Lecture 17: Co-Analytic Sets

In the previous lecture we saw how to translate set theoretic definitions of sets of reals into second order arithmetic. One can ask the converse question – does definability in second order arithmetic imply constructibility? We will see that this is indeed true for Σ^1_2 definable reals. Along the way, we will prove a number of interesting results about Π^1_1 and Σ^1_2 sets.

Normal forms

Analytic sets are projections of closed sets. Closed sets are in $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ are infinite paths through *trees on* $\mathbb{N} \times \mathbb{N}$, i.e. *two-dimensional trees*.

Definition 17.1: A set $T \subseteq \mathbb{N}^{<\mathbb{N}} \times \mathbb{N}^{<\mathbb{N}}$ is a **two-dimensional tree** if

(i) $(\sigma, \tau) \in T$ implies $|\sigma| = |\tau|$ and

(ii) $(\sigma, \tau) \in T$ implies $(\sigma \mid n, \tau \mid n) \in T$ for all $n \leq |\sigma|$.

An **infinite branch** of T is a pair $(\alpha, \beta) \in \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ so that

$$\forall n \in \mathbb{N}\, (\alpha \mid n, \beta \mid n) \in T.$$

As in the one-dimensional case, we use $[T]$ to denote the set of all infinite paths through T. It follows that $A \subseteq \mathbb{N}^{\mathbb{N}}$ is analytic if and only if there exists a two-dimensional tree T on $\mathbb{N} \times \mathbb{N}$ such that

$$\begin{aligned} \alpha \in A \quad &\Leftrightarrow \quad \exists \beta\, (\alpha, \beta) \in [T] \\ &\Leftrightarrow \quad \exists \beta\, \forall n\, (\alpha \mid n, \beta \mid n) \in T. \end{aligned}$$

Another way to write this is to put, for given T and $\alpha \in \mathbb{N}^{\mathbb{N}}$,

$$T(\alpha) = \{\tau : (\alpha \mid |\tau|, \tau) \in T\}.$$

Then we have, with T witnessing that A is analytic,

$$\alpha \in A \quad \Leftrightarrow \quad T(\alpha) \text{ has an infinite path} \quad \Leftrightarrow \quad T(\alpha) \text{ is not well-founded.}$$

We obtain the following *normal form* for co-analytic sets.

Proposition 17.2: *A set $A \subseteq \mathbb{N}^{\mathbb{N}}$ is $\mathbf{\Pi}^1_1$ if and only if there exists a two-dimensional tree T such that*

$$\alpha \in A \quad \Leftrightarrow \quad T(\alpha) \text{ is well-founded.}$$

If A is (lightface) Π^1_1, then there exists a recursive such T, and the mapping $\alpha \mapsto T(\alpha)$ is computable, as a mapping between reals and trees (which can be coded by reals). This relativizes, i.e. for a $\Pi^1_1(\gamma)$ set, the mapping $\alpha \mapsto T(\alpha)$ is computable in γ. Since computable mappings are continuous, we obtain that the in the above proposition, the mapping $\alpha \mapsto T(\alpha)$ is continuous.

$\mathbf{\Pi}^1_1$-complete sets

How does one show that a specific set is *not* Borel? A related question is: Given a definition of a set in second order arithmetic, how can we tell that there is not an easier definition (in the sense that it uses less quantifier changes, no function quantifiers etc.)? The notion of *completeness* for classes in Polish spaces provides a general method to answer such questions.

Definition 17.3: Let X, Y be Polish spaces. We say a set $A \subseteq X$ is **Wadge reducible** to $B \subseteq Y$, written $A \leq_W B$, if there exists a continuous function $f : X \to Y$ such that

$$x \in A \quad \Leftrightarrow \quad f(x) \in B.$$

The important fact about Wadge reducibility is that it preserves classes closed under continuous preimages.

Proposition 17.4: *Let Γ be a family of subsets in various Polish spaces (such as the classes of the Borel or projective hierarchy). If Γ is closed under continuous preimages, then $A \leq_W B$ and $B \in \Gamma$ implies $A \in \Gamma$.*

Proof. If $A \leq_W B$ via f, then $A = f^{-1}(B)$. □

Definition 17.5: A set $A \subseteq X$ is **Γ-complete** is $A \in \Gamma$ and for all $B \in \Gamma$, $B \leq_W A$.

Γ-complete sets can be seen as the most complicated members of Γ. For instance, a $\mathbf{\Pi}^1_1$-complete set cannot be Borel, since otherwise every $\mathbf{\Pi}^1_1$ set would be Borel, which we have seen is not true. More generally if Γ is any class in the Borel or projective hierarchy, and A is Γ-complete, then A is not in $\neg\Gamma$. For suppose $B \in \Gamma \setminus \neg\Gamma$. Then $B \leq_W A$. If A were also in $\neg\Gamma$, then $B \in \neg\Gamma$, a contradiction.

If $A \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is $\mathbb{N}^{\mathbb{N}}$-universal for come class Γ in the Borel or projective hierarchy, then the set

$$\{\langle \alpha, \beta \rangle : (\alpha, \beta) \in A\}$$

is Γ-complete, where $\langle .,. \rangle$ here denotes the pairing function for reals

$$\langle \alpha, \beta \rangle(n) = \begin{cases} \alpha(k) & n = 2k, \\ \beta(k) & n = 2k+1. \end{cases}$$

Since $\langle .,. \rangle$ is continuous, and $B \in \Gamma$ if and only if $B = A_\gamma$ for some $\gamma \in \mathbb{N}^\mathbb{N}$, we have in that case that $B \leq_\mathrm{W} A$ via the mapping

$$f(\beta) = \langle \gamma, \beta \rangle.$$

It follows that complete sets exist for all levels of the Borel and projective hierarchy. However, the universal sets they are based on are rather abstract objects. Complete sets are most useful when we can show that a *specific property* implies completeness. We will encounter next an important example for the class of co-analytic sets.

Well-founded relations and well-orderings

In the last lecture we encountered the property of a real coding a well-founded relation: Recall that given $\beta \in \mathbb{N}^\mathbb{N}$, $E_\beta(m,n)$ if and only if $\beta(\langle m,n \rangle) = 0$. Let

$$\mathrm{WF} = \{\beta \in \mathbb{N}^\mathbb{N} \colon E_\beta \text{ is well-founded}\}.$$

Then

$$\beta \in \mathrm{WF} \quad \Leftrightarrow \quad \forall \gamma \in \mathbb{N}^\mathbb{N} \, \exists n \, \forall m \, [\gamma(n) E_\beta \gamma(m)],$$

and hence WF is Π^1_1. A closely related set is

$$\mathrm{WOrd} = \{\beta \in \mathbb{N}^\mathbb{N} \colon E_\beta \text{ is a well-ordering}\}.$$

Then

$$\beta \in \mathrm{WOrd} \quad \Leftrightarrow \quad \beta \in \mathrm{WF} \text{ and } E_\beta \text{ is a linear ordering.}$$

Coding a linear order is easily seen $\boldsymbol{\Sigma}^1_1$, hence WOrd is Π^1_1, too.

Theorem 17.6: *The sets* WF *and* WOrd *are* $\boldsymbol{\Pi}^1_1$*-complete.*

Proof. We have seen in Lecture 4 that a tree has an infinite path if and only if the inverse prefix ordering is ill-founded. Trees can be coded as reals, and hence Proposition 17.2 yields immediately that WF is Π^1_1-complete.

For WOrd we use the Kleene-Brouwer ordering (see Lecture 4) and Proposition 4.5. □

The theorem lets us gain further insights in the structure of co-analytic sets. If $\alpha \in \mathbb{N}^{\mathbb{N}}$ codes a well-ordering on $\mathbb{N}$, let

$$\|\alpha\| = \text{ order type of well-ordering coded by } \alpha.$$

It is clear that $\|\alpha\| < \omega_1$. For a fixed ordinal $\xi < \omega_1$, we let

$$\mathrm{WOrd}_\xi = \{\alpha \in \mathrm{WOrd}\colon \ \|\alpha\| \leq \xi\}.$$

Lemma 17.7: *For any $\xi < \omega_1$, the set* WOrd_ξ *is Borel.*

Proof. Let $\alpha \in \mathbb{N}^{\mathbb{N}}$. We say $m \in \mathbb{N}$ is in the *domain* of E_α, $m \in \mathrm{dom}(E_\alpha)$, if

$$\exists n\, [mE_\alpha n \ \vee \ nE_\alpha m].$$

It is clear from the definition of E_α that $\mathrm{dom}(E_\alpha)$ is Borel. For $\xi < \omega_1$, let

$$B_\xi = \{(\alpha, n)\colon E_\alpha \mid \{m\colon mE_\alpha n\} \text{ is a well-ordering of order type} \leq \xi\}$$

We show by transfinite induction that every B_ξ is Borel. Suppose B_ζ is Borel for all $\zeta < \xi$. Then, since ξ is countable, $\bigcup_{\zeta<\xi} B_\zeta$ is Borel, too. But

$$(\alpha, n) \in B_\xi \quad \Longleftrightarrow \quad \forall m\, [mE_\alpha n \Rightarrow (\alpha, m) \in \bigcup_{\zeta<\xi} B_\zeta],$$

and from this it follows that B_ξ is Borel. Finally, note that

$$\alpha \in \mathrm{WOrd}_\xi \quad \Longleftrightarrow \quad \forall n\, [n \in \mathrm{dom}(E_\alpha) \Rightarrow (\alpha, n) \in B_\xi],$$

which implies that WOrd_ξ is Borel. $\square$

Corollary 17.8: *Every $\mathbf{\Pi}^1_1$ set is a union of $\aleph_1$ many Borel sets.*

Proof. Since WOrd is $\mathbf{\Pi}^1_1$-complete, every co-analytic set A is the preimage of WOrd for some continuous function f. We have

$$\mathrm{WOrd} = \bigcup_{\xi<\omega_1} \mathrm{WOrd}_\xi,$$

and hence

$$A = \bigcup_{\xi<\omega_1} f^{-1}(\mathrm{WOrd}_\xi).$$

Since continuous preimages of Borel sets are Borel, the result follows. $\square$

If we work instead with the set

$$C_\xi = \{(\alpha \colon \alpha \in \mathrm{WOrd}_\xi \ \text{ or } \exists n \in \mathrm{dom}(E_\alpha) \\ [E_\alpha \mid \{m \colon m E_\alpha n\} \text{ is a well-ordering of order type } \xi]\},$$

then we get that $\mathrm{WOrd} = \bigcap_{\xi < \omega_1} C_\xi$, and hence

Corollary 17.9: *Every $\mathbf{\Pi}^1_1$ set can be obtained as a union or intersection of $\aleph_1$-many Borel sets. Consequently, the same holds for every $\mathbf{\Sigma}^1_1$ set.*

Finally, the previous results allow us to solve the cardinality problem of co-analytic sets at least partially.

Corollary 17.10: *Every $\mathbf{\Pi}^1_1$ set is either countable, of cardinality $\aleph_1$, or of cardinality $2^{\aleph_0}$.*