Problem 1: W.l.o.g. we can assume that $0 \in U$.

If $h(z) = g(z^n)$ is analytic, then it has a power series representation

$$h(z) = \sum_{m=0}^{\infty} h_m z^m$$

that converges in a nbhd of 0.

We want to show that $h_m = 0$ if m is not a multiple of n, because then we have

$$h(z) = h_n z^n + h_{2n} z^{2n} + h_{3n} z^{3n} + \dots \qquad n \geq 0$$

and $g(z) = h_n z + h_{2n} z^2 + h_{3n} z^3 + \dots$

so g is analytic, since it is representable as a convergent power series.

Let $\xi = e^{2\pi i/n}$. Then $h(\xi z) = g(\underbrace{\xi^n}_{=1} \cdot z^n) = g(z^n) = h(z)$.

We now compute h_m via the Cauchy integral formula:

$$h_m = \frac{1}{2\pi i} \oint_{|z|=\varepsilon} \frac{h(z)}{z^{m+1}} \, dz$$

Where ε is sufficiently small.

Now substitute $z \longmapsto w \cdot \xi$. Then

$$h_m = \frac{1}{2\pi i} \oint_{|w|=\varepsilon} \frac{h(w \cdot \xi)}{(w \cdot \xi)^{m+1}} \cdot \xi \, dw = \frac{1}{2\pi i} \oint_{|w|=\varepsilon} \frac{h(w)}{w^{m+1}} \cdot \frac{1}{\xi^m} \, dw$$

$$= \frac{h_m}{\xi^m}$$

Hence $h_m (1 - \xi^m) = 0$.

We have that $\xi^m = 1 \iff m$ is a multiple of n,

so $h_m = 0$ if m is not a multiple of n.

$\square$

Problem 2: If f is entire, then the Taylor series of f around 0 has radius of convergence ∞.

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

Termwise differentiation yields:

$$f''(z) = \sum_{n=0}^{\infty} \frac{f^{(n+2)}(0)}{n!} z^n$$

If $f(z) + f''(z) = 0$, then by comparing coefficients we get

$$f^{(n)}(0) = - f^{(n+2)}(0) \quad \text{f. all } n \geq 0.$$

This allows us to write

$$f(z) = \sum_{n=0}^{\infty} \frac{f(0)(-1)^n}{2n!} z^{2n} + \sum_{n=0}^{\infty} \frac{f'(0)(-1)^n}{(2n+1)!} z^{2n+1}$$

$$= f(0)\cos z + f'(0)\sin z.$$

Hence ~~all~~ the entire functions f s.t. $f + f'' = 0$ are precisely the functions of the form

$$\alpha \cos z + \beta \sin z \qquad \alpha, \beta \in \mathbb{C}$$