Homework 1 for
MATH 574, Topics in Logic

Due: Wednesday Jan 29

Problem 1

We work in $A^{\mathbb{N}}$ with A discrete. An elementary *cylinder set* is a set of the form

$$[a]_i = \{x \in A^{\mathbb{N}} \colon x_i = a\}$$

for $a \in A$. These $[a]_i$ form a *subbase* of the product topology. That means that the collection of *finite intersections* of cylinders forms a basis of the topology. Finite intersections of cylinders have the following form:

$$\{x \in A^{\mathbb{N}} \colon x_{i_1} = a_1 \ldots, x_{i_n} = a_n\}, \tag{*}$$

where $a_1, \ldots, a_n \in A$. Every open set in $A^{\mathbb{N}}$ can be written as a union of such sets.

However, we can choose a simpler basis: The basic open cylinder set $[\sigma]$ is defined as

$$[\sigma] = \{x \in A^{\mathbb{N}} \colon \sigma \sqsubseteq x\},$$

where σ is a string over A. (Hence $[\sigma]$ is a special set of the form $(*)$, with $i_1 = 0$, $i_n = |\sigma|$, and $a_i = \sigma_i$.)

Prove that the basic open cylinder sets form indeed a basis, i.e. every open set that can be obtained as a union of sets of the form $(*)$ can be obtained as a union of $[\sigma]$'s.

Problem 2

Again assume A is discrete.

(a) **Show that $A^{\mathbb{N}}$ is compact if and only if A is finite.**

(b) Assume $T \subseteq A^{<\mathbb{N}}$ is an infinite tree. **Show that if A is finite, then T has a path.** Does this also hold when A is infinite?