Homework 7 for MATH 185

Due: Wednesday March 14, 3:10 pm in class

Problem 1

Let $D \subseteq \mathbb{C}$ be open, and let $M \subseteq D$. Show that the following are equivalent:

(i) M is discrete in D.

(ii) For each $p \in M$ there exists an $\varepsilon > 0$ such that $U_\varepsilon(p) \cap M = \{p\}$ and M is closed in D (i.e. there exists a closed set $A \subseteq \mathbb{C}$ such that $M = A \cap D$).

(iii) For each compact set $K \subseteq D$, the set $M \cap K$ is finite.

(iv) For each $z \in D$ there exists an $\varepsilon > 0$ such that $U_\varepsilon(z) \subseteq D$ and $M \cap U_\varepsilon(z)$ is finite.

Problem 2

Let D be a domain, and let $M \subseteq D$ be discrete in D. Show that $D \setminus M$ is a domain.

Problem 3

Let f be the analytic function given by

$$f(z) = \sum_{n=1}^{\infty} z^{n!}.$$

Show that f cannot be analytically extended to any domain D properly containing the unit disk $\mathbb{E} = \{z : |z| < 1\}$.

Problem 4

Determine whether there exists an analytic function $f : U_\varepsilon(0) \to \mathbb{C}$, $\varepsilon > 0$ such that for all $n \in \mathbb{N}$,

(a)

$$f(\tfrac{1}{n}) = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even;} \end{cases}$$

(b)

$$f(\tfrac{1}{n}) = \begin{cases} \frac{1}{n+1} & \text{if } n \text{ is odd,} \\ \frac{1}{n} & \text{if } n \text{ is even;} \end{cases}$$

(c)

$$f(\tfrac{1}{n}) = \frac{n}{n+1}$$

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