The topology of random graphons

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April 29, 2015

joint work with Cameron Freer (MIT)

Homogeneous structures

- A countable (relational) structure $\mathcal{M}$ is *homogeneous* if every isomorphism between finite substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.
- **Fraissé**: Any homogeneous structure arises as a *amalgamation process* of finite structures over the same language (Fraissé limits).
- Examples:
  - $(\mathbb{Q}, <)$,
  - the Rado (random) graph
  - the universal K_n-free graphs, $n \geq 3$ (Henson)

Randomnized constructions

- Many universal homogeneous structures can obtained (almost surely) by adding new points according to a randomized process.
  - $(\mathbb{Q}, <)$: add the n-th point between (or at the ends) of any existing point with uniform probability $1/n$.
  - Rado graph: add the n-th vertex and connect to every previous vertex with probability p (uniformly and independently).
  - Vershik: Urysohn space, Droste and Kuske: universal poset
  - Henson graph: ???

Constructions "from below"

- A naive approach to "randomize" the construction of the Henson graph would be as follows:
  - In the n-th step of the construction, pick a one-vertex extension uniformly among all possible extensions that preserve K_n-freeness.
- However: **Erdös, Kleitman, and Rothschild** showed that this asymptotically almost surely yields a bipartite graph (in fact, the *random* countable bipartite graph)
  - The Henson graph(s), in contrast, has to contain every finite K_n-free graph as an induced subgraph, in particular, C_5 and hence cannot be bipartite.

Constructions "from above"

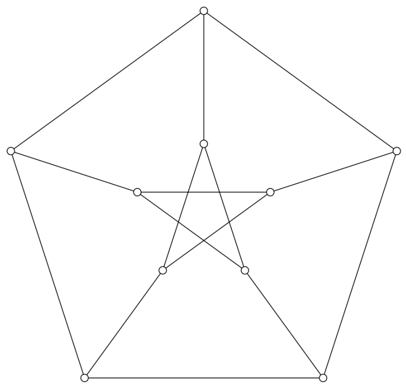
- **Petrov and Vershik** (2010) showed how to construct universal K_n-free graphs probabilistically by *sampling them from a continuous graph*.
- These continuous graphs, known as **graphons**, have been studied extensively over the past decade.
  - See, for example the recent book by Lovasz, *Large networks and graph limits* (2012).

Graphons

- One basic motivation behind graphons is to capture the asymtotic behavior of growing sequences of dense graphs, e.g. with respect to subgraph densities.
- While the Rado graph can be seen as the limit object of a sequence (G_n) of finite random graphs, it does not distinguish between the distributions with which the edges are produced.
- For any $0 < p < 1$, $\mathbb{G}(n, p)$ converges almost surely to (an isomorphic copy of) the Rado graph.
  - However, $p_1 \ll p_2$, $\mathbb{G}(n, p_1)$ will exhibit very different subgraph densities than $\mathbb{G}(n, p_2)$

Graphons and graph limits

Basic idea: "**pixel pictures**"



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0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
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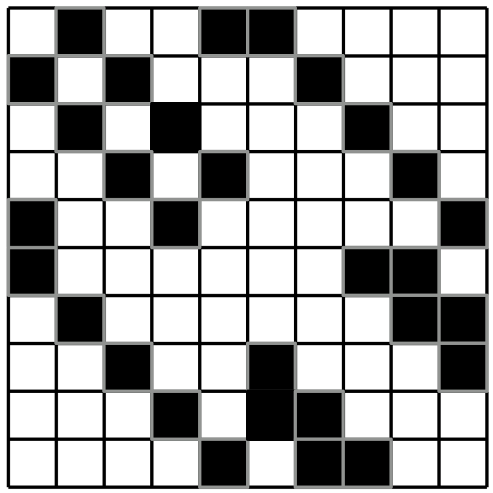
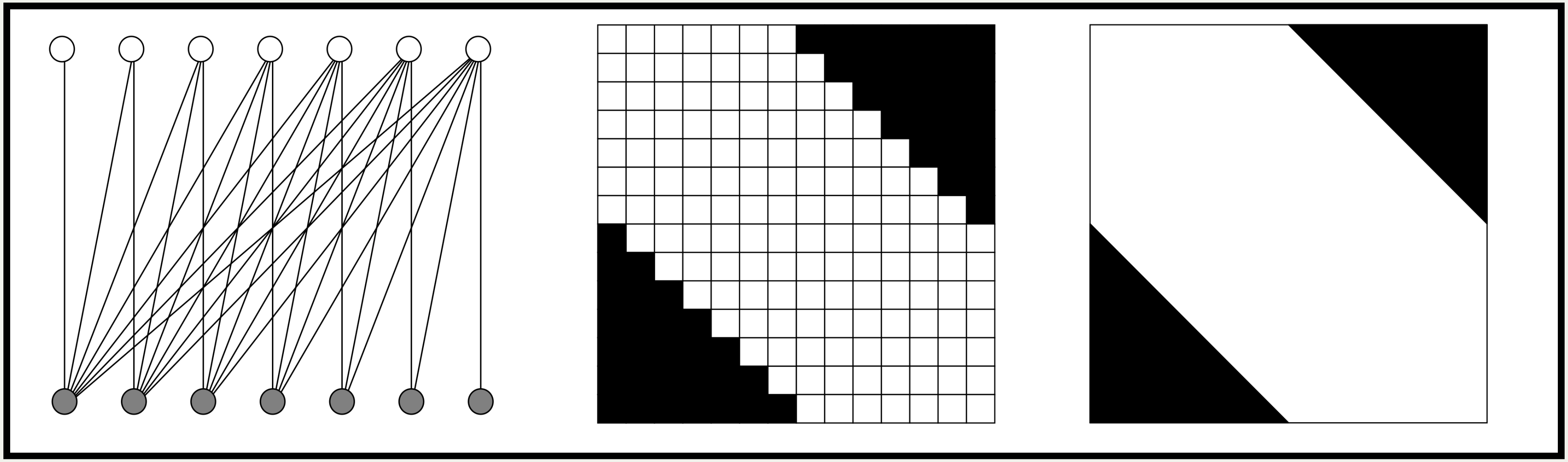
\end{matrix}$$



from Lovasz (2012), Large networks and graph limits

Graphons and graph limits

Convergence of pixel pictures



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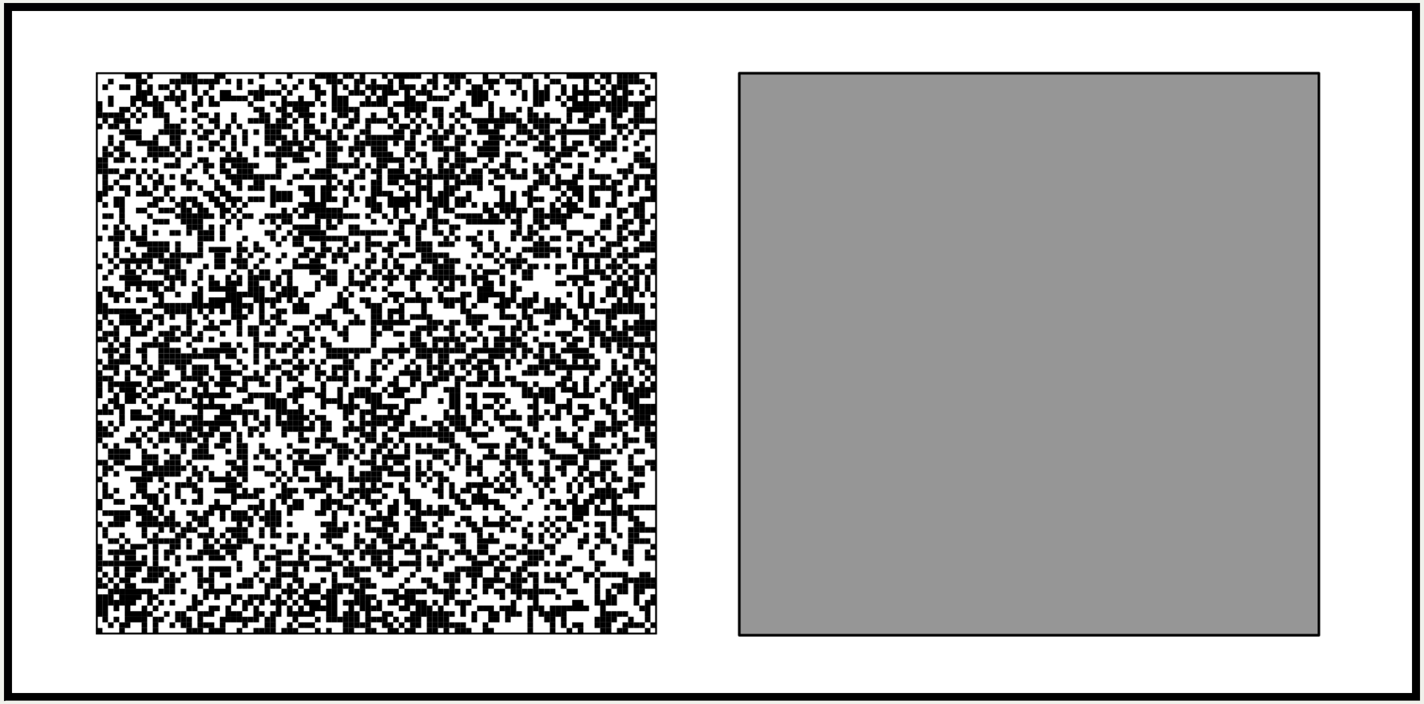
Graphons and graph limits

Convergence of pixel pictures



from Lovasz (2012), Large networks and graph limits

Convergence

- Let (G_n) be a graph sequence with $|V(G_n)| \to \infty$.
- We say (G_n) **converges** if

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for every finite graph F, the relative number $t_i(F, G_n)$ of embeddings of F into G_n converges.

Graphons

- $W : [0,1]^2 \to [0,1]$ measurable, and for all x, y,
$$W(x,x) = 0 \text{ and } W(x,y) = W(y,x).$$
- Think: $W(x,y)$ is the probability there is an edge between x and y.
- Subgraph densities:
  - edges: $\int W(x,y)\,dx\,dy$
  - triangles: $\int W(x,y)W(y,z)W(z,x)\,dx\,dy\,dz$
  - this can be generalized to define $t_i(F, W)$.

The limit graphon

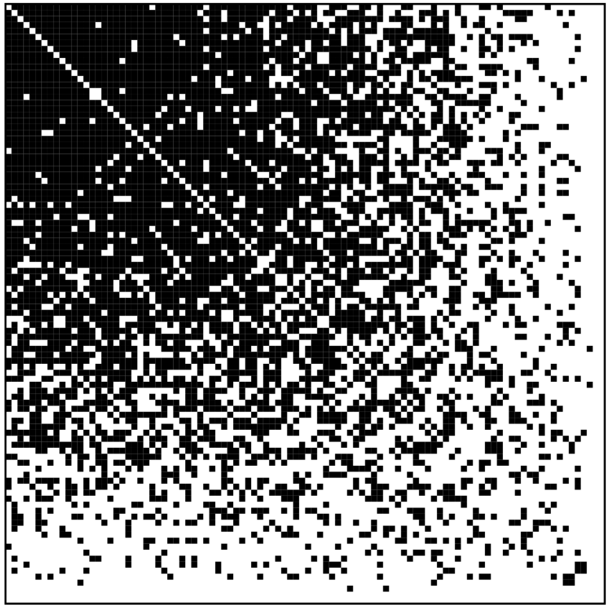
THM: For every convergent graph sequence (G_n) there exists (up to weak isomorphim) exactly one graphon W such that for all finite F:

$$t_i(F, G_n) \longrightarrow t_i(F, W) \ .$$

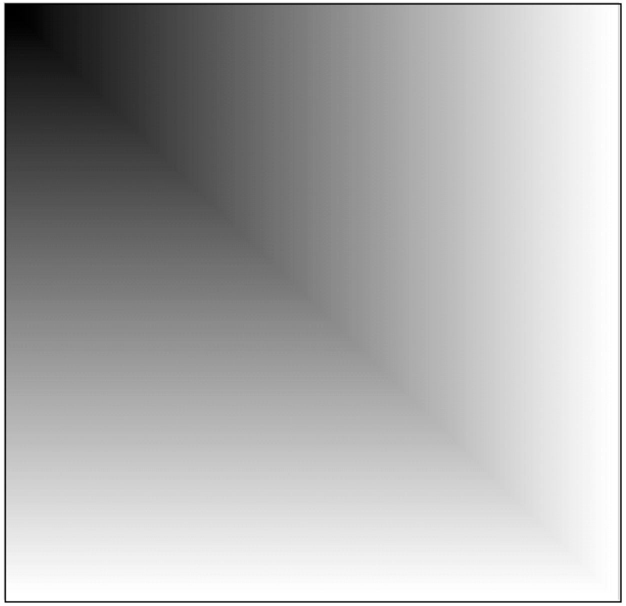
The limit graphon

Example: Uniform attachment graphs

uniform attachment graph:
add new node,
connect any pair of non-adjacent nodes with prob. 1/n



graphon:
$W(x,y) = 1 - \max(x,y)$



from: L. Lovász, Large networks and graph limits (2012)

from Lovasz (2012), Large networks and graph limits

A compatible metric

- *Edit distance*: $d_1(F, G) = \| A_F - A_G \|_1$.

- *Cut distance*: $d_\square(F, G) = \| A_F - A_G \|_\square$, where $\|.\|_\square$ is the **cut norm**

$$\|A\|_\square = \frac{1}{n^2} \max_{S,T \subseteq [n]} \Big| \sum_{i \in S, j \in T} A_{ij} \Big|.$$

- $d_\square$ can be extended to graphs of different order...

- ... and to graphons:

$$\|W\|_\square = \sup_{S,T \subseteq [0,1]} \int_{S \times T} W(x, y)\, dx\, dy.$$

A compatible metric

- A sequence (G_n) converges iff it is a Cauchy sequence with respect to $d_\square$.

- $G_n \to W \quad \text{iff} \quad d_\square(G_n, W) \to 0$

Sampling from graphons

- We can obtain a finite graph $\mathbb{G}(n, W)$ from W by (independently) sampling n points $x_1, \ldots, x_n$ from W and filling edges according to probabilities $W(x_i, x_j)$.
  - almost surely, we get a sequence with $\mathbb{G}(n, W) \to W$.
- If we sample ω-many points from $W(x, y) \equiv 1/2$, we almost surely get the random graph.

The Petrov-Vershik graphon

- **Petrov and Vershik** (2010) constructed, for each $n \geq 3$, a graphon W such that we almost surely sample a universal K_n-free graph.
  - The graphons are (necessarily) $0, 1$-valued.
  - Such graphons are called **random-free**.
  - The constructions resembles a finite extension construction with simple geometric forms, where each step satisfies a new type requiring attention.
  - The method can also be used to construct random-free graphons universal for *all* finite graphs.

Universal graphons

- A random-free graphon is *countably universal* if for every set of distinct points from $[0, 1]$, $x_1, x_2, \dots, x_n$, $y_1, \dots, y_m$, the intersection
$$\bigcap_{i,j} E_{x_i} \cap E_{y_j}^C$$
has non-empty interior.
    - Here $E_x = \{y\colon W(x,y) = 1\}$ is the neighborhood of x.

- For *countably K_n-free universal* graphs, we require this to hold only for such tuples where the induced subgraph by the x_i has no induced K_{n-1}-subgraph,
    - additionally, require that there are no n-tuples in X which induce a K_n

which induce a K_n.

The topology of graphons

- Neighborhood distance:
  $r_W(x, y) = \| \ W(x, .\ ) - W(y, .\ ) \ \|_1 = \int |W(x, z) - W(y, z$
  and mod out by $r_W(x, y) = 0$.
- Example: $W(x, y) \equiv p$ is a singleton space.
- **THM: (Freer & R.)** (informal) If W is a random-free universal graphon obtained via a "tame" extension method, then W is n compact in the r_W topology.

"Tame" extensions

- **DEF:** A random-free graphon W has *continuous realization of extensions* if there exists a function
$$f : (x_1, \ldots, x_n), (y_1, \ldots, y_m) \mapsto (l, r)$$
that is continuous a.e. such that for all $\vec{x}, \vec{y}$,
$$[l, r] \subseteq \bigcap_{i,j} E_{x_i} \cap E_{y_j}^C.$$
- Here $E_x = \{y: W(x, y) = 1\}$ is the neighborhood of x.
- The Petrov-Vershik graphons have uniformly continuous realization of extensions.

Non-compactness

THM: *If a countably (K_n-free) universal graphon has uniformly continuous realization of extensions, then it is not compact in the r_W-topology.*

Features of the proof

- Building a "Cantor sequence" in W.
- Apply the Szemeredi regularity lemma to pass to a sequence of stepfunctions that approximate the graphon *uniformly*.
- Use universality to find the next splitting.
- Uniform continuity guarantees that the Szemeredi "squares" are filled with the right measure.

Regularity lemma

- For every $\epsilon > 0$ there is an $S(\epsilon) \in \mathbb{N}$ such that every graph G with at least $S(\epsilon)$ vertices has an equitable partition of V into k pieces ($1/\varepsilon \le k \le S(\varepsilon)$) such that for all but ϵk^2 pairs of indices i, j, the bipartite graph $G[V_i, V_j]$ is ϵ-regular.

- For every graphon W and $k \geq 1$ there is stepfunction U with k steps such that

$$d_{\square}(W, U) < \frac{2}{\sqrt{\log k}} \parallel W \|_2$$

Complexity of universal graphons

construction:	**fully random**	**tame deterministic**	**general deterministic**
complexity of graphon	low (singleton)	high (non-compact, infinite Minkowski dimension)	?