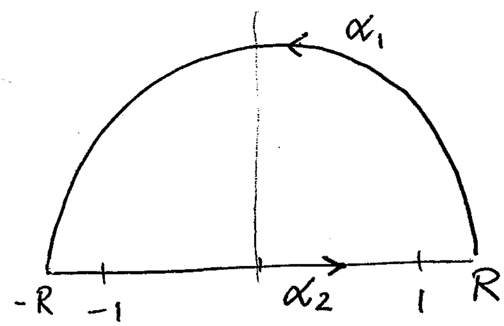
Problem 2: The integration path is



Partial fraction decomposition of f:

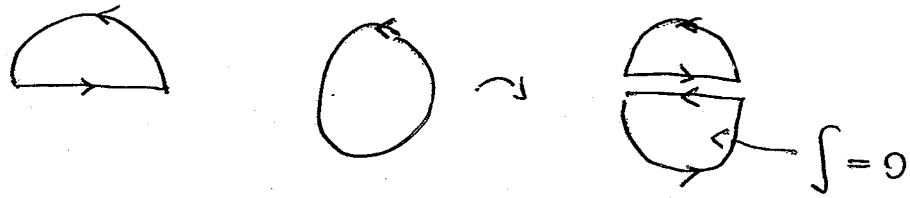
$$f(z) = \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)$$

Hence $$\int_{\alpha_1 \oplus \alpha_2} f(\xi)\, d\xi = \frac{1}{2i}\left(\int_{\alpha_1 \oplus \alpha_2} \frac{1}{\xi - i}\, d\xi - \int_{\alpha_1 \oplus \alpha_2} \frac{1}{\xi + i}\, d\xi\right)$$

The mapping $z \longmapsto \frac{1}{z+i}$ is analytic in a star-shaped domain containing the full upper semi-disk of radius R.
Hence, by the Cauchy integral thm., $\int_{\alpha_1 \oplus \alpha_2} \frac{1}{\xi + i}\, d\xi = 0$.

We argue that $$\int_{\alpha_1 \oplus \alpha_2} \frac{1}{\xi - i}\, d\xi = \oint_{|\xi| = R} \frac{1}{\xi - i}\, d\xi .$$

This can be justified by an argument similar to the one in class:



Thus, $\displaystyle\int_{\alpha_1 \oplus \alpha_2} f(\xi)\, d\xi = \frac{1}{2i} \int_{|\xi|=R} \frac{1}{\xi - i} = \frac{2\pi i}{2i} = \pi$.

To prove the second assertion, we use the standard estimate.

$$\left| \int_{\alpha_1^{(R)}} f(\xi)\, d\xi \right| \leq \ell(\alpha_1^{(R)}) \cdot \max \{ |f(z)| : z \text{ on } \alpha_1^{(R)} \}$$

$$= \pi R \cdot \frac{1}{\min\{ |(Re^{it})^2 + 1| : 0 \leq t \leq \pi \}}$$

$$\leq \pi R \cdot \frac{1}{R^2 - 1} \xrightarrow{R \to \infty} 0$$

Combining the two results, we obtain

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{1}{1+t^2}\, dt = \lim_{R\to\infty} \Bigg[\underbrace{\int_{\alpha_2^{(R)}} \frac{1}{1+\xi^2}\, d\xi}_{= \pi} + \underbrace{\int_{\alpha_1^{(R)}} \frac{1}{1+\xi^2}\, d\xi}_{\xrightarrow{R\to\infty} 0} \Bigg]$$

$$= \lim_{R\to\infty} \pi = \pi$$

$\square$

Problem 3: (a) partial fraction decomposition:

$$\frac{z^2-1}{z^2+1} = 1 + \frac{i}{z-i} - \frac{i}{z+i}$$

Hence

$$\oint_{|\xi|=2} \frac{\xi^2-1}{\xi^2+1}\, d\xi = \underbrace{\oint_{|\xi|=2} 1}_{=0} + i \underbrace{\int_{|\xi|=2} \frac{1}{\xi - i}}_{=2\pi i} - i \underbrace{\int_{|\xi|=2} \frac{1}{\xi + i}}_{=2\pi i} = 0$$

(b)

$$\oint_{|\xi|=1} \frac{\sin(\exp(\xi))}{\xi}\, d\xi = \sin(\exp(0)) \cdot 2\pi i = 2\pi i \sin(1)$$

(c)

$$\oint_{|\xi-1|=1} \left(\frac{\xi}{\xi-1}\right)^n d\xi = f^{(n-1)}(1) \frac{2\pi i}{(n-1)!} \quad \text{with } f(z) = z^n$$

$$= n! \cdot 1 \cdot \frac{2\pi i}{(n-1)!} = 2\pi i\, n$$

$\square$

Problem 4: Proof by induction.

$n=1$: The verification of the identity

$$(*) \quad \frac{F_1(z)-F_1(a)}{z-a} - \frac{1}{2\pi i}\int_\alpha \frac{\varphi(\xi)}{(\xi-a)^2}\,d\xi = \frac{z-a}{2\pi i}\int_\alpha \frac{\varphi(\xi)}{(\xi-a)^2(\xi-z)}$$

is straightforward. (Note that the identity

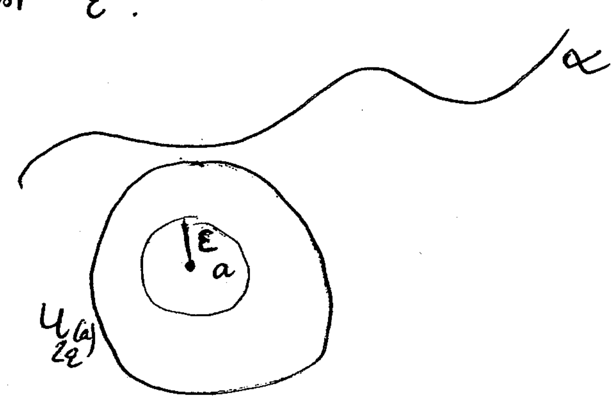
$$(**) \quad \frac{1}{(\xi-z)^m} = \frac{1}{(\xi-z)^{m-1}(\xi-a)} + \frac{z-a}{(\xi-z)^m(\xi-a)}$$

also holds for $m=2$.)

It remains to show that the RHS of $(*)$ goes to 0 for $z \to a$.
For this, it is sufficient to show that the integral is bounded by a constant as $z \to a$.

Note that φ is continuous, so $|\varphi(\xi)|$ attains a maximum M on α.
Furthermore, $\alpha([0,1])$ is closed, so if $a \notin \mathrm{Im}(\alpha)$, then
$= \mathrm{Image}(\alpha)$
there exists an $\varepsilon > 0$ s.t. $\mathcal{U}_{2\varepsilon}(a) \cap \mathrm{Im}(\alpha) = \emptyset$.

Hence, f. all $z \in \mathcal{U}_\varepsilon(a)$, the distance of z to the curve is at least ε.



Hence we can infer

$$\left| \frac{\varphi(\xi)}{(\xi-a)^2(\xi-z)} \right| \leq \frac{M}{\varepsilon^3} \qquad \text{f. all } z \in U_\varepsilon(a), \ \xi \text{ on } \alpha.$$

This yields that

$$\frac{z-a}{2\pi i} \int_\alpha \frac{\varphi(\xi)}{(\xi-a)^2(\xi-z)}\, d\xi \xrightarrow{z \to a} 0$$

$n \rightsquigarrow n+1$:

$$\frac{F_{n+1}(z) - F_{n+1}(a)}{z-a} = \frac{1}{z-a}\left[\frac{1}{2\pi i}\int_\alpha \frac{\varphi(\xi)}{(\xi-z)^{n+1}}\, d\xi - \frac{1}{2\pi i}\int_\alpha \frac{\varphi(\xi)}{(\xi-a)^{n+1}}\, d\xi\right]$$

$$\overset{(**)}{=} \frac{1}{z-a}\left[\frac{1}{2\pi i}\int_\alpha \frac{\varphi(\xi)}{(\xi-z)^n(\xi-a)}\, d\xi + \frac{z-a}{2\pi i}\int_\alpha \frac{\varphi(\xi)}{(\xi-z)^{n+1}(\xi-a)} - \frac{1}{2\pi i}\int_\alpha \frac{\varphi(\xi)}{(\xi-a)^{n+1}}\, d\xi\right]$$

$$= \frac{1}{z-a}\left[\frac{1}{2\pi i}\int_\alpha \frac{\varphi(\xi)/(\xi-a)}{(\xi-z)^n}\, d\xi - \frac{1}{2\pi i}\int_\alpha \frac{\varphi(\xi)/(\xi-a)}{(\xi-a)^n}\, d\xi\right] + \frac{1}{2\pi i}\int_\alpha \frac{\varphi(\xi)/\xi-a}{(\xi-z)^{n+1}}\, d\xi$$

Let $\tilde{\varphi}(\xi) = \dfrac{\varphi(\xi)}{\xi-a}$. This mapping is continuous on α, since a is not on α. Write $\tilde{F}_n(z) = \frac{1}{2\pi i}\int_\alpha \frac{\tilde{\varphi}(\xi)}{(\xi-z)^n} d\xi$. Thus we have

$$\frac{F_{n+1}(z) - F_{n+1}(a)}{z-a} = \frac{\tilde{F}_n(z) - \tilde{F}_n(a)}{z-a} + \tilde{F}_{n+1}(z) \qquad (+)$$

Using the induction hypothesis on $\tilde{F}_n$, we see that

$$\frac{\tilde{F}_n(z) - \tilde{F}_n(a)}{z-a} \xrightarrow{z \to a} \tilde{F}_n'(a) = n \cdot \tilde{F}_{n+1}(a)$$

$$= n \cdot \int_\alpha \frac{\varphi(\xi)}{(\xi-a)^{n+2}} d\xi = n\, F_{n+2}(a)$$

It remains to show that $\tilde{F}_{n+1}(z)$ is continuous in a, because this implies

$$\tilde{F}_{n+1}(z) \xrightarrow{z \to a} \tilde{F}_{n+1}(a) = \int_\alpha \frac{\varphi(\xi)}{(\xi-a)^{n+2}} d\xi = F_{n+2}(a),$$

and so we have for $(+)$

$$\frac{F_{n+1}(z) - F_{n+1}(a)}{z-a} \xrightarrow{z \to a} n \cdot F_{n+2}(a) + F_{n+2}(a) =$$

$$(n+1)\, F_{n+2}(a)$$

It is clear that it suffices to show that F_{n+1} is continuous in a, since p is an arbitrary contin. function on $\operatorname{Im}(\alpha)$.

Applying identity $(*)$ once more, we get

$$F_{n+1}(z) - F_{n+1}(a) = \frac{1}{2\pi i} \int_\alpha \frac{\varphi(\xi)}{(\xi-z)^{n+1}} d\xi - \frac{1}{2\pi i} \int_\alpha \frac{\varphi(\xi)}{(\xi-a)^{n+1}} d\xi$$

$$= \frac{1}{2\pi i} \int_\alpha \varphi(\xi) \left(\frac{1}{(\xi-z)^n (\xi-a)} + \frac{z-a}{(\xi-z)^{n+1}(\xi-a)} \right) d\xi - \frac{1}{2\pi i} \int_\alpha \frac{\varphi(\xi)}{(\xi-a)^{n+1}} d\xi$$

$$= \frac{1}{2\pi i}\int_\alpha \frac{\tilde\varphi(\xi)}{(\xi-z)^n} - \frac{1}{2\pi i}\int_\alpha \frac{\tilde\varphi(\xi)}{(\xi-a)^n} + \frac{(z-a)}{2\pi i}\int_\alpha \frac{\varphi(\xi)}{(\xi-z)^{n+1}(\xi-a)}d\xi$$

$$= \tilde F_n(z) - \tilde F_n(a) + \frac{z-a}{2\pi i}\int_\alpha \frac{\varphi(\xi)}{(\xi-z)^{n+1}(\xi-a)}d\xi$$

The inductive hypoth. yields that $\tilde F_n(z) \xrightarrow{z\to a} \tilde F_n(a)$.

Furthermore, a similar argument as in the case $n=1$ bounds the integral $\left|\int_\alpha \frac{\varphi(\xi)}{(\xi-z)^{n+1}(\xi-a)}d\xi\right|$ by $\frac{M}{\varepsilon^{n+2}}\cdot \ell(\alpha)$, so the second term goes to 0 as $z\to a$.

This proves the continuity.

$\square$

Now we use the lemma to infer the Cauchy integral formula:

The case $n=0$ has been proved in class:

$\underline{n \rightsquigarrow n+1}$:
$$\frac{f^{(n)}(z)-f^{(n)}(a)}{z-a} = \frac{1}{z-a}\left[\frac{n!}{2\pi i}\int_\alpha \frac{f(\xi)}{(\xi-z)^{n+1}}d\xi - \frac{n!}{2\pi i}\int_\alpha \frac{f(\xi)}{(\xi-a)^{n+1}}\right]$$

$$= \frac{n!}{z-a}\left[F_{n+1}(z) - F_{n+1}(a)\right] \quad (\text{where } \varphi := f)$$

$$\xrightarrow[\text{lemma}]{z\to a} n!\, F_{n+1}'(a) = n!\,(n+1)\,F_{n+2}(a) = \frac{(n+1)!}{2\pi i}\int_\alpha \frac{f(\xi)}{(\xi-a)^{n+2}}d\xi$$

$\square$

Problem 5: (a) Assume $|f(z)| \geq e^{|z|}$ f. all $z \in \mathbb{C}$.

Then $\left|\frac{1}{f(z)}\right| \leq e^{-|z|} \leq 1$, and since f does not have any zeros, $\frac{1}{f}$ is an entire function.
Since it is bounded, $\frac{1}{f}$ must be constant, by Liouville's thm.
Hence f is constant, too, contradiction.

(b) If A is empty, we have $|f(z)| \geq 1$ f. all $z \in \mathbb{C}$.
Apply the same reasoning as in (a) to conclude that f must be constant.

(c) Assume $A \subseteq \overline{U_r(0)}$. If f does not have a zero, $\frac{1}{f}$ is entire. We have that

$$z \notin \overline{U_r(0)} \implies |f(z)| \geq 1, \text{ hence}$$

$$z \notin \overline{U_r(0)} \implies \frac{1}{|f(z)|} \leq 1$$

On the other hand, $\frac{1}{f}$ is continuous, so the image of the compact set $\overline{U_r(0)}$ under $\frac{1}{f}$ is bounded, so $\frac{1}{f}$ is bounded on all of $\mathbb{C}$. Again, Liouville's thm implies that f is constant.

(d) Consider for example $f(z) = \exp(z)$.

$\square$