Algorithmic Equivalence Relations

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Borel Equivalence Relations

- The study of definable equivalence relations in Polish spaces has been a major topic in several branches of mathematics.
    - (Descriptive) Set Theory
    - Ergodic Theory
    - Operator Algebras
    - Representation Theory
    - Recursion Theory
        - Harrington-Kechris-Louveau (Effective Descriptive Set Theory)
        - Slaman-Steel
        - Martin's Conjecture

Borel Equivalence Relations

- Let X be a Polish space, i.e. a topological space that is completely metrizable and has a countable dense subset.
  - Standard examples: 2^ω (Cantor space), ω^ω (Baire space)
  - In fact, every Polish space is the continuous image of ω^ω.
- A relation $R \subseteq X \times X$ is Borel, if it is a Borel subset of the product space $X \times X$.
- If R is an equivalence relation, we say R is countable if every equivalence class is countable.

Borel Equivalence Relations

Motivation:

- Some important objects in mathematics (orbit spaces, moduli spaces) do not carry a Polish structure, but can be represented in the form X/E, where X is Polish and E is an analytic equivalence relation.
- Classification problems: Describe objects up to isomorphism. Assign invariants.
  - Example: Bernoulli shifts up to conjugacy → entropy (Ornstein)
- Effective/definable cardinality.

Borel Equivalence Relations

- Examples
  - id_X – identity on X
  - E_0 – Vitali equivalence on 2^ω, ω^ω

$$A \; E_0 \; B \iff \exists m \; \forall n>m \; [\, A(n) = B(n) \,]$$

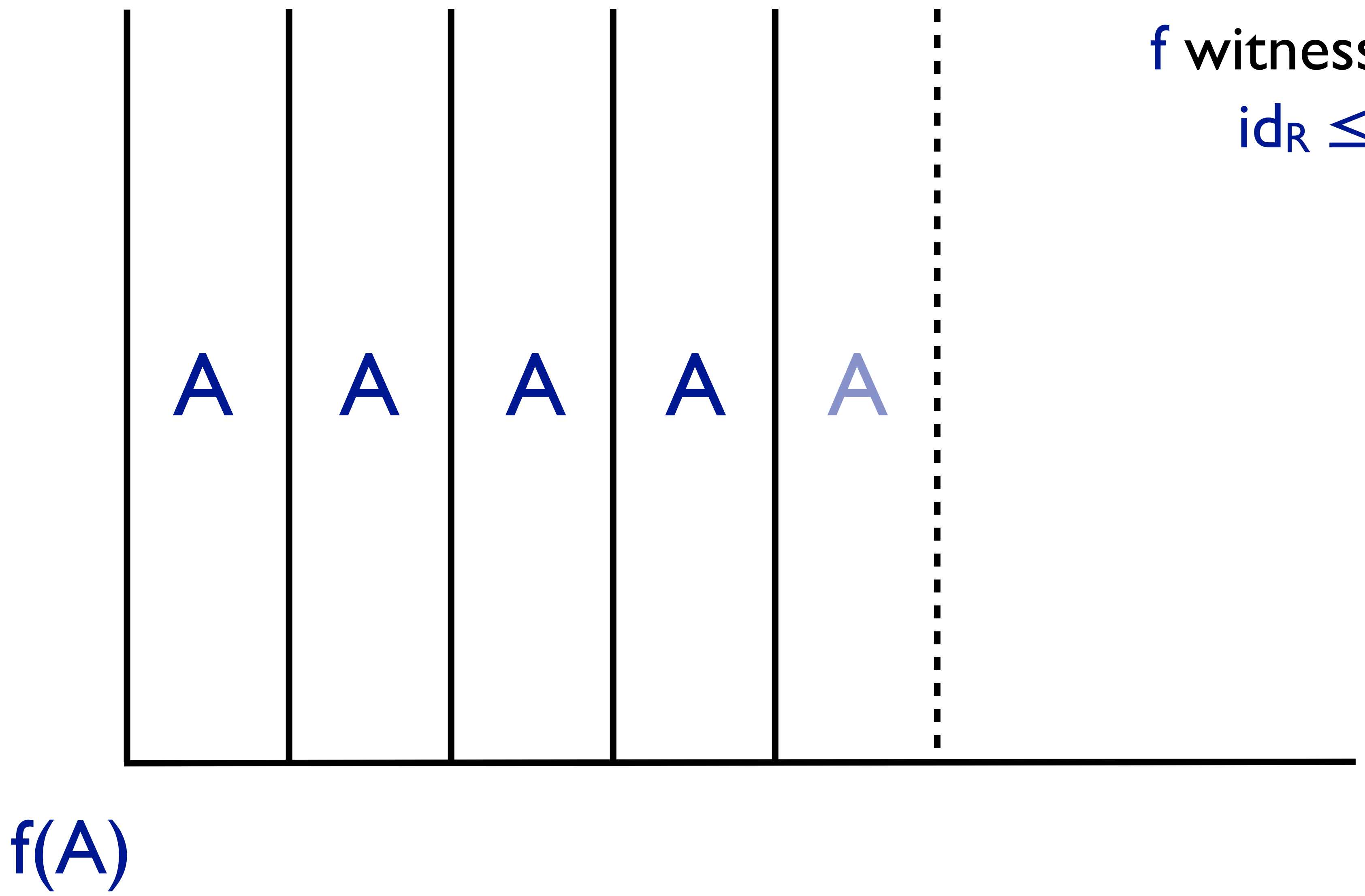
- Compare equivalence relations:
  E on X is Borel reducible ($\leq_B$) to F on Y if there exists Borel $f: X \to Y$ such that

$$x_0 \; E \; x_1 \quad \text{iff} \quad f(x_0) \; F \; f(x_1)$$

- Example:

  $id_{\mathbf{R}} \leq_B E_0$

The Column Method



f witnesses that $id_R \leq_B E_0$

Borel Reductions

- A Borel reduction, i.e. $f: X \to Y$ such that

$$x \; E \; y \quad \text{iff} \quad f(x) \; F \; f(y)$$

  can be seen as a definable injection of X/E into Y/F.

- If a reduction cannot be reversed, this means Y/F has higher definable cardinality than X/E.

- Example: $\mathrm{id}_{\mathbf{R}} \not\geq_B E_0$.

  - Suppose f is a reduction. We may assume $f(2^\omega) \subseteq [0,1]$.

  - The preimages $f^{-1}[0,1/2]$, $f^{-1}[1/2,1]$ are Borel tail sets, so by Zero-One Law one of them has measure 1.

  - Continue splitting, obtain that f is constant almost everywhere – contradiction.

Dichotomies for Equivalence Relations

- Silver [1980]: For any Borel equivalence relation E, E has either countably many classes or there exists a perfect set of mutually E-inequivalent elements.
  - Corollary: For any Borel E

    either $E \leq_B id_N$ or $id_{\mathbf{R}} \leq_B E$.

  - CH holds for Borel (even co-analytic) equivalence relations.

- Harrington-Kechris-Louveau [1990]: For any Borel E

  either $E \leq_B id_{\mathbf{R}}$ or $E_0 \leq_B E$.

A Universal Countable Relation

- Any action of a group G on a space X gives rise to an orbit equivalence

$$x \sim_G y \quad \Leftrightarrow \quad \exists g \, (y = g \cdot x).$$

- Feldman-Moore [1977]: Any countable Borel equivalence relation can be represented as an orbit equivalence of a Borel action.

- If G is a group, G acts on 2^G via the shift-operation:

$$(g \cdot A)(h) = A(g^{-1} \cdot h)$$

- Let F_2 be the free group on two generators.

- Shift-equivalence on 2^{F_2} is a universal countable Borel equivalence relation, E_∞.

The Picture

$$\mathrm{id}_1 <_B \mathrm{id}_2 < \cdots < \mathrm{id}_N < \mathrm{id}_R < E_0 < \qquad < E_\infty$$

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$$\mathrm{id}_1 <_B \mathrm{id}_2 < \cdots < \mathrm{id}_N < \mathrm{id}_R < E_0 < \qquad < E_\infty$$

Vitali equivalence on 2^ω

The Picture

Shift-equivalence on subsets of F_2

$$\mathrm{id}_1 <_B \mathrm{id}_2 < \cdots < \mathrm{id}_N < \mathrm{id}_R < E_0 < \qquad\qquad < E_\infty$$

Vitali equivalence on 2^ω

The Picture

Shift-equivalence on subsets of F_2

$$\mathrm{id}_1 <_B \mathrm{id}_2 < \cdots < \mathrm{id}_N < \mathrm{id}_R < E_0 < \qquad\qquad < E_\infty$$

Vitali equivalence on 2^ω

$2^{\aleph_0}$ many incomparable [Adams-Kechris]

The Picture

Shift-equivalence on subsets of F_2

$\equiv_r$?

$$\mathrm{id}_1 <_B \mathrm{id}_2 < \cdots < \mathrm{id}_N < \mathrm{id}_R < E_0 < \qquad < E_\infty$$

Vitali equivalence on 2^ω

$2^{\aleph_0}$ many incomparable [Adams-Kechris]

Classifying Turing Equivalence

- An equivalence relation is hyperfinite if it can be written as an increasing sequence of finite Borel equivalence relations.
  - Equivalently, a hyperfinite equivalence relation is one that is induced by a Z-action.

- Slaman-Steel [1988]: Turing equivalence is not hyperfinite.
  - Corollary: No countable Borel equivalence relation coarser than 1-equivalence is hyperfinite.

    Proof: Use $X \equiv_T Y$ iff $X' \equiv_1 Y'$.

Classifying Turing Equivalence

- Kechris [1991]:

  $\equiv_T$ is not amenable.

- Jackson-Kechris-Louveau [2000]:

  $\equiv_T$ is not treeable.

- Thomas [2007]:

  not generated by a free group action

  $\equiv_T$ is not essentially free.

- Question/Conjecture [Kechris]:

  $\equiv_T$ is universal (?)

Universality of Algorithmic Relations

- Permutation equivalence: Let $n \in \{1,2,3, \ldots\} \cup \{\omega\}, S \leq S_\infty$.

  For $X,Y \in n^\omega$: $X \equiv_S Y \Leftrightarrow \exists \pi \in S \, \forall n \, [Y(n) = X(\pi(n))]$

- Dougherty-Kechris [1991]:

  Recursive isomorphism on ω^ω is universal.

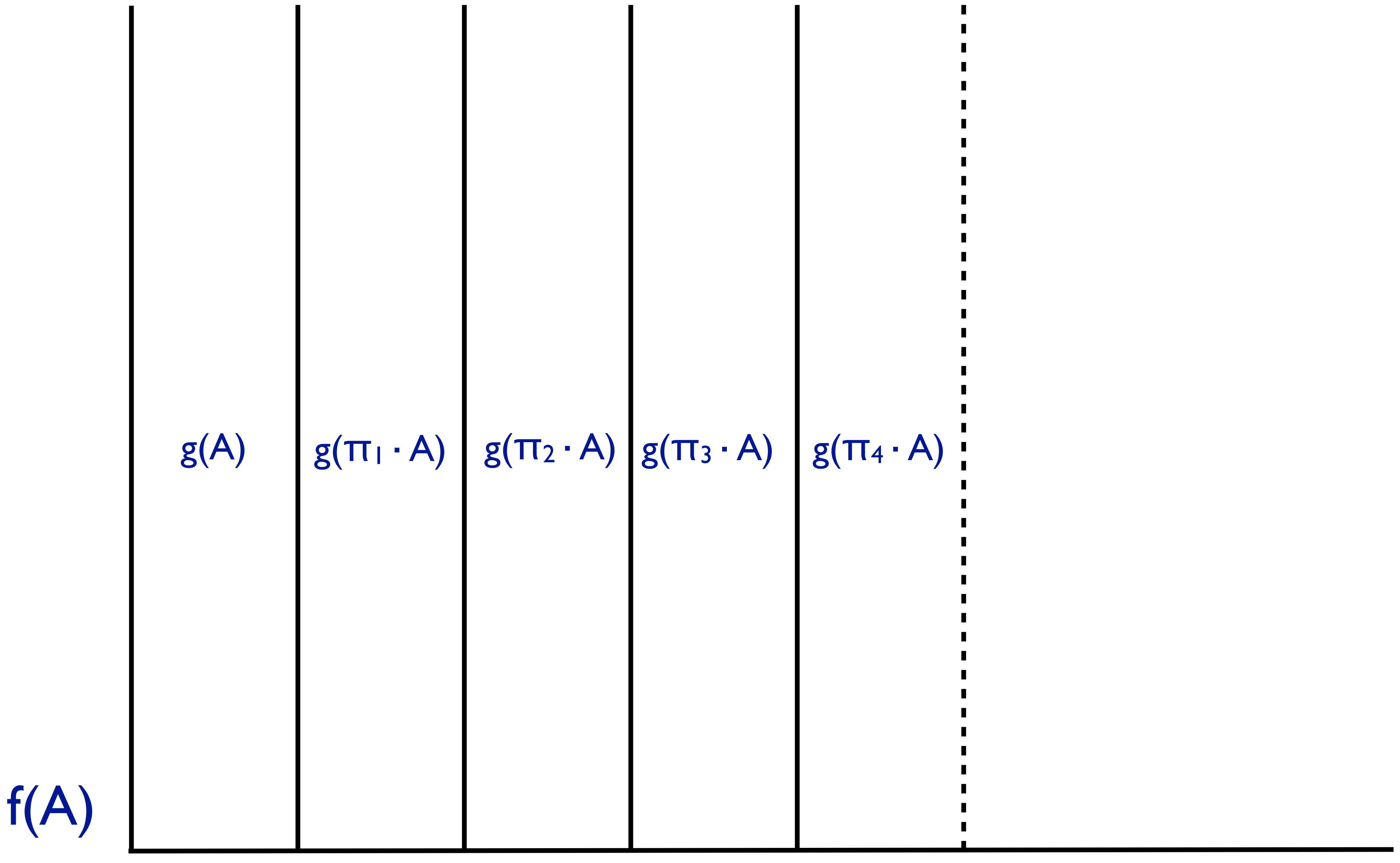
- They prove something more general:

  There is a recursive countable group $S_0 \leq S_\infty$ such that for any countable group S with $S_0 \leq S \leq S_\infty$, S-permutation equivalence on ω^ω is universal.

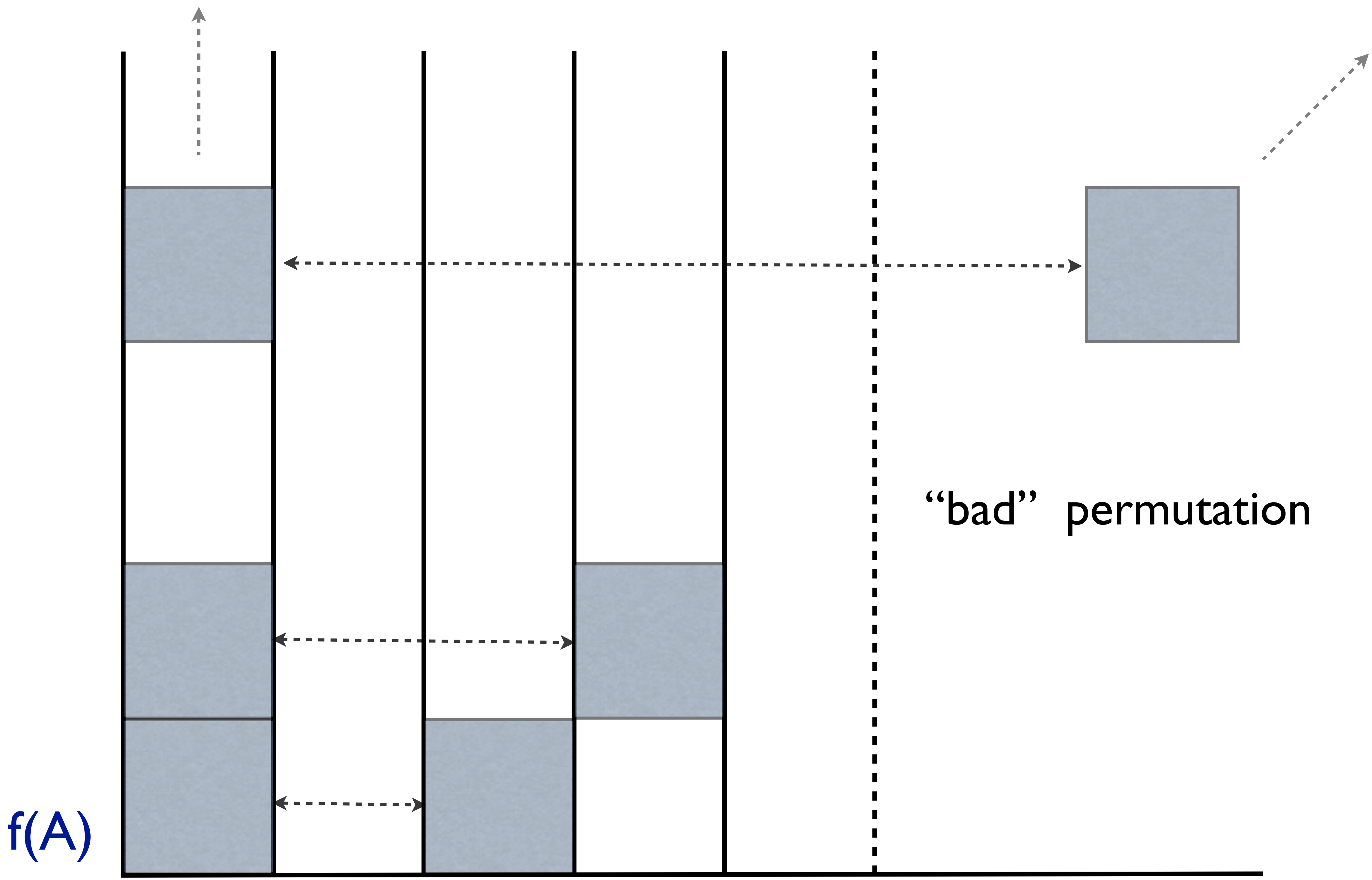
- Andretta-Camerlo-Hjorth [2001]:

  Recursive isomorphism on 5^ω is universal.

The Reduction



The Reduction



Universality of Algorithmic Relations

- Marks [2011]:

  Recursive isomorphism on 3^ω is universal.

  - The proof is based on the Dougherty-Kechris approach.
  - Marks also identified combinatorial obstacles why this cannot easily be extended to $k = 2$.

Weak Reducibilities

- Slaman-Steel [unpublished]:

  Arithmetic equivalence $\equiv_A$ on 2^ω is universal.

  - As before, reduce shift equivalence on 2^{F_2}.
  - Basic idea: Code into an arithmetic degree based on distances in the Cayley graph –

    the further away Y is from X shift-wise,
    the more jumps are needed to recover $F(Y)$ from $F(X)$.

  - Use the technique of jump-coding.

- The method can be adapted to variants of Turing reducibility, like polynomially bounded T-equivalence.

Jump Coding

- A frequently used way to make $X \leq_T Y'$ is to ensure

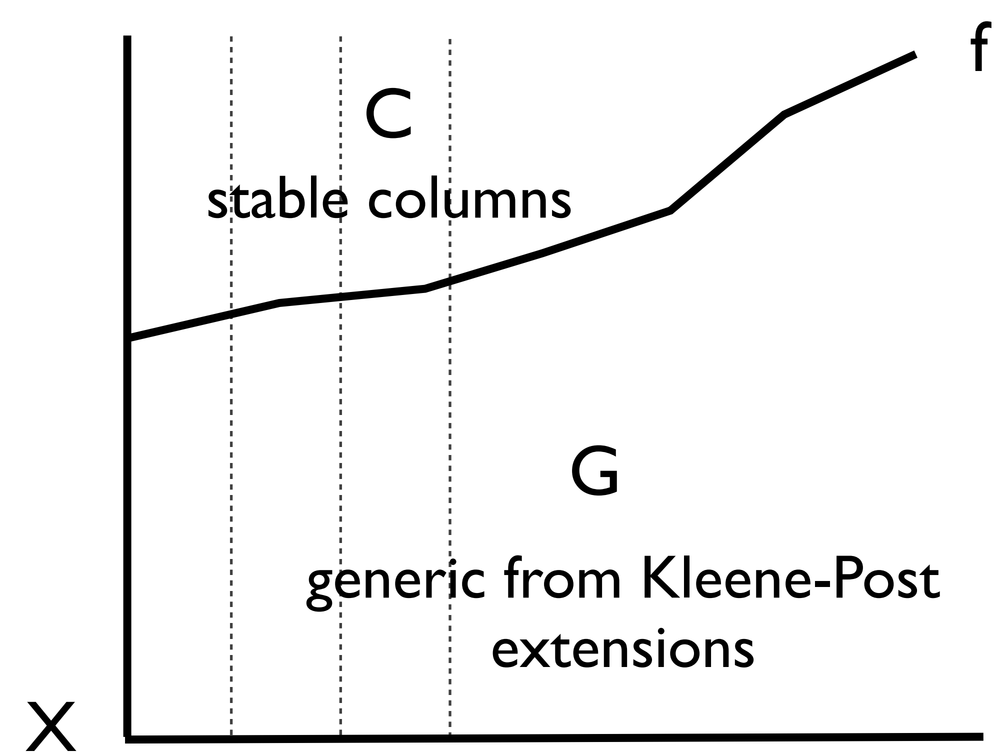
$$X(n) = \lim_s Y(n,s)$$

- This can be combined with a finite extension strategy (Cohen conditions).

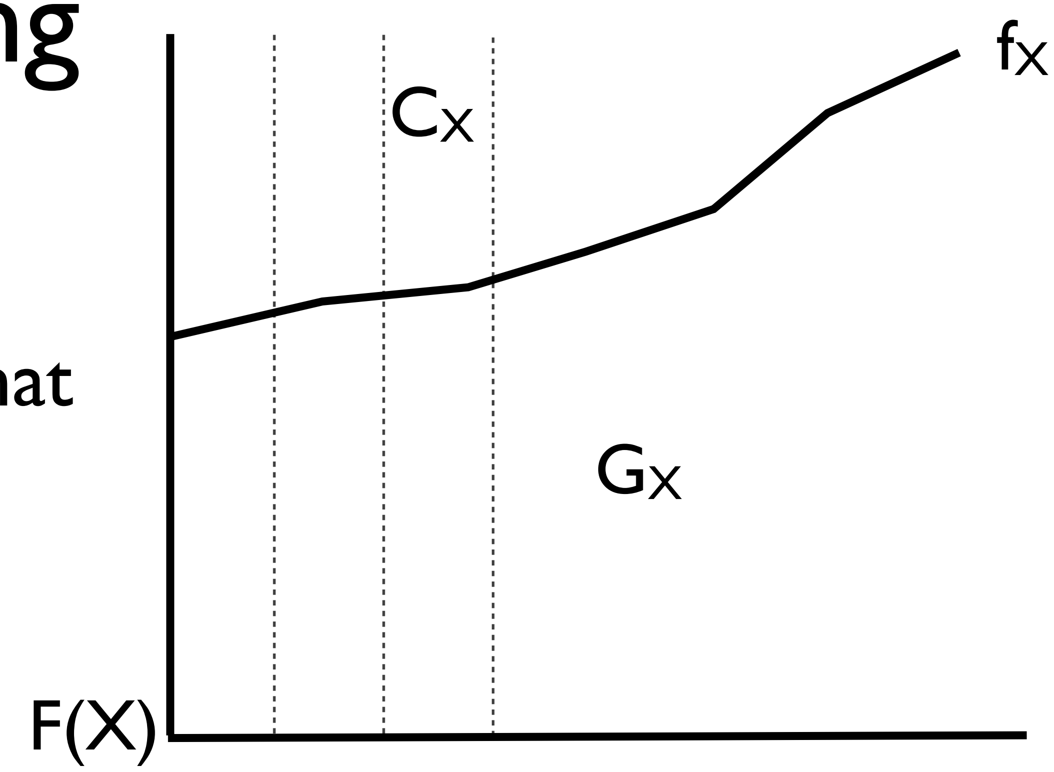
- The accordant forcing notion has the property that

$$X' \equiv_T (G \oplus f)' \oplus C$$

we code C into the jump of X



Jump Coding



- Construct Borel $F: 2^{F_2} \to 2^{\mathbb{N}}$ such that

$$F(X)' \equiv_T G_X \oplus f_X \oplus C_X \oplus 0'$$

where

$$C_X = F(aX) \oplus F(bX) \oplus F(a^{-1}X) \oplus F(b^{-1}X)$$

- Iterated jump coding

$$F(X)^{(k)} \equiv_T \left(\bigoplus_{\substack{w \in F_2 \\ |w| < k}} G_{w \cdot X} \oplus f_{w \cdot X} \oplus C_{w \cdot X} \right) \oplus \left(\bigoplus_{\substack{w \in F_2 \\ |v| = k}} F(v \cdot X) \right) \oplus 0^{(k)}.$$

Turing Equivalence

- Why is the case of Turing equivalence so hard?
- It is tied to a deep problem in recursion theory: What are the definable (Turing) degree invariant functions?
- A function $f: 2^\omega \to 2^\omega$ is degree invariant if

  $$X \equiv_T Y \quad \text{implies} \quad f(X) \equiv_T f(Y).$$
- Standard examples:
  - identity, jump operator, iterates of the jump operator
- Question: are there other (natural) examples?

Martin's Conjecture

1. If f is a Borel, Turing degree invariant function, then either f is increasing on a cone or f is constant on a cone.

2. The relation

$$f \leq_M g \Leftrightarrow \{X\colon f(X) \leq_T g(X)\} \text{ contains a cone}$$

prewellorders the degree-invariant functions which are increasing on a cone. If f has $\leq_M$-rank α, then f' has rank $\alpha+1$, where $f'(X) = (f(X))'$.

Martin's Conjecture

- If $\equiv_T$ is universal, then Martin's Conjecture fails.
  - If $\equiv_T$ is universal, then $\equiv_T \times \equiv_T \leq_B \equiv_T$.
  - It follows there exists a degree-invariant pairing function $\langle .,. \rangle$
  - Then $X \rightarrow \langle 0,X \rangle$ and $X \rightarrow \langle 0',X \rangle$ are degree-invariant Borel injections, hence not constant on a cone.
  - If MC holds, then both must be increasing on a cone.
  - Then their ranges are disjoint, cofinal Borel sets, contradicting Turing-Determinancy.

Uniform Reductions

- If E, F are countable Borel equivalence relations represented by orbit equivalences E_G and F_H, respectively, then a reduction $E_G \leq_B F_H$ given by f is uniform iff there exists $t: G \to H$ such that

$$g \cdot x = y \quad \Leftrightarrow \quad t(g) \cdot f(x) = f(y).$$

- All known universality proofs so far are uniform.

Uniform Reductions

- Montalban-R.-Slaman: Shift-equivalence on 2^{F_2} is not uniformly reducible to $\equiv_T$:

  There do not exist Borel $f: 2^{F_2} \to 2^N$ and $t: F_2 \to N$ such that

  $$g \cdot x = y \quad \Leftrightarrow \quad f(x) \equiv_T f(y)$$

  via reduction $t(g) = (\Phi_d, \Phi_e)$.

- The proof extends the game-theoretic approach used by Slaman and Steel to show that Martin's Conjecture holds for uniformly degree invariant functions.

Excursion: Cocycles

- The question of uniform reductions is related to the notion of a cocyle.
- Suppose $f: X \to Y$ is a Borel reduction $E \leq_B F$.
  Suppose further E, F are generated by groups G, H, respectively.
- Suppose further the action is free,
  i.e. $g \cdot x \neq x$ for all $g \neq 1$ and all $x \in X$.
- Then $\alpha(g,x) =$ unique $h \in H$ such that $f(g \cdot x) = h \cdot f(x)$ is defined.
- The function α is a typical example of a Borel cocyle.
- If the reduction is uniform, α depends only on g and becomes a group homomorphism.
- Popa [2007] showed that cocylces on free Bernoulli actions of sufficiently mixing groups are essentially homomorphisms.

Essentially Free Equivalence Relations

- Question: Can we recover G from the equivalence relation E it induces?
- If so, we could hope to gain insights on the complexity of E by studying the complexity of G.
- Two necessary requirements:
  - The action must be free.
  - There must exist a G-invariant probability measure on X.
- A countable Borel equivalence relation is essentially free if it is Borel equivalent to an equivalence relation generated by a free group action.

Essentially Free Equivalence Relations

- Essentially free equivalence relations
  - are closed downwards under Borel reducibility,
  - are closed under refinement.
- Thomas [2009]: The class of essentially free equivalence relations does not admit a universal element.
  - The proof uses Popa's superrigidity result.
- Corollary: $\equiv_T$ is not essentially free.
  - If it were, then the shift equivalence on 2^{F_2}, seen as a refinement of $\equiv_T$, were essentially free and universal.

Weak Universality

- The closure of essentially free equivalence relations under refinements gives rise to a weaker notion of reducibility

  E is weakly reducible to F if there is a countable-to-one Borel homomorphism from E to F.

- E is weakly reducible to F iff there exists a countable equivalence relation $R \subseteq F$ such that $E \leq_B R$.

- Thomas [2009]:

  - No weakly universal equivalence relation is essentially free.

  - Turing equivalence is weakly universal.

- Question: Is every weakly universal equivalence relation universal?

Ergodicity

- If G acts on X and μ is a G-invariant probability measure on X, then the action of G is said to be ergodic if every G-invariant subset of X has either measure 0 or 1.
- An equivalence relation E is ergodic if some action that induces it is ergodic.
- Generalization: Suppose E, F are equivalence relations on X, Y, respectively, and μ is an E-invariant measure on X.
    - A Borel homomorphism from E to F is a Borel function f: $X \to Y$ such that $x \mathrel{E} y$ implies $f(x) \mathrel{F} f(y)$.
    - E is called F-ergodic if there exists $Z \subseteq X$ such that $\mu(Z) = 1$ and $f|_Z$ is F-constant.
- Ergodicity in this framework becomes $\mathrm{id}_{\mathbf{R}}$-ergodicity.

Ergodicity

- If E is ergodic it cannot reduce to $id_{\mathbb{R}}$: This would imply the existence of an equivalence class of measure 1, which is impossible.
- Martin measure on $2^{\mathbb{N}}/\equiv_T$:
  $m(A) = 1$ if A contains a cone, 0 otherwise.
  - This is indeed a Borel measure by Borel-Turing Determinacy.
- Martin measure is ergodic:
  Every degree invariant mapping to the reals is constant on a cone.

Ergodicity

- If Martin's Conjecture (MC) holds, $\equiv_T$ satisfies an even stronger ergodicity property.

- Thomas [2009]: Suppose MC is true. If E is any countable Borel equivalence relation, then exactly one of the following conditions holds:

(a) E is weakly universal.

(b) $\equiv_T$ is E-ergodic.

Further Directions

- Prove Martin's Conjecture
- Extend Slaman-Steel technique to other reducibilities
  - interesting candidate: LR-reducibility
- Extend universality to other strong reducibilities
  - Recent work by Marks
- Prove further ergodicity results (w/o assuming MC)
  - Maybe easier for arithmetic equivalence.
- Study possible group representations for $\equiv_T$