# Homework 10 for **MATH 185**

Brief sketches to solutions

## Problem 1 [**]

Find the Laurent series expansion of the following functions around $a = 0$:

(a) $\sin(1/z)$ (b) $1/(z(z+1))$ (c) $z/(z+1)$ (d) $e^z/z^2$.

*Solution.* (a) $\sum_{n=-\infty}^{0} \frac{(1)^{n+1}}{(2n+1)!} z^{-n-1}$ (b) $\sum_{n=-1}^{\infty} (-1)^{n+1} z^n$ (c) $\sum_{n=1}^{\infty} (-1)^n z^n$ (d) $\sum_{n=-2}^{\infty} \frac{1}{(n+2)!} z^n$. ■

## Problem 2 [**]

Find the Laurent series of the function

$$f : \mathbb{C} \setminus \{1, -2\} \to \mathbb{C}, \quad z \mapsto \frac{1}{(z-1)(z+2)}$$

for the annuli

(a) $\mathcal{A}_1 = \{z : 0 < |z| < 1\}$ (b) $\mathcal{A}_2 = \{z : 1 < |z| < 2\}$ (c) $\mathcal{A}_3 = \{z : 2 < |z|\}$

*Solution.* (a) $\sum_{n=0}^{\infty} \left(\frac{(-1)^n}{2^{n+1}} - 1\right) z^n$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n + \sum_{n=-\infty}^{-1} z^n$ (c) $\sum_{n=-\infty}^{-1} [(-1)^{n+1} 2^{n+1} + 1] z^n$ ■

## Problem 3 [**]

Let $0 < \varepsilon < 2\pi$ and define $f : \dot{U}_\varepsilon(0) \to \mathbb{C}$ by

$$f(z) = \frac{\sin(z)}{\cos(z^3) - 1}.$$

Show that $f$ has a pole of order 5 in 0 (i.e. $\mathrm{ord}(f; 0) = -5$) and determine the coefficient $a_{-1}$ of the Laurent series of $f$ around 0.

*Solution.* We have the representations

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} \quad \text{and} \quad \cos(z^3) - 1 = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} z^{3n}.$$

We can write $f$ as

$$\frac{\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} \pm \cdots}{\frac{z^6}{2!} - \frac{z^{12}}{4!} + \frac{z^{18}}{6!} \pm \cdots} = \frac{z}{z^6} \frac{\frac{1}{1!} - \frac{z^2}{3!} + \frac{z^4}{5!} \pm \cdots}{\frac{1}{2!} - \frac{z^6}{4!} + \frac{z^12}{6!} \pm \cdots}$$

The function

$$h(z) = \frac{\frac{1}{1!} - \frac{z^2}{3!} + \frac{z^4}{5!} \pm \cdots}{\frac{1}{2!} - \frac{z^6}{4!} + \frac{z^12}{6!} \pm \cdots}$$

has a removable singularity at 0 and $h(0) \neq 0$, so we conclude 0 is a pole of order 5 (since $h(z) = f(z)z^5$).

To compute the coefficient $a_{-1}$ of the Laurent series of $f$, we start by observing that for this power series it must hold that

$$\sin(z) = (\cos(z^3) - 1) \sum_{n=-5}^{\infty} a_n z^n.$$

Substituting the power series representations, we get

$$\frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} \pm \cdots = \left(\frac{z^6}{2!} - \frac{z^{12}}{4!} + \frac{z^{18}}{6!} \pm \cdots\right) \left(a_{-5} z^{-5} + a_{-4} z^{-4} + \cdots + a_{-1} z^{-1} + a_0 + \cdots\right)$$

Now we can expand the right hand side and obtain, by comparing coefficients with the series on the left hand side, linear equations for the unknowns $a_n$. Because of the special structure of the series for $\cos(z^3) - 1$, these equations will be of a very simple nature:

$$\frac{1}{1!} = a_{-5} \frac{1}{2!} \quad 0 = a_{-4} \frac{1}{2!} \quad \frac{1}{3!} = a_{-3} \frac{1}{2!} \quad 0 = a_{-2} \frac{1}{2!} \quad \frac{1}{5!} = a_{-1} \frac{1}{2!}.$$

Hence we have $a_{-1} = 1/60$. ■

**Problem 4 [**]**

Let $D \subseteq \mathbb{C}$ be open. Suppose $a \in D$ and $f : D \setminus \{a\} \to \mathbb{C}$ is analytic and one-one. Prove the following statements.

(a) $f$ has in $a$ a non-essential singularity.

*Solution.* If $a$ were an essential singuarity, by Picard's Big Theorem the image of any $U_\varepsilon(a)$ under $f$ is all of $\mathbb{C}$ with the exception of at most one point. This means that for a given (suitably small) $\varepsilon$ there exist $b, c \in \mathbb{C}$ such that $\mathbb{C} \setminus \{b, c\} \subseteq f(U_{\varepsilon/2}(a)) \subseteq f(U_\varepsilon(a))$. But this implies that there must exist infinitely many points in $U_\varepsilon(a) \setminus U_{\varepsilon/2}(a)$ that are mapped to $f(U_{\varepsilon/2}(a))$, which contradicts the injectivity of $f$. ■

(b) If $f$ has a pole in $a$, then it is a pole of order 1.

*Solution.* Let $k$ be the order of the pole. Then $h(z) = f(z)(z-a)^k$ can be analytically extended to $a$, and $h(a) \neq 0$. We fix an $r$ sufficiently small such that $h(z) \neq 0$ in $U_r(a)$. Then there exists an analytic $h_0$ such that $h(z) = (h_0(z))^k$ for all $z \in U_r(a)$. But then $f$ can be written as $f(z) = [h_0(z)/(z-a)]^k$. If $k > 1$, this would contradict the injectivity of $f$, since $z \mapsto z^k$ is not one-one in this case. (Note that $h_0$ is locally conformal around $a$, so $h_0(z)/(z-a)$ maps a small neighborhood of $a$ to a "neigborhood" of $\infty$.) ■

(c) If $f$ has a removable singularity in $a$, then the analytic extension of $f$ to $D$ is one-one, too.

*Solution.* Assume the extension is not one-one, so there exists $z \in D$ such that $c = f(a) = f(z)$. Choose $r$ such that $U_r(a) \cap U_r(z) = \emptyset$. By the open mapping theorem, $f(U_r(a))$ and $f(U_r(z))$ are both open and contain $c$. But the intersection of two open sets is open, so there is a whole neighborhood around $c$ contained in $f(U_r(a)) \cap f(U_r(z))$,contradicting the injectivity of $f$. ■