Lecture 4: Trees

Let A be a set. The set of all finite sequences over A is denoted by $A^{<\mathbb{N}}$.

Definition 4.1: A **tree** on A is a set $T \subseteq A^{<\mathbb{N}}$ that is *closed under prefixes*, that is

$$\forall \sigma, \tau \, [\tau \in T \,\&\, \sigma \subseteq \tau \;\Rightarrow\; \sigma \in T]$$

We call the elements of T *nodes*.

A sequence $\alpha \in A^{\mathbb{N}}$ is a **infinite path through** or **infinite branch of** T if for all n, $\alpha\restriction_n = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in T$. We denote the set of infinite paths through T by $[T]$.

An important criterion for a tree to have infinite paths is the following.

Theorem 4.2 (König's Lemma)**:** *If T is* finite branching*, i.e. each node has at most finitely many immediate extensions, then*

$$T \text{ infinite} \quad \Rightarrow \quad T \text{ has an infinite path}.$$

Proof sketch. We construct an infinite path inductively. Let T_σ denote the tree "above" σ, i.e. $T_\sigma = \{\tau \in A^{<\mathbb{N}} \colon \sigma^\frown\tau \in T\}$. If T is finite branching, by the *pigeonhole principle*, at least one of the sets T_σ for $|\sigma| = 1$ must be infinite. Pick such a σ and let $\alpha\restriction_1 = \sigma$. Repeat the argument for $T = T_\sigma$ and continue inductively. This yields a sequence $\alpha \in [T]$. □

If $[T] = \emptyset$, we call T **well-founded**. The motivation behind this is that T is well-founded if and only if the *inverse prefix* relation

$$\sigma \preceq \tau \quad :\Leftrightarrow \quad \sigma \supseteq \tau$$

is well-founded, i.e. it does not have an infinite descending chain.

If $T \neq \emptyset$ is well-founded, we can assign T an ordinal number, its **rank** $\rho(T)$.

- If σ is a *terminal node*, i.e. σ has no extensions in T, then let $\rho_T(\sigma) = 0$.
- If σ is not terminal, and $\rho_T(\tau)$ has been defined for all $\tau \supset \sigma$, we set $\rho_T(\sigma) = \sup\{\rho_T(\tau) + 1 \colon \tau \in T, \tau \supset \sigma\}$.
- Finally, set $\rho(T) = \sup\{\rho_T(\sigma) + 1 \colon \sigma \in T\} = \rho_T(\varnothing) + 1$, where $\varnothing$ denotes the empty string.

Orderings on trees

If A itself is linearly ordered, we can extend the inverse prefix ordering to a total ordering on $A^{<\mathbb{N}}$ So suppose $\leq$ is a linear ordering of A. The (partial) **lexicographical ordering** $\leq_{\text{lex}}$ of $A^{<\mathbb{N}}$ is defined as

$$\sigma \leq_{\text{lex}} \tau \quad \text{iff} \quad \sigma = \tau \ \text{ or } \ \exists i < \min\{|\sigma|, |\tau|\},\ [(\forall j < i)\, \sigma_j = \tau_j \,\&\, \sigma_i < \tau_i]$$

This ordering extends to $A^{\mathbb{N}}$ in a natural way.

Proposition 4.3: *If $\leq$ is a well-ordering of A and T is a tree on A with $[T] \neq \emptyset$, then $[T]$ has a $\leq_{\text{lex}}$-minimal element, the **leftmost branch**.*

Proof. We *prune* the tree T by deleting any node that is not on an infinite branch. This yields a subtree $T' \subseteq T$ with $[T'] = [T]$. Let $T'_n = \{\sigma \in T' : |\sigma| = n\}$. Since $\leq$ is a well-ordering on A, T'_1 must have a $\leq_{\text{lex}}$-least element. Denote it by $\alpha\restriction_1$. Since T' is pruned, $\alpha\restriction_1$ must have an extension in T, and we can repeat the argument to obtain $\alpha\restriction_2$. Continuing inductively, we define an infinite path α through T', and it is straightforward to check that α is a $\leq_{\text{lex}}$-minimal element of $[T']$ and hence of $[T]$. □

We can combine the $\leq_{\text{lex}}$-ordering with the inverse prefix order to obtain a linear ordering of $A^{<\mathbb{N}}$. This ordering has the nice property that if A is well-ordered and T is well-founded, then the ordering restricted to T is a well-ordering.

Definition 4.4: The **Kleene-Brouwer ordering** $\leq_{\text{KB}}$ of $A^{<\mathbb{N}}$ is defined as follows.

$$\sigma \leq_{\text{KB}} \tau \quad \text{iff} \quad \sigma \supseteq \tau \ \text{ or } \ \sigma \leq_{\text{lex}} \tau$$

This means σ is smaller than τ if it is a proper extension of τ or “to the left” of τ.

We now have

Proposition 4.5: *Assume $(A, \leq)$ is a well-ordered set. Then for any tree T on A,*

$$T \text{ is well-founded} \quad \Longleftrightarrow \quad \leq_{\text{KB}} \text{ restricted to } T \text{ is a well-ordering.}$$

Proof. Suppose T is not well-founded. Let $\alpha \in [T]$. Then $\alpha\restriction_0, \alpha\restriction_1, \dots$ is an infinite descending sequence with respect to $\leq_{\text{KB}}$.

Conversely, suppose $\sigma_0 >_{\text{KB}} \sigma_1 >_{\text{KB}} \dots$ is an infinite descending sequence on T. Then $\sigma_1(0) \geq \sigma_2(0) \geq \dots$ as a sequence in A. Since A is well-ordered, this sequence must eventually be constant, say $\sigma_n(0) = a_0$ for all $n \geq n_0$. Since the σ_n are descending, by the definition of $\leq_{\mathrm{KB}}$ it follows that $|\sigma_n| \geq 2$ for $n > n_0$. Hence we can consider the sequence $\sigma_{n_0+1}(1) \geq \sigma_{n_0+2}(1) \geq \dots$ in A. Again, this must be constant $= a_1$ eventually. Inductively, we obtain a sequence $\alpha = (a_0, a_1, a_2, \dots) \in [T]$, i.e. T is not well-founded. $\square$

Note however that the order type of a well-founded tree under $\leq_{\mathrm{KB}}$ is not the same as its rank $\rho(T)$.

Of course we can also define an ordering on $A^{<\mathbb{N}}$ via an injective mapping from $A^{<\mathbb{N}}$ to some linearly ordered set A. We will use this repeatedly for the case $A = \mathbb{N}$ and $A = \{0, 1\}$.

For $A = \mathbb{N}$, we can use the standard coding mapping

$$\pi : (a_0, a_1, \dots, a_n) \mapsto p_0^{a_0+1} p_1^{a_1} \cdots p_n^{a_n},$$

where p_k is the kth prime number. This embeds $\mathbb{N}^{<\mathbb{N}}$ into $\mathbb{N}$, and we can well-order $\mathbb{N}^{<\mathbb{N}}$ by letting $\sigma < \tau$ if and only if $\pi(\sigma) < \pi(\tau)$.

For $A = \{0, 1\}$ we set

$$\pi : (b_0, b_1, \dots, b_n) \mapsto \sum_{i=0}^{n} 2^{b_i}.$$

These two embedding allows us henceforth to see *trees as subsets of the natural numbers*. If we optimize the coding suitably, we can make it onto, and henceforth also assume that every subset of $\mathbb{N}$ codes a tree (on $\{0, 1\}$ or $\mathbb{N}$, depending on the circumstances). This will be an important component in exploring the relation between *topological and arithmetical complexity*.

Trees and closed sets

Let A be a set with the discrete topology. Consider $A^{\mathbb{N}}$ with the product topology defined in Lecture 2.

Proposition 4.6: *A set $F \subseteq A^{\mathbb{N}}$ is closed if and only if there exists a tree T on A such that $F = [T]$.*

Proof. Suppose F is closed. Let

$$T_F = \{\sigma \in A^{<\mathbb{N}} : \sigma \subset \alpha \text{ for some } \alpha \in F\}.$$

Then clearly $F \subset [T_F]$. Suppose $\alpha \in [T_F]$. This means for any n, $\alpha\restriction_n \in T_F$, which implies that there exists $\beta_n \in F$ such that $\alpha_n \subset \beta_n$. The sequence (β_n) converges to α, and since F is closed, $\alpha \in F$.

For the other direction, suppose $F = [T]$. Let $\alpha \in A^{\mathbb{N}} \setminus F$. Then there exists an n such that $\alpha\restriction_n \notin T$. Since a tree is closed under prefixes, since implies that no extension of $\alpha\restriction_n$ can be in T. This in turn implies $N_{\alpha\restriction_n} \subseteq A^{\mathbb{N}} \setminus F$, and hence $A^{\mathbb{N}} \setminus F$ is open. □

Trees and continuous mappings

Let $f : A^{\mathbb{N}} \to A^{\mathbb{N}}$ be continuous. We define a mapping $\varphi : A^{<\mathbb{N}} \to A^{<\mathbb{N}}$ by setting

$$\varphi(\sigma) = \text{ the longest } \tau \text{ such that } N_\sigma \subseteq f^{-1}(N_\tau).$$

This mapping has the following properties:

(1) It is *monotone*, i.e. $\sigma \subseteq \tau$ implies $\varphi(\sigma) \subseteq \varphi(\tau)$.

(2) For any $\alpha \in A^{\mathbb{N}}$ we have $\lim_n |\varphi(\alpha\restriction_n)| = \infty$. This follows directly from the continuity of f: For any neighborhood N_τ of $f(\alpha)$ there exists a neighborhood N_σ of α such that $f(N_\sigma) \subseteq N_\tau$. But τ has to be of the form $\tau = f(\alpha)\restriction_m$, and σ of the form $\alpha\restriction_n$. Hence for any m there must exist an n such that $\varphi(\alpha\restriction_n) \supseteq f(\alpha)\restriction_m$.

On the other hand, if a function $\varphi : A^{<\mathbb{N}} \to A^{<\mathbb{N}}$ satisfies (1) and (2), it induces a function $\varphi^* : A^{\mathbb{N}} \to A^{\mathbb{N}}$ by letting

$$\varphi^*(\alpha) = \lim_n \varphi(\alpha\restriction_n) = \text{ the unique sequence extending all } \varphi(\alpha\restriction_n).$$

This φ^* is indeed continuous: The preimage of N_τ under φ^* is given by

$$(\varphi^*)^{-1}(N_\tau) = \bigcup \{N_\sigma : \varphi(\sigma) \supseteq \tau\},$$

which is an open set.

We have shown

Proposition 4.7: *A mapping $f : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is continuous if and only if there exists a mapping φ satisfying (1) and (2) such that $f = \varphi^*$.*

Again, note that we can completely describe a topological concept, continuity, through a relation between finite strings.