Homework 2 for MATH 104

Due: Tuesday, September 19, 9:30am in class

Problem 1

Verify the following statements by induction: For all $n \in \mathbb{N}$,

(1) $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$,

(2) $(1 + x)^n \geqslant 1 + nx$, whenever $x \geqslant -1$.

Problem 2

Investigate the following sequences. Determine whether they converge, and if so, determine their limit.

(1) $(a_n)_{n \in \mathbb{N}}$ with $a_n = \frac{5n+2}{3n+4}$;

(2) $(b_n)_{n \in \mathbb{N}}$ with $b_n = \sqrt{n^2 + n} - n$;

(3) $(c_n)_{n \in \mathbb{N}}$ with $c_n = \frac{1^3 + 2^3 + \cdots + n^3}{n^4}$.

Problem 3

Define a sequence (s_n) inductively by letting $s_1 = 1$, and $s_{n+1} = \sqrt{s_n + 1}$. Prove that $\lim_n s_n = \frac{1+\sqrt{5}}{2}$. (*Hint:* For the limit s it must hold that $s = \sqrt{s+1}$. Why?)

Problem 4

Let (a_n) be a convergent sequence of real numbers with limit a. Define the sequence (b_n) by

$$b_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \tag{*}$$

Prove that (b_n) converges and that $\lim_n b_n = a$. (*Hint:* Reduce the general case to the case $b = 0$.)

Find a divergent sequence (a_n) such that the sequence (b_n) defined as in (*) is convergent.

Bonus Problem

Deduce formally from the axioms for complete ordered fields the existence of square roots. That is, prove that for every nonnegative $x \in \mathbb{R}$ there exists a $y \in \mathbb{R}$ such that $y^2 = x$. Can you generalize your argument to the case of n-th roots, i.e. prove that for every nonnegative $x \in \mathbb{R}$ there exists a $y \in \mathbb{R}$ such that $y^n = x$?