Homework 8 for MATH 104

Brief solutions to selected exercises

Problem 1

Find two sequences of real functions $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ from some $S \subseteq \mathbb{R}$ into $\mathbb{R}$ such that $f_n \to f$ uniformly and $g_n \to g$ uniformly, but $f_n g_n$ does not converge uniformly to fg.

Solution. Consider $f_n(x) = x$ for all n, and $g_n \equiv 1/n$. Then obviously $f_n \to f$ uniformly, where $f(x) = x$, and $g_n \to 0$ uniformly. However, $f_n g_n = x/n \to 0 = fg$ only pointwise, since $\sup\{|f_n g_n(x) - 0| : x \in \mathbb{R}\} = \infty$. ∎

Problem 2

Let $P_0 = 0$ and define for $n \in \mathbb{N}$,

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Prove that $P_n \to |x|$ uniformly on $[-1, 1]$.

[*Hint:* Use the identity

$$|x| - P_{n+1}(x) = (|x| - P_n(x))\left(1 - \frac{|x| + P_n(x)}{2}\right)$$

to prove that $0 \leqslant P_n(x) \leqslant P_{n+1}(x) \leqslant |x|$ for $|x| \leqslant 1$, and that

$$|x| - P_n(x) \leqslant |x|\left(1 - \frac{|x|}{2}\right)^n < \frac{2}{n+1}$$

for $|x| \leqslant 1$.]

Problem 3

(a) Assume $\sum a_n$ and $\sum b_n$ are absolutely convergent series. Define $c_n = \sum_{k=0}^n a_k b_{n-k}$. Show that $\sum c_n$ converges and that

$$\left(\sum_{n=0}^\infty a_n\right)\left(\sum_{n=0}^\infty b_n\right) = \sum_{n=0}^\infty c_n.$$

(b) Find an example of two convergent but not absolutely convergent series such that the multiplication property from (a) does not hold.

[Here is a hint on how to proceed in (a): Let $A_n = \sum_{k=0}^n a_k$, $B_n = \sum_{k=0}^n b_k$, $C_n = \sum_{k=0}^n c_k$. Argue that it suffices to show that $\lim_n |C_{2n} - A_n B_n| = 0$ and $\lim_n |C_{2n+1} - A_{n+1}B_n| = 0$. To prove the first assertion, deduce the identity

$$\begin{aligned} C_{2n} - A_n B_n = {} & a_0(b_{n+1} + b_{n+2} + \cdots + b_{2n}) + a_1(b_{n+1} + b_{n+2} + \cdots + b_{2n-1}) + a_{n-1}b_{n+1} \\ & + a_{n+1}(b_0 + b_1 + \cdots + b_{n-1}) + a_{n+2}(b_0 + b_1 + \cdots + b_{n-2}) + \cdots + a_{2n}b_0. \end{aligned}$$

Now use the fact that $\sum_{k=0}^n |a_k|$ and $\sum_{k=0}^n |b_k|$ are bounded, together with the Cauchy criterion for convergent series. To prove the second assertion, derive a similar identity for $C_{2n+1} - A_{n+1}B_n$ and proceed analogously.]

Problem 4

(a) Use Euler's formula to deduce the *addition theorems* for sin and cos.

$$\begin{aligned} \cos(x+y) &= \cos x \cos y - \sin x \sin y \\ \sin(x+y) &= \sin x \cos y + \cos x \sin y \end{aligned}$$

Solution. By Euler's formula, we know that

$$\cos(x+y)+i\sin(x+y)=e^{i(x+y)}=e^{ix}e^{iy}=(\cos x+i\sin x)(\cos y+i\sin y).$$

Multiplying out the right hand side, and comparing the real and imaginary parts yields the addition theorems. ∎

(b) Prove the identity $\cos(x-y)-\cos(x+y)=2\sin x\sin y$ and use it to show that cos is decreasing in the interval $[0, \frac{\pi}{2}]$.

Solution. By the addition theorems,

$$\cos(x-y)-\cos(x+y)=\cos x\cos(-y)-\sin x\sin(-y)-\cos x\cos y+\sin x\sin y=2\sin x\sin y,$$

for $\cos(-y)=\cos y$ and $\sin(-y)=-\sin y$.

We set $u:=x+y$ and $v:=x-y$, so we have

$$\cos v-\cos u=2\sin\frac{u+v}{2}\sin\frac{u-v}{2}.$$

Let $\pi/2>u>v\geqslant 0$. Then

$$\frac{u+v}{2},\frac{u-v}{2}\in(0,\tfrac{\pi}{2}].$$

But sin is positive on $(0,\frac{\pi}{2}]$, so

$$\cos v>\cos u.$$

∎