Homework 11 for **MATH 185**

Brief sketches to solutions

Problem 1

Let $D \subseteq \mathbb{C}$ be a domain, $a \in D$, and suppose $f, g : D \setminus \{a\} \to \mathbb{C}$ are analytic functions with non-essential singularities in a. Show that the following assertions hold.

(a) If a is a pole of order k (i.e. $\operatorname{ord}(f; a) = -k$), then

$$\operatorname{Res}(f; a) = \lim_{z \to a} \frac{h^{(k-1)}(z)}{(k-1)!}, \quad \text{where } h(z) = (z-a)^k f(z).$$

Solution. f has a Laurent series near a of the form

$$f(z) = \sum_{n=-k}^{\infty} a_n (z-a)^n.$$

The function $h(z)(z-a)^k$ has removable singularity at a, and for the Talyor series of h near a it holds that

$$h(z) = \sum_{n=0}^{\infty} a_{n-k} (z-a)^n.$$

But we also know that the Taylor series of an analytic function is of the form

$$h(z) = \sum_{n=0}^{\infty} \frac{h^{(n)}(a)}{n!} (z-a)^n.$$

Now the desired equality follows by comparing coefficients. ∎

(b) If $\operatorname{ord}(f; a) = l$ and $\operatorname{ord}(g; a) = l+1$, $l \geq 0$, then

$$\operatorname{Res}(f/g; a) = (l+1) \frac{f^{(l)}(a)}{g^{(l+1)}(a)}.$$

Solution. Near a, we have $f(z) = \sum_{n=l}^{\infty} a_n (z-a)^n$ and $g(z) = \sum_{n=l+1}^{\infty} b_n (z-a)^n$, where $a_l, b_{l+1} \neq 0$. The function $h(z) = z f(z)$ has a removable singularity in a and $h(0) \neq 0$, so f/g has a pole of order 1 in a. Hence $\operatorname{Res}(f/g; a) = h(a)$. It holds that

$$h(a) = \frac{a_l}{b_{l+1}} = \frac{f^{(l)}(a)/l!}{g^{(l+1)}(a)/(l+1)!} = (l+1) \frac{f^{(l)}(a)}{g^{(l+1)}(a)}.$$

∎

(c) If $f \not\equiv 0$, then $\operatorname{Res}(f'/f; a) = \operatorname{ord}(f; a)$.

Solution. Assume first $\operatorname{ord}(f; a) = 0$. Then $\operatorname{ord}(f'; a) = 0$, and f'/f is analytically extendable to a, hence $\operatorname{Res}(f'/f; a) = 0$.

Assume now $\operatorname{ord}(f; a) = k > 0$. Then $\operatorname{ord}(f'; a) = k-1$. We can apply part (b) with $f = f'$ and $g = f$ and obtain $\operatorname{Res}(f'/f; a) = k f'(a)/f'(a) = k = \operatorname{ord}(f; a)$.

Finally, assume $\operatorname{ord}(f; a) = -k$, $k > 0$. Then the Laurent representation of f' in a punctured disk around a is

$$f'(z) = -k a_{-k} (z-a)^{-k-1} - \cdots - a_{-1} (z-a)^{-2} + a_1 + a_2 (z-a) + \cdots$$

where $a_{-k}(z-a)^{-k} - \cdots - a_{-1}(z-a)^{-1} + a_0 + a_1(z-a) + \cdots$ is the Laurent series for f. We have that

$$h(z) := (z-a) \frac{f'(z)}{f(z)} = \frac{(z-a)^{k+1}}{(z-a)^k} \frac{f'(z)}{f(z)} = \frac{-k a_{-k} + (-k+1) a_{-k+1} (z-a) + \cdots}{a_{-k} + a_{-k+1}(z-a) + \cdots},$$

which has a removable singularity in a. Hence

$$\operatorname{Res}(f'/f; a) = h(a) = \frac{-k a_{-k}}{a_{-k}} = -k = \operatorname{ord}(f; a).$$

∎

Problem 2

Compute the residues of the following functions at the indicated points:

$$(a)\ \frac{\exp(z^2)}{z-1}, a = 1 \qquad (d)\ \frac{z^2}{z^4-1}, a = \exp(\pi i/2) \qquad (g) \frac{z+2}{z^2-2z}, a = 0$$

$$(b)\ \frac{\exp(z^2)}{(z-1)^2}, a = 1 \qquad (e)\ \frac{\exp(z)-1}{\sin(z)}, a = 0 \qquad (h) \frac{1+\exp(z)}{z^4}, a = 0$$

$$(c)\ \left(\frac{\cos(z)-1}{z}\right)^2, a = 0 \qquad (f)\ \frac{1}{\exp(z)-1}, a = 0 \qquad (i) \frac{\exp(z)}{(z^2-1)^2}, a = 1$$

Solution.

(a) Pole of order 1 in 1, hence residue given by $h(1)$ where $h(z) = f(z)(z-1)$, thus $\mathrm{Res}(f;1) = \exp(1) = e$.

(b) Pole of order 2 in 1, hence residue given by $h'(1)$, so $\mathrm{Res}(f;1) = \exp(1^2)\,2 = 2e$.

(c) Pole of order 2 in 0, hence residue given by $h'(1) = 2(\cos(1)-1)\sin(1)$, so $\mathrm{Res}(f;0) = 0$.

(d) Pole of order 1 in i, hence residue given by $h(i)$, so $\mathrm{Res}(f;1) = 1/[(i-1)(i+1)(2i)] = 1/4i$.

(e) $\mathrm{ord}(\sin, 0) = 1$, $\mathrm{ord}(\exp(z)-1;0) \geq 0$, hence residue given by $g(0)/h'(0) = (\exp(0)-1)/\cos(0) = 0$.

(f) $\mathrm{ord}(\exp(z)-1;0) = 1$, so $\mathrm{Res}(f;0) = 1/\exp(0) = 1$.

(g) Pole of order 1 in 0, so $\mathrm{Res}(f;0) = (0+2)/(0-2) = -1$.

(h) Pole of order 4 in 0, so $\mathrm{Res}(f;0) = h^(3)(0)/3!$ where $h(z) = \exp(z)+1$. Thus $\mathrm{Res}(f;0) = \exp(0)/3! = 1/6$.

(i) Pole of order 2 in 0, so $\mathrm{Res}(f;1) = h'(1)$ where $h(z) = \exp(z)/(z+1)^2$. Hence $\mathrm{Res}(f;1) = e(2^2 - 2(1+1))/2^4 = 0$.

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Problem 3

Evaluate the integral

$$\oint_{|z|=7} \frac{1+z}{1-\cos(z)} dz.$$

Solution. $1-\cos(z)$ is 0 if and only if z is an integer multiple of 2π. From the Taylor series for $\cos(z)$ we conclude that $2\pi k$, $k \in \mathbb{Z}$, is a zero of order 2 for $1-\cos(z)$, hence a pole of order 2 for $f(z) := (1+z)/(1-\cos(z))$. Only the poles $a_1 = -2\pi, a_2 = 0, a_3 = 2\pi$ lie inside the circle of radius 7 around 0.

We now compute the residue of f at these points. Let

$$f(z) = a_{-2}(z-a_j)^{-2} + a_{-1}(z-a_j)^{-1} + a_0 + \cdots$$

be the Laurent series of f around a_j. It the holds that

$$1 + z = (1+a_j) + (z-a_j) = (a_{-2}(z-a_j)^{-2} + a_{-1}(z-a_j)^{-1} + a_0 + \cdots)((z-a_j)^2/2! - (z-a_j)^4/4! \pm \cdots)$$

Exapnding the right hand side and comparing coefficients, we obtain

$$2 = a_{-1} = \mathrm{Res}(f;a_j).$$

Hence, by the residue theorem (the winding number is clearly 1),

$$\oint_{|z|=7} \frac{1+z}{1-\cos(z)} dz = 2\pi i(2+2+2) = 12\pi i.$$

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Problem 4

Do exercise III.6.2 on page 172. Use the hint. Justify your steps carefully and precisely.

Solution. The winding number at a is defined as

$$\chi(\alpha; a) = \frac{1}{2\pi i}\int_\alpha \frac{1}{z-a}dz.$$

Define the function $G : [0,1] \to \mathbb{C}$ by

$$G(t) = \int_0^t \frac{\alpha'(s)}{\alpha(s)-a}ds.$$

By definition of the path integral in $\mathbb{C}$, $G(1) = 2\pi i\chi(\alpha; a)$.

Furthermore, define $F(t) := (\alpha(t) - a)\exp(-G(t))$. F is differentiable, since α is smooth and G is differentiable by the fundamental theorem of calculus.

It holds that

$$F'(t) = \alpha'(t)\exp(-G(t)) + (\alpha(t) - a)\exp(-G(t))(-G'(t)).$$

The fundmental theorem of calculus yields that

$$G'(t) = \frac{d}{dt}\int_0^t \frac{\alpha'(s)}{\alpha(s)-a}ds = \frac{\alpha'(s)}{\alpha(s)-a}.$$

This yields $F'(t) = 0$ for all $t \in [0,1]$. Since $[0,1]$ is connected, F is constant. In particular, it holds that $F(0) = F(1)$. By definition of G, $G(0) = 0$, so $F(0) = (\alpha(0) - a)$. Since α is a closed curve, we have $\alpha(0) = \alpha(1)$, and thus

$$(\alpha(0) - a) = F(0) = F(1) = (\alpha(0) - a)\exp(-G(1)).$$

Since $\alpha(0) - a \neq 0$, we must have $\exp(-G(1)) = 1$, which means $G(1)$ is an integer multiple of $2\pi i$. From this it follows immediately that $\chi(\alpha; a) = G(1)/2\pi i$ is an integer. ■ Since $a \notin \mathrm{Image}(\alpha)$, and α is a smooth curve, the function $t \mapsto$ is well-defined and integrable.