Lecture 12: Analytic Sets

Definition 12.1: A subset A of a Polish space X is **analytic** if it is empty or there exists a continuous function $f : \mathbb{N}^{\mathbb{N}} \to X$ such that $f(\mathbb{N}^{\mathbb{N}}) = A$.

We will later see that the analytic sets correspond to the sets definable by means of Σ^1_1 formulas, that is formulas in the language of second order arithmetic that have *one existential function quantifier*. Therefore, we will denote the analytic subsets of X also by

$$\boldsymbol{\Sigma}^1_1(X).$$

Here are some simple properties of analytic sets.

Proposition 12.2:

(i) Every Borel set is analytic.

(ii) A continuous image of analytic set is analytic.

(iii) Countable unions of analytic sets are analytic.

Proof. (i) This follows directly from Corollary 11.3.

(ii) The composition of continuous mappings is continuous.

(iii) Let A_n be analytic and $f_n : \mathbb{N}^{\mathbb{N}} \to X$ such that $f_n(\mathbb{N}^{\mathbb{N}}) = A_n$. Define $f : \mathbb{N}^{\mathbb{N}} \to X$ by

$$f(m, \alpha) = f_n(\alpha).$$

Then f is continuous and $f(\mathbb{N}^{\mathbb{N}}) = \bigcup_n A_n$. $\square$

We can use our previous results about Borel sets to give various equivalent characterizations of analytic sets.

Proposition 12.3: *For a subset A of a Polish space X, the following are equivalent.*

(i) *A is analytic,*

(ii) *A is empty or there exists a Polish space Y and a continuous $f : Y \to X$ such that $f(Y) = A$,*

(iii) *A is empty or there exists a Polish space Y, a Borel set $B \subseteq Y$ and a continuous $f : Y \to X$ such that $f(B) = A$.*

(iv) *A is the* projection *of a closed set $F \subseteq \mathbb{N}^{\mathbb{N}} \times X$ along $\mathbb{N}^{\mathbb{N}}$,*

(v) *A is the projection of a $\mathbf{\Pi}^0_2$ set $G \subseteq 2^{\mathbb{N}} \times X$ along $2^{\mathbb{N}}$,*

(vi) *A is the projection of a Borel set $B \subseteq X \times Y$ along Y, for some Polish space Y.*

Proof. (i) $\Leftrightarrow$ (ii): Follows from Theorem 2.6 and Proposition 12.2 (ii).

(ii) $\Leftrightarrow$ (iii): Follows from Corollary 11.3 and Proposition 12.2 (ii).

(i) $\Rightarrow$ (iv): Let $f : \mathbb{N}^{\mathbb{N}} \to X$ be continuous, $f(\mathbb{N}^{\mathbb{N}}) = A$. Then

$$x \in A \quad \Leftrightarrow \quad \exists \alpha \, (\alpha, x) \in \mathrm{Graph}(f),$$

hence A is the projection of the closed set Graph(f) along $\mathbb{N}^{\mathbb{N}}$.

(iv) $\Rightarrow$ (iii): Clear, since projections are continuous.

(iv) $\Rightarrow$ (v): $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to a $\mathbf{\Pi}^0_2$ subset of $2^{\mathbb{N}}$. (Exercise!)

(v) $\Rightarrow$ (vi), (vi) $\Rightarrow$ (iii): Obvious. □

The Lusin Separation Theorem

In a course on computability theory one learns that there are *effectively inseparable* disjoint r.e. sets. i.e. disjoint r.e. sets $W, Z \subseteq \mathbb{N}$ for which no recursive set A exists with $W \subseteq A$ and $A \cap Z = \emptyset$.

In contrast to this, disjoint analytic sets can always be separated by a Borel set, they are **Borel separable**.

Theorem 12.4 (Lusin)**:** *Let $A, B \subseteq X$ be disjoint analytic sets. Then there exists a Borel $C \subseteq X$ such that*

$$A \subseteq C \quad \textit{and} \quad B \cap C = \emptyset,$$

Proof. Let $f : \mathbb{N}^{\mathbb{N}} \to A$ and $g : \mathbb{N}^{\mathbb{N}} \to B$ be continuous surjections.

We argue by contradiction. The key idea is: if A and B are Borel inseparable, then, for some $i, j \in \mathbb{N}$, $A_{\langle i \rangle} = f(N_{\langle i \rangle})$ and $B_{\langle j \rangle} = g(N_{\langle j \rangle})$ are Borel inseparable.

This follows from the observation

($*$) if the sets $R_{m,n}$ separate the sets P_m, Q_n (for each m, n), then $R = \bigcup_m \bigcap_n R_{m,n}$ separates the sets $P = \bigcup_m P_m$, $Q = \bigcup_n Q_n$.

So, by using (∗) repeatedly, we can construct sequences $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$ such that for all n, $A_{\alpha|n}$ and $B_{\beta|n}$ are Borel inseparable, where

$$A_\sigma = f(N_\sigma) \quad \text{and} \quad B_\sigma = g(N_\sigma).$$

Then we have $f(\alpha) \in A$ and $g(\beta) \in B$, and since A and B are disjoint, $f(\alpha) \neq g(\beta)$. Let U, V be disjoint open sets such that $f(\alpha) \in U$, $g(\beta) \in V$. Since f and g are continuous, there exists N such that $f(N_{\alpha|N}) \subseteq U$, $g(N_{\beta|N}) \subseteq V$, hence U separates $A_{\alpha|N}$ and $B_{\beta|N}$, contradiction. □

The Separation Theorem yields a nice characterization of the Borel sets.

Theorem 12.5 (Souslin): *If a set A and its complement $\neg A$ are both analytic, then A is Borel.*

Proof. In Theorem 12.4, chose $A_0 = A$ and $A_1 = \neg A$. □

Sets whose complement is analytic are called **co-analytic**. Analogous to the levels of the Borel hierarchy, the co-analytic subsets of a Polish space X are denoted by

$$\mathbf{\Pi}^1_1(X).$$

If we define, again analogy to the Borel hierarchy,

$$\mathbf{\Delta}^1_1(X) = \mathbf{\Sigma}^1_1(X) \cap \mathbf{\Pi}^1_1(X),$$

then Souslin's Theorem states that

$$\text{Borel}(X) = \mathbf{\Delta}^1_1(X).$$

The Souslin operation

Souslin schemes give an alternative presentation of analytic sets which will be useful later.

Definition 12.6: A **Souslin scheme** on a Polish space X is a family $P = (P_\sigma)_{\sigma \in \mathbb{N}^{<\mathbb{N}}}$ of subsets of X indexed by $\mathbb{N}^{<\mathbb{N}}$.

The **Souslin operation** $\mathcal{A}$ for a Souslin scheme is given by

$$\mathcal{A}P = \bigcup_{\alpha \in \mathbb{N}^{\mathbb{N}}} \bigcap_{n \in \mathbb{N}} P_{\alpha|n}.$$

This means

$$x \in \mathcal{A}P \quad \Longleftrightarrow \quad \exists \alpha \in \mathbb{N}^{\mathbb{N}}\ \forall n \in \mathbb{N}\ x \in P_{\alpha\restriction n}. \tag{∗}$$

The analytic sets are precisely the sets that can be obtained by Souslin operations on closed sets. If a Γ is a class of sets in various Polish spaces, we let

$$\mathcal{A}\Gamma = \{\mathcal{A}P\colon\ P = (P_\sigma) \text{ is a Souslin scheme with } P_\sigma \in \Gamma \text{ for all } \sigma\}.$$

Theorem 12.7:

$$\mathbf{\Sigma}^1_1(X) \ = \ \mathcal{A}\,\mathbf{\Pi}^0_1(X).$$

Proof. Suppose $f : \mathbb{N}^{\mathbb{N}} \to X$ is continuous with $f(\mathbb{N}^{\mathbb{N}}) = A$. Then

$$x \in A \quad \Longleftrightarrow \quad \exists \alpha \in \mathbb{N}^{\mathbb{N}}\ \forall n \in \mathbb{N}\ x \in \overline{f(N_{\alpha|n})}.$$

Hence if we let $P_\sigma = \overline{f(N_\sigma)}$, then

$$A = \mathcal{A}\,P,$$

for the Souslin scheme $P = (P_\sigma)$.

To see that any set A in $\mathcal{A}\,\mathbf{\Pi}^0_1(X)$ is analytic, consider $(*)$. If the P_σ are closed, the condition

$$(\alpha, x) \in F \quad \Longleftrightarrow \quad \forall n \in \mathbb{N}\ x \in P_{\alpha\restriction n}$$

defines a closed subset of $\mathbb{N}^{\mathbb{N}} \times X$ such that A is the projection of F along $\mathbb{N}^{\mathbb{N}}$. $\square$

Note that the Souslin scheme (P_σ) used in the previous proof has the additional property that

$$\sigma \subseteq \tau \quad \Rightarrow \quad P_\sigma \supseteq P_\tau.$$

Such Souslin schemes are called **regular**.