Homework 2 for MATH 185

Solutions to selected exercises

Problem 1

Verify the identities

$$\cos(z) = \frac{\exp(iz) + \exp(-iz)}{2} \qquad \sin(z) = \frac{\exp(iz) - \exp(-iz)}{2i}$$

and use them to show that $\sin(\mathbb{C}) = \mathbb{C}$ and $\cos(\mathbb{C}) = \mathbb{C}$.

Solution. We use the fact that $\exp(\mathbb{C}) = \mathbb{C}^\bullet$. Given $z \in \mathbb{C}$, we want to find $w \in CC$ such that

$$z = \sin(w) = \frac{\exp(iw) - \exp(-iw)}{2i}.$$

Substitute $a = \exp(iw)$, hence $\exp(-iw) = a^{-1}$. The equation

$$a - a^{-1} = 2iz$$

transforms into $a^2 - 2iza - 1 = 0$ (note that $a \neq 0$), which has a solution $a \neq 0$. The mapping $z \mapsto iz$ is a bijection of $\mathbb{C}$, so there exists a w such that $\exp(iw) = a$. Then it holds that $sin(w) = z$. ∎

Determine all $z \in \mathbb{C}$ such that $\sin(z) = 12i/5$.

Solution. It holds that $\sin(z) = 12i/5$ iff $\exp(iz) - \exp(-iz) = -24/5$. Substitution $exp(iz) \mapsto a$ yields the equation $a^2 + 24a/5 - 1 = 0$, which has solutions $1/5$ and -5. Hence

$$\begin{aligned}
\sin(z) = 12i/5 &\iff [\exp(iz) = 1/5 \text{ or } \exp(iz) = -5] \\
&\iff [iz = \log|1/5| + i\,\mathrm{Arg}(1/5) + 2\pi ik \text{ or } iz = \log|-5| + i\,\mathrm{Arg}(-5) + 2\pi ik \text{ for some } k \in \mathbb{Z}] \\
&\iff [z = i\log(5) - 2\pi k \text{ or } z = i\log(5) - \pi - 2\pi k \text{ for some } k \in \mathbb{Z}]
\end{aligned}$$

∎

Problem 2

Determine all points at which the function

$$f : \mathbb{C} \to \mathbb{C}\,, \qquad z = x + iy \mapsto f(z) := x^3y^2 + ix^2y^3\,, \qquad x, y \in \mathbb{R}\,,$$

is complex differentiable.

Solution. Obviously, we have $u(x, y) = x^3y^2$ and $v(x, y) = x^2y^3$. The partial derivatives are $\partial_1 u = \partial_2 v = 3x^2y^2$, $\partial_2 u = 2x^3y$ and $\partial_1 v = 2xy^3$. f is obviously differentiable in the sense of real analysis, so f is complex differentiable in exactly those points where the Cauchy-Riemann differential equations hold. This is true here iff $\partial_2 u(x, y) = -\partial_1 v(x, y)$, or $2xy^3 = -2x^3y$, which is easily seen to hold iff $x = 0$ or $y = 0$. So f is complex differentiable exactly on the coordinate axes $\mathbb{R}$ and $i\mathbb{R}$. ∎

Does there exist a non-empty open set $D \subseteq \mathbb{C}$ on which f is analytic?

Solution. No, since this would require f to be complex differentiable on a full disc $U_\delta(z)$ for some $z \in \mathbb{C}$. But obviously, the coordinate axes do not contain a full open disc for any radius δ. ∎

Problem 3

Show that the function $f : \mathbb{C} \to \mathbb{C}$,

$$f(z) = \begin{cases} \exp(-1/z^4) & \text{for } z \neq 0, \\ 0 & \text{for } z = 0, \end{cases}$$

satisfies the Cauchy-Riemann equations for all $z \in \mathbb{C}$ and is complex differentiable for all $z \in \mathbb{C}^\bullet = \mathbb{C}\backslash\{0\}$, but not at the origin.

Solution. If $z \neq (0,0)$, then f is analytic in z as a composition of two analytic functions, so the Cauchy-Riemann equations hold.

If $z = (0,0)$, we compute the partial derivatives: In all cases (note $i^4 = 1$), this reduces to computing the limit

$$\lim_{h\to 0} \frac{\exp(-1/h^4)}{h}.$$

But it is known from calculus/analysis that this goes to 0, hence the Cauchy-Riemann hold for $z = (0,0)$, too.

Nevertheless, f is not differentiable at the origin, since this would imply that f is bounded in a neighborhood of 0. But consider the directional limit

$$\lim_{t\searrow 0} f(t(1+i)).$$

Since $(1+i)^4 = -4$, we have $f(t(1+i)) = \exp(1/(4t^4))$, which clearly is unbounded if $t \searrow 0$. ■

Problem 4

(a) Let $D = \mathbb{C}^\bullet$ and $u : D \to \mathbb{R}$ with $u(x,y) = \frac{x}{x^2+y^2}$. Show that u is harmonic and find an analytic function $f : D \to \mathbb{C}$ with $\mathrm{Re}(f) = u$.

Solution. Verification that $\Delta(u) = 0$ is a little cumbersome, but routine. A possible f is $1/z$. ■

(b) Given two harmonic functions $u_1, u_2 : \mathbb{R}^2 \to \mathbb{R}$, prove or disprove (counterexample) the following statements.

1.) $u_1 + u_2$ is harmonic.

2.) $u_1 \cdot u_2$ is harmonic.

Solution. $u_1 + u_2$ is harmonic, because the partial differential operator is linear. Consider $u_1 = u_2 = x$. Then $\Delta(u_1) = \Delta(u_2) = 0$, but $u_1 \cdot u_2 = x^2$, which is not harmonic. ■