Sample Final for MATH 104

Problem 1

[Review all important definition and results of the relevant material.]

Problem 2

If the followings statements are true, answer "TRUE". If not, give a brief explanation why.

(1) If F is a field and $x, y \in F$, then $x \cdot y = 0$ implies $x = 0$ or $y = 0$.

(2) If $S, T \subseteq \mathbb{R}$ are bounded and $\sup S = \inf T$, then $S \cap T \neq \emptyset$.

(3) If (s_n) is a sequence of real numbers, and for some $k \geqslant 1$, $\lim_n (s_{n+k} - s_n) \to 0$, then (s_n) is a Cauchy sequence.

(4) The series $\sum \frac{n!}{n^n}$ converges absolutely.

(5) If a set contains no interior points, it is closed.

(6) The set of nondecreasing functions from $\mathbb{Q}$ into $\{0, 1\}$ is countable.

(7) If f is differentiable and $f(-x) = f(x)$, then $f'(-x) = -f'(x)$.

(8) If $f : [0, 1] \to [0, 1]$ is bijective, and $f(0) = 0$ and $f(1) = 1$, then f is continuous on $[0, 1]$.

(9) If $0 \leqslant f(x) \leqslant g(x)$ for all $x \in [a, b]$, and g is Riemann integrable on $[a, b]$, then f is Riemann integrable on $[a, b]$.

(10) If f and g are not differentiable at $x = 0$, then $f \cdot g$ is not differentiable at 0.

Problem 3

Show that if E is a compact subset of $\mathbb{R}$, then $\sup E$ and $\inf E$ belong to E.

Problem 4

Show that if f is differentiable on (a, b) and $f'(x) < 0$ for all $x \in (a, b)$, then f' is strictly decreasing on (a, b).

Problem 5

Show that there does not exist a sequence (p_n) of polynomials that converges uniformly to e^x on $\mathbb{R}$.

Problem 6

Suppose $0 < t < 1$. Let $s_1 = 1$ and $s_{n+1} = t(s_n + 1)$. Show that (s_n) converges (hint: bounded and monotone) and calculate $\lim_n s_n$.

Problem 7

Let $f : (a, b) \to \mathbb{R}$ be differentiable. Suppose $\lim_{x \to b} f(x) = \infty$. Show that $\lim_{x \to b} f'(x) = \infty$, provided that the limit exists.

Problem 8

Suppose that f is a continuous function on $[a, b]$ and that $f(x) \geqslant 0$ for all $x \in [a, b]$. Show that if $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.