Lecture 16: Constructible Reals

In this lecture we transfer the results about L to the projective hierarchy. The main idea is to relate sets of reals that are defined by set theoretic formulas to sets defined in second order arithmetic.

The effective projective hierarchy

We have seen that the Borel sets of finite order correspond to the sets definable (from parameters) by formulas using only number quantifiers (*arithmetical formulas*). A similar relation holds between projective sets and sets definable by formulas using both number and function quantifiers. In fact, the way we defined the projective hierarchy makes this easy to see.

Historically, however, the topological approach and the definability approach happened separably, the former devised by the Russian school of Souslin, Lusin, and others, while the effective approach was pursued by Kleene. Kleene named the sets definable over second order arithmetic the *analytical* sets, which to this day is a source of much confusion.

Definition 16.1 (Kleene)**:** A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Σ^1_n if there exists an arithmetical formula $\varphi(\alpha, \beta_1, \dots, \beta_n)$ such that

$$\alpha \in A \quad \Longleftrightarrow \quad \exists \beta_1 \forall \beta_2 \dots Q\beta_n \, \varphi(\alpha, \beta_1, \dots, \beta_n)$$

where Q is $\exists$ if n is odd and Q is $\forall$ if n is even. Similarly, $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Π^1_n if there exists an arithmetical formula $\varphi(\alpha, \beta_1, \dots, \beta_n)$ such that

$$\alpha \in A \quad \Longleftrightarrow \quad \forall \beta_1 \exists \beta_2 \dots Q\beta_n \, \varphi(\alpha, \beta_1, \dots, \beta_n)$$

where Q is $\forall$ if n is odd and Q is $\exists$ if n is even. A set that is Σ^1_n and Π^1_n at the same time is called Δ^1_n. A set A is **analytical** if it is Σ^1_n or Π^1_n.

It is often useful only having to deal with analytical predicates of certain form. Kleene provided normal forms of analytical predicates.

Proposition 16.2 (Kleene)**:** *Every analytical predicate $A(\alpha)$ is equivalent to one of the following forms:*

$$\begin{array}{llll} & \exists \beta \forall m \psi(\alpha, \beta, m) & \exists \beta \forall \gamma \exists m \psi(\alpha, \beta, \gamma, m) & \dots \\ \theta(\alpha, \beta, \gamma, m) & & & \\ & \forall \beta \exists m \psi(\alpha, \beta, m) & \forall \beta \exists \gamma \forall m \psi(\alpha, \beta, \gamma, m) & \dots \end{array}$$

where θ is arithmetic, and ψ is a formula whose quantifiers (if any) are bounded number quantifiers.

The Normal Form is proved by applying a sequence of quantifier manipulations. We provide a sufficient list:

$$\begin{array}{rcl} \forall m\, \exists \alpha\, \varphi(\alpha, m) & \Leftrightarrow & \exists \alpha\, \forall m\, \varphi((\alpha)_m, m) \\ \exists m\, \varphi(m) & \Leftrightarrow & \exists \alpha\, \varphi(\alpha(0)) \\ \exists \alpha\, \exists \beta\, \varphi(\alpha, \beta) & \Leftrightarrow & \exists \gamma\, \varphi((\gamma)_0, (\gamma)_1) \\ \exists m\, \exists n\, \varphi(m, n) & \Leftrightarrow & \exists k\, \varphi((k)_0, (k)_1) \end{array}$$

Each rule has a dual obtained by flipping the quantifiers. The quantifier manipulations can also be used to show

Proposition 16.3 (Kleene)**:** *A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is definable over second order arithmetic if it is analytical.*

The following theorem complements Theorem 9.9. It is an immediate consequence of the definition of the classes $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$.

Theorem 16.4: *A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is $\mathbf{\Sigma}^1_n$ ($\mathbf{\Pi}^1_n$) if and only if it is definable in γ by a Σ^1_n (Π^1_n) formula, for some $\gamma \in \mathbb{N}^{\mathbb{N}}$.*

The set of constructible reals

What is the complexity of the set $\mathbb{N}^{\mathbb{N}} \cap L$? In particular, is it in the projective hierarchy? The set of all constructible reals is defined by a Σ_1 formula over set theory:

$$\varphi(x_0) \exists y\ [y \text{ is an ordinal } \wedge\ x_0 \in L_y\ \wedge\ x_0 \text{ is a set of natural numbers }].$$

We would like to replace this formula by an "equivalent" one in the language of second order arithmetic. In particular, we would like to replace the quantifier $\exists y$ by a quantifier over a real number.

The key for doing this is the fact that every constructible real shows up at a countable stage of L: Since CH holds in L, $L \cap \mathcal{P}(\omega) \subseteq L_{\omega_1}$. Hence if $\alpha \in L \cap \mathbb{N}^{\mathbb{N}}$, there exists a countable ξ such that $\alpha \in L_\xi$. Since $|\xi| = |L_\xi|$, L_ξ is countable, too. Hence we can hope to replace L_ξ by something like "*there exists a real that codes a model that looks like L_ξ*".

A set theoretic structure is simply a set X with a binary relation (the interpretation of $\in$). If X is countable (infinite), we can assume $X = \omega$, and then any $\alpha \in \mathbb{N}^{\mathbb{N}}$ codes the set theoretic structure

$$(\omega, E_\alpha) \qquad \text{where } E_\alpha = \{\langle m,n\rangle : \alpha(\langle m,n\rangle) = 0\}.$$

We know from the previous lecture that there exists a sentence $\varphi_{\mathsf{V=L}}$ so that if Y is a transitive set, $Y \models \varphi_{\mathsf{V=L}}$ if and only if $Y = L_\delta$ for some limit δ. But for an arbitrary real α, E_α does not need to look anything like a set. It may even fail to be well-founded as a relation. However, if E_α *is* well-founded and *extensional*, then it looks very much like a (transitive) set.

The Mostowski collapse

Let E be a binary relation on a set X. Think of (X,P) as an intended model of set theory. We would like E to behave like the $\in$-relation for sets. For this purpose, let for each $x \in M$

$$\mathrm{ext}_E(x) = \{y \in X : y \, E \, x\}$$

If E behaves "set-like", then it will respect the *Axiom of Extensionality*, i.e. two sets are identical if and only if they have the same elements. Therefore we say that E is **extensional** if

$$x, z \in X,\ x \neq z \quad \text{implies} \quad \mathrm{ext}_E(x) \neq \mathrm{ext}_E(z).$$

Furthermore, we want to exclude infinite descending E-chains. We say that E is **well-founded** if

every non-empty set $Y \subseteq X$ has an E-minimal subset.

Theorem 16.5: *If E is an extensional and well-founded relation on a set X, then there exists a transitive set S and a bijection $\pi : X \to S$ such that*

$$x \, E \, y \quad \Longleftrightarrow \quad \pi(x) \in \pi(y) \quad \textit{for all} x, y \in X.$$

Moreover, S and π are unique.

Proof. We construct π and $S = \mathrm{im}(\pi)$ by recursion on E, which is possible since it is well-founded. (For details on recursion and induction on well-founded relations, see Jech [2003].) For each $x \in X$, let

$$\pi(x) = \{\pi(y) : y \, E \, x\},$$

and set $S = \mathrm{im}(\pi)$.

The injectivity of π follows from the extensionality of π by induction along E: Suppose we have shown

$$\forall z\, (zEx \to \forall y \in X(\pi(z) = \pi(y) \to z = y)).$$

and we have to show that it holds for x. Assume $\pi(x) = \pi(y)$ for some $y \in X$. Then

$$\begin{aligned} cEx \quad &\to \quad \pi(c) \in \pi(x) = \pi(y) \\ &\to \quad \pi(c) = \pi(z) \ \text{ for some } zEy \\ &\to \quad c = z \ \text{ by Ind. Hyp., since } cEx \\ &\to \quad cEy. \end{aligned}$$

Similarly, we get $cEy \to cEx$, hence $x = y$ as desired due to the extensionality of E. Finally we have

$$\begin{aligned} \pi(x) \in \pi(y) \quad &\to \quad \pi(x) = \pi(c) \ \text{ for some } cEy \\ &\to \quad x = c \ \text{ since } \pi \text{ is injective} \\ &\to \quad xEy. \end{aligned}$$

Thus π is an isomorphism.

To see the uniqueness of π and S, assume ρ, T are such that the statement of the Theorem is satisfied. Then $\pi \circ \rho^{-1}$ is an isomorphism between $(T, \in)$ and $(S, \in)$.

Lemma 16.6: *Suppose X, Y are sets, and θ is an isomorphism between $(X, \in)$ and $(Y, \in)$. Then $X = Y$ and $\theta(x) = x$ for all $x \in X$.*

Proof. By induction on the well-founded relation $\in$. Assume that $\theta(z) = z$ for all $z \in x$ and let $y = \theta(x)$. We have $x \subseteq y$ because if $z \in x$, then $z = \theta(z) \in \theta(x) = y$. We also have $y \subseteq x$: Let $t \in y$. Since $y \in Y$, there is $z \in X$ with $\theta(z) = t$. Since $\theta(z) \in y$, we have $z \in x$, and thus $t = \theta(z) = z \in x$. Hence $x = y$, and this also implies $\theta(x) = x$. □

The lemma, applied to $\pi \circ \rho^{-1}$, yields $S = T$ and $\pi \circ \rho^{-1} = \mathrm{id}$, hence $\pi = \rho$ □

Arithmetizing the satisfaction relation

We can now reformulate membership in L for reals as follows:

$$\alpha \in L \cap \mathbb{N}^{\mathbb{N}} \quad \Leftrightarrow \quad \exists \beta \exists m\, [E_\beta \text{ is an extensional and well-founded relation} \\ \wedge\ (\omega, E_\beta) \models \varphi_{\mathsf{V=L}}\ \wedge\ \pi_\beta(m) = \alpha],$$

where π_β is the Isomorphism of the Mostowski collapse of E_β.

It remains to show that the notion occurring inside the square brackets are definable in second order arithmetic.

Proposition 16.7:

(a) *For any $n \in \mathbb{N}$,*

$$\{(m, \sigma, \gamma) \in \mathbb{N} \times \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} : m = \ulcorner \varphi \urcorner \wedge \varphi \textit{ is } \Sigma_n \wedge (\omega, E_\gamma) \models \varphi[\sigma]\}$$

is Σ^0_n.

(b) *If $\alpha \in \mathbb{N}^{\mathbb{N}}$ and E_α is well-founded and extensional, then*

$$\{(m, \gamma) \in \mathbb{N} \times \mathbb{N}^{\mathbb{N}} : \pi_\alpha(m) = \gamma\}$$

is arithmetical in α.

Proof. See Kanamori [2003]. □

The complexity of well-foundedness

It remains to define the properties "*E_β is extensional*" and "*E_β is well-founded*"

For the first one, notice that

$$E_\beta \text{ is extensional} \quad \Leftrightarrow \quad \forall m, n\, [\forall k (k E_\beta m \leftrightarrow k E_\beta n) \rightarrow m = n].$$

Hence it is arithmetical. On the other hand,

$$E_\beta \text{ is well-founded} \quad \Leftrightarrow \quad \forall \gamma \in \mathbb{N}^{\mathbb{N}}\, \exists n\, \forall m\, [\gamma(n) E_\beta \gamma(m)].$$

Hence being well-founded is a Π^1_1 property. Putting everything together we now have the following.

Theorem 16.8: *The set $L \cap \mathbb{N}^{\mathbb{N}}$ is Σ^1_2.*

In similar way we can show

Theorem 16.9: *The set* $\{(\alpha,\beta) \in (L \cap \mathbb{N}^{\mathbb{N}})^2 : \alpha <_L \beta\}$ *is* Σ^1_2.

If $\mathsf{V} = \mathsf{L}$, then the set is actually Δ^1_2, since then

$$\alpha <_L \beta \quad \Longleftrightarrow \quad \alpha \neq \beta \wedge \neg(\beta <_L \alpha).$$

Finally, since $\mathsf{V} = \mathsf{L}$ implies CH, we can use Proposition 14.4 to show the existence of non-measurable sets under $\mathsf{V} = \mathsf{L}$.

Corollary 16.10: *If* $\mathsf{V} = \mathsf{L}$*, then there exists a* Δ^1_2 *set that is not Lebesgue-measurable and does not have the Baire property.*