Homework 5 for MATH 104

Brief solutions to selected exercises

Problem 1

For $x, y \in \mathbb{R}$, define

$$d_1(x,y) = (x-y)^2 \qquad d_3(x,y) = |x^2 - y^2| \qquad d_5(x,y) = \frac{|x-y|}{1+|x-y|}$$

$$d_2(x,y) = \sqrt{|x-y|} \qquad d_4(x,y) = |x-2y|$$

Determine, for each of these, whether it is a metric on $\mathbb{R}$ or not. Justify your answer.

Problem 2

Consider the real line $\mathbb{R}$ with the standard metric $d(x,y) = |x-y|$.

(a) Prove that the set $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is compact directly from the definition (without using the Heine-Borel Theorem).

Solution. Let $\mathcal{U}$ be an open cover of $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. Let $U_0 \in \mathcal{U}$ be such that $0 \in U_0$. Since U_0 is open, there exists an $r > 0$ such that $B_r(0) = (-r, r) \subseteq U_0$. Since $\lim_n \frac{1}{n} = 0$, there exists $N \in \mathbb{N}$ such that for all $n > N$, $\frac{1}{n} < r$. For $1 \leqslant i \leqslant N$, choose $U_i \in \mathcal{U}$ such that $\frac{1}{i} \in U_i$. Then, $\{U_0, U_1, \ldots, U_N\}$ is a finite subcover of $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$. ■

(b) Give an example of a compact subset K of $\mathbb{R}$ such that K has countably many limit points.

Solution. The set $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is countable, closed and bounded. By the Heine-Borel Theorem, it is compact, and has countably many limit points. ■

Problem 3

Let (X, d) be a metric space.

(a) Define the *diameter* $\delta(A)$ of a set $A \subseteq X$ as

$$\delta(A) = \sup\{d(x,y) : x, y \in A\}$$

Consider the metric space $(\mathbb{R}^n, d_2)$, where d_2 denotes the Euclidean metric. Show that for any compact subset K of $\mathbb{R}^n$, there exists $x, y \in K$ such that $\delta(K) = d(x,y)$.

Solution. If K is compact, then it is closed and bounded by the Heine-Borel Theorem. Therefore, $\delta(A) = \sup\{d(x,y) : x, y \in A\}$ is finite. For any $n \in \mathbb{N}$, there exist $x_n, y_n \in K$ such that $\delta(K) - \frac{1}{n} \leqslant d(x_n, y_n) \leqslant \delta(K)$. Since (x_n) is in K, it is bounded. By the Bolzano-Weierstrass Theorem, there exists a convergent subsequence (x_{n_k}) of (x_n). Apply the Bolzano-Weierstrass Theorem again to obtain a convergent subsequence $(y_{n_{k_l}})$ of (y_{n_k}). Assume the limits of the subsequences are x and y, respectively. Then $\lim_l d(x_{n_{k_l}}, y_{n_{k_l}}) = d(x,y) = \delta(K)$. But, by closedness of K, K contains all its limit points, and hence $x, y \in K$. ■

(b) Define the *distance* $\delta(A, B)$ of two sets as

$$\delta(A, B) = \inf\{d(x, y) : x \in A, y \in B\}.$$

Give an example of two closed subsets of $\mathbb{R}$ (with respect to the standard metric) such that $\delta(A, B) < d(x, y)$ for all $x \in A$, $y \in B$. (As always, justify your answer.)

Solution. Consider the sets $A = \{n : n \in \mathbb{N}\}$ and $B = \{n + \frac{1}{2n} : n \in \mathbb{N}\}$. Then $\delta(A, B) = 0$, but $d(a, b) > 0$ for all $a \in A, b \in B$, as is easily seen. Also, A and B are both closed, since their complements are clearly open. ■

Problem 4

Let (X, d) be a metric space. A set $D \subset X$ is called *dense* if every point in X is a limit point of D. A metric space (X, d) is called *separable* if it contains a countable dense subset.

(a) Prove that $(\mathbb{R}^n, d_2)$ is separable. (Hint: consider the set $\mathbb{Q}^n = \{(q_1, \ldots, q_n) : q_i \in \mathbb{Q}$ for $1 \leqslant i \leqslant n\}$.)

Solution. Idea: Use the density of $\mathbb{Q}$ in $\mathbb{R}$ and the fact that a sequence $(x^{(k)})$ in $\mathbb{R}^n$ converges iff each sequence $(x_i^{(k)})$ converges in $\mathbb{R}$. ■

(b) Show that every compact metric space is separable.

Solution. Let (X, d) be a compact metric space. For each $n \in \mathbb{N}$, consider the family of open sets $\mathcal{U}_n = \{B_{1/n}(x) : x \in X\}$. Obviously, $\mathcal{U}_n$ is an open cover of X. By compactness, for each n there exists a finite subcover $\{U_1^{(n)}, \ldots, U_{n_k}^{(n)}\}$ of $\mathcal{U}_n$.

Let $\mathcal{V}$ be the set of all such $U_i^{(n)}$. Note that $\mathcal{V}$ is countable. Furthermore, each $U_i^{(n)}$ is an open ball, so let D be the set of all centers of these balls. Now it is clear that D is countable, and it is straightforward to verify that D is dense in X. ■