Homework 5 for
MATH 561, Set Theory

Due: Thursday Mar 22

Problem 1 – Reflection in L

Jech, Exercise 13.22 (p. 198)

Problem 2 – The "hereditary" hierarchy, III

Show that if $\mathsf{V} = \mathsf{L}$, then $L_\kappa = H_\kappa$ for all infinite cardinals κ.

Problem 3 – The minimal model

Let $\ulcorner\mathsf{ZF}\urcorner$ be the set of all Gödel numbers of axioms of ZF. Let SM bet the sentence

$$\exists M\ (M \text{ is transitive } \wedge\ \forall e \in \ulcorner\mathsf{ZF}\urcorner\,\mathrm{Sat}(M, e)).$$

We abbreviate the latter by $M \models \ulcorner\mathsf{ZF}\urcorner$. Show that one can prove in $\mathsf{ZF} + \mathsf{SM}$ that there exists a δ such that

$$L_\delta \models \ulcorner\mathsf{ZF}\urcorner \quad \wedge \quad \forall M\,((M \text{ transitive } \wedge\ M \models \ulcorner\mathsf{ZF}\urcorner) \rightarrow\ L_\delta \subseteq M).$$

Further, show that this L_δ is a model for $\mathsf{ZFC} + \mathsf{V} = \mathsf{L} + \neg\mathsf{SM}$. Infer that $\mathrm{Con}(\mathsf{ZF}) \rightarrow \mathrm{Con}(\mathsf{ZF} + \neg\mathsf{SM})$

Problem 4 – Definability without parameters[**]

Suppose $\alpha > \omega$ is a limit ordinal, and suppose that

$$\mathcal{P}(\omega) \cap L_{\alpha+1} \neq \mathcal{P}(\omega) \cap L_\alpha.$$

Then every element of L_α is definable in L_α *without* parameters, i.e. for every $a \in L_\alpha$ there exists a formula $\varphi(x_0)$ such that for all $b \in L_\alpha$, $L_\alpha \models \varphi[b]$ if and only if $b = a$.

Hint: Let X be the set of elements of all elements of L_α that are definable without parameters. Argue that X is an elementary substructure. (It is the Skolem hull of a subset of L_α under definable Skolem functions for L. Why do these exist?) Apply the Condensation Lemma (to a transitive collapse of X), and argue that for the resulting L_β we must have $\beta = \alpha$: The real that is constructed at α is already constructed at β. If $\beta \neq \alpha$, this would lead to a contradiction.