Homework 2 for MATH 104

Due: Tuesday, September 19, 9:30am in class

Problem 1

Verify the following statements by induction: For all $n \in \mathbb{N}$,

(1) $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$,

Solution.

$n = 1$: Obviously, $1^3 = (1)^2$.

$n \curvearrowright n+1$: $1^3 + \cdots + n^3 + (n+1)^3 \overset{\text{Ind.Hyp.}}{=} (1 + \cdots + n)^2 + (n+1)^3 = \frac{1}{4}n^2(n+1)^2 + (n+1)^3 = \frac{1}{4}(n+1)^2(n^2+4n+4) = \frac{1}{4}(n+1)^2(n+2)^2 = (1 + \cdots + n + (n+1))^2$.

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(2) $(1+x)^n \geqslant 1 + nx$, whenever $x \geqslant -1$.

Solution.

$n = 1$: Obviously, $(1+x) \geqslant 1 + x$ holds.

$n \curvearrowright n+1$: We have $(1+x)^{n+1} = (1+x)^n(1+x)$, which, by inductive hypothesis, is $\geqslant (1+nx)(1+x)$. (Note that here it is important that $1+x \geqslant 0$, i.e. $x \geqslant -1$.) The last expression evaluates to $1 + (n+1)x + nx^2$. Since $nx^2 \geqslant 0$ for all n, it follows that $(1+x)^{n+1} \geqslant 1 + (n+1)x$, as desired.

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Problem 2

Investigate the following sequences. Determine whether they converge, and if so, determine their limit.

(1) $(a_n)_{n \in \mathbb{N}}$ with $a_n = \frac{5n+2}{3n+4}$;

Solution. We write a_n as $(5 + \frac{2}{n}/(3 + \frac{4}{n})$. Let $s_n = 5 + \frac{2}{n}$ and $t_n = 3 + \frac{4}{n}$. Since $(\frac{1}{n}) \to 0$, it follows from the limit theorems for sequences that $s_n \to 5$ and $t_n \to 3$. Another application of the limit theorems (for the sequences (s_n) and $(1/t_n)$) yields $\lim_n(a_n) = \lim(s_n/t_n) = \lim_n s_n \cdot \lim_n 1/t_n = \lim_n s_n \cdot 1/\lim_n t_n = 5/3$. ■

(2) $(b_n)_{n \in \mathbb{N}}$ with $b_n = \sqrt{n^2+n} - n$;

Solution. We first tranform the expression s_n into a different form that will make it easy to apply the limit theorems.

$$\sqrt{n^2+n} - n = \frac{(\sqrt{n^2+n} - n)(\sqrt{n^2+n} + n)}{\sqrt{n^2+n} + n} = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{n}{\sqrt{n^2+n}+n}$$
$$= \frac{1}{1 + \frac{\sqrt{n^2+n}}{n}} = \frac{1}{1 + \sqrt{\frac{n^2+n}{n^2}}} = \frac{1}{1 + \sqrt{1 + \frac{1}{n}}}.$$

Now we can apply the limit theorems: $(1/n)$ converges to 0, so $1+\frac{1}{n} \to 1$, and

$$\frac{1}{1+\sqrt{1+\frac{1}{n}}} \to \frac{1}{2}.$$

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(3) $(c_n)_{n\in\mathbb{N}}$ with $c_n = \frac{1^3+2^3+\cdots+n^3}{n^4}$.

Solution. We know from Problem 1.(1) that $1^3+2^3+\cdots+n^3 = (1+2+\cdots+n)^2$, so we can write

$$\frac{1^3+2^3+\cdots+n^3}{n^4} = \frac{(n^2(n+1)^2)}{4n^4} = \frac{n^4+2n^3+n^2}{4n^4} = \frac{1+\frac{2}{n}+\frac{1}{n^2}}{4}.$$

Since $(2/n) \to 0$ and $(1/n^2) \to 0$, the limit theorems imply that $(c_n) \to 1/4$. ■

Problem 3

Define a sequence (s_n) inductively by letting $s_1 = 1$, and $s_{n+1} = \sqrt{s_n+1}$. Prove that $\lim_n s_n = \frac{1+\sqrt{5}}{2}$. (*Hint:* For the limit s it must hold that $s = \sqrt{s+1}$. Why?)

Solution. We first show that the sequence converges by showing that it is bounded and monotone. These properties will be shown inductively.

(s_n) *is nondecreasing:* We have to show that for all n, $s_{n+1} \geqslant s_n$. For $n = 1$, we have $s_2 = \sqrt{2} \geqslant 1+s_1$, so the claim holds for $n = 1$. Assume now it holds for n, that is, $s_{n+1} = \sqrt{s_n+1} \geqslant s_n$. This implies that $\sqrt{s_n+1}+1 \geqslant s_n+1$, and, since for nonnegative $a, b \in \mathbb{R}$, $a \geqslant b$ implies $\sqrt{a} \geqslant \sqrt{b}$, it also follows that $\sqrt{\sqrt{s_n+1}+1} \geqslant \sqrt{s_n+1}$. Therefore, $s_{n+2} \geqslant s_{n+1}$, and the claim follows by induction.

(s_n) *is bounded from above by* 2*:* For $n = 1$, this is obviously true. Assume now the claim is true for n, then $s_{n+1} = \sqrt{s_n+1} \leqslant \sqrt{2+1} = \sqrt{3} \leqslant 2$. Again, the claim follows by induction.

We conclude that (s_n) converges. Let s be the limit of s_n. We argue that for s it must hold that $s = \sqrt{s+1}$. As (s_n) converges, it is easy to see that the sequence (s_{n+1}) converges to s, too. Hence we have $s = \lim s_{n+1} = \lim \sqrt{s_n+1}$. By the limit theorems, we can infer that $s = \sqrt{\lim s_n + 1} = \sqrt{s+1}$.

The limit s must therefore satisfy the equation $s^2 - s - 1 = 0$. This equation has two solutions, $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$. But s must also satisfy $1 \leqslant s \leqslant 2$, since it is nondecreasing and bounded by 2. Therefore $s = (1+\sqrt{5})/2$. ■

Problem 4

Let (a_n) be a convergent sequence of real numbers with limit a. Define the sequence (b_n) by

$$b_n = \frac{a_1+a_2+\cdots+a_n}{n} \tag{*}$$

Prove that (b_n) converges and that $\lim_n b_n = a$. (*Hint:* Reduce the general case to the case $b = 0$.)

Solution. We first assume $a = 0$. Let $\varepsilon > 0$. Then there exists some $N \in \mathbb{N}$ such that for all $n > N$, $|a_n| < \varepsilon/2$. Furthermore, by the Archimedean Principle, there exists a natural number $N' \geqslant N$ such that

$$\frac{|a_1|+\cdots+|a_N|}{N'} < \frac{\varepsilon}{2}.$$

It now follows, using the triangle-inequality, that for all $n > N'$,

$$|b_n| = \frac{|a_1+\cdots+a_n|}{n} \leqslant \frac{|a_1|+\cdots+|a_n|}{n} = \frac{|a_1|+\cdots+|a_N|}{n} + \frac{|a_{N+1}|+\cdots+|a_n|}{n}$$
$$< \frac{\varepsilon}{2} + \left(\frac{n-N}{n}\right)\frac{\varepsilon}{2} < \varepsilon.$$

Hence (b_n) converges. The general case reduces to this case by replacing the sequence (a_n) by the sequence $a_n' = a_n - a$. This converges to 0 if and only if a_n converges to a. ■

Find a divergent sequence (a_n) such that the sequence (b_n) defined as in (*) is convergent.

Solution. The sequence $a_n = (-1)^n$ is divergent. For the sequence b_n we then have

$$b_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ -1/n & \text{if } n \text{ is odd.} \end{cases}$$

It is easy to see that $b_n \to 0$. ■

Bonus Problem

Deduce formally from the axioms for complete ordered fields the existence of square roots. That is, prove that for every nonnegative $x \in \mathbb{R}$ there exists a $y \in \mathbb{R}$ such that $y^2 = x$. Can you generalize your argument to the case of n-th roots, i.e. prove that for every nonnegative $x \in \mathbb{R}$ there exists a $y \in \mathbb{R}$ such that $y^n = x$?