Lesson 3

Dynamical Systems

Math 574 - Topics in Logic
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Jan Reimann

PENNSTATE
Eberly College of Science

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Subshifts

Topological Dynamical System

$$(X, T)$$

compact metric space

Continuous map

$T : X \to X$

Shift system: $X = A^{\mathbb{N}}$ A finite

$T : A^{\mathbb{N}} \to A^{\mathbb{N}}$ shift map

$T : x \longmapsto T(x)$

$$\boxed{T(x)_n = x_{n+1}}$$

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$$x = x_0 x_1 x_2 \dots$$

$$Tx = x_1 x_2 x_3 \dots$$

Two-sided shift: $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$

$$T(x)_n = x_{n+1} \qquad n \in \mathbb{Z}$$

Verify: T is continuous

$$\dots x_{-1} x_0 x_1 x_2 x_3 \dots$$

$(A^{\mathbb{N}}, T)$ Full shift (one-sided)

$(A^{\mathbb{Z}}, T)$ Full two-sided shift

Subshift: $S \subseteq A^{\mathbb{N}}$ $(A^{\mathbb{Z}})$ s.t.

1) S closed (topologically)

2) one sided: $T(S) \subseteq S$ (closed under shift)

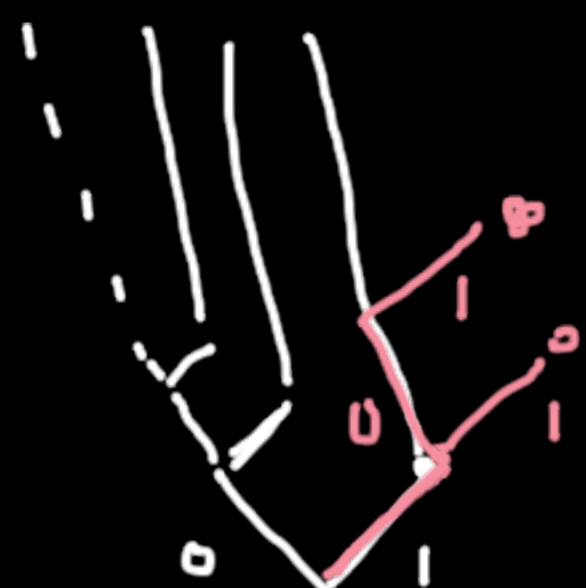
two sided: $T(S) = S$ (shift-invariant)

EXA: $S = A^{\mathbb{N}}$

$S = \{0^\infty\}$

$S = \{0101010\ldots,\ 1010101\ldots\}$

$S = \{x : x \text{ contains at most one } 1\}$



forbidden substring

$11,\ 101,\ \ldots$

Prop: $S \subseteq A^{\mathbb{N}}$ is a subshift

Exercise

iff there exists $\underline{W} \subseteq A^{<\mathbb{N}}$ s.t.
forbidden words

$S = \{ x : \text{ no } w \in W \text{ is a } \underline{\text{substring}} \text{ of } x \}$

w substring of x $:\Longleftrightarrow$

$\exists k \quad x_k \ldots x_{k+|w|-1} = w$

Same characterization holds for two-sided shifts

Ex: Given $W \subseteq A^{<\mathbb{N}}$, write

$$S(W) = \{x : x \text{ has no substring in } W\}$$

factors

- $W = \{11, 101, 1001, \ldots\}$

  $S(W) = \{x : x \text{ has at most one } 1\}$

- $W = \{10^{2n+1}1 : n \geq 0\}$

  $S(W) = \{x : \text{between two } 1\text{'s is an an even } \# \text{ of } 0\text{'s}\}$

even shift

Classes of Subshifts

- classified via complexity of set W (forbidden words)

- Shifts of finite type (SFT)

  W finite

- Sofic shifts

  W regular + additional condition