Homework 4 for MATH 104

Due: Tuesday, October 10, 9:30am in class

Problem 1

Determine if the following series converge. Justify your answer.

(a) $\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})$,

(b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$,

(c) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$, where x is an arbitrary real number.

(d) $\sum_{n=1}^{\infty} \frac{\sin(n\pi/6)}{n}$.

Problem 2

Find an absolutely convergent series $\sum_n a_n$ such that

$$\limsup_n \frac{a_{n+1}}{a_n} = \infty.$$

Justify your answer.

Problem 3

Assume $\sum_n a_n$ converges and $a_n \geqslant 0$ for all $n \in \mathbb{N}$. Show that

$$\sum_n \frac{\sqrt{a_n}}{n}$$

converges.

Problem 4

Suppose $a_n > 0$ for all n, and suppose $\sum a_n$ converges. Set

$$r_n = \sum_{k=n}^{\infty} a_k.$$

(a) Prove that if $m < n$, then

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}.$$

Deduce that $\sum \frac{a_n}{r_n}$ diverges.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}).$$

Deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.