Lesson 2

Computability

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Jan Reimann

PENNSTATE
Eberly College of Science

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Undecidability of the Halting Problem

$$K = \{ \langle e, w \rangle : M_e(w)\downarrow \}$$

Note that K is semidecidable (recursively enumerable) via a universal TM U

$$U(\langle e, w \rangle) = M_e(w)$$

We will show that the diagonal HP

$$K_d = \{ x : M_x(x)\downarrow \}$$

is undecidable.

(Undecidability of K_d implies undec. of K)

THM: K_d is undecidable.

Pf: Assume for a contradiction there ex. TM M that decides K_d, i.e.

note: M halts on every input

$$M(x) = \begin{cases} 1 & \text{if } M_x(x)\downarrow \\ 0 & \text{if } M_x(x)\uparrow \end{cases}$$

Using Church-Turing Thesis, we can use M to define a new TM $\overline{M}$ that behaves as follows:

$$\overline{M}(x) = \begin{cases} \uparrow & \text{if } M(x) = 1 \\ 1 & \text{if } M(x) = 0 \end{cases}$$

$\overline{M}$ is a Turing machine, hence it has an index (GN), say $\overline{e}$.

i.e. $\overline{M} = M_{\overline{e}}$

<u>Q</u>: Is $\overline{e} \in K_d$ or not?

Does $M_{\overline{e}}(\overline{e}) \downarrow$ or $\uparrow$

$$M_{\bar{e}}(x) = \overline{M}(x) = \begin{cases} \uparrow & \text{if } M(x) = 1 \\ 1 & \text{if } M(x) = 0 \end{cases}$$

cannot exist $\longrightarrow$ $M(x) = \begin{cases} 1 & \text{if } M_x(x) \downarrow \\ 0 & \text{if } M_x(x) \uparrow \end{cases}$

Supp $\bar{e} \in K_d \Rightarrow M(\bar{e}) = 1 \Rightarrow \overline{M}(\bar{e}) \uparrow$

$\Rightarrow M_{\bar{e}}(\bar{e}) \uparrow$

$\Rightarrow \bar{e} \notin K_d$

$\bar{e} \notin K_d \Rightarrow M(\bar{e}) = 0 \Rightarrow \overline{M}(\bar{e}) = 1$

$\Rightarrow M_{\bar{e}}(\bar{e}) \downarrow = 1$

$\Rightarrow \bar{e} \in K_d$ $\lightning$

The Halting Problem was the first problem shown to be algorithmically undecidable. (Turing 1936)

Many more examples:

- Solvability of diophantine equations Hilbert's 10th problem
- Word problem for groups
- Kolmogorov complexity