Effective Baire Category Concepts

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Abstract

Mehlhorn [Me73] introduced an effective Baire category concept designed for measuring the size of classes of computable sets. This concept is based on effective extension functions. By considering partial extension functions, we introduce a stronger concept. Similar resource-bounded concepts have been previously introduced by Ambos-Spies et al. [AFH88] and Ambos-Spies [Am96]. By defining a new variant of the Banach-Mazur game, we give a game theoretical characterization of our category concept.

1 Introduction

The classical Baire category concept gives a classification of the subclasses of the Cantor space 2^ω (and other topological spaces) according to their size. The small classes in this sense are the *meager* classes. All finite classes are meager, any subclass of a meager class is meager, and the countable union of meager classes is meager again. The consistency of the system follows from Baire's theorem which implies that the complement of a meager class is not meager.

Since the class **REC** of the recursive (computable) sets is countable, hence meager, Baire category is too coarse for classifying classes of computable sets. Mehlhorn [Me73] has shown, however, that the category concept can be effectivized to obtain a classification schema for classes of recursive sets, which has similar properties if we replace arbitrary countable unions by uniformly computable unions (in an appropriate sense). Mehlhorn's concept is based on the characterization of Baire category in terms of total extension functions and it admits only recursive extension functions.

The extension function characterization of Baire category shows the close relations between this concept and the finite extension method, a fundamental diagonalization technique in computability theory (see e.g. Odifreddi [Od89], Chapter V.3). Correspondingly, Mehlhorn's concept can be viewed as a formalization of the effective finite extension method.

It turned out, however, that the effectivization of certain finite extension arguments requires slightly more complex diagonalization arguments, namely slow diagonalizations which are also called effective wait and see arguments. The typical feature of these arguments is that, in contrast to a standard finite extension argument, the requirements are not met in order. In computational complexity theory these more flexible diagonalizations become necessary for studying almost everywhere complexity and the related bi-immunity concepts (see [Am96] for details).

An effective Baire category concept capturing slow diagonalizations can be obtained by admitting partial extension functions. For resource-bounded diagonalizations, such concepts have been introduced by Ambos-Spies, Fleischhack and Huwig [AFH88] and Ambos-Spies [Am96]. In particular, the latter contains a detailed discussion of the role of the above diagonalization concepts in complexity theory and their relation to Baire category. There, the interest focused on complexity classes, however, whence the general recursive case was not discussed.

Here we introduce the corresponding effective category concept, called extended effective (e.e.) category. We prove some basic facts for this new concept. In particular we show that it properly extends Mehlhorn's concept. Moreover, by introducing a new variant of the Banach-Mazur game, we give a game theoretical characterization of e.e. category.

The outline of the paper is as follows. In Section 2 we review Mehlhorn's effective category concept. There we give a new characterization of the effectively meager classes in terms of the arithmetical hierarchy of classes which easily implies many of Mehlhorn's results on effective meagerness.

In Section 3 we introduce our new e.e. category concept. By showing that **REC** is not e.e. meager, we show that this concept induces a consistent category notion for classes of computable sets. Moreover, we show that e.e. category is a proper refinement of effective category.

Section 4 is devoted to the game theoretical characterization of Baire category in terms of Banach-Mazur games. We introduce a new variant of this game, in which the second player is allowed to cut back certain moves of the first player, and we characterize e.e. meagerness in terms of determinateness of the effective variant of this extended Banach-Mazur game.

Finally, in Section 5 we shortly comment on resource-bounded category.

We conclude this section by introducing some notation.

Let ω be the set of natural numbers, let $2^{<\omega}$ be the set of (finite) binary strings, and let 2^ω be the set of infinite binary sequences (the Cantor space). Lower case letters $\ldots, v, w, x, y, z$ from the end of the alphabet will denote strings while the other letters denote numbers with the exception of f, g, h and l which are reserved for functions. For a function $f : \omega \times 2^{<\omega} \to 2^{<\omega}$, f_n will denote the n-th branch of f: $f_n(x) = f(n, x)$. A subset of ω is called a *set* while sets of sets are called *classes*. Capital letters denote sets, boldface capital letters denote classes.

For a string x, $x(m)$ denotes the $(m+1)$th bit in x, i.e., $x = x(0)\dots x(n-1)$, where $n = |x|$ is the length of x. λ is the empty string and $x\hat{\ }y$ denotes the concatenation of x and y. The initial segment of x of length n is denoted by $x\restriction n = x(0)\dots x(n-1)$. If y extends x then we write $x \sqsubseteq y$ and $x \sqsubset y$ expresses that the extension is proper.

The notation for strings is canonically extended to infinite sequences. Moreover, we identify a set A with its characteristic sequence, i.e., $n \in A$ iff $A(n) = 1$ and $n \notin A$ iff $A(n) = 0$. Hence a class is a subset of the Cantor space. The class of all sets extending a string x is denoted by $\mathbf{B}_x = \{A : x \sqsubset A\}$.

We assume familiarity with the basic concepts of recursion theory. From complexity theory we will only consider the complexity classes $\mathbf{P}$ and $\mathbf{NP}$ of deterministic respectively nondeterministic polynomial-time. Here $A \in \mathbf{P}$ ($A \in \mathbf{NP}$) if there is a (non)deterministic Turing machine M computing A, where the running time of M on input n is bounded by $f(|n|)$, where $|n|$ is the length of the binary representation of n and f is some polynomial.

2 Mehlhorn's Effective Category Concept

The classical Baire category concept for the Cantor space 2^ω can be defined in terms of extension functions. An *extension function* f is a function $f : 2^{<\omega} \to 2^{<\omega}$ such that $x \sqsubseteq f(x)$ for all strings x. A subclass $\mathbf{C}$ of 2^ω is *nowhere dense* via f if $\mathbf{B}_{f(x)} \cap \mathbf{C} = \emptyset$ for all strings x. Then a class $\mathbf{C}$ is *meager* if it is contained in the countable union of nowhere dense classes, and the complement of a meager class is *comeager*. Intuitively, meager classes are small and, correspondingly, comeager classes are large. This intuition is justified by the following facts.

(2.1) Every finite class is meager.

(2.2) Every subclass of a meager class is meager.

(2.3) The countable union of meager classes is meager.

(2.4) 2^ω is not meager.

By (2.1) and (2.3) every countable class is meager whence Baire category is too coarse for classifying the size of classes of recursive sets. Mehlhorn [Me73], however, introduced an effectivization of the category concept which provides such a classification. By considering only computable extension functions, Mehlhorn defines effectively nowhere dense classes and he lets the effectively meager classes be the countable unions of *uniformly* effectively nowhere dense classes.

Definition 2.1 *(Mehlhorn [Me73])*

(a) An effective extension function f *is a total recursive function* $f : 2^{<\omega} \to 2^{<\omega}$ *such that* $x \sqsubseteq f(x)$ *for all strings* x*. A class* $\mathbf{C}$ *is* effectively nowhere dense via f *if* f *is an effective extension function and* $\mathbf{B}_{f(x)} \cap \mathbf{C} = \emptyset$ *for*

all strings x, and $\mathbf{C}$ *is* effectively nowhere dense *if* $\mathbf{C}$ *is effectively nowhere dense via some function* f.

(b) *An* effective system of extension functions *is a recursive function* $f : \omega \times 2^{<\omega} \to 2^{<\omega}$ *such that* f_e *is an effective extension function for all* $e \geq 0$. *A class* $\mathbf{C}$ *is* effectively meager via f *if* f *is an effective system of extension functions and* $\mathbf{C}$ *is the union of classes* $\mathbf{C}_e$, $e \geq 0$, *where* $\mathbf{C}_e$ *is effectively nowhere dense via* f_e. $\mathbf{C}$ *is* effectively meager *if* $\mathbf{C}$ *is effectively meager via some function* f, *and* $\mathbf{C}$ *is* effectively comeager *if the complement* $\overline{\mathbf{C}}$ *of* $\mathbf{C}$ *is effectively meager.*

Mehlhorn's category concept refines the classical concept in the sense that every effectively nowhere dense (meager, comeager) class is nowhere dense (meager, comeager), but the converse fails. In particular the countable class **REC** of the recursive sets is not effectively meager, whence the effective category concept relativized to **REC** yields a classification schema for classes of recursive sets.

Mehlhorn has shown that every recursively presentable class of recursive sets is effectively meager but he also gave examples of effectively meager classes containing nonrecursive sets. In order to describe the strength of Mehlhorn's concept we characterize it in terms of Σ_2^0-classes.

Definition 2.2 *A class* $\mathbf{C}$ *is a* Σ_2^0-class *if there is a total recursive* 0-1-*valued functional* $\Phi :\ 2^\omega \times \omega \times \omega \to \{0,1\}$ *such that*

$$A \in \mathbf{C} \ \Leftrightarrow\ (\exists m)\,(\forall n)\,(\Phi^A(m,n) = 1).$$

For more details on the arithmetical hierarchy of classes we refer to Rogers [Ro67], Chapter 15.

Theorem 2.3 *For any class* $\mathbf{C}$ *the following are equivalent.*

(i) $\mathbf{C}$ *is effectively meager.*

(ii) *There is a* Σ_2^0*-class* $\mathbf{D}$ *and a recursive set* A *such that* $\mathbf{C} \subseteq \mathbf{D}$, $\mathbf{D}$ *is c.f.v., and* $A \notin \mathbf{D}$.

(iii) *There is a* Σ_2^0*-class* $\mathbf{D}$ *and a set* A *such that* $\mathbf{C} \subseteq \mathbf{D}$ *and, for any finite variant* $\hat{A}$ *of* A, $\hat{A} \notin \mathbf{D}$.

For the proof of Theorem 2.3 we need the following observations.

Proposition 2.4 *Let* f *be an effective system of extension functions. There is a greatest class which is meager via* f, *namely*

$$\mathbf{M}(f) = \{A :\ (\exists e)\,(\forall n)\,(f_e(A\restriction n) \not\sqsubseteq A)\}$$

Proof. Straightforward. $\square$

Proposition 2.5 *For any effective system of extension functions f, $\mathbf{M}(f)$ is a Σ^0_2-class.*

Proof. Immediate by definition. $\square$

Proposition 2.6 *For any effective system of extension functions f, $\mathbf{REC} \not\subseteq \mathbf{M}(f)$.*

Proof. Given f, by an effective finite extension argument inductively define a recursive set $A \notin \mathbf{M}(f)$ as follows. Enumerate $A \upharpoonright l(s)$ in stages, where $l : \omega \to \omega$ is a strictly increasing function, and $l(s)$ and the finite extension $A \upharpoonright l(s)$ of $A \upharpoonright l(s-1)$ are defined at stage s: Given s and $A \upharpoonright l(s-1)$ (where $l(-1) = 0$), let $A \upharpoonright l(s) = f_s(A \upharpoonright l(s-1))$. Then A is recursive and the initial segment $A \upharpoonright l(s)$ ensures that A is not effectively nowhere dense via f_s. $\square$

Proposition 2.7 *For any effective system of extension functions f there is an effective system of extension functions $\hat{f}$ such that $\mathbf{M}(f) \subseteq \mathbf{M}(\hat{f})$ and $\mathbf{M}(\hat{f})$ is closed under finite variants.*

Proof. Given f the function $\hat{f}$ is obtained by considering all finite variants of the extension functions of the system f in the following sense. For strings x and y such that $|x| \geq |y|$ let $var(x, y)$ be the string obtained from x by replacing it's initial segment of length $|y|$ by y. Then, for any number e and strings y and z, the extension function $\hat{f}_{\langle e,y,z\rangle}$ extends a string x with $|x| \geq |y|$ in the same way as f_e extends the string $var(x, y)$ and it maps shorter strings to an extension determined by z. To be more precise, for x with $|x| \geq |y|$, $\hat{f}_{\langle e,y,z\rangle}(x) = x\hat{\ }v$, where v is the unique string satisfying $f_e(var(x, y)) = var(x, y)\hat{\ }v$, and $\hat{f}_{\langle e,y,z\rangle}(x) = x\hat{\ }0^{|y|-|x|}\hat{\ }z$ otherwise. $\square$

Proof of Theorem 2.3. Since the implication *(ii)* $\Rightarrow$ *(iii)* is obvious, it suffices to show *(i)* $\Rightarrow$ *(ii)* and *(iii)* $\Rightarrow$ *(i)*.

For a proof of *(i)* $\Rightarrow$ *(ii)* assume that $\mathbf{C}$ is effectively meager via f. Then, for $\hat{f}$ as in Proposition 2.7 and for $\mathbf{D} = \mathbf{M}(\hat{f})$, $\mathbf{C} \subseteq \mathbf{D}$ (by Proposition 2.4), $\mathbf{D}$ is a c.f.v. Σ^0_2- class (by Propositions 2.5 and 2.7) and there is a recursive set $A \notin \mathbf{D}$ (by Proposition 2.6).

For a proof of the implication *(iii)* $\Rightarrow$ *(i)* let $\mathbf{D}$ be a Σ^0_2-class, say

$$X \in \mathbf{D} \Leftrightarrow (\exists m)\,(\forall n)\,(\Phi^X(m, n) = 1),$$

and let A be a set such that, $\hat{A} \notin \mathbf{D}$ for all finite variants of A. By Proposition 2.4, it suffices to define an effective system of extension functions f such that $\mathbf{D} \subseteq \mathbf{M}(f)$. Note that, by assumption on A, for any number m and for any string x there is a string $y \sqsupseteq x$ such that $\Phi^y(m, n) = 0$ for some number n. So, by letting $f_m(x)$ be the least such string y, f is an effective system of extension functions as required. $\square$

The above characterization of effective meagerness easily implies the previously mentioned results of Mehlhorn on his concept.

Corollary 2.8 *(Mehlhorn [Me73])* **REC** *is not effectively meager.* □

Recall that a class $\mathbf{C}$ is *recursively presentable* (*r.p.*) if there is a recursive set $U \subseteq \omega \times \omega$ such that $\mathbf{C} = \{U_e : e \geq 0\}$. A class of recursive sets is called *bounded* if it is contained in an r.p. class.

Corollary 2.9 *(Mehlhorn [Me73]) Every bounded class of recursive sets is effectively meager.*

Proof. By Theorem 2.3 it suffices to show that every r.p. class $\mathbf{C}$ is a Σ^0_2-class. But given a recursive set U such that $\mathbf{C} = \{U_e : e \geq 0\}$,

$$X \in \mathbf{C} \iff (\exists e)\,(\forall n)\,(U(e,n) = X(n))$$

whence $\mathbf{C}$ is a Σ^0_2-class. □

By Corollary 2.9, the standard complexity classes are effectively meager. An interesting example of an effectively meager class containing nonrecursive sets is the following.

Corollary 2.10 *(Mehlhorn [Me73]) The class* $\mathbf{C} = \{A : \mathbf{P}^A = \mathbf{NP}^A\}$ *is effectively meager.*

Proof (Sketch). Let $\{M_e : e \geq 0\}$ and $\{N_e : e \geq 0\}$ be recursive enumerations of the deterministic respectively nondeterministic polynomial time bounded oracle Turing machines. Then, for any oracle set A, the set

$$K^A = \{\langle e, n, 2^m \rangle : N_e \text{ accepts } n \text{ in less than } m \text{ steps}\}$$

is $\mathbf{NP}^A$-complete (see Baker et al. [BGS75]). It follows that

$$\mathbf{C} = \{A : K^A \in \mathbf{P}^A\} = \{A : (\exists n)\,(\forall m)\,(K^A(m) = M_n(m))\}$$

whence $\mathbf{C}$ is a Σ^0_2-class. Moreover, for $A =^* B$, $\mathbf{P}^A = \mathbf{P}^B$ and $\mathbf{NP}^A = \mathbf{NP}^B$ whence $\mathbf{C}$ is c.f.v. Finally, Baker et al. [BGS75] have shown that there is a set $A \notin \mathbf{C}$. Hence effective meagerness of $\mathbf{C}$ follows from Theorem 2.3. □

Mayordomo [May94] pointed out some limitations of the effective category concept. For any class $\mathbf{C}$, a set A is $\mathbf{C}$-*bi-immune* if neither A nor its complement $\overline{A}$ contain any infinite set $X \in \mathbf{C}$ as a subset.

Theorem 2.11 *(Mayordomo [May94]) The class* $\overline{\mathbf{BI}(\mathbf{P})} \cap \mathbf{REC}$ *of the recursive sets which are not* $\mathbf{P}$*-bi-immune is not effectively meager though* $\overline{\mathbf{BI}(\mathbf{P})}$ *is meager in the classical sense.*

This theorem can be interpreted as follows: The effective category concept formalizes the effective finite extension method. A finite extension construction of a **P**-bi-immune set, however, cannot be effective, i.e. the finite extension method yields only nonrecursive **P**-bi-immune sets. On the other hand, however, Balcázar and Schöning [BS85] have constructed recursive **P**-bi-immune sets by a *slow diagonalization* or *wait and see argument.* The extension of Mehlhorn's category concept introduced in the next section is designed to formalize this more general diagonalization technique. For a detailed discussion of this topic we refer to Ambos-Spies [Am96].

3 The Extended Effective Category Concept

The extension of Mehlhorn's effective category concept which we will consider here is based on partial extension functions. For such a function f we say that a class $\mathbf{C}$ is nowhere dense via f if

$$(\forall A \in \mathbf{C})\ (\exists^{\infty} n)\ (f(A\restriction n)\downarrow)$$

and

$$(\forall x)\ (f(x)\downarrow\ \Rightarrow\ \mathbf{B}_{f(x)} \cap \mathbf{C} = \emptyset).$$

Note that we can extend f to a total extension function $\hat{f}$ such that $\mathbf{C}$ is nowhere dense via $\hat{f}$ by letting $\hat{f}(x) = f(y)$ for the least $y \sqsupseteq x$ such that $f(y)\downarrow$ if such a string y exists and by letting $\hat{f}(x) = x$ otherwise. Hence, in the classical case, partial extension functions lead to the same category concept. In case of effective extensions, however, we obtain a stronger concept as we shall show below.

Definition 3.1 *(a) An* effective partial extension function *f is a partial recursive function $f : 2^{<\omega} \to 2^{<\omega}$ with recursive domain where $x \sqsubseteq f(x)$ for all strings x such that $f(x)$ is defined. An effective partial extension function f is* dense along *a set A if $f(A\restriction n)\downarrow$ for infinitely many numbers n. A set A* meets *f if $f(A\restriction n) \sqsubset A$ for some number n such that $f(A\restriction n)\downarrow$; and A* avoids *f otherwise. A class $\mathbf{C}$ is* extended effectively (e.e.) nowhere dense via *f if f is an effective partial extension function such that f is dense along all sets in $\mathbf{C}$ and $\mathbf{B}_{f(x)} \cap \mathbf{C} = \emptyset$ for all strings x such that $f(x)\downarrow$ (i.e. all sets in $\mathbf{C}$ avoid f). $\mathbf{C}$ is* e.e. nowhere dense *if $\mathbf{C}$ is e.e. nowhere dense via some f.*

(b) An effective system of partial extension functions *is a partial recursive function $f : \omega \times 2^{<\omega} \to 2^{<\omega}$ with recursive domain such that f_e is an effective partial extension function for all e. A class $\mathbf{C}$ is* e.e. meager *via f if f is an effective system of partial extension functions and $\mathbf{C}$ is the union of classes $\mathbf{C}_e$ which are e.e. nowhere dense via f_e. $\mathbf{C}$ is* e.e.

meager *if* **C** *is* e.e. *meager via some* f, *and* **C** *is* e.e. comeager *if* $\overline{\mathbf{C}}$ *is e.e. meager.*

Obviously, any class which is effectively nowhere dense via f is also e.e. nowhere dense via f, whence the above concept extends the effective category concept of Mehlhorn. As there we can show that, for every effective system f of partial extension functions, there is a largest class which is e.e. meager via f, namely

$$\mathbf{M}(f) = \{A : (\exists e)\ (f_e \text{ is dense along } A \ \&\ A \text{ avoids } f_e)\}.$$

Hence, to show that **REC** is not e.e. meager it suffices to show that $\mathbf{REC} \not\subseteq \mathbf{M}(f)$ for all effective systems of partial extension functions f.

Theorem 3.2 **REC** *is not e.e. meager.*

Proof. Let f be an effective system of partial extension functions. It suffices to construct a recursive set A such that $A \notin \mathbf{M}(f)$, i.e., such that A meets the requirements

$$R_e : (\exists^\infty n)\ (f_e(A{\restriction}n){\downarrow}) \Rightarrow (\exists n)\ (f_e(A{\restriction}n) \sqsubseteq A)$$

for all e. The set A is constructed by a slow diagonalization where $A(n)$ is defined at stage n. At any stage of the construction action is taken for the sake of the requirement R_e with the least index e which has not yet been completely satisfied and which now can become further or completely satisfied. Formally, given $A{\restriction}n$ fix $e \leq n$ and $m \leq n$ minimal (in this order) such that

- there is no number $k \leq n$ such that $f_e(A{\restriction}k){\downarrow}$ and $f_e(A{\restriction}k) \sqsubseteq A{\restriction}n$, and
- $f_e(A{\restriction}m){\downarrow}$ and $A{\restriction}n \sqsubset f_e(A{\restriction}m)$,

let $A(n) = f_e(A{\restriction}m)(n)$ and say that R_e is active at stage n (if such numbers e and m do not exist, let $A(n) = 0$). Then, by induction on e, every requirement R_e will be active at most finitely often and, if the hypothesis of R_e holds, there will be a number m such that R_e will become active at the consecutive stages $m, \ldots |f_e(A{\restriction}m)|$thereby ensuring $f_e(A{\restriction}m) \sqsubset A$. □

Remark. For the proof of Theorem 3.2 it is crucial that the domain of the system f is recursive. If we drop this requirement, we can easily define a partial recursive system f such that **REC** is meager via f. In fact, given a recursive enumeration $\{W_e : e \geq 0\}$ of the recursively enumerable sets, the class of r.e. sets will be meager via the partial recursive function f defined by $f_{2n}(x) = x\hat{\ }1^n$ for all x and $f_{2n+1}(x) = x\hat{\ }0$ if $|x| \in W_n$ and $f_{2n+1}(x){\uparrow}$ otherwise. Namely, any finite set will be nowhere dense via f_{2n} for all sufficiently large n and $\{W_n\}$ will be nowhere dense via f_{2n+1} if W_n is infinite.

By Theorem 3.2 the extended effective category concept relativized to **REC** is not trivial. To show that this concept is strictly stronger than Mehlhorn's effective category, we observe that, in contrast to Theorem 2.3 the class $\mathbf{BI(C)}$ of the **C**-bi-immune sets is e.e. comeager for all r.p. classes **C**.

Theorem 3.3 *Let* $\mathbf{C}$ *be r.p. Then* $\mathbf{BI}(\mathbf{C})$ *is e.e. comeager.*

Proof. It suffices to define an effective system of partial extension functions f such that for every set $A \notin \mathbf{BI}(\mathbf{C})$ there is an index e such that f_e is dense along A and $f_e(A{\restriction}n) \not\sqsubseteq A$ whenever $f_e(A{\restriction}n)$ is defined. Given a recursive set U such that $\mathbf{C} = \{U_e : e \geq 0\}$ such a system f is obtained by letting $f_{2e+i}(x) = x\hat{\ }i$ if $|x| \in U_e$ and $f_{2e+i}(x)\uparrow$ otherwise $(e \geq 0,\ i \leq 1)$. Namely, if $A \notin \mathbf{BI}(\mathbf{C})$ then there is an infinite set $U_e$such that $U_e \subseteq A$ or $U_e\overline{A}$ whence $\{A\}$ is e.e. nowhere dense via f_{2e} or f_{2e+1}, respectively. □

Corollary 3.4 *There is a class* $\mathbf{C}$ *of recursive sets such that* $\mathbf{C}$ *is e.e. meager but not effectively meager.*

Proof. By Theorems 3.3 and 2.11, $\overline{\mathbf{BI}(\mathbf{P})} \cap \mathbf{REC}$ is e.e. meager but not effectively meager. □

4 Banach-Mazur Games

Banach-Mazur games allow a game theoretic approach to the concepts of Baire category. They were originally proposed by Mazur for characterizing Baire category on the real line, but the games can be defined on any topological space [Ox57]. Here we only consider the Cantor space version.

Given a class $\mathbf{C} \subseteq 2^\omega$, the *Banach-Mazur game* $BM(\mathbf{C})$ for $\mathbf{C}$ is a two person game. In a *play* the players alternate in extending an initial segment by nonempty strings:

I	x_0		x_1		
					...
II		y_0		y_1	

The *result* or *outcome* of the play is the unique set $R \in 2^\omega$ defined by the initial segments

$$R{\restriction}r_n = x_0\hat{\ }y_0\hat{\ }\ldots\hat{\ }x_n\hat{\ }y_n$$

constructed by the end of round n $(n \geq 0)$. Player I *wins* if $R \in \mathbf{C}$. Otherwise Player II *wins*.

A *strategy* for Player I (or Player II) is some rule according to which Player I, knowing *all* his and his opponent's previous moves (classical Banach-Mazur games are games of *perfect information* [GS53]), determines his next move. A *winning strategy* is a strategy with which the player wins every play in $BM(\mathbf{C})$. If there is a winning strategy for one of the players, the game is called *determined*. Formally, strategies can be defined as *trees* [Ke95]. However, the concept of a strategy can be simplified. For the players it is sufficient to know the current initial segment $R{\restriction}r_n$, as defined above. The information which parts of the current initial segment were played by which player is not relevant for the strategy. So strategies for the players are just proper extension functions (i.e. extension functions f with $x \sqsubset f(x)$ for all strings x).

Definition 4.1 *Let* $\mathbf{C} \subseteq 2^\omega$*. A* strategy *for Player I or Player II in the* Banach-Mazur game $BM(\mathbf{C})$ *for* $\mathbf{C}$ *is a proper extension function. The outcome of the play of the game where the players use the strategies* g *and* h*, respectively, is the unique set* $R = R(g,h) \in 2^\omega$ *defined by the initial segments:*

$$R \restriction r_n = (h \circ g)^n(\lambda)$$

Player I wins *if* $R \in \mathbf{C}$*, otherwise Player II* wins*. The strategy* h *of Player II is a* winning strategy *if* $R(g,h) \notin \mathbf{C}$ *for all strategies* g *of Player I.*

The main theorem for Banach-Mazur games shows the connection between the determinacy of a game and Baire category.

Theorem 4.2 *(Banach and Mazur, Oxtoby) Player II has a winning strategy for* $BM(\mathbf{C})$ *if and only if* $\mathbf{C}$ *is meager.*

For a proof see e.g. [Ox57]. Mazur stated one direction of this theorem (namely, if $\mathbf{C}$ is meager, then Player II has a winning strategy) first in the *Scottish Book* (see [Mau81], Problem 43). Banach observed that the converse implication holds, too, but his proof was never published.

In order to characterize effective Baire category, the classical version of the game has to be modified. While the role of Player I is unchanged, Player II has to choose his moves effectively. In terms of strategies this can be expressed by requiring that this player uses an effective extension strategy. Lisagor [Li81] introduced this modified version of the game for the Baire space ω^ω and, subsequently, Lutz [Lu90] and Fenner [Fe91] did the same for the Cantor space.

Definition 4.3 *Given a class* $\mathbf{C} \in 2^\omega$*,* $BM(\mathbf{C}, \mathbf{REC})$ *denotes the* effective Banach-Mazur game *for the class* $\mathbf{C}$*. This game differs from the classical Banach-Mazur game for* $\mathbf{C}$ *in requiring Player II to use an effective extension function as a strategy.*

The main theorem on Banch-Mazur games carries over to the effective case:

Theorem 4.4 *(Lisagor [Li81], Lutz [Lu90], Fenner [Fe91]) Player II has a winning strategy for* $BM(\mathbf{C}, \mathbf{REC})$ *if and only if* $\mathbf{C}$ *is effectively meager.*

A proof can be found in [Fe91].

For the above characterization of classical and effective Baire category in terms of Banach-Mazur games it is crucial that these category concepts are based on total extension functions. Hence, for a game theoretic characterization of the extended effective category concept which is based on partial extension functions, the game has to be extended.

The basic idea for the extended Banach-Mazur game introduced below is to allow Player II to cut back the current move of Player I. Without any restrictions, however, this will give Player II complete control over the outcome of the play. So the possibility of Player II for cutting Player I's move is confined by making Player II pay for the cut: In any round where Player II does not cut he receives one token and in any round in which he chooses to cut he has to pay back one token. In particular, Player II cannot cut the first move of Player I in round 0 since his start capital is 0.

Hence a strategy for Player II does not only consist of an extension function but it also has to provide a decision procedure telling Player II whether he should cut and, if so, how much of Player I's move is to be cut. This decision should be based on the values of the possible cuts in relation to the current capital of the player. We formalize this by defining a weight function $h^w : 2^{<\omega} \to \omega$ which assigns a weight to any initial segment and, in round $n+1$, we let Player II cut the move x_{n+1} of Player I to the longest string $x \sqsubseteq x_{n+1}$ such that the weight w of $(R\restriction r_n)\hat{}x$ is minimal provided that the current capital of Player II exeeds w. The motivation for these rules is as follows: The weight function is chosen in a way that the lower the weight of an extension the more desirable this extension is for Player II. Hence, if Player II cuts a move, he will always choose a cut with the lowest possible weight. Moreover, if in some round all possible cuts have relatively high weights, whence the advantage of a cut is relatively low, Player II will spend a token for a cut only if his capital is relatively high. This will help to avoid the situation that, in some round of the play, there is the chance for a cut of low weight, but Player II has no tokens left for performing the cut, since he spent all of them on cuts of higher weights in the preceeding rounds.

By further requiring that the strategy of Player II is effective we obtain the effective extended Banach-Mazur game which can be formally defined in terms of strategies as follows.

Definition 4.5 *For any class* $\mathbf{C}$ *the* effective extended Banach-Mazur game $EBM(\mathbf{C}, \mathbf{REC})$ *is defined as follows. A* strategy g *for Player I is a proper extension function while a* strategy h *for Player II is a pair* $h = (h^e, h^w)$ *where* $h^e : 2^{<\omega} \to 2^{<\omega}$ *is a proper effective extension function and* $h^w : 2^{<\omega} \to \omega$ *is a total recursive function* (weight function). *The* result *or* outcome $R = R(g,h)$ *of the play of* $EBM(\mathbf{C}, \mathbf{REC})$ *in which the players use the strategies* g *and* h*, respectively, is the unique set* R *defined by the initial segments* $R\restriction r_n$ *describing the outcomes of the play up to round* n*. Here* $R\restriction r_n$ *and the* current capital $c(n)$ *of Player II at the end of round* n *are inductively given by:*

$$R\restriction r_0 \;=\; h^e(g(\lambda)) \quad \textit{and} \quad c(0) = 1$$

and, for $n \geq 1$, $L = \{|R\restriction r_n|, \ldots, |g(R\restriction r_n)|\}$,

$$R\restriction r_{n+1} \;=\; h^e(g(R\restriction r_n)\restriction l)$$

where l is the greatest number in L such that

$$(\forall l' \in L)(h^w(g(R\restriction r_n)\restriction l) \leq h^w(g(R\restriction r_n)\restriction l')) \text{ and}$$
$$h^w(g(R\restriction r_n)\restriction l) < c(n)$$

if such a number exists, $l = |g(R\restriction r_n)|$ otherwise, and

$$c(n+1) = \begin{cases} c(n) - 1 & \text{if } l < |g(R\restriction r_n)| \\ c(n) + 1 & \text{otherwise} \end{cases}$$

Player II wins *the play if $R(g,h) \notin \mathbf{C}$ and h is a* winning strategy *for Player II if by using strategy h Player II wins all plays no matter what strategy g Player I employs.*

Our main result shows that this extended game concept characterizes the e.e. category concept.

Theorem 4.6 *For any class $\mathbf{C}$ the following are equivalent.*

(i) Player II has a winning strategy for $EBM(\mathbf{C}, \mathbf{REC})$.

(ii) $\mathbf{C}$ is extended effectively meager.

Proof. For a proof of *(i)* $\Rightarrow$ *(ii)* let $h = (h^e, h^w)$ be a winning strategy of Player II for $EBM(\mathbf{C}, \mathbf{REC})$. It suffices to define an effective system of partial extension functions f such that $\mathbf{C} \subseteq \mathbf{M}(f)$. Let the system f consist of the following components, where $k, l \in \omega$:

$$f_l^k(x) = \begin{cases} h^e(x) & \text{if } |x| > l \;\&\; h^w(x) = k \\ \uparrow & \text{otherwise} \end{cases}$$

$$f_l^\infty(x) = \begin{cases} h^e(x) & \text{if } |x| > l \\ \uparrow & \text{otherwise} \end{cases}$$

Now, to show that $\mathbf{C} \subseteq \mathbf{M}(f)$, fix $A \notin \mathbf{M}(f)$. It suffices to define a strategy g for Player I such that $R(g,h) = A$. For the definition of g distinguish the following two cases:

Case 1: $(\exists k)\,(\overset{\infty}{\exists}\, n)\,(h^w(A\restriction n) = k)$.

Let k_0 be the least such k and fix l_0 such that $h^w(A\restriction n) \geq k_0$ for $n \geq l_0$. Then $f_l^{k_0}$ is dense along A for all $l \geq 0$ whence, by $A \notin \mathbf{M}(f)$, A meets all these extension functions. So for $x \sqsubset A$ we may fix $n = n(x)$ minimal such that $f^{k_0}_{\max(|x|,l_0)}(A\restriction n(x)) \sqsubset A$. Note that $x \sqsubset A\restriction n(x)$, $h^w(A\restriction n(x)) = k_0$ and $h^e(A\restriction n(x)) = f^{k_0}_{\max(|x|,l_0)}(A\restriction n(x))$. Now define g as follows: For $x \sqsubset A$ let $g(x) = A\restriction n(x)$; otherwise let $g(x) = x\hat{\ }0$. To show that $R(g,h) = A$, by induction on m show that, for $R = R(g,h)$, $R\restriction r_m \sqsubset A$. Since Player II cannot cut the first move of Player I,

$$R\restriction r_0 = h^e(g(\lambda))$$

and, by definition of g,

$$h^e(g(\lambda)) = h^e(A\upharpoonright n(\lambda)) = f_{l_0}^{k_0}(A\upharpoonright n(\lambda)) \sqsubset A$$

and $r_0 > l_0$. For the inductive step let $x = R\upharpoonright r_m$ and assume that $x \sqsubset A$. Then $g(R\upharpoonright r_m) = A\upharpoonright n(x)$, $h^w(A\upharpoonright n(x)) = k_0$ and, by $r_m \geq r_0 > l_0$, $h^w(y) \geq k_0$ for all strings y with $R\upharpoonright r_m \sqsubseteq y \sqsubseteq g(R\upharpoonright r_m)$. Hence, Player II will not cut the move of Player I in round $m+1$, whence

$$R\upharpoonright r_{m+1} = h^e(g(R\upharpoonright r_m)) = f_{\max(|x|,l_0)}^{k_0}(A\upharpoonright n(x)) \sqsubset A.$$

Case 2: Otherwise.
Then there is a nondecreasing function $l : \omega \to \omega$ such that, for any $k \in \omega$, $l(k) > k$ and $h^w(A\upharpoonright n) > k$ for all $n \geq l(k)$. Since the functions f_l^∞ are dense along all sets, for $x \sqsubset A$ we can fix a number $n(x) > |x|$ such that $f_{l(|x|)}^\infty(A\upharpoonright n(x)) \sqsubset A$. Define the extension function g by letting $g(x) = A\upharpoonright n(x)$ if $x \sqsubset A$ and $g(x) = x\hat{}0$ otherwise. To show that $R(g,h) = A$, by induction on m show that, for $R = R(g,h)$, $R\upharpoonright r_m \sqsubset A$ and $r_m > l(m)$. For $m = 0$ this follows as in Case 1. So fix m, let $x = R\upharpoonright r_m$ and assume that $R\upharpoonright r_m \sqsubset A$ and $r_m > l(m)$. Then $g(x) = A\upharpoonright n(x)$ and, by $r_m > l(m)$, $h^w(y) > m$ for all y with $x \sqsubseteq y \sqsubseteq A\upharpoonright n(x)$. So Player II cannot cut this move since $c(m) \leq m$. It follows that

$$R\upharpoonright r_{m+1} = h^e(g(x)) = f_{l(|x|)}^\infty(A\upharpoonright n(x)) \sqsubseteq A$$

and, since $|x| \geq r_m \geq m+1$, $r_{m+1} > n(x) > l(|x|) \geq l(m+1)$ by definition of $n(x)$.

For a proof of *(ii)* $\Rightarrow$ *(i)* assume that $\mathbf{C}$ is e.e. meager, say $\mathbf{C} \subseteq \mathbf{M}(f)$ for an effective system of partial extension functions f. It suffices to give a winning strategy $h = (h^e, h^w)$ for Player II in the game $EBM(\mathbf{M}(f), \mathbf{REC})$. Note that for $A \in \mathbf{M}(f)$ there is an index k sucht that f_k is dense along A and A avoids f_k. So, if for the outcome $R = R(g,h)$ of some play, f_k is dense along R then we have to ensure that $f_k(R\upharpoonright n) \sqsubset R$ for some number n. This is achieved by ensuring that if $f_k(R\upharpoonright n)$ is defined infinitely often then in some round the extension function f_k will be simulated. Here smaller indices are given higher priority i.e. will be simulated first (in case of a conflict).

Roughly speaking this is achieved by letting $h^w(x) = k$ and $h^e(x) = f_k(x)$ for the least index k such that $f_k(x)\downarrow$ and f_k is not yet met by x, i.e. $f_k(y) \not\sqsubseteq x$ for all $y \sqsubseteq x$ with $f_k(y)\downarrow$. The actual definition of h is somewhat more involved, however, since by the above procedure a strategy $f_{k'}$ with $k' < k$, which is not defined on x but on some y with $x \sqsubset y \sqsubset f_k(x)$, might be dense along R but will possibly never be met. This problem is avoided by the following inductive search for k.

Given a string x, inductively define $h_k^w(x)$ and $h_k^e(x)$ for $k = |x|+1, |x|, \ldots, 0$ as follows and let $h^w(x) = h_0^w(x)$ and $h^e(x) = h_0^e(x)$. For $k = |x|+1$ let $h_k^w(x) = k$ and $h_k^e(x) = x\hat{}0$. Given $k \leq |x|$, $h_{k+1}^w(x)$ and $h_{k+1}^e(x)$, we say that k *requires attention at* x if

(4.1) $\quad (\forall y \sqsubseteq x)\,(f_k(y)\!\downarrow\, \Rightarrow\, f_k(y) \not\sqsubseteq x)$

(4.2) $\quad (\exists z)\,(x \sqsubseteq z \sqsubseteq h^e_{k+1}(x)\ \&\ f_k(z)\!\downarrow)$

If k requires attention at x fix the least z as above and let $h^w_k(x) = k$ and $h^e_k(x) = f_k(z)$. Otherwise, let $h^w_k(x) = h^w_{k+1}(x)$ and $h^e_k(x) = h^e_{k+1}(x)$. This completes the definition of the strategy h.

For the remainder of the proof fix a strategy g for Player I and let $R = R(g,h)$. To show that $R \notin \mathbf{M}(f)$, we have to show that for every index k such that f_k is dense along R, R will meet f_k. This will be established by the following claims.

We say that k *requires attention in round* $m+1$ if there is a string x with $R\restriction r_m \sqsubseteq x \sqsubseteq g(R\restriction r_m)$ and k requires attention at x; and k *receives attention in round* $m+1$ if there is such a string x and $R\restriction r_{m+1} = f_k(x) = h^e(x)$.

Claim 1. If k receives attention in round m then R meets f_k and k does not require attention after round m.

Proof. Straightforward.

Claim 2. Every k requires attention in at most finitely many rounds.

Proof. For a contradiction fix k minimal which requires attention infinitely often. Then, by Claim 1, k will never receive attention. Fix m_0 such that no $k' < k$ requires attention after round m_0 and fix $m_1 > m_0 + k$ minimal such that k requires attention in round m_1. Note that if in some round m the move of Player I is cut back to a string x with $h^w(x) = k'$ then k' receives attention in this round. So, for m with $m > m_0$, either a number $k' > k$ receives attention in round m, whence $c(m) \geq c(m-1) - 1 > k' - 1 \geq k$, or $c(m) = c(m-1) + 1$. Since $m_1 > m_0 + k$ it follows that $c(m_1 - 1) \geq k$. Hence k will receive attention in round m_1 contrary to Claim 1.

Claim 3. If f_k is dense along R then R meets f_k.

Proof. For a contradiction assume that f_k is dense along R and R does not meet f_k. By Claim 2 fix $m_0 > k$ such that no k' with $k' \leq k$ requires attention after round m_0, and, by density, fix $m > m_0$ such that $f_k(z)\!\downarrow$ for some string z with $R\restriction r_m \sqsubseteq z \sqsubset R\restriction r_{m+1}$. Since R avoids f_k, it follows that k requires attention at z and $h^w(z) \leq k$. On the other hand, by choice of m_0 and by definition of the strategy h, there is a number $k' > k$ and a string x with $R\restriction r_m \sqsubseteq x \sqsubseteq g(R\restriction r_m)$ such that

$$h^w(x) = h^w_{k+1}(x) = h^w_{k'}(x) = k' \text{ and } R\restriction r_{m+1} = h^e(x) = h^e_{k+1}(x) = h^e_{k'}(x).$$

Since Player II cuts back the move of a Player I to a string of lowest possible weight, it follows that

$$R\restriction r_m \sqsubseteq x \sqsubseteq g(R\restriction r_m) \sqsubset z \sqsubseteq h^e(x) = R\restriction r_{m+1}.$$

Since $h^e(x) = h^e_{k+1}(x)$, by definition of the strategy h this implies that k requires attention at x whence $h^w(x) \leq h^w_k(x) = k$ contrary to $h^w(x) = k'$. □

5 Resource-Bounded Category

Resource-bounded Baire Category concepts have been introduced and studied e.g. in [AFH88], [Lu90], [Fe91], [Fe95] and [Am96]. Most of these concepts can be defined in terms of (partial) extension functions though some of them were originally defined in terms of condition sets. In general, time and space bounds were considered and frequently the interest focused on polynomial time computable extension functions which, by $|A \restriction n| = n \approx 2^{|n|}$, correspond to diagonalizations over exponential time objects, i.e., diagonalizations over the class $\mathbf{E} = \mathbf{DTIME}(2^{lin})$. The concept of Lutz [Lu90] is the analog of Mehlhorn's concept in this setting. It is based on total extension functions comparable within the given resource bound. As Fenner [Fe91] observed this concept is quite weak. Namely, the resource bounds implicitly impose bounds on the length of the extensions. Hence, for instance polynomial-time extension functions are too short for diagonalizing over polynomial-time reductions (see [Fe91], [Am96]). Fenner [Fe91] proposed a stronger category concept overcoming these limitations in part which is based on (totally defined) extension functions which only specify a part of the extension. For obtaining a full analog of effective category in the complexity setting, however, one has to consider extension functions which are *locally* computable within the given resource bound (whence the resource bound does not bound the length of the extension). This was independently observed by Fenner [Fe95] and Ambos-Spies [Am96]. In particular, for the total locally polynomial-time computable extension functions introduced in [Fe95], the corresponding p-category concept has a characterization by Σ^0_2-classes corresponding to Theorem 2.3: A class $\mathbf{C}$ is p-meager iff there is a Σ^0_2-class $\mathbf{D}$ and a set $A \in \mathbf{E}$ such that $\mathbf{C} \subseteq \mathbf{D}$, $\mathbf{D}$ is c.f.v. and $A \notin \mathbf{D}$ (see [Fe95] and [Am96]). In [Am96] the resource-bounded version of e.e. category based on partial extension functions is introduced. For a comparison of all these concepts and their relations to the fundamental diagonalization techniques in complexity theory we refer to [Am96].

Lutz [Lu90] and Fenner [Fe91], [Fe95] gave game theoretical characterizations of their concepts based on resource bounded variants of the Banach-Mazur games. The papers [AFH88] and [Am96], where resource-bounded category concepts based on partial extensions were introduced, focus on generic sets, i.e. sets G such that the singleton $\{G\}$ is not nowhere dense or, equivalently, not meager (in the corresponding setting).Though in [Am96] it is pointed out how meagerness can be defined in terms of genericity, meager classes were not explicitely studied. In particular no game theoretical characterization was given. Reimann [Re97] has shown, however, that the extended Banach-Mazur game introduced here can be adapted to describe the category concepts of [AFH88] and [Am96].

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