Homework 2 for **MATH 574**

Due: Wednesday February 9

Problem 1

Prove Proposition 3.4: Let $\mathcal{U}$ be a Polish metric space that contains an isometric image of every separable metric space. Then $\mathcal{U}$ is Urysohn universal if and only if every isometry between finite subsets of $\mathcal{U}$ extends to an isometry of $\mathcal{U}$ onto itself.

If we do not require the strong homogeneity properties of the Urysohn space, there are various universal spaces.

Problem 2

Recall the *Hilbert cube* $\mathbb{I} = [0, 1]^{\mathbb{N}}$ with the product topology. Show that every separable metric space is homeomorphic to a subspace of $\mathbb{I}$.

Problem 3

Show that every compact metric space is a continuous image of Cantor space $2^{\mathbb{N}}$.

One can even find spaces that are universal with respect to isometries.

Problem 4

Let l^∞ be the Banach space of bounded sequences in $\mathbb{R}$, equipped with the sup-norm. Show that every separable metric space embeds isometrically into l^∞.

Bonus: Show, however, that l^∞ is not Urysohn universal.

Problem 5

A mapping $\phi : A^{<\mathbb{N}} \to A^{<\mathbb{N}}$ is *monotone* if it preserves the (inverse) prefix relation:

$$\sigma \supseteq \tau \quad \Rightarrow \quad \phi(\sigma) \supseteq \phi(\tau).$$

Let S, T be well-founded trees over A. Show that $\rho(S) \leqslant \rho(T)$ if and only if there exists a monotone mapping $\phi : S \to T$.

(*Hint*: Proceed by induction on $\rho(T)$. Note that ρ_T is an order preserving mapping from T into the ordinals.)