# The Strength of the Besicovitch-Davies Theorem

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# Measure-Theoretic Regularity

- For Lebesgue measure $\lambda$, and any measurable $E$

Inner regularity

$$\lambda(E) = \sup\{\lambda(K) : K \subseteq E, K \text{ compact}\}$$
$$= \inf\{\lambda(U) : U \supseteq E, U \text{ open}\}.$$

Outer regularity

- Holds more generally for any positive Borel measure on a σ-compact Hausdorff space in which any compact set has finite measure.

- Not true in general for arbitrary Borel measures.

  Example: $s$-dimensional Hausdorff measure $\mathcal{H}^s$, $0 < s < 1$.

  Open sets $U \subseteq \mathbb{R}$ have infinite measure.

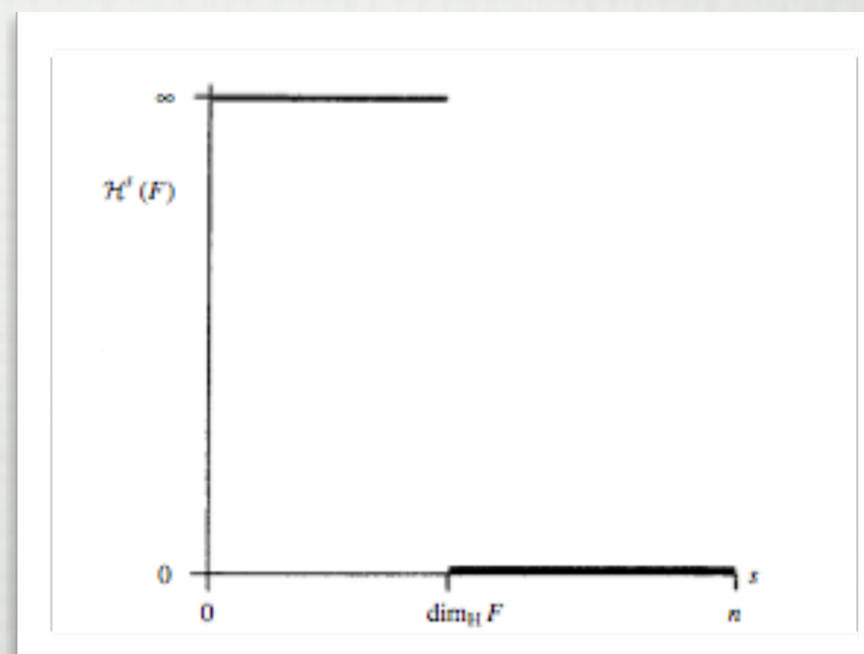
# Hausdorff Measures

- Fix a non-negative real $s$. Given a parameter $\varepsilon > 0$ , cover a set as well as possible with open sets of diameter at most $\varepsilon$ :

$$\mathcal{H}^s_\varepsilon(E) = \inf\left\{\sum_i \operatorname{diam}(U_i)^s : \bigcup_i U_i \supseteq E,\ \operatorname{diam}(U_i) < \varepsilon\right\}$$

- Then let $\varepsilon \to \infty$:

$$\mathcal{H}^s(E) = \lim_{\varepsilon \to \infty} \mathcal{H}^s_\varepsilon(E) \quad \in [0, \infty]$$




- The restriction to the measurable sets yields a Borel measure.

- Hausdorff dimension:

$$\dim_H(E) = \inf\{s : \mathcal{H}^s(E) = 0\}$$

# Besicovitch-Davies Theorem

- **Thm:** [Besicovitch, 1952; Davies, 1952]

  For any analytic set $E \subseteq \mathbf{R}^n$ and for any $s \geq 0$,

  $$\mathcal{H}^s(E) = \sup\{\mathcal{H}^s(K) \colon K \subseteq E, K \text{ compact}\}.$$

- Howroyd [1995]: true in any compact metric space, hence in particular in Cantor space $2^\omega$.

- Result is a fundamental tool in the study of fractals, since closed sets of finite measure are generally well-behaved.

# Our Motivation

- Does this result hold effectively? In other words, given a $\Sigma^0_n$ set of reals of positive Hausdorff s-measure, does there exist an effectively closed subset of positive measure?

- To keep notation simple we concentrate on a simpler result:

  Every analytic set of positive Hausdorff dimension has a closed subset of positive dimension.

- Motivation: Studying the set of reals MIN of minimal Turing degree.

  - Has Hausdorff dimension 1. [Greenberg-Miller, using Kumabe-Lewis forcing]

  - Is a $\Pi^0_4$ set of reals.

# Our Motivation

- By Besicovitch-Davies, MIN has a closed subset of dimension arbitrarily close to 1.

- In particular, this set supports a “nice” probability measure (close to Lebesgue measure), and hence contains reals that “almost” look like Martin-Löf random reals.

- But no real of minimal degree can be ML-random, since, by van Lambalgen, it splits into two relatively random reals.

- However, this set cannot be effectively closed, since every effectively closed set of reals contains a real of r.e. Turing degree.

- Effective descriptive set theory allows us to give a more specific answer:

The closed subset is quite complicated.

# Complexity of Index Sets

- We first look at the index set complexity of deciding whether a set has positive Hausdorff dimension.
- Problem: Given an index for an effective (lightface) Borel set of reals, decide whether the class is nonempty / of positive dimension / of positive Lebesgue measure.

| Family | Nonempty? | Positive Hausdorff dimension? | Positive Lebesgue measure? |
|---|---|---|---|
| $\Sigma^0_1$ | $\Sigma^0_1$-complete | $\Sigma^0_1$-complete | $\Sigma^0_1$-complete |
| $\Pi^0_1$ | $\Pi^0_1$-complete | $\Sigma^0_2$-complete | $\Sigma^0_2$-complete |
| $\Pi^0_2$ | $\Sigma^1_1$-complete | $\Sigma^1_1$-complete | $\Sigma^0_3$-complete |

# Bounding Parameters

- "Fundamental Theorem" of effective descriptive set theory:

$$\boldsymbol{\Sigma}^0_n = \{E \subseteq 2^\omega : \text{ exists } X \in 2^\omega \text{ such that } E \text{ is } \Sigma^0_n(X)\}.$$

  (same for $\boldsymbol{\Pi}^0_n$).

- Question: Given a $\Sigma^1_1$ set of reals, how complex is the parameter $X$ that gives a $\Pi^0_1(X)$ subset of positive measure?

# Bounding Parameters (Lebesgue)

- For Lebesgue measure: interesting connection with algorithmic randomness.

- **THM**: [Simpson, building on work by Kjos-Hanssen, Miller, and Solomon]

  For any real $X$,

  every $\Sigma^0_{\alpha+2}$ set of reals has a $\Sigma^0_2(X)$ subset of the same Lebesgue measure

  iff $\varnothing^{(\alpha)} \leq_{LR} X$

  ($Y \leq_{LR} X$ means: every $X$-random is also $Y$-random; $X$ detects at least as much non-randomness as $Y$ does.)

# Bounding Parameters (Hausdorff)

- Given a $\Sigma^1_1$ set $U$ of reals of positive Hausdorff dimension, let

$$S(U) = \{X \in 2^\omega : \text{exists } E \subseteq U,\ E \text{ is } \Pi^0_1(X),\ \dim_H(E) > 0\}.$$

- **THM:** Let $Y$ be a real.

(1) If for some $U$, every member of $S(U)$ computes $Y$, then $Y$ is hyperarithmetic.

(2) There is a $\Pi^0_2$ set $U$ such that if $Y$ is hyperarithmetic, then every member of $S(U)$ computes $Y$.

(3) If $Y$ is $\Pi^1_1$ complete, then $Y$ computes a member of $S(U)$, for any $U$.

  - In particular, we can always find an effectively closed subset of positive dimension relative to Kleene's $\mathcal{O}$.

# Beyond Analytic

- One can show (in ZFC) that there exists a set of reals of dimension 1/2 that does not have a closed subset of positive dimension.
- One proves this by constructing a universal measure zero set of positive dimension. [Zindulka]
    - Universal measure zero: does not support a continuous probability measure.
    - Any such set cannot have a closed subset of positive dimension, since such a set would be uncountable and closed and hence have a perfect subset, over which we could continuously distribute a unit mass.
- Construction is based on result by Grzegorek: There exists a u.m.z. set $Z$ such that

$$|Z| = \text{non}\,\mathcal{L}$$

least cardinality of non-measurable set