Lecture 3: Excursion – The Urysohn Space

Recall that a mapping $e : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is an **isometry** if

$$d_Y(f(x), f(y)) = d_X(x, y) \quad \text{for all } x, y \in X,$$

that is, an isometry is a mapping that preserves distances. f is also called an *isometric embedding* of X into Y. X and Y are **isometric** if there exists a bijective isometry between them.

It is a remarkable fact that there exists a "universal" Polish space – a complete, separable metric space that contains an *isometric copy* of any other Polish metric space.

Theorem 3.1: *There exists a Polish metric space $\mathbb{U}$ such that every Polish metric space isometrically embeds into $\mathbb{U}$.*

A concrete example of such a space is $\mathcal{C}[0,1]$, the set of all continuous real-valued functions on $[0,1]$ with the sup-metric (see exercises). But this space is not quite what we have in mind. There exists another space with a stronger, more "intrinsic" universality property, **Urysohn space**. It was constructed by Urysohn [1927].

The construction features an *amalgamation principle* that has surfaced in other areas like model theory or graph theory. It has recently attracted increased attention, which has also led to renewed interest in the Urysohn space.

Extensions of finite isometries and Urysohn universality

We first sketch the basic idea for constructing the Urysohn space. Suppose X is a Polish metric space. Let $D = \{x_1, x_2, \dots\}$ be a countable, dense subset. We first observe that it is sufficient to isometrically embed D into $\mathbb{U}$.

Lemma 3.2: *If Y is Polish, then any isometric embedding e of D into Y extends to an isometric embedding e^* of X into Y.*

Proof. Given $z \in X$, let (x_{i_n}) be a sequence in D converging to z. Since (x_{i_n}) converges, it is Cauchy. e is an isometry, and thus $y_n := e(x_{i_n})$ is Cauchy, and since Y is Polish, (y_n) converges to some $y \in Y$. Put $e^*(z) = y$. To see that this mapping is well-defined, let (x_{j_n}) be another sequence with $x_{j_n} \to z$.

Then $d(x_{i_n}, x_{j_n}) \to 0$, and hence $d(e(x_{i_n}), e(x_{j_n}) = d(y_n, e(x_{j_n})) \to 0$, implying $e(x_{j_n}) \to y$. Furthermore, suppose $w = \lim x_{k_n}$ is another point in X. Then (since a metric is a continuous mapping from $Y \times Y \to \mathbb{R}$)

$$d(e * (z), e^*(w)) = \lim d(e(x_{i_n}), e(x_{k_n})) = \lim d(x_{i_n}, x_{k_n}) = d(z, w).$$

Thus e^* is an isometry. □

In order to embed D, we can now exploit the *inductive structure* of $\mathbb{N}$ and reduce the task to *extending finite isometries*.

Suppose we have constructed an isometry e between $F_N = \{x_1, \dots, x_N\} \subset D$ and $\mathbb{U}$. We would like to extend the isometry to include x_{N+1}. For this we have to find an element $y \in \mathbb{U}$ such that for all $i \leq N$

$$d_{\mathbb{U}}(y, e(x_i)) = d_X(x_{N+1}, x_i).$$

This extension property gives rise to the following definition.

Definition 3.3: A Polish space $(\mathbb{U}, d_{\mathbb{U}})$ is **Urysohn universal** if for every finite subspace $F \subset \mathbb{U}$ and any extension $F^* = F \sqcup \{x^*\}$ with metric d^* such that

$$d^*|_{X \times X} = d_{\mathbb{U}},$$

there exists a point $u \in \mathbb{U}$ such that

$$d_{\mathbb{U}}(u, x) = d^*(x^*, x) \quad \text{for all } x \in F.$$

One can show that any two Urysohn universal spaces are isometric. We will show here that this unique (up to isometry) space actually exists, the **Urysohn space** $\mathbb{U}$.

The extension property also implies a strong intrinsic extension property for the Urysohn space itself.

Proposition 3.4: *Let $\mathcal{U}$ be a separable and complete metric space that contains an isometric image of every separable metric space. Then $\mathcal{U}$ is Urysohn universal if and only if every isometry between finite subsets of $\mathcal{U}$ extends to an isometry of $\mathcal{U}$ onto itself.*

Constructing the Urysohn space – a first approximation

We first give a construction of a space that has the extension property, but is not Polish. After that we will take additional steps to turn it into a Polish space.

The crucial idea is to observe that if X is a metric space and $x \in X$, then the mapping $f_x : X \to \mathbb{R}^{\geq 0}$ given by

$$f_x(y) = d_X(x, y)$$

is 1-*Lipschitz*. Recall that a function g between metric spaces X and Y is L**-Lipschitz**, $L > 0$ if for every $x, y \in X$,

$$d(g(x), g(y)) \leq L\, d(x, y).$$

Let $\mathrm{Lip}_1(X)$ be the set of 1-Lipschitz mappings from X to $\mathbb{R}$. We endow $\mathrm{Lip}_1(X)$ with the *supremum* metric

$$d(f, g) = \sup\{|f(x) - g(x)| : x \in X\}.$$

If $\mathrm{diam}(X) \leq \mathrm{d}$ and f, g are 1-Lipschitz, then $d(f, g)$ is indeed finite. However, we will need that the resulting space is also bounded. Let $\mathrm{Lip}_1^{\mathrm{d}}(X)$ be the space of all 1-Lipschitz functions from X to $[0, \mathrm{d}]$. Clearly, $\mathrm{diam}(\mathrm{Lip}_1^{\mathrm{d}}(X)) \leq \mathrm{d}$.

With this metric, the mapping $x \mapsto f_x(y) = d(x, y)$ becomes an isometry: We have

$$d(f_x, f_z) = \sup\{|d(x, y) - d(z, y)| : y \in X\}.$$

By the reverse triangle inequality, this is always $\leq d(x,z)$. On the other hand, setting $z = x$ yields $d(f_x, f_z) \geq d(x,z)$. This embedding of X into $\mathrm{Lip}_1^{\mathrm{d}}(X)$ is called the **Kuratowski embedding**.

We use this fact as follows: If $X^* = X \sqcup \{x^*\}$ and d^* is an extension of d_X, then f_{x^*} is an element of $\mathrm{Lip}_1^{\mathrm{d}}(X)$, and as above, for any $x \in X$

$$d(f_{x^*}, f_x) = d^*(x^*, x).$$

Hence $\mathrm{Lip}_1^{\mathrm{d}}(X)$ has an extension property of the kind we are looking for.

Iterative construction: Let X_0 be any non-empty Polish space with finite diameter $\mathrm{d} > 0$. Given X_n, let $\mathrm{d}(n) = \mathrm{diam}(X_n)$ and set $X_{n+1} = \mathrm{Lip}_1^{2\mathrm{d}(n)}(X_n)$. Finally, put $X_\infty = \bigcup_n X_n$. Note that X_∞ inherits a well-defined metric d from the X_n, which embed isometrically into it.

We claim that X_∞ has the extension property needed to be Urysohn universal. Let F be a finite subset of X_∞. There exists N such that $F \subset X_N$. Suppose $F^* = F \sqcup \{x^*\}$ and d^* is an extension of d to F^*. Let $\mathrm{d}^* = \mathrm{diam}(F^*)$. Note that $\mathrm{diam}(X_n) = 2^n\mathrm{d}$. Choose M so that $M \geq N$ and $\mathrm{diam}(X_M) \geq \mathrm{d}^*$. The next lemma ensures that we can find $f \in X_{M+1}$ such that $f(x) = d^*(x^*, x)$ for all $x \in F$.

Lemma 3.5 (McShane-Whitney)**:** *Let X be a metric space with* $\mathrm{diam}(X) \leq \mathrm{d}$*, $A \subseteq X$, and $f \in \mathrm{Lip}_1^{\mathrm{d}}(A)$, then f can be extended to an* 1*-Lipschitz function f^* on all of X such that*

$$f^*|_A = f \quad \text{and} \quad f^* \in \mathrm{Lip}_1^{2\mathrm{d}}(X).$$

Proof. For each $a \in A$ define $f_a : X \to \mathbb{R}$ as

$$f_a(x) = f(a) + d(a, x).$$

Then f_a is 1-Lipschitz, by the reverse triangle inequality. Let

$$f^*(x) = \inf\{f_a(x) : a \in A\}.$$

Then $f^*(a) = f(a)$ for all $a \in A$. Let $x, y \in X$ and $\varepsilon > 0$. Wlog assume $f^*(y) \geq f^*(x)$. Pick $a \in A$ so that $f_a(x) \leq f^*(x) + \varepsilon$. Then

$$\begin{aligned}|f^*(x) - f^*(y)| = f^*(y) - f^*(x) &\leq f^*(y) - f_a(x) + \varepsilon \\ &\leq f_a(y) - f_a(x) + \varepsilon \leq Ld(x, y) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we have $|f^*(x) - f^*(y)| \leq Ld(x, y)$.

Finally, we have $f(a) \leq f_a(x) \leq f(a) + \mathrm{d}$ and thus $0 \leq f^*(x) \leq f_a(x) \leq 2\mathrm{d}$. $\square$

Mending the construction

The set X_∞ we constructed has two deficiencies with respect to our goal of constructing a Urysohn universal space: X_∞ is not necessarily separable, and X_∞ is not necessarily complete.

To make X_∞ separable, we observe that if X is compact, then the set $\mathrm{Lip}_1^{\mathrm{d}}(X)$ is closed in $\mathcal{C}(X)$ (the set of all real-valued continuous functions on X), bounded, and *equicontinuous*. By the *Arzelà-Ascoli Theorem*, $\mathrm{Lip}_1^{\mathrm{d}}(X)$ is compact. Every compact metric space is separable: For every $\varepsilon > 0$, there exists a finite covering of the space with sets of diam $< \varepsilon$. Letting ε traverse all positive rationals and picking a point from each set in an ε-covering yields a countable dense subset.

Hence if we start with X_0 compact, each X_n will be compact, too. A countable union of separable spaces is separable, thus X_∞ is separable.

To obtain a complete space, we can pass from X_∞ to its *completion* $\overline{X_\infty}$. First note that if a metric space X is separable, so is its completion $\overline{X}$. However, we also have to ensure that $\overline{X_\infty}$ retains the universality property of X_∞.

Lemma 3.6: *If a complete metric space (Y, d) admits a dense Urysohn universal subspace $\mathcal{U}$, then Y is Urysohn universal.*

Proof. We follow Gromov [1999]. Let $F = \{x_1, \ldots, x_n\} \subset Y$ and assume $F^* = F \sqcup \{x^*\}$ is an extension with metric d^*.

We first note that Y is *aproximately universal*. This means that for any $\varepsilon > 0$, there exists a point $y^* \in Y$ such that

$$|d(y*, x) - d^*(x^*, x)| < \varepsilon \quad \text{for all } x \in F. \qquad (*)$$

This can be seen as follows. Since $\mathcal{U}$ is dense in Y, we can find a finite set $F_\varepsilon = \{z_1, \ldots, z_n\} \subset \mathcal{U}$ such that

$$d(x_i, z_i) < \varepsilon \quad \text{for } 1 \leq i \leq n.$$

To keep the proof technically simple, wlog we assume ε is much smaller than the individual distances between the x_i. Consider the extension $F^*_\varepsilon = F_\varepsilon \sqcup \{x^*\}$ with metric

$$e^*(x^*, z_i) = d^*(x^*, x_i) + d(x_i, z_i).$$

Since $\mathcal{U}$ has the finite extension property, we can find $y^* \in \mathcal{U}$ such that

$$d(y^*, z_i) = e^*(x^*, z_i)$$

Hence

$$\begin{aligned} |d(y^*, x_i) - d^*(x^*, x_i)| &= |e^*(x^*, z_i) - d^*(x^*, x_i)| \\ &= |d^*(x^*, x_i) + d(x_i, z_i) - d^*(x^*, x_i)| < \varepsilon. \end{aligned}$$

We use this approximate universality to construct a Cauchy sequence (y_k) in Y of 'approximate' extension points that satisfy $(*)$ for smaller and smaller ε.

Let $0 < \delta = \max\{d^*(x^*, x_i) \colon 1 \leq i \leq n\}$. The formal requirements for the sequence (y_i) are as follows.

(i) $|d(y_k, x_i) - d^*(x^*, x_i)| \leq 2^{-k}\delta$.

(ii) $d(x_{k+1}, x_k) \le 2^{-k}\delta$.

The sequence necessarily converges in Y and the limit point must be a true extension point, due to (i).

Suppose we have already constructed $y_1, \dots, y_k$ satisfying (i), (ii). Add an (abstract) point y^*_{k+1} to $F_k = F \cup \{y_1, \dots, y_k\}$. Let $F^*_{k+1} = F_k \sqcup \{y^*_{k+1}\}$.

We want to use approximate universality on F^*_{k+1}. To this end we have to define a metric e^* on F^*_{k+1} that has the following properties

$$e^*|_{F_k} = d|_{F_k} \tag{+}$$
$$e^*(y^*_{k+1}, x_i) = d^*(x^*, x_i) \quad (1 \le i \le n) \tag{++}$$
$$e^*(y^*_{k+1}, y_k) = 2^{-k-1}\delta \tag{+++}$$

Indeed such a metric exists: The condition (+) already defines a metric on the set F_k. (+)-(+++) also define a metric on $F \cup \{y_k, y^*_{k+1}\}$. The only thing left to check for this is the triangle inequality for y_k, y^*_{k+1}.

$$|e^*(x_i, y_k) - e^*(y^*_{k+1}, x_i)| = |d(x_i, y_k) - d^*(x^*, x_i)| \le 2^{-k}\delta = e^*(y_k, y^*_{k+1}),$$

by (i). These metrics agree on the set

$$F_k \cap (F \cup \{y_k, y^*_{k+1}\}) = F \cup \{y_k\}.$$

Therefore, we can "merge" them to a metric on all of F^*_{k+1} by letting

$$e^*(y^*_{k+1}, y_j) = \inf\{e^*(y^*_{k+1}, z) + e^*(z, y_j) \colon z \in \{y_1, \dots, y_{k-1}\}\}.$$

Now choose $\varepsilon < 2^{-k-1}\delta$ and apply approximate universality to F^*_{k+1}. This yields a point $y_{k+1} \in Y$ such that

$$|d(y_{k+1}, z) - e^*(y^*_{k+1}, z)| < 2^{-k-1}\delta$$

for all $z \in F_k$. By definition of e^*, we have

$$|d(y_{k+1}, x_i) - d^*(y^*_{k+1}, z)| < 2^{-k-1}\delta$$

for $1 \le i \le n$, and (+++) yields

$$d(y_{k+1}, y_k) < e^*(y^*_{k+1}, y_k) + \varepsilon \le 2^{-k-1}\delta + 2^{-k-1}\delta = 2^{-k}\delta$$

as required. □