Lecture 21: Co-analytic Ranks

In the previous lecture we learned about how Π^1_1 set can be analyzed in terms of countable ordinals. In this lecture we will deepen this analysis. We will develop the theory of Π^1_1**-ranks**, which is a powerful tool in descriptive set theory. We can view the recursive function f that we constructed in the proof of Theorem 20.2 as the central fact:

$$\text{If } R_e \text{ is well-founded, then } \rho(R_e) \leq |f(e)|_{\mathcal{O}} \qquad (*)$$

Boundedness Principles

We start by picking up the observation made in Lemma 20.1. It states that r.e. subsets of $\mathcal{O}$ are **uniformly bounded**: Given an index e of an r.e. subset of $\mathcal{O}$, we can compute uniformly in e a ordinal bounding all ordinals denoted by W_e. We can strengthen this to Σ^1_1 sets.

Theorem 21.1 (Spector): *If $X \subseteq \mathcal{O}$ is Σ^1_1, then there exists $b \in \mathcal{O}$ such that*

$$\forall x \in X \quad |x|_{\mathcal{O}} < |b|_{\mathcal{O}}.$$

Proof. Let t be a reduction from $\mathcal{O}$ to $\mathrm{WF}_{\mathbb{N}}$, that is t is recursive such that

$$x \in \mathcal{O} \quad \Leftrightarrow \quad R_{t(x)} \text{ is well-founded.}$$

The idea is that if X is unbounded in $\mathcal{O}$, then we can characterize $\mathcal{O}$ by a Σ^1_1 formula, contradicting Corollary 20.7. If the desired b does not exist, then, for each $x \in \mathcal{O}$, we can find a $y \in X$ such that there exists an embedding of $R_{t(x)}$ into $\mathcal{O}$ below y. Using the proof of Theorem 20.2, we can formulate this as a property $P(x)$,

$$P(x) \quad \Leftrightarrow \quad \exists y\ [y \in X \ \wedge\ \exists \gamma \forall z_0, z_1\, (R_{t(x)}(z_0, z_1) \ \Rightarrow\ \langle \gamma(z_0), \gamma(z_1)\rangle \in W_{g(z)})],$$

where g is a recursive function so that $W_{g(z)} = \{\langle x, y\rangle : x <_{\mathcal{O}} y <_{\mathcal{O}} z\}$ (see Proposition 19.7). If X is Σ^1_1, then P is Σ^1_1.

If $x \in \mathcal{O}$, then $R_{t(x)}$ is well-founded, hence by $(*)$, $\rho(R_{t(x)}) \leq |f(t(x))|_{\mathcal{O}}$, and thus if X is unbounded in $\mathcal{O}$, $P(x)$ holds. If $P(x)$ holds on the other hand, then $R_{t(x)}$ must be well-founded (otherwise such a mapping would not exist), and thus $x \in \mathcal{O}$. Hence P would be a Σ^1_1 characterization of $\mathcal{O}$. $\square$

Corollary 21.2: *If $X \subseteq \mathbb{N}$ is Δ^1_1, and h is recursive such that $x \in X$ if and only if $h(x) \in \mathcal{O}$, then there exists a $b \in \mathcal{O}$ such that*

$$\forall x \in X \quad |h(x)| < |b|_{\mathcal{O}}.$$

A similar statement holds with $\mathrm{WF}_{\mathbb{N}}$ in place of $\mathcal{O}$.

Boundedness for sets of reals

The key to Spector's theorem is the fact that $\mathrm{WF}_{\mathbb{N}}$ and $\mathcal{O}$ are m-complete for the class of Π^1_1 sets of natural numbers.

We have seen (Theorem 17.6) that the set $\mathrm{WOrd}, \mathrm{WF} \subseteq \mathbb{N}^{\mathbb{N}}$ are $\boldsymbol{\Pi}^1_1$-complete with respect to *Wadge-reducibility*. This lets us obtain a similar result for Σ^1_1 sets of reals.

Theorem 21.3 (Σ^1_1-boundedness for reals)**:** *Let $A \subseteq \mathrm{WOrd}$ be Σ^1_1. Then there exists a $\xi < \omega_1^{\mathrm{CK}}$ such that*

$$\forall \alpha \in A \quad \|\alpha\| < \xi,$$

where $\|\alpha\|$ denotes the order type of the well-ordering coded by α.

An analogous statement holds for WF, with respect to the rank function ρ of a well-founded relation.

Proof. If such a ξ did not exist, then

$$\alpha \in \mathrm{WOrd} \quad \Longleftrightarrow \quad \exists \beta \, [\beta \in A \wedge \mathrm{WOrd}_\beta].$$

The right-hand side is Σ^1_1, and hence WOrd would be Σ^1_1, contradiction. □

Rank analysis of co-analytic sets

The previous results constitute a powerful technique when analyzing the complexity of sets. In particular, they give us a method to show that a Π^1_1 set is *not* Borel, besides proving that they are Π^1_1-complete.

If $A \subseteq \mathbb{N}^{\mathbb{N}}$ is Π^1_1, then there exists a recursive tree T such that

$$\alpha \in A \quad \Longleftrightarrow \quad T(\alpha) \text{ is well-founded.}$$

Every well-founded $T(\alpha)$ has a rank $\rho(T(\alpha))$. Σ^1_1-boundedness tells us that if A is moreover Δ^1_1, then the *spectrum* of these ranks is **bounded by a computable ordinal**. This means that we can show that A is not Δ^1_1 by showing that its ordinal spectrum $\{\rho(T(\alpha)) \colon \alpha \in A\}$ is unbounded in ω_1^{CK}.

These observations generalize (using relativization) to $\mathbf{\Pi}^1_1$ sets: Ranks of Borel sets are bounded by an ordinal $\xi < \omega_1$.

The downside of this method is that the tree T associated with a Π^1_1 set is a rather generic object, stemming from the canonical representation of Π^1_1 sets, and it may be rather difficult to prove anything about the ordinals $\rho(T(\alpha))$.

In many cases one can replace the canonical rank function with a "custom" one that better reflects the structure of a set.

Given a set S, a **rank** on S is a map $\varphi : S \to \mathrm{Ord}$. A rank is called **regular** if $\varphi(S)$ is an ordinal, i.e. $\varphi(S)$ is an initial segment of Ord.

Each rank gives rise to a **prewellordering** $\leq_\varphi$:

$$x \leq_\varphi y \quad \Leftrightarrow \quad \varphi(x) \leq \varphi(y).$$

A prewellordering is a binary relation on S that is reflexive, transitive, and *connected* (any two elements are comparable), and every non-empty subset of S has a $\leq_\varphi$-minimal element.

Under AC every set can be well-ordered, which means that every set admits a regular rank function that is one-one. However, we would like a rank function to reflect the complexity and structure of the set. In particular, we would like to preserve the boundedness properties of Σ^1_1 sets. For those to hold it was crucial that the initial segments WOrd_ξ, $\xi < \omega_1$ (and similarly for $\mathcal{O}$) are Borel.

We formulate a similar property that ensures the same for general rank functions.

Definition 21.4: Let X be a Polish space, and suppose $A \subseteq X$. A rank $\varphi : A \to \mathrm{Ord}$ is a **$\mathbf{\Pi}^1_1$-rank** if there exists a $\mathbf{\Sigma}^1_1$ relation $\leq^\Sigma_\varphi$ and a $\mathbf{\Pi}^1_1$ relation $\leq^\Pi_\varphi$ such that for $y \in A$,

$$\begin{aligned}\{x \in A \colon \varphi(x) \leq \varphi(y)\} &= \{x \in X : x \leq^\Sigma_\varphi y\} \\ &= \{x \in X : x \leq^\Pi_\varphi y\}.\end{aligned}$$

In other words, the initial segments $\leq_\varphi$ below a given $y \in A$ are uniformly $\mathbf{\Delta}^1_1$.

Theorem 21.5: *Every $\mathbf{\Pi}^1_1$ set $A \subseteq \mathbb{N}^\mathbb{N}$ admits a $\mathbf{\Pi}^1_1$-rank.*

Proof. We first show that WOrd admits a $\mathbf{\Pi}^1_1$-rank. The function φ is obviously $\varphi(\alpha) = \|\alpha\|$. We have to express $\|\alpha\| \le \|\beta\|$ in a $\Sigma 1$ and a $\mathbf{\Pi}^1_1$ way.

For the $\mathbf{\Sigma}^1_1$ relation $\le^\Sigma_\varphi$, let

$$\begin{aligned}\alpha \le^\Sigma_\varphi \beta \quad &\Leftrightarrow \quad E_\alpha \text{ is a linear ordering and} \\ &\qquad \exists \gamma\, [\gamma \text{ is a one-one, relation preserving mapping } \gamma : E_\alpha \to E_\beta] \\ &\Leftrightarrow \quad E_\alpha \text{ is a linear ordering and } \exists \gamma \forall m,n\, [m\, E_\alpha\, n \;\Rightarrow\; \gamma(m)\, E_\beta \gamma(n)].\end{aligned}$$

Recall that "E_α is a linear ordering" is Π^0_1, hence $\le^\Sigma_\varphi$ is Σ^1_1.

For the $\mathbf{\Sigma}^1_1$ relation $\le^\Pi_\varphi$, let

$$\begin{aligned}\alpha \le^\Pi_\varphi \beta \quad &\Leftrightarrow \quad E_\alpha \text{ is a well-ordering and} \\ &\qquad \text{there is no relation preserving mapping of } E_\beta \text{ onto an initial segment of } E_\alpha \\ &\Leftrightarrow \quad \alpha \in \text{WOrd and } \forall \gamma \neg \exists k \forall m,n\, [m\, E_\beta\, n \;\Rightarrow\; \gamma(m)\, E_\alpha \gamma(n)\, E_\alpha k].\end{aligned}$$

Since WOrd is Π^1_1, $\le^\Pi_\varphi$ is Π^1_1, too.

Now we have for $\beta \in$ WOrd,

$$\alpha \le^\Sigma_\varphi \beta \quad \Leftrightarrow \quad \alpha \le^\Pi_\varphi \beta \quad \Leftrightarrow \quad \|\alpha\| \le \|\beta\|,$$

as desired. □

Theorem 21.6 (Boundedness for arbitrary rank functions)**:** *Suppose $A \subseteq X$ is $\mathbf{\Pi}^1_1$ but not Borel and $\varphi : A \to \mathrm{Ord}$ is a $\mathbf{\Pi}^1_1$-rank on A. If $B \subseteq A$ is $\mathbf{\Sigma}^1_1$, then there is an $x_0 \in A$ such that*

$$\varphi(x) \le \varphi(x_0) \quad \textit{for all } x \in B.$$

Proof. If not, then

$$x \in A \quad \Leftrightarrow \quad \exists y\, [y \in B \wedge x \le^\Sigma_\varphi y],$$

and thus A would be $\mathbf{\Sigma}^1_1$, and thus Borel, a contradiction. □

Corollary 21.7: *Suppose $A \subseteq X$ is $\mathbf{\Pi}^1_1$ and $\varphi : A \to \mathrm{Ord}$ is a regular $\mathbf{\Pi}^1_1$-rank. Then*

(a) $\varphi(A) \le \omega_1$*;*

(b) A is Borel if $\varphi(A) < \omega_1$;

(c) if $B \subseteq A$ is $\mathbf{\Sigma}^1_1$, then $\sup\{\varphi(x) : x \in B\} < \omega_1$.

The Cantor-Bendixson Rank

We illustrate the concept of $\mathbf{\Pi}^1_1$-ranks with a rank function that is different from the canonical rank function.

Suppose T is a tree on $\{0,1\}$. Define the **Cantor-Bendixson derivative** of T as

$$T' = \{\sigma \in T : \ \sigma \text{ has at least two incompatible extensions}\}.$$

We can iterate this derivative along the ordinals:

$$\begin{aligned} T^{(\xi+1)} &= (T^{(\xi)})' \quad \text{and} \\ T^{(\lambda)} &= \bigcup_{\xi<\lambda} T^{(\xi)} \quad \text{for } \lambda \text{ limit.} \end{aligned}$$

We clearly have $T^{(\zeta)} \subseteq T^{(\xi)}$ for $\zeta < \xi$. There must exist an ordinal ξ_0 such that $(T^{(\xi_0)})' = T^{(\xi_0)}$. Since T is countable, $\xi_0 < \omega_1$. We call the least such ξ_0 the **Cantor-Bendixson rank** of T, $\| T \|_{\mathrm{CB}}$.

The following is not hard to see.

Proposition 21.8: *For any tree T,*

(a) *if $[T^{\|T\|_{\mathrm{CB}}}] \neq \emptyset$, then $[T^{\|T\|_{\mathrm{CB}}}]$ is a perfect subset of $\mathbb{N}^{\mathbb{N}}$;*

(b) *$T^{\|T\|_{\mathrm{CB}}} = \emptyset$ if and only if $[T]$ is countable.*

We hence have a new proof of the Cantor-Bendixson Theorem 2.5 for $2^{\mathbb{N}}$.

One can show that $\| \,.\, \|_{\mathrm{CB}}$ is indeed a $\mathbf{\Pi}^1_1$-rank on the set of all countable compact subsets of $2^{\mathbb{N}}$. This follows from the theory of **Borel derivatives**, which generalizes the Cantor-Bendixson derivative to other settings (see Kechris [1995]).

Since for any given ordinal $\xi < \omega_1$, we can find a tree $T \subseteq 2^{<\mathbb{N}}$ with $\| T \|_{\mathrm{CB}} = \xi$, it follows that the set

$$K_\omega(2^{\mathbb{N}}) = \{K \subseteq 2^{\mathbb{N}} : K \text{ countable}\}$$

is not Borel.

Using a different derivative, Kechris and Woodin [1986] showed that the set

$$\mathrm{Diff} = \{f \in \mathcal{C}[0,1] : f \text{ differentiable on } [0,1]\}$$

is not Borel.