Homework 9 for MATH 104

Brief solutions to selected problems

Problem 1

(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}$. Show that

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x). \tag{*}$$

Solution. Assume $h_n \to 0$, $h_n \neq 0$. We have to show that

$$\lim_{n\to\infty} \frac{f(x+h_n) - f(x-h_n)}{2h_n} = f'(x).$$

We write

$$\frac{f(x+h_n) - f(x-h_n)}{2h_n} = \frac{f(x+h_n) - f(x) + f(x) - f(x-h_n)}{2h_n} = \frac{f(x+h_n) - f(x)}{2h_n} + \frac{f(x) - f(x-h_n)}{2h_n}.$$

Since $h_n \to 0$, $x + h_n \to x$ and $x - h_n \to x$. Since f' exists at x, it follows that

$$\lim_{n\to\infty} \frac{f(x+h_n) - f(x)}{2h_n} = \frac{f'(x)}{2} \quad \text{and} \quad \lim_{n\to\infty} \frac{f(x) - f(x-h_n)}{2h_n} = \frac{f'(x)}{2}.$$

Therefore,

$$\lim_{n\to\infty} \frac{f(x+h_n) - f(x-h_n)}{2h_n} = \frac{f'(x)}{2} + \frac{f'(x)}{2} = f'(x).$$

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(b) Find an example of a function $g : \mathbb{R} \to \mathbb{R}$ such that the limit in (*) exists for some $x \in \mathbb{R}$ but g is not even continuous at x.

Solution. One can take any differentiable function f and make it non-continuous at some point z. For example, define

$$f(x) = \begin{cases} x & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

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Problem 2

Consider the functions

$$f(x) = \sin(\tfrac{1}{x}) \quad g(x) = x\sin(\tfrac{1}{x}) \quad h(x) = x^2\sin(\tfrac{1}{x}) \quad \text{for } x \neq 0,$$

and set $g(0) = h(0) = 0$.

(a) Show that f cannot be extended continuously to $x = 0$, i.e. show that there is no continuous function $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ such that $\widetilde{f}(x) = f(x)$ for all $x \neq 0$.

Solution. By a theorem proved earlier, it suffices to show that f is not uniformly continuous on $(0, 1)$. Let $\varepsilon = 1/2$. Given an arbitrary $\delta > 0$, pick n such that $1/n < \delta$. Then $x = 1/2n\pi$ and $y < 1/(2n\pi + \pi/2) < \delta$. Hence

$$|x - y| < \delta \quad \text{but} \quad |f(x) - f(y)| = 1 > \varepsilon.$$

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(b) Show that g is continuous but not differentiable at $x = 0$.

Solution. Since $|g(x)| \leqslant |x|$, it follows immediately that $\lim_{x\to 0} g(x) = 0$, hence g is continuous at 0.

Furthermore,

$$\frac{g(x) - g(0)}{x - 0} = \sin(\tfrac{1}{x}).$$

But it follows from (a) that $\lim_{x\to 0} \sin(\frac{1}{x})$ does not exist. ■

(c) Show that h is differentiable at $x = 0$ but h' is not continuous at $x = 0$.

Solution. Differentiabilty of h follows from continuity of g at 0. Applying the product rule, we get $h'(x) = 2x\sin(\frac{1}{x}) + x^2\cos(\frac{1}{x})(-\frac{1}{x^2})$ for $x \neq 0$. For $x = 0$, we have

$$h'(0) = \lim_{x\to 0} \frac{x^2 \sin(\frac{1}{x})}{x} = 0.$$

But is easy to see that $\lim_{x\to 0} h'(x)$ does not exist. ■

Problem 3

(a) Use the mean value theorem to show that

$$\sqrt{1+x} < 1 + \frac{x}{2} \quad \text{for all } x > 0.$$

Solution. By the mean value theorem, there exists a $1 < y < 1 + x$ such that

$$\frac{1}{2\sqrt{y}} = \frac{\sqrt{1+x} - \sqrt{1}}{x}.$$

Hence, since $y > 1$,

$$\frac{x}{2} + 1 > \frac{x}{2\sqrt{y}} + 1 = \sqrt{x+1}.$$

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(b) Suppose f that differentiable on $\mathbb{R}$, that $1 \leqslant f'(x) \leqslant 2$ for all $x \in \mathbb{R}$ and that $f(0) = 0$. Show that $x \leqslant f(x) \leqslant 2x$ for all $x \geqslant 0$.

Solution. If $x > 0$, then by the mean value theorem there exists a $0 < z < x$ such that

$$f'(z) = \frac{f(x)}{x}.$$

By the assumption on f' it follows that

$$x \leqslant f(x) \leqslant 2x.$$

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Problem 4

Let f be differentiable on $\mathbb{R}$ with $a = \sup\{|f'(x)| : x \in \mathbb{R}\} < 1$. Select $s_0 \in \mathbb{R}$ and define $s_n = f(s_{n-1})$ for $n \geqslant 1$. Show that (s_n) converges.

[*Hint*: Prove the inequality $|s_{n+1} - s_n| \leqslant a|s_n - s_{n-1}|$.]

Solution. Wlog we can assume that $s_{n+1} \neq s_n$ for all n (otherwise (s_n) converges, because the sequence is constant then from some n on).

We have $|s_{n+1} - s_n| = |f(s_n) - f(s_{n-1})|$. By the mean value theorem, there exists a z such that

$$f'(z) = \frac{f(s_n) - f(s_{n-1})}{s_n - s_{n-1}}.$$

Hence, by assumption,

$$|s_{n+1} - s_n| = |f(s_n) - f(s_{n-1})| = |f'(z)||s_n - s_{n-1}| \leqslant a|s_n - s_{n-1}|.$$

Now it follows from Homework 3, Problem 2 that (s_n) is a Cauchy sequence. ■