# Sample Midterm 2 for MATH 104

### Problem 1

[Review all important definition and results of the relevant material. You will be asked to state a few of them precisely. Among those are: *series, convergence, absolute convergence, Cauchy criterion, comparison test, root test, ratio test, alternating series, metric spaces, completenes, open and closed sets, compact sets, Heine-Borel, continuity, continuous functions on compact sets, intermediate value theorem, uniform continuity, continuity on metric spaces, power series, radius of convergence, pointwise and uniform convergence of power series, continuity and uniform convergence, Weierstrass M-test, uniform convergence of power series, Abel's theorem.*]

### Problem 2

If the followings statements are true, answer "TRUE". If not, give a brief explanation why.

(1) $\sum_{n=2}^{\infty}(n+(-1)^n)^{-2}$ converges. – **TRUE**

(2) $d(x,y)=|x_1-y_1|+|x_2-y_2|$ is a metric on $\mathbb{R}^2$. – **TRUE**

(3) $[0,1]\setminus\{\frac{1}{n} : n\in\mathbb{N}\}$ is compact. – **FALSE:** The set is not closed, since e.g. 1/2 is a limit point of the set, so the set is not closed, hence not compact by the Heine-Borel Theorem.

(4) If $f_n \to f$ pointwise on a compact set, and all $f_n$ and $f$ are continuous, then $f_n \to f$ uniformly. – **FALSE:** Consider for example the sequence of functions $f_n : [0,1] \to \mathbb{R}$ given by

$$f_n = \begin{cases} nx & \text{if } 0 \leqslant x \leqslant 1/n, \\ -nx+2 & \text{if } 1/n < x \leqslant 2/n, \\ 0 & \text{if } 2/n < x \leqslant 1. \end{cases}$$

(5) A power series converges uniformly on its interval of convergence. – **FALSE:** Consider for example $\sum x^n/2^n$.

### Problem 3

Let $K \subseteq \mathbb{R}$ be compact. Show that $\sup K$ is finite and belongs to $K$.

*Solution.* By the Heine-Borel Theorem, $K$ is closed and bounded, hence $\sup K$ is finite. For every $n$ we can pick an $x_n \in K$ such that $x_n \geqslant \sup K - 1/n$ (otherwise $\sup K$ would not be a least upper bound). Obviously, $x_n \to \sup K$. Since $K$ is closed, $\sup K \in K$. ■

## Problem 4

Let $f_n(x) = x^n/n$. Show that $f_n$ converges uniformly on $[-1, 1]$ and specify the limit function.

*Solution.* We use the $d_{\text{sup}}$-characterization of uniform convergence. We claim the limit function is $f \equiv 0$. For, given $\varepsilon > 0$, choose $N \in \mathbb{N}$ such that for all $n > N$, $1/n < \varepsilon$. Then for every $x \in [-1, 1]$,

$$|x^n/n - 0| = |x|^n/n \leqslant 1/n < \varepsilon.$$

Hence $f_n \to 0$ uniformly. ■

## Problem 5

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that there exist $a, b \in \mathbb{R}$ such that $f(a)f(b) < 0$. Show that there exists an $x$ such that $f(x) = 0$.

*Solution.* Since $f(a)f(b) < 0$, we have either $f(a) < 0$ and $f(b) > 0$, or $f(a) > 0$ and $f(b) < 0$. Suppose $f(a) < 0$ and $f(b) > 0$. Then by the Intermediate Value Theorem there exists an $x$ such that $a < x < b$ and $f(x) = 0$. The case $f(a) > 0$ and $f(b) < 0$ is handled analogously. ■