# Lecture 2: Polish Spaces

The proofs in the previous lecture are quite general, that is, they make little use of specific properties of $\mathbb{R}$. If we scan the arguments carefully, we see that we can replace $\mathbb{R}$ by any metric space that is *complete and contains a countable basis of the topology*.

## Review of some concepts from topology

Let $(X, \mathcal{O})$ be a topological space. A family $\mathcal{B} \subseteq \mathcal{O}$ of subsets if $X$ is a **basis** for the topology if every open set from $\mathcal{O}$ is the union of elements of $\mathcal{B}$. For example, the open intervals with rational endpoints form a basis of the standard topology of $\mathbb{R}$. (We used this fact in Lecture 1.) $\mathcal{S} \subseteq \mathcal{O}$ is a **subbasis** if the set of finite intersections of sets in $\mathcal{S}$ is a basis for the topology. Finally, if $\mathcal{S}$ is any family of subset of $X$, the **topology generated by** $\mathcal{S}$ is the smallest topology on $X$ containing $\mathcal{S}$. It consists of all unions of finite intersections of sets in $\mathcal{S} \cup \{X, \emptyset\}$.

A set $D \subset X$ is **dense** if for open $U \neq \emptyset$ there exists $z \in D \cap U$. If a topological space $(X, \mathcal{O})$ has a countable dense subset, the space is called **separable**.

If $(X_i)_{i \in I}$ is a family of topological spaces, one defines the **product topology** on $\Pi_{i \in I} X_i$ to be the topology generated by the sets $\pi_i^{-1}(U)$, where $i \in I$, $U \subseteq X_i$ is open, and $\pi_i : \Pi_{i \in I} X_i \to X_i$ is the $i$th projection.

Now suppose $(X, d)$ is a metric space. With each point $x \in X$ and every $\varepsilon > 0$ we associate an **$\varepsilon$-neighborhood** or **$\varepsilon$-ball**

$$U_\varepsilon(x) = \{y \in X : d(x, y) < \varepsilon\}.$$

The $\varepsilon$-neighborhoods form the basis of a topology, called the **topology** of the metric space $(X, d)$. If this topology agrees with a given topology $\mathcal{O}$ on $X$, we say the metric $d$ is **compatible** with the topology $\mathcal{O}$. If for a topological space $(X, \mathcal{O})$ there exists a compatible metric, $(X, \mathcal{O})$ is called **metrizable**[2].

If a topological space $(X, \mathcal{O})$ is separable and metrizable, then the balls with center in a countable dense subset $D$ and rational radius form a *countable base of the topology*.

[2]Note that a compatible metric is not necessarily unique.

## Polish spaces

**Definition 2.1:** A **Polish space** is a separable topological space $X$ for which exists a compatible metric $d$ such that $(X, d)$ is a complete metric space.

As mentioned before, there may be many different compatible metrics that make $X$ complete. If $X$ is already given as a complete metric space with countable dense subset, then we call $X$ a **Polish metric space**.

The standard example is, of course, $\mathbb{R}$, the set of real numbers. One can obtain other Polish spaces using the following basic observations.

**Proposition 2.2:**

1) *A closed subset of a Polish space is Polish.*
2) *The product of a countable (in particular, finite) sequence of Polish spaces is Polish.*

Hence we can conclude that $\mathbb{R}^n$, $\mathbb{C}$, $\mathbb{C}^n$, the unit interval $[0, 1]$, the unit circle $\mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$, and the infinite dimensional spaces $\mathbb{R}^{\mathbb{N}}$ and $[0, 1]^{\mathbb{N}}$ (the *Hilbert cube*) are Polish spaces.

Any countable set with the **discrete topology** is Polish, by means of the **discrete metric** $d(x, y) = 1 \iff x \neq y$.

Some subsets of Polish spaces are Polish but not closed. For example, $(0, 1)$, the open unit interval, is a Polish space, of course with a different metric. We will later characterize all subsets of Polish spaces that are Polish themselves.

## Product spaces

In a certain sense, the most important Polish spaces are of the form $A^{\mathbb{N}}$, where $A$ is a countable set carrying the discrete topology. The standard cases are

$$2^{\mathbb{N}}, \text{ the Cantor space} \qquad \text{and} \qquad \mathbb{N}^{\mathbb{N}}, \text{ the Baire space.}$$

We will, for now, denote elements from $A^{\mathbb{N}}$ by lower case greek letters from the beginning of the alphabet. The $n$-th term of $\alpha$ we will denote by either $\alpha(n)$ or $\alpha_n$, whichever is more convenient.

We endow $A$ with the discrete topology. The product topology on these spaces has a convenient characterization. Given a set $A$, let $A^{<\mathbb{N}}$ be the sets of all finite binary sequences over $A$. Given $\sigma, \tau \in A^{\mathbb{N}}$, we write $\sigma \subset \tau$ to indicate that $\sigma$ is an initial segment of $\tau$. $\subset$ means the initial segment is proper. This notation extends naturally to hold between elements of $2^{<\mathbb{N}}A$ and $A^{\mathbb{N}}$, $\sigma \subset \alpha$ meaning that $\sigma$ is a finite initial segment of $\alpha$.

A basis for the product topology on $A^{\mathbb{N}}$ is given by the **cylinder sets**

$$[\sigma] = \{\alpha \in A^{\mathbb{N}} : \sigma \subset \alpha\},$$

that is, the set of all infinite sequences extending $\sigma$. The complement of a cylinder is a finite union of cylinders and hence open. Therefore, each set $[\sigma]$ is clopen.

A compatible metric is given by

$$d(\alpha, \beta) = \begin{cases} 2^{-N} & \text{where } N \text{ is least such that } \alpha_N \neq \beta_N \\ 0 & \text{if } \alpha = \beta. \end{cases}$$

The representation of the topology via cylinders (which are characterized by finitary objects) allows for a combinatorial treatment of many questions and will be essential later on.

**Proposition 2.3** (Topological properties of $A^{\mathbb{N}}$)**:** *Let $A$ be a countable set, equipped with the discrete topology. Suppose $A^{\mathbb{N}}$ is equipped with the product topology. Then the following hold.*

1) *$A^{\mathbb{N}}$ is Polish.*
2) *$A^{\mathbb{N}}$ is* zero-dimensional*, i.e. it has a basis of* clopen *sets.*
3) *$A^{\mathbb{N}}$ is compact if and only if $A$ is finite.*

Via the mapping

$$\alpha \mapsto \sum_{i=0}^{\infty} \frac{2\alpha_i}{3^{i+1}},$$

$2^{\mathbb{N}}$ is homeomorphic to the middle-third Cantor set in $\mathbb{R}$, whereas the **continued fraction** mapping

$$\beta \mapsto \beta_0 + \cfrac{1}{\beta_1 + \cfrac{1}{\beta_2 + \cfrac{1}{\beta_3 + \dots}}}$$

provides a homeomorphism between $\mathbb{N}^{\mathbb{N}}$ and the irrational real numbers.

The universal role played by the discrete product spaces is manifested in the following results.

**Theorem 2.4:** *Every perfect Polish space contains a homeomorphic embedding of the Cantor space,* $2^{\mathbb{N}}$.

The proof is similar to the proof of Theorem 1.1. Note that the proof actually constructs an embedding of $2^{\mathbb{N}}$. The continuity of the mapping is straightforward.

**Theorem 2.5:** *Every Polish space $X$ is the continuous image of the Baire space,* $\mathbb{N}^{\mathbb{N}}$.

*Proof.* Let $d$ be a compatible metric on $X$, and let $D = \{x_i : i \in \mathbb{N}\}$ be a countable dense subset of $X$. Every point in $X$ is the limit of a sequence in $D$. Define a mapping $g : \mathbb{N}^{\mathbb{N}} \to X$ by putting

$$\alpha = \alpha(0)\,\alpha(1)\,\alpha(2)\ldots \mapsto \lim_n x_{\alpha(n)}.$$

The problem is, of course, that the limit on the right hand side not necessarily exists. Besides, even if it exists, the mapping may not be continuous at that point, since we made no additional assumptions about the set $D$.

To remedy the situation, we proceed more carefully. Given $\alpha \in \mathbb{N}$, we define iteratively $y_0^\alpha = x_{\alpha(0)}$ and

$$y_{n+1}^\alpha = \begin{cases} x_{\alpha(n+1)} & \text{if } d(y_n^\alpha, x_{\alpha(n+1)}) < 2^{-n}, \\ y_n^\alpha & \text{otherwise .} \end{cases}$$

The resulting sequence $(y_n^\alpha)$ is clearly Cauchy in $X$, and hence converges to some point $y^\alpha \in X$, by completeness. We define

$$f(\alpha) = y^\alpha.$$

$f$ is continuous, since if $\alpha$ and $\beta$ agree up to length $N$ (that is, their distance is at most $2^{-N}$ with respect to the above metric), then the sequences $(y_n^\alpha)$ and $(y_n^\beta)$ will agree up to index $N$, and all further terms are within $2^{-N}$ of $y_N^\alpha$ and $y_N^\beta$, respectively.

Finally, since $D$ is dense in $X$, $f$ is a surjection. □

We will later combine the techniques for both theorems to prove a strengthening of Theorem 2.5.