

Ergodic  $\Leftrightarrow$

$$\overline{f \circ T(x)} = \overline{f(x)} \quad \mu\text{-a.e.} \Rightarrow f \text{ const } \mu\text{-a.e.}$$

$$\Rightarrow: \quad \overline{f \circ T(x)} = \overline{f(x)} \quad \mu\text{-a.e.}$$

WTS:  $f$  const.  $\mu$ -a.e.

Idea: "Pin down"  $f$  by smaller and smaller intervals.

Then show for a  $\mu$ -measure 1 set,  $f(x)$  must be in the same smaller and smaller intervals.

$$k \in \mathbb{Z}, n > 0$$

$$\begin{aligned} \text{Put } \underline{X_{k,n}} &= \left\{ x: \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \right\} \\ &= f^{-1} \left( \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right) \right) \end{aligned}$$

$$X = \bigcup_{k \in \mathbb{Z}} X_{k,n} \quad \text{disjoint} \quad n \text{ fixed}$$

Show:  $\int \mu \left( T^{-1} X_{k,n} \Delta X_{k,n} \right) = 0$

using assumption  $f \circ T = f$   $\mu$ -almost everywhere.

Then use ergodicity to infer sth. about  
possible meas. of  $X_{k,n}$

Then find a  $\mu$ -large set  $\bigcirc \bigcup$  on which  
 $f$  is constant (using the measure-fact

Note:  $\bigcap$  of countable family of meas. 1 sets has meas. 1  
for the  $X_{k,n}$ )

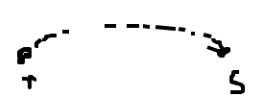
# 3

$(p, M)$  stationary Markov  
 $\uparrow$  stochastic matrix

$$Q = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} M^i \quad \text{exists}$$

i.e.  $a_{ij}$  exists for all  $i, j$ .

i steps



Use ergodic thm.  $\frac{1}{n} \sum f(T^i x) \rightarrow f^*(x)$

Problem: Bring erg thm into right form

$f$  will be  $\chi_{[i]}$   $\leftarrow$  char. function of cylinder  $[i]$

$T$  shift.

is fixed  
a.s.  $\frac{1}{n} \sum_{i=0}^{n-1} \underbrace{\chi_{[i]} \left( \frac{\cdot}{n} \right)}_{(*) \quad A_n(x)} \longrightarrow \chi_{[i]}^*$

Show.  $\int A_n(x) d\mu(x) \longrightarrow \int \chi_{[i]}^* d\mu(x)$

Dominated Convergence Thm

What does

$\int \underbrace{A_n(x) d\mu}_{\text{finite sum}} \text{ actually describe?}$

$\dots = \frac{1}{n} \sum_{i=0}^{n-1} \sum_k p_k \cdot \underline{\underline{\mu^{(i)}_{kj}}}$

Want:  $\frac{1}{n} \sum \mu^{(i)}_{kj}$  for single  $k$  | multiplication by  $\chi_{[i]}$