Homework 6 for
MATH 497A, Introduction to Ramsey Theory

Due: Monday October 3

Problem 1

Failure of Ramsey's Theorem for infinite colorings

Show that for any infinite cardinal κ, $2^\kappa \not\longrightarrow (3)^2_\kappa$.

Problem 2

Failure of Ramsey's Theorem for power sets

Show that for any cardinal κ, $2^\kappa \not\longrightarrow (\kappa^+)^2_2$.

Problem 3

Increasing sequences of real numbers

Show that for any ordinal $\beta < \omega_1$, there exists an increasing sequence of reals of length β, i.e. a sequence $\{a_\xi : \xi < \beta\}$ such that for any $\xi < \beta$, $a_\xi <_{\mathbb{R}} a_{\xi+1}$.

Problem 4

No Banach-Tarski Paradox for $\mathbb{Z}$

An easy reformulation of the Banach-Tarski paradox says that the unit ball C in $\mathbb{R}^3$ can be *paradoxically decomposed*, i.e. there exist disjoint sets A, B such that $C = A \cup B$ and each set A, B can be further partitioned into finitely many pairwise disjoint sets $A_1, \dots, A_m$ and $B_1, \dots, B_n$ such that

$$A = \bigcup_{k=1}^{m} A_k \qquad B = \bigcup_{l=1}^{n} B_l$$

and there exist rigid motions $\alpha_1, \dots, \alpha_m$, $\beta_1, \dots, \beta_n$ such that

$$C = \bigcup_{k=1}^{m} \alpha_k(A_k) = \bigcup_{l=1}^{n} \beta_l(B_l).$$

Show that such a decomposition is impossible for $\mathbb{Z}$, if as motions we only allow translations by integers, i.e. translations of a set A in the form $A + j = \{a + j : a \in A\}$ for some $j \in \mathbb{Z}$.

Problem 5

BONUS: The measure problem for non-atomic measures

Translation invariance of a measure on $\mathbb{R}$, $\mu(A+r) = \mu(A)$, implies $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$. Measures with the latter property are called *non-atomic*. Does the measure problem become solvable if we only require non-atomicity? That is, does there exist a function $\mu : \mathcal{P}(\mathbb{R}) \to [0,\infty]$ such that

(1) $\mu([0,1]) = 1$,

(2) $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$,

(3) $\mu(\bigcup A_n) = \sum \mu(A_n)$ for any sequence of pairwise disjoint sets $A_n \subseteq \mathbb{R}$?