Homework 8 for **MATH 435**

Due: Friday Oct 29

Problem 1

Book, p. 206, Exercise 2.114 (viii)-(xi), and p. 229, Exercise 3.1 (i)-(vii).
(Give the reasons.)

Problem 2

The finite simple groups are in a certain sense “building blocks” of all finite groups since they cannot be split any further into a normal subgroup and a quotient group. Therefore, mathematicians have made a huge effort to classify all finite simple groups, which is now believed to be complete. See for instance the article

`http://www.ams.org/notices/200407/fea-aschbacher.pdf`

We have seen that the abelian simple groups are precisely the groups of prime order (i.e. $\mathbb{Z}_p$). Can there be other simple p-groups? The following exercise shows that this is impossible.

> Do exercise 2.118 on page 207.

Then show that

> S_4 is not simple.

Problem 3

This exercise continues the previous one. It gives a precise meaning to the interpretation of being simple as *indecomposable* mentioned above. Let G be a finite abelian group. Show that there exists a sequence of subgroups

$$G = H_0 \geqslant H_1 \geqslant \cdots \geqslant H_r = \{1\},$$

such that H_{i+1} is normal in H_i for $i = 0, \ldots, r-1$, and $|H_i/H_{i+1}|$ is prime.

Remark: This looks very much like the definition of solvable. In fact, one can show that for a finite group G, this is equivalent to being solvable. You are asked here to prove it in the abelian case.

The exercise shows that you can decompose any abelian group into a sequence of proper normal subgroups, ending with the trivial group $\{1\}$. Moreover, what you need to ‘add’ to H_{i+1} to get H_i is a ‘very simple’ group, namely some group isomorphic to $\mathbb{Z}_p$.

Problem 4

This is an easy exercise to become acquainted with the ring axioms:

> Give a detailed proof (in the style of Proposition 3.5) that Lemma 3.2 and Corollary 3.3 hold in every commutative ring.

Problem 5

In a commutative ring R, some elements $a \neq 0$ may have a *multiplicative inverse*, i.e. there exists $b \in R$ such that $ab = 1$. Such elements are called *units*. Let $U(R)$ be the set of all units.

(a) Show that the set of units forms a group under the ring multiplication.

(b) Determine all units of $\mathbb{Z}$, $\mathbb{Z}_{10}$, $\mathcal{F}(\mathbb{R})$, and $\mathbb{Z}[i]$.

Units are “nice” elements of a ring since we can always divide by them: If $b \in R$ and a is a unit, then $b = b(a^{-1}a) = (ba^{-1})a$, and hence we can interpret (ba^{-1}) as the result of dividing b by a.