leq \infty$, then $\|\mathbb{E}(f|\mathcal{X}')\|_{L^p} \leq \|f\|_{L^p}$. (Hint: do the p=1 and $p = \infty$ cases first.) This implies in particular that conditional expectation has a unique continuous extension to $L^p(X, \mathcal{X}, \mu)$ for $1 \leq p \leq \infty$ (the $p = \infty$ case is exceptional, but note that L^∞ is contained in L^2 since μ is finite). $\diamond$

For applications to ergodic theory, we will only be interested in taking conditional expectations with respect to a *shift-invariant* sub-σ-algebra $\mathcal{X}'$, thus T and T^{-1} preserve $\mathcal{X}'$. In that case T preserves $L^2(X, \mathcal{X}', \mu)$, and thus T commutes with conditional expectation, or in other words that

$$\mathbb{E}(T^n f|\mathcal{X}') = T^n\mathbb{E}(f|\mathcal{X}') \quad (15)$$

a.e. for all $f \in L^2(X, \mathcal{X}, \mu)$ and all n.

Now we connect conditional expectation to the mean ergodic theorem. Let $\mathcal{X}^T := \{E \in \mathcal{X} : TE = E \hbox{ a.e.}\}$ be the set of essentially shift-invariant sets. One easily verifies that this is a shift-invariant sub-σ-algebra of $\mathcal{X}$.

Exercise 7. Show that if E lies in $\mathcal{X}^T$, then there exists a set $F \in \mathcal{X}$ which is genuinely invariant (TF=F) and which differs from E only by a set of measure zero. Thus it does not matter whether we deal with shift-invariance or essential shift-invariance here. (More generally, it will not make any significant difference if we modify any of the sets in our σ-algebras by null sets.) $\diamond$

The relevance of this algebra to the mean ergodic theorem arises from the following identity:

Exercise 8. Show that $L^2(X, \mathcal{X}, \mu)^T = L^2(X, \mathcal{X}^T, \mu)$. $\diamond$

As a corollary of this and Corollary 1, we have

> **Corollary 2.** (Mean ergodic theorem, again) Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. Then for any $f \in L^2(X, \mathcal{X}, \mu)$, the averages $\frac{1}{N}\sum_{n=0}^{N-1} T^n f$ converge in L^2 norm to $\mathbb{E}(f|\mathcal{X}^T)$.

Exercise 9. Show that Corollary 2 continues to hold if L^2 is replaced throughout by L^p for any $1 \leq p < \infty$. (Hint: for the case $p < 2$, use that L^2 is dense in L^p. For the case \$p>2\$, use that L^∞ is dense in L^p.) What happens when $p = \infty$? $\diamond$

Let us now give another proof of Corollary 2 (leading to a fourth proof of the mean ergodic theorem). The key here will be the decomposition $f = f_{U^\perp} + f_U$, where $f_{U^\perp} := \mathbb{E}(f|\mathcal{X}^T)$ is the "structured" part of f (at least as far as the mean ergodic theorem is concerned) and $f_U := f - f_{U^\perp}$ is the "random" part. (The subscripts $U^\perp, U$ stand for "anti-uniform" and "uniform" respectively; this notation is not standard.) As $f_{U^\perp}$ is shift-invariant, we clearly have

$$\frac{1}{N}\sum_{n=0}^{N-1} T^n f_{U^\perp} = f_{U^\perp} \quad (16)$$

so it suffices to show that

$$\|\frac{1}{N}\sum_{n=0}^{N-1} T^n f_U\|_{L^2}^2 \to 0 \quad (17)$$

as $N \to \infty$. But we can expand out the left-hand side (using the unitarity of T) as

$$\langle F_N, f_U \rangle := \int_X F_N \overline{f_U}\ d\mu \quad (18)$$

where F_N is the *dual function* of f_U, defined as

$$F_N := \frac{1}{N^2}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} T^{n-m} f_U. \quad (19)$$

Now, from the triangle inequality we know that the sequence of dual functions F_N is uniformly bounded in L^2 norm, and so by Cauchy-Schwarz we know that the inner products $\langle F_N, f_U \rangle$ are bounded. If they converge to zero, we are done; otherwise, by the Bolzano-Weierstrass theorem, we have $\langle F_{N_j}, f_U \rangle \to c$ for some subsequence N_j and some non-zero c.

(One could also use ultrafilters instead of subsequences here if desired, it makes little difference to the argument.) By the Banach-Alaoglu theorem (or more precisely, the sequential version of this in the separable case), there is a further subsequence $F_{N'_j}$ which converges weakly (or equivalently in this Hilbert space case, in the weak-* sense) to some limit $F_\infty \in L^2(X, \mathcal{X}, \mu)$. Since c is non-zero, F_∞ must also be non-zero. On the other hand, from telescoping series one easily computes that $\|TF_N - F_N\|_{L^2}$ decays like O(1/N) as $N \to \infty$, so on taking limits we have $TF_\infty - F_\infty = 0$. In other words, F_∞ lies in $L^2(X, \mathcal{X}^T, \mu)$.

On the other hand, by construction of f_U we have $\mathbb{E}(f_U|\mathcal{X}^T) = 0$. From (15) and linearity we conclude that $\mathbb{E}(F_N|\mathcal{X}^T) = 0$ for all N, so on taking limits we have $\mathbb{E}(F_\infty|\mathcal{X}^T) = 0$. But since F_∞ is already in $L^2(X, \mathcal{X}^T, \mu)$, we conclude $F_\infty = 0$, a contradiction.

Remark 5. This argument is lengthier than some of the other proofs of the mean ergodic theorem, but it turns out to be fairly robust; it demonstrates (using the compactness properties of certain "dual functions") that a function f_U with sufficiently strong "mixing" properties (in this case, we require that $\mathbb{E}(f_U|\mathcal{X}^T) = 0$) will cancel itself out when taking suitable ergodic averages, thus reducing the study of averages of f to the study of averages of $f_U = \mathbb{E}(f|\mathcal{X}^T)$. In the modern jargon, this means that $\mathcal{X}^T$ is (the σ-algebra induced by) a characteristic factor of the ergodic average $f \mapsto \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N T^n f$. We will see further examples of characteristic factors for other averages later in this course. $\diamond$

Exercise 10. Let $(\Gamma, \cdot)$ be a countably infinite discrete group. A *Følner sequence* is a sequence of increasing finite non-empty sets F_n in Γ with $\bigcup_n F_n = \Gamma$ with the property that for any given finite set $S \subset \Gamma$, we have $|(F_n \cdot S)\Delta F_n|/|F_n| \to 0$ as $n \to \infty$, where $F_n \cdot S := \{fs : f \in F_n, s \in S\}$ is the product set of F_n and S, $|F_n|$ denotes the cardinality of F_n, and Δ denotes symmetric difference. (For instance, in the case $\Gamma = \mathbb{Z}$, the sequence $F_n := \{-n, \ldots, n\}$ is a Følner sequence.) If Γ acts (on the left) in a measure-preserving manner on a probability space $(X, \mathcal{X}, \mu)$, and $f \in L^2(X, \mathcal{X}, \mu)$, show that $\frac{1}{|F_n|}\sum_{\gamma\in F_n} f \circ \gamma^{-1}$ converges in L^2 to $\mathbb{E}(f|\mathcal{X}^\Gamma)$, where $\mathcal{X}^\Gamma$ is the collection of all measurable sets which are Γ-invariant modulo null sets, and $f \circ \gamma^{-1}$ is the function $x \mapsto f(\gamma^{-1}x)$. $\diamond$

[*Update*, Jan 30: exercise corrected, another exercise added.]

[*Update*, Feb 1: Some corrections.]

[*Update*, Feb 4: Ergodic averages changed to sum over 0 to N-1 rather than over 1 to N.]

[*Update*, Feb 11: Discussion of the ergodic theorem in weak topologies (Proposition 2) added.]

[*Update*, Feb 21: Exercise 10 corrected.]

16 comments

Comments feed for this article

30 January, 2008 at 6:41 pm

Lior

254A, Lecture 9: Ergodicity

4 February, 2008 in 254A - ergodic theory, math.DS

Tags: disintegration, ergodic decomposition, ergodic theorem, ergodicity, maximal inequality, probability kernel

We continue our study of basic ergodic theorems, establishing the maximal and pointwise ergodic theorems of Birkhoff. Using these theorems, we can then give several equivalent notions of the fundamental concept of ergodicity, which (roughly speaking) plays the role in measure-preserving dynamics that minimality plays in topological dynamics. A general measure-preserving system is not necessarily ergodic, but we shall introduce the *ergodic decomposition*, which allows one to express any non-ergodic measure as an average of ergodic measures (generalising the decomposition of a permutation into disjoint cycles).

– The maximal ergodic theorem –

Just as we derived the mean ergodic theorem from the more abstract von Neumann ergodic theorem in the previous lecture, we shall derive the maximal ergodic theorem from the following abstract maximal inequality.

Theorem 1. (Dunford-Schwartz maximal inequality) Let $(X, \mathcal{X}, \mu)$ be a probability space, and let $P: L^1(X, \mathcal{X}, \mu) \to L^1(X, \mathcal{X}, \mu)$ be a linear operator with P1=1 and $P^*1 = 1$ (i.e. $\int_X Pf\ d\mu = \int_X f\ d\mu$ for all $f \in L^1(X, \mathcal{X}, \mu)$. Then the maximal function $Mf := \sup_{N>0} \frac{1}{N} \sum_{n=1}^N P^n f$ obeys the inequality

$$\lambda\mu(\{Mf > \lambda\}) \leq \int_{Mf>\lambda} f\ d\mu \quad (1)$$

for any $\lambda \in \mathbb{R}$.

Proof. We can rewrite (1) as

$$\int_{Mf-\lambda>0} (f-\lambda)\ d\mu \geq 0. \quad (2)$$

Since $Mf - \lambda = M(f-\lambda)$, we thus see (by replacing f with $f - \lambda$) that we can reduce to proving (2) in the case $\lambda = 0$.

For every $m \geq 1$, consider the modified maximal function $F_m := \sup_{0 \leq N \leq m} \sum_{n=0}^{N-1} P^n f$. Observe that $Mf(x) > 0$ if and only if $F_m(x) > 0$ for all sufficiently large m. By the dominated convergence theorem, it thus suffices to show that

$$\int_{F_m>0} f\ d\mu \geq 0 \quad (3)$$

for all m. But observe from definition of F_m (and the positivity preserving nature of P) that we have the pointwise recursive inequality

$$F_m(x) \leq F_{m+1}(x) = \max(0, f + PF_m(x)). \quad (4)$$

Integrating this on the region $F_m > 0$ and using the non-negativity of F_m, we obtain

$$\int_X F_m\ d\mu \leq \int_{F_m>0} f + \int_X PF_m\ d\mu. \quad (6)$$

Since $F_m \in L^1(X, \mathcal{X}, \mu)$ and $P^*1 = 1$, the claim follows. $\Box$

Applying this in the case when P is a shift operator, and replacing f by |f|, we obtain

Corollary 1. (Maximal ergodic theorem) Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. Then for any $f \in L^1(X, \mathcal{X}, \mu)$ and $\lambda > 0$ one has

$$\mu(\{\sup_N \frac{1}{N} \sum_{n=0}^{N-1} |T^n f| > \lambda\}) \leq \frac{1}{\lambda} \|f\|_{L^1(X,\mathcal{X},\mu)}. \quad (7)$$

Note that this inequality implies Markov's inequality

$$\mu(\{|f| > \lambda\}) \leq \frac{1}{\lambda} \int_X |f|\ d\mu. \quad (8)$$

as a special case. Applying the real interpolation method, one also easily deduces the maximal inequality

$$\| \sup_N \frac{1}{N} \sum_{n=0}^{N-1} |T^n f| \|_{L^p(X,\mathcal{X},\mu)} \leq C_p \|f\|_{L^p(X,\mathcal{X},\mu)} \quad (9)$$

for all $1 < p \leq \infty$, where the constant C_p depends on p (it blows up like $O(1/(p-1))$ in the limit $p \to 1$).

Exercise 1 (Rising sun inequality). If $f \in l^1(\mathbb{Z})$, and $f^*(m) := \sup_N \frac{1}{N} \sum_{n=0}^{N-1} f(m+n)$, establish the *rising sun inequality*

$$\lambda |\{m \in \mathbb{Z} : f^*(m) > \lambda\}| \leq \sum_{m \in \mathbb{Z}} f^*(m) \quad (10)$$

for any $\lambda > 0$. (*Hint*: one can either adapt the proof of Theorem 1, or else partition the set appearing in (10) into disjoint intervals. The latter proof also leads to a proof of Corollary 1 which avoids the Dunford-Schwartz trick of introducing the functions F_m. The terminology "rising sun" comes from seeing how these intervals interact with the graph of the partial sums of f, which resembles the shadows cast on a hilly terrain by a rising sun.) $\diamond$

Exercise 2. (Transference principle) Show that Corollary 1 can be deduced directly from (10). (*Hint*: given $f \in L^1(X, \mathcal{X}, \mu)$, apply (10) to the functions $f_x(n) := T^n f(x)$ for each $x \in X$ (truncating the integers to a finite set if necessary), and then integrate in x using Fubini's theorem.) This is an example of a *transference principle* between maximal inequalities on $\mathbb{Z}$ and maximal inequalities on measure-preserving systems. $\diamond$

Exercise 3 (Stein-Stromberg maximal inequality). Derive a continuous version of the Dunford-Schwartz maximal inequality, in which the operators P^n are replaced by a semigroup P_t acting on both L^1 and L^∞, in which the underlying measure space is only assumed to be σ-finite rather than a probability space, and the averages $\frac{1}{N}\sum_{n=0}^{N-1} P^n$ are replaced by $\frac{1}{T}\int_0^T P^t\ dt$. Apply this continuous version with $P_t := e^{t\Delta}$ equal to the heat operator on $\mathbb{R}^d$ for $d \geq 1$ to deduce the *Stein-Stromberg maximal inequality*

$$m(\{x \in \mathbb{R}^d : \sup_{R>0} \frac{1}{m(B(x,R))} \int_{B(x,r)} |f|\ dm > \lambda\}) \leq \frac{Cd}{\lambda}\|f\|_{L^1(\mathbb{R}^d,dm)} \quad (11)$$

for all $\lambda > 0$ and $f \in L^1(\mathbb{R}^d, dm)$, where m is Lebesgue measure, $B(x,R)$ is the Euclidean ball of radius R centred at x, and the constant C is absolute (independent of d). This improves upon the Hardy-Littlewood maximal inequality, which gives the same estimate but with Cd replaced by C^d. It is an open question whether the dependence on d can be removed entirely; the estimate (11) is still the best known in high dimension. For d=1, the best constant C is known to be $\frac{11+\sqrt{61}}{12} = 1.567\ldots$, a result of Melas. $\diamond$

Remark 1. The study of maximal inequalities in ergodic theory is, of course, a subject in itself; a classical reference is this monograph of Stein. $\diamond$

— The pointwise ergodic theorem –

Using the maximal ergodic theorem and a standard limiting argument we can now deduce

Theorem 2 (Pointwise ergodic theorem). Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let $f \in L^1(X, \mathcal{X}, \mu)$. Then for μ-almost every $x \in X$, $\frac{1}{N}\sum_{n=0}^{N-1} T^n f(x)$ converges to $\mathbb{E}(f|\mathcal{X}^T)(x)$.

Proof. By subtracting $\mathbb{E}(f|\mathcal{X}^T)$ from f if necessary, it suffices to show that

$$\limsup_{N\to\infty} |\frac{1}{N}\sum_{n=0}^{N-1} T^n f(x)| = 0 \quad (12)$$

a.e. whenever $\mathbb{E}(f|\mathcal{X}^T) = 0$. By telescoping series, (12) is already true when f takes the form $f = Tg - g$ for some $g \in L^\infty(X, \mathcal{X}, \mu)$. So by the arguments used to prove the von Neumann ergodic theorem from the previous lecture, we have already established the claim for a dense class of functions f in $L^2(X, \mathcal{X}, \mu)$ with $\mathbb{E}(f|\mathcal{X}^T) = 0$, and thus also for a dense class of functions in $L^1(X, \mathcal{X}, \mu)$ with $\mathbb{E}(f|\mathcal{X}^T) = 0$ (since the latter space is dense in the former, and the L^2 norm controls the L^1 norm by the Cauchy-Schwarz inequality).

Now we use a standard limiting argument. Let $f \in L^1(X, \mathcal{X}, \mu)$ with $\mathbb{E}(f|\mathcal{X}^T) = 0$. Then we can find a sequence f_j in the above dense class which converges in L^1 to f. For almost every x, we thus have

$$\lim_{N\to\infty} |\frac{1}{N}\sum_{n=0}^{N-1} T^n f_j(x)| = 0 \quad (13)$$

for all j, and so by the triangle inequality we have

$$\limsup_{N\to\infty} |\frac{1}{N}\sum_{n=0}^{N-1} T^n f(x)| \leq \sup_N \frac{1}{N}\sum_{n=0}^{N-1} T^n|f-f_j|(x). \quad (14)$$

But by Corollary 1 we see that the right-hand side of (14) converges to zero in measure as $j \to \infty$. Since the left-hand side does not depend on j, it must vanish almost everywhere, as required. $\Box$

Remark 2. More generally, one can derive a pointwise convergence result on a class of rough functions by first establishing convergence for a dense subclass of functions, and then establishing a maximal inequality which is strong enough to allow one to take limits and establish pointwise convergence for all functions in the larger class. Conversely, principles such as Stein's maximal principle indicate that in many cases this is in some sense the *only* way to establish such pointwise convergence results for rough functions. $\diamond$

Remark 3. Using the dominated convergence theorem (starting first with bounded functions f in order to get the domination), one can deduce the mean ergodic theorem from the pointwise ergodic theorem. But the converse is significantly more difficult; pointwise convergence for various ergodic averages is often a much harder result to establish than the corresponding norm convergence result (in particular, many of the techniques discussed in this course appear to be of sharply limited utility for pointwise convergence problems), and many questions in this area remain open. $\diamond$

Exercise 4 (Lebesgue differentiation theorem). Let $f \in L^1(\mathbb{R}^d, dm)$ with Lebesgue measure dm. Show that for almost every $x \in \mathbb{R}^d$, we have $\lim_{r\to 0^+} \frac{1}{m(B(x,r))}\int_{B(x,r)} |f(y)-f(x)|\ dx = 0$, and in particular that $\lim_{r\to 0^+} \frac{1}{m(B(x,r))}\int_{B(x,r)} f(y)\ dx = f(x)$. $\diamond$

– Ergodicity –

Combining the mean ergodic theorem with the pointwise ergodic theorem (and with Exercises 7, 8 from the previous lecture) we have

Theorem 3 (Characterisations of ergodicity) Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. Then the following are equivalent:

1. Any set $E \in \mathcal{X}$ which is invariant (thus TE=E) has either full measure $\mu(E) = 1$ or zero measure $\mu(E) = 0$.
2. Any set $E \in \mathcal{X}$ which is almost invariant (thus TE differs from E by a null set) has either full measure or zero measure.
3. Any measurable function f with $Tf = f$ a.e. is constant a.e.
4. For any $1 < p < \infty$ and $f \in L^p(X, \mathcal{X}, \mu)$, the averages $\frac{1}{N}\sum_{n=0}^N T^n f$ converge in L^p norm to $\int_X f$.
5. For any two $f, g \in L^\infty(X, \mathcal{X}, \mu)$, we have $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N \int_X (T^n f) g\ d\mu = (\int_X f\ d\mu)(\int_X g\ d\mu)$.
6. For any two measurable sets E and F, we have $\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N \mu(T^n E \cap F) = \mu(E)\mu(F)$.

7. For any $f \in L^1(X, \mathcal{X}, \mu)$, the averages $\frac{1}{N}\sum_{n=0}^N T^n f$ converge pointwise almost everywhere to $\int_X f\, d\mu$.

A measure-preserving system with any (and hence all) of the above properties is said to be *ergodic*.

Remark 4. Strictly speaking, ergodicity is a property that applies to a measure-preserving system $(X, \mathcal{X}, \mu, T)$. However, we shall sometimes abuse notation and apply the adjective "ergodic" to a single component of a system, such as the measure μ or the shift T, when the other three components of the system are clear from context. $\diamond$

Here are some simple examples of ergodicity:

Example 1. If X is finite with uniform measure, then a shift map $T: X \to X$ is ergodic if and only if it is a cycle. $\diamond$

Example 2. If a shift T is ergodic, then so is T^{-1}. However, from Example 1 we see that it is not necessarily true that T^n is ergodic for all n (this latter property is also known as *total ergodicity*). $\diamond$

Exercise 5. Show that the circle shift $(\mathbb{R}/\mathbb{Z}, x \mapsto x + \alpha)$ (with the usual Lebesgue measure) is ergodic if and only if α is irrational. (*Hint*: analyse the equation $Tf = f$ for (say) $f \in L^2(X, \mathcal{X}, \mu)$ using Fourier analysis. *Added*, Feb 21: As pointed out to me in class, another way to proceed is to use the Lebesgue density theorem (or Lebesgue differentiation theorem) combined with Exercise 14 from Lecture 6.) $\diamond$

Exercise 6. Let $(\Omega, \mathcal{B}, \mu)$ be a probability space. Show that the Bernoulli shift on the product system $(\Omega^{\mathbb{Z}}, \mathcal{B}^{\mathbb{Z}}, \mu^{\mathbb{Z}})$ is ergodic. (*Hint*: first establish property 6 of Theorem 3 when E and F each depend on only finitely many of the coordinates of $\Omega^{\mathbb{Z}}$.) $\diamond$

Exercise 7. Let $(X, \mathcal{X}, \mu, T)$ be an ergodic system. Show that if λ is an eigenvalue of $T: L^2(X, \mathcal{X}, \mu) \to L^2(X, \mathcal{X}, \mu)$, then $|\lambda| = 1$, the eigenspace $\{f \in L^2(X, \mathcal{X}, \mu) : Tf = \lambda f\}$ is one-dimensional, and that every eigenfunction f has constant magnitude |f| a.e.. Show that the the eigenspaces are orthogonal to each other in $L^2(X, \mathcal{X}, \mu)$, and the set of all eigenvalues of T forms an at most countable subgroup of the unit circle S^1. $\diamond$

Now we give a less trivial example of an ergodic system.

Proposition 1. (Ergodicity of skew shift) Let $\alpha \in \mathbb{R}$ be irrational. Then the skew shift $((\mathbb{R}/\mathbb{Z})^2, (x, y) \mapsto (x + \alpha, y + x))$ is ergodic.

Proof. Write the skew shift system as $(X, \mathcal{X}, \mu, T)$. To simplify the notation we shall omit the phrase "almost everywhere" in what follows.

We use an argument of Parry. If the system is not ergodic, then we can find a non-constant $f \in L^2(X, \mathcal{X}, \mu)$ such that Tf = f. Next, we use Fourier analysis to write $f = \sum_m f_m$, where $f_m(x, y) := \int_{\mathbb{R}/\mathbb{Z}} f(x, y + \theta) e^{-2\pi i m\theta}\, d\theta$. Since f is T-invariant, and the vertical rotations

$(x, y) \mapsto (x, y + \theta)$ commute with T, we see that the f_m are also T-invariant. The function f_0 depends only on the x variable, and so is constant by Exercise 5. So it suffices to show that f_m is zero for all non-zero m.

Fix m. We can factorise $f_m(x, y) = F_m(x)e^{2\pi i m y}$. The T-invariance of f_m now implies that $F_m(x + \alpha) = e^{-2\pi i m x} F_m(x)$. If we then define $F_{m,\theta} := F_m(x + \theta)\overline{F_m(x)}$ for $\theta \in \mathbb{R}$, we see that $F_{m,\theta}(x + \alpha) = e^{-2\pi i m \theta} F_{m,\theta}(x)$, thus $F_{m,\theta}$ is an eigenfunction of the circle shift with eigenvalue $e^{-2\pi i m \theta}$. But this implies (by Exercise 7) that $F_{m,\theta}$ is orthogonal to $F_{m,0}$ for θ close to zero. Taking limits we see that $F_{m,0}$ is orthogonal to itself and must vanish; this implies that F_m and hence f_m vanish as well, as desired. □

Exercise 8. Show that for any irrational α and any $d \geq 1$, the iterated skew shift system $(\mathbb{R}/\mathbb{Z}^d, (x_1, \ldots, x_d) \to (x_1 + \alpha, x_2 + x_1, \ldots, x_d + x_{d-1}))$ is ergodic. ◇

– Generic points –

Now let us suppose that we have a topological measure preserving system $(X, \mathcal{F}, \mu, T)$, i.e. a measure-preserving system $(X, \mathcal{X}, \mu, T)$ which is also a topological dynamical system $(X, \mathcal{F}, T)$, with $\mathcal{X}$ the Borel σ-algebra of T. Then we have the space C(X) of continuous (real or complex-valued) functions on X, which is dense inside $L^2(X)$. From the Stone-Weierstrass theorem we also see that C(X) is separable.

A sequence $x_1, x_2, x_3, \ldots$ in X is said to be *uniformly distributed* with respect to μ if we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^N f(x_n) = \int_X f\, d\mu \quad (15)$$

for all $f \in C(X)$. A point x in X is said to be *generic* if the forward orbit $x, Tx, T^2x, \ldots$ is uniformly distributed.

Exercise 9. Let $(X, \mathcal{F}, \mu)$ be a compact metrisable space with a Borel probability measure μ, and let $x_1, x_2, \ldots$ be a sequence in X. Show that this sequence is uniformly distributed if and only if $\lim_{N\to\infty} \frac{1}{N}|\{1 \leq i \leq N : x_i \in U\}| = \mu(U)$ for all open sets U in X. ◇

From Theorem 3 and the separability of C(X) we obtain

Proposition 2. A topological measure-preserving system is ergodic if and only if almost every point is generic.

A topological measure-preserving system is said to be uniquely ergodic if *every* point is generic. The following exercise explains the terminology:

Exercise 10. Show that a topological measure-preserving system $(X, \mathcal{F}, \mu, T)$ is uniquely ergodic if and only if the only T-invariant Borel probability measure on T is μ. (Hint: use Lemma 1 from Lecture 7.) Because of this fact, one can sensibly define what it means for a topological dynamical system $(X, \mathcal{F}, T)$ to be uniquely ergodic, namely that it has a unique T-invariant Borel probability measure. ◇

It is not always the case that an ergodic system is uniquely ergodic. For instance, in the Bernoulli system $\{0,1\}^{\mathbb{Z}}$ (with uniform measure on $\{0,1\}$, say), the point $0^{\mathbb{Z}}$ is not generic. However, for more algebraic systems, it turns out that ergodicity and unique ergodicity are largely equivalent. We illustrate this with the circle and skew shifts:

Exercise 11. Show that the circle shift $(\mathbb{R}/\mathbb{Z}, x \mapsto x+\alpha)$ (with the usual Lebesgue measure) is uniquely ergodic if and only if α is irrational. (*Hint*: first show in the circle shift system that any translate of a generic point is generic.) $\diamond$

Proposition 3. (Unique ergodicity of skew shift) Let $\alpha \in \mathbb{R}$ be irrational. Then the skew shift $((\mathbb{R}/\mathbb{Z})^2, (x,y) \mapsto (x+\alpha, y+x))$ is uniquely ergodic.

Proof. We use an argument of Furstenberg. We again write the skew shift as $(X, \mathcal{X}, \mu, T)$. Suppose this system was not uniquely ergodic, then by Exercise 10 there is another shift-invariant Borel probability measure $\mu' \neq \mu$. If we push μ and μ' down to the circle shift system $(\mathbb{R}/\mathbb{Z}, x \mapsto x+\alpha$ by the projection map $(x,y) \mapsto x$, then by Exercises 10, 11 we must get the same measure. Thus μ and μ' must agree on any set of the form $A \times (\mathbb{R}/\mathbb{Z})$.

Let E denote the points in X which are generic with respect to μ; note that this set is Borel measurable. By Proposition 2, this set has full measure in μ. Also, since the vertical rotations $(x,y) \mapsto (x, y+\theta)$ commute with T and preserve μ, we see that E must be invariant under such rotations; thus they are of the form $A \times (\mathbb{R}/\mathbb{Z})$ for some A. By the preceding discussion, we conclude that E also has full measure in μ'. But then (by the pointwise or mean ergodic theorem for $(X, \mathcal{X}, \mu', T)$) we conclude that $\mathbb{E}_{\mu'}(f|\mathcal{X}^T) = \int_X f\ d\mu$ μ'-almost everywhere for every continuous f, and thus on integrating with respect to μ' we obtain $\int_X f\ d\mu' = \int_X f\ d\mu$ for every continuous f. But then by the Riesz representation theorem we have $\mu = \mu'$, a contradiction. $\square$

Corollary 2. If $\alpha \in \mathbb{R}$ is irrational, then the sequence $(\alpha n^2 \bmod 1)_{n \in \mathbb{N}}$ is uniformly distributed in $\mathbb{R}/\mathbb{Z}$ (with respect to uniform measure).

Exercise 11a. Show that the systems considered in Exercise 8 are uniquely ergodic. Conclude that the exponent 2 in Corollary 2 can be replaced by any positive integer d. $\diamond$

Note that the topological dynamics theory developed in Lecture 6 only establishes the weaker statement that the above sequence is dense in $\mathbb{R}/\mathbb{Z}$ rather than uniformly distributed. More generally, it seems that ergodic theory methods can prove topological dynamics results, but not vice versa. Here is another simple example of the same phenomenon:

Exercise 12. Show that a uniquely ergodic topological dynamical system is necessarily minimal. (The converse is not necessarily true, as already mentioned in Remark 6 of Lecture 7.) $\diamond$

– The ergodic decomposition –

Just as not every topological dynamical system is minimal, not every measure-preserving system is ergodic. Nevertheless, there is an important decomposition that allows one to represent non-ergodic measures as averages of ergodic measures. One can already see this in the finite case, when X is a finite set with the discrete σ-algebra, and $T: X \to X$ is a permutation on X, which can be decomposed as the disjoint union of cycles on a partition $X = C_1 \cup \ldots \cup C_m$ of X. In this case, all shift-invariant probability measures take the form

$$\mu = \sum_{j=1}^m \alpha_j \mu_j \quad (16)$$

where μ_j is the uniform probability measure on the cycle C_j, and α_j are non-negative constants adding up to 1. Each of the μ_j are ergodic, but no non-trivial linear combination of these measures is ergodic. Thus we see in the finite case that every shift-invariant measure can be uniquely expressed as a convex combination of ergodic measures.

It turns out that a similar decomposition is available in general, at least if the underlying measure space is a compact topological space (or more generally, a Radon space). This is because of the following general theorem from measure theory.

Definition 1 (Probability kernel). Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be measurable spaces. A *probability kernel* $y \mapsto \mu_y$ is an assignment of a probability measure μ_y on X to each $y \in Y$ in such a way that the map $y \mapsto \int_X f\ d\mu_y$ is measurable for every bounded measurable $f: X \mapsto \mathbb{C}$.

Example 3. Every measurable map $\phi: Y \to X$ induces a probability kernel $y \mapsto \delta_{\phi(y)}$. Every probability measure on X can be viewed as a probability kernel from a point to X. If $y \mapsto \mu_y$ and $x \mapsto \nu_x$ are two probability kernels from Y to X and from X to Z respectively, their composition $x \mapsto (\mu \circ \nu)_x := \int_X \mu_y\ d\nu_x(y)$ is also a probability kernel, where $\int_X \mu_y\ d\nu_x(y)$ is the measure that assigns $\int_X \mu_y(E)\ d\nu_x(y)$ to any measurable set E in Z. Thus one can view the class of measurable spaces and their probability kernels as a category, which includes the class of measurable spaces and their measurable maps as a subcategory. $\diamond$

Definition 2. (Regular space) A measurable space $(X, \mathcal{X})$ is said to be *regular* if there exists a compact metrisable topology $\mathcal{F}$ on X for which $\mathcal{X}$ is the Borel σ-algebra.

Example 4. Every topological measure-preserving system is regular. $\diamond$

Remark 5. Measurable spaces $(X, \mathcal{X})$ in which $\mathcal{X}$ is the Borel σ-algebra of a topological space generated by a separable complete metric space (i.e. a Polish space) are known as *standard Borel spaces*. It is a non-trivial theorem from descriptive set theory that up to measurable isomorphism, there are only three types of standard Borel spaces: finite discrete spaces, countable discrete spaces, and the unit interval [0,1] with the usual Borel σ-algebra. From this one can see that regular spaces are the same as standard Borel spaces, though we will not need this fact here. $\diamond$

Theorem 4 (Disintegration theorem). Let $(X, \mathcal{X}, \mu)$ and $(Y, \mathcal{Y}, \nu)$ be probability spaces, with $(X, \mathcal{X})$ regular. Let $\pi : X \to Y$ be a morphism (thus $\nu = \pi_\# \mu$). Then there exists a probability kernel $y \mapsto \mu_y$ such that

$$\int_X f(g \circ \pi)\ d\mu = \int_Y (\int_X f\ d\mu_y) g(y)\ d\nu(y) \quad (17)$$

for any bounded measurable $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$. Also, for any such g, we have

$$g \circ \pi = g(y)\ \mu_y\text{-a.e.} \quad (18)$$

for ν-a.e. y.

Furthermore, this probability kernel is unique up to ν-almost everywhere equivalence, in the sense that if $y \mapsto \mu'_y$ is another probability kernel with the same properties, then $\mu_y = \mu'_y$ for ν-almost every y.

We refer to the probability kernel $y \mapsto \mu_y$ generated by the above theorem as the *disintegration* of μ relative to the factor map π.

Proof. We begin by proving uniqueness. Suppose we have two probability kernels $y \mapsto \mu_y, y \mapsto \mu'_y$ with the above properties. Then on subtraction we have

$$\int_Y (\int_X f\ d(\mu_y - \mu'_y)) g(y)\ d\nu(y) = 0 \quad (19)$$

for all bounded measurable $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$. Specialising to $f = 1_E$ for some measurable set $E \in \mathcal{X}$, we conclude that $\mu_y(E) = \mu'_y(E)$ for ν-almost every y. Since $\mathcal{X}$ is regular, it is separable and we conclude that $\mu_y = \mu'_y$ for ν-almost every y, as required.

Now we prove existence. The pullback map $\pi^\# : L^2(Y, \mathcal{Y}, \nu) \to L^2(X, \mathcal{X}, \mu)$ defined by $g \mapsto g \circ \pi$ has an adjoint $\pi_\# : L^2(X, \mathcal{X}, \mu) \to L^2(Y, \mathcal{Y}, \nu)$, thus

$$\int_X f(g \circ \pi)\ d\mu = \int_Y (\pi_\# f) g\ d\nu \quad (20)$$

for all $f \in L^2(X, \mathcal{X}, \mu)$ and $g \in L^2(Y, \mathcal{Y}, \nu)$. It is easy to see from duality that we have $\|\pi_\# f\|_{L^\infty(Y, \mathcal{Y}, \nu)} \leq \|f\|_{C(X)}$ for all $f \in C(X)$ (where we select a compact metrisable topology that generates the regular σ-algebra $\mathcal{X}$). Recall that $\pi_\# f$ is not quite a measurable function, but is instead an equivalence class of measurable functions modulo ν-almost everywhere equivalence. Since C(X) is separable, we find a measurable representative $\tilde \pi_\# f : Y \to \mathbb{C}$ of $\pi_\# f$ to every $f \in C(X)$ which varies linearly with f, and is such that $|\tilde \pi_\# f(y)| \leq \|f\|_{C(X)}$ for all y outside of a set E of ν-measure zero and for all $f \in C(X)$. For all such y, we can then apply the Riesz representation theorem to obtain a Borel probability measure μ_y such that

$$\tilde \pi_\# f(y) = \int_X f\ d\mu_y \quad (21)$$

for all such y. We set μ_y equal to some arbitrarily fixed Borel probability measure for $y \in E$. We then observe that the required properties (including the measurability of $y \mapsto \int_X f\ d\mu_y$) are already obeyed for $f \in C(X)$. To generalise this to bounded measurable f, observe that the class $\mathcal{C}$ of f obeying the required properties is closed under dominated pointwise convergence, and so contains the indicator functions of open sets (by Urysohn's lemma). Applying dominated pointwise convergence again, together with linearity, we see that the sets whose indicator functions lie in $\mathcal{C}$ form a σ-algebra and so contain all Borel sets. Thus all simple measurable functions lie in $\mathcal{C}$, and on taking uniform limits we obtain the claim.

Finally, we prove (18). From two applications of (17) we have

$$\int_Y (\int_X f(g \circ \pi)\ d\mu_y) h(y)\ d\nu(y) = \int_Y (\int_X f g(y)\ d\mu_y) h(y)\ d\nu(y) \quad (22)$$

for all bounded measurable $f : X \to \mathbb{C}$ and $h : Y \to \mathbb{C}$. The claim follows (using the separability of the space of all f). $\diamond$

Exercise 13. Let the notation and assumptions be as in Theorem 4. Suppose that $\mathcal{Y}$ is also regular, and that the map $\pi : X \to Y$ is continuous with respect to some compact metrisable topologies that generate $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then show that for ν-almost every y, the probability measure ν_y is supported in $\pi^{-1}(\{y\})$. $\diamond$

Proposition 4 (Ergodic decomposition). Let $(X, \mathcal{X}, \mu, T)$ be a regular measure-preserving system. Let $(Y, \mathcal{Y}, \nu, S)$ be the system defined by $Y := X$, $\mathcal{Y} := \mathcal{X}^T$, $\nu := \mu\ \lfloor_{\mathcal{Y}}$, and $S := T$, and let $\pi : X \to Y$ be the identity map. Let $y \mapsto \mu_y$ be the disintegration of μ with respect to the factor map π. Then for ν-almost every y, the measure μ_y is T-invariant and ergodic.

Proof. Observe from the T-invariance $\mu = T_\# \mu$ of μ (and of $\mathcal{X}^T$) that the probability kernel $y \mapsto T_\# \mu_y$ would also be a disintegration of μ. Thus we have $\mu_y = T_\# \mu_y$ for ν-almost every y.

Now we show the ergodicity. As the space of bounded measurable $f : X \to \mathbb{C}$ is separable, it suffices by Theorem 3 and a limiting argument to show that for any fixed such f, the averages $\frac{1}{N} \sum_{n=1}^N T^n f$ converge pointwise μ_y-a.e. to $\int_X f\ d\mu_y$ for ν-a.e. y.

From the pointwise ergodic theorem, we already know that $\frac{1}{N} \sum_{n=1}^N T^n f$ converges to $\mathbb{E}(f|\mathcal{X}^T)$ outside of a set of μ-measure zero. By (17), this set also has μ_y-measure zero for ν-almost every y. Thus it will suffice to show that $\mathbb{E}(f|\mathcal{X}^T)$ is μ_y-a.e. equal to $\int_X f\ d\mu_y$ for ν-a.e. y. Now observe that $\mathbb{E}(f|\mathcal{X}^T)(x) = \pi_\# f(\pi(x))$, so the claim follows from (18) and (21). $\diamond$

Exercise 14. Let $(X, \mathcal{X})$ be a separable measurable space, and let T be bimeasurable bijection $T : X \to X$. Let $M(\mathcal{X})$ denote the Banach space of all finite measures on $\mathcal{X}$ with the total variation norm. Let $\Pr(\mathcal{X})^T \subset M(\mathcal{X})$ denote the collection of probability measures on $\mathcal{X}$ which are T-invariant. Show that this is a closed convex subset of $M(\mathcal{X})$, and the extreme points of $\Pr(\mathcal{X})^T$ are precisely the ergodic probability measures (which also form a closed subset of $M(\mathcal{X})$). This allows one to prove a variant of Proposition 4 using Choquet's theorem. $\diamond$

Exercise 15. Show that a topological measure-preserving system $(X, \mathcal{F}, T, \mu)$ is uniquely ergodic if and only if the only ergodic shift-invariant Borel probability measure on X is μ. $\diamond$

[*Update*, Feb 6: Some corrections; new exercises added.]

[*Update*, Feb 23: More exercises added.]

9 comments

Comments feed for this article

5 February, 2008 at 7:04 am

Lior

In the proof of Thm 1, we should "[o]bserve that $Mf(x) > 0$ …" and not as written. Ex.11 duplicates Ex. 5 (should ask to show the circle shift is uniquely ergodic). Def. 1 should note that μ_y is a measure on X. Example 3 should have $y \mapsto \nu_y$ and not as written; "composition" there is a form of convolution.

5 February, 2008 at 11:53 am

Terence Tao

Thanks, Lior, for the corrections!

5 February, 2008 at 1:26 pm

Lior

Of course my remark on Example 3 is wrong — sorry about that. I too should re-read what I write.. To Exercize 14 you can add showing that the set of ergodic measures is closed (eliminating the annoyance with the meaning of the "support" of the measure you get from Choquet's theorem).

13 February, 2008 at 1:39 pm

254A, Lecture 11: Compact systems « What's new

[…] measure-preserving systems are isomorphic); the notion of regularity was defined in Definition 2 of Lecture 9. Hint: take a countable shift-invariant family of sets that generate (thus T acts on this space by […]

22 February, 2008 at 2:39 pm

254A, Lecture 10: The Furstenberg correspondence principle

10 February, 2008 in 254A - ergodic theory, math.CO, math.DS

Tags: averaging argument, correspondence principle, density Hales-Jewett, multiple recurrence, polynomial recurrence, Szemeredi's theorem

In this lecture, we describe the simple but fundamental *Furstenberg correspondence principle* which connects the "soft analysis" subject of ergodic theory (in particular, recurrence theorems) with the "hard analysis" subject of combinatorial number theory (or more generally with results of "density Ramsey theory" type). Rather than try to set up the most general and abstract version of this principle, we shall instead study the canonical example of this principle in action, namely the equating of the *Furstenberg multiple recurrence theorem* with Szemerédi's theorem on arithmetic progressions.

In 1975, Szemerédi established the following theorem, which had been conjectured in 1936 by Erdős and Turán:

Theorem 1. (Szemerédi's theorem) Let $k \geq 1$ be an integer, and let A be a set of integers of positive upper density, thus $\limsup_{N\to\infty} \frac{1}{2N+1}|A \cap \{-N,\ldots,N\}| > 0$. Then A contains a non-trivial arithmetic progression $n, n+r, \ldots, n+(k-1)r$ of length k. (By "non-trivial" we mean that $r \neq 0$.) [More succinctly: every set of integers of positive upper density contains arbitrarily long arithmetic progressions.]

This theorem is trivial for k=1 and k=2. The first non-trivial case is k=3, which was proven by Roth in 1953 and will be discussed in a later lecture. The k=4 case was also established by Szemerédi in 1969.

In 1977, Furstenberg gave another proof of Szemerédi's theorem, by establishing the following equivalent statement:

Theorem 2. (Furstenberg multiple recurrence theorem) Let $k \geq 1$ be an integer, let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let E be a set of positive measure. Then there exists $r > 0$ such that $E \cap T^{-r}E \cap \ldots \cap T^{-(k-1)r}E$ is non-empty.

Remark 1. The negative signs here can be easily removed because T is invertible, but I have placed them here for consistency with some later results involving non-invertible transformations, in which the negative sign becomes important. $\diamond$

Exercise 1. Prove that Theorem 2 is equivalent to the apparently stronger theorem in which "is non-empty" is replaced by "has positive measure", and "there exists $r > 0$" is replaced by "there exist infinitely many $r > 0$". $\diamond$

Note that the k=1 case of Theorem 2 is trivial, while the k=2 case follows from the Poincaré recurrence theorem (Theorem 1 from Lecture 8). We will prove the higher k cases of this theorem in later lectures.

In this one, we will explain why, for any fixed k, Theorem 1 and Theorem 2 are equivalent.

Let us first give the easy implication that Theorem 1 implies Theorem 2. This follows immediately from

> **Lemma 1.** Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let E be a set of positive measure. Then there exists a point x in X such that the recurrence set $\{n \in \mathbb{Z} : T^n x \in E\}$ has positive upper density.

Indeed, from Lemma 1 and Theorem 1, we obtain a point x for which the set $\{n \in \mathbb{Z} : T^n x \in E\}$ contains an arithmetic progression of length k and some step r, which implies that $E \cap T^r E \cap \ldots \cap T^{(k-1)r} E$ is non-empty.

Proof of Lemma 1. Observe (from the shift-invariance of μ) that

$$\int_X \frac{1}{2N+1} \sum_{n=-N}^N 1_{T^n E}\, d\mu = \mu(E). \quad (1)$$

On the other hand, the integrand is at most 1. We conclude that for each N, the set $\{x : \frac{1}{2N+1} \sum_{n=-N}^N 1_{T^n E}(x) \geq \mu(E)/2\}$ must have measure at least $\mu(E)/2$. The claim now follows from the Borel-Cantelli lemma. ◇

Now we show how Theorem 2 implies Theorem 1. If we could pretend that "upper density" was a probability measure on the integers, then this implication would be immediate by applying Theorem 2 to the dynamical system $(\mathbb{Z}, n \mapsto n+1)$. Of course, we know that the integers do not admit a shift-invariant probability measure (and upper density is not even additive, let alone a probability measure). So this does not work directly. Instead, we need to first lift from the integers to a more abstract universal space and use a standard "compactness and contradiction" argument in order to be able to build the desired probability measure properly.

More precisely, let A be as in Theorem 1. Consider the topological boolean Bernoulli dynamical system $2^{\mathbb{Z}}$ with the product topology and the shift $T : B \mapsto B+1$. The set A can be viewed as a point in this system, and the orbit closure $X := \overline{\{A+n : n \in \mathbb{Z}\}}$ of that point becomes a subsystem of that Bernoulli system, with the relative topology.

Suppose for contradiction that A contains no non-trivial progressions of length k, thus $A \cap A+r \cap \ldots \cap A+(k-1)r = \emptyset$ for all $r > 0$. Then, if we define the cylinder set $E := \{B \in X : 0 \in B\}$ to be the collection of all points in X which (viewed as sets of integers) contain 0, we see (after unpacking all the definitions) that $E \cap T^r E \cap \ldots T^{(k-1)r} E = \emptyset$ for all $r > 0$.

In order to apply Theorem 2 and obtain the desired contradiction, we need to find a shift-invariant Borel probability measure μ on X which assigns a positive measure to E.

For each integer N, consider the measure μ_N which assigns a mass of $\frac{1}{2N+1}$ to the points $T^{-n} A$ in X for $-N \leq n \leq N$, and no mass to the rest of X. Then we see that $\mu_N(E) = \frac{1}{2N+1} |A \cap \{-N, \ldots, N\}|$. Thus, since A has positive upper density, there exists some sequence N_j going to infinity such that $\liminf_{j \to \infty} \mu_{N_j}(E) > 0$. On the other hand, by vague sequential compactness (Lemma 1 of Lecture 7) we know that some subsequence of μ_{N_j} converges in the vague topology to a probability measure μ, which then assigns a positive measure to the (clopen) set E. As the μ_{N_j} are asymptotically shift invariant, we see that μ is invariant also (as in the proof of Corollary 1 of Lecture 7). As μ now has all the required properties, we have completed the deduction of Theorem 1 from Theorem 2.

Exercise 2. Show that Theorem 2 in fact implies a seemingly stronger version of Theorem 1, in which the conclusion becomes the assertion that the set $\{n : n, n+r, \ldots, n+(k-1)r \in A\}$ has positive upper density for infinitely many r. $\diamond$

Exercise 3. Show that Theorem 1 in fact implies a seemingly stronger version of Theorem 2: If $E_1, E_2, E_3, \ldots$ are sets in a probability space with uniformly positive measure (i.e. $\inf_n \mu(E_n) > 0$), then for any k there exists positive integers n, r such that $\mu(E_n \cap E_{n+r} \cap \ldots \cap E_{n+(k-1)r}) > 0$. $\diamond$

– Varnavides type theorems –

A similar "compactness and contradiction" argument (combined with a preliminary averaging-over-dilations trick of Varnavides) allows us to use Theorem 2 to imply the following apparently stronger statement (observed by Bergelson, Host, McCutcheon, and Parreau):

Theorem 3. (Uniform Furstenberg multiple recurrence theorem) Let $k \geq 1$ be an integer and $\delta > 0$. Then for any measure-preserving system $(X, \mathcal{X}, \mu, T)$ and any measurable set E with $\mu(E) \geq \delta$ we have

$$\frac{1}{N} \sum_{r=0}^{N-1} \mu(E \cap T^r E \cap \ldots \cap T^{(k-1)r} E) \geq c(k, \delta) \quad (2)$$

for all $N \geq 1$, where $c(k, \delta) > 0$ is a positive quantity which depends only on k and δ (i.e. it is uniform over all choices of system and of the set E with measure at least δ).

Exercise 4. Assuming Theorem 3, show that if N is sufficiently large depending on k and δ, then any subset of $\{1, \ldots, N\}$ with cardinality at least δN will contain at least $c'(k, \delta) N^2$ non-trivial arithmetic progressions of length k, for some $c'(k, \delta) > 0$. (This result for k=3 was first established by Varnavides via an averaging argument from Roth's theorem.) Conclude in particular that Theorem 3 implies Theorem 1. $\diamond$

It is clear that Theorem 3 implies Theorem 2; let us now establish the converse. We first use an averaging argument of Varnavides to reduce Theorem 3 to a weaker statement, in which the conclusion (2) is not asserted to hold for all N, but instead one asserts that

$$\frac{1}{N_0} \sum_{r=1}^{N_0-1} \mu(E \cap T^r E \cap \ldots \cap T^{(k-1)r} E) \geq c(k, \delta) \quad (2')$$

is true for some $N_0 = N_0(k, \delta) > 0$ depending only on k and δ (note that the r=0 term in (2′) has been dropped, otherwise the claim is trivial). To see why one can recover (2) from (2′), observe by replacing the shift T with a power T^a that we can amplify (2′) to

$$\frac{1}{N_0} \sum_{r=1}^{N_0-1} \mu(E \cap T^{ar} E \cap \ldots \cap T^{(k-1)ar} E) \geq c(k, \delta) \quad (2")$$

for all a. Averaging (2") over $1 \leq a \leq N$ we easily conclude (2).

It remains to prove that (2) holds under the hypotheses of Theorem 3. Our next reduction is to observe that for it suffices to perform this task for the boolean Bernoulli system $X_0 := 2^{\mathbb{Z}}$ with the cylinder set $E_0 := \{B \in X_0 : 0 \in B\}$ as before. To see this, recall from Example 5 of Lecture 2 that there is a morphism $\phi : X \to X_0$ from any measure-preserving system $(X, \mathcal{X}, \mu, T)$ with a distinguished set E to the system X_0 with the product σ-algebra $\mathcal{X}_0$, the usual shift T_0, and the set E_0, and with the push-forward measure $\mu_0 := \phi_\# \mu$. Specifically, ϕ sends any point x in X to its recurrence set $\phi(x) := \{n \in \mathbb{Z} : T^n x \in E\}$. Using this morphism it is not difficult to show that the claim (2) for $(X, \mathcal{X}, \mu, T)$ and E would follow from the same claim for $(X_0, \mathcal{X}_0, \mu_0, T_0)$ and E_0.

We still need to prove (2) for the boolean system. The point is that by lifting to this universal setting, the dynamical system $(X, \mathcal{X}, T)$ and the set E have been canonically fixed; the only remaining parameter is the probability measure μ. But now we can exploit vague sequential compactness again as follows.

Suppose for contradiction that Theorem 3 failed for the boolean system. Then by carefully negating all the quantifiers, we can find $\delta > 0$ such that for any N_0 there is a sequence of shift-invariant probability measures μ_j on X with $\mu_j(E) \geq \delta$,

$$\frac{1}{N_0} \sum_{r=1}^{N_0 - 1} \mu_j(E \cap T^r E \cap \ldots \cap T^{(k-1)r} E) \to 0 \quad (3)$$

as $j \to \infty$. Note that if (3) holds for one value of N_0, then it also holds for all smaller values of N_0. A standard diagonalisation argument then allows us to build a sequence μ_j as above, but which obeys (3) for *all* $N_0 \geq 1$.

Now we are finally in a good position to apply vague sequential compactness. By passing to a subsequence if necessary, we may assume that μ_j converges vaguely to a limit μ, which is a shift-invariant probability measure. In particular we have $\mu(E) \geq \delta > 0$, while from (3) we see that

$$\frac{1}{N_0} \sum_{r=1}^{N_0 - 1} \mu(E \cap T^r E \cap \ldots \cap T^{(k-1)r} E) = 0 \quad (4)$$

for all $N_0 \geq 1$; thus the sets $E \cap T^r E \cap \ldots \cap T^{(k-1)r} E$ all have zero measure for $r > 0$. But this contradicts Theorem 2 (and Exercise 1). This completes the deduction of Theorem 3 from Theorem 2.

– Other recurrence theorems and their combinatorial counterparts –

The Furstenberg correspondence principle can be extended to relate several other recurrence theorems to their combinatorial analogues. We give some representative examples here (without proofs). Firstly, there is a multidimensional version of Szemerédi's theorem (compare with Exercise 7 from Lecture 4):

Theorem 4 (Multidimensional Szemerédi theorem) Let $d \geq 1$, let $v_1, \ldots, v_k \in \mathbb{Z}^d$, and let $A \subset \mathbb{Z}^d$ be a set of upper Banach density (which means that $\limsup_{N \to \infty} |A \cap B_N| / |B_N| > 0$, where $B_N := \{-N, \ldots, N\}^d$). Then A contains a pattern of the form $n + rv_1, \ldots, n + rv_k$ for some $n \in \mathbb{Z}^d$ and $r > 0$.

Note that Theorem 1 corresponds to the special case when $d = 1$ and $v_i = i - 1$.

This theorem was first proven by Furstenberg and Katznelson, who deduced it via the correspondence principle from the following generalisation of Theorem 2:

Theorem 5 (Recurrence for multiple commuting shifts) Let $k \geq 1$ be an integer, let $(X, \mathcal{X}, \mu)$ be a probability space, let $T_1, \ldots, T_k : X \to X$ be measure-preserving bimeasurable maps which commute with each other, and let E be a set of positive measure. Then there exists $r > 0$ such that $T_1^r E \cap T_2^r E \cap \ldots \cap T_k^r E$ is non-empty.

Exercise 5. Show that Theorem 4 and Theorem 5 are equivalent. $\diamond$

Exercise 6. State an analogue of Theorem 3 for multiple commuting shifts, and prove that it is equivalent to Theorem 5. $\diamond$

There is also a polynomial version of these theorems (cf. Theorem 1 from Lecture 5), which we will also state in general dimension:

Theorem 6 (Multidimensional polynomial Szemerédi theorem) Let $d \geq 1$, let $P_1, \ldots, P_k : \mathbb{Z} \to \mathbb{Z}^d$ be polynomials with $P_1(0) = \ldots = P_k(0) = 0$, and let $A \subset \mathbb{Z}^d$ be a set of upper Banach density. Then A contains a pattern of the form $n + P_1(r), \ldots, n + P_k(r)$ for some $n \in \mathbb{Z}^d$ and $r > 0$.

This theorem was established by Bergelson and Leibman, who deduced it from

Theorem 7 (Polynomial recurrence for multiple commuting shifts) Let k, $(X, \mathcal{X}, \mu)$, $T_1, \ldots, T_k : X \to X$, E be as in Theorem 5, and let $P_1, \ldots, P_k$ be as in Theorem 6. Then there exists $r > 0$ such that $T^{-P_1(r)}E \cap T^{-P_2(r)}E \cap \ldots \cap T^{-P_k(r)}E$ is non-empty, where we adopt the convention $T^{(a_1,\ldots,a_k)} := T_1^{a_1} \ldots T_k^{a_k}$ (thus we are making the action of $\mathbb{Z}^d$ on X explicit).

Exercise 7. Show that Theorem 6 and Theorem 7 are equivalent. $\diamond$

Exercise 8. State an analogue of Theorem 3 for polynomial recurrence for multiple commuting shifts, and prove that it is equivalent to Theorem 7. (Hint: first establish this in the case that each of the P_j are monomials, in which case there is enough dilation symmetry to use the Varnavides averaging trick. Interestingly, if one only restricts attention to one-dimensional systems k=1, it does not seem possible to deduce the uniform polynomial recurrence theorem from the non-uniform polynomial recurrence theorem, thus indicating that the averaging trick is less universal in its applicability than the correspondence principle.) $\diamond$

In the above theorems, the underlying action was given by either the integer group $\mathbb{Z}$ or the lattice group $\mathbb{Z}^d$. It is not too difficult to generalise these results to the semigroups $\mathbb{N}$ and $\mathbb{N}^d$ (thus dropping the assumption that the shift maps are invertible), by using a trick similar to that used in Exercise 9 of Lecture 4, or by using the correspondence principle back and forth a few times. A bit more surprisingly, it is possible to extend these results to even weaker objects than semigroups. To describe this we need some more notation.

Define a *partial semigroup* $(G, \cdot)$ to be a set G together with a partially defined multiplication operation $\cdot : \Omega \to G$ for some subset $\Omega \subset G \times G$, which is associative in the sense that whenever $(a \cdot b) \cdot c$ is defined, then $a \cdot (b \cdot c)$ is defined and equal to $(a \cdot b) \cdot c$, and vice versa. A good example of a partial semigroup is the finite subsets $\binom{S}{<\omega} := \{A \subset S : |A| < \infty\}$ of a fixed set S, where the multiplication operation $A \cdot B$ is disjoint union, or more precisely $A \cdot B := A \cup B$ when A and B are disjoint, and $A \cdot B$ is undefined otherwise.

Remark 2. One can extend a partial semigroup to be a genuine semigroup by adjoining a new element err to G, and redefining multiplication $a \cdot b$ to equal err if it was previously undefined (or if one of a or b was already equal to err). However, we will avoid using this trick here, as it tends to complicate the notation a little. $\diamond$

One can take Cartesian products of partial semigroups in the obvious manner to obtain more partial semigroups. In particular, we have the partial semigroup $\binom{\mathbb{N}}{<\omega}^d$ for any $d \geq 1$, defined as the collection of d-tuples $(A_1, \ldots, A_d)$ of finite sets of natural numbers (not necessarily disjoint), with the partial semigroup law $(A_1, \ldots, A_d) \cdot (B_1, \ldots, B_d) := (A_1 \cup B_1, \ldots, A_d \cup B_d)$ whenever A_i and B_i are disjoint for each $1 \leq i \leq d$.

If $(X, \mathcal{X}, \mu)$ is a probability space and $(G, \cdot)$ is a partial semigroup, we define a *measure-preserving action* of G on X to be an assignment of a measure-preserving transformation $T^g : X \to X$ (not necessarily invertible) to each $g \in G$, such that $T^{g \cdot h} = T^g T^h$ whenever $g \cdot h$ is defined.

An action T of $\binom{\mathbb{N}}{<\omega}$ on X is known as an *IP system* on X; it is generated by a countable number $T_1, T_2, \ldots$ of commuting measure-preserving transformations, with $T^A := \prod_{i \in A} T^i$. (Admittedly, it is possible that the action of the empty set is not necessarily the identity, but this turns out to have a negligible impact on matters.) An action T of $\binom{\mathbb{N}}{<\omega}^d$ is then a collection of d simultaneously commuting IP systems.

Furstenberg and Katznelson showed the following generalisation of Theorem 5:

Theorem 8 (IP multiple recurrence theorem) Let T be an action of $\binom{\mathbb{N}}{<\omega}^d$ on a probability space $(X, \mathcal{X}, \mu)$. Then there exists a non-empty set $A \in \binom{\mathbb{N}}{<\omega}$ such that $E \cap (T^{A_1})^{-1}(E) \cap \ldots \cap (T^{A_d})^{-1}(E)$ is non-empty, where $A_i := (\emptyset, \ldots, \emptyset, A, \emptyset, \ldots, \emptyset)$ is the group element which equals A in the i^{th} position and is the empty set otherwise.

It has a number of combinatorial consequences, such as the following strengthening of Szemerédi's theorem:

Theorem 9. (IP Szemerédi theorem) Let A be a set of integers of positive upper density, let $k \geq 1$, and let $B \subset \mathbb{N}$ be infinite. Then A contains an arithmetic progression $n, n+r, \ldots, n+(k-1)r$ of length k in which r lies in FS(B), the set of finite sums of B (cf. Hindman's theorem from Lecture 5).

(There is also a multidimensional version of this theorem, but it requires a fair amount of notation to state properly.)

Exercise 9. Deduce Theorem 9 from Theorem 8. ◇

Exercise 10. Using Theorem 9, show that for any k, and any set of integers A of positive upper density, the set of steps r which occur in the arithmetic progressions in A of length k is syndetic. ◇

Exercise 11. Using Theorem 8, show that if $\mathbb{F}$ is a finite field, and $\mathbb{F}^{<\omega} := \bigcup_{n=0}^\infty \mathbb{F}^n$ is the canonical vector space over $\mathbb{F}$ spanned (in the algebraic sense) by a countably infinite number of basis vectors, show that any subset A of $\mathbb{F}^{<\omega}$ of positive upper Banach density (which means that $\limsup_{n\to\infty} |A \cap \mathbb{F}^n|/|\mathbb{F}^n| > 0$ contains affine subspaces of arbitrarily high dimension. ◇

The IP recurrence theorem is already very powerful, but even stronger theorems are known. For instance, Furstenberg and Katznelson established the following deep strengthening of the Hales-Jewett theorem (Theorem 8 from Lecture 5), as well as of Exercise 11 above:

Theorem 10 (Density Hales-Jewett theorem) Let A be a finite alphabet. If E is a subset of $A^{<\omega}$ of positive upper Banach density, then E contains a combinatorial line.

This theorem was deduced (via an advanced form of the correspondence principle) by a somewhat complicated recurrence theorem which we will not state here; rather than the action of a group, semigroup, or partial semigroup, one instead works with an ensemble of sets (as in Exercise 3), and furthermore one regularises the system of the probability space and set ensemble (which can collectively be viewed as a random process) to be what Furstenberg and Katznelson call a *strongly stationary process*, which (very) roughly means that the statistics of this process look "the same" when restricted to any combinatorial subspace of a fixed dimension.

Remark 3. Similar correspondence principles can be established connecting property testing results for graphs and hypergraphs to the measure theory of exchangeable measures: see this paper of myself, and of myself and Austin, for details. There is also a correspondence principle connecting ergodic convergence theorems with a (rather complicated looking) finitary analogue; see the papers of Avigad-Gerhardy-Towsner and of myself on this subject. Finally, we have implicitly been using a similar correspondence principle between topological dynamics and colouring Ramsey theorems in our previous lectures (in particular Lecture 3, Lecture 4, and Lecture 5). $\diamond$

Remark 4. The Furstenberg correspondence principle also comes tantalisingly close to deducing my theorem with Ben that the primes contain arbitrarily long arithmetic progressions from Szemerédi's theorem. More precisely, they show that any subset A of a *genuinely* random set of integers with logarithmic-type density B, with A having positive *relative* upper density with respect to B, contains arbitrarily long arithmetic progressions; see this unpublished note of myself. Unfortunately, the almost primes are not known to quite obey enough "correlation conditions" to behave sufficiently pseudorandomly that these arguments apply to the primes, though perhaps there is still a "softer" way to prove our theorem than the way we did it (there is for instance some recent work by Trevisan, Tulsiani, and Vadhan in this direction). $\diamond$

4 comments

Comments feed for this article

11 February, 2008 at 9:14 pm

254A, Lecture 11: Compact systems « What's new

[…] and the next few will be to give a proof of the Furstenberg recurrence theorem (Theorem 2 from the previous lecture). Along the way we will develop a structural theory for measure-preserving […]

22 February, 2008 at 2:39 pm

254A, Lecture 12: Weakly mixing systems « What's new

[…] can then immediately establish the k=3 case of Furstenberg's theorem (Theorem 2 from Lecture 10) by combining the above result with the ergodic decomposition (Proposition 4 from Lecture 9). The […]

1 March, 2008 at 11:05 pm

254A, Lecture 13: Compact extensions « What's new

254A, Lecture 11: Compact systems

11 February, 2008 in 254A - ergodic theory, math.CA, math.DS

Tags: almost periodicity, compact systems, Kronecker systems, multiple recurrence
The primary objective of this lecture and the next few will be to give a proof of the Furstenberg recurrence theorem (Theorem 2 from the previous lecture). Along the way we will develop a structural theory for measure-preserving systems.

The basic strategy of Furstenberg's proof is to first prove the recurrence theorems for very simple systems - either those with "almost periodic" (or *compact*) dynamics or with "weakly mixing" dynamics. These cases are quite easy, but don't manage to cover all the cases. To go further, we need to consider various combinations of these systems. For instance, by viewing a general system as an extension of the maximal compact factor, we will be able to prove Roth's theorem (which is equivalent to the k=3 form of the Furstenberg recurrence theorem). To handle the general case, we need to consider compact extensions of compact factors, compact extensions of compact extensions of compact factors, etc., as well as weakly mixing extensions of all the previously mentioned factors.

In this lecture, we will consider those measure-preserving systems $(X, \mathcal{X}, \mu, T)$ which are *compact* or *almost periodic*. These systems are analogous to the equicontinuous or isometric systems in topological dynamics discussed in Lecture 6, and as with those systems, we will be able to characterise such systems (or more precisely, the ergodic ones) algebraically as Kronecker systems, though this is not strictly necessary for the proof of the recurrence theorem.

— Almost periodic functions –

We begin with a basic definition.

Definition 1. Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. A function $f \in L^2(X, \mathcal{X}, \mu)$ is *almost periodic* if the orbit closure $\overline{\{T^n f : n \in \mathbb{Z}\}}$ is compact in $L^2(X, \mathcal{X}, \mu)$.

Example 1. If f is periodic (i.e. $T^n f = f$ for some $n > 0$) then it is clearly almost periodic. In particular, any shift-invariant function (such as a constant function) is almost periodic. ◇

Example 2. In the circle shift system $(\mathbb{R}/\mathbb{Z}, x \mapsto x + \alpha)$, every function $f \in L^2(\mathbb{R}/\mathbb{Z})$ is almost periodic, because the orbit closure lies inside the set $\{f(\cdot + \theta) : \theta \in \mathbb{R}/\mathbb{Z}\}$, which is the continuous image of a circle $\mathbb{R}/\mathbb{Z}$ and therefore compact. ◇

Exercise 1. Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let $f \in L^2(X, \mathcal{X}, \mu)$. Show that f is almost periodic in the ergodic theory sense (i.e. Definition 1 above) if and only if it is almost periodic in the topological dynamical systems sense (see Lecture 3), i.e. if the sets

$\{n \in \mathbb{Z} : \|T^n f - f\|_{L^2(X,\mathcal{X},\mu)} \leq \varepsilon\}$ are syndetic for every $\varepsilon > 0$. (*Hint:* if f is almost periodic in the ergodic theory sense, show that the orbit closure is an isometric system and thus a Kronecker system, at which point Theorem 2 from Lecture 3 can be applied. For the converse implication, use the Heine-Borel theorem and the isometric nature of T on L^2.) ◇

Exercise 2. Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. Show that the space of almost periodic functions in $L^2(X, \mathcal{X}, \mu)$ is a closed shift-invariant subspace which is also closed under the pointwise operations $f, g \mapsto \max(f, g)$ and $f, g \mapsto \min(f, g)$. Similarly, show that the space of almost periodic functions in $L^\infty(X, \mathcal{X}, \mu)$ is a closed subspace which is also an algebra (closed under products) as well as closed under $\max$ and $\min$. ◇

Exercise 3. Show that in any Bernoulli system $\Omega^{\mathbb{Z}}$, the only almost periodic functions are the constants. (Hint: first show that if $f \in L^2(X, \mathcal{X}, \mu)$ has mean zero, then $\lim_{n\to\infty} \int_X f T^n f\ d\mu = 0$, by first considering elementary functions. ◇

Let us now recall the Furstenberg recurrence theorem, which we now phrase in terms of functions rather than sets:

> **Theorem 1.** (Furstenberg recurrence theorem) Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, let $k \geq 1$, and let $f \in L^\infty(X, \mathcal{X}, \mu)$ be a non-negative function with $\int_X f\ d\mu > 0$. Then we have $\liminf_{N\to\infty} \frac{1}{N} \sum_{r=0}^{N-1} \int_X f T^r f \ldots T^{(k-1)r} f > 0$.

Exercise 4. Show that Theorem 1 is equivalent to Theorem 2 from the previous lecture. ◇

We can now quickly establish this theorem in the almost periodic case:

> **Proposition 1.** Theorem 1 is true whenever f is almost periodic.

Proof. Without loss of generality we may assume that f is bounded a.e. by 1. Let $\varepsilon > 0$ be chosen later. Recall from Exercise 1 that $T^n f$ lies within ε of f in the L^2 topology for a syndetic set of n. For all such n, one also has $\|T^{(j+1)n} f - T^{jn} f\|_{L^2(X,\mathcal{X},\mu)} \leq \varepsilon$ for all j, since T acts isometrically. By the triangle inequality, we conclude that $T^{jn} f$ lies within $O_k(\varepsilon)$ of f in L^2 for $0 \leq j \leq k$. On the other hand, from Hölder's inequality we see that on the unit ball of $L^\infty(X, \mathcal{X}, \mu)$ with the L^2 topology, pointwise multiplication is Lipschitz. Applying this fact repeatedly, we conclude that for n in this syndetic set, $f T^n f \ldots T^{(k-1)n} f$ lies within $O_k(\varepsilon)$ in L^2 of f^k. In particular,

$$\int_X f T^n f \ldots T^{(k-1)n} f\ d\mu = \int_X f^k\ d\mu + O_k(\varepsilon). \quad (1)$$

On the other hand, since $\int f\ d\mu > 0$, we must have $\int_X f^k\ d\mu > 0$. Choosing ε sufficiently small, we thus see that the left-hand side of (1) is uniformly bounded away from zero in a syndetic set, and the conclusion of Theorem 1 follows. ◇

Remark 1. Because f lives in a Kronecker system, one can also obtain the above result using various multiple recurrence theorems from topological dynamics, such as Proposition 3 from Lecture 6 or the Birkhoff multiple recurrence theorem from Lecture 4 (though to get the full strength of the results, one needs to use either syndetic van der Waerden theorem, see Exercise 15.3 from Lecture 5, or the Varnavides averaging trick from the previous lecture). We leave the details to the reader. $\diamond$

– Kronecker systems and Haar measure –

We have seen how nice almost periodic functions are. Motivated by this, we define

> **Definition 2.** A measure-preserving system $(X, \mathcal{X}, \mu, T)$ is said to be compact if every function in $L^2(X, \mathcal{X}, \mu)$ is almost periodic.

Thus, for instance, by Example 2, the circle shift system is compact, but from Example 3, no non-trivial Bernoulli system is compact. From Proposition 1 we know that the Furstenberg recurrence theorem is true for compact systems.

One source of compact systems comes from *Kronecker systems*, as introduced in Lecture 6. As such systems are topological rather than measure-theoretic, we will need to endow them with a canonical measure - Haar measure - first.

Let G be a compact metrisable topological group (not necessarily abelian). Without an ambient measure, we cannot yet define the convolution $f * g$ of two continuous functions $f, g \in C(G)$. However, we can define the convolution $\mu * f$ of a finite Borel measure μ on G and a continuous function $f \in C(G)$ to be the function

$$\mu * f(x) := \int_G f(y^{-1}x)\ d\mu(y), \quad (2)$$

which (by the uniform continuity of f) is easily seen to be another continuous function. We similarly define

$$f * \mu(x) := \int_G f(xy^{-1})\ d\mu(y). \quad (2')$$

Also, one can define the convolution $\mu * \nu$ of two finite Borel measures to be the finite Borel measure defined as

$$\mu * \nu(E) := \int_G \nu(y^{-1} \cdot E)\ d\mu(y) \quad (3)$$

for all Borel sets E. For instance, the convolution $\delta_x * \delta_y$ of two Dirac masses is another Dirac mass δ_{xy}. Fubini's theorem tells us that the convolution of two finite measures is another finite measure. Convolution is also bilinear and associative (thus $(\mu * \nu) * \rho = \mu * (\nu * \rho)$, $f * (\mu * \nu) = (f * \mu) * \nu$, $(\mu * f) * \nu = \mu * (f * \nu)$, and $(\mu * \nu) * f = \mu * (\nu * f)$ for measures μ, ν, ρ and continuous f); in particular, left-convolution and right-convolution commute. Also observe that the convolution of two Borel probability measures is again a Borel probability measure. Convolution also has a powerful *smoothing effect* that can upgrade weak convergence to strong convergence. Specifically, if μ_n converges in the vague sense to μ, and f is continuous, then an easy application of compactness of the underlying group G reveals that $\mu_n * f$ converges in the uniform sense to $\mu * f$.

Let us say that a number c is a *left-mean* (resp. *right-mean*) of a continuous function $f \in C(G)$ if there exists a probability measure μ such that $\mu * f$ (resp. $f * \mu_n$) is equal to a constant c. For compact metrisable groups G, this mean is well defined:

Lemma 1. (Existence and uniqueness of mean) Let G be a compact metrisable topological group, and let $f \in C(G)$. Then there exists a unique constant c which is both a left-mean and right-mean of f.

Proof. Without loss of generality we can take f to be real-valued. Let us first show that there exists a left-mean. Define the oscillation of a real-valued continuous function to be the difference between its maximum and minimum. By the vague sequential compactness of probability measures (Lemma 1 from Lecture 7), one can find a probability measure μ which minimises the oscillation of $\mu * f$. If this oscillation is zero, we are done. If the oscillation is non-zero, then (using the compactness of the group and the transitivity of the group action) it is not hard to find a finite number of left rotations of $\mu * f$ whose average has strictly smaller oscillation than that of $\mu * f$ (basically by rotating the places where $\mu * f$ is near its maximum to cover where it is near its mimum). Thus we have a finitely supported probability ν with $\nu * \mu * f$ having smaller oscillation than $\mu * f$, a contradiction. We thus see that a left-mean exists. Similarly, a right-mean exists. But since left-convolution commutes with right-convolution, we see that all left-means are equal to all right-means, and the claim follows. $\diamond$

The map $f \mapsto c$ from a continuous function to its mean is a bounded non-negative linear functional on C(G) which preserves constants, and thus by the Riesz representation theorem is given by a unique probability measure μ; since left- and right-convolution commute, we see that this measure is both left- and right- invariant. Conversely, given any such measure μ we easily see (again using the commutativity of left-and right-convolution) that $f * \mu = \mu * f = c$. We have thus shown

Corollary 1. (Existence and uniqueness of Haar measure) If G is a compact metrisable topological group, then there exists a unique Borel probability measure μ on G which is both left and right invariant.

In particular, every topological Kronecker system $(K, x \mapsto x + \alpha)$ can be canonically converted into a measure-preserving system, which is then compact by the same argument used to establish Example 2. (Actually this observation works for non-abelian Kronecker systems as well as abelian ones.)

Remark 2. One can also build left- and right- Haar measures for locally compact groups; these measures are locally finite Radon measures rather than Borel probability measures, and are unique up to constants; however it is no longer the case that such measures are necessarily equal to each other except in special cases, such as when the group is abelian or compact. These measures play an important role in the harmonic analysis and representation theory of such groups, but we will not discuss these topics further here. $\diamond$

– Classification of compact systems –

We have just seen that every Kronecker system is a compact system. The converse is not quite true; consider for instance the disjoint union of two Kronecker systems from different groups (with the probability measure being split, say, 50-50, between the two components). The situation is similar to that in Lecture 6, in which every Kronecker system was equicontinuous and isometric, but the converse only held under the additional assumption of minimality. There is a similar situation here, but first we need to define the notion of *equivalence* of two measure-preserving systems.

Define an *abstract measure-preserving system* $(\mathcal{X}, \mu, T)$ to be an abstract separable σ-algebra $\mathcal{X}$ (which, by Stone's theorem, can be viewed as consisting of subsets of some ambient space X, though for our purposes it is better here to stay in the abstract world), together with an abstract probability measure $\mu: \mathcal{X} \to [0,1]$ and an abstract invertible shift $T: \mathcal{X} \to \mathcal{X}$ which preserves the measure μ (but does not necessarily come from an invertible map $T: X \to X$ on some ambient space). (An abstract measure space $(\mathcal{X}, \mu)$ is sometimes also known as a *measure algebra*.) There is an obvious notion of a morphism $\Phi: (\mathcal{X}, \mu, T) \to (\mathcal{Y}, \nu, S)$ between abstract measure-preserving systems, in which $\Phi: \mathcal{Y} \to \mathcal{X}$ (note the contravariance) is a σ-algebra homomorphism with $\nu = \mu \circ \Phi$ and $S \circ \Phi = \Phi \circ T$. This makes the class of abstract measure-preserving systems into a category. In particular we have a notion of two abstract measure-preserving systems being isomorphic.

Example 5. Let $(X, \mathcal{X}, \mu, T)$ be a skew shift $(y, z) \mapsto (y + \alpha, z + y)$ and let $(Y, \mathcal{Y}, \nu, S)$ be the underlying circle shift $y \mapsto y + \alpha$. These systems are of course non-isomorphic, although there is a factor map $\pi: X \to Y$ which is a morphism. If however we consider the σ-algebra $\pi^\#(\mathcal{Y}) \subset \mathcal{X}$ (which are the Cartesian products of horizontal Borel sets with the vertical circle $\mathbb{R}/\mathbb{Z}$), we see that π induces an isomorphism between the abstract measure-preserving systems $(\pi^\#(\mathcal{Y}), \mu, T)$, and $(\mathcal{Y}, \nu, S)$. $\diamond$

Given a concrete measure-preserving system $(X, \mathcal{X}, \mu, T)$, we can define its *abstraction* $(\mathcal{X}/\sim, \mu, T)$, where $\sim$ is the equivalence relation of almost everywhere equivalence modulo μ. In category theoretic language, abstraction is a covariant functor from the category of concrete measure-preserving systems to the category of abstract measure-preserving systems. We say that two concrete measure-preserving systems are *equivalent* if their abstractions are isomorphic. Thus for instance, in Example 5 above, $(X, \pi^\#(\mathcal{Y}), \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ are equivalent; there is no concrete isomorphism between the these two systems, but once one abstracts away the underlying sets X and Y, we can recover an equivalence. As another example, we see that if we add or remove a null set to a measure-preserving system, we obtain an abstractly equivalent measure-preserving system.

Remark 3. Up to null sets, we can also identify an abstract measure-preserving system $(\mathcal{X}, \mu, T)$ with its abelian von Neumann algebra $L^\infty(\mathcal{X}, \mu)$ (which acts on the Hilbert space $L^2(\mathcal{X}, \mu)$ by pointwise multiplication), together with an automorphism T of that algebra; conversely, one can recover the algebra $\mathcal{X}$ as the idempotents 1_E of the von Neumann algebra, and the measure $\mu(E)$ of a set being the trace of the idempotent 1_E. A significant portion of ergodic theory can in fact be rephrased in terms of von Neumann algebras (which, in particular, naturally suggests a non-commutative generalisation of the subject), although we will not adopt this perspective here. $\diamond$

Many results and notions about concrete measure-preserving systems $(X, \mathcal{X}, \mu, T)$ can be rephrased to not require knowledge of the underlying space X (and to be stable under modification by null sets), and so can be converted to statements about abstract measure-preserving systems; for instance, the

Furstenberg recurrence theorem is of this form once one replaces "non-empty" with "positive measure" (see Exercise 1 from Lecture 10). The notion of ergodicity is also of this form. In particular, such results and notions automatically become preserved under equivalence. In view of this, the following classification result is of interest:

Theorem 2. (Classification of ergodic compact systems) Every ergodic compact system is equivalent to an (abelian) Kronecker system.

To prove this theorem, it is convenient to use a harmonic analysis approach. Define an *eigenfunction* of a measure-preserving system $(X, \mathcal{X}, \mu, T)$ to be a bounded measurable function f, not a.e. zero, such that $Tf = \lambda f$ a.e..

Let $\mathcal{Z}_1 \subset \mathcal{X}$ denote the σ-algebra generated by all the eigenfunctions. Note that this contains $\mathcal{Z}_0 := \mathcal{X}^T$, which is the σ-algebra generated by the eigenfunctions with eigenvalue 1. We have the following fundamental result:

Proposition 2. (Description of the almost periodic functions) Let $(X, \mathcal{X}, \mu, T)$ be an ergodic measure-preserving system, and let $f \in L^2(X, \mathcal{X}, \mu)$. Then f is almost periodic if and only if it lies in $L^2(X, \mathcal{Z}_1, \mu)$, i.e. if it is $\mathcal{Z}_1$-measurable (note that $\mathcal{Z}_1$ contains all null sets of $\mathcal{X}$).

Remark 4. One can view $(X, \mathcal{Z}_1, \mu, T)$ as the maximal compact factor of $(X, \mathcal{X}, \mu, T)$, in much the same way that $(X, \mathcal{Z}_0, \mu, T)$ is the maximal factor on which the system is essentially trivial (every function is essentially invariant). $\diamond$

Proof. It is clear that every eigenfunction is almost periodic. From repeated application of Exercise 2 we conclude that the indicator of any set in $\mathcal{Z}_1$ is also almost periodic, and thus (by more applications of Exercise 2) every function in $L^2(X, \mathcal{Z}_1, \mu)$ is almost periodic.

Conversely, suppose $f \in L^2(X, \mathcal{X}, \mu)$ is almost periodic. Then the orbit closure $Y_f \subset L^2(X, \mathcal{X}, \mu)$ of f is an isometric system; the orbit of f is clearly dense in Y_f, and thus by isometry the orbit of every other point is also dense. Thus Y_f is minimal, and therefore Kronecker by Proposition 1 from Lecture 6; thus we have an isomorphism $\phi : K \to Y_f$ from a group rotation $(K, x \mapsto x + \alpha)$ to Y_f. By rotating if necessary we may assume that $\phi(0) = f$.

By Corollary 1, K comes with an invariant probability measure ν. The theory of Fourier analysis on compact abelian groups then says that $L^2(K, \nu)$ is spanned by an (orthonormal) basis of characters χ. In particular, the Dirac mass at 0 (the group identity of K) can be expressed as the weak limit of finite linear combinations of such characters.

Now we need to move this information back to X. For this we use the operator $S : L^2(K, \nu) \to L^2(X, \mathcal{X}, \mu)$ defined by $Sh := \int_K \phi(y) h(y)\ d\nu(y)$; one checks from Minkowski's integral inequality that this is a bounded linear map. Because ϕ is a morphism, and each character is an eigenfunction of the group rotation $x \mapsto x + \alpha$, one easily checks that the image $S\chi$ of a character χ is an eigenfunction. Since the image of the Dirac mass is (formally) just f, we thus conclude that f is the weak limit of finite linear combinations of characters. (One can in fact use compactness and continuity to make this a strong limit, but this is not necessary here.) In particular, f is equivalent a.e. to a $\mathcal{Z}_1$-measurable function, as desired. □

Exercise 5. (Spectral description of Kronecker factor) Show that the product of two eigenfunctions is again an eigenfunction. Using this and Proposition 2, conclude that $L^2(X, \mathcal{Z}_1, \mu)$ is in fact equal to $\mathbb{H}_{pp}$, the closed subspace of the Hilbert space $\mathbb{H} := L^2(X, \mathcal{X}, \mu)$ generated by the eigenfunctions of the shift operator T. ◇

Exercise 6. Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let $f \in L^2(X, \mathcal{X}, \mu)$. We say that f is quasiperiodic if the orbit $\{T^n f : n \in \mathbb{Z}\}$ lies in a finite-dimensional space. Show that a function is quasiperiodic if and only if it is a finite linear combination of eigenfunctions. Deduce that a function is almost periodic if and only if it is the limit in L^2 of quasiperiodic functions. ◇

Exercise 7. The purpose of this exercise is to show how abstract measure-preserving systems, and the morphisms between them, can be satisfactorily modeled by concrete systems and morphisms.

1. Let $(\mathcal{X}, \mu, T)$ be an abstract measure-preserving system. Show that there exists a concrete regular measure-preserving system $(X', \mathcal{X}', \mu', T')$ which is equivalent to $(\mathcal{X}, \mu, T)$ (thus after omitting X' and quotienting out both σ-algebras by null sets, the two resulting abstract measure-preserving systems are isomorphic); the notion of regularity was defined in Definition 2 of Lecture 9. *Hint:* take a countable shift-invariant family of sets that generate $\mathcal{X}$ (thus T acts on this space by permutation), and use this to create a σ-algebra morphism from $\mathcal{X}$ to $\mathcal{X}'$, the product σ-algebra of some boolean space $X' := 2^{\mathbb{Z}}$, endowed with a permutation action T'.
2. Let $\phi : (\mathcal{X}, \mu, T) \to (\mathcal{Y}, \nu, S)$ be an abstract morphism. Show that there exist regular measure-preserving systems $(X', \mathcal{X}', \mu', T')$ and $(Y', \mathcal{Y}', \nu', S')$ equivalent to $(\mathcal{X}, \mu, T)$ and $(\mathcal{Y}, \nu, S)$, together with a concrete morphism $\phi' : X' \to Y'$, such that obvious commuting square connecting the abstract σ-algebras $\mathcal{X}, \mathcal{Y}, \mathcal{X}', \mathcal{Y}'$ quotiented out by null sets does indeed commute. ◇

Remark 6. Exercise 7 (and various related results) show that the distinction between concrete and abstract measure-preserving systems are very minor in practice. There are however other areas of mathematics in which taking an abstract or "point-less" approach by deleting (or at least downplaying) the underlying space can lead to non-trivial generalisations or refinements of the original concrete concept, for instance when moving from varieties to schemes. ◇

Proof of Theorem 2. Note that if f is an eigenfunction then T|f|=|f|, and so (if the system is ergodic) |f| is a.e. constant (which implies also that the eigenvalue lies on the unit circle). In particular, any eigenfunction is invertible. The quotient of two eigenfunctions of the same eigenvalue is then T-invariant and thus constant a.e. by ergodicity, which shows that all eigenspaces have geometric multiplicity 1 modulo null sets. As T is unitary, any eigenfunctions of different eigenvalues are orthogonal to each other; as $L^2(X, \mathcal{X}, \mu)$ is separable, we conclude that the number of eigenfunctions (up to constants and a.e. equivalence) is at most countable.

Let $(\phi_n)_{n \in A}$ be a collection of representative eigenfunctions for some at most countable index set A with eigenvalues λ_n; we can normalise $|\phi_n| = 1$ a.e.. By modifying each eigenfunction on a set of measure zero (cf. Exercise 7 of Lecture 8) we can assume that $T\phi_n = \lambda_n \phi_n$ and $|\phi_n| = 1$ *everywhere* rather than just almost everywhere. Then the map $\Phi : x \mapsto (\log \phi_n(x))_{n \in A}$ is a morphism from $(X, \mathcal{X}, \mu, T)$ to the torus $(\mathbb{R}/\mathbb{Z})^A$ with the product σ-algebra $\mathcal{B}$, the push-forward measure $\Phi_\# \mu$, and the shift $x \mapsto x + \alpha$, where $\alpha := (\log \lambda_n)_{n \in A}$. From Proposition 2 we see that every measurable set in $\mathcal{X}$ differs by a null set from a set in the pullback σ-algebra $\Phi^\#(\mathcal{B})$. From this it is not hard to see that $(X, \mathcal{X}, \mu, T)$ is equivalent to the system $(\mathbb{R}/\mathbb{Z})^A, \mathcal{B}, \Phi_\# \mu, x \mapsto x + \alpha)$.

Now, $\mathbb{R}/\mathbb{Z})^A$ is a compact metrisable space. The orbit closure K of α inside this space is thus also compact metrisable. The support of $\Phi_\# \mu$ is shift-invariant and thus K-invariant; but from the ergodicity of μ we conclude that the support must in fact be a single translate of K. In particular, $\Phi_\# \mu$ is just a translate of Haar measure on K. From this one easily concludes that $(\mathbb{R}/\mathbb{Z})^A, \mathcal{B}, \Phi_\# \mu, x \mapsto x + \alpha)$ is equivalent to the Kronecker system $(K, x \mapsto x + \alpha)$ with the Borel σ-algebra and Haar measure, and the claim follows. $\square$

Exercise 8. Let $(X, \mathcal{X}, \mu, T)$ be a compact system which is not necessarily ergodic, and let $y \mapsto \mu_y$ be the ergodic decomposition of μ relative to the projection $\pi : (X, \mathcal{X}, \mu, T) \to (Y, \mathcal{Y}, \nu, S)$ given by Proposition 4 of Lecture 9. Show that $(X, \mathcal{X}, \mu_y, T)$ is a compact ergodic system for ν-almost every y. From this and Theorem 2, we conclude that every compact system can be disintegrated into ergodic Kronecker systems (cf. the discussion after Proposition 2 of Lecture 6). $\diamond$

Remark 7. We comment here on finitary versions of the above concepts. Consider the cyclic group system $(\mathbb{Z}/N\mathbb{Z}, x \mapsto x+1)$ with the discrete σ-algebra and uniform probability measure. Strictly speaking, every function on this system is periodic with period N and thus almost periodic, and so this is a compact system. But suppose we consider N as a large parameter going to infinity (in which case one can view these systems, together with some function $f = f_N$ on these systems, "converging" to some infinite system with some limit function f, as in the derivation of Theorem 3 from Theorem 2 in the previous lecture). Then we would be interested in *uniform* control on the almost periodicity of the function or the compactness of the system, i.e. quantitative bounds involving expressions such as O(1) which are bounded uniformly in N. With such a perspective, the analogue of a quasiperiodic function (see Exercise 6) is a function $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ which is a linear combination of at most O(1) characters (i.e. its Fourier transform is non-zero at only O(1) frequencies), whilst an almost periodic function f is one which is approximable in L^2 by quasiperiodic functions, thus for every $\varepsilon > 0$ one can find a function with only $O_\varepsilon(1)$ frequencies which lies within ε of f in L^2 norm. Most functions on $\mathbb{Z}/N\mathbb{Z}$ for large N are not like this, and so the cyclic shift system is not compact in the asymptotic limit $N \to \infty$; however if one coarsens the underlying σ-algebra significantly one can recover compactness, though unfortunately one has to replace exact shift-invariance by approximate shift-invariance when one does so. For instance if one considers a σ-algebra $\mathcal{B}$ generated by a bounded (O(1)) number of Bohr sets $\{n \in \mathbb{Z}/N\mathbb{Z} : \|\frac{\xi n}{N} - a\|_{\mathbb{R}/\mathbb{Z}} \leq \varepsilon\}$, then $\mathcal{B}$ is no longer shift-invariant in general, but all the functions which are measurable with respect to this algebra are uniformly almost periodic in the above sense. For some further developments of these sorts of "quantitative ergodic theory" ideas, see these papers of Ben Green and myself. $\diamond$

[*Update*, Feb 12: Another exercise and remark added.]

254A, Lecture 12: Weakly mixing systems

21 February, 2008 in 254A - ergodic theory, math.DS, math.FA

Tags: Cauchy-Schwarz, compactness, Hilbert-Schmidt, randomness, Roth's theorem, strong mixing, van der Corput, weak mixing

In the previous lecture, we studied the recurrence properties of compact systems, which are systems in which all measurable functions exhibit almost periodicity - they almost return completely to themselves after repeated shifting. Now, we consider the opposite extreme of *mixing systems* - those in which all measurable functions (of mean zero) exhibit *mixing* - they become orthogonal to themselves after repeated shifting. (Actually, there are two different types of mixing, *strong mixing* and *weak mixing*, depending on whether the orthogonality occurs individually or on the average; it is the latter concept which is of more importance to the task of establishing the Furstenberg recurrence theorem.)

We shall see that for weakly mixing systems, averages such as $\frac{1}{N}\sum_{n=0}^{N-1} T^n f \ldots T^{(k-1)n} f$ can be computed very explicitly (in fact, this average converges to the constant $(\int_X f\ d\mu)^{k-1}$). More generally, we shall see that weakly mixing components of a system tend to average themselves out and thus become irrelevant when studying many types of ergodic averages. Our main tool here will be the humble Cauchy-Schwarz inequality, and in particular a certain consequence of it, known as the *van der Corput lemma*.

As one application of this theory, we will be able to establish Roth's theorem (the k=3 case of Szemerédi's theorem).

— Mixing functions –

Much as compact systems were characterised by their abundance of almost periodic functions, we will characterise mixing systems by their abundance of mixing functions (this is not standard terminology). To define and motivate this concept, it will be convenient to introduce a weak notion of convergence (this notation is also not standard):

> **Definition 1.** (Cesàro convergence) A sequence c_n in a normed vector space is said to *converge in the Cesàro sense* to a limit c if the averages $\frac{1}{N}\sum_{n=0}^{N-1} c_n$ converge strongly to c, in which case we write $C-\lim_{n\to\infty} c_n = c$. We also write $C-\sup_{n\to\infty} c_n := \limsup_{N\to\infty} \|\frac{1}{N}\sum_{n=0}^{N-1} c_n\|$ (thus $C-\lim_{n\to\infty} c_n = 0$ if and only if $C-\sup_{n\to\infty} c_n = 0$).

Example 1. The sequence $0, 1, 0, 1, \ldots$ has a Cesàro limit of 1/2. $\diamond$

Exercise 1. Let c_n be a bounded sequence of *non-negative* numbers. Show that the following three statements are equivalent:

1. $C-\lim_{n\to\infty} c_n = 0$.
2. $C-\lim_{n\to\infty} |c_n|^2 = 0$.
3. c_n converges to zero in density. [We say c_n *converges in density* to c if for any $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |c_n - c| > \varepsilon\}$ has upper density zero.]

Which of the implications between 1, 2, 3 remain valid if c_n is not bounded? $\diamond$Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let $f \in L^2(X, \mathcal{X}, \mu)$ be a function. We consider the *correlation coefficients* $\langle T^n f, f\rangle := \int_X T^n f \overline{f}\ d\mu$ as n goes to infinity. Note that we have the symmetry $\langle T^n f, f\rangle = \overline{\langle T^{-n} f, f\rangle}$, so we only need to consider the case when n is positive. The mean ergodic theorem (Corollary 2 from Lecture 8) tells us the Cesàro behaviour of these coefficients. Indeed, we have

$$C-\lim_{n\to\infty}\langle T^n f, f\rangle = \langle \mathbb{E}(f|\mathcal{X}^T), f\rangle = \|\mathbb{E}(f|\mathcal{X}^T)\|^2_{L^2(X,\mathcal{X},\mu)} \quad (1)$$

where $\mathcal{X}^T$ is the σ-algebra of essentially shift-invariant sets. In particular, if the system is ergodic, and f has mean zero (i.e. $\int_X f\ d\mu = 0$), then we have

$$C-\lim_{n\to\infty}\langle T^n f, f\rangle = 0, \quad (2)$$

thus the correlation coefficients go to zero in the Cesàro sense. However, this does not necessarily imply that these coefficients go to zero pointwise. For instance, consider a circle shift system $(\mathbb{R}/\mathbb{Z}, x \mapsto x+\alpha)$ with α irrational (and with uniform measure), thus this system is ergodic by Exercise 5 from Lecture 9. Then the function $f(x) := e^{2\pi i x}$ has mean zero, but one easily computes that $\langle T^n f, f\rangle = e^{2\pi i n\alpha}$. The coefficients $e^{2\pi i n\alpha}$ converge in the Cesàro sense to zero, but have magnitude 1 and thus do not converge to zero pointwise.

Definition 2. (Mixing) Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. A function $f \in L^2(X, \mathcal{X}, \mu)$ is *strongly mixing* if $\lim_{n\to\infty}\langle T^n f, f\rangle = 0$, and *weakly mixing* if $C-\lim_{n\to\infty}|\langle T^n f, f\rangle| = 0$.

Remark 1. Clearly strong mixing implies weak mixing. From (1) we also see that if f is weakly mixing, then $\mathbb{E}(f|\mathcal{X}^T)$ must vanish a.e. $\diamond$

Exercise 2. Show that if f is both almost periodic and weakly mixing, then it must be 0 almost everywhere. In particular, in a compact system, the only weakly mixing function is 0 (up to a.e. equivalence). $\diamond$

Exercise 3. In any Bernoulli system $\Omega^{\mathbb{Z}}$ with the product σ-algebra and a product measure, and the standard shift, show that any function of mean zero is strongly mixing. (*Hint*: first do this for functions that depend on only finitely many of the variables.) $\diamond$

Exercise 4. Consider a skew shift system $((\mathbb{R}/\mathbb{Z})^2, (x,y) \mapsto (x+\alpha, y+x))$ with the usual Lebesgue measure and Borel σ-algebra, and with α irrational. Show that the function $f(x,y) := e^{2\pi i x}$ is neither strongly mixing nor weakly mixing, but that the function $g(x,y) := e^{2\pi i y}$ is both strongly mixing and weakly mixing. $\diamond$

Exercise 5. Let $X := \mathbb{C}^{\mathbb{Z}}$ be given the product Borel σ-algebra $\mathcal{X}$ and the shift $T: (z_n)_{n \in \mathbb{Z}} \to (z_{n+1})_{n \in \mathbb{Z}}$. For each $d \geq 1$, let μ_d be the probability distribution in X of the random sequence $(z_n)_{n \in \mathbb{Z}}$ given by the rule

$$z_n := \frac{1}{2^{d/2}} \sum_{\omega_1,\ldots,\omega_d \in \{0,1\}} w_{\omega_1,\ldots,\omega_d} e^{2\pi i \sum_{j=1}^d \omega_j n/100^j}, \quad (3)$$

where the $w_{\omega_1,\ldots,\omega_d}$ are iid standard complex Gaussians (thus each w has probability distribution $e^{-\pi|w|^2}\ dw$). Show that each μ_d is shift invariant. If μ is a vague limit point of the sequence μ_d, and $f: X \to \mathbb{C}$ is the function defined as $f((z_n)_{n \in \mathbb{Z}}) := \mathrm{sgn}(\mathrm{Re} z_0)$, show that f is weakly mixing but not strongly mixing (and more specifically, that $\langle T^{100^j} f, f \rangle$ stays bounded away from zero) with respect to the system $(X, \mathcal{X}, \mu, T)$. $\diamond$

Remark 2. Exercise 5 illustrates an important point, namely that stationary processes yield a rich source of measure-preserving systems (indeed the two notions are almost equivalent in some sense, especially after one distinguishes a specific function f on the measure-preserving system). However, we will not adopt this more probabilistic perspective to ergodic theory here. $\diamond$

Remark 3. We briefly discuss the finitary analogue of the weak mixing concept in the context of functions $f: \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$ on a large cyclic group $\mathbb{Z}/N\mathbb{Z}$ with the usual shift $x \mapsto x+1$. Then one can compute

$$C-\lim_{n \to \infty} |\langle T^n f, f \rangle|^2 = \sum_{\xi \in \mathbb{Z}/N\mathbb{Z}} |\hat{f}(\xi)|^4 \quad (4)$$

where $\hat{f}(\xi) := \frac{1}{N} \sum_{x \in \mathbb{Z}/N\mathbb{Z}} f(x) e^{-2\pi i x \xi/N}$ are the Fourier coefficients of f. Comparing this against the Plancherel identity $\|f\|_{L^2}^2 = \sum_{\xi \in \mathbb{Z}/N\mathbb{Z}} |\hat{f}(\xi)|^2$ we thus see that a function f bounded in L^2 norm should be considered "weakly mixing" if it has no large Fourier coefficients. Contrast this with Remark 7 from Lecture 11. $\diamond$

Now let us see some consequences of the weak mixing property. We need the following lemma, which gives a useful criterion as to whether a sequence of bounded vectors in a Hilbert space converges in the Cesàro sense to zero.

Lemma 1 (van der Corput lemma). Let $v_1, v_2, v_3, \ldots$ be a bounded sequence of vectors in a Hilbert space H. If

$$C-\lim_{h \to \infty} C-\sup_{n \to \infty} \langle v_n, v_{n+h} \rangle = 0 \quad (5)$$

then $C-\lim_{n \to \infty} v_n = 0$.

Informally, this lemma asserts that if each vector in a bounded sequence tends to be orthogonal to nearby elements in that sequence, then the vectors will converge to zero in the Cesàro sense. This formulation of the lemma is essentially the version in this paper by Bergelson, except that we have made the minor change of replacing one of the Cesàro limits with a Cesàro supremum.

Proof. We can normalise so that $\|v_n\| \leq 1$ for all n. In particular, we have $v_n = O(1)$, where O(1) denotes a vector of bounded magnitude. For any h and $N \geq 1$, we thus have the telescoping identity

$$\frac{1}{N}\sum_{n=0}^{N-1} v_{n+h} = \frac{1}{N}\sum_{n=0}^{N-1} v_n + O(|h|/N); \quad (6)$$

averaging this over all h from 0 to H-1 for some $H \geq 1$, we obtain

$$\frac{1}{N}\sum_{n=0}^{N-1} \frac{1}{H}\sum_{h=0}^{H-1} v_{n+h} = \frac{1}{N}\sum_{n=0}^{N-1} v_n + O(H/N); \quad (7)$$

by the triangle inequality we thus have

$$\|\frac{1}{N}\sum_{n=0}^{N-1} v_n\| \leq \frac{1}{N}\sum_{n=0}^{N-1} \|\frac{1}{H}\sum_{h=0}^{H-1} v_{n+h}\| + O(H/N) \quad (8)$$

where the O() terms are now scalars rather than vectors. We square this (using the crude inequality $(a+b)^2 \leq 2a^2 + 2b^2$) and apply Cauchy-Schwarz to obtain

$$\|\frac{1}{N}\sum_{n=0}^{N-1} v_n\|^2 \leq O(\frac{1}{N}\sum_{n=0}^{N-1} \|\frac{1}{H}\sum_{h=0}^{H-1} v_{n+h}\|^2) + O(H^2/N^2) \quad (9)$$

which we rearrange as

$$\|\frac{1}{N}\sum_{n=0}^{N-1} v_n\|^2 \leq O(\frac{1}{H^2}\sum_{0\leq h,h'<H} \frac{1}{N}\sum_{n=0}^{N-1}\langle v_{n+h}, v_{n+h'}\rangle) + O(H^2/N^2). \quad (10)$$

We take limits as $N \to \infty$ (keeping H fixed for now) to conclude

$$\limsup_{N\to\infty} \|\frac{1}{N}\sum_{n=0}^{N-1} v_n\|^2$$

$$\leq O(\frac{1}{H^2}\sum_{0\leq h,h'<H} C-\sup_{n\to\infty}\langle v_{n+h}, v_{n+h'}\rangle). \quad (11)$$

Another telescoping argument (and symmetry) gives us

$$C-\sup_{n\to\infty}\langle v_{n+h}, v_{n+h'}\rangle = C-\sup_{n\to\infty}\langle v_{n+|h-h'|}, v_n\rangle \quad (12)$$

and so

$$\limsup_{N\to\infty} \|\frac{1}{N}\sum_{n=0}^{N-1} v_n\|^2 \leq O(\frac{1}{H}\sum_{0\leq h<H} C-\sup_{n\to\infty}\langle v_{n+h}, v_n\rangle). \quad (13)$$

Taking limits as $H \to \infty$ and using (5) we obtain the claim. □

Exercise 6. Let $P: \mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ be a polynomial with at least one irrational non-constant coefficient. Using Lemma 1 (in the scalar case $H = \mathbb{C}$) and an induction on degree, show that $C-\lim_{n\to\infty} e^{2\pi i P(n)} = 0$. Conclude that the sequence $(P(n))_{n\in\mathbb{N}}$ is uniformly distributed with respect to uniform measure (see Lecture 9 for a definition of uniform distribution). ◇

Exercise 7. Using Exercise 6, give another proof of Theorem 1 from Lecture 6. ◇

We now apply the van der Corput lemma to weakly mixing functions.

Corollary 1. Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let $f \in L^2(X, \mathcal{X}, \mu)$ be weakly mixing. Then for any $g \in L^2(X, \mathcal{X}, \mu)$ we have $C-\lim_{n\to\infty} |\langle T^n f, g\rangle| = 0$ and $C-\lim_{n\to\infty} |\langle f, T^n g\rangle| = 0$.

Proof. We just prove the first claim, as the second claim is similar. By Exercise 1, it suffices to show that

$$\frac{1}{N}\sum_{n=0}^{N-1} |\langle T^n f, g\rangle|^2 \to 0 \quad (14)$$

as $N \to \infty$. The left-hand side can be rewritten as

$$\langle \frac{1}{N}\sum_{n=0}^{N-1} \langle g, T^n f\rangle T^n f, g\rangle \quad (15)$$

so by Cauchy-Schwarz it suffices to show that

$$C-\lim_{N\to\infty} \langle g, T^n f\rangle T^n f = 0. \quad (16)$$

Applying the van der Corput lemma and discarding the bounded coefficients $\langle g, T^n f\rangle$, it suffices to show that

$$C-\lim_{H\to\infty} C-\sup_{n\to\infty} |\langle T^{n+h} f, T^n f\rangle| = 0. \quad (17)$$

But $\langle T^{n+h} f, T^n f\rangle = \langle T^h f, f\rangle$, and the claim now follows from the weakly mixing nature of f. $\square$

– Weakly mixing systems –

Now we consider systems which are full of mixing functions.

Definition 3. (Mixing systems) A measure-preserving system $(X, \mathcal{X}, \mu, T)$ is *weakly mixing* (resp. *strongly mixing*) if every function $f \in L^2(X, \mathcal{X}, \mu)$ with mean zero is weakly mixing (resp. strongly mixing).

Example 2. From Exercise 2, we know that any system with a non-trivial Kronecker factor is not weakly mixing (and thus not strongly mixing). On the other hand, from Exercise 3, we know that any Bernoulli system is strongly mixing (and thus weakly mixing also). From Remark 1 we see that any strongly or weakly mixing system must be ergodic. $\diamond$

Exercise 8. Show that the system in Exercise 5 is weakly mixing but not strongly mixing. $\diamond$

Here is another characterisation of weak mixing:

Exercise 9. Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system. Show that the following are equivalent:

1. $(X, \mathcal{X}, \mu, T)$ is weakly mixing.
2. For every $f, g \in L^2(X, \mathcal{X}, \mu)$, $\langle T^n f, g\rangle$ converges in density to $(\int_X f\ d\mu)(\int_X g\ d\mu)$. (See Exercise

1 for a definition of convergence in density.)
3. For any measurable E, F, $\mu(T^n E \cap F)$ converges in density to $\mu(E)\mu(F)$.
4. The product system $(X \times X, \mathcal{X} \times \mathcal{X}, \mu \times \mu, T \times T)$ is ergodic.

[Hints: To equate 1 and 2, use the decomposition $f = (f - \int_X f\ d\mu) + \int_X f\ d\mu$ of a function into its mean and mean-free components. To equate 2 and 4, use the fact that the space $L^2(X \times X, \mathcal{X} \times \mathcal{X}, \mu \times \mu)$ is spanned (in the topological vector space sense) by tensor products $(x, y) \mapsto f(x)g(y)$ with $f, g \in L^2(X, \mathcal{X}, \mu)$.] $\diamond$

Exercise 10. Show that the equivalences between 1, 2, 3 in Exercise 9 remain if "weak mixing" and "converges in density" are replaced by "strong mixing" and "converges" respectively. $\diamond$

Exercise 11. Let $(X, \mathcal{F}, T)$ be any minimal topological system with Borel σ-algebra $\mathcal{B}$, and let μ be a shift invariant Borel probability measure. Show that if $(X, \mathcal{B}, \mu, T)$ is weakly mixing (resp. strongly mixing), then $(X, \mathcal{F}, T)$ is topologically weakly mixing (resp. topologically mixing), as defined in Definition 3 and Exercise 12 of Lecture 7. $\diamond$

Exercise 12. If $(X, \mathcal{X}, \mu, T)$ is weakly mixing, show that $(X, \mathcal{X}, \mu, T^n)$ is weakly mixing for any non-zero n. $\diamond$

Exercise 13. Let $(X, \mathcal{X}, \mu, T)$ be a measure preserving system. Show that the following are equivalent:

1. $(X, \mathcal{X}, \mu, T)$ is weakly mixing.
2. Whenever $(Y, \mathcal{Y}, \nu, S)$ is ergodic, the product system $(X \times Y, \mathcal{X} \times \mathcal{Y}, \mu \times \nu, T \times S)$ is ergodic.

(Hint: To obtain 1 from 2, use Exercise 9. To obtain 2 from 1, repeat the *methods* used to prove Exercise 9.) $\diamond$

Exercise 14. Show that the product of two weakly mixing systems is again weakly mixing. (Hint: use Exercises 9 and 13.) $\diamond$

Now we come to an important type of observation for the purposes of establishing the Furstenberg recurrence theorem: in weakly mixing systems, functions of mean zero are negligible as far as multiple averages are concerned.

Proposition 1. Let $a_1, \ldots, a_k \in \mathbb{Z}$ be distinct non-zero integers for some $k \geq 1$. Let $(X, \mathcal{X}, \mu, T)$ be weakly mixing, and let $f_1, \ldots, f_k \in L^\infty(X, \mathcal{X}, \mu)$ be such that at least one of $f_1, \ldots, f_k$ has mean zero. Then we have

$$C-\lim_{n\to\infty} T^{a_1 n} f_1 \ldots T^{a_k n} f_k = 0 \quad (18)$$

in $L^2(X, \mathcal{X}, \mu)$.

Proof. We induct on k. When k=1 the claim follows from the mean ergodic theorem (recall from Example 2 that all weakly mixing systems are ergodic).

Now let $k \geq 2$ and suppose that the claim has already been proven for k-1. Without loss of generality we may assume that it is f_1 which has mean zero. Applying the van der Corput lemma (Lemma 1), it suffices to show that

$$C-\sup_{n \to \infty} \langle T^{a_1(n+h)} f_1 \ldots T^{a_k(n+h)} f_k, T^{a_1 n} f_1 \ldots T^{a_k n} f_k \rangle \quad (19)$$

converges in density to zero as $h \to \infty$. But the left-hand side can be rearranged as

$$C-\sup_{n \to \infty} \int_X T^{(a_1-a_k)n} f_{1,h} \ldots T^{(a_{k-1}-a_k)n} f_{k-1,h} f_{k,h}\ d\mu \quad (20)$$

where $f_{j,h} := T^{(a_j-a_k)h} f_j \overline{f_j}$. Applying Cauchy-Schwarz, it suffices to show that

$$C-\sup_{n \to \infty} T^{(a_1-a_k)n} f_{1,h} \ldots T^{(a_{k-1}-a_k)n} f_{k-1,h} \quad (21)$$

converges in density to zero as $h \to \infty$.

Since $(X, \mathcal{X}, \mu, T)$ is weakly mixing, the mean-zero function f_1 is weakly mixing, and so the mean of $f_{1,h}$ goes to zero in density as $h \to \infty$. As all functions are assumed to be bounded, we can thus subtract the mean from $f_{1,h}$ in (21) without affecting the desired conclusion, leaving behind the mean-zero component $f_{1,h} - \int_X f_{1,h}\ d\mu$. But then the contribution of this expression to (21) vanishes by the induction hypothesis. $\square$

Remark 4. The key point here was that functions f of mean zero were weakly mixing and thus had the property that $T^h f \overline{f}$ almost had mean zero, and were thus almost weakly mixing. One could iterate this further to investigate the behaviour of "higher derivatives" of f such as $T^{h+h'} f \overline{T^h f T^{h'} f} f$. Pursuing this analysis further leads to the Gowers-Host-Kra seminorms, which are closely related to the Gowers uniformity norms in additive combinatorics. $\diamond$

> **Corollary 2.** Let $a_1, \ldots, a_k \in \mathbb{Z}$ be distinct integers for some $k \geq 1$, let $(X, \mathcal{X}, \mu, T)$ be a weakly mixing system, and let $f_1, \ldots, f_k \in L^\infty(X, \mathcal{X}, \mu)$. Then $\int_X T^{a_1 n} f_1 \ldots T^{a_k n} f_k\ d\mu$ converges in the Cesáro sense to $(\int_X f_1\ d\mu) \ldots (\int_X f_k\ d\mu)$.

Note in particular that this establishes the Furstenberg recurrence theorem (Theorem 1 from Lecture 11) in the case of weakly mixing systems.

Proof. We again induct on k. The k=1 case is trivial, so suppose $k > 1$ and the claim has already been proven for k-1. If any of the functions f_j is constant then the claim follows from the induction hypothesis, so we may subtract off the mean from each function and suppose that all functions have mean zero. By shift-invariance we may also fix a_k (say) to be zero. The claim now follows from Proposition 1 and Cauchy-Schwarz. $\square$

Exercise 15. Show that the Cesáro convergence in Corollary 1 can be strengthened to convergence in density. (Hint: first reduce to the mean zero case, then apply Exercise 14 to work with the product system instead.) ◇

Exercise 16. Let $(X, \mathcal{X}, \mu, T)$ be a weakly mixing system, and let $f \in L^\infty(X, \mathcal{X}, \mu)$ have mean zero. Show that $T^{n^2} f$ converges in the Cesáro sense in $L^2(X, \mathcal{X}, \mu)$ to zero. (Hint: use van der Corput and Proposition 1 or Corollary 2.) ◇

Exercise 17. Show that Corollary 2 continues to hold if the linear polynomials $a_1 n, \ldots, a_k n$ are replaced by arbitrary polynomials $P_1(n), \ldots, P_k(n)$ from the integers to the integers, so long as the difference between any two of these polynomials is non-constant. (Hint: you will need the "PET induction" machinery from Exercise 3 of Lecture 5. This result was first established by Bergelson.) ◇

– Hilbert-Schmidt operators –

We have now established the Furstenberg recurrence theorem for two distinct types of systems: compact systems and weakly mixing systems. From Example 2 we know that these systems are indeed quite distinct from each other. Here is another indication of "distinctness":

Exercise 18. In any measure-preserving system $(X, \mathcal{X}, \mu, T)$, show that almost periodic functions and weakly mixing functions are always orthogonal to each other. ◇

On the other hand, there are certainly systems which are neither weakly mixing nor compact (e.g. the skew shift). But we have the following important dichotomy (cf. Theorem 3 from Lecture 7):

Theorem 1. Suppose that $(X, \mathcal{X}, \mu, T)$ is a measure-preserving system. Then exactly one of the following statements is true:

1. (Structure) $(X, \mathcal{X}, \mu, T)$ has a non-trivial compact factor.
2. (Randomness) $(X, \mathcal{X}, \mu, T)$ is weakly mixing.

[Note: in ergodic theory, a *factor* of a measure-preserving system is simply a morphism from that system to some other measure-preserving system. Unlike the case with topological dynamics, we do not need to assume surjectivity of the morphism, since in the measure-theoretic setting, the image of a morphism always has full measure.]

In Example 2 we have already shown that 1 and 2 cannot be both true; the tricky part is to show that lack of weak mixing implies a non-trivial compact factor.

In order to prove this result, we recall some standard results about Hilbert-Schmidt operators on a separable Hilbert space. (As usual, the hypothesis of separability is not absolutely essential, but is convenient to assume throughout; for instance, it assures that orthonormal bases always exist and are at most countable.) We begin by recalling the notion of a tensor product of two Hilbert spaces:

Proposition 2. Let H, H' be two separable Hilbert spaces. Then there exists another separable Hilbert space $H \otimes H'$ and a bilinear tensor product map $\otimes : H \times H' \to H \otimes H'$ such that

$$\langle v \otimes v', w \otimes w' \rangle_{H \otimes H'} = \langle v, w \rangle_H \langle v', w' \rangle_{H'} \quad (22)$$

for all $v, w \in H$ and $v', w' \in H'$. Furthermore, the tensor products $(e_n \otimes e'_{n'})_{n \in A, n' \in A'}$ between any orthonormal bases $(e_n)_{n \in A}$, $(e'_{n'})_{n' \in A'}$ of H and H' respectively, form an orthonormal basis of $H \otimes H'$.

It is easy to see that $H \otimes H'$ is unique up to isomorphism, and so we shall abuse notation slightly and refer to $H \otimes H'$ as **the** tensor product of H and H'.

Example 3. The tensor product of $L^2(X, \mathcal{X}, \mu)$ and $L^2(Y, \mathcal{Y}, \nu)$ is $L^2(X \times Y, \mathcal{X} \times \mathcal{Y}, \mu \times \nu)$, with the tensor product operation $f \otimes g(x, y) := f(x)g(y)$. The tensor product of $\mathbb{C}^m$ and $\mathbb{C}^n$ is $\mathbb{C}^{n \times m}$, which can be thought of as the Hilbert space of $n \times m$ (or $m \times n$) matrices, with the inner product $\langle A, B \rangle := \mathrm{tr}(AB^\dagger) = \mathrm{tr}(A^\dagger B)$. $\diamond$

Proof. Take any orthonormal bases $(e_n)_{n \in A}$ and $(e'_{n'})_{n' \in A'}$ of H and H' respectively, and let $H \otimes H'$ be the Hilbert space generated by declaring the formal quantities $e_n \otimes e'_{n'}$ to be an orthonormal basis. If one then defines

$$(\sum_n c_n e_n) \otimes (\sum_{n'} c'_{n'} e_{n'}) := \sum_n \sum_{n'} c_n c'_{n'} e_n \otimes e_{n'} \quad (23)$$

for all square-summable sequences c_n and $c'_{n'}$, one easily verifies that $\otimes$ is indeed a bilinear map that obeys (22). in particular, if $(f_m)_{m \in B}$ and $(f'_{m'})_{m' \in B'}$ are some other orthonormal bases of H, H' respectively, then from (22) $(f_m \otimes f'_{m'})_{m \in B, m' \in B'}$ is an orthonormal set, and one can approximate any element $e_n \otimes e'_{n'}$ in the original orthonormal basis to arbitrary accuracy by linear combinations from this orthonormal set, and so this set is in fact an orthonormal basis as required. $\square$

Given a Hilbert space H, define its *complex conjugate* $\overline{H}$ to be the same set as H, but with the conjugated scalar multiplication structure $z, v \mapsto \overline{z}v$ and the conjugated inner product $\langle z, w \rangle_{\overline{H}} := \overline{\langle z, w \rangle_H} = \langle w, z \rangle_H$, but with all other structures unchanged. This is also a Hilbert space. (Of course, for real Hilbert spaces rather than complex, the notion of complex conjugation is trivial.)

Example 4. The conjugation map $f \mapsto \overline{f}$ is a Hilbert space isometry between the Hilbert space $L^2(X, \mathcal{X}, \mu)$ and its complex conjugate. $\diamond$

Every element $K \in \overline{H} \otimes H'$ induces a bounded linear operator $T_K : H \to H'$, defined via duality by the formula

$$\langle T_K v, v' \rangle_{H'} := \langle K, v \otimes v' \rangle \quad (24)$$

for all $v \in H, v' \in H'$. We refer to K as the *kernel* of T_K. Any operator $T = T_K$ that arises in this manner is called a *Hilbert-Schmidt operator* from H to H'. The Hilbert space structure on the space $\overline{H} \otimes H'$ of kernels induces an analogous Hilbert space structure on the Hilbert-Schmidt operators, leading to the Hilbert-Schmidt norm $\|T\|_{HS}$ and inner product $\langle S, T \rangle_{HS}$ for such operators. Here are some other characterisations of this concept:

Exercise 19. Let H, H' be Hilbert spaces with orthonormal bases $(e_n)_{n\in A}$ and $(e'_{n'})_{n'\in A'}$ respectively, and let $T : H \to H'$ be a bounded linear operator. Show that the following are equivalent:

1. T is a Hilbert-Schmidt operator.
2. $\sum_{n\in A} \|Te_n\|_{H'}^2 < \infty$.
3. $\sum_{n\in A} \sum_{n'\in A'} |\langle Te_n, e'_{n'}\rangle_{H'}|^2 < \infty$.

Also, show that if $T, S : H \to H'$ are Hilbert-Schmidt operators, then

$$\langle T, S\rangle_{HS} = \sum_{n\in A} \langle Te_n, Se_n\rangle_{H'} \quad (25)$$

and

$$\|T\|_{HS}^2 = \sum_{n\in A} \|Te_n\|_{H'}^2 = \sum_{n\in A} \sum_{n'\in A'} |\langle Te_n, e'_{n'}\rangle_{H'}|^2. \quad (26)$$ ◇

As one consequence of the above exercise, we see that the Hilbert-Schmidt norm controls the operator norm, thus $\|Tv\| \leq \|T\|_{HS}\|v\|$ for all vectors v.

Remark 5. From this exercise and Fatou's lemma, we see in particular that the limit (in either the norm, strong or weak operator topologies) of a sequence of Hilbert-Schmidt operators with uniformly bounded Hilbert-Schmidt norm, is still Hilbert-Schmidt. We also see that the composition of a Hilbert-Schmidt operator with a bounded operator is still Hilbert-Schmidt (thus the Hilbert-Schmidt operators can be viewed as a closed two-sided ideal in the space of bounded operators). ◇

Example 5. An operator $T : L^2(X, \mathcal{X}, \mu) \to L^2(Y, \mathcal{Y}, \nu)$ is Hilbert-Schmidt if and only if it takes the form $Tf(y) := \int_X K(x, y) f(x)\ d\mu(x)$ for some kernel $K \in L^2(X \times Y, \mathcal{X} \times \mathcal{Y}, \mu \times \nu)$, in which case the Hilbert-Schmidt norm is $\|K\|_{L^2(X\times Y, \mathcal{X}\times\mathcal{Y}, \mu\times\nu)}$. The Hilbert-Schmidt inner product is defined similarly. ◇

Example 6. The identity operator on an infinite-dimensional Hilbert space is never Hilbert-Schmidt, despite being bounded. On the other hand, every finite rank operator is Hilbert-Schmidt. ◇

One of the key properties of Hilbert-Schmidt operators which will be relevant to us is the following.

Lemma 2. If $T : H \to H'$ is Hilbert-Schmidt, then it is compact (i.e. the image of any bounded set is precompact).

Proof. Let $\varepsilon > 0$ be arbitrary. By Exercise 19 and monotone convergence, we can find a finite orthonormal set $e_1, \ldots, e_N$ such that $\sum_{n=1}^N \|Te_n\|_{H'}^2 \geq \|T\|_{HS}^2 - \varepsilon^2$, and in particular that $\|Te_{n+1}\|_{H'} \leq \varepsilon$ for any e_{n+1} orthogonal to $e_1, \ldots, e_n$. As a consequence, the image of the unit ball of H under T lies within ε of the image of the unit ball of the finite-dimensional space $\mathrm{span}(e_1, \ldots, e_N)$. This image is therefore totally bounded and thus precompact. □

The following exercise may help illuminate the distinction between bounded operators, Hilbert-Schmidt operators, and compact operators:

Exercise 20. Let λ_n be a sequence of complex numbers, and consider the diagonal operator $T : (z_n)_{n\in\mathbb{N}} \mapsto (\lambda_n z_n)_{n\in\mathbb{N}}$ on $l^2(\mathbb{N})$.

1. Show that T is a well-defined bounded linear operator on $l^2(\mathbb{N})$ if and only if the sequence (λ_n) is bounded.
2. Show that T is Hilbert-Schmidt if and only if the sequence (λ_n) is square-summable.
3. Show that T is compact if and only if the sequence (λ_n) goes to zero as $n \to \infty$. $\diamond$

Now we apply the above theory to establish Theorem 1. Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let $f \in L^2(X, \mathcal{X}, \mu)$. The rank one operators $g \mapsto \langle g, T^n f\rangle T^n f$ can easily be verified to have a Hilbert-Schmidt norm of $\|f\|_{L^2}^2$, and so by the triangle inequality, their averages $S_{f,N} : g \mapsto \frac{1}{N}\sum_{n=0}^{N-1}\langle g, T^n f\rangle T^n f$ have a Hilbert-Schmidt norm of at most $\|f\|_{L^2}^2$. On the other hand, from the identity

$$\langle S_{f,N} g, h\rangle = \frac{1}{N}\sum_{n=0}^{N-1}\langle g\otimes\overline{h}, (T\otimes T)^n(f\otimes\overline{f})\rangle \quad (27)$$

and the mean ergodic theorem (applied to the product space) we see that $S_{f,N}$ converges in the weak operator norm to some limit S_f, which is then also Hilbert-Schmidt by Remark 4, and thus compact by Lemma 2. (Actually, $S_{f,N}$ converges to S_f in the Hilbert-Schmidt norm, and thus also in the operator norm and in the strong topology: this is another application of the mean ergodic theorem, which we leave as an exercise. Since each of the $S_{f,N}$ is clearly finite rank, this gives a direct proof of the compactness of S_f.) Also, it is easy to see that S_f is self-adjoint and commutes with T. As a consequence, we conclude that for any $g \in L^2(X, \mathcal{X}, \mu)$, the image $S_f g$ is almost periodic (since $\{T^n S_f g : n \in \mathbb{Z}\} = S_f\{T^n g : n \in \mathbb{Z}\}$ is the image of a bounded set by the compact operator S_f and therefore precompact).

On the other hand, observe that

$$\langle S_f f, f\rangle = C\text{-}\lim_{n\to\infty} |\langle T^n f, f\rangle|^2. \quad (28)$$

Thus by Definition 2 (and Exercise 1), we see that $\langle S_f f, f\rangle \neq 0$ whenever f is not weakly mixing. In particular, f is not orthogonal to the almost periodic function $S_f f$. From this and Exercise 18, we have thus shown

Proposition 3. (Dichotomy between structure and randomness) Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. A function $f \in L^2(X, \mathcal{X}, \mu)$ is weakly mixing if and only if it is orthogonal to all almost periodic functions (or equivalently, orthogonal to all eigenfunctions).

Remark 6. Interestingly, essentially the same result appears in the spectral and scattering theory of linear Schrödinger equations, which in that context is known as the "RAGE theorem" (after Ruelle, Amrein-Georgescu, and Enss). $\diamond$

Remark 7. The finitary analogue of the expression $S_f f$ is the *dual function* (of order 2) of f (the dual function of order 1 was briefly discussed in Lecture 8). If we are working on $\mathbb{Z}/N\mathbb{Z}$ with the usual shift, then S_f can be viewed as a Fourier multiplier which multiplies the Fourier coefficient at ξ by $|\hat{f}(\xi)|^2$; informally, S_f filters out all the low amplitude frequencies of f, leaving only a handful of high-amplitude frequencies. $\diamond$

Recall from Proposition 2 and Exercise 5 of Lecture 11 that a function $f \in L^2(X, \mathcal{X}, \mu)$ is almost periodic if and only if it is $\mathcal{Z}_1$-measurable, or if it lies in the pure point component $\mathbb{H}_{pp}$ of the shift operator T. We thus have

Corollary 3. (Koopman-von Neumann theorem) Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let $f \in L^2(X, \mathcal{X}, \mu)$. Let $\mathcal{Z}_1$ be the σ-algebra generated by the eigenfunctions of T.

1. f is almost periodic if and only if $f \in L^2(X, \mathcal{Z}_1, \mu)$ if and only if $f \in \mathbb{H}_{pp}$.
2. f is weakly mixing if and only if $\mathbb{E}(f|\mathcal{Z}_1) = 0$ a.e. if and only if $f \in \mathbb{H}_c = \mathbb{H}_{sc} + \mathbb{H}_{ac}$ (corresponding to the continuous spectrum of T).
3. In general, f has a unique decomposition $f = f_{U^\perp} + f_U$ into an almost periodic function $f_{U^\perp}$ and a weakly mixing function f_U. Indeed, $f_{U^\perp} = \mathbb{E}(f|\mathcal{Z}_1)$ and $f_U = f - \mathbb{E}(f|\mathcal{Z}_1)$.

Theorem 1 follows immediately from this Corollary. Indeed, if a system is not weakly mixing, then by the above Corollary we see that $\mathcal{Z}_1$ is non-trivial, and the identity map from $(X, \mathcal{X}, \mu, T)$ to $(X, \mathcal{Z}_1, \mu, T)$ yields a non-trivial compact factor.

– Roth's theorem –

As a quick application of the above machinery we give a proof of Roth's theorem. We first need a variant of Corollary 1, which is proven by much the same means:

Exercise 21. Let $(X, \mathcal{X}, \mu, T)$ be an ergodic measure-preserving system, let a_1, a_2, a_3 be distinct integers, and let $f_1, f_2, f_3 \in L^\infty(X, \mathcal{X}, \mu)$ with at least one of f_1, f_2, f_3 weakly mixing. Show that $C-\lim_{n\to\infty} \int_X T^{a_1 n} f_1 T^{a_2 n} f_2 T^{a_3 n} f_3\ d\mu = 0$. $\diamond$

Theorem 2 (Roth's theorem). Let $(X, \mathcal{X}, \mu, T)$ be an ergodic measure-preserving system, and let $f \in L^\infty(X, \mathcal{X}, \mu)$ be non-negative with $\int_X f\ d\mu > 0$. Then

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_X f T^n f T^{2n} f\ d\mu > 0. \quad (29)$$

Proof. We decompose $f = f_{U^\perp} + f_U$ as in Corollary 3. The contribution of f_U is negligible by Exercise 21, so it suffices to show that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_X f_{U^\perp} T^n f_{U^\perp} T^{2n} f_{U^\perp}\ d\mu > 0. \quad (30)$$

But as $f_{U^\perp}$ is almost periodic, the claim follows from Proposition 1 of Lecture 11. □

One can then immediately establish the k=3 case of Furstenberg's theorem (Theorem 2 from Lecture 10) by combining the above result with the ergodic decomposition (Proposition 4 from Lecture 9). The k=3 case of Szemerèdi's theorem (i.e. Roth's theorem) then follows from the Furstenberg correspondence principle (see Lecture 10).

Exercise 22. Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let $f \in L^2(X, \mathcal{X}, \mu)$ be non-negative. Show that for every $\varepsilon > 0$, one has $\langle T^n f, f \rangle \geq \int_X (f\ d\mu)^2 - \varepsilon$ for infinitely many n. (Hint: first show this when f is almost periodic, and then use Corollary 1 and Corollary 3 to prove the general case.) This is a simplified version of the *Khintchine recurrence theorem*, which asserts that the set of such n is not only infinite, but is also syndetic. Analogues of the Khintchine recurrence theorem hold for double recurrence but not for triple recurrence; see this paper of Bergelson, Host, and Kra for details. ◇

[*Update*, Mar 3: Added the observation that as $S_{f,N}$ converges to S_f in the operator norm, S_f is the limit of finite rank operators and thus clearly compact. Thanks to Guatam Zubin for this remark.]

4 comments

Comments feed for this article

23 February, 2008 at 2:26 pm

David Speyer

I am not sure I see how Theorem 1 follows from corollary 3. The obvious conclusion from cor. 3 is that, if your dynamical system is not weakly mixing, then there is a nonconstant almost periodic function. We have to use this to build a compact factor. Looking at Theorem 3 in Lecture 7, I think the chain of reasoning is the following:

Since X is not weakly mixing, X \times X is also not weakly mixing so
there is a nonconstant almost periodic function f on X \times X. View f as a function g:X –> L^2(X), then g commutes with the translation T. Moreover, f is continuous (why?) and g(X) is compact (why?). It's not even clear to me which of these two points uses the fact that f is almost periodic (although I am betting on the second.)

Any hints are appreciated!

23 February, 2008 at 2:34 pm

Terence Tao

Dear David,

254A, Lecture 13: Compact extensions

27 February, 2008 in 254A - ergodic theory, math.DS, math.OA

Tags: almost periodicity, compact extensions, Hilbert modules, multiple recurrence, structure, van der Waerden's theorem

In Lecture 11, we studied *compact* measure-preserving systems - those systems $(X, \mathcal{X}, \mu, T)$ in which every function $f \in L^2(X, \mathcal{X}, \mu)$ was almost periodic, which meant that their orbit $T^n f : n \in \mathbb{Z}$ was precompact in the $L^2(X, \mathcal{X}, \mu)$ topology. Among other things, we were able to easily establish the Furstenberg recurrence theorem (Theorem 1 from Lecture 11) for such systems.

In this lecture, we generalise these results to a "relative" or "conditional" setting, in which we study systems which are compact relative to some factor $(Y, \mathcal{Y}, \nu, S)$ of $(X, \mathcal{X}, \mu, T)$. Such systems are to compact systems as isometric extensions are to isometric systems in topological dynamics. The main result we establish here is that the Furstenberg recurrence theorem holds for such compact extensions whenever the theorem holds for the base. The proof is essentially the same as in the compact case; the main new trick is to not to work in the Hilbert spaces $L^2(X, \mathcal{X}, \mu)$ over the complex numbers, but rather in the Hilbert module $L^2(X, \mathcal{X}, \mu|Y, \mathcal{Y}, \nu)$ over the (commutative) von Neumann algebra $L^\infty(Y, \mathcal{Y}, \nu)$. (Modules are to rings as vector spaces are to fields.) Because of the compact nature of the extension, it turns out that results from topological dynamics (and in particular, van der Waerden's theorem) can be exploited to good effect in this argument.

[Note: this operator-algebraic approach is not the only way to understand these extensions; one can also proceed by disintegrating μ into fibre measures μ_y for almost every $y \in Y$ and working fibre by fibre. We will discuss the connection between the two approaches below.]

– Hilbert modules –

Let $X = (X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let $\pi : X \to Y$ be a factor map, i.e. a morphism from X to another system $Y = (Y, \mathcal{Y}, \nu, S)$. The algebra $L^\infty(Y)$ can be viewed (using π) as a subalgebra of $L^\infty(X)$; indeed, it is isomorphic to $L^\infty(X, \pi^\#(\mathcal{Y}), \mu)$, where $\pi^\#(\mathcal{Y}) := \{\pi^{-1}(E) : E \in \mathcal{Y}\}$ is the pullback of $\mathcal{Y}$ by π.

Example 1. Throughout these notes we shall use the *skew shift* as our running example. Thus, in this example, $X = (\mathbb{R}/\mathbb{Z})^2$ with shift $T : (y, z) \mapsto (y + \alpha, z + x)$ for some fixed α (which can be either rational or irrational), $Y = \mathbb{R}/\mathbb{Z}$ with shift $S : y \mapsto y + \alpha$, with factor map $\pi : (y, z) \mapsto y$. In this case, $L^\infty(Y)$ can be thought of (modulo equivalence on null sets, of course) as the space of bounded functions on $(\mathbb{R}/\mathbb{Z})^2$ which depend only on the first variable. $\diamond$

Example 2. Another (rather trivial) example is when the factor system Y is simply a point. In this case, $L^\infty(Y)$ is the space of constants and can be identified with $\mathbb{C}$. At the opposite extreme, another example is when Y is equal to X. It is instructive to see how all of the concepts behave in each of these two extreme cases, as well as the typical intermediate case presented in Example 1. $\diamond$

The idea here will be to try to "relativise" the machinery of Hilbert spaces over $\mathbb{C}$ to that of Hilbert modules over $L^\infty(Y)$. Roughly speaking, all concepts which used to be complex or real-valued (e.g.

inner products, norms, coefficients, etc.) will now take values in the algebra $L^\infty(Y)$. The following table depicts the various concepts that will be relativised:

Absolute / unconditional	Relative / conditional
Constants $\mathbb{C}$	Factor-measurable functions $L^\infty(Y)$
Expectation $\mathbb{E}f = \int_X f\ d\mu \in \mathbb{C}$	Conditional expectation $\mathbb{E}(f\|Y) \in L^\infty(Y)$
Inner product $\langle f, g\rangle_{L^2(X)} = \mathbb{E}f\overline{g}$	Conditional inner product $\langle f, g\rangle_{L^2(X\|Y)} = \mathbb{E}(f\overline{g}\|Y)$
Hilbert space $L^2(X)$	Hilbert module $L^2(X\|Y)$
Finite-dimensional subspace $\{\sum_{j=1}^d c_j f_j : c_1, \ldots, c_d \in \mathbb{C}\}$	Finitely generated module $\{\sum_{j=1}^d c_j f_j : c_1, \ldots, c_d \in L^\infty(Y)\}$
Almost periodic function	Conditionally almost periodic function
Compact system	Compact extension
Hilbert-Schmidt operator	Conditionally Hilbert-Schmidt operator
Weakly mixing function	Conditionally weakly mixing function
Weakly mixing system	Weakly mixing extension

(The last few notions in this table will be covered in the next lecture, rather than this one.)

Remark 1. In information-theoretic terms, one can view Y as representing all the observables in the system X that have already been "measured" in some sense, so that it is now permissible to allow one's "constants" to depend on that data, and only study the remaining information present in X conditioning on the observed values in Y. Note though that once we activate the shift map T, the data in Y will similarly shift (by S), and so the various fibres of π can interact with each other in a non-trivial manner, so one should take some caution in applying information-theoretic intuition to this setting. $\diamond$

We have already seen that the factor Y induces a sub-σ-algebra $\pi^\#(\mathcal{Y})$ of $\mathcal{X}$. We therefore have a conditional expectation map $f \mapsto \mathbb{E}(f|Y)$ defined for all absolutely integrable f by the formula

$$\mathbb{E}(f|Y) := \mathbb{E}(f|\pi^\#(\mathcal{Y})). \quad (1)$$

In general, this expectation only lies in $L^1(Y)$, though for the functions we shall eventually study, the expectation will always lie in $L^\infty(Y)$ when needed.

As stated in the table, conditional expectation will play the role in the conditional setting that the unconditional expectation $\mathbb{E}f = \int_X f\ d\mu$ plays in the unconditional setting. Note though that the conditional expectation takes values in the algebra $L^\infty(Y)$ rather than in the complex numbers. We recall that conditional expectation is linear over this algebra, thus

$$\mathbb{E}(cf + dg|Y) = c\mathbb{E}(f|Y) + d\mathbb{E}(g|Y) \quad (2)$$

for all absolutely integrable f, g and all $c, d \in L^\infty(Y)$.

Example 3. Continuing Example 1, we see that for any absolutely integrable f on $(\mathbb{R}/\mathbb{Z})^2$, we have $\mathbb{E}(f|Y)(y,z) = \int_{\mathbb{R}/\mathbb{Z}} f(y,z')\ dz'$ almost everywhere. $\diamond$

Let $L^2(X|Y)$ be the space of all $f \in L^2(X, \mathcal{X}, \mu)$ such that the conditional norm

$$\|f\|_{L^2(X|Y)} := \mathbb{E}(\|f\|^2|Y)^{1/2} \quad (3)$$

lies in $L^\infty(Y)$ (rather than merely in $L^2(Y)$, which it does automatically). Thus for instance we have the inclusions

$$L^\infty(X) \subset L^2(X|Y) \subset L^2(X). \quad (4)$$

The space $L^2(X|Y)$ is easily seen to be a vector space over $\mathbb{C}$, and moreover (thanks to (2)) is a module over $L^\infty(Y)$.

Exercise 1. If we introduce the inner product

$$\langle f, g\rangle_{L^2(X|Y)} := \mathbb{E}(f\overline{g}|Y) \quad (5)$$

(which, initially, is only in $L^1(Y)$), establish the pointwise Cauchy-Schwarz inequality

$$|\langle f, g\rangle_{L^2(X|Y)}| \leq \|f\|_{L^2(X|Y)} \|g\|_{L^2(X|Y)} \quad (6)$$

almost everywhere. In particular, the inner product lies in $L^\infty(Y)$. (Hint: repeat the standard proof of the Cauchy-Schwarz inequality verbatim, but with $L^\infty(Y)$ playing the role of the constants $\mathbb{C}$.) $\diamond$

Example 4. Continuing Examples 1 and 3, $L^2(X|Y)$ consists (modulo null set equivalence) of all measurable functions $f(y,z)$ such that $\|f\|_{L^2(X|Y)} = (\int_{\mathbb{R}/\mathbb{Z}} |f(y,z)|^2\ dz)^{1/2}$ is bounded a.e. in y, with the relative inner product

$$\langle f, g\rangle_{L^2(X|Y)}(y) := \int_{\mathbb{R}/\mathbb{Z}} f(y,z)\overline{g(y,z)}\ dz \quad (7)$$

defined a.e. in y. Observe in this case that the relative Cauchy-Schwarz inequality (6) follows easily from the standard Cauchy-Schwarz inequality. $\diamond$

Exercise 2. Show that the function $f \mapsto \| \|f\|_{L^2(X|Y)} \|_{L^\infty(Y)}$ is a norm on $L^2(X|Y)$, which turns that space into a Banach space. (*Hint*: you may need to "relativise" one of the standard proofs that $L^2(X)$ is complete. You may also want to start with the skew shift example to build some intuition.) Because of this completeness, we refer to $L^2(X|Y)$ as a *Hilbert module* over $L^\infty(Y)$. $\diamond$

As π is a morphism, one can easily check the intertwining relationship

$$\mathbb{E}(T^n f|Y) = S^n \mathbb{E}(f|Y) \quad (7)$$

for all $f \in L^1(X)$ and integers n. As a consequence we see that the map T (and all of its powers) preserves the space $L^2(X|Y)$, and furthermore is conditionally unitary in the sense that

$$\langle T^n f, T^n g\rangle_{L^2(X|Y)} = S^n \langle f, g\rangle_{L^2(X|Y)} \quad (8)$$

for all $f, g \in L^2(X|Y)$ and integers n.

In the Hilbert space $L^2(X)$ one can create finite dimensional subspaces $\{c_1 f_1 + \ldots + c_d f_d : c_1, \ldots, c_d \in \mathbb{C}\}$ for any $f_1, \ldots, f_d \in L^2(X)$. Inside such subspaces we can create the bounded finite-dimensional zonotopes $\{c_1 f_1 + \ldots + c_d f_d : c_1, \ldots, c_d \in \mathbb{C}, |c_1|, \ldots, |c_d| \leq 1\}$. Observe (from the Heine-Borel theorem) that a subset E of $L^2(X)$ is precompact if and only if it can be approximated by finite-dimensional zonotopes in the sense that for every $\varepsilon > 0$, there exists a finite-dimensional zonotope Z of $L^2(X)$ such that E lies within the ε neighbourhood of Z.

Remark 2. There is nothing special about zonotopes here; just about any family of bounded finite-dimensional objects would suffice for this purpose. In fact, it seems to be slightly better (for the purposes of quantitative analysis, and in particular in controlling the dependence on dimension d) to work instead with octahedra, in which the constraint $|c_1|, \ldots, |c_d| \leq 1$ is replaced by $|c_1| + \ldots + |c_d| = 1$; see this paper of mine in which this perspective is used. ◇

Inspired by this, let us make some definitions. A *finitely generated module* of $L^2(X|Y)$ is any submodule of $L^2(X|Y)$ of the form $\{c_1 f_1 + \ldots + c_d f_d : c_1, \ldots, c_d \in L^\infty(Y)\}$, where $f_1, \ldots, f_d \in L^2(X|Y)$. Inside such a module we can define a *finitely generated module zonotope* $\{c_1 f_1 + \ldots + c_d f_d : c_1, \ldots, c_d \in L^\infty(Y); \|c_1\|_{L^\infty(Y)}, \ldots, \|c_d\|_{L^\infty(Y)} \leq 1\}$.

Definition 1. A subset E of $L^2(X|Y)$ is said to be *conditionally precompact* if for every $\varepsilon > 0$, there exists a finitely generated module zonotope Z of $L^2(X|Y)$ such that E lies within the ε-neighbourhood of Z (using the norm from Exercise 2).

A function $f \in L^2(X|Y)$ is said to be *conditionally almost periodic* if its orbit $\{T^n f : n \in \mathbb{Z}\}$ is conditionally precompact.

A function $f \in L^2(X|Y)$ is said to be *conditionally almost periodic in measure* if every $\varepsilon > 0$ there exists a set E in Y of measure at most ε such that $f 1_{E^c}$ is conditionally almost periodic.

The system X is said to be a compact extension of Y if every function in $L^2(X|Y)$ is conditionally almost periodic in measure.

Example 5. Any bounded subset of $L^\infty(Y)$ is conditionally precompact (though note that it need not be precompact in the topological sense, using the topology from Exercise 2). In particular, every function in $L^\infty(Y)$ is conditionally almost periodic. ◇

Example 6. Every system is a compact extension of itself. A system is a compact extension of a point if and only if it is a compact system. ◇

Example 7. Consider the skew shift (Examples 1, 3, 4), and consider the orbit of the function $f(y, z) := e^{2\pi i z}$. A computation shows that

$$T^n f(y, z) = e^{2\pi i \frac{n(n-1)}{2} \alpha y} f \quad (9)$$

which reveals (for α irrational) that f is not almost periodic in the unconditional sense. However, observe that all the shifts $T^n f$ lie in the zonotope $\{cf : c \in L^\infty(Y), \|c\|_{L^\infty(Y)} \leq 1\}$ generated by a single generator f, and so f is *conditionally* almost periodic. ◇

Exercise 3. Consider the skew shift (Examples 1,3,4,7). Show that a sequence $f_n \in L^\infty(X)$ is conditionally precompact if and only if the sequences $f_n(y, cdot) \in L^\infty(\mathbb{R}/\mathbb{Z})$ are precompact in $L^2(\mathbb{R}/\mathbb{Z})$ (with the usual Lebesgue measure) for almost every y. ◇

Exercise 4. Show that the space of conditionally almost periodic functions in $L^2(X|Y)$ is a shift-invariant $L^\infty(Y)$ module, i.e. it is closed under addition, under multiplication by elements of $L^\infty(Y)$, and under powers T^n of the shift operator. ◇

Exercise 5. Consider the skew shift (Examples 1,3,4,7 and Exercise 3) with α irrational, and let $f \in L^2(X|Y)$ be the function defined by setting $f(y,z) := e^{2\pi i n z}$ whenever $n \geq 1$ and $y \in (1/(n+1), 1/n]$. Show that f is conditionally almost periodic in measure, but not conditionally almost periodic. Thus the two notions can be distinct even for bounded functions (a subtlety that does not arise in the unconditional setting). ◇

Exercise 6. Let $\mathcal{Z}_{X|Y}$ denote the collection of all measurable sets E in X such that 1_E is conditionally almost periodic in measure. Show that $\mathcal{Z}_{X|Y}$ is a shift-invariant sub-σ-algebra of $\mathcal{X}$ that contains $\pi^\# \mathcal{Y}$, and that a function $f \in L^2(X|Y)$ is conditionally almost periodic in measure if and only if it is $\mathcal{Z}$-measurable. (In particular, $(X, \mathcal{Z}_{X|Y}, \mu, T)$ is the maximal compact extension of Y.) [*Hint*: you may need to truncate the generators $f_1, \ldots, f_d$ of various module zonotopes to be in $L^\infty(X)$ rather than $L^2(X|Y)$.] ◇

Exercise 7. Show that the skew shift (Examples 1, 3, 4, 7 and Exercises 3,5) is a compact extension of the circle shift. (Hint: Use Example 7 and Exercise 6. Alternatively, approximate a function on the skew torus by its vertical Fourier expansions. For each fixed horizontal coordinate y, the partial sums of these vertical Fourier series converge (in the vertical L^2 sense) to the original function, pointwise in y. Now apply Egorov's theorem.) ◇

Exercise 8. Show that each of the iterated skew shifts (Exercise 8 from Lecture 9) are compact extensions of the preceding skew shift. ◇

Exercise 9. Let $(Y, \mathcal{Y}, \nu, S)$ be a measure-preserving system, let G be a compact metrisable group with a closed subgroup H, let $\sigma: Y \to G$ be measurable, and let $Y \times_\sigma G/H$ be the extension of Y with underlying space $Y \times G/H$, with measure equal to the product of ν and Haar measure, and shift map $T: (y, \zeta) \mapsto (Sy, \sigma(y)\zeta)$. Show that $Y \times_\sigma G/H$ is a compact extension of Y. ◇

— Multiple recurrence for compact extensions –

Let us say that a measure-preserving system $(X, \mathcal{X}, \mu, T)$ obeys the *uniform multiple recurrence* (UMR) property if the conclusion of the Furstenberg multiple recurrence theorem holds for this system, thus for all $k \geq 1$ and all non-negative $f \in L^\infty(X)$ with $\int_X f\, d\mu > 0$, we have

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_X f T^n f \ldots T^{(k-1)n} f\, d\mu > 0. \quad (10)$$

Thus in Lecture 11 we showed that all compact systems obey UMR, and in Lecture 12 we showed that all weakly mixing systems obey UMR. The Furstenberg multiple recurrence theorem asserts, of course, that *all* measure-preserving systems obey UMR.

We now establish an important further step (and, in many ways, the *key* step) towards proving that theorem:

Theorem 1. Suppose that $X = (X, \mathcal{X}, \mu, T)$ is a compact extension of $Y = (Y, \mathcal{Y}, \nu, S)$. If Y obeys UMR, then so does X.

Note that the converse implication is trivial: if a system obeys UMR, then all of its factors automatically do also.

Proof of Theorem 1. Fix $k \geq 1$, and fix a non-negative function $f \in L^\infty(X)$ with $\int_X f\, d\mu > 0$. Our objective is to show that (10) holds. As X is a compact extension, f is conditionally almost periodic in measure; by definition (and uniform integrability), this implies that f can be bounded from below by another conditionally almost periodic function which is non-negative with positive mean. Thus we may assume without loss of generality that f is conditionally almost periodic.

We may normalise $\|f\|_{L^\infty(X)} = 1$ and $\int_X f\ d\mu = \delta$ for some $0 < \delta < 1$. The reader may wish to follow this proof using the skew shift example as a guiding model.

Let $\varepsilon > 0$ be a small number (depending on k and δ) to be chosen later. If we set $E := \{y \in Y : \mathbb{E}(f|Y) > \delta/2\}$, then E must have measure at least $\delta/2$.

Since f is almost periodic, we can find a finitely generated module zonotope $\{c_1 f_1 + \ldots + c_d f_d : \|c_1\|_{L^\infty(Y)}, \ldots, \|c_d\|_{L^\infty(Y)} \leq 1\}$ whose ε-neighbourhood contains the orbit of f. In other words, we have an identity of the form

$$T^n f = c_{1,n} f_1 + \ldots + c_{d,n} f_d + e_n \quad (11)$$

for all n, where $c_{1,n}, \ldots, c_{d,n} \in L^\infty(Y)$ with norm at most 1, and $e_n \in L^2(X,Y)$ is an error with $\|e_n\|_{L^2(X|Y)} = O(\varepsilon)$ almost everywhere.

By splitting into real and imaginary parts (and doubling d if necessary) we may assume that the $c_{j,n}$ are real-valued. By further duplication we can also assume that $\|f_i\|_{L^2(X|Y)} \leq 1$ for each i. By rounding off $c_{j,n}(y)$ to the nearest multiple of ε/d for each y (and absorbing the error into the e_n term) we may assume that $c_{j,n}(y)$ is always a multiple of ε/d. Thus each $c_{j,n}$ only takes on $O_{\varepsilon,d}(1)$ values.

Let K be a large integer (depending on k, d, δ, ε) to be chosen later. Since the factor space Y obeys UMR, and E has positive measure in Y, we know that

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_Y 1_E T^n 1_E \ldots T^{(K-1)n} 1_E\ d\nu > 0. \quad (12)$$

In other words, there exists a constant $c > 0$ such that

$$\nu(\Omega_n) > c \quad (13)$$

for a set of n of positive lower density, where Ω_n is the set

$$\Omega_n := E \cap T^n E \cap \ldots \cap T^{(K-1)n} E. \quad (14)$$

Let n be as above. By definition of Ω_n and E (and (8)), we see that

$$\mathbb{E}(T^{an} f|Y)(y) \geq \delta/2 \quad (15)$$

for all $y \in \Omega_n$ and $0 \leq a < K$. Meanwhile, from (11) we have

$$\|T^{an} f - c_{1,an} f_1 - \ldots - c_{d,an} f_d\|_{L^2(X|Y)}(y) = O(\varepsilon) \quad (16)$$

for all $y \in \Omega_n$ and $0 \leq a < K$.

Fix y. For each $0 \leq a < K$, the d-tuple $\vec{c}_{an}(y) := (c_{1,an}(y), \ldots, c_{d,an}(y))$ ranges over a set of cardinality $O_{d,\varepsilon}(1)$. One can view this as a colouring of $\{0, \ldots, K-1\}$ into $O_{d,\varepsilon}(1)$ colours. Applying van der Waerden's theorem (Exercise 3 from Lecture 4), we can thus find (if K is sufficiently large depending on

d, ε, k) an arithmetic progression $a(y), a(y)+r(y), \ldots, a(y)+(k-1)r(y)$ in $\{0, \ldots, K-1\}$ for each y such that

$$\vec{c}_{a(y)n}(y) = \vec{c}_{(a(y)+r(y))n}(y) = \ldots = \vec{c}_{(a(y)+(k-1)r(y))n}(y). \quad (17)$$

The quantities a(y) and r(y) can of course be chosen to be measurable in y. By the pigeonhole principle, we can thus find a subset Ω'_n of Ω_n of measure at least $\sigma > 0$ for some σ depending on c, K, d, ε but independent of n, and an arithmetic progression $a, a+r, \ldots, a+(k-1)r$ in $\{0, \ldots, K-1\}$ such that

$$\vec{c}_{an}(y) = \vec{c}_{(a+r)n}(y) = \ldots = \vec{c}_{(a+(k-1)r)n}(y) \quad (18)$$

for all $y \in \Omega'_n$. (The quantities a and r can still depend on n, but this will not be of concern to us.)

Fix these values of a, r. From (16), (18) and the triangle inequality we see that

$$\|T^{(a+jr)n} f - T^{an} f\|_{L^2(X|Y)}(y) = O(\varepsilon) \quad (19)$$

for all $1 \leq j \leq k$ and $y \in \Omega'_n$. Recalling that f was normalised to have $L^\infty(X)$ norm 1, it is then not hard to conclude (by induction on k and the relative Cauchy-Schwarz inequality) that

$$\|T^{an} f T^{(a+r)n} f \ldots T^{(a+(k-1)r)n} f - (T^{an} f)^k\|_{L^2(X|Y)}(y) = O_k(\varepsilon) \quad (20)$$

and thus (by another application of relative Cauchy-Schwarz)

$$\mathbb{E}(T^{an} f T^{(a+r)n} f \ldots T^{(a+(k-1)r)n} f)(y) \geq \mathbb{E}((T^{an} f)^k | Y)(y) - O_k(\varepsilon). \quad (21)$$

But from (14), (15) and relative Cauchy-Schwarz again we have

$$\mathbb{E}(T^{an} f | Y)(y) \geq \delta/2 - O(\varepsilon) \quad (22)$$

and so by several more applications of relative Cauchy-Schwarz we have

$$\mathbb{E}((T^{an} f)^k | Y)(y) \geq c(k, \delta) > 0 \quad (23)$$

for some positive quantity $c(k, \delta)$ (if ε is sufficiently small depending on k, δ). From (21), (23) we conclude that

$$\mathbb{E}(T^{an} f T^{(a+r)n} f \ldots T^{(a+(k-1)r)n} f)(y) \geq c(k, \delta)/2 \quad (24)$$

for $y \in \Omega'_n$, again if ε is small enough. Integrating this in y and using the shift-invariance we conclude that

$$\int_X f T^{nr} f \ldots T^{(k-1)nr} f \, d\mu \geq c(k, \delta)\sigma/2. \quad (25)$$

The quantity r depends on n, but ranges between 1 and K-1, and so (by the non-negativity of f)

$$\sum_{s=1}^{K-1} \int_X f T^{ns} f \ldots T^{(k-1)ns} f \, d\mu \geq c(k, \delta)\sigma/2 \quad (26)$$

for a set of n of positive lower density. Averaging this for n from 1 to N (say) one obtains (10) as desired. $\square$

Thus for instance we have now established UMR for the skew shift as well as higher iterates of that shift, thanks to Exercises 7 and 8.

Remark 3. One can avoid the use of Hilbert modules, etc. by instead appealing to the theory of disintegration of measures (Theorem 4 from Lecture 9). We sketch the details as follows. First, one has to restrict attention to those spaces X which are regular, though an inspection of the Furstenberg correspondence principle (Lecture 10) shows that this is in fact automatic for the purposes of such tasks as proving Szemerédi's theorem. Once one disintegrates μ with respect to ν, the situation now resembles the concrete example of the skew shift, with the fibre measures μ_y playing the role of integration along vertical fibers $\{(y,z): z \in \mathbb{R}/\mathbb{Z}\}$. It is then not difficult (and somewhat instructive) to convert the above proof to one using norms such as $L^2(X, \mathcal{X}, mu_y)$ rather than the module norm $L^2(X|Y)$. We leave the details to the reader (who can also get them from Furstenberg's book). $\diamond$

Remark 4. It is an intriguing question as to whether there is any interesting non-commutative extension of the above theory, in which the underlying von Neumann algebra $L^\infty(Y, \mathcal{Y}, \nu)$ is replaced by a non-commutative von Neumann algebra. While some of the theory seems to extend relatively easily, there does appear to be some genuine difficulties with other parts of the theory, particularly those involving multiple products such as $fT^n fT^{2n} f$. $\diamond$

Remark 5. Just as ergodic compact systems can be described as group rotation systems (Kronecker systems), it turns out that ergodic compact extensions can be described as (inverse limits of) group quotient extensions, somewhat analogously to Lemma 2 from Lecture 6. Roughly speaking, the idea is to first use some spectral theory to approximate conditionally almost periodic functions by conditionally *quasiperiodic* functions - those functions whose orbit lies on a finitely generated module zonotope (as opposed to merely being close to one). One can then use the generators of that zonotope as a basis from which to build the group quotient extension, and then use some further trickery to make the group consistent across all fibres. The precise machinery for this is known as *Mackey theory*; it is of particular importance in the deeper structural theory of dynamical systems, but we will not describe it in detail here, instead referring the reader to the papers of Furstenberg and of Zimmer. $\diamond$

[*Update*, Mar 5: Exercise added.]

[*Update*, Mar 12: Definition of conditionally almost periodic measure tweaked (to an equivalent definition) in order to simplify the proof of Theorem 1.]

3 comments

Comments feed for this article

2 March, 2008 at 9:24 pm

254A, Lecture 14: Weakly mixing extensions « What's new

[…] Hilbert-Schmidt operators, randomness, weak mixing Having studied compact extensions in the previous lecture, we now consider the opposite type of extension, namely that of a weakly mixing extension. Just as […]

6 March, 2008 at 11:21 am

Anonymous

All the formulas in this post seem to lack backlashases before TeX commands.

6 March, 2008 at 1:28 pm

254A, Lecture 14: Weakly mixing extensions

2 March, 2008 in 254A - ergodic theory, math.DS, math.OA

Tags: almost periodicity, Hilbert-Schmidt operators, randomness, weak mixing

Having studied compact extensions in the previous lecture, we now consider the opposite type of extension, namely that of a *weakly mixing extension*. Just as compact extensions are "relative" versions of compact systems, weakly mixing extensions are "relative" versions of weakly mixing systems, in which the underlying algebra of scalars $\mathbb{C}$ is replaced by $L^\infty(Y)$. As in the case of unconditionally weakly mixing systems, we will be able to use the van der Corput lemma to neglect "conditionally weakly mixing" functions, thus allowing us to lift the uniform multiple recurrence property (UMR) from a system to any weakly mixing extension of that system.

To finish the proof of the Furstenberg recurrence theorem requires two more steps. One is a relative version of the dichotomy between mixing and compactness: if a system is not weakly mixing relative to some factor, then that factor has a non-trivial compact extension. This will be accomplished using the theory of conditional Hilbert-Schmidt operators in this lecture. Finally, we need the (easy) result that the UMR property is preserved under limits of chains; this will be accomplished in the next lecture.

– Conditionally weakly mixing functions –

Recall that in a measure-preserving system $X = (X, \mathcal{X}, \mu, T)$, a function $f \in L^2(X) = L^2(X, \mathcal{X}, \mu)$ is said to be *weakly mixing* if the squared inner products $|\langle T^n f, f\rangle_X|^2 := (\int_X T^n f \overline{f}\ d\mu)^2$ converge in the Cesàro sense, thus

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} |\int_X T^n f\overline{f}\ d\mu|^2 = 0. \quad (1)$$

Now let $Y = (Y, \mathcal{Y}, \nu, S)$ be a factor of X, so that $L^\infty(Y)$ can be viewed as a subspace of $L^\infty(X)$. Recall that we have the conditional inner product $\langle f, g\rangle_{X|Y} := \mathbb{E}(f\overline{g}|Y)$ and the Hilbert module $L^2(X|Y)$ of functions f for which $\langle f, f\rangle_{X|Y}$ lies in $L^\infty(Y)$. We shall say that a function $f \in L^2(X|Y)$ is *conditionally weakly mixing* relative to Y if the L^2 norms $\|\langle T^n f, f\rangle_{X|Y}\|^2_{L^2(Y)}$ converge to zero in the Cesàro sense, thus

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \int_X |\mathbb{E}(T^n f\overline{f}|Y)|^2\ d\mu = 0. \quad (2)$$

Example 1. If $X = Y \times Z$ is a product system of the factor space $Y = (Y, \mathcal{Y}, \nu, S)$ and another system $Z = (Z, \mathcal{Z}, \rho, R)$, then a function $f(y, z) = f(z)$ of the vertical variable $z \in Z$ is weakly mixing relative to Y if and only if f(z) is weakly mixing in Z. $\diamond$

Much of the theory of weakly mixing systems extends easily to the conditionally weakly mixing case. For instance:

Exercise 1. By adapting the proof of Corollary 2 from Lecture 12, show that if $f \in L^2(X|Y)$ is conditionally weakly mixing and $g \in L^2(X|Y)$, then $\|\langle T^n f, g\rangle_{X|Y}\|^2_{L^2(Y)}$ and $\|\langle f, T^n g\rangle_{X|Y}\|^2_{L^2(Y)}$ converge to zero in the Cesàro sense. (*Hint*: you will need to show that expressions such as $\langle g, T^n f\rangle_{X|Y} T^n f$ converge in $L^2(X)$ in the Cesàro sense. Apply the van der Corput lemma and use the the fact that $\langle g, T^n f\rangle_{X|Y}$ are uniformly bounded in $L^\infty(Y)$ by conditional Cauchy-Schwarz.) $\diamond$

Exercise 2. Show that the space of conditionally weakly mixing functions in $L^2(X|Y)$ is a module over $L^\infty(Y)$ (i.e. it is closed under addition and multiplication by the "scalars" $L^\infty(Y)$), which is also shift-invariant and topologically closed in the topology of $L^2(X|Y)$ (see Exercise 2 from Lecture 13). $\diamond$

Let us now see the first link between conditional weak mixing and conditional almost periodicity (cf. Exercise 18 from Lecture 12):

Lemma 1. If $f \in L^2(X|Y)$ is conditionally weakly mixing and $g \in L^2(X|Y)$ is conditionally almost periodic, then $\langle f, g\rangle_{X|Y} = 0$ a.e.

Proof. Since $\langle f, g\rangle_{X|Y} = T^{-n}\langle T^n f, T^n g\rangle_{X|Y}$, it will suffice to show that

$$C-\sup_{n\to\infty} |\langle T^n f, T^n g\rangle_{X|Y}|_{L^2(Y)} = 0. \quad (3)$$

Let $\varepsilon > 0$ be arbitrary. As g is conditionally almost periodic, one can find a finitely generated module zonotope $\{c_1 f_1 + \ldots + c_d f_d : \|c_1\|_{L^\infty(Y)}, \ldots, \|c_d\|_{L^\infty(Y)} \leq 1\}$ with $f_1, \ldots, f_d \in L^2(X|Y)$ such that all the shifts $T^n g$ lie within ε (in $L^2(X|Y)$) of this zonotope. Thus (by conditional Cauchy-Schwarz) we have

$$\|\langle T^n f, T^n g\rangle_{X|Y}\|_{L^2(Y)} = \|\langle T^n f, c_{1,n} f_1 + \ldots + c_{d,n} f_d\rangle_{X|Y}\|_{L^2(Y)} + O(\varepsilon) \quad (4)$$

for all n and some $c_{1,n}, \ldots, c_{d,n} \in L^\infty(Y)$ with norm at most 1. We can pull these constants out of the conditional inner product and bound the left-hand side of (4) by

$$\|\langle T^n f, f_1\rangle\|_{L^2(Y)} + \ldots + \|\langle T^n f, f_1\rangle\|_{L^2(Y)} + O(\varepsilon). \quad (5)$$

By Exercise 1, the Cesàro supremum of (5) is at most $O(\varepsilon)$. Since ε is arbitrary, the claim (3) follows. $\Box$

Since all functions in $L^\infty(Y)$ are conditionally almost periodic, we conclude that every conditionally weakly mixing function f is orthogonal to $L^\infty(Y)$, or equivalently that $\mathbb{E}(f|Y) = 0$ a.e. Let us say that f has relative mean zero if the latter holds.

Definition 1. A system X is a *weakly mixing extension* of a factor Y if every $f \in L^2(X|Y)$ with relative mean zero is relatively weakly mixing.

Exercise 3. Show that a product $X = Y \times Z$ of a system Y with a weakly mixing system Z is always a weakly mixing extension of Y. ◇

Remark 1. If X is regular, then we can disintegrate the measure μ as an average $\mu = \int_Y \mu_y d\nu(y)$, see Theorem 4 from Lecture 9. It is then possible to construct a relative product system $X \times_Y X$, which is the product system $X \times X$ but with the measure $\mu \times_\nu \mu := \int_Y \mu_y \times \mu_y d\nu(y)$ instead of $\mu \times \mu$. It can then be shown (cf. Exercise 9 from Lecture 12) that X is a weakly mixing extension of Y if and only if $X \times_Y X$ is ergodic; see for instance Furstenberg's book for details. However, in these notes we shall focus instead on the more abstract operator-algebraic approach which avoids the use of disintegrations. ◇

Now we show that the uniform multiple recurrence property (UMR) from Lecture 13 is preserved under weakly mixing extensions (cf. Theorem 1 from Lecture 13).

Theorem 1. Suppose that $X = (X, \mathcal{X}, \mu, T)$ is a weakly mixing extension of $Y = (Y, \mathcal{Y}, \nu, S)$. If Y obeys UMR, then so does X.

The proof of this theorem rests on the following analogue of Proposition 1 from Lecture 12:

Proposition 1. Let $a_1, \ldots, a_k \in \mathbb{Z}$ be distinct integers for some $k \geq 1$. Let $X = (X, \mathcal{X}, \mu, T)$ is a weakly mixing extension of $Y = (Y, \mathcal{Y}, \nu, S)$, and let $f_1, \ldots, f_k \in L^\infty(X)$ be such that at least one of $f_1, \ldots, f_k$ has relative mean zero. Then

$$C-\lim_{n\to\infty} T^{a_1 n} f_1 \ldots T^{a_k n} f_k = 0 \quad (6)$$

in $L^2(X, \mathcal{X}, \mu)$.

Exercise 4. Prove Proposition 1. (*Hint*: modify (or "relativise") the proof of Proposition 1 from Lecture 12.) ◇

Corollary 1. Let $a_1, \ldots, a_k \in \mathbb{Z}$ be distinct integers for some $k \geq 1$. Let $X = (X, \mathcal{X}, \mu, T)$ is a weakly mixing extension of $Y = (Y, \mathcal{Y}, \nu, S)$, and let $f_1, \ldots, f_k \in L^\infty(X)$ be such that at least one of $f_1, \ldots, f_k$ has relative mean zero. Then

$$C-\lim_{n\to\infty} \int_X T^{a_1 n} f_1 \ldots T^{a_k n} f_k \, d\mu$$

$$- \int_X T^{a_1 n}\mathbb{E}(f_1|Y) \ldots T^{a_k n}\mathbb{E}(f_k|Y) \, d\mu = 0. \quad (7)$$

Exercise 5. Prove Corollary 1. (Hint: adapt the proof of Corollary 2 from Lecture 12.) ◇

Proof of Theorem 1. Let $f \in L^\infty(X)$ be non-negative with positive mean. Then $\mathbb{E}(f|Y) \in L^\infty(Y)$ is also non-negative with positive mean. Since Y obeys UMR, we have

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mathbb{E}(f|Y) T^n \mathbb{E}(f|Y) \ldots T^{(k-1)n}\mathbb{E}(f|Y) > 0. \quad (8)$$

Applying Corollary 1 we see that the same statement holds with $\mathbb{E}(f|Y)$ replaced by f, and the claim follows. □

Remark 2. As the above proof shows, Corollary 1 lets us replace functions in the weakly mixing extension X by their expectations in Y for the purposes of computing k-fold averages. In the notation of Furstenberg and Weiss, Corollary 1 asserts that Y is a *characteristic factor* of X for the average (7). The deeper structural theory of such characteristic factors (and in particular, on the minimal characteristic factor for any given average) is an active and difficult area of research, with surprising connections with Lie group actions (and in particular with flows on nilmanifolds), as well as the theory of inverse problems in additive combinatorics (and in particular to inverse theorems for the Gowers norms); see for instance the ICM paper of Kra for a survey of recent developments. The concept of a characteristic factor (or more precisely, finitary analogues of this concept) also is fundamental in my work with Ben Green on primes in arithmetic progression. ◇

– The dichotomy between structure and randomness –

The remainder of this lecture is devoted to proving the following "relative" generalisation of Theorem 1 from Lecture 12, and which is a fundamental ingredient in the proof of the Furstenberg recurrence theorem:

Theorem 2. Suppose that $X = (X, \mathcal{X}, \mu, T)$ is an extension of a system $Y = (Y, \mathcal{Y}, \nu, S)$. Then exactly one of the following statements is true:

1. (Structure) X has a factor Z which is a non-trivial compact extension of Y.
2. (Randomness) X is a weakly mixing extension of Y.

As in Lecture 12, the key to proving this theorem is to show

Proposition 2. Suppose that $X = (X, \mathcal{X}, \mu, T)$ is an extension of a system $Y = (Y, \mathcal{Y}, \nu, S)$. Then a function $f \in L^2(X|Y)$ is relatively weakly mixing if and only if $\langle f, g \rangle_{X|Y} = 0$ a.e. for all relatively almost periodic g.

The "only if" part of this proposition is Lemma 1; the harder part is the "if" part, which we will prove shortly. But for now, let us see why Proposition 2 implies Theorem 2.

From Lemma 1, we already know that no non-trivial function can be simultaneously conditionally weakly mixing and conditionally almost periodic, which shows that cases 1 and 2 of Theorem 2 cannot simultaneously hold. To finish the proof of Theorem 2, suppose that X is not a weakly mixing extension of Y, thus there exists a function $f \in L^2(X|Y)$ of relative mean zero which is not weakly mixing. By Proposition 2, there must exist a relatively almost periodic $g \in L^2(X|Y)$ such that $\langle f, g \rangle_{X|Y}$ does not vanish a.e.. Since f is orthogonal to all functions in $L^\infty(Y)$, we conclude that g is *not* in $L^\infty(Y)$, thus we have a single relatively almost periodic function. From Exercise 6 of Lecture 13, this shows that the maximal compact extension of Y is non-trivial, and the claim follows.

It thus suffices to prove the "if" part of Proposition 2; thus we need to show that every non-conditionally-weakly-mixing function correlates with some conditionally almost periodic function. But observe that if $f \in L^2(X|Y)$ is not conditionally weakly mixing, then by definition we have

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} |\mathbb{E}(T^n f \overline{f}|Y)|^2_{L^2(Y)} > 0. \quad (9)$$

We can rearrange this as

$$\limsup_{N\to\infty} \langle S_{f,N} f, f\rangle_X > 0. \quad (10)$$

where $S_{f,N} : L^2(X|Y) \to L^2(X|Y)$ is the operator

$$S_{f,N} g := \frac{1}{N}\sum_{n=0}^{N-1} \mathbb{E}(g\overline{T^n f}|Y) T^n f. \quad (11)$$

To prove Proposition 2, it thus suffices (by weak compactness) to show that

Proposition 3. (Dual functions are almost periodic) Suppose that $X = (X, \mathcal{X}, \mu, T)$ is an extension of a system $Y = (Y, \mathcal{Y}, \nu, S)$, and let $f \in L^2(X|Y)$. Let S_f be any limit point of $S_{f,N}$ in the weak operator technology. Then $S_f f$ is relatively almost periodic.

Remark 3. By applying the mean ergodic theorem to the dynamical system $X \times_Y X$, one can show that the sequence D_N is in fact convergent in the weak or strong operator topologies (at least when X is regular). But to avoid some technicalities we shall present an argument that does not rely on existence of a strong limit. $\diamond$

As one might expect from the experience with unconditional weak mixing, the proof of Proposition 3 relies on the theory of conditionally Hilbert-Schmidt operators on $L^2(X|Y)$. We give here a definition of such operators which is suited for our needs.

Definition 2. Let X, Y be as above. A *sub-orthonormal set* in $L^2(X|Y)$ is any at most countable sequence $e_\alpha \in L^2(X|Y)$ such that $\langle e_\alpha, e_\beta\rangle_{X|Y} = 0$ a.e. for all $\alpha \neq \beta$ and $\langle e_\alpha, e_\alpha\rangle_{X|Y} \leq 1$ a.e. for all α. A linear operator $A : L^2(X|Y) \to L^2(X|Y)$ is said to be a *conditionally Hilbert-Schmidt operator* if we have the module property

$$A(cf) = cAf \text{ for all } c \in L^\infty(Y) \quad (12)$$

and the bound

$$\sum_\alpha \sum_\gamma |\langle Ae_\alpha, f_\beta\rangle_{X,Y}|^2 \leq C^2 \text{ a.e.} \quad (13)$$

for all sub-orthonormal sets $\{e_\alpha\}$, $\{f_\beta\}$ and some constant $C > 0$; the best such C is called the *(uniform) conditional Hilbert-Schmidt norm* $\| \|A\|_{HS(X|Y)} \|_{L^\infty(Y)}$ of A.

Remark 4. As in Lecture 12, one can also set up the concept of a tensor product of two Hilbert modules, and use that to define conditionally Hilbert-Schmidt operators in a way which does not require sub-orthonormal sets. But we will not need to do so here. One can also define a pointwise conditional Hilbert-Schmidt norm $\|A\|_{HS(X|Y)}(y)$ for each $y \in Y$, but we will not need this concept. $\diamond$

Example 2. Suppose Y is just a finite set (with the discrete σ-algebra), then X splits into finitely many fibres $\pi^{-1}(\{y\})$ with the conditional measures μ_y, and $L^2(X|Y)$ can be direct sum (with the l^∞ norm) of the Hilbert spaces $L^2(\mu_y)$. A conditional Hilbert-Schmidt operator is then equivalent to a family of Hilbert-Schmidt operators $A_y : L^2(\mu_y) \to L^2(\mu_y)$ for each y, with the A_y uniformly bounded in Hilbert-Schmidt norm. $\diamond$

Example 3. In the skew shift example $X = (\mathbb{R}/\mathbb{Z})^2 = \{(y,z) : y, z \in \mathbb{R}/\mathbb{Z}\}$, $Y = (\mathbb{R}/\mathbb{Z})$, one can show that an operator A is conditionally Hilbert-Schmidt if and only if it takes the form $Af(y,z) = \int_{\mathbb{R}/\mathbb{Z}} K_y(z,z') f(y,z')\ dz'$ a.e. for all $f \in L^2(X|Y)$, with $\| \|A\|_{HS(X|Y)} \|_{L^\infty(Y)} = \sup_y (\int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} |K_y(z,z')|^2 dz dz')^{1/2}$ finite. $\diamond$

Exercise 6. Let $f_1, f_2 \in L^2(X|Y)$ with $\|f_1\|_{L^2(X|Y)}, \|f_2\|_{L^2(X|Y)} \leq 1$ a.e.. Show that the rank one operator $g \mapsto \langle g, f_1 \rangle_{X|Y} f_2$ is conditionally Hilbert-Schmidt with norm at most 1. $\diamond$

Observe from (11) that the $S_{f,N}$ are averages of rank one operators arising from the functions $T^n f$, and so by Exercise 6 and the triangle inequality we see that the $S_{f,N}$ are uniformly conditionally Hilbert-Schmidt. Taking weak limits using (13) (and Fatou's lemma) we conclude that S_f is also conditionally Hilbert-Schmidt.

Next, we observe from the telescoping identity that for every h, $T^h S_{f,N} - S_{f,N} T^h$ converges to zero in the weak operator topology (and even in the operator norm topology) as $N \to \infty$; taking limits, we see that S_f commutes with T. To show that $S_f f$ is conditionally almost periodic, it thus suffices to show the following analogue of Lemma 2 from Lecture 12:

Lemma 2. Let $A : L^2(X|Y) \to L^2(X|Y)$ be a conditionally Hilbert-Schmidt operator. Then the image of the unit ball of $L^2(X|Y)$ under A is conditionally precompact.

Proof. We shall prove this lemma by establishing a sort of conditional singular value decomposition for A. We can normalise A to have uniform conditional Hilbert-Schmidt norm 1. We fix $\varepsilon > 0$, and we will also need an integer k and a small quantity $\delta > 0$ depending on ε to be chosen later.

We first consider the quantities $|\langle Ae_1, f_1 \rangle_{X|Y}|^2$ where e_1, f_1 ranges over all sub-orthonormal sets of cardinality 1. On the one hand, these quantities are bounded pointwise by 1, thanks to (13). On the other hand, observe that if $|\langle Ae_1, f_1 \rangle_{X|Y}|^2$ and $|\langle Ae'_1, f'_1 \rangle_{X|Y}|^2$ are of the above form, then so is the join $\max(|\langle Ae_1, f_1 \rangle_{X|Y}|^2, |\langle Ae'_1, f'_1 \rangle_{X|Y}|^2)$, as can be seen by taking $tildee_1 := e_1 1_E + e'_1 1_{E^c}$ and

$tilde f_1 := f_1 1_E + f_1' 1_{E^c}$, where E is the set where $|\langle Ae_1, f_1\rangle_{X|Y}|^2$ exceeds $|\langle Ae_1', f_1'\rangle_{X|Y}|^2$. By using a maximising sequence for the quantity $\int_Y |\langle Ae, f\rangle_{X|Y}|^2\ d\nu$ and applying joins repeatedly, we can thus (on taking limits) find a pair e_1, f_1 which is near-optimal in the sense that $|\langle Ae_1, f_1\rangle_{X|Y}|^2 \geq (1-\delta)|\langle Ae_1', f_1'\rangle_{X|Y}|^2$ a.e. for all competitors e_1', f_1'.

Now fix e_1, f_1, and consider the quantity $|\langle Ae_2, f_2\rangle_{X|Y}|^2$, where $\{e_1, e_2\}$ and $\{f_1, f_2\}$ are sub-orthonormal sets. By arguing as before we can find an e_2, f_2 which is near optimal in the sense that $|\langle Ae_2, f_2\rangle_{X|Y}|^2 \geq (1-\delta)|\langle Ae_2', f_2'\rangle_{X|Y}|^2$ a.e. for all competitors e_2', f_2'.

We continue in this fashion k times to obtain sub-orthonormal sets $\{e_1, \ldots, e_k\}$ and $\{f_1, \ldots, f_k\}$ with the property that $|\langle Ae_i, f_i\rangle_{X|Y}|^2 \geq (1-\delta)|\langle Ae_i', f_i'\rangle_{X|Y}|^2$ whenever $\{e_1, \ldots, e_{i-1}, e_i'\}, \{f_1, \ldots, f_{i-1}, f_i'\}$ are sub-orthonormal sets. On the other hand, from (13) we know that $\sum_i |\langle Ae_i, f_i\rangle_{X|Y}|^2 \leq 1$. From these two facts we soon conclude that $|\langle Ae, f\rangle_{X|Y}|^2 \leq 1/k + O_k(\delta)$ a.e. whenever $\{e_1, \ldots, e_k, e\}$ and $\{f_1, \ldots, f_k, f\}$ are sub-orthonormal. If k, δ are chosen appropriately we obtain $|\langle Ae, f\rangle_{X|Y}| \leq \varepsilon$ a.e. Thus (by duality) A maps the unit ball of the orthogonal complement of the span of $\{e_1, \ldots, e_k\}$ to the ε-neighbourhood of the span of $\{f_1, \ldots, f_k\}$ (with notions such as orthogonality, span, and neighbourhood being defined conditionally of course, using the $L^\infty(Y)$-Hilbert module structure of $L^2(X|Y)$). From this it is not hard to establish the desired precompactness. $\Box$

[Update, Mar 1: Typo corrected.]

6 comments

Comments feed for this article

3 March, 2008 at 7:43 am

Lior

I may be confused, but in the proof of Thm. 2 shouldn't the almost periodic function g have *non-zero* correlation with f?

3 March, 2008 at 8:49 am

Terence Tao

Oops! You're right of course; thanks for the correction!

5 March, 2008 at 8:46 pm

254A, Lecture 15: The Furstenberg-Zimmer structure theorem and the Furstenberg recurrence theorem « What's new

254A, Lecture 15: The Furstenberg-Zimmer structure theorem and the Furstenberg recurrence theorem

5 March, 2008 in 254A - ergodic theory, math.DS

Tags: Furstenberg recurrence theorem, randomness, structure, Zorn's lemma
In this lecture - the final one on general measure-preserving dynamics - we put together the results from the past few lectures to establish the Furstenberg-Zimmer structure theorem for measure-preserving systems, and then use this to finish the proof of the Furstenberg recurrence theorem.

– The Furstenberg-Zimmer structure theorem –

Let $X = (X, \mathcal{X}, \mu, T)$ be a measure-preserving system, and let $Y = (Y, \mathcal{Y}, \nu, S)$ be a factor. In Theorem 2 of the previous lecture, we showed that X was not a weakly mixing extension of Y, then we could find a non-trivial compact extension Z of Y (thus $L^2(Z)$ is a non-trivial superspace of $L^2(Y)$). Combining this with Zorn's lemma (and starting with the trivial factor Y = pt), one obtains

Theorem 1. (Furstenberg-Zimmer structure theorem) Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. Then there exists an ordinal α and a factor $Y_\beta = (Y_\beta, \mathcal{Y}_\beta, \nu_\beta, S_\beta)$ for every $\beta \leq \alpha$ with the following properties:

1. $Y_\emptyset$ is a point.
2. For every successor ordinal $\beta + 1 \leq \alpha$, $Y_{\beta+1}$ is a compact extension of Y_β.
3. For every limit ordinal $\beta \leq \alpha$, Y_β is the inverse limit of the Y_γ for the $\gamma < \beta$, in the sense that $L^2(Y_\beta)$ is the closure of $\bigcup_{\gamma<\beta} L^2(Y_\gamma)$.
4. X is a weakly mixing extension of Y_α.

This theorem should be compared with Furstenberg's structure theorem for distal systems in topological dynamics (Theorem 2 from Lecture 7). Indeed, in analogy to that theorem, the factors Y_β are known as *distal measure-preserving systems*. The result was proven independently by Furstenberg and by Zimmer.

Exercise 1. Deduce Theorem 1 from Theorem 2 of the previous lecture. $\diamond$

Remark 1. Since the Hilbert spaces $L^2(Y_\beta)$ are increasing inside the separable Hilbert space $L^2(X)$, it is not hard to see that the ordinal α must be at most countable. Conversely, a result of Beleznay and Foreman shows that every countable ordinal can appear as the minimal length of a Furstenberg tower of a given system. Thus, in some sense, the complexity of a system can be as great as any countable ordinal.

This is because the structure theorem roots out every last trace of structure from the system, so much so that every remaining function orthogonal to the final factor $L^2(Y_\alpha)$ is weakly mixing. But in many applications one does not need so much weak mixing; for instance to establish k-fold recurrence for a function f, it would be enough to obtain weak mixing control on just a few combinations of f (such as $T^h f \overline{f}$), as we already saw in the proof of Roth's theorem in Lecture 12. In fact, it is not hard to show that to prove Furstenberg's recurrence theorem for a fixed k, one only needs to analyse the first k-2 steps of the Furstenberg tower. As one consequence of this, it is possible to avoid the use of Zorn's lemma (and the axiom of choice) in the proof of the recurrence theorem. $\diamond$

Remark 2. Analogues of the structure theorem exist for other actions, such as the action of $\mathbb{Z}^d$ on a measure space (which can equivalently be viewed as the action of d commuting shifts $T_1, \ldots, T_d: X \to X$). There is a new feature in this case, though: instead of having a tower of purely compact extensions, followed by one weakly mixing extension at the end, one instead has a tower of hybrid extensions (known as primitive extensions), each one of which is compact along one subgroup of $\mathbb{Z}^d$ and weakly mixing along a complementary subgroup. See for instance Furstenberg's book for details. $\diamond$

– The Furstenberg recurrence theorem –

The Furstenberg recurrence theorem asserts that every measure-preserving system $(X, \mathcal{X}, \mu, T)$ has the uniform multiple recurrence (UMR) property, thus

$$\liminf_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_X f T^n f \ldots T^{(k-1)n} f \, d\mu > 0 \quad (1)$$

whenever $k \geq 1$ and $f \in L^\infty(X)$ is non-negative with positive mean. The UMR property is trivially true for a point, and we have already shown that UMR is preserved by compact extensions (Theorem 1 of Lecture 13) and by weakly mixing extensions (Theorem 1 of Lecture 14). The former result lets us climb the successor ordinal steps of the tower in Theorem 1, while the latter lets us jump from the final distal system Y_α to X. But to clinch the proof of the recurrence theorem, we also need to deal with the limit ordinals. More precisely, we need to prove

Theorem 2. (Limits of chains) Let $(Y_\beta)_{\beta \in B}$ be a totally ordered family of factors of a measure-preserving system X (thus $L^2(Y_\beta)$ is increasing with β, and let Y be the inverse limit of the Y_β. If each of the Y_β obeys the UMR, then Y does also.

With this theorem, the Furstenberg recurrence theorem (Theorem 1 from Lecture 11) follows from the previous theorems and transfinite induction.

The main difficulty in establishing Theorem 2 is that while each Y_β obeys the UMR separately, we do not know that this property holds uniformly in β. The main new observation needed to establish the theorem is that there is another way to leverage the UMR from a factor to an extension… if the support of the function f is sufficiently "dense". We motivate this by first considering the unconditional case.

Proposition 1. (UMR for densely supported functions) Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system, let $k \geq 1$ be an integer, and let $f \in L^\infty(X)$ be a non-negative function whose support $\{x : f(x) > 0\}$ has measure greater than $1 - 1/k$. Then (1) holds.

Proof. By monotone convergence, we can find $\varepsilon > 0$ such that $f(x) > \varepsilon$ for all x outside of a set E of measure at most $1/k - \varepsilon$. For any n, this implies that $f(x)T^n f(x) \ldots T^{(k-1)n} f(x) > \varepsilon^k$ for all x outside of the set $E \cup T^n E \cup \ldots \cup T^{(k-1)n} E$, which has measure at most $1 - k\varepsilon$. In particular we see that

$$\int_X f T^n f \ldots T^{(k-1)n} f \ d\mu > k\varepsilon^{k+1} \quad (2)$$

for all n, and the claim follows. □

As with the other components of the proof of the recurrence theorem, we will need to upgrade the above proposition to a "relative" version:

Proposition 2. (UMR for relatively densely supported functions) Let $(X, \mathcal{X}, \mu, T)$ be an extension of a factor $(Y, \mathcal{Y}, \nu, S)$ with the UMR, let $k \geq 1$ be an integer, and let $f \in L^\infty(X)$ be a non-negative function whose support $\Omega := \{x : f(x) > 0\}$ is such that the set $\{y \in Y : \mathbb{E}(1_\Omega|Y) > 1 - 1/k\}$ has positive measure in Y. Then (1) holds.

Proof. By monotone convergence again, we can find $\varepsilon > 0$ such that the set $E := \{x : f(x) > \varepsilon\}$ is such that the set $F := \{y \in Y : \mathbb{E}(1_E|Y) > 1 - 1/k + \varepsilon\}$ has positive measure. Since Y has the UMR, this implies that (1) holds for 1_F. In other words, there exists $c > 0$ such that

$$\nu(F \cap T^n F \cap \ldots \cap T^{(k-1)n} F) > c \quad (3)$$

for all n in a set of positive lower density.

Now we turn to f. We have the pointwise lower bound $f(x) \geq \varepsilon 1_E(x)$, and so

$$f T^n f \ldots T^{(k-1)n} f(x) \geq \varepsilon^k 1_{E \cap T^n E \cap \ldots \cap T^{(k-1)n} E}(x). \quad (4)$$

We have the crude lower bound

$$1_{E \cap T^n E \cap \ldots \cap T^{(k-1)n} E}(x) \geq 1 - \sum_{j=0}^{k-1} 1_{T^{jn} E^c}(x); \quad (5)$$

inserting this into (4) and taking conditional expectations, we conclude

$$\mathbb{E}(f T^n f \ldots T^{(k-1)n} f|Y)(y) \geq \varepsilon^k (1 - \sum_{j=0}^{k-1} \mathbb{E}(1_{T^{jn} E^c}|Y)(y)) \quad (6)$$

a.e. On the other hand, we have

$$\mathbb{E}(1_{T^{jn} E^c}|Y) = 1 - \mathbb{E}(1_{T^{jn} E}|Y) = 1 - T^{jn} \mathbb{E}(1_E|Y). \quad (7)$$

By definition of F, we thus see that if y lies in $F \cap T^n F \cap \ldots \cap T^{(k-1)n} F$, then

$$\mathbb{E}(fT^n f \ldots T^{(k-1)n} f|Y)(y) \geq \varepsilon^k \times k\varepsilon. \quad (8)$$

Integrating this and using (3), we obtain

$$\int_X fT^n f \ldots T^{(k-1)n} f \, d\mu \geq c\varepsilon^k \times k\varepsilon \quad (9)$$

for all n in a set of positive lower density, and (1) follows. □

Proof of Theorem 2. Let $f \in L^\infty(Y)$ be non-negative with positive mean $\int_X f \, d\mu = c > 0$; we may normalise f to be bounded by 1. Since Y is the inverse limit of the Y_β, we see that the orthogonal projections $\mathbb{E}(f|Y_\beta)$ converge in $L^2(X)$ norm to $\mathbb{E}(f|Y) = f$. Thus, for any ε, we can find β such that

$$\|f - \mathbb{E}(f|Y_\beta)\|_{L^2(X)} \leq \varepsilon. \quad (10)$$

Now $\mathbb{E}(f|Y_\beta)$ has the same mean c as f, and is also bounded by 1. Thus the set $E := \{y : \mathbb{E}(f|Y_\beta)(y) \geq c/2\}$ must have measure at least c/2 in Y_β. Now if $\Omega := \{x : f(x) > 0\}$, then we have the pointwise bound

$$|f - \mathbb{E}(f|Y_\beta)| \geq \tfrac{c}{2} 1_{\Omega^c} 1_E; \quad (11)$$

squaring this and taking conditional expectations we obtain

$$\mathbb{E}(|f - \mathbb{E}(f|Y_\beta)|^2)(y) \geq \tfrac{c^2}{4}(1 - \mathbb{E}(1_\Omega|Y_\beta)(y)) 1_E(y), \quad (12)$$

and so by (10) and Markov's inequality we see that $1 - \mathbb{E}(1_\Omega|Y_\beta)(y) 1_E(y) < 1/k$ on a set of measure $O_c(\varepsilon^2)$. Choosing ε sufficiently small depending on c, we conclude (from the lower bound $\mu(E) \geq c/2$) that $\mathbb{E}(1_\Omega|Y_\beta)(y) > 1 - 1/k$ on a set of positive measure. The claim now follows from Proposition 2. □

The proof of the Furstenberg recurrence theorem (and thus Szemerédi's theorem) is finally complete.

Remark 3. The same type of argument yields many further recurrence theorems, and thus (by the correspondence principle) many combinatorial results also. For instance, in the original paper of Furstenberg it was noted that the above arguments allow one to strengthen (1) to

$$\liminf_{N \to \infty} \inf_M \frac{1}{N} \sum_{n=M}^{M+N-1} \int_X fT^n f \ldots T^{(k-1)n} f \, d\mu > 0, \quad (13)$$

which allows one to conclude that in a set A of positive upper density, the set of n for which $A \cap (A+n) \cap \ldots \cap (A+(k-1)n)$ has positive upper density is syndetic for every k. One can also extend the argument to higher dimensions, and to polynomial recurrence without too many changes in the structure of the proof. But some more serious modifications to the argument are needed for other recurrence results involving IP systems or Hales-Jewett type results; see Lecture 10 for more discussion. ◇

[*Update*, Mar 6: Statement of Proposition 1 corrected.]

254A, Lecture 16: A Ratner-type theorem for nilmanifolds

9 March, 2008 in 254A - ergodic theory, math.DS, math.GR

Tags: dichotomy, ergodicity, nilmanifolds, nilpotent groups, Ratner's theorem, structure

The last two lectures of this course will be on Ratner's theorems on equidistribution of orbits on homogeneous spaces. Due to lack of time, I will not be able to cover all the material here that I had originally planned; in particular, for an introduction to this family of results, and its connections with number theory, I will have to refer readers to my previous blog post on these theorems. In this course, I will discuss two special cases of Ratner-type theorems. In this lecture, I will talk about Ratner-type theorems for discrete actions (of the integers $\mathbb{Z}$) on nilmanifolds; this case is much simpler than the general case, because there is a simple criterion in the nilmanifold case to test whether any given orbit is equidistributed or not. Ben Green and I had need recently to develop quantitative versions of such theorems for a number-theoretic application. In the next and final lecture of this course, I will discuss Ratner-type theorems for actions of $SL_2(\mathbb{R})$, which is simpler in a different way (due to the semisimplicity of $SL_2(\mathbb{R})$, and lack of compact factors).

— Nilpotent groups –

Before we can get to Ratner-type theorems for nilmanifolds, we will need to set up some basic theory for these nilmanifolds. We begin with a quick review of the concept of a nilpotent group - a generalisation of that of an abelian group. Our discussion here will be purely algebraic (no manifolds, topology, or dynamics will appear at this stage).

Definition 1. (Commutators) Let G be a (multiplicative) group. For any two elements g,h in G, we define the commutator [g,h] to be $[g,h] := g^{-1}h^{-1}gh$ (thus g and h commute if and only if the commutator is trivial). If H and K are subgroups of G, we define the commutator $[H,K]$ to be the group generated by all the commutators $\{[h,k] : h \in H, k \in K\}$.

For future reference we record some trivial identities regarding commutators:

$$gh = hg[g,h] = [g^{-1},h^{-1}]hg \quad (1)$$

$$h^{-1}gh = g[g,h] = [h,g^{-1}]g \quad (2)$$

$$[g,h]^{-1} = [h,g]. \quad (3)$$

Exercise 1. Let H, K be subgroups of a group G.

1. Show that [H,K] = [K,H].
2. Show that H is abelian if and only if [H,H] is trivial.
3. Show that H is central if and only if [H,G] is trivial.
4. Show that H is normal if and only if $[H,G] \subset H$.
5. Show that [H,G] is always normal.
6. If $L \lhd H, K$ is a normal subgroup of both H and K, show that $[H,K]/([H,K] \cap L) \equiv [H/L, K/L]$.
7. Let HK be the group generated by $H \cup K$. Show that $[H,K]$ is a normal subgroup of HK, and when one quotients by this subgroup, H/[H,K] and K/[H,K] become abelian. $\diamond$

Exercise 2. Let G be a group. Show that the group G/[G,G] is abelian, and is the universal abelianisation of G in the sense that every homomorphism $\phi : G \to H$ from G to an abelian group H can be uniquely factored as $\phi = \tilde\phi \circ \pi$, where $\pi : G \to G/[G,G]$ is the quotient map and $\tilde\phi : G/[G,G] \to H$ is a homomorphism. $\diamond$

Definition 2. (Nilpotency) Given any group G, define the lower central series

$$G = G_0 = G_1 \rhd G_2 \rhd G_3 \rhd \ldots \quad (4)$$

by setting $G_0, G_1 := G$ and $G_{i+1} := [G_i, G]$ for $i \geq 1$. We say that G is nilpotent of step s if G_{s+1} is trivial (and G_s is non-trivial).

Examples 1. A group is nilpotent of step 0 if and only if it is trivial. It is nilpotent of step 1 if and only if it is non-trivial and abelian. Any subgroup or homomorphic image of a nilpotent group of step s is nilpotent of step at most s. The direct product of two nilpotent groups is again nilpotent, but the semi-direct product of nilpotent groups is merely solvable in general. If G is any group, then G/G_{s+1} is nilpotent of step at most s. $\diamond$

Example 2. Let $n \geq 1$ be an integer, and let

$$U_n(\mathbb{R}) = \begin{pmatrix} 1 & \mathbb{R} & \ldots & \mathbb{R} \\ 0 & 1 & \ldots & \mathbb{R} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \quad (5)$$

be the group of all upper-triangular $n \times n$ real matrices with 1s on the diagonal (i.e. the group of unipotent upper-triangular matrices). Then $U_n(\mathbb{R})$ is nilpotent of step n. Similarly if $\mathbb{R}$ is replaced by other fields. $\diamond$

Exercise 3. Let G be an arbitrary group.

1. Show that each element G_i of the lower central series is a characteristic subgroup of G, i.e. $\phi(G_i) = G_i$ for all automorphisms $\phi : G \to G$. (Specialising to inner automorphisms, this shows that the G_i are all normal subgroups of G.)

2. Show the filtration property $[G_i, G_j] \subset G_{i+j}$ for all $i, j \geq 0$. (Hint: induct on i+j; then, holding i+j fixed, quotient by G_{i+j}, and induct on (say) i. Note that once one quotients by G_{i+j}, all elements of $[G_{i-1}, G_j]$ are central (by the first induction hypothesis), while G_{i-1} commutes with $[G, G_j]$ (by the second induction hypothesis). Use these facts to show that all the generators of $[G, G_{i-1}]$ commute with G_j.) ◇

Exercise 4. Let G be a nilpotent group of step 2. Establish the identity

$$g^n h^n = (gh)^n [g, h]^{\binom{n}{2}} \quad (6)$$

for any integer n and any $g, h \in G$, where $\binom{n}{2} := \frac{n(n-1)}{2}$. (This can be viewed as a discrete version of the first two terms of the Baker-Campbell-Hausdorff formula.) Conclude in particular that the space of *Hall-Petresco sequences* $n \mapsto g_0 g_1^n g_2^{\binom{n}{2}}$, where $g_i \in G_i$ for $i = 0, 1, 2$, is a group under pointwise multiplication (this group is known as the *Hall-Petresco group* of G). There is an analogous identity (and an analogous group) for nilpotent groups of higher step; see for instance this paper of Leibman for details. The Hall-Petresco group is rather useful for understanding multiple recurrence and polynomial behaviour in nilmanifolds; we will not discuss this in detail, but see Exercise 5 below for a hint as to the connection. ◇

Exercise 5. (Arithmetic progressions in nilspaces are constrained) Let X be a nilspace of step $s \leq 2$, and consider two arithmetic progressions $x, gx, \ldots g^{s+1}x$ and $y, hy, \ldots, h^{s+1}y$ of length s+2 in X, where $x, y \in X$ and $g, h \in G$. Show that if these progressions agree in the first s+1 places (thus $g^i x = h^i y$ for all $i = 0, \ldots, s$ then they also agree in the last place. (Hint: the only tricky case is s=2. For this, either use direct algebraic computation, or experiment with the group of Hall-Petresco sequences from the previous exercise. The claim is in fact true for general s, because the Hall-Petresco group exists in every step.) ◇

Remark 1. By Exercise 3.2, the lower central series is a filtration with respect to the commutator operation $g, h \mapsto [g, h]$. Conversely, if G admits a filtration $G = G_{(0)} = G_{(1)} \geq \ldots$ with $[G_{(i)}, G_{(j)}] \subset G_{(i+j)}$ and $G_{(j)}$ trivial for $j > s$, then it is nilpotent of step at most s. It is sometimes convenient for inductive purposes to work with filtrations rather than the lower central series (which is the "minimal" filtration available to a group G); see for instance my paper with Ben Green on this topic. ◇

Remark 2. Let G be a nilpotent group of step s. Then $[G, G_s] = G_{s+1}$ is trivial and so G_s is central (by Exercise 1), thus abelian and normal. By another application of Exercise 1, we see that G/G_s is nilpotent of step s-1. Thus we see that any nilpotent group G of step s is an *abelian extension* of a nilpotent group G/G_s of step s-1, in the sense that we have a short exact sequence

$$0 \to G_s \to G \to G/G_s \to 0 \quad (6)$$

where the kernel G_s is abelian. Conversely, every abelian extension of an s-1-step nilpotent group is nilpotent of step at most s. In principle, this gives a recursive description of s-step nilpotent groups as an s-fold iterated tower of abelian extensions of the trivial group. Unfortunately, while abelian groups are of course very well understood, abelian *extensions* are a little inconvenient to work with algebraically; the sequence (6) is not quite enough, for instance, to assert that G is a semi-direct product of G_s and G/G_s (this would require some means of embedding G/G_s back into G, which is not available in general). One can identify G (using the axiom of choice) with a product set $G/G_s \times G_s$ with a group law $(g,n) \cdot (h,m) = (gh, nm\phi(g,h))$, where $\phi: G/G_s \times G/G_s \to G_s$ is a map obeying various cocycle-type identities, but the algebraic structure of ϕ is not particularly easy to exploit. Nevertheless, this recursive tower of extensions seems to be well suited for understanding the *dynamical* structure of nilpotent groups and their quotients, as opposed to their *algebraic* structure (cf. our use of recursive towers of extensions in our previous lectures in dynamical systems and ergodic theory). $\diamond$

In our applications we will not be working with nilpotent groups G directly, but rather with their homogeneous spaces X, i.e. spaces with a transitive left-action of G. (Later we will also add some topological structure to these objects, but let us work in a purely algebraic setting for now.) Such spaces can be identified with group quotients $X \equiv G/\Gamma$ where $\Gamma \leq G$ is the stabiliser $\Gamma = \{g \in G : gx = x\}$ of some point x in X. (By the transitivity of the action, all stabilisers are conjugate to each other.) It is important to note that in general, Γ is not normal, and so X is not a group; it has a left-action of G but not right-action of G. Note though that any central subgroup of G acts on either the left or the right.

Now let G be s-step nilpotent, and let us temporarily refer to $X = G/\Gamma$ as an *s-step nilspace*. Then G_s acts on the right in a manner that commutes with the left-action of G. If we set $\Gamma_s := G_s \cap \Gamma \lhd G_s$, we see that the right-action of Γ_s on G/Γ is trivial; thus we in fact have a right-action of the abelian group $T_s := G_s/\Gamma_s$. (In our applications, T_s will be a torus.) This action can be easily verified to be free. If we let $\overline{X} := X/T_s$ be the quotient space, then we can view X as a principal T_s-bundle over $\overline{X}$. It is not hard to see (cf. the isomorphism theorems) that $\overline{X} \equiv \pi(G)/\pi(\Gamma)$, where $\pi: G \to G/G_s$ is the quotient map. Observe that $\pi(G)$ is nilpotent of step s-1, and $\pi(\Gamma)$ is a subgroup. Thus we have expressed an arbitrary s-step nilspace as a principal bundle (by some abelian group) over an s-1-step nilspace, and so s-step nilspaces can be viewed as towers of abelian principal bundles, just as s-step nilpotent groups can be viewed as towers of abelian extensions.

– Nilmanifolds –

It is now time to put some topological structure (and in particular, Lie structure) on our nilpotent groups and nilspaces.

Definition 3 (Nilmanifolds). An s-step nilmanifold is a nilspace G/Γ, where G is a finite-dimensional Lie group which is nilpotent of step s, and Γ is a discrete subgroup which is cocompact or uniform in the sense that the quotient G/Γ is compact.

Remark 3. In the literature, it is sometimes assumed that the nilmanifold G/Γ is connected, and that the group G is connected, or at least that its group $\pi_0(G) := G/G^\circ$ of connected components ($G^\circ \lhd G$ being the identity component of G) is finitely generated (one can often easily reduce to this case in applications). It is also convenient to assume that G° is simply connected (again, one can usually reduce to this case in applications, by passing to the universal cover of G° if necessary), as this implies (by the Baker-Campbell-Hausdorff formula) that the nilpotent Lie group G° is *exponential*, i.e. the exponential map $\exp: \mathfrak{g} \to G^\circ$ is a homeomorphism. $\diamond$

Example 3. (Skew torus) If we define

$$G := \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}; \quad \Gamma := \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \quad (7)$$

(thus G consists of the upper-triangular unipotent matrices whose middle right entry is an integer, and Γ is the subgroup in which all entries are integers) then G/Γ is a 2-step nilmanifold. If we write

$$[x, y] := \begin{pmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Gamma \quad (8)$$

then we see that G/Γ is isomorphic to the square $\{[x, y] : 0 \leq x, y \leq 1\}$ with the identifications $[x, 1] \equiv [x, 0]$ and $[0, y] := [1, y + x \bmod 1]$. (Topologically, this is homeomorphic to the ordinary 2-torus $(\mathbb{R}/\mathbb{Z})^2$, but the skewness will manifest itself when we do dynamics.) $\diamond$

Example 4. (Heisenberg nilmanifold) If we set

$$G := \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}; \quad \Gamma := \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix} \quad (9)$$

then G/Γ is a 2-step nilmanifold. It can be viewed as a three-dimensional cube with the faces identified in a somewhat skew fashion, similarly to the skew torus in Example 3. $\diamond$

Let $\mathfrak{g}$ be the Lie algebra of G. Every element g of G acts linearly on $\mathfrak{g}$ by conjugation. Since G is nilpotent, it is not hard to see (by considering the iterated commutators of g with an infinitesimal perturbation of the identity) that this linear action is unipotent, and in particular has determinant 1. Thus, any constant volume form on this Lie algebra will be preserved by conjugation, which by basic differential geometry allows us to create a volume form (and hence a measure) on G which is invariant under both left and right translation; this Haar measure is clearly unique up to scalar multiplication. (In other words, nilpotent Lie groups are unimodular.) Restricting this measure to a fundamental domain of G/Γ and then descending to the nilmanifold we obtain a left-invariant Haar measure, which (by compactness) we can normalise to be a Borel probability measure. (Because of the existence of a left-invariant probability measure μ on G/Γ, we refer to the discrete subgroup Γ of G as a lattice.) One can show that this left-invariant Borel probability measure is unique.

Definition 4. (Nilsystem) An *s-step nilsystem (or nilflow)* is a topological measure-preserving system (i.e. both a topological dynamical system and a measure-preserving system) with underlying space G/Γ a s-step nilmanifold (with the Borel σ-algebra and left-invariant probability measure), with a shift T of the form $T : x \mapsto gx$ for some $g \in G$.

Example 5. The Kronecker systems $x \mapsto x + \alpha$ on compact abelian Lie groups are 1-step nilsystems. $\diamond$

Example 6. The skew shift system $(x, y) \mapsto (x + \alpha, y + x)$ on the torus $(\mathbb{R}/\mathbb{Z})^2$ can be identified with a nilflow on the skew torus (Example 3), after identifying (x,y) with [x,y] and using the group element

$$g := \begin{pmatrix} 1 & \alpha & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad (10)$$

to create the flow. $\diamond$

Example 7. Consider the Heisenberg nilmanifold (Example 4) with a flow generated by a group element

$$g := \begin{pmatrix} 1 & \gamma & \beta \\ 0 & 1 & \alpha \\ 0 & 0 & 1 \end{pmatrix} \quad (11)$$

for some real numbers α, β, γ. If we identify

$$[x, y, z] := \begin{pmatrix} 1 & z & y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \Gamma \quad (12)$$

then one can verify that

$$T^n : [x, y, z] \mapsto [\{x + n\alpha\}, y + n\beta + \frac{n(n+1)}{2}\alpha\gamma - \lfloor x$$

$$+ n\alpha\rfloor(x + n\gamma) \mod 1, latexz + n\gamma \mod 1] \quad (13)$$

where $\lfloor\rfloor$ and $\{\}$ are the integer part and fractional part functions respectively. Thus we see that orbits in this nilsystem are vaguely quadratic in n, but for the presence of the not-quite-linear operators $\lfloor\rfloor$ and $\{\}$. (These expressions are known as *bracket polynomials*, and are intimately related to the theory of nilsystems.) $\diamond$

Given that we have already seen that nilspaces of step s are principal abelian bundles of nilspaces of step s-1, it should be unsurprising that nilsystems of step s are abelian extensions of nilsystems of step s-1. But in order to ensure that topological structure is preserved correctly, we do need to verify one point:

Lemma 1. Let G/Γ be an s-step nilmanifold, with G connected and simply connected. Then $\Gamma_s := G_s \cap \Gamma$ is a discrete cocompact subgroup of G_s. In particular, $T_s := G_s/\Gamma_s$ is a compact connected abelian Lie group (in other words, it is a torus).

Proof. Recall that G is exponential and thus identifiable with its Lie algebra $\}$. The commutators G_i can be similarly identified with the Lie algebra commutators $\}_i$; in particular, the G_i are all connected, simply connected Lie groups.

The key point to verify is the cocompact nature of Γ_s in G_s; all other claims are straightforward. We first work in the abelianisation G/G_2, which is identifiable with its Lie algebra and thus isomorphic to a vector space. The image of Γ under the quotient map $G \to G/G_2$ is a cocompact subgroup of this vector space; in particular, it contains a basis of this space. This implies that Γ contains an "abelianised" basis $e_1, \ldots, e_d$ of G in the sense that every element of G can be expressed in the form $e_1^{t_1} \ldots e_d^{t_d}$ modulo an element of the normal subgroup G_2 for some real numbers $t_1, \ldots, t_d$, where we take advantage of the exponential nature of G to define real exponentiation $g^t := \exp(t \log(g))$. Taking commutators s times (which eliminates all the "modulo G_2" errors), we then see that G_s is generated by expressions of the form $[e_{i_1}, [e_{i_2}, [\ldots, e_{i_s}] \ldots]^t$ for $i_1, \ldots, i_s \in \{1, \ldots, d\}$ and real t. Observe that these expressions lie in Γ_s if t is integer. As G_s is abelian, we conclude that each element in G_s can be expressed as an element of Γ_s , times a bounded number of elements of the form $[e_{i_1}, [e_{i_2}, [\ldots, e_{i_s}] \ldots]^t$ with $0 \leq t < 1$. From this we conclude that the quotient map $G_s \mapsto G_s/\Gamma_s$ is already surjective on some bounded set, which we can take to be compact, and so G_s/Γ_s is compact as required. $\square$

As a consequence of this lemma, we see that if $X = G/\Gamma$ is an s-step nilmanifold with G connected and simply connected, then X/T_s is an s-1-step nilmanifold (with G still connected and simply connected), and that X is a principal T_s-bundle over X/T_s in the topological sense as well as in the purely algebraic sense. One consequence of this is that every s-step nilsystem (with G connected and simply connected) can be viewed as a *toral extension* (i.e. a group extension by a torus) of an s-1-step nilsystem (again with G connected and simply connected). Thus for instance the skew shift system (Example 6) is a circle extension of a circle shift, while the Heisenberg nilsystem (Example 7) is a circle extension of an abelian 2-torus shift.

Remark 4. One should caution though that the converse of the above statement is not necessarily true; an extension $X \times_\phi T$ of an s-1-step nilsystem X by a torus T using a cocycle $\phi : X \to T$ need not be isomorphic to an s-step nilsystem (the cocycle ϕ has to obey an an additional equation (or more precisely, a system of equations when $s > 2$), known as the *Conze-Lesigne equation*, before this is the case. See for instance this paper of Ziegler for further discussion. $\diamond$

Exercise 6. Show that Lemma 1 continues to hold if we relax the condition that G is connected and simply connected, to instead require that G/Γ is connected, that G/G° is finitely generated, and that G° is simply connected. (I believe that all three of these hypotheses are necessary, but haven't checked this carefully.) $\diamond$

Exercise 7. Show that Lemma 1 continues to hold if G_s and Γ_s are replaced by G_i and $\Gamma_i = G_i \cap \Gamma$ for any $0 \leq i \leq s$. In particular, setting i=2, we obtain a projection map $\pi : X \to X_2$ from X to the Kronecker nilmanifold $X_2 = (G/G_2)/(\Gamma G_2/G_2)$. $\diamond$

Remark 5. One can take the structural theory of nilmanifolds much further, in particular developing the theory of *Mal'cev bases* (of which the elements $e_1, \ldots, e_d$ used to prove Lemma 1 were a very crude prototype). See the foundational paper of Mal'cev (or its English translation) for details, as well as the later paper of Leibman which addresses the case in which G is not necessarily connected. $\diamond$

– A criterion for ergodicity –

We now give a useful criterion to determine when a given nilsystem is ergodic.

Theorem 1. Let $(X,T) = (G/\Gamma, x \mapsto gx)$ be an s-step nilsystem with G connected and simply connected, and let (X_2, T_2) be the underlying Kronecker factor, as defined in Exercise 7. Then X is ergodic if and only if X_2 is ergodic.

This result is originally due to Leon Green, using spectral theory methods. We will use an argument of Parry (and adapted by Leibman), relying on "vertical" Fourier analysis and topological arguments, which we have already used for the skew shift in Proposition 1 of Lecture 9.

Proof. If X is ergodic, then the factor X_2 is certainly ergodic. To prove the converse implication, we induct on s. The case $s \leq 1$ is trivial, so suppose $s > 1$ and the claim has already been proven for s-1. Then if X_2 is ergodic, we already know from induction hypothesis that X/T_s is ergodic. Suppose for contradiction that X is not ergodic, then we can find a non-constant shift-invariant function on X. Using Fourier analysis (or representation theory) of the vertical torus T_s as in Proposition 1 of Lecture 9, we may thus find a non-constant shift-invariant function f which has a *single vertical frequency* χ in the sense that one has $f(g_s x) = \chi(g_s) f(x)$ for all $x \in X$, $g_s \in G_s$, and some character $\chi: G_s \to S^1$. If the character χ is trivial, then f descends to a non-constant shift-invariant function on X/T_s, contradicting the ergodicity there, so we may assume that χ is non-trivial. Also, |f| descends to a shift-invariant function on X/T_s and is thus constant by ergodicity; by normalising we may assume $|f| = 1$.

Now let $g_{s-1} \in G_{s-1}$, and consider the function $F_{g_{s-1}}(x) := f(g_{s-1}x)\overline{f(x)}$. As G_s is central, we see that $F_{g_{s-1}}$ is G_s-invariant and thus descends to X/T_s. Furthermore, as f is shift-invariant (so $f(gx) = f(x)$), and $[g_{s-1}, g] \in G_s$, some computation reveals that $F_{g_{s-1}}$ is an eigenfunction:

$$F_{g_{s-1}}(gx) = \chi([g_{s-1}, g]) F_{g_{s-1}}(x). \quad (14)$$

In particular, if $\chi([g_{s-1}, g]) \neq 1$, then $F_{g_{s-1}}$ must have mean zero. On the other hand, by continuity (and the fact that |f|=1) we know that $F_{g_{s-1}}$ has non-zero mean for g_{s-1} close enough to the identity. We conclude that $\chi([g_{s-1}, g]) = 1$ for all g_{s-1} close to the identity; as the map $g_{s-1} \mapsto \chi([g_{s-1}, g])$ is a homomorphism, we conclude in fact that $\chi([g_{s-1}, g]) = 1$ for all g_{s-1}. In particular, from (14) and ergodicity we see that $F_{g_{s-1}}$ is constant, and so $f(g_{s-1}x) = c(g_{s-1}) f(x)$ for some $c(g_{s-1}) \in S^1$.

Now let $h \in G$ be arbitrary. Observe that

$$\int_G f(hg_{s-1}x)\overline{f(x)}\ d\mu = \int_G f(hy)\overline{f(g_{s-1}^{-1}y)}\ d\mu$$

$$= c(g_{s-1}) \int_G f(hy)\overline{f(y)}\ d\mu$$

$$= \int_G f(g_{s-1}hy)\overline{f(y)}\ d\mu$$

$$= \chi([g_{s-1}, h]) \int_G f(hg_{s-1}y)\overline{f(y)}\ d\mu. \quad (15)$$

For h and g_{s-1} close enough to the identity, the integral is non-zero, and we conclude that $\chi([g_{s-1}, h]) = 1$ in this case. The map $(g_{s-1}, h) \mapsto \chi([g_{s-1}, h])$ is a homomorphism in each variable and so is constant. Since $G_s = [G_{s-1}, G]$, we conclude that χ is trivial, a contradiction. $\Box$

Remark 6. The hypothesis that G is connected and simply connected can be dropped; see the paper of Leibman for details. $\diamond$

One pleasant fact about nilsystems, as compared with arbitrary dynamical systems, is that ergodicity can automatically be upgraded to unique ergodicity:

Theorem 2. Let (X,T) be an ergodic nilsystem. Then (X,T) is also uniquely ergodic. Equivalently, for every $x \in X$, the orbit $(T^n x)_{n\in\mathbb{Z}}$ is equidistributed.

Exercise 8. By inducting on step and adapting the proof of Proposition 3 from Lecture 9, prove Theorem 2. $\diamond$

– a Ratner-type theorem –

A subnilsystem of a nilsystem $(X, T) = (G/\Gamma, T)$ is a compact subsystem (Y, S) which is of the form $Y = Hx$ for some $x \in X$ and some closed subgroup $H \leq G$. One easily verifies that a subnilsystem is indeed a nilsystem.

From the above theorems we quickly obtain

Corollary 1 (Dichotomy between structure and randomness) Let (X,T) be a nilsystem with group G connected and simply connected, and let $x \in X$. Then exactly one of the following statements is true:

1. The orbit $(T^n x)_{n\in\mathbb{Z}}$ is equidistributed.
2. The orbit $(T^n x)_{n\in\mathbb{Z}}$ is contained in a proper subnilsystem (Y,S) with group H connected and simply connected, and with dimension strictly smaller than that of G.

Proof. It is clear that 1. and 2. cannot both be true. Now suppose that 1. is false. By Theorem 2, this means that (X,T) is not ergodic; by Theorem 1, this implies that the Kronecker system (X_2, T_2) is not ergodic. Expanding functions on $X_2 \equiv G/G_2$ into characters and using Fourier analysis, we conclude that there is a non-trivial character $\chi : G/G_2 \to S^1$ which is T_2-invariant. If we let $\pi : G \to G/G_2$ be the canonical projection, then $\chi : G \to S^1$ is a continuous homomorphism, and the kernel H is a closed connected subgroup of G of strictly lower dimension. Furthermore, Hx is equal to a level set of χ and is thus compact. Since χ is T_2 invariant, we see that $T^n x \in Hx$ for all n, and the claim follows. $\Box$

Iterating this, we obtain

Corollary 2 (Ratner-type theorem for nilmanifolds) Let (X,T) be a nilsystem with group G connected and simply connected, and let $x \in X$. Then the orbit $(T^n x)_{n\in\mathbb{Z}}$ is equidistributed in some subnilmanifold (Y,S) of (X,T). (In particular, this orbit is dense in Y.) Furthermore, Y= Hx for some closed connected subgroup H of G.

Remark 7. Analogous claims also hold when G is not assumed to be connected or simply connected, and if the orbit $(T^n x)_{n \in \mathbb{Z}}$ is replaced with a polynomial orbit $(T^{p(n)} x)_{n \in \mathbb{Z}}$; see this paper of Leibman for details (and this followup paper for the case of $\mathbb{Z}^d$-actions. In a different direction, such discrete Ratner-type theorems have been extended to other unipotent actions on finite volume homogeneous spaces by Shah. Quantitative versions of this theorem have also been obtained by Ben Green and myself. $\diamond$

No comments

Comments feed for this article

254A, Lecture 17: A Ratner-type theorem for SL_2(R) orbits

15 March, 2008 in 254A - ergodic theory, math.DS, math.RT

Tags: Lie algebras, Lie groups, Mautner phenomenon, Ratner's theorem, unipotence
In this final lecture, we establish a Ratner-type theorem for actions of the special linear group $SL_2(\mathbb{R})$ on homogeneous spaces. More precisely, we show:

> **Theorem 1.** Let G be a Lie group, let $\Gamma < G$ be a discrete subgroup, and let $H \leq G$ be a subgroup isomorphic to $SL_2(\mathbb{R})$. Let μ be an H-invariant probability measure on G/Γ which is ergodic with respect to H (i.e. all H-invariant sets either have full measure or zero measure). Then μ is *homogeneous* in the sense that there exists a closed connected subgroup $H \leq L \leq G$ and a closed orbit $Lx \subset G/\Gamma$ such that μ is L-invariant and supported on Lx.

This result is a special case of a more general theorem of Ratner, which addresses the case when H is generated by elements which act unipotently on the Lie algebra $\mathfrak{g}$ by conjugation, and when G/Γ has finite volume. To prove this theorem we shall follow an argument of Einsiedler, which uses many of the same ingredients used in Ratner's arguments but in a simplified setting (in particular, taking advantage of the fact that H is semisimple with no non-trivial compact factors). These arguments have since been extended and made quantitative by Einsiedler, Margulis, and Venkatesh.

– Representation theory of $SL_2(\mathbb{R})$ –

Theorem 1 concerns the action of $H \equiv SL_2(\mathbb{R})$ on a homogeneous space G/Γ. Before we are able to tackle this result, we must first understand the linear actions of $H \equiv SL_2(\mathbb{R})$ on real or complex vector spaces - in other words, we need to understand the representation theory of the Lie group $SL_2(\mathbb{R})$ (and its associated Lie algebra $\mathfrak{sl}_2(\mathbb{R})$).

Of course, this theory is very well understood, and by using the machinery of weight spaces, raising and lowering operators, etc. one can completely classify all the finite-dimensional representations of $SL_2(\mathbb{R})$; in fact, all such representations are isomorphic to direct sums of symmetric powers of the standard representation of $SL_2(\mathbb{R})$ on $\mathbb{R}^2$. This classification quickly yields all the necessary facts we will need here. However, we will use only a minimal amount of this machinery here, to obtain as direct and elementary a proof of the results we need as possible.

The first fact we will need is that finite-dimensional representations of $SL_2(\mathbb{R})$ are completely reducible.

Lemma 1. (Complete reducibility) Let $SL_2(\mathbb{R})$ act linearly (and smoothly) on a finite-dimensional real vector space V, and let W be a $SL_2(\mathbb{R})$-invariant subspace of V. Then there exists a complementary subspace W' to W which is also $SL_2(\mathbb{R})$-invariant (thus V is isomorphic to the direct sum of W and W').

Proof. We will use Weyl's unitary trick to create the complement W', but in order to invoke this trick, we first need to pass from the non-compact group $SL_2(\mathbb{R})$ to a compact counterpart. This is done in several stages.

First, we linearise the action of the Lie group $SL_2(\mathbb{R})$ by differentiating to create a corresponding linear action of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ in the usual manner.

Next, we complexify the action. Let $V^{\mathbb{C}} := V \otimes \mathbb{C}$ and $W^{\mathbb{C}} := W \otimes \mathbb{C}$ be the complexifications of V and W respectively. Then the complexified Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ acts on both $V^{\mathbb{C}}$ and $W^{\mathbb{C}}$, and in particular the special unitary Lie algebra $\mathfrak{su}_2(\mathbb{C})$ does also.

Since the special unitary group

$$SU_2(\mathbb{C}) = \{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}; |\alpha|^2 + |\beta|^2 = 1\} \quad (1)$$

is topologically equivalent to the 3-sphere S^3 and is thus simply connected, a standard homotopy argument allows one to exponentiate the $\mathfrak{su}_2(\mathbb{C})$ action to create a $SU_2(\mathbb{C})$ action, thus creating the desired compact action. (This trick is not restricted to $\mathfrak{sl}_2(\mathbb{R})$, but can be generalised to other semisimple Lie algebras using the Cartan decomposition.)
Now we can apply the unitary trick. Take any Hermitian form $\langle , \rangle$ on $V^{\mathbb{C}}$. This form need not be preserved by the $SU_2(\mathbb{C})$ action, but if one defines the averaged form

$$\langle u, v \rangle_{SU_2} := \int_{SU_2(\mathbb{C})} \langle gu, gv \rangle\ dg \quad (2)$$

where dg is Haar measure on the compact Lie group $SU_2(\mathbb{C})$, then we see that $\langle , \rangle_{SU_2}$ is a Hermitian form which is $SU_2(\mathbb{C})$-invariant; thus this form endows $V^{\mathbb{C}}$ with a Hilbert space structure with respect to which the $SU_2(\mathbb{C})$-action is unitary. If we then define $(W')^{\mathbb{C}}$ to be the orthogonal complement of $W^{\mathbb{C}}$ in this Hilbert space, then this vector space is invariant under the $SU_2(\mathbb{C})$ action, and thus (by differentiation) by the $\mathfrak{su}_2(\mathbb{C})$ action. But observe that $\mathfrak{su}_2(\mathbb{C})$ and $\mathfrak{sl}_2(\mathbb{R})$ have the same complex span (namely, $\mathfrak{sl}_2(\mathbb{C})$); thus the complex vector space $(W')^{\mathbb{C}}$ is also $\mathfrak{sl}_2(\mathbb{R})$-invariant.

The last thing to do is to undo the complexification. If we let W' be the space of real parts of vectors in $(W')^{\mathbb{C}}$ which are real modulo $W^{\mathbb{C}}$, then one easily verifies that W' is $\mathfrak{sl}_2(\mathbb{R})$-invariant (hence $SL_2(\mathbb{R})$-invariant, by exponentiation) and is a complementary subspace to W, as required. $\Box$

Remark 1. We can of course iterate the above lemma and conclude that every finite-dimensional representation of $SL_2(\mathbb{R})$ is the direct sum of irreducible representations, which explains the term "complete reducibility". Complete reducibility of finite-dimensional representations of a Lie algebra (over a field of characteristic zero) is equivalent to that Lie algebra being semisimple. The situation is slightly more complicated for Lie groups, though, if such groups are not simply connected. $\diamond$

An important role in our analysis will be played by the one-parameter unipotent subgroup $U := \{u^t : t \in \mathbb{R}\}$ of $SL_2(\mathbb{R})$, where

$$u^t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}. \quad (3)$$

Clearly, the elements of U are unipotent when acting on $\mathbb{R}^2$. It turns out that they are unipotent when acting on all other finite-dimensional representations also:

> **Lemma 2.** Suppose that $SL_2(\mathbb{R})$ acts on a finite-dimensional real or complex vector space V. Then the action of any element of U on V is unipotent.

Proof. By complexifying V if necessary we may assume that V is complex. The action of the Lie group $SL_2(\mathbb{R})$ induces a Lie algebra homomorphism $\rho : \mathfrak{sl}_2(\mathbb{R}) \to \mathrm{End}(V)$. To show that the action of U is unipotent, it suffices to show that $\rho(\log u)$ is nilpotent, where

$$\log u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (4)$$

is the infinitesimal generator of U. To show this, we exploit the fact that $\log u$ induces a raising operator. We introduce the diagonal subgroup $D := \{d^t : t \in \mathbb{R}\}$ of $SL_2(\mathbb{R})$, where

$$d^t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}. \quad (5)$$

This group has infinitesimal generator

$$\log d = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (6)$$

Observe that $[\log d, \log u] = 2 \log u$, and thus (since ρ is a Lie algebra homomorphism)

$$[\rho(\log d), \rho(\log u)] = 2\rho(\log u). \quad (7)$$

We can rewrite this as

$$(\rho(\log d) - \lambda - 2)\rho(\log u) = \rho(\log u)(\rho(\log d) - \lambda) \quad (8)$$

for any $\lambda \in \mathbb{C}$, which on iteration implies that

$$(\rho(\log d) - \lambda - 2r)^m \rho(\log u)^r = \rho(\log u)^r (\rho(\log d) - \lambda)^m \quad (9)$$

for any non-negative integers m, r. But this implies that $\rho(\log u)^r$ raises generalised eigenvectors of $\rho(\log d)$ of eigenvalue λ to generalised eigenvectors of $\rho(\log d)$ of eigenvalue $\lambda + 2m$. But as V is finite dimensional, there are only finitely many eigenvalues of $\rho(\log d)$, and so $\rho(\log u)$ is nilpotent on each of the generalised eigenvectors of $\rho(\log d)$. By the Jordan normal form, these generalised eigenvectors span V, and we are done. $\square$

Exercise 1. By carrying the above analysis further (and also working with the adjoint of U to create lowering operators) show (for complex V) that $\rho(\log d)$ is diagonalisable, and the eigenvalues are all integers. For an additional challenge: deduce from this that the representation is isomorphic to a direct sum of the representations of $SL_2(\mathbb{R})$ on the symmetric tensor powers $\mathrm{Sym}^k(\mathbb{R}^2)$ of $\mathbb{R}^2$ (or, if you wish, on the space of homogeneous polynomials of degree k on 2 variables). Of course, if you are stuck, you can turn to any book on representation theory (I recommend Fulton and Harris). $\square$

The group U is merely a subgroup of the group $SL_2(\mathbb{R})$, so it is not a priori evident that any vector (in a space that $SL_2(\mathbb{R})$ acts on) which is U-invariant, is also $SL_2(\mathbb{R})$-invariant. But, thanks to the highly non-commutative nature of $SL_2(\mathbb{R})$, this turns out to be the case, even in infinite dimensions, once one restricts attention to *continuous unitary* actions:

Lemma 3 (Mautner phenomenon). Let $\rho : SL_2(\mathbb{R}) \to U(V)$ be a continuous unitary action on a Hilbert space V (possibly infinite dimensional). Then any vector $v \in V$ which is fixed by U, is also fixed by $SL_2(\mathbb{R})$.

Proof. We use an argument of Margulis. We may of course take v to be non-zero. Let $\varepsilon > 0$ be a small number. Then even though the matrix $w^\varepsilon := \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$ is very close to the identity, the double orbit $Uw^\varepsilon U$ can stray very far away from U. Indeed, from the algebraic identity

$$\begin{pmatrix} e^t & 0 \\ \varepsilon & e^{-t} \end{pmatrix} = u^{(e^t-1)/\varepsilon} w^\varepsilon u^{(e^{-t}-1)/\varepsilon} \quad (10)$$

which is valid for any $t \in \mathbb{R}$, we see that this double orbit in fact comes very close to the diagonal group D. Applying (10) to the U-invariant vector v and taking inner products with v, we conclude from unitarity that

$$\langle \rho(\begin{pmatrix} e^t & 0 \\ \varepsilon & e^{-t} \end{pmatrix})v, v\rangle = \langle \rho(w^\varepsilon)v, v\rangle. \quad (11)$$

Taking limits as $\varepsilon \to 0$ (taking advantage of the continuity of ρ) we conclude that $\langle \rho(d^t)v, v\rangle = \langle v, v\rangle$. Since $\rho(d^t)v$ has the same length as v, we conclude from the converse Cauchy-Schwarz inequality that $\rho(d^t)v = v$, i.e. that v is D-invariant. Since U and D generate $SL_2(\mathbb{R})$, the claim follows. $\square$

Remark 2. The key fact about U being used here is that its Lie algebra is not trapped inside any proper ideal of $sl_2(\mathbb{R})$, which, in turn, follows from the fact that this Lie algebra is simple. One can do the same thing for semisimple Lie algebras provided that the unipotent group U is non-degenerate in the sense that it has non-trivial projection onto each simple factor. $\diamond$

This phenomenon has an immediate dynamical corollary:

Corollary 1 (Moore ergodic theorem). Suppose that $SL_2(\mathbb{R})$ acts in a measure-preserving fashion on a probability space $(X, \mathcal{X}, \mu)$. If this action is ergodic with respect to $SL_2(\mathbb{R})$, then it is also ergodic with respect to U.

Proof. Apply Lemma 3 to $L^2(X, \mathcal{X}, \mu)$. $\Box$

– Proof of Theorem 1 –

Having completed our representation-theoretic preliminaries, we are now ready to begin the proof of Theorem 1. The key is to prove the following dichotomy:

Proposition 1. (Lack of concentration implies additional symmetry) Let G, H, μ, Γ be as in Theorem 1. Suppose there exists a closed connected subgroup $H \leq L \leq G$ such that μ is L-invariant. Then exactly one of the following statements hold:

1. (Concentration) μ is supported on a closed orbit Lx of L.
2. (Additional symmetry) There exists a closed connected subgroup $L < L' \leq G$ such that μ is L'-invariant.

Iterating this proposition (noting that the dimension of L' is strictly greater than that of L) we will obtain Theorem 1. So it suffices to establish the proposition.

We first observe that the ergodicity allows us to obtain the concentration conclusion (1) as soon as μ assigns any non-zero mass to an orbit of L:

Lemma 4. Let the notation and assumptions be as in Proposition 1. Suppose that $\mu(Lx_0) > 0$ for some x_0. Then Lx_0 is closed and μ is supported on Lx_0.

Proof. Since Lx_0 is H-invariant and μ is H-ergodic, the set Lx_0 must either have full measure or zero measure. It cannot have zero measure by hypothesis, thus $\mu(Lx_0) = 1$. Thus, if we show that Lx_0 is closed, we automatically have that μ is supported on Lx_0.

As G/Γ is a homogeneous space, we may assume without loss of generality (conjugating L if necessary) that x_0 is at the origin, then $Lx_0 \equiv L/(\Gamma \cap L)$. The measure μ on this set can then be pulled back to a measure m on L by the formula

$$\int_L f(g)\ dm(g) = \int_{L/(\Gamma\cap L)} \sum_{g\in x(\Gamma\cap L)} f(g)\ d\mu(x). \quad (12)$$

By construction, m is left L-invariant (i.e. a left Haar measure) and right $(\Gamma\cap L)$-invariant. From uniqueness of left Haar measure up to constants, we see that for any g in L there is a constant $c(g) > 0$ such that $m(Eg) = c(g)m(E)$ for all measurable E. It is not hard to see that $c: L \to \mathbb{R}^+$ is a character, i.e. it is continuous and multiplicative, thus $c(gh) = c(g)c(h)$ for all g, h in L. Also, it is the identity on $(\Gamma\cap L)$ and thus descends to a continuous function on $L/(\Gamma\cap L)$. Since μ is L-invariant, we have

$$\int_{L/(\Gamma\cap L)} c(g)\ d\mu(g) = \int_{L/(\Gamma\cap L)} c(hg)\ d\mu(g) = \int_{L/(\Gamma\cap L)} c(h)c(g)\ d\mu(g) \quad (13)$$

for all h in L, and thus c is identically 1 (i.e. L is unimodular). Thus m is right-invariant, which implies that μ obeys the right-invariance property $\mu(Kx_0) = \mu(Kgx_0)$ for any g in L and any sufficiently small compact set $K \subset L$ (small enough to fit inside a single fundamental domain of $L/(\Gamma\cap L)$).

Recall that $\mu(Lx_0) = 1$. By partitioning L into countably many small sets as above, we can thus find a small compact set $K \subset L$ such that $\mu(Kx_0) > 0$. Now consider a maximal set of disjoint translates $Kg_1x_0, Kg_2x_0, \ldots, Kg_kx_0$ of Kx_0; since all of these sets have the same positive measure, such a maximal set exists and is finite. Then for any g in L, Kgx_0 must intersect one of the sets Kg_ix_0, which implies that $Lx_0 = \bigcup_{i=1}^k K^{-1}Kg_ix_0$. But the right-hand side is compact, and so Lx_0 is closed as desired. □

We return to the proof of Proposition 1. In view of Lemma 4, we may assume that μ is totally non-concentrated on L-orbits in the sense that

$$\mu(Lx) = 0 \hbox{ for all } x \in G/\Gamma. \quad (14)$$

In particular, for μ-almost every x and y, y does not lie in the orbit Lx of x and vice versa; informally, the group elements in G that are used to move from x to y should be somehow "transverse" to L.

On the other hand, we are given that μ is ergodic with respect to H, and thus (by Corollary 1) ergodic with respect to U. This implies (cf. Proposition 2 from Lecture 9) that μ-almost every point x in G/Γ is *generic* (with respect to U) in the sense that

$$\int_{G/\Gamma} f\ d\mu = \lim_{T\to+\infty} \frac{1}{T}\int_0^T f(u^t x)\ dt. \quad (15)$$

for all continuous compactly supported $f: G/\Gamma \to \mathbb{R}$.

Exercise 2. Prove this claim. (Hint: obtain continuous analogues of the theory from Lecture 8 and Lecture 9.) ◇

The equation (15) (and the Riesz representation theorem) lets us describe the measure μ in terms of the U-orbit of a generic point. On the other hand, from (14) and the ensuing discussion we see that any two generic points are likely to be separated from each other by some group element "transverse" to L. It is the interaction between these two facts which is going to generate the additional symmetry needed for Proposition 1. We illustrate this with a model case, in which the group element centralises U:

Proposition 2 (central case). Let the notation and assumptions be as in Proposition 1. Suppose that x, y are generic points such that $y = gx$ for some $g \in G$ that centralises U (i.e. it commutes with every element of u). Then μ is invariant under the action of g.

Proof. Let $f : G/\Gamma \to \mathbb{R}$ be continuous and compactly supported. Applying (15) with x replaced by y=gx we obtain

$$\int_{G/\Gamma} f\ d\mu = \lim_{T\to+\infty} \frac{1}{T} \int_0^T f(u^t gx)\ dt. \quad (16)$$

Commuting g with u^t and using (15) again, we conclude

$$\int_{G/\Gamma} f\ d\mu = \int_{G/\Gamma} f(gy)\ d\mu(y) \quad (17)$$

and the claim follows from the Riesz representation theorem. □

Of course, we don't just want invariance under one group element g; we want a whole group L' of symmetries for which one has invariance. But it is not hard to leverage the former to the latter, provided one has enough group elements:

Lemma 5. Let the notation and assumptions be as in Proposition 1. Suppose one has a sequence g_n of group elements tending to the identity, such that the action of each of the g_n preserve μ, and such that none of the g_n lie in L. Then there exists a closed connected subgroup $L < L' \leq G$ such that μ is L-invariant.

Proof. Let S be the stabiliser of μ, i.e. the set of all group elements g whose action preserves μ. This is clearly a closed subgroup of G which contains L. If we let L' be the identity connected component of S, then L' is a closed connected subgroup containing L which will contain g_n for all sufficiently large n, and in particular is not equal to L. The claim follows. □

From Proposition 2 and Lemma 5 we see that we are done if we can find pairs $x_n, y_n = g_n x_n$ of nearby generic points with g_n going to the identity such that $g_n \not\in L$ and that g_n centralises U. Now we need to consider the non-central case; thus suppose for instance that we have two generic points x, y=gx in which g is close to the identity but does not centralise U. The key observation here is that we can use the U-invariance of the situation to pull x and y slowly apart from each other. More precisely, since x and y are generic, we observe that $u^t x$ and $u^t y$ are also generic for any t, and that these two points differ by the conjugated group element $g^t := u^t g u^{-t}$. Taking logarithms (which are well-defined as long as g^t stays close to the identity), we can write

$$\log(g^t) = u^t \log(g) u^{-t} = \exp(t \mathrm{ad}(\log u)) \log(g) \quad (18)$$

where ad is the adjoint representation. From Lemma 2, we know that $\mathrm{ad}(\log u) : \mathfrak{g} \to \mathfrak{g}$ is nilpotent, and so (by Taylor expansion of the exponential) $\log(g^t)$ depends polynomially on t. In particular, if g does not centralise U, then $\log(g^t)$ is non-constant and thus must diverge to infinity as $t \to +\infty$. In particular, given some small ball B around the origin in $\mathfrak{g}$ (with respect to some arbitrary norm), then whenever $\log g$

lies inside B around the origin and is not central, there must be a first time $t = t_g$ such that $\log g^{t_g}$ reaches the boundary ∂B of this ball. We write $g^* := g^{t_g} \in \partial B$ for the location of g when it escapes. We now have the following variant of Proposition 2:

Proposition 3 (non-central case). Let the notation and assumptions be as in Proposition 1. Suppose that $x_n, y_n \in G$ are generic points such that $y_n = g_n x_n$ for some $g_n \in G$ which do not centralise u, but such that g_n converge to the identity (in particular, $g_n \in B$ for all sufficiently large n). Suppose furthermore that x_n, y_n are *uniformly generic* in the sense that for any continuous compactly supported $f : G/\Gamma \to \mathbb{R}$, the convergence of (15) (with x replaced by x_n or y_n) is uniform in n. Then μ is invariant under the action of any limit point $g^* \in \partial B$ of the g_n^*.

Proof. By passing to a subsequence if necessary we may assume that g_n^* converges to g^*. For each sufficiently large n, we write $T_n := t_{g_n}$, thus $g_n^t \in B$ for all $0 \leq t \leq T_n$, and $g_n^{T_n} = g_n^*$. We rescale this by defining the functions $h_n : [0,1] \to B$ by $h_n(s) := g_n^{sT_n}$. From the unipotent nature of U, these functions are polynomial (with bounded degree), and also bounded (as they live in B), and are thus equicontinuous (since all norms are equivalent on finite dimensional spaces). Thus, by the Arzelà-Ascoli theorem, we can assume (after passing to another subsequence) that h_n is uniformly convergent to some limit f, which is another polynomial. Since we already have $h_n(1) = g_n^*$ converging to g_*, this implies that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $h_n(s) = g_* + O(\varepsilon)$ for all $1 - \delta \leq s \leq 1$ and all sufficiently large n. In other words, we have

$$u^t g_n u^{-t} = g_* + O(\varepsilon) \quad (19)$$

for sufficiently large n, whenever $(1-\delta)T_n \leq t \leq T_n$.

This is good enough to apply a variant of the Proposition 2 argument. Namely, if $f : G/\Gamma \to \mathbb{R}$ is continuous and compactly supported, then by uniform genericity we have for T sufficiently large that

$$\int_{G/\Gamma} f\ d\mu = \frac{1}{\delta T} \int_{(1-\delta)T}^{T} f(u^t y_n)\ dt + O(\varepsilon) \quad (20)$$

for all n. Applying (19) we can write $u^t y_n = g_* u^t x_n + O(\varepsilon)$ on the support of f, and so by uniform continuity of f

$$\int_{G/\Gamma} f\ d\mu = \frac{1}{\delta T} \int_{(1-\delta)T}^{T} f(g_* u^t x_n)\ dt + o(1) \quad (21)$$

where o(1) goes to zero as $\varepsilon \to 0$, uniformly in n. Using (15) again and then letting $\varepsilon \to 0$, we obtain the g_*-invariance of μ as desired. $\Box$

Now we have all the ingredients to prove Proposition 1, and thus Theorem 1.

Proof of Proposition 1. We know that μ-almost every point is generic. Applying Egoroff's theorem, we can find sets $E \subset G/\Gamma$ of measure arbitrarily close to 1 (e.g. $\mu(E) \geq 0.9$) on which the points are uniformly generic.

Now let V be a small neighbourhood the origin in L. Observe from the Fubini-Tonelli theorem that

$$\int_X \frac{1}{m(V)} \int_V 1_E(x) 1_E(gx) \quad dm(g) d\mu(x) \geq 2\mu(E) - 1 \geq 0.8 \quad (22)$$

where m is the Haar measure on the unimodular group L, from which one can find a set $E' \subset E$ of positive measure such that $m(\{g \in V : gx \in E\}) = 0.7m(V)$ for all $x \in E'$; one can view E' as "points of density" of E in some approximate sense (and with regard to the L action).

Since E' has positive measure, and using (14), it is not hard to find sequences $x_n, y_n \in E'$ with $y_n \notin Lx_n$ for any n and with $\mathrm{dist}(x_n, y_n) \to 0$ (using some reasonable metric on G/Γ).

Exercise 3. Verify this. (*Hint*: G/Γ can be covered by countably many balls of a fixed radius.) $\diamond$

Next, recall that $H \equiv SL_2(\mathbb{R})$ acts by conjugation on the Lie algebra $\mathfrak{g}$ of G, and also leaves the Lie algebra $\mathfrak{l} \subset \mathfrak{g}$ of L invariant. By Lemma 1, this implies there is a complementary subspace W of $\mathfrak{l}$ in $\mathfrak{g}$ which is also H-invariant (and in particular, U-invariant). From the inverse function theorem, we conclude that for any group element g in G sufficiently close to the identity, we can factor $g = \exp(w)l$ where $l \in L$ is also close to the identity, and $w \in W$ is small (in fact this factorisation is unique). We let $\pi_L : g \mapsto l$ be the map from g to l; this is well-defined and smooth near the identity.

Let n be sufficiently large, and write $y_n = g_n x_n$ where g_n goes to the identity as n goes to infinity. Pick $l_n \in V$ at random (using the measure m conditioned to V). Using the inverse function theorem and continuity, we see that the random variable $\pi_L(l_n g_n)$ is supported in a small neighbourhood of V, and that its distribution converges to the uniform distribution of V (in, say, total variation norm) as $n \to \infty$. In particular, we see that $y'_n := l_n y_n \in E$ with probability at least 0.7 and $x'_n := \pi_L(l_n g_n) x_n \in E$ with probability at least 0.6 (say) if n is large enough. In particular we can find an $l_n \in V$ such that y'_n, x'_n both lie in E. Also by construction we see that $y'_n = \exp(w_n) x'_n$ for some $w_n \in W$; since $y_n \notin Lx_n$, we see that w_n is non-zero. On the other hand, since W is transverse to $\mathfrak{l}$ and the distance between x_n, y_n go to zero, we see that w_n goes to zero.

There are now two cases. If $\exp(w_n)$ centralises U for infinitely many n, then from Proposition 2 followed by Lemma 5 we obtain conclusion 2 of Proposition 1 as required. Otherwise, we may pass to a subsequence and assume that none of the $\exp(w_n)$ centralise U. Since W is preserved by U, we see that the group elements $\exp(w_n)^*$ also lie in $\exp(K)$ for some compact set K in W, and also on the boundary of B. This space is compact, and so by Proposition 3 we see that μ is invariant under some group element $g \in \exp(K) \cap \partial B$, which cannot lie in L. Since the ball B can be chosen arbitrarily small, we can thus apply Lemma 5 to again obtain conclusion 2 of Proposition 1 as required. $\square$

2 comments

Comments feed for this article

17 March, 2008 at 5:30 am

www.smasra.com