Homework 9 for **MATH 185**

Brief sketches to solutions

Problem 1 [**]

For the following functions, determine the kind of singularity (removable, pole (with order), or essential) in a.

$$\text{(a) } f(z) = \frac{z^3 + 3z - 2i}{z^2 + 1}, a = i; \quad \text{(b) } f(z) = \frac{z}{e^z - 1}, a = 0; \quad \text{(c) } f(z) = \exp(\exp(-1/z)), a = 0.$$

Solution. (a) i is a solution of $z^3 + 3z - 2i$, so the expression reduces to $\frac{(z+2i)(z-i)}{z+i}$. Now it is immediately clear that the singularity in i is removable.

(b) Expanding e^z into a Taylor series, the expression reduces to $1/(1 + \frac{z}{2} + \frac{z^2}{6} + \cdots)$. Obviously, the singularity in 0 is removable.

(c) The image of $\exp(\exp(-1/z))$, $z \neq \dot{U}_r(0)$ where $r > 0$ arbitrary, is $\mathbb{C}^\bullet$ (periodicity of the exponential function!). Hence, by Casorati-Weierstrass, the singularity is essential. ∎

Problem 2 [*]

Let a be a non-essential singularity of the analytic functions $f, g : D \to \mathbb{C}$, where D is a non-empty domain.

Show that a is also a non-essential singularity of the functions

$$f \pm g, \quad f \cdot g, \quad f/g, \text{ if } g(z) \neq 0 \text{ for all } z \in D \setminus \{a\},$$

and that the following hold:

$$\begin{aligned} \operatorname{ord}(f \pm g; a) &\geq \min\{\operatorname{ord}(f; a), \operatorname{ord}(g; a)\}, \\ \operatorname{ord}(f \cdot g; a) &= \operatorname{ord}(f; a) + \operatorname{ord}(g; a) \\ \operatorname{ord}(f/g; a) &= \operatorname{ord}(f; a) - \operatorname{ord}(g; a) \end{aligned}$$

Solution. (1) Let $\operatorname{ord}(f; a) = -k$ and $\operatorname{ord}(g; a) = -l$. Wlog $k \geq l$, so $-k = \min\{\operatorname{ord}(f; a), \operatorname{ord}(g; a)\}$. We have to show that $\operatorname{ord}(f + g; a) \geq -k$. But since $k \geq l$, the function

$$f(z)(z-a)^k + g(z)(z-a)^k = (f(z) + g(z))(z-a)^k$$

has a removable singularity at a. It follows that $\operatorname{ord}(f + g; a) \geq -k$. The proof for $f - g$ is completely analogous.

(2) Again, assume $\operatorname{ord}(f; a) = -k$ and $\operatorname{ord}(g; a) = -l$. Then the function

$$f(z)(z-a)^k g(z)(z-a)^l = (f(z)g(z))(z-a)^{k+l}$$

has a removable singularity in a. This yields $\operatorname{ord}(f \cdot g; a) \geq \operatorname{ord}(f; a) + \operatorname{ord}(g; a)$. Furthermore, we know that for $h_f(z) = f(z)(z-a)^k$, and $h_g(z) = g(z)(z-a)^l$, the analytic extensions satisfy $h_f(a) \neq 0$ and $h_g(a) \neq 0$. If we set $h_{fg}(z) = (f(z)g(z))(z-a)^{k+l}$, this yields that $h_{fg}(a) \neq 0$. Therefore, $\operatorname{ord}(f \cdot g; a) = \operatorname{ord}(f; a) + \operatorname{ord}(g; a)$

(3) The last assertion is proved analogously to (2). ∎

Problem 3 [**]

Let $F_1, F_2 \subset \mathbb{E}$ be finite, and suppose $f : \mathbb{E} \setminus F_1 \to \mathbb{E} \setminus F_2$ is a bijective mapping such that f and f^{-1} are analytic. (Such a function is also called *bianalytic* or *biholomorphic*.)

(a) Show that there exists a unique extension of f to a biholomorphic function $\widetilde{f} : \mathbb{E} \to \mathbb{E}$.

Solution. (1) Since F_1 is finite and $\mathbb{E}$ is open, $\mathbb{E} \setminus F_1$ is open, so all points in F_1 are isolated singularities of f. If $a \in F_1$ and $r > 0$ is such that $\dot{U}_r(a) \subseteq \mathbb{E}$, then $f(\dot{U}_r(a)) \subseteq \mathbb{E} \setminus F_2$, which is obviously is a bounded set. By the Riemann removability theorem, f can be analytically extended to a function $g : \mathbb{E} \to \mathbb{C}$.

(2) Since g is continous and $a \in \mathbb{E}$, we know that $|g(a)| \leq 1$. But g is analytic, so the image $g(\mathbb{E})$ is open by the open mapping theorem. This implies that $|g(a)| < 1$ (the image cannot have boundary points), and so $g(\mathbb{E}) \subseteq \mathbb{E}$.

(3) By assumption, the same reasoning can be applied to f^{-1}, yielding an analytic extension $h : \mathbb{E} \to \mathbb{E}$.

(4) Since g and h agree with f and f^{-1} on $\mathbb{E} \setminus F_1$ and $\mathbb{E} \setminus F_2$, respectively, we have that $h(g(z)) = z$ for all $z \in \mathbb{E} \setminus F_1$ and $g(h(z)) = z$ for all $z \in \mathbb{E} \setminus F_2$. Since $h \circ g$ and $g \circ h$ are analytic functions on $\mathbb{E}$, and the sets F_1, F_2 are discrete in $\mathbb{E}$, we conclude by the identity theorem that the identities hold on all of $\mathbb{E}$. It follows that $g : \mathbb{E} \to \mathbb{E}$ is biholomorphic, and that $h = g^{-1}$.

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(b) Deduce that F_1 and F_2 have the same cardinality.

Solution. From part (a) it follows that g, as an extension of f, is a bijection between $\mathbb{E} \setminus F_1$ and $\mathbb{E} \setminus F_2$. Since g is also a bijection bewteen $\mathbb{E}$ and $\mathbb{E}$, it follows that g is a bijection between F_1 and F_2. ■

Problem 4 [***]

Let $D_1, D_2, D_3 \subseteq \mathbb{C}$ be domains, $f : D_1 \to D_2$, $g : D_2 \to D_3$, and suppose f is analytic and onto, and $h = g \circ f$ is analytic. Show that g then must be analytic, too.

Solution. (1) We show that the preimage $g^{-1}(U)$ of an open set $U \subseteq D_3$ is open in D_2. $h^{-1}(U) \subseteq D_1$ is open, since h is analytic. As f is onto, we have that $g^{-1}(U) = f(h^{-1}(U))$. But the latter set is open due to the open mapping theorem.

(2) Let $F := \{z \in D_1 : f'(z) = 0\}$. By the lemma on which the identity theorem is based, F is discrete in D_1 (f' is analytic), so $D_1 \setminus F$ is a domain (see homework 8, problem 2). (By the open mapping theorem $f(D_1 \setminus F)$ is a domain.)

(3) Let $z \in D_1 \setminus F$. Part 1 of the implicit function theorem implies that there exists a dotted disk $\dot{U}_r(z) \subseteq D_1 \setminus F$ such that $f|\dot{U}_r(z)$ is one-one. Part 2 of the implicit function theorem now yields a local biholomorphism between $\dot{U}_r(z)$ and $f(\dot{U}_r(z))$, which is an open set. Using the identity theorem these local biholomorphisms for any $z \in D_1 \setminus F$ combine into a bianalytic mapping $D_1 \setminus F \to f(D_1 \setminus F)$. Now, on $f(D_1 \setminus F)$ we can write $g = h \circ f^{-1}$, which as a composition of analytic functions is analytic.

(4) Now let $w \in D_2 \setminus f(D_1 \setminus F)$. Since f is onto, there exists a $z_0 \in D_1$ such that $f(z_0) = w$. Obviously, $z_0 \in F$. Since F is discrete in D_1, we can find $\varepsilon > 0$ such that $\dot{U}_\varepsilon(z_0) \subseteq D_1 \setminus F$. Then $f(\dot{U}_\varepsilon(z_0)) \subseteq f(D_1 \setminus F)$ and open, so we can find a $\delta > 0$ such that $\dot{U}_\delta(w) \subseteq f(\dot{U}_\varepsilon(z_0))$. Now consider $g(\dot{U}_\delta(w))$. We know that $f^{-1}(g(\dot{U}_\delta(w))$ is contained in $\dot{U}_\varepsilon(z_0)$. By analyticity of h, $h(\dot{U}_\varepsilon(z_0))$ is bounded, so $g(\dot{U}_\delta(w)) \subseteq h(\dot{U}_\varepsilon(z_0))$ is bounded. Now the Riemann removability condition implies that g can be analytically extended to w.

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