Lesson 3
Dynamical Systems
3-2: Measurable Dynamics

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Stochastic Processes

A finite alphabet, $(X_n)_{n\in\mathbb{N}}$ A-valued process.

- Recall: joint distribution (P_n), where

$$P_n(X_0 = a_0, \dots, X_n = a_n) \quad \text{(short: } P_n(a_0,\dots,a_n)\text{)}$$

  describes the distribution of the process up to time n, and is subject to the consistency requirement

$$P_n(a_0,\dots,a_n) = \sum_{a\in A} P_{n+1}(a_0,\dots,a_n,a).$$

- Kolmogorov Extension Theorem: Constructs an underlying measure μ on $A^{\mathbb{N}}$ such that

$$\mu[\sigma] = P_{|\sigma|}(\sigma).$$

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Stationary Processes

In many cases, the joint distributions of the process will not depend on the choice of time origin, i.e. for all $m \leqslant n$, $a_i \in A$,

$$\mathrm{Prob}(X_i = a_i,\ m \leqslant i \leqslant n) = \mathrm{Prob}(X_{i+1} = a_i,\ m \leqslant i \leqslant n)$$

Such a process is called stationary.

i.i.d.

uniform distribution

$A = \{0, 1\}$ $\quad P(X_i = 0) = P(X_i = 1) = \frac{1}{2}$

$P(0) = p$

$P(1) = 1 - p$

. . . .

$X_0 \quad X_1 \quad X_2 \quad X_3 \quad \cdots$

From Stationarity to Shift-Invariance

Let (X_n) be stationary, and let $(A^{\mathbb{N}}, \mathcal{B}, \mu)$ its Kolmogorov extension ($\mathcal{B}$ Borel sets on $A^{\mathbb{N}}$). Then μ is shift-invariant.

- Let T be the shift-map on $A^{\mathbb{N}}$. Then

$\{x : x_k = a_0 \cdots x_{k+n-1} = a_{n-1}\}$

$$T^{-1}[a_0 \dots a_{n-1}]_k = [a_0 \dots a_{n-1}]_{k+1}.$$

Hence the preimage of a cylinder is a cylinder, and it follows that $T^{-1}(B)$ is Borel for any Borel set $B \subseteq A^{\mathbb{N}}$. $\Rightarrow$ T is measurable.

- Since (X_n) is stationary,

$$\mu[a_0 \dots a_{n-1}]_k = \mu[a_0 \dots a_{n-1}]_{k+1} = \mu(T^{-1}[a_0 \dots a_{n-1}]_k).$$

This in turn extends to all Borel sets:

$$\mu(T^{-1}(B)) = \mu(B) \quad \text{for all } B \subseteq A^{\mathbb{N}} \text{ Borel.}$$

T is a μ-preserving transformation

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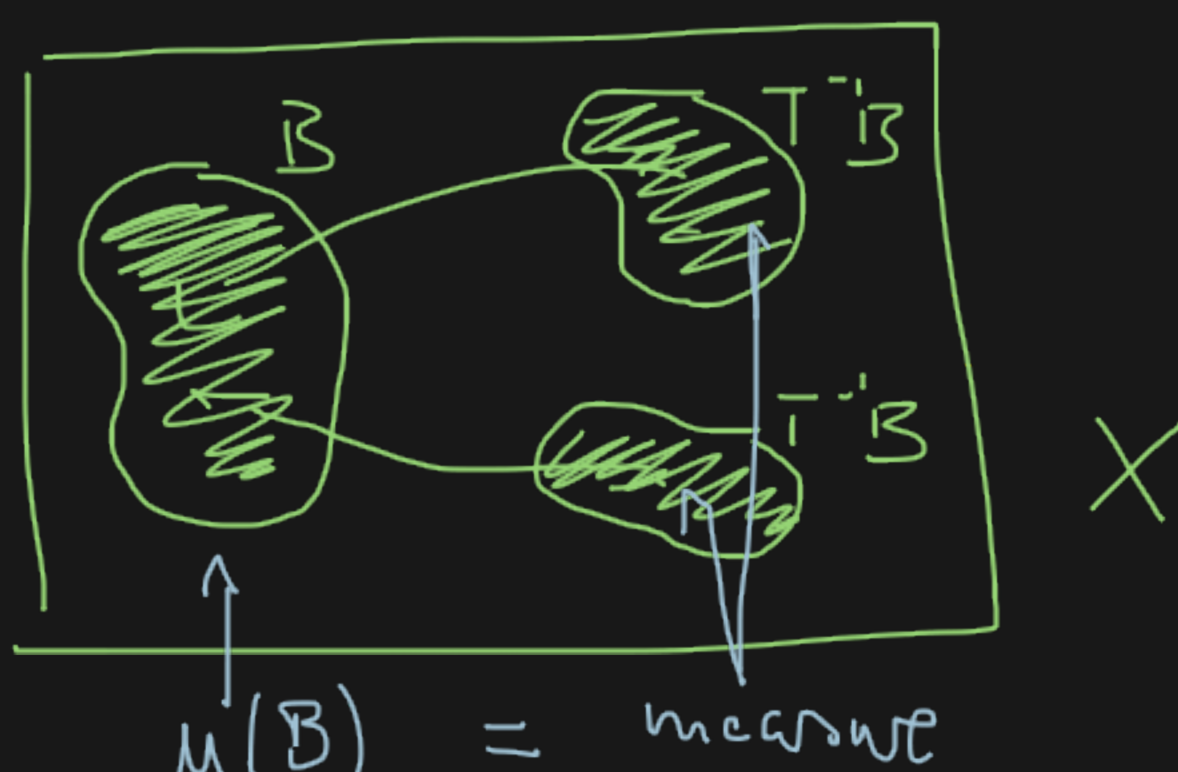
Measure-theoretic Dynamical Systems

Together with the shift map T, the Kolmogorov measure space $(A^{\mathbb{N}}, \mathcal{B}, \mu)$ corresponding to a stationary process (X_n) forms a measure-preserving dynamical system.

In general, such a system is a tuple $(X, \mathcal{A}, \mu, T)$, where $(X, \mathcal{A}, \mu)$ is any probability space ($\mathcal{A}$ is a σ-algebra), and $T : X \to X$ is measurable and measure-preserving, i.e.

$$\mu(T^{-1}(B)) = \mu(B) \qquad \text{for all } B \in \mathcal{A}.$$

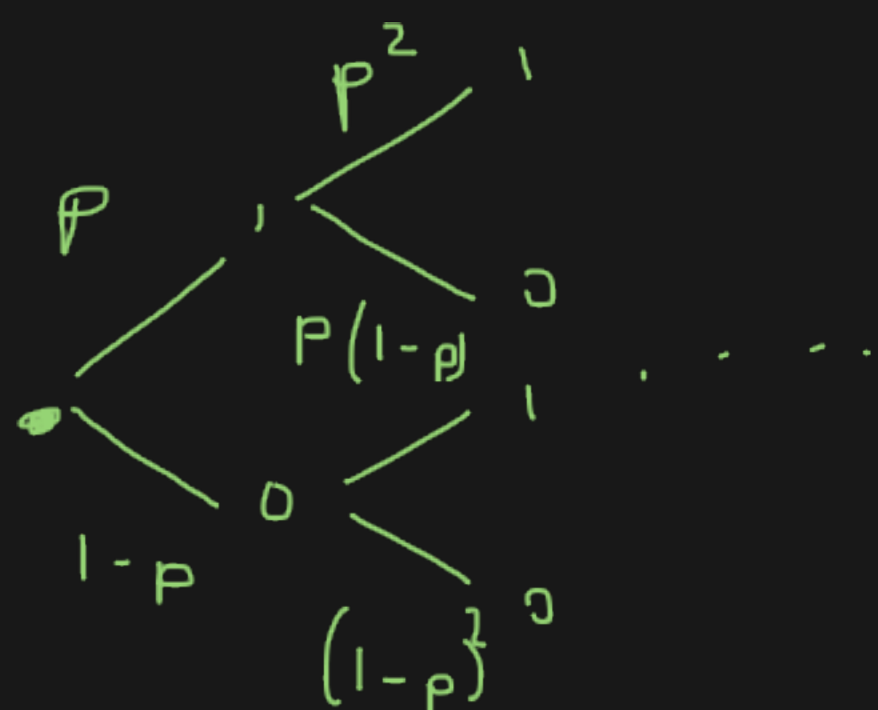


Example: Bernoulli Shifts

Let $A = \{0, 1\}$, and $0 \leqslant p \leqslant 1$. Then the measure μ_p given by

$$\mu_p[\sigma] = p^N (1-p)^{|\sigma|-N},$$

where $N = \#\{i \colon \sigma(i) = 1\}$, is shift-invariant. The system $(A^{\mathbb{N}}, \mathcal{B}, \mu_p, T)$ is the most simple example of a Bernoulli shift.



- We can also look at the two-sided Bernoulli shift on $A^{\mathbb{Z}}$. This has the advantage that the shift is invertible.

Example: Bernoulli Shifts

More generally, if $(X, \mathcal{A}, \mu)$ is a probability space, then

$$(Y, \mathcal{F}, \nu) = \prod_{i=-\infty}^{\infty} (X, \mathcal{A}, \mu)$$

is invariant under the shift $T : Y \to Y$, where $T(y) = T((x_n)_{n\in\mathbb{Z}}) = (z_n)_{n\in\mathbb{Z}}$ with $z_n = x_{n+1}$. This is called the (two-sided) Bernoulli shift with state space $(X, \mathcal{A}, \mu)$.

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From Shifts to Processes

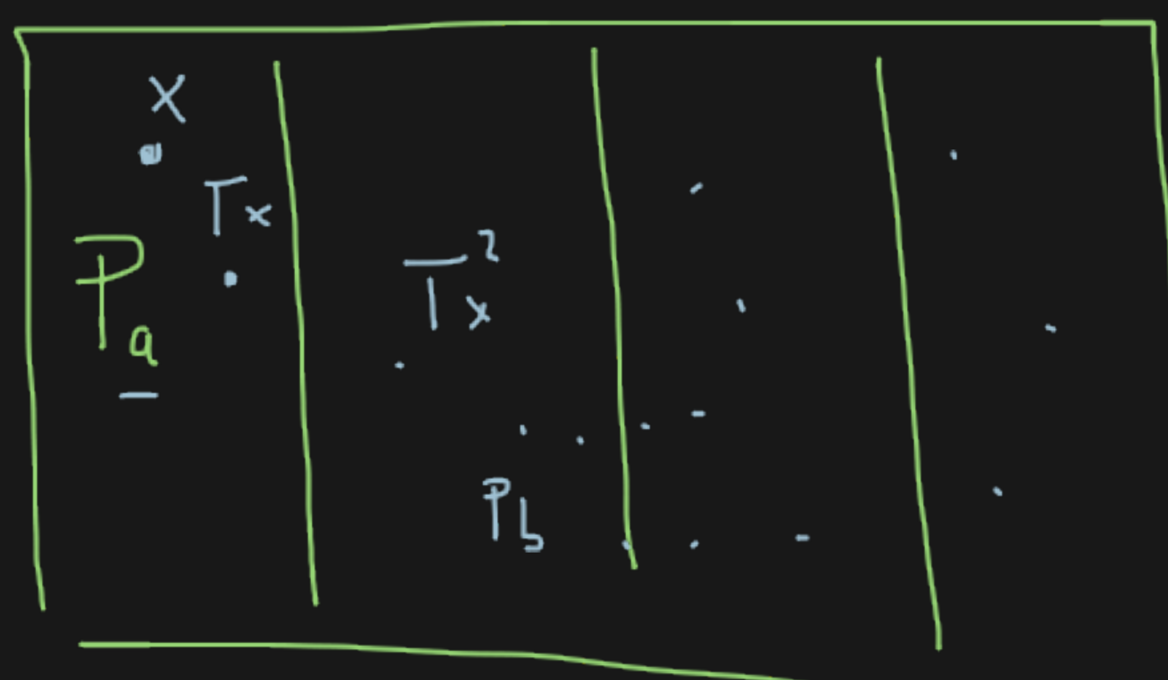
If μ is a shift-invariant measure on $A^{\mathbb{N}}$, then we can derive a stationary process from it as follows:

$= [a]$

- Partition $A^{\mathbb{N}}$ into $\mathcal{P} = \{P_a : a \in A\}$, where $P_a = \{x : x_0 = a\}$.
- Define the A-valued random variable $X_{\mathcal{P}}$ by mapping $x \in A^{\mathbb{N}} \mapsto X_{\mathcal{P}}(x) = a$ where a is such that $x \in P_a$.
- The random variable X_n is then given by

Kolmogorov partition

$$X_n(x) = X_{\mathcal{P}}(T^n(x)) \quad (n \geqslant 1).$$



$X = A^{\mathbb{N}}$