Homework 5 for MATH 185

Due: Wednesday February 28, 3:10 pm in class

Problem 1

Redo homework 3. This time you may use all the theorems we proved since then.

Problem 2

We can use complex integration to compute real integrals:

Let $f(z) = 1/(1+z^2)$. Let $\alpha_1^{(R)}$ be the the upper half-circle around 0 of radius $R > 0$ (traversed counter-clockwise), and let $\alpha_2^{(R)}$ be the line segment from $-R$ to R. Show that

$$\int_{\alpha_1^{(R)} \oplus \alpha_2^{(R)}} f(\xi) d\xi = \int_{\alpha_1^{(R)}} f(\xi) d\xi + \int_{\alpha_2^{(R)}} f(\xi) d\xi = \pi.$$

and

$$\lim_{R\to\infty} \left| \int_{\alpha_2^{(R)}} f(\xi) d\xi \right| = 0.$$

Deduce that $\int_{-\infty}^{\infty} 1/(1+t^2) dt$ (as a real integral) has value π. (Do *not* use the fact that, as a real function, $1/(1+x^2)$ has arctan as an antiderivative.)

Problem 3

Compute the following integrals:

(a) $\oint_{|\xi|=2} (\xi^2 - 1)/(\xi^2 + 1) d\xi$,

(b) $\oint_{|\xi|=1} \sin(\exp(\xi))/\xi d\xi$,

(c) $\oint_{|\xi-1|=1} (\xi/\xi - 1)^n d\xi$, $n \in \mathbb{N}$.

Problem 4

Prove the lemma given in Problem 10 on page 103. Use it to prove the generalized Cauchy integral formula.

Problem 5

Let $f : \mathbb{C} \to \mathbb{C}$ be analytic, non-constant.

(a) Show that it cannot hold that for all $z \in \mathbb{C}$,

$$|f(z)| \geqslant e^{|z|}.$$

(b) Let $A := f^{-1}(\mathbb{E}) = \{z : |f(z)| < 1\}$. Show that A is not empty.

(c) Show that if A is bounded (i.e. is contained in some ball), then there exists some z_0 such that $f(z_0) = 0$.

(d) Find an example where A is not bounded.