Homework 3 for MATH 104

Due: Tuesday, September 26, 9:30am in class

Problem 1

Given a sequence $(s_n)_{n\in\mathbb{N}}$, let $(-s_n)$ be the sequence defined as $(-s_1, -s_2, -s_3, \dots)$. Show that $\liminf_{n\to\infty}(s_n) = -\limsup_{n\to\infty}(-s_n)$.

Solution. We discard the case where $\liminf_n s_n = \pm\infty$.

So assume $\liminf_n s_n = u \in \mathbb{R}$. We first show that for any set $S \subseteq \mathbb{R}$ bounded from below, $\inf S = -\sup -S$, where $-S = \{-s : s \in S\}$. Assume that $a \leqslant \inf S$. Then $a \leqslant s$ for all $s \in S$. This implies $-a \geqslant -s$ for all $s \in S$, hence $-a \geqslant \sup -S$, hence $a \leqslant -\sup -S$. On the other hand, let $a > \inf S$. Then there exists some $s \in S$ such that $s < a$. This implies $-a < -s$, and therefore $-a < \sup -S$. Hence $a > -\sup -S$. The claim follows.

$\liminf_n s_n$ is defined as $\lim_N \inf\{s_n : n > N\}$. It follows that

$$\liminf_n s_n = \lim_N \inf\{s_n : n > N\} = \lim_N -\sup\{-s_n : n > N\}.$$

By the limit theorems, this is equal to $-\limsup_{n\to\infty}(-s_n)$. ∎

Problem 2

Let $(q_n)_{n\in\mathbb{N}}$ be a sequence. Suppose that there exists an $r \in \mathbb{R}$, $0 < r < 1$ such that for all $n \in \mathbb{N}$,

$$|q_{n+2} - q_{n+1}| \leqslant r\,|q_{n+1} - q_n|.$$

Show that (q_n) is a Cauchy sequence.

Solution. 1) A simple induction shows that for all $n \in \mathbb{N}$,

$$|q_{n+1} - q_n| \leqslant r\,|q_2 - q_1|.$$

2) Let $m > n$. Then it follows that

$$\begin{aligned}
|q_m - q_n| &= |q_m - q_{m-1} + q_{m-1} - q_{m-2} + \cdots - q_{n+1} - q_n| \\
&\leqslant |q_m - q_{m-1}| + |q_{m-1} - q_{m-2}| + \cdots + |q_{n+1} - q_n| \\
&\leqslant (r^{m-2} + \cdots + r^{n-1})|q_2 - q_1| \\
&= r^{n-1}(r^{m-n-1} + \cdots + r + 1)|q_2 - q_1| \\
&= r^{n-1}\left(\frac{1 - r^{m-n}}{1-r}\right)|q_2 - q_1| \\
&\leqslant r^{n-1}\left(\frac{1}{1-r}\right)|q_2 - q_1|.
\end{aligned}$$

3) For any $r \in \mathbb{R}$ with $0 < r < 1$, the sequence (r^n) converges to 0. A proof can be found in Ross. Since $|q_2 - q_1|/(1-r)$ is a constant, it follows that $r^{n-1}\left(\frac{1}{1-r}\right)|q_2 - q_1|$ converges to 0, too.

4) Now a standard ε-argument yields that (q_n) is a Cauchy sequence. ∎

Problem 3

A real number x is called *algebraic* if it is the solution of a polynomial with integer coefficients, i.e. if there exists a natural number n and integers $a_n, a_{n-1}, \dots, a_1, a_0$ with $a_n \neq 0$ such that

$$a_n x^n + \cdots + a_1 x + a_0 = 0.$$

Show that the set of all algebraic real numbers is countable.

Solution. We first prove a lemma which says that the countable union of countable sets is countable.

Lemma: Let $\{E_i : i \in \mathbb{N}\}$ be a countable family such that each E_i is countable. Then

$$\bigcup_{i \in \mathbb{N}} E_i = \{x : \text{ exists } i \in \mathbb{N} \text{ such that } x \in E_i\}$$

is countable.

Proof. Since each E_i is at most countable, for all $i \in \mathbb{N}$ there exists a bijection $f_i : \mathbb{N} \to E_i$. Then $\bigcup_{i \in \mathbb{N}} E_i = \{f_i(j) : i, j \in \mathbb{N}\}$. The mapping $(i, j) \mapsto f_i(j)$ defines a surjection from $\mathbb{N} \times \mathbb{N}$ onto $\{f_i(j) : i, j \in \mathbb{N}\}$. But we already proved that $\mathbb{N} \times \mathbb{N}$ is countable, and that the surjective image of a countable set is at most countable. But since none of the E_i is finite, the union cannot be finite, hence the union is countable.

Now let P be the set of all polynomials with integer coefficients. As a polynomial is uniquely determined by its coefficients, there is a bijection between P and the set $\{(a_0, a_1, \ldots, a_n) : a_i \in \mathbb{Z}, |, a_n \neq 0\}$. Since $\mathbb{Z}$ and $\mathbb{Z} \setminus \{0\}$ are countable, P is of the same cardinality as the set F of all finite sequences of natural numbers. Furthermore, we have

$$F = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n,$$

where $\mathbb{N}^n$ is the set of all sequences of natural numbers of length n. Since $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$, an easy induction shows that $\mathbb{N}^n \sim \mathbb{N}$ for all $n \in \mathbb{N}$. We apply the lemma and obtain that F, and hence P is countable.

(An alternative proof to show that F is countable is as follows: Let p_n be the n-th prime number. Given $(n_1, \ldots, n_k) \in F$, map this to $p_1^{n_1} \cdots p_k^{n_k}$. By the uniqueness of prime decomposition, this mapping is injective from F into $\mathbb{N}$.)

Given a polynomial $p \in P$, let A_p be the set of its roots. Each A_p is finite. Obviously,

$$A = \bigcup_{p \in P} A_p.$$

Since we know that P is countable, we can apply the lemma again to obtain the countability of A. ■

Problem 4

Show that the set $\mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}$ has the same cardinality as $\mathbb{R}$.

Solution. We know that $\mathbb{R} \sim D$, where $D = \{(d_n) : d_n \in \{0, 1, \ldots, 9\}\}$. It therefore suffices to show that $D \times D \sim D$.

Define a mapping $f : D \times D \to D$ by letting

$$f((c_n), (d_n)) = (e_n) \text{ where } e_n = \begin{cases} c_{n/2}, & \text{if } n \text{ is even,} \\ d_{n/2-1}, & \text{if } n \text{ is odd.} \end{cases}$$

It is easy to see that this mapping is 1-1 and onto. ■