Homework 9 for **MATH 185**

Due: Wednesday April 11, 3:10 pm in class

Problem 1 [**]

For the following functions, determine the kind of singularity (removable, pole (with order), or essential) in a.

$$\text{(a) } f(z) = \frac{z^3 + 3z - 2i}{z^2 + 1}, a = i; \quad \text{(b) } f(z) = \frac{z}{e^z - 1}, a = 0; \quad \text{(c) } f(z) = \exp(\exp(-1/z)), a = 0.$$

Problem 2 [*]

Let a be a non-essential singularity of the analytic functions $f, g : D \to \mathbb{C}$, where D is a non-empty domain.

Show that a is also a non-essential singularity of the functions

$$f \pm g, \quad f \cdot g, \quad f/g, \text{ if } g(z) \neq 0 \text{ for all } z \in D \setminus \{a\},$$

and that the following hold:

$$\begin{aligned} \operatorname{ord}(f \pm g; a) &\geq \min\{\operatorname{ord}(f; a), \operatorname{ord}(g; a)\}, \\ \operatorname{ord}(f \cdot g; a) &= \operatorname{ord}(f; a) + \operatorname{ord}(g; a) \\ \operatorname{ord}(f/g; a) &= \operatorname{ord}(f; a) - \operatorname{ord}(g; a) \end{aligned}$$

Problem 3 [**]

Let $F_1, F_2 \subset \mathbb{E}$ be finite, and suppose $f : \mathbb{E} \setminus F_1 \to \mathbb{E} \setminus F_2$ is a bijective mapping such that f and f^{-1} are analytic. (Such a function is also called *bianalytic* or *biholomorphic*.)

(a) Show that there exists a unique extension of f to a biholomorphic function $\widetilde{f} : \mathbb{E} \to \mathbb{E}$.

(b) Deduce that F_1 and F_2 have the same cardinality.

[*Hint:* (1) Use the Riemann removability theorem to show that f can be extended to an analytic function g on $\mathbb{E}$. (2) Use continuity and the open mapping theorem to show that $g(\mathbb{E}) \subseteq \mathbb{E}$. (3) Argue the same is possible for f^{-1}, yielding an extension $h : \mathbb{E} \to \mathbb{E}$. (4) Use the identity theorem to conclude that h is an analytic inverse of g on all of $\mathbb{E}$ (the points of F_1, F_2 are isolated!).]

Problem 4 [***]

Let $D_1, D_2, D_3 \subseteq \mathbb{C}$ be domains, $f : D_1 \to D_2$, $g : D_2 \to D_3$, and suppose f is analytic and onto, and $h = g \circ f$ is analytic. Show that g then must be analytic, too.

[*Hint:* (1) Apply the open mapping theorem to f to show that g is continuous. (2) If $F := \{z \in D_1 : f'(z) = 0\}$ show that $f(D_1 \setminus F)$ is a domain (identity theorem!) (3) Show that f is locally biholomorphic on $D_1 \setminus F$ and conclude that g is analytic on $f(D_1 \setminus F)$. (4) Now try to use the Riemann removability theorem to deal with the set F.]