Homework 4 for MATH 104

Brief solutions to selected exercises

Problem 1

Determine if the following series converge. Justify your answer.

(a) $\sum_{n=1}^{\infty}(\sqrt{n+1}-\sqrt{n})$,

Solution. This is a telescoping sum. It holds that $S_n = \sum_{k=1}^{n}(\sqrt{n+1}-\sqrt{n}) = \sqrt{n+1}-\sqrt{1}$. Since $\sqrt{n+1} \to \infty$, it follows that the series diverges. ■

(b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$,

Solution. We want to use the ratio test. $a_n = \frac{n!}{n^n}$, and a simple calculation yields that $\frac{a_{n+1}}{a_n} = (\frac{n}{n+1})^n$. We know (midterm!) that $\lim_n(\frac{n}{n+1})^n = e$, and a simple estimate yields that $2 \leqslant e \leqslant 3$. Therefore, by the limit theorems, we have

$$\limsup_n \left|\frac{a_{n+1}}{a_n}\right| = \lim_n(\frac{n}{n+1})^n = \frac{1}{e} < 1.$$

Therefore, the series converges by the ratio test. ■

(c) $\sum_{n=1}^{\infty} \frac{x^n}{n!}$, where x is an arbitrary real number.

Solution. The ratio test yields $a_{n+1}/a_n = x/(n+1)$ since $x/(n+1) \to 0$ for all $x \in \mathbb{R}$, we conclude that the series converges for all $x \in \mathbb{R}$. ■

(d) $\sum_{n=1}^{\infty} \frac{\sin(n\pi/6)}{n}$.

Solution. Let $a_n = \sin(n\pi/6)$. Then $a_n = a_{n+12}$ and $a_n = -a_{n+6}$.

Define b_n as

$$b_n = \frac{a_1}{6n+1} + \frac{a_2}{6n+2} + \cdots + \frac{a_6}{6n+6}.$$

It follows from the alternating series theorem that $\sum_n(-1)^n b_n$ converges. Thus, the subsequence S_{6n} of the partial sums of $\sum_{n=1}^{\infty} \frac{\sin(n\pi/6)}{n}$ is convergent. To show that $\sum_{n=1}^{\infty} \frac{\sin(n\pi/6)}{n}$ converges, we apply the Cauchy criterion for series, by noting that the values of the n-th partial sum S_n cannot differ much from S_m, where m is the closest number to n of the form $6k$. ■

Problem 2

Find an absolutely convergent series $\sum_n a_n$ such that

$$\limsup_n \frac{a_{n+1}}{a_n} = \infty.$$

Justify your answer.

Solution. The series $\sum_n \frac{n}{2^n}$ converges absolutely, as can be easily seen by the ratio test. Define a sequence a_n by

$$a_n = \begin{cases} 2^{-n} & \text{if } n \text{ is odd,} \\ n2^{-n} & \text{if } n \text{ is even.} \end{cases}$$

The series $\sum_n a_n$ converges absolutely, as verified by the comparison criterion. For odd n, we have

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{2}.$$

Therefore, $\limsup_n a_{n+1}/a_n = \infty$. ■

Problem 3

Assume $\sum_n a_n$ converges and $a_n \geqslant 0$ for all $n \in \mathbb{N}$. Show that

$$\sum_n \frac{\sqrt{a_n}}{n}$$

converges.

Solution. If $\sqrt{a_n}/n > a_n$ then it follows that $1/n > \sqrt{a_n}$, and hence $1/n^2 > \sqrt{a_n}/n$. Therefore, for all n, $\sqrt{a_n}/n \leqslant \max\{a_n, 1/n^2\}$. It remains to show that if $\sum_n a_n$ and $\sum_n b_n$ converge, and $a_n, b_n \geqslant 0$, then $\sum_n c_n$ with $c_n = \max\{a_n, b_n\}$ converges. But this is easily verified via the comparison criterion, since $c_n \leqslant a_n + b_n$. ■

Problem 4

Suppose $a_n > 0$ for all n, and suppose $\sum a_n$ converges. Set

$$r_n = \sum_{k=n}^{\infty} a_k.$$

(a) Prove that if $m < n$, then

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}.$$

Deduce that $\sum \frac{a_n}{r_n}$ diverges.

Solution. If $m < n$, then $r_m > r_n$, since $a_n > 0$ for all n. This implies

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_m} = \frac{r_m - r_{n+1}}{r_m} = 1 - \frac{r_{n+1}}{r_m} > 1 - \frac{r_n}{r_m}$$

Suppose now $\sum \frac{a_n}{r_n}$ converges, then, by the Cauchy criterion for series, there must be an $N \in \mathbb{N}$ such that for all $m, n > N$,

$$\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} < \frac{1}{2}.$$

It follows that for all n,

$$1 - \frac{r_n}{r_{N+1}} < 1/2,$$

or, equivalently, $r_n > r_{N+1}/2$. But this is impossible, since the convergence of $\sum_n a_n$ implies that $r_n \to 0$. ■

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}}).$$

Deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Solution.

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

is equivalen to

$$\frac{a_n}{\sqrt{r_n}}(\sqrt{r_n} + \sqrt{r_{n+1}} < 2(r_n - r_{n+1}).$$

$r_n - r_{n+1}$ evaluates to a_n, hence the last inequality holds iff

$$a_n \left(1 + \frac{\sqrt{r_{n+1}}}{\sqrt{r_n}}\right) < 2a_n,$$

which in turn holds iff $\frac{\sqrt{r_{n+1}}}{\sqrt{r_n}} < 1$. But this holds since $r_{n+1} < r_n$.

The convergence of $\sum \frac{a_n}{\sqrt{r_n}}$ now easily follows from a telescoping sum argument. It holds that $\sum_{k=1}^{n} 2(\sqrt{r_k} - \sqrt{r_{k+1}}) = 2\sqrt{r_1} - 2\sqrt{r_{n+1}}$, hence $\sum_n \sum_n 2(\sqrt{r_n} - \sqrt{r_{n+1}})$ converges, and by the comparison test, $\sum \frac{a_n}{\sqrt{r_n}}$ converges. ■