Homework 1 for MATH 185

Due: Wednesday January 31, 3:10 pm in class

Problem 1

Let $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ be the *upper half plane*, and $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ be the *(open) unit disk*.

Show that the mapping $f : \mathbb{H} \to \mathbb{C}$ defined by

$$z \mapsto \frac{z - i}{z + i}$$

is one-one and it holds that $f(\mathbb{H}) = \mathbb{E}$ (i.e. f is a *bijection* between $\mathbb{H}$ and $\mathbb{E}$).

Problem 2

Show that a quadratic equation $z^2 + pz + q = 0\ ,\ p, q \in \mathbb{C}$ always has two solutions in $\mathbb{C}$ (counting multiplicity). What can you say about the solutions if both p and q are real numbers?

Problem 3

Let $n \in \mathbb{N}$, $\zeta_n := \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} \in \mathbb{C}$. Show that for all $k \in \mathbb{N}$,

$$1 + \zeta_n^k + \zeta_n^{2k} + \cdots + \zeta_n^{(n-1)k} = \begin{cases} n, & \text{if } n \text{ divides } k \\ 0, & \text{otherwise.} \end{cases}$$

Problem 4

Let U be an open subset of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be a continuous function. Assume there exists $a \in U$ such that $f(a) \neq 0$. Prove that there is an open ball B containing a such that $f(z) \neq 0$ for all z in B.