Sample Midterm 2 for **MATH 185**

Problem 1

If the followings statements are true, answer "TRUE". If not, give a brief explanation why.

(1) If $f(z)$ is analytic on a domain $D \subseteq \mathbb{C}$, and α is a closed path in D, then $\int_\alpha f(z)dz = 0$.

Solution. FALSE. If D is not elementary, this is not necessarily true, e.g. $1/z$ on $D = \mathbb{C}^{\cdot}$. ■

(2) If f is analytic on the unit disk $\mathbb{E} = \{z : |z| < 1\}$, then there exists an $a \in \mathbb{E}$ such that $|f(a)| \geq |f(0)|$.

Solution. TRUE. If f is constant, this is true. If f is non-constant, by the maximal modulus principle, f cannot take a maximal modulus on $\mathbb{E}$, in particular not in 0. ■

(3) If $\sum_n a_n z^n$ has radius of convergence R, then $\sum_n \mathrm{Re}(a_n) z^n$ has radius of convergence $\geq R$.

Solution. TRUE. Since $|\mathrm{Re}(a_n)| \leq |a_n|$, so $\limsup_n \sqrt[n]{|\mathrm{Re}(a_n)|} \leq \limsup_n \sqrt[n]{|a_n|}$. ■

(4) If f and g are analytic on D, and if they agree on a non-empty set S which is closed in D, then $f = g$ in D.

Solution. FALSE. S might not have an accumulation point in D. E.g. $D = \mathbb{E}$, $S = \{0\}$. Then $f(z) = z$ and $f(z) = z^2$ agree on S, but are not identical on D. ■

Problem 2

Compute the integral

$$\oint_{|z|=3} \frac{\cos(\pi z)}{z^2 - 1}.$$

Solution. A partial fraction decomposition yields

$$\frac{\cos(\pi z)}{z^2 - 1} = \frac{1}{2}\left[\frac{\cos(\pi z)}{z-1} - \frac{\cos(\pi z)}{z+1}\right].$$

The Cauchy integral theorem yields

$$\oint_{|z|=3} \frac{\cos(\pi z)}{z-1} = 2\pi i \cos(\pi)$$

and

$$\oint_{|z|=3} \frac{\cos(\pi z)}{z-1} = 2\pi i \cos(-\pi) = 2\pi i \cos(\pi),$$

so the value of the integral is 0. ■

Problem 3

Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant, entire function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$, i.e. for every $a \in \mathbb{C}$ and for every $\varepsilon > 0$, $U_\varepsilon(a)$ contains a point from $f(\mathbb{C})$.

Solution. Let $a \in \mathbb{C}$. Assume there exists an $\varepsilon > 0$ such that $U_\varepsilon(a) \cap f(\mathbb{C}) = \emptyset$. Consider the function $g(z) = 1/(f(z) - a)$. Obviously, g is entire. Since $|f(z) - a| \geq \varepsilon$ for all $z \in \mathbb{C}$, we have $|g(z)| \leq 1/\varepsilon$, so g is bounded, hence constant by Liouville's Thm. Suppose $g \equiv c$, $c \in \mathbb{C}$. But then $f(z) = \frac{1}{c} - a$ is constant, too – contradiction. ■

Problem 4

Expand $\frac{1}{z^2-1}$ in a Taylor series around $z = 0$ and determine the radius of convergence.

Solution. Obviously,

$$\frac{1}{z^2 - 1} = -\frac{1}{1 - z^2} = -\sum_{n=0}^{\infty} z^{2n}.$$

The radius of convergence is 1. ■