# Homework 4 for **MATH 574**

Due: Wednesday February 23

## Problem 1

Let $X$ be a topological space. Let $K(X)$ be the set of compact subsets of $X$. The **Vietoris topology** on $K(X)$ is the topology generated by the following sets.

$$\{K \in K(X) \colon K \cap U \neq \emptyset\}$$
$$\{K \in K(X) \colon K \subseteq U\}$$

The Vietoris topology is an example of a *hit-and-miss* topology, because the basic neighborhoods consist of all those compact sets that *hit* a given finite number of open sets and *miss* a given closed set.

(a) Show that $\emptyset$ is isolated in $K(X)$ with the Vietoris topology.

Now assume $d$ is a metric on $X$ with $d \leqslant 1$. Define the *Hausdorff metric* $d_H$ on $K(X)$ as follows.

$$d_H(K, L) = \inf\{\varepsilon > 0 \; K \subseteq L_\varepsilon \;\&\; L \subseteq K_\varepsilon\},$$

where for a subset $Y \subseteq X$,

$$Y_\varepsilon = \{z \in X \colon d(z, Y) = \inf\{d(z, y) \colon y \in Y\} < \varepsilon\}.$$

(b) Show that the Hausdorff metric is compatible with the Vietoris topology.

One can show that if $X$ is Polish, so is $K(X)$, and if $X$ is compact, so is $K(X)$.

Finally, assume $X$ is Polish.

(c) Show that
$$K_p = \{K \in K(X) \colon K \text{ is perfect}\}$$
is $G_\delta$ in $K(X)$.

(d) Show that if $X$ is perfect, then $K_p(X)$ is dense in $K(X)$. Conclude that $K_p(X)$ is comeager in $K(X)$, that is, from the point of view of Baire category, most compact subsets of $X$ are perfect.

## Problem 2

Show that the non-measurable Vitali set does not have the Baire property.

## Problem 3

Show that $\mathsf{DC}$ implies $\mathsf{AC}_\omega$.