Homework 2 for MATH 185

Due: Wednesday February 7, 3:10 pm in class

Problem 1

Verify the identities

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} \qquad \sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}$$

and use them to show that $\sin(\mathbb{C}) = \mathbb{C}$ and $\cos(\mathbb{C}) = \mathbb{C}$.

Determine all $z \in \mathbb{C}$ such that $\sin(z) = 12i/5$.

Problem 2

Determine all points at which the function

$$f : \mathbb{C} \to \mathbb{C}\ , \qquad z = x + iy \mapsto f(z) := x^3y^2 + ix^2y^3\ , \qquad x, y \in \mathbb{R}\ ,$$

is complex differentiable.

Does there exist a non-empty open set $D \subseteq \mathbb{C}$ on which f is analytic?

Problem 3

Show that the function $f : \mathbb{C} \to \mathbb{C}$,

$$f(z) = \begin{cases} \exp(-1/z^4) & \text{for } z \neq 0, \\ 0 & \text{for } z = 0, \end{cases}$$

satisfies the Cauchy-Riemann equations for all $z \in \mathbb{C}$ and is complex differentiable for all $z \in \mathbb{C}^\bullet = \mathbb{C}\setminus\{0\}$, but not at the origin.

Problem 4

(a) Let $D = \mathbb{C}^\bullet$ and $u : D \to \mathbb{R}$ with $u(x, y) = \frac{x}{x^2+y^2}$. Show that u is harmonic and find an analytic function $f : D \to \mathbb{C}$ with $\mathrm{Re}(f) = u$.

(b) Given two harmonic functions $u_1, u_2 : \mathbb{R}^2 \to \mathbb{R}$, prove or disprove (counterexample) the following statements.

1.) $u_1 + u_2$ is harmonic.

2.) $u_1 \cdot u_2$ is harmonic.