Lesson 3
Dynamical Systems
3-5: Ergodicity

Jan Reimann

Math 574, Topics in Logic
Penn State, Spring 2014

PENNSTATE
Eberly College of Science

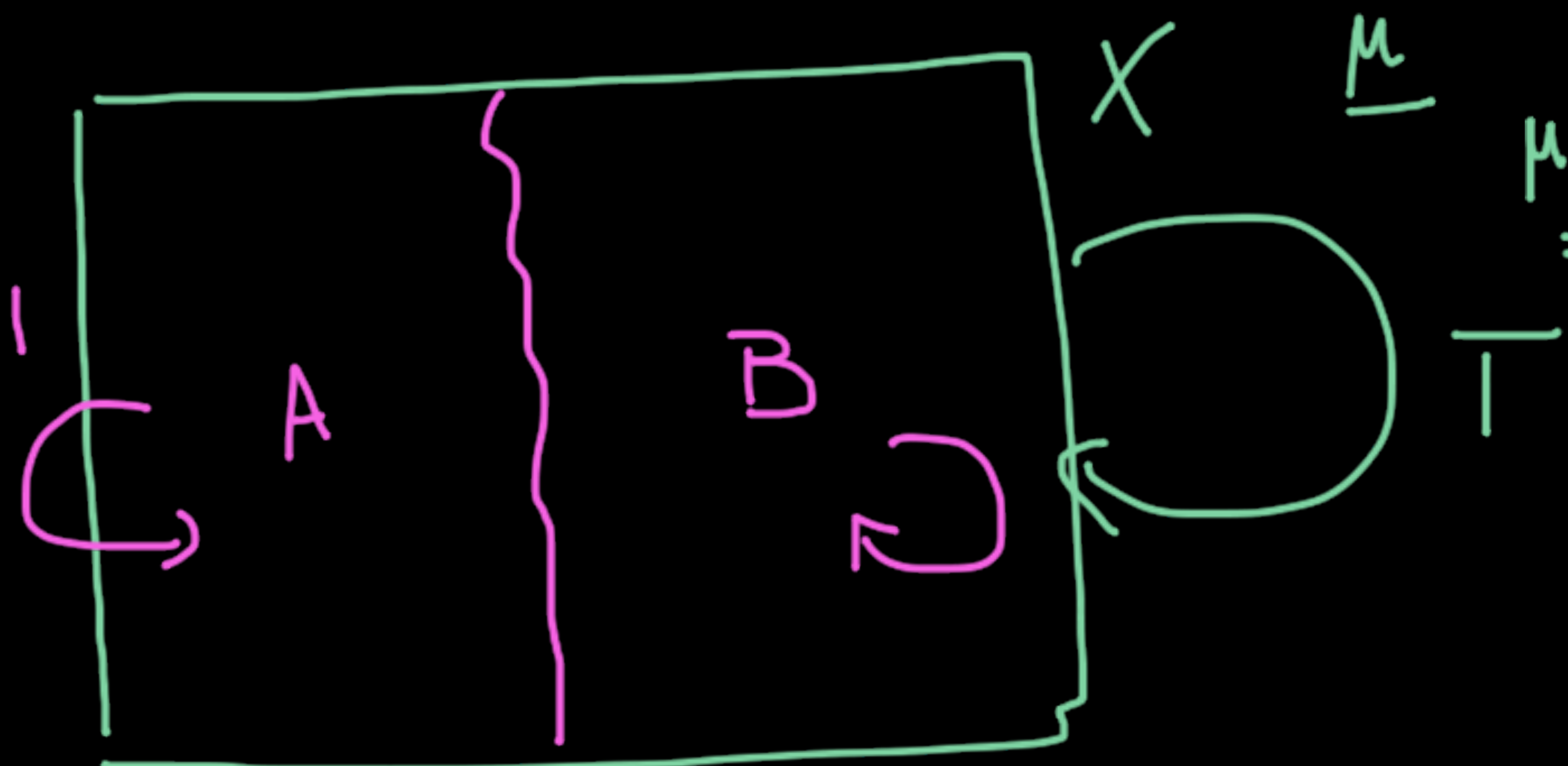
Ergodic Systems

Irreducible Markov shifts: can reach any state from any other.

Cannot be ``broken up'' into smaller systems.

Ergodicity: measure-theoretic version of irreducibility.

$0 < \mu(A) < 1$



$\mu(A) = \mu(T^{-1}A)$

$TA \subseteq A$ $\qquad$ $TB \subseteq B$

Ergodic Systems

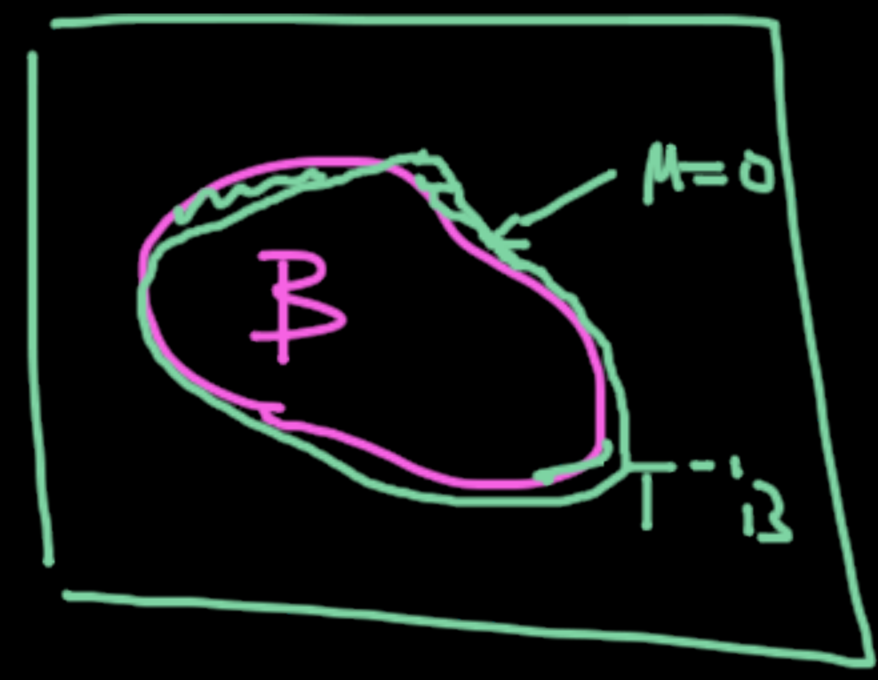
$(X, \mathcal{B}, \mu, T)$ measure-theoretic dynamical system.

- $A \subseteq X$ is T-invariant if $TA \subseteq A$.
- X is T-decomposable if there exist disjoint, T-invariant $B_1, B_2 \in \mathcal{B}$ of positive measure so that $X = B_1 \cup B_2$.

  In this case we have $T^{-1}B_i = B_i$.

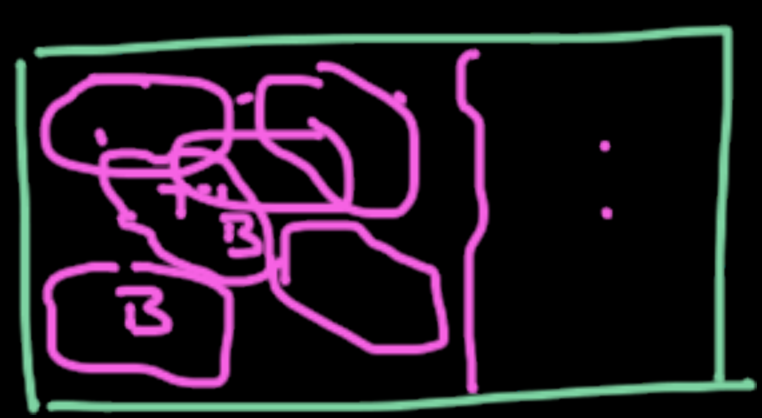
- T is ergodic if X is T-indecomposable.

Equivalent Formulations



The following are equivalent:

- T is ergodic
- For all $B \in \mathcal{B}$ with $T^{-1}B = B$, $\mu(B) = 0$ or $\mu(B) = 1$.
- $\mu(T^{-1}B \triangle B) = 0$ implies $\mu(B) = 0$ or $\mu(B) = 1$.
- If $\mu(B) > 0$, then $\mu\left(\bigcup_n T^{-n}B\right) = 1$.



PENNSTATE
Eberly College of Science

The Operator Theoretic View

A measure preserving transformation T induces an isometry of $L^p(X)$.

Recall:

$$L^p(X, \mathcal{B}, \mu) = \left\{ f : X \to \mathbb{C} : \ f \text{ measurable and } \int |f|^p d\mu < \infty \right\}.$$

$L^0(X) =$ measurable functions on X

Operator $U_T : L^p(X) \to L^p(X)$ defined by letting

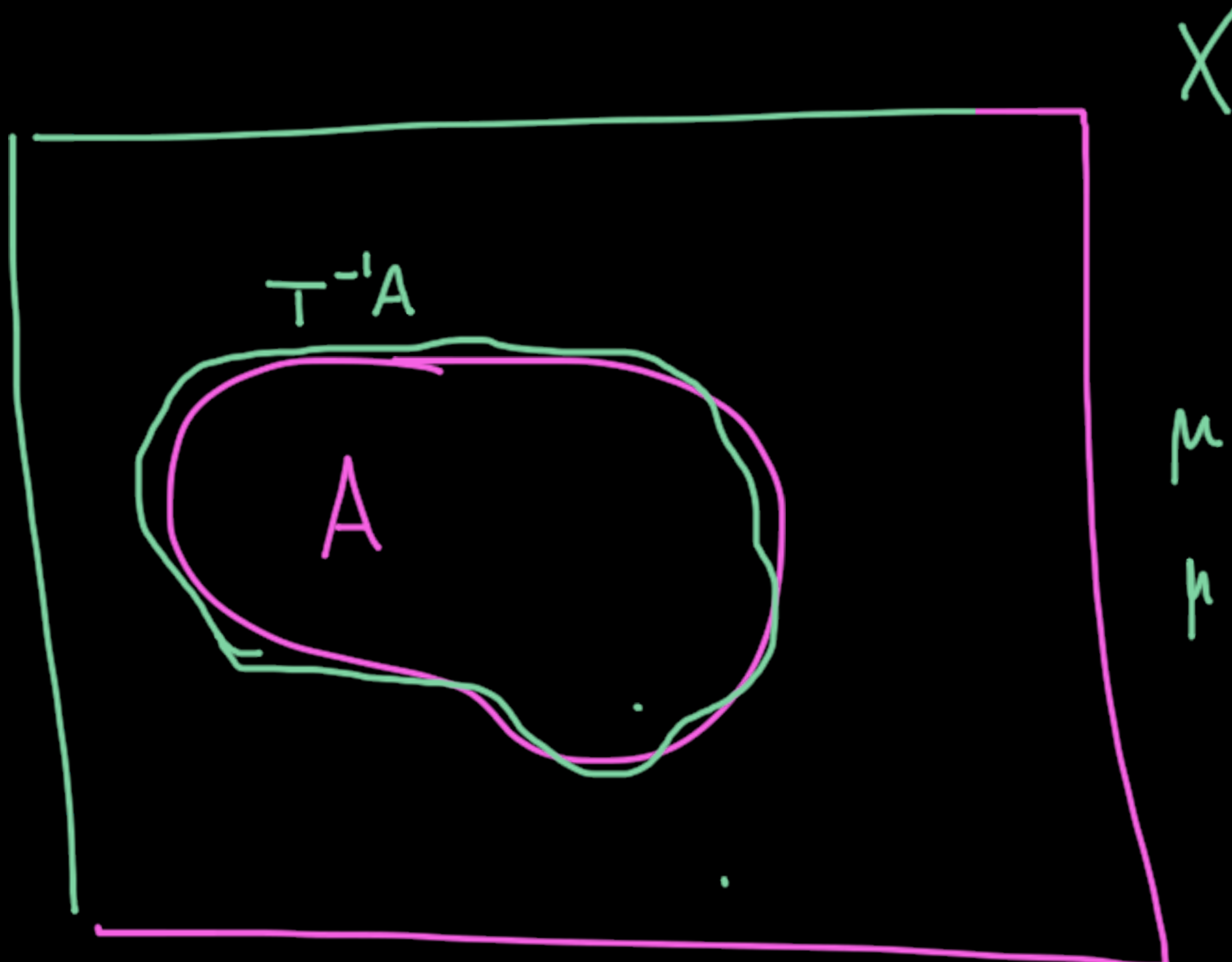
$$U_T(f) = f \circ T$$

T measure preserving implies U_T is isometry with respect to the usual p-norm.

Ergodicity can then be expressed as: for all f measurable,

$$U_T(f) = f \ \mu\text{-a.e. implies } f \text{ constant } \mu\text{-a.e.}$$

PENNSTATE Eberly College of Science



$\mu(A) = 0$

$\mu(A) = 1$

$T^{-1}A = A$

$f = \chi_A$ μ-a.e.

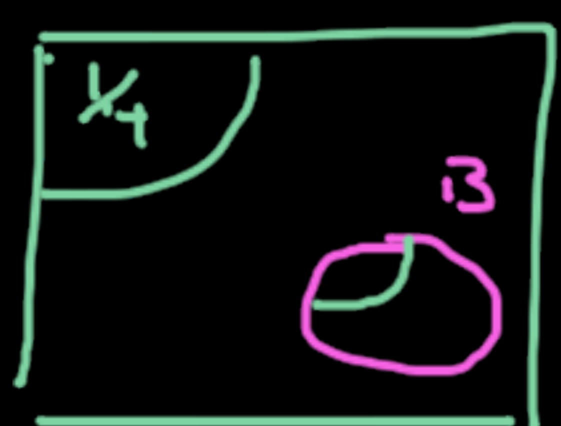
$f \circ T = U_T f = f$ μ-a.e. $\Rightarrow f$ const

Ergodicity of IID Processes

We call a stationary process (X_n) ergodic if the shift map is ergodic with respect to its Kolmogorov representation.

If (X_n) is i.i.d., then the corresponding dynamical system is not only ergodic, but (strongly) mixing:

$$\lim_n \mu(T^{-n}A \cap B) = \mu(A)\mu(B) \quad (A, B \in \mathcal{B}).$$



$$\frac{\mu(T^{-n}A \cap B)}{\mu(B)} \longrightarrow \mu(A)$$

Mixing implies ergodicity: Assume $T^{-1}B = B$ and consider the mixing condition with $A = B$.

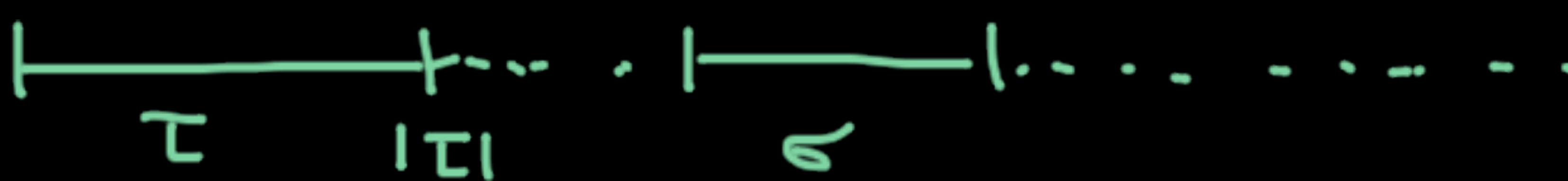
$$\lim_n \underbrace{\mu(\underbrace{T^{-n}B}_{B} \cap B)}_{\mu(B)} = \mu(B)^2$$

$$\Rightarrow \mu(B) = 0 \text{ or } = 1$$

Mixing for IID Processes

To establish mixing for all sets $A, B \in \mathcal{B}$, it suffices to show it for cylinder sets.

- Observe: $T^{-n}[\sigma] = [\sigma]_n$ $= \{x : x_n = \sigma_0 \quad x_{n+1} = \sigma_1 \cdots \}$
- If $n > |\tau|$, then $[\sigma]_n$ fixes positions independently of τ.



- Since μ is a product measure, it follows that $\mu([\sigma]_n \cap [\tau]) = \mu[\sigma]_n \mu[\tau]$.

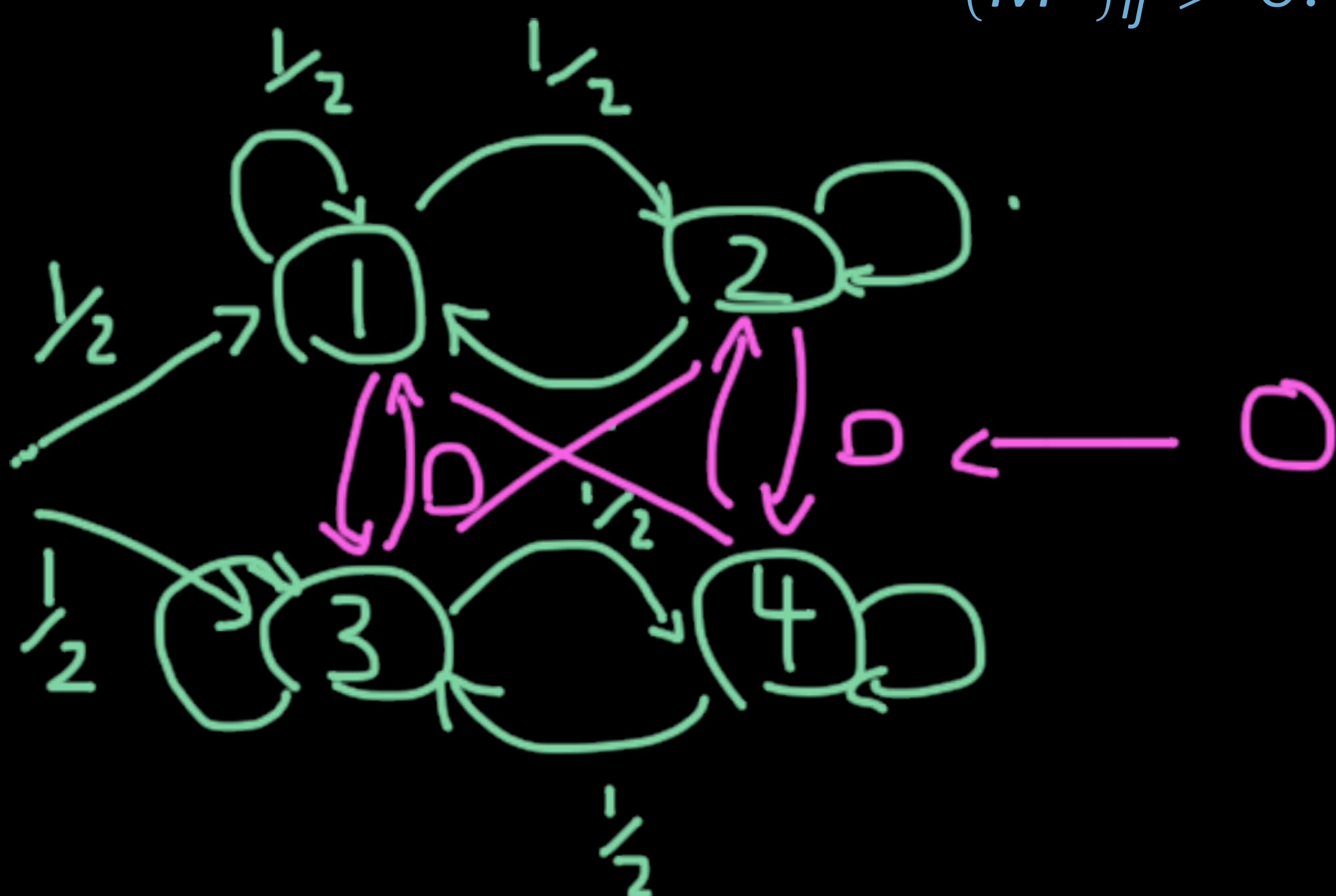
Ergodicity for Markov Chains

When is a Markov chain ergodic?

- For Markov chains, ergodicity can be thought of as a measure theoretic version of irreducibility.

Stochastic irreducibility: A stochastic matrix M is irreducible if for any pair i, j there exists n s.t.

$$(M^n)_{ij} > 0.$$



$$M = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$= M^2 = M^3 = \dots$$

Ergodicity for Markov Chains

THM: For a stationary Markov chain $(\vec{p}, M)$ given by a stochastic matrix M and an initial distribution $\vec{p} = (p_1, \ldots, p_n)$ with $p_i > 0$, the following are equivalent.

(i) $(\vec{p}, M)$ is ergodic,

(ii) M is irreducible,

(iii) 1 is a simple eigenvalue of M.