T ergodic $\Leftrightarrow$

$f \circ T(x) = f(x)$ μ-a.e. $\Rightarrow$ f const μ a.e.

χ_A ——— —·—

$\Rightarrow$: $f \circ T(x) = f(x)$ μ-a.e.

WTS: f const. μ-a.e.

Idea: "Pin down" f by smaller and smaller intervals.

Then show for a μ-measure 1 set, $f(x)$ must be in the same smaller and smaller intervals.

$k \in \mathbb{Z}$, $n > 0$

Put $$X_{k,n} = \left\{ x : \frac{k}{2^n} \le f(x) < \frac{k+1}{2^n} \right\}$$

$$= f^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)$$

disjoint

$$X = \dot{\bigcup_{k \in \mathbb{Z}}} X_{k,n} \qquad n \text{ fixed}$$

Show: $\mu(T^{-1}X_{k,n} \,\Delta\, X_{k,n}) = 0$.

using assumption $f \circ T = f$ μ-almost everywhere.

Then use ergodicity for infer s.th. about possible meas. of $X_{k,n}$

Then find a μ-large set V on which f is constant (using the measure-fact for the $X_{k,n}$)

Note: $\bigcap$ of countable family of meas. 1 sets has meas. 1

3 $\quad (p, M)$ stationary Markov

$\quad\quad\quad\quad\uparrow$ stochastic matrix

$$\underline{Q} = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} \underline{M}^i \quad \text{exists}$$

i.e. Q_{ij} exists for all i,j.

$$r \xrightarrow{i \text{ steps}} s$$

Use ergodic thm. $\quad \frac{1}{n} \sum f(T^i x) \longrightarrow f^*(x)$

Problem: Bring erg thm into right form

f will be $\chi_{[j]}$ $\leftarrow$ char. function of cylinder $[j]$

T shift.

j fixed

a.s. $\quad \underbrace{\frac{1}{n} \sum_{i=0}^{n-1} \underline{\chi_{[j]}} (T^i x)}_{(*) \quad A_n(x)} \longrightarrow \chi^*_{[j]}$

Show: $\quad \int A_n^{(x)} d\mu(x) \longrightarrow \int \chi^*_{[j]} \, d\mu(x)$

Dominated Convergence Thm

What does

$\int \underbrace{A_n(x)}_{\text{finite sum}} d\mu \quad$ actually describe ?

$$= \frac{1}{n} \sum_{i=0}^{n-1} \underline{\sum_k p_k \cdot (M^{(i)})_{kj}}$$

Want: $\quad \underline{\frac{1}{n} \sum M^{(i)}_{kj}}$ for single k $\Big|$ multiplication by $\chi_{[k]}$