Homework 1 for MATH 104 – Solutions

Due: Tuesday, September 12, 9:40am in class

Problem 1

Determine whether the following sets are bounded (from below, above, or both). If so, determine their infimum and/or supremum and find out whether these infima/suprema are actually minima/maxima.

(1) $S_1 = \{\ 1 + (-1)^n : n \in \mathbb{N}\}$;

Solution. For even n, we have $1 + (-1)^n = 2$, for odd n, we have $1 + (-1)^n = 0$. Therefore, $S_1 = \{0, 2\}$, and hence the set is finite, and hence bounded, with $\min S_1 = 0$ and $\max S_1 = 2$. ■

(2) $S_2 = \{\frac{1}{m} + \frac{1}{n} : m, n \in \mathbb{N}\}$;

Solution. Since $\frac{1}{n}$ is positive whenever n is positive, it follows that S_2 is bounded from below, 0 being a lower bound. We claim that the same holds for the set S_2. Suppose $\inf S_2 > 0$. Since the sequence $\frac{1}{n}$ convereges to 0, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \frac{\sup S_2}{2}$. Then $\frac{1}{n} + \frac{1}{n} \in S_2$, and $\frac{1}{n} + \frac{1}{n} < \sup S_2$, a contradiction. 0 is an infimum which is not contained in the set, so S_2 does not have a minimum.

The set S_2 is also bounded from above, since the sequence $(\frac{1}{n})$ is decreasing. 2 is an upper bound, which is also a maximum. ■

(3) $S_3 = \{x \in \mathbb{R} : x^2 + x + 1 \geqslant 0\}$;

Solution. We claim that $S_3 = \mathbb{R}$. To prove this, note that $x^2 + x + 1 = (x + \frac{1}{2})^2 + \frac{3}{4}$. Hence $x \in S_3$ iff $(x + \frac{1}{2})^2 + \frac{3}{4} \geqslant 0$. But $(x + \frac{1}{2})^2$ is always nonnegative, and $\frac{3}{4} > 0$, so this holds for any x. Therefore, S_3 is neither bounded from below nor from above. By the convention concerning $\infty, -\infty$, we have $\inf S_3 = -\infty$, and $\sup S_3 = \infty$. ■

(4) $S_4 = \{\cos(\frac{n\pi}{3}) : n \in \mathbb{N}\}$.

Solution. It is known from calculus (we may pove this formally later) that for all $x \in \mathbb{R}$, $-1 \leqslant \cos(x) \leqslant 1$. Therefore S_4 is bounded from below by -1, and from above by 1. Furthermore, for $n = 0$ we have $\cos(\frac{0\pi}{3}) = \cos(0) = 1$, and for $n = 3$ we have $\cos(\frac{3\pi}{3}) = \cos(\pi) = -1$. Therefore, $\inf S_4 = \min S_4 = -1$, and $\sup S_4 = \max s_4 = 1$. ■

Problem 2

Prove that in any ordered field F, the following hold:

(1) $0 < 1$;

Solution. If $1 \leqslant 0$, then $0 \leqslant -1$ by Theorem 3.2 (i). It follows from Theorem 3.2 (iv) that $0 \leqslant (-1)^2$. Theorem 3.1 (iv) yields $(-1)^2 = 1^2 = 1$. Hence $0 = 1$, in contradiction to the property of F being a field (which means that 0 and 1 must be distinct). ■

(2) if $0 < a < b$, then $0 < b^{-1} < a^{-1}$ for $a, b \in F$.

Solution. Suppose $0 < a < b$. It follows from Theorem 3.2 (vi) that $a^{-1}, b^{-1} > 0$. Hence we can use (O5) to infer

$$a < b \Rightarrow aa^{-1} \leqslant ba^{-1} \Rightarrow b^{-1} \leqslant a^{-1}bb^{-1} \Rightarrow b^{-1} \leqslant a^{-1}.$$

It remains to show that $b^{-1} \neq a^{-1}$. If $b^{-1} = a^{-1}$, we can infer

$$b^{-1} = a^{-1} \Rightarrow b^{-1}a = a^{-1}a \Rightarrow b^{-1}a = 1 \Rightarrow ab^{-1}b = b \Rightarrow a = b,$$

contradicting $a < b$. ■

Problem 3

Let A and B sets of real numbers such that

(i) $A \cup B = \mathbb{R}$,

(ii) if a is in A and b is in B, then $a < b$,

(iii) A contains no largest element (maximum).

Prove that B contains a smallest element (minimum).

Solution. It follows from (ii) that every $a \in A$ is a lower bound for the set B. In particular, B is bounded from below, and by completeness of $\mathbb{R}$ there exists a real number $b_0 = \inf B$.

We have to show that $b_0 \in B$. Suppose $b_0 \notin B$. By (i), it follows that $b_0 \in A$. Since A does not have a maximal element (iii), there exists some $a_0 \in A$ with $b_0 < a_0$. But now (ii) implies that a_0 is a lower bound for B, in contradiction to $b_0 = \inf B$. ■

Problem 4

Let A and B nonempty sets of reals which are both bounded from above. Define the set $A + B$ as

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

Show that $\sup A + B = \sup A + \sup B$.

Solution. We first show that $A + B$ is bounded from above. We claim that $\sup A + \sup B$ is an upper bound on $A + B$. Let $c \in A + B$, i.e. $c = a + b$ for some $a \in A$, $b \in B$. Then $a \leqslant \sup A$ and $b \leqslant \sup B$ and hence $c = a + b \leqslant \sup A + \sup B$. It follows from this that $\sup A + B \leqslant \sup A + \sup B$.

Suppose that $\sup A + B < \sup A + \sup B$. By the density of the rational numbers, we can choose some (rational) r such that $\sup A + B < r < \sup A + \sup B$. This implies that $r - \sup A < \sup B$. It follows from the definition of sup that there must exist some $b \in B$ such that $r - \sup A < b \leqslant \sup B$. (Otherwise, $r - \sup A$ would be an upper bound on B less than $\sup B$.) It follows that $r - b < \sup A$. The same reasoning as before yields the existence of some $a \in A$ with $r - b < a \leqslant \sup A$. Hence we have $r < a + b$ with $a \in A$, $b \in B$. But this contradicts $\sup A + B < r$, which implies that for all $a \in A$, $b \in B$.

■