Review Worksheet for **MATH 574**

Problem 1

Show that for any $\xi < \omega_1$, $\mathbf{\Sigma}^0_\xi$ is closed under countable unions, finite intersections, and projections along $\mathbb{N}$.

(*Hint:* Proceed by induction along Ord.)

Problem 2

Show that for a function $f : X \to Y$ between Polish spaces, the following are equivalent:

(i) f is Borel,

(ii) the graph of f is Borel,

(iii) the graph of f is analytic.

Problem 3

Show that for two disjoint closed subsets $A, B \subseteq \mathbb{N}^\mathbb{N}$, there exists a clopen $C \subseteq \mathbb{N}^\mathbb{N}$ such that

$$A \subseteq C \qquad \text{and} \qquad B \cap C = \emptyset.$$

Is the same true for open sets?

Remark: We have shown a similar property for analytic sets (the Lusin Separation Theorem). What can we say about separability for other Borel classes?

Problem 4

The Perfect Subset Property for Analytic Sets: Show that if $A \subseteq \mathbb{N}^\mathbb{N}$ is analytic and uncountable, then it contains a perfect subset.

Hint: Since A is analytic, there exists a continuous mapping $f : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ such that $A = f(\mathbb{N}^\mathbb{N})$. Construct an embedding of $2^\mathbb{N}$ into A. Show that we can find two disjoint open sets U_0, U_1 whose intersection with $A = f(\mathbb{N}^\mathbb{N})$ is uncountable. The preimages of the U_i are disjoint open subsets with uncountable images. Show that this process can be continued and defines in the limit an injection of $2^\mathbb{N}$ into A.

Problem 5

Show that the classes of the projective hierarchy $\mathbf{\Sigma}^1_n, \mathbf{\Pi}^1_n$ are closed under the Souslin operation for $n \geqslant 2$.

Problem 6

Show that if $\xi \geqslant \omega$ and $X \subseteq \xi$ is constructible, then $X \in L_\eta$, where η is the least cardinal greater than ξ.

Problem 7

Define the relation IS_L on $\mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$ by

$$\mathrm{IS}_L(\alpha, \beta) \qquad \Leftrightarrow \qquad \{(\alpha)_n : n \in \mathbb{N}\} = \{\gamma \in \mathbb{N}^\mathbb{N} : \gamma <_L \beta\},$$

where $(\alpha)_n$ is the nth column of α. Show that IS_L is Σ^1_2.

Problem 8

Let $A \subseteq \mathbb{N}^\mathbb{N}$ be Δ^1_1. Show that there exists a computable function $\pi : \mathbb{N}^\mathbb{N} \to \mathbb{N}^\mathbb{N}$ and a Π^0_1 set P such that $\pi(P) = A$.

Hint: Fix a computable Wadge-reduction form A to WOrd. Then the range of f is bounded by some ordinal $\xi < \omega_1^{CK}$. We can express membership $\alpha \in A$ as "there exists an order preserving mapping from $E_{f(\alpha)}$ onto an initial segment of ξ". We can bring this statement in normal form and obtain a suitable Π^0_1 predicate from it.

Problem 9

Let

$$\delta^1_1 = \sup\{\|\alpha\| : \alpha \in \text{WOrd and the set } \{\langle m, n\rangle : \alpha(m) = n\} \text{ is } \Delta^1_1\}.$$

Show that $\delta^1_1 = \omega_1^{CK}$.

Hint: Show that if a some Δ^1_1 $\beta \in$ WOrd were "unreachable" by recursive well-orderings, then by boundedness one could show that some properly Π^1_1 set is Δ^1_1.

Problem 10

A *path* in $\mathcal{O}$ is a subset of $\mathcal{O}$ that is linearly ordered by $<_O$ and is closed downward under $<_O$. A path Z can be extended if there exists $x \in O$ such that $\forall z \in Z$, $z <_O x$. Show that there exists a path of order type $< \omega_1^{CK}$ which cannot be extended.

Problem 11

Show that for any $\xi < \omega_1$ there exists a tree T with $\|T\|_{CB} = \xi$.

Problem 12

An alternative way to define the H-sets is

$$\begin{aligned} H^*_1 &= 0, \\ H^*_{2^x} &= (H_x)', \\ H^*_{3\cdot 5^x} &= \{\langle n, m\rangle : m \in H_{\varphi_x(n)}\}. \end{aligned}$$

Show that for any $x \in \mathcal{O}$, $H_x \equiv_T H^*_x$.