Homework 3 for MATH 104

Due: Tuesday, September 26, 9:30am in class

Problem 1

Given a sequence $(s_n)_{n\in\mathbb{N}}$, let $(-s_n)$ be the sequence defined as $(-s_1, -s_2, -s_3, \dots)$. Show that $\liminf_{n\to\infty}(s_n) = -\limsup_{n\to\infty}(-s_n)$.

Problem 2

Let $(q_n)_{n\in\mathbb{N}}$ be a sequence. Suppose that there exists an $r \in \mathbb{R}$, $0 < r < 1$ such that for all $n \in \mathbb{N}$,

$$|q_{n+2} - q_{n+1}| \leqslant r\,|q_{n+1} - q_n|.$$

Show that (q_n) is a Cauchy sequence.

(*Remark:* You will have to use a fact about the convergence of the sequence r^n. You are required to prove this fact.)

Problem 3

A real number x is called *algebraic* if it is the solution of a polynomial with integer coefficients, i.e. if there exists a natural number n and integers $a_n, a_{n-1}, \dots, a_1, a_0$ with $a_n \neq 0$ such that

$$a_n x^n + \cdots + a_1 x + a_0 = 0.$$

Show that the set of all algebraic real numbers is countable.

Problem 4

Show that the set $\mathbb{R} \times \mathbb{R} = \{(x, y) : \; x, y \in \mathbb{R}\}$ has the same cardinality as $\mathbb{R}$.

(*Remark:* This implies that the plane has the same cardinality as the real line!)