Homework 4 for
MATH 574, Topics in Logic

Due: Wednesday March 16

Problem 1

Let $A, B \subseteq \mathbb{N}$. We say A is *many-one reducible* to B, $A \leq_m B$, if there exists a computable function f such that

$$f(A) \subseteq B \text{ and } f(\mathbb{N} \setminus A) \subseteq \mathbb{N} \setminus B, \text{ i.e. } x \in A \Leftrightarrow f(x) \in B.$$

(a) Show that if B is decidable and $A \leq_m B$, then A is dedicable, too.

(b) Show that the relation $A \equiv_m B$ defined as $A \leq_m B$ & $B \leq_m A$ is an *equivalence relation*.

(c) Let $A \oplus B = \{2n : n \in A\} \cup \{2n+1 : n \in B\}$. Show that $A \oplus B$ is a *least upper bound* of A, B with respect to $\leq_m$, that is, show that $A \leq_m A \oplus B$ and $B \leq_m A \oplus B$, and if $A \leq_m C$ and $B \leq_m C$, then $A \oplus B \leq_m C$.

Problem 2

Two sets $A, B \subseteq \mathbb{N}$ are called *computably inseperable* if there does not exist a decidable set C such that $A \subseteq C$ and $B \cap C = \emptyset$.

(a) Show that the sets $A = \{n : M_n(n) \downarrow= 0\}$ and $B = \{n : M_n(n) \downarrow= 1\}$ are computably inseperable.[1]

(b) Use part (a) to show that there exists a partial computable function f with the folowing property: there does not exist a Turing machine M such that (1) M halts on all inputs, and (2) whenever $f(n)$ is defined, $M(n) = f(n)$. This means there exists a partial computable function that cannot be extended to a (total) computable function.

Problem 3

Prove the characterization of subshifts via forbidden substrings mentioned in Lesson 3-1:
A set $S \subseteq A^{\mathbb{N}}$ is a subshift if and only if there exists a set $W \subseteq A^{<\mathbb{N}}$ such that

$$S = \{x \in A^{<\mathbb{N}} : \text{ no } w \in W \text{ appears as a substring in } x\}.$$

Problem 4

Let $(X, \mathcal{A}, \mu)$ be a probability space and suppose $T : X \to X$ is measure preserving. Let $E \subseteq X$ be measurable and of positive measure, $\mu E > 0$. Show that there exists a measurable set $F \subseteq E$ with $\mu(E \setminus F) = 0$ such that

$$\forall x \in F \; \exists n \geq 1 \; T^n(x) \in F,$$

that is, almost every point in E returns to E at some point.

[1] Recall that M_k denotes the partial function computed by the k-th Turing machine, and $M_k(m) \downarrow$ means the k-th machine halts on input m