Lecture 19: Recursive Ordinals and Ordinal Notations

We have seen that the property "*α codes a well-ordering of* $\mathbb{N}$" is important for the study of co-analytic sets. If A is $\mathbf{\Pi}^1_1$, then there exists a tree T on $\mathbb{N} \times \mathbb{N}$ such that

$$\alpha \in A \quad \Longleftrightarrow \quad T(\alpha) \text{ is well-founded.}$$

If $T(\alpha)$ is well-founded, then the Kleene-Brouwer ordering restricted to T is a well-ordering. Since $T(\alpha)$ is a tree on $\mathbb{N}$, it constitutes an ordering on $\mathbb{N}$, using a standard bijection between strings and natural numbers.

If A is moreover Π^1_1, then there is a *recursive* such tree and the tree $T(\alpha)$ is recursive in α. If α is recursive and $\alpha \in A$, then $T(\alpha)$ encodes a *recursive well-ordering*.

In general, we say an ordinal $\xi < \omega_1$ is **recursive** if there exists a recursive $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that

$$E_\alpha = \{(m,n) \in \mathbb{N} \times \mathbb{N} \colon \alpha(\langle m,n \rangle) = 0\}$$

is a well-ordering of order type ξ.

Proposition 19.1: *The recursive ordinals form a countable initial segment of the class* Ord *of all ordinals.*

Proof. Suppose ξ is a recursive ordinal. Let $\alpha \in \mathbb{N}^{\mathbb{N}}$ be recursive so that the order type of E_α is ξ. Let $\eta < \xi$. Since $\eta \in \xi$, there exists n such that E_α restricted to

$$\{m \colon m E_\alpha n\}$$

has order type η. Hence η is recursive via the relation

$$\{(k,m) \colon k, m E_\alpha n \;\&\; k E_\alpha m\}.$$

Thus the set of all recursive ordinals forms an initial segment of Ord. The initial segment is countable since there are only countably many recursive relations. □

There must exist a least non-recursive ordinal (which is countable). This ordinal is called ω_1^{CK}. The CK stands for *Church-Kleene*.

Ordinal Notations

The definition of a recursive ordinal is rather from *outside*. As we will see later, deciding whether a recursive relation defines a well-ordering is quite difficult.

To get a better handle on recursive ordinals, we will construct them from *inside*. The idea is that if we have constructed ξ, then we also know how to construct $\xi + 1$. If we have a sequence of ordinals (ξ_n) previously constructed, we can also construct their limit, provided the sequence itself is constructive.

To make this precise, we introduce *ordinal notations*. A notation system for ordinals assigns ordinals to natural numbers in a way that reflects how each ordinal is built up from its predecessors. Our exposition in this part follows Rogers [1987].

Definition 19.2 (Kleene)**:** A **system of notation** S is a mapping ν_S from a set $D_S \subseteq \mathbb{N}$ onto an initial segment of Ord such that

(a) there exists a partial recursive function k_S such that

$$\begin{aligned} \nu_S(x) = 0 \quad &\Rightarrow \quad k_S(x) = 0 \\ \nu_S(x) \text{ successor} \quad &\Rightarrow \quad k_S(x) = 1 \\ \nu_S(x) \text{ limit} \quad &\Rightarrow \quad k_S(x) = 2; \end{aligned}$$

(b) there exists a partial recursive function p_S such that

$$\nu_S(x) \text{ successor} \quad \Rightarrow \quad [p_s(x)\downarrow \ \& \ \nu_S(x) = \nu_S(p_S(x)) + 1];$$

(c) there exists a partial recursive function q_S such that

$$\begin{aligned} \nu_S(x) \text{ limit} \quad \Rightarrow \quad [q_S(x)\downarrow \ \& \ \varphi_{q_S(x)} \text{ total } \ \& \ &(\nu_S(\varphi_{q_S(x)}(n)))_{n\in\mathbb{N}} \\ &\text{is an increasing sequence with limit } \nu_S(x)]. \end{aligned}$$

Given an ordinal notation $x \in D_S$ we can hence decide whether x codes a successor or a limit ordinal (or 0), and we can determine a notation for the predecessor of x (if x is a successor), or an index for a sequence of notations of ordinals converging to the ordinal denoted by x.

Note that the conditions ensure that the ordinals with a notation in S actually form an initial segment of Ord. This follows by induction. Note further that we do not require ν_S to be one-one. An ordinal may receive multiple notations.

Definition 19.3: An ordinal ξ is **constructive** if there exists a system of notation that assigns at least one notation to ξ.

Of course there are many different systems of notation. We would like to have one that encompasses *all constructive ordinals*, that is, a universal system.

Definition 19.4: A system of notation S is **universal** if for any system S' there is a partial recursive function φ such that $\varphi(D_{S'}) \subseteq D_S$ and

$$x \in D_{S'} \quad \Rightarrow \quad \nu_{S'}(x) \leq \nu_S(\varphi(x))$$

Since systems are closed downwards, this means that S assigns a notation to $\nu_{S'}(x)$, too.

It is a remarkable result due to Kleene that universal systems exist.

The system S_1

We define the system S_1 recursively.

- 0 receives notation 1.
- If all ordinals $< \xi$ have received their notations then
  (a) if $\xi = \eta + 1$, ξ receives the notations $\{2^x : \ x \text{ is a notation for } \eta\}$,
  (b) if ξ is limit, ξ receives the notation $3 \cdot 5^y$ for each y such that for all n, $\varphi_y(n)$ is a notation and the ordinals denoted by the $\varphi_y(n)$ form a sequence with limit ξ.

The functions $k_{S_1}, p_{S_1}, q_{S_1}$ are easily defined as

$$k_{S_1}(x) = \begin{cases} 0 & x = 1 \\ 1 & x = 2^y \\ 2 & x = 3 \cdot 5^y \\ \uparrow & \text{otherwise} \end{cases}$$

and

$$p_{S_1}(2^x) = x \qquad q_{S_1}(3 \cdot 5^y) = y,$$

where $p_{S_1}(z)$ and $q_{S_1}(z)$ are undefined in all other cases.

One can show that S_1 is universal (see Rogers [1987]). We will impose additionally an ordering on the ordinal notations of S_1. This will be useful later. The result is the system $\mathcal{O}$.

The system $\mathcal{O}$

We define simultaneously a system of notations and an ordering $<_{\mathcal{O}}$ on notations.

- 0 receives notation 1.
- Suppose all ordinals $< \xi$ have received their notations, and assume that $<_{\mathcal{O}}$ has been defined on these notations.
  - (a) if $\xi = \eta + 1$, ξ receives the notations $\{2^x : \; x \text{ is a notation for } \eta\}$, and set $z <_{\mathcal{O}} 2^x$ for $z = x$ or $z <_{\mathcal{O}} x$.
  - (b) if ξ is limit, ξ receives the notation $3 \cdot 5^y$ for each y such that for all n, $\varphi_y(n)$ is a notation and the ordinals denoted by the $\varphi_y(n)$ form a sequence with limit ξ, and for all $i < j$, $\varphi_y(i) <_{\mathcal{O}} \varphi_y(j)$. Furthermore, for each such y, set $z <_{\mathcal{O}} 3 \cdot 5^y$ for any z with $z < \varphi_y(n)$ for some n.

The function $k_{\mathcal{O}}, p_{\mathcal{O}}, q_{\mathcal{O}}$ are identical with $k_{S_1}, p_{S_1}, q_{S_1}$.

We will denote $D_{\mathcal{O}}$ by $\mathcal{O}$, too. Instead of $\nu_{\mathcal{O}}(x)$ we write $|x|_{\mathcal{O}}$. $<_{\mathcal{O}}$ is a partial ordering on $\mathcal{O}$. Its transitivity follows from the definition of $<_{\mathcal{O}}$. An effective limit of ordinals with notations in $\mathcal{O}$ can (and does) have many possible indices. This makes the ordering non-linear. This is reflected in the following diagram of the initial structure of $<_{\mathcal{O}}$.

$$1 <_{\mathcal{O}} 2 <_{O} 2^2 <_{\mathcal{O}} \dots \quad \begin{cases} 3 \cdot 5^{y_1} <_{O} 2^{3 \cdot 5^{y_1}} <_{\mathcal{O}} 2^{2^{3 \cdot 5^{y_1}}} <_{\mathcal{O}} \dots \\ 3 \cdot 5^{y_2} <_{O} 2^{3 \cdot 5^{y_2}} <_{\mathcal{O}} 2^{2^{3 \cdot 5^{y_2}}} <_{\mathcal{O}} \dots \\ \vdots \end{cases}$$

$3 \cdot 5^{y_1}$ and $3 \cdot 5^{y_2}$ are two of the infinitely many notations for ω. Any index y of the recursive function that maps x to x-many iterations of $n \mapsto 2^n$ constitutes a notation $3 \cdot 5^y$ of ω. For this reason $|x|_{\mathcal{O}} < |y|_{\mathcal{O}}$ does not necessarily imply $x <_{\mathcal{O}} y$.

However, it is easy to see that $x <_{\mathcal{O}} y$ implies $|x|_{\mathcal{O}} < |y|_{\mathcal{O}}$. Since an infinite descending sequence in $<_{\mathcal{O}}$ would induce an infinite descending sequence in Ord, we have

Proposition 19.5: *The relation $<_{\mathcal{O}}$ is well-founded.*

This allows us to prove facts about $<_{\mathcal{O}}$ via induction along a well-founded relation.

Proposition 19.6: *Let $y \in \mathcal{O}$. Then*

(a) *the restriction of $<_{\mathcal{O}}$ to $\{x : x <_{\mathcal{O}} y\}$ is linear;*

(b) *the restriction of $\mathcal{O}$ to $<_{\mathcal{O}}$ to $\{x : x <_{\mathcal{O}} y\}$ is one-one.*

Proof. (a) We proceed by induction along $<_{\mathcal{O}}$. Suppose $x_1, x_2 <_{\mathcal{O}} y$. If $y = 2^z$, then $z <_{\mathcal{O}} y$ and by definition of $<_{\mathcal{O}}$ if $v <_{\mathcal{O}} 2^z$ then $v \leq_{\mathcal{O}} z$. Hence $x_1, x_2 \leq_{\mathcal{O}} z$ and we can apply the induction hypothesis. If $y = 3 \cdot 5^z$, then by definition of $<_{\mathcal{O}}$ there exist n_1, n_2 such that $x_1 <_{\mathcal{O}} \varphi_z(n_1)$ and $x_2 <_{\mathcal{O}} \varphi_z(n_2)$. Wlog $n_1 < n_2$. Then, by the condition for y to be notation we have $\varphi_z(n_1) <_{\mathcal{O}} \varphi_z(n_2)$, and hence $x_1, x_2 <_{\mathcal{O}} \varphi_z(n_2)$, and we can apply the induction hypothesis.

(b) This is an easy induction – each step in the definition of $\mathcal{O}, <_{\mathcal{O}}$ defines a notation for an ordinal larger than all ordinals having received a notation before. □

We can also show

Proposition 19.7: *The restriction of $<_{\mathcal{O}}$ to $\{y : y <_{\mathcal{O}} x\}$ is uniformly r.e., i.e. there exists a recursive function f such that for all x, if $x \in \mathcal{O}$ then $W_{f(x)} = \{y : y <_{\mathcal{O}} x\}$.*

We defer the proof for a while to discuss the use of the Recursion Theorem.

Effective Transfinite Recursion

The Recursion Theorem plays an essential role in computations with ordinal notations. To see why, consider the following problem. We would like to introduce a (partial) recursive function $+_{\mathcal{O}}$ that mirrors the addition of ordinals on the notational side. More specifically we would like a function $+_{\mathcal{O}}$ such that for all $x, y \in \mathcal{O}$,

(a) $x +_{\mathcal{O}} y \in \mathcal{O}$,

(b) $|x +_{\mathcal{O}} y|_{\mathcal{O}} = |x|_{\mathcal{O}} + |y|_{\mathcal{O}}$, and

(c) $y \neq 1$ implies $x <_{\mathcal{O}} x +_{\mathcal{O}} y$.

The obvious way to define such a function $+_{\mathcal{O}}$ is by recursion. Suppose we fix x and try to define $x +_{\mathcal{O}} y$. It is clear that $x +_{\mathcal{O}} 1 = x$. If $y = 2^z$, we define $x +_{\mathcal{O}} y = 2^{x +_{\mathcal{O}} z}$. Now suppose $y = 3 \cdot 5^z$. To match the definition of ordinal addition, we have to put $x +_{\mathcal{O}} y$ to be the "*limit*" of the notations $x +_{\mathcal{O}} \varphi_z(n)$. In other words, we have to set $x +_{\mathcal{O}} y = 3 \cdot 5^e$, where e is an index of the computable mapping $n \mapsto x +_{\mathcal{O}} \varphi_z(n)$. Hence to determine e, we need an index for the very function we are trying to build!

This is where the Recursion Theorem is indispensable. It ensures us that we know such an index "beforehand". The following theorem captures this possibility of **effective transfinite recursion**.

Theorem 19.8 (Effective transfinite recursion)**:** *Let R be a well-founded relation defined on a subset of $\mathbb{N}$. Suppose $F : \mathbb{N} \to \mathbb{N}$ is a (total) recursive function. Suppose further that for all $e \in \mathbb{N}$ and $x \in \operatorname{dom}(R)$,*

$$\forall y \, R \, x \; \varphi_e(y) \downarrow \quad \Rightarrow \quad \varphi_{F(e)}(x) \downarrow .$$

Then there exists a $c \in \mathbb{N}$ such that

$$\forall x \in \operatorname{dom}(R) \; \varphi_c(x) \downarrow \quad \textit{and} \quad \varphi_c = \varphi_{F(c)}.$$

The idea is that if we have efficiently constructed a function (i.e. an index e) below x, and given this index we effectively compute an extension to x (via $\varphi_{F(e)}$), then we actually succeeded in effectively constructing a function defined on all of $\operatorname{dom}(R)$. This is precisely the situation we are facing in the definition of $x +_{\mathcal{O}} y$.

Proof. By the Recursion Theorem there exists a c such that $\varphi_c = \varphi_{F(c)}$. If $\varphi_c(x)$ were undefined for some $x \in \operatorname{dom}(R)$, then, since R is well-founded, there must exist an R-minimal such x. This implies that $\varphi_c(y)$ is defined for all $y \, R \, x$, and hence by assumption, $\varphi_{F(c)}(x) \downarrow$. Since $\varphi_{F(c)} = \varphi_c$, this is a contradiction. □

Armed with effective transfinite recursion, we can give a formal construction of the function $+_{\mathcal{O}}$. Using the S-m-n Theorem, we can fix an injective function $h : \mathbb{N}^3 \to \mathbb{N}$ such that

$$\varphi_{h(e,x,d)}(y) = \varphi_e(x, \varphi_d(y)) \quad \text{for all } e, x, d, y \in \mathbb{N}.$$

Let F be a recursive function such that for all e,

$$\varphi_{F(e)}(x,y) \simeq \begin{cases} x & \text{if } y = 1, \\ 2^{\varphi_e(x,z)} & \text{if } y = 2^z, \\ 3 \cdot 5^{h(e,x,z)} & \text{if } y = 3 \cdot 5^z, \\ 7 & \text{otherwise.} \end{cases}$$

Let c be a fixed point of F and put $x +_{\mathcal{O}} y = \varphi_c(x,y)$. It is straightforward to verify that this definition of $+_{\mathcal{O}}$ has the desired properties. Note that for the definition to work, it is essential that we can distinguish effectively between codes for 0, successor, and limit ordinals (the recursiveness of $k_{\mathcal{O}}$).

Maybe surprisingly, $+_{\mathcal{O}}$ turns out to be total. Suppose $\langle x,y\rangle$ is minimal (with respect to the usual ordering of $\mathbb{N}$) so that $x +_{\mathcal{O}} y$ is undefined. Since h is total, the only way for $x +_{\mathcal{O}} y$ to be undefined is for y to be of the form 2^z. But this means $\varphi_c(x,z)\uparrow$, and hence $x +_{\mathcal{O}} z$ is undefined for some lesser pair $\langle x,z\rangle$, under the standard pair coding function.

We can use $+_{\mathcal{O}}$ to prove the universality of $\mathcal{O}$.

Proposition 19.9: *$\mathcal{O}$ is a universal system of notation.*

Proof. Let S be a system of notation. Again, we use effective transfinite recursion. Let h be a recursive function such that

$$\varphi_{h(e)}(0) = \varphi_e(\varphi_{q_S(x)}(0)),$$
$$\varphi_{h(e)}(x+1) = \varphi_{h(e)}(x) +_{\mathcal{O}} +\varphi_e(\varphi_{q_S(x)}(x+1)).$$

Recall that $+_{\mathcal{O}}$ is total. Define a recursive function F such that

$$\varphi_{F(e)}(x) \simeq \begin{cases} 1 & \text{if } k_S(x) = 0, \\ 2^{\varphi_e(p_S(x))} & \text{if } k_S(x) = 1, \\ 3 \cdot 5^{h(e)} & \text{if } y = 3 \cdot 5^z, \\ \uparrow & \text{otherwise.} \end{cases}$$

The Recursion Theorem yields a fixed point $\varphi_{F(c)} = \varphi_c$. Then φ_c is the desired reduction. Suppose not. Then there exists an least ξ such that $\nu_S(x) = \xi$ for some x, but $|\varphi_c(x)|_{\mathcal{O}} < \nu_S(x)$. If $\xi = \eta + 1$, then $\varphi_c(x) = 2^{\varphi_c(p_S(x))}$, and $|\varphi_c(p_S(x))|_{\mathcal{O}} < |\varphi_c(x)|_{\mathcal{O}} \leq \eta = \nu_S(p_S(x))$, contradicting the fact that ξ was chosen minimal. The case that ξ is limit is similar. $\square$

Finally, we give a proof of Proposition 19.7.

Proof of Proposition 19.7. We follow Sacks [1990]. We claim that there exists a recursive function f such that

$$\begin{aligned} W_{f(1)} &= \emptyset, \\ W_{f(2^x)} &= W_{f(x)} \cup \{x\}, \\ W_{f(3\cdot 5^x)} &= \bigcup \{W_{f(\varphi_x(n))} : \varphi_x(n)\downarrow\}. \end{aligned}$$

It follows by induction along $<_{\mathcal{O}}$ that such f satisfies the assertion of the theorem. Choose an index e_0 and recursive functions h_0, h_1 such that

$$\begin{aligned} W_{e_0} &= \emptyset, \\ W_{h_0(e,x)} &= W_{\varphi_e(x)} \cup \{x\}, \\ W_{h_1(e,x)} &= \bigcup \{W_{\varphi_e(\varphi_x(n))} : n \in \mathbb{N}\}. \end{aligned}$$

Here $W_{\varphi_e(x)} = \emptyset$ if $\varphi_e(x)\uparrow$; similarly for $W_{\varphi_e(\varphi_x(n))}$. There exists a recursive function F such that

$$\varphi_{F(e)}(x) \simeq \begin{cases} e_0 & \text{if } x = 1, \\ h_0(e,z) & \text{if } x = 2^z, \\ h_1(e,z) & \text{if } x = 3\cdot 5^z, \\ 0 & \text{otherwise.} \end{cases}$$

Let c be a fixed point of F and define $f(x) = \varphi_c(x)$. Note that f is total because h_0, h_1 are. □

The last two result puts us in a position to prove

Theorem 19.10: *A constructive ordinal is recursive.*

Proof. Let α be a constructive ordinal. Since $\mathcal{O}$ is universal, it assigns α a notation, say x. By Proposition 19.7 the set set $\mathcal{O}_x = \{y : y <_{\mathcal{O}} x\}$ is r.e. A slight variation of the proof of Proposition 19.7 yields that the set $\mathcal{O}_x^< = \{\langle y, z\rangle : y <_{\mathcal{O}} z <_{\mathcal{O}} x\}$ is r.e., too. We may assume that $\mathcal{O}_x$ is infinite. By Proposition 19.6 $\mathcal{O}_x^<$ is well-founded and linear, hence a well-ordering. An easy induction shows that the order type of $\mathcal{O}_x^<$ is $|x|_{\mathcal{O}} = \alpha$. Let f be recursive, one-one such that $\operatorname{ran}(f) = \mathcal{O}_x$. Put

$$m R n \quad \Leftrightarrow \quad \langle f(m), f(n) \rangle \in \mathcal{O}_x^<.$$

Since $\mathcal{O}_x = \operatorname{ran}(f)$ is the domain of $\mathcal{O}_x^<$ and $\mathcal{O}_x^<$ is r.e., it follows that R is a recursive well-ordering of order type α. $\square$

We will see in the next lecture that the converse is also true. This will be a consequence of the **completeness properties of** $\mathcal{O}$, which we study next.