Lesson 1

Sequence Spaces,
Random Variables,
And Probailistic Processes

Math 574 - Topics in Logic
Penn State, Spring 2014

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Sequence Spaces

Infinite Sequences

A alphabet

$A^{\mathbb{N}}$ infinite sequences over A

$x = x_0 x_1 x_2 x_3 \ldots \qquad x_i \in A$

$= x(0)\, x(1)\, x(2)\, x(3) \ldots \qquad x(i) \in A$

alternative notation

x is a function $\mathbb{N} \to A$

$A^{\mathbb{Z}}$ bi-infinite sequences over A

$x = \ldots x_{-2} x_{-1} x_0 x_1 x_2 \ldots$

important in dynamics context

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$A^{\mathbb{N}}$ as a Product Space

Topology: What are the open sets?

Topology on A: standard topology for $A = \mathbb{R}$, $[0,1]$

For A finite, or $A = \mathbb{N}, \mathbb{Z}$ we use the discrete topology: every subset is open

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Product topology on $A^{\mathbb{N}}$: smallest topology s.t.

projections $\pi_i : A^{\mathbb{N}} \to A$, $x \mapsto x_i$ are continuous.

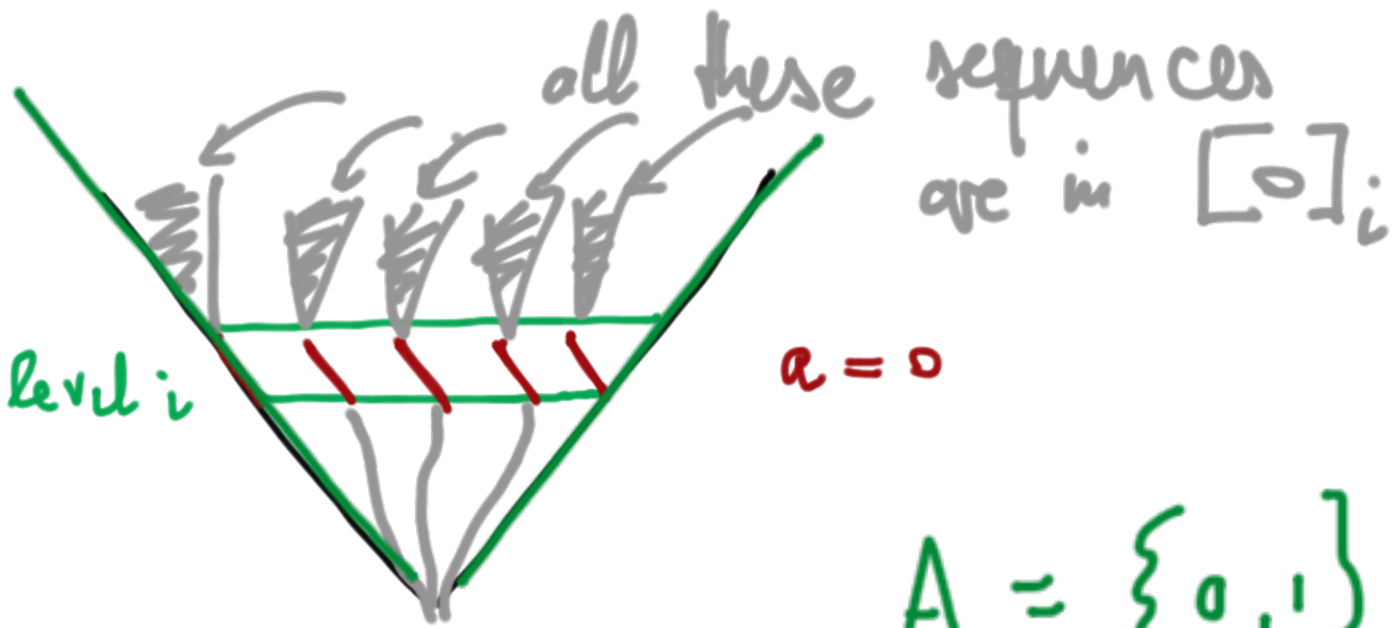
This means that for $U \subseteq A$ open, $\pi_i^{-1}(U)$ must be open, for $F \subseteq A$ closed, $\pi_i^{-1}(F)$ must be closed.

For discrete topology, $\{a\} \subseteq A$ is open.

$$\pi_i^{-1}(\{a\}) = \{x \in A^{\mathbb{N}} : x_i = a\}$$

Cylinder sets: Fix $a \in A$, $i \in \mathbb{N}$

$$[a]_i = \{ x \in A^{\mathbb{N}} : x_i = a \} = \pi_i^{-1}(\{a\})$$



$A = \{0, 1\}$

Basic open cylinders

Smallest topology generated by cylinder sets $[a]_i$:

Need to close under

- finite intersections
- arbitrary unions

This means the $[a]_i$ form a subbase of the topology

Finite intersections of cylinders:

$$[a]_i \cap [b]_j = \{ x \in A^{\mathbb{N}} : x_i = a \ \& \ x_j = b \}$$

$\to$ Basic open cylinders: Finitely many positions are fixed

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Basic Open Cylinders

Any such basic open cylinder can be obtained from cylinder of the following form:

$$[\sigma]_n = \{ x \in A^{\mathbb{N}} : x_n = \sigma_0 , \dots , x_{n+k-1} = \sigma_{k-1} \}$$

string of length k

(using unions and finite intersections)

For $n=0$ we drop subscript and write simply

$[\sigma]$ Basic open cylinder given by σ

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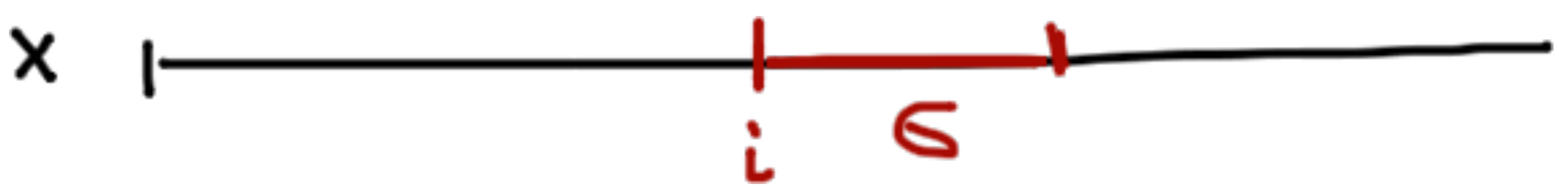
Summary:

Open sets in $A^{\mathbb{N}}$ are precisely the ones that can be written as a union of basic open cylinders $[\sigma]_i$



Note: development for $A^{\mathbb{Z}}$ is similar

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