Kolmogorov Complexity and Diophantine Approximation

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  - An outcome of an (unbiased) coin toss has high complexity. (There is no shorter description than the sequence itself.)

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- Rigorous Formulation: Kolmogorov complexity.

Kolmogorov Complexity

- **Kolmogorov complexity:** U a universal Turing-machine. Def. for a binary string $\sigma \in \{0,1\}^*$,

$$C(\sigma) = C_U(\sigma) = \min\{|p| : p \in \{0,1\}^*, U(p) = \sigma\},$$

i.e. $C(\sigma)$ is the length of the shortest program (for U) that outputs σ.

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  i.e. $C(\sigma)$ is the length of the shortest program (for U) that outputs σ.
- Kolmogorov's invariance theorem: C is independent of U (up to a constant).
- The pigeonhole principle yields that for any length there are incompressible strings, $C(\sigma) \geq |\sigma|$ (in fact, most of them are).

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- Therefore, N can be described by a program of length $O(n \log \log N)$. (Identify natural numbers with their binary representation).
- Yields a contradiction if N is incompressible ($C(N) \geq \log N$).

Kolmogorov complexity and Coding

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- Kraft-Chaitin Theorem: $\{\sigma_i\}_{i\in\mathbb{N}}$ set of strings, $\{l_i, l_2, \ldots\}$ sequence of natural numbers ('lengths') such that

$$\sum_{i\in\mathbb{N}} 2^{-l_i} \leq 1,$$

then one can construct (primitive recursively) a prefix-free TM M and strings $\{\tau_i\}_{i\in\mathbb{N}}$, such that

$$|\tau_i| = l_i \quad \text{and} \quad M(\tau_i) = \sigma_i.$$

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- Identify randomness with incompressibility: Say an infinite binary sequence ξ is random if for all n, $K(\xi\restriction_n) \geq n - c$, for some constant c.

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- Such a series of rationals is obtained by the continued fraction expansion:

$$\alpha = [a_0, a_1, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}} \qquad a_i \in \mathbb{N} \tag{6}$$

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- The expansion is finite if and only if α is rational.

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- A number β is badly approximable if there exists a K such that
$$\forall p/q \in \mathbb{Q}\, |\beta - p/q| \geq K/q^2.$$
- Examples of badly approximable numbers:
  - Golden mean $(1+\sqrt{5})/2 = [1,1,1,1,\ldots]$. $0 < K < \sqrt{5}$.
  - $\sqrt{2}/2 = [1,2,2,2,\ldots]$, in fact all irrational square roots.
  - $e \mod 1 = [1,2,1,1,4,1,1,6,\ldots]$ not badly approximable.
  - $\pi \mod 1 = [7,15,1,292,1,1,\ldots]$???

Diophantine Approximation

- Algebraic numbers are close to badly approximable:
  Roth's Theorem: For any algebraic α, for any $\varepsilon > 0$,

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  has only finitely many solutions.
- However, there are numbers which are very well approximable.
  A Liouville number is an irrational α for which

$$\forall n \exists \frac{p}{q} \left| \alpha - \frac{p}{q} \right| \leq \frac{1}{q^n}.$$

  Example: $\sum 10^{-n!}$.

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  Khintchine's Theorem: Let $\psi : \mathbb{N} \to \mathbb{R}^+$ be such that $\lim_n \psi(n) = 0$. Define

  $$W_\psi = \{\alpha : \overset{\infty}{\exists}(p/q)\, |\alpha - (p/q)| < \psi(q)\}.$$

  Then it holds, for Lebesgue measure λ,

  $$\lambda W_\psi = \begin{cases} 0, & \text{if } \sum k\psi(k) < \infty, \\ 1, & \text{if } \sum k\psi(k) = \infty. \end{cases}$$

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- So, well-approximable numbers are rather rare. Can we tell how rare?

Hausdorff Measures

- Caratheodory-Hausdorff construction on metric spaces: let $A \subseteq X$, X some seperable metric space, $h : \mathbb{R} \to \mathbb{R}$ a monotone, increasing, continous on the right function with $h(0) = 0$, and let $\delta > 0$.

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- Define a set function

$$\mathcal{H}^h_\delta(A) = \text{inf} \left\{ \sum_i h(\text{diam}(U_i)) : A \subseteq \bigcup_i U_i, \ \text{diam}(U_i) \leq \delta \right\}.$$

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- The h-dimensional Hausdorff measure $\mathcal{H}^h$ is defined as

$$\mathcal{H}^h(A) = \lim_{\delta \to 0} \mathcal{H}^h_\delta(A)$$

Properties of Hausdorff Measures

- $\mathcal{H}^h$ is Borel regular:
  all Borel sets are measurable and for every $Y \subseteq X$ there is a Borel set $B \subseteq Y$ such that $\mathcal{H}^h(B) = \mathcal{H}^h(Y)$.

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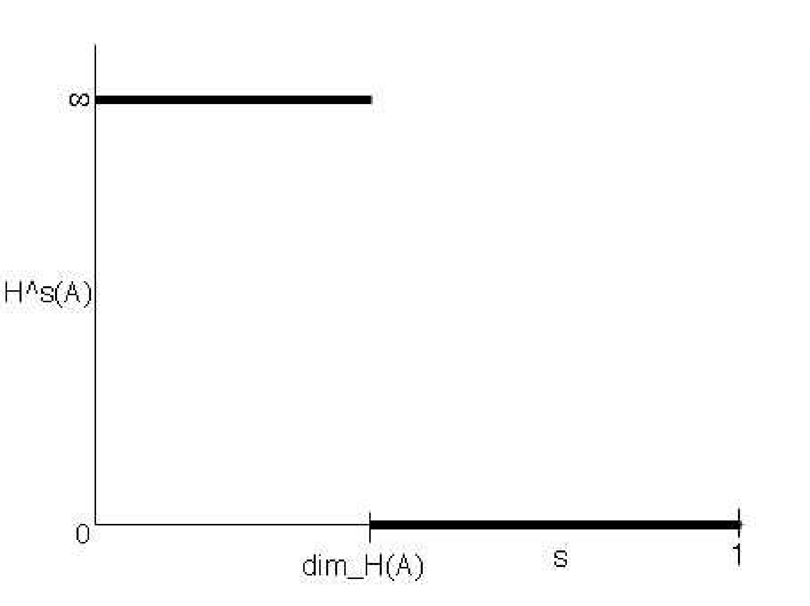
- For $s = 1$, $\mathcal{H}^1$ is the usual Lebesgue measure λ on$2^{\mathbb{N}}$.
- For $0 \leq s < t < \infty$ and $Y \subseteq X$,

$$\mathcal{H}^s(Y) < \infty \text{ implies } \mathcal{H}^t(Y) = 0,$$

$$\mathcal{H}^t(Y) > 0 \text{ implies } \mathcal{H}^s(Y) = \infty.$$

Hausdorff dimension

- The situation is depicted in the following graph:



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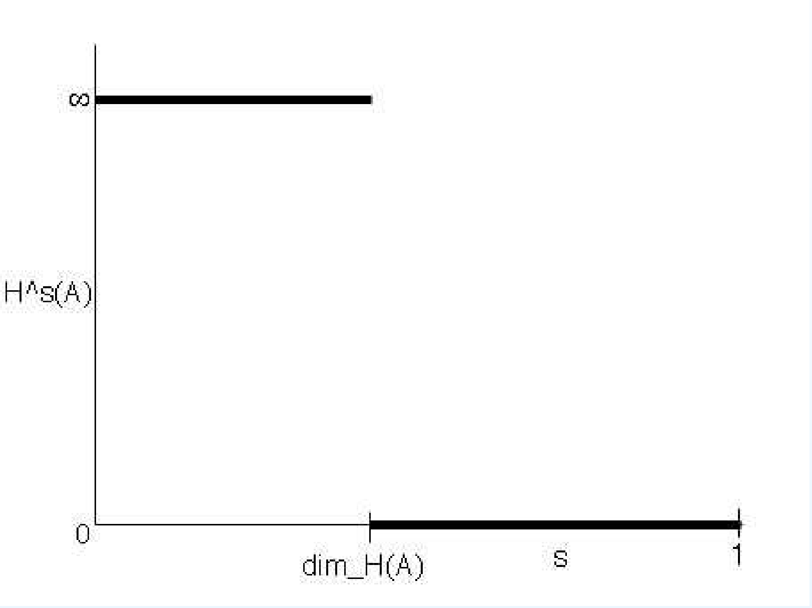

- The Hausdorff dimension of A is

$$\begin{aligned} \dim_{\mathrm{H}}(A) &= \inf\{s \geq 0 : \mathcal{H}^s(A) = 0\} \\ &= \sup\{t \geq 0 : \mathcal{H}^t(A) = \infty\} \end{aligned}$$

Hausdorff Dimension and Diophantine Approximation

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- In particular, the set of Liouville numbers has dimension zero.

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- Interesting from a different point of view: What is the complexity of a function from natural numbers to natural numbers?
- In the following, identify an initial segment $[a_1, \ldots, a_n]$ of a continued fraction with the n-convergent

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots \cfrac{1}{a_n}}}}.$$

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- Using the ergodic theorem, one can show, for instance, that

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- Entropy of the Gauss map $x \mapsto \frac{1}{x} \mod 1$.

Measure on Baire Space

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- Gives rise to a Borel measure in the usual way (extension from algebra to σ-algebra). Equivalently, Hausdorff measures.

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- Let $X = 2^{\mathbb{N}}$ or $X = \mathbb{N}^{\mathbb{N}}$, let $s \geq 0$ be rational. $A \subseteq X$ is Σ_1-$\mathcal{H}^s$ null, Σ^0_1-$\mathcal{H}^s(A) = 0$, if there is a recursive sequence (C_n) of enumerable sets such that for each n,

$$A \subseteq \bigcup_{w \in C_n} [w] \quad \text{and} \quad \sum_{w \in C_n} \operatorname{diam}[w] < 2^{-n}.$$

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- For $s = 1$, one obtains an effective vesion of Lebesgue measure λ.
- Theorem: (Schnorr) A sequence $\xi \in 2^{\mathbb{N}}$ is random if and only if $\{\xi\}$ is not Σ_1-λ null.

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- Theorem: (Reimann, Gacs) A continued fraction α is not Σ_1-λ-null if and only if

$$\sup_n \{-K(\langle a_1, \ldots, a_n \rangle) - \log \lambda[a_1, \ldots, a_n]\} < \infty$$

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- Problem: The continued fraction expansion might code things more efficiently.

Random Continued Fractions

- Surprisingly, it does not.

  Theorem: (Lochs) For $\xi \in 2^{\mathbb{N}}$, denote by $\pi_n(\xi)$ the number of partial convergents a_i of the continued fraction expansion of ξ obtained from the first n digits of ξ. Then it holds for almost every ξ,

$$\lim_{n\to\infty} \frac{|\pi_n(\xi)|}{n} = \frac{6\log^2 2}{\pi^2}.$$

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- Proof requires some effort to avoid the use of the ergodic theorem (which is not an effective law of probability).

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- Theorem: An irrational α is badly approximable if and only if its continued fraction expansion is bounded.
- Hence, from our point of view, badly approximable numbers must be compressible.

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- Theorem: For any sequence $\xi \in 2^{\mathbb{N}}$ it holds that

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Then, for $n \geq 8$, $1 - \frac{4}{n \log 2} \leq \dim_H E_n \leq 1 - \frac{1}{8n \log n}$.

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- Later, Bumby and Hensley gave good approximations of $\dim_H E_n$ for smaller values of n, e.g. $\dim_H E_2 = 0.5312050 \ldots$.

Badly approximable numbers

- Badly approximable numbers are compressible.
- Jarnik's Theorem: Let

$$E_n = \{\alpha = [a_0, a_1, \ldots] \in [0,1] : \forall i\ a_i \leq n\}.$$

  Then, for $n \geq 8$, $1 - \frac{4}{n \log 2} \leq \dim_H E_n \leq 1 - \frac{1}{8n \log n}$.
- Later, Bumby and Hensley gave good approximations of $\dim_H E_n$ for smaller values of n, e.g. $\dim_H E_2 = 0.5312050\ldots$.
- Using complexity theoretic characterization of dimension, we can give a simpler, essentially combinatorial proof.