Lesson 3
Dynamical Systems
3-3: Markov Shifts

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Markov Shifts

In the following: A finite alphabet $\{0, 1, \ldots, k-1\}$.

Markov Shift: Infinite paths through a finite state system (finite directed graph).

$A = \{0, 1\}$

$0101110\ldots$

$1111111\ldots$

Topological Markov Shifts

Mathematical description using matrices:

- G directed graph on vertices $\{0, 1, \ldots, k-1\}$.
- Let $M = M^{(G)}$ be its adjacency matrix, i.e. $M_{ij} \in \{0, 1\}$ and

$$M_{ij} = 1 \quad \Longleftrightarrow \quad \text{there is an edge between } i \text{ and } j.$$

Wlog: no zero rows, i.e. no dead ends.

$$M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

- Then the Markov shift given by M is defined as

$$X_M = X_G = \{x \in A^{\mathbb{N}} \colon \forall i \geqslant 0 \; M_{x_i x_{i+1}} = 1\}.$$

We can define a two-sided shift based on M in a similar way.

Observation: X_M is a subshift.

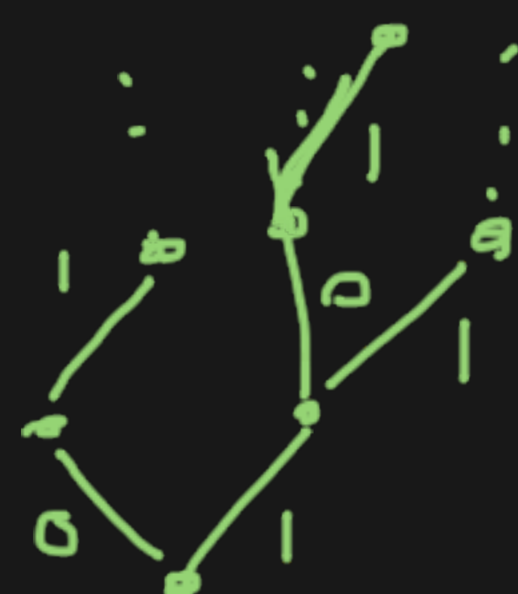
Markov Shifts

Observation: X_M is a subshift.

- Closed under shift.
- Topologically closed: Consider the set of allowed words,

$$\mathcal{U}(X_M) = \{u \in A^{<\mathbb{N}} : \text{ u is a substring of some } x \in X_M\}.$$

Then X_M is the set of infinite paths through a tree, where the n-the lavel of the tree is given by $\mathcal{U}_n(X_M)$, the allowed words of length n.



Two Observations

- $(M^n)_{ij}$ is the number of allowed words of length n beginning with i and ending with j.

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

- Markov shifts are shifts of finite type (SFT).

↑ finite set of forbidden words

$u = u_0 \dots u_{\ell-1}$

Is $u^\frown i$ allowed? $\iff$ $u_{\ell-1}{}^\frown i$ is allowed

Ex: 00 forbidden iff $M_{u_{\ell-1} i} = 1$

generalized concept: multi-step Markov shifts

Irreducible Shifts

Some graphs give rise to a shift with several ``components''.

On the matrix side, this means that there are indices i, j such that $(M^n)ij = 0$ for all n.

If the graph G is strongly connected, i.e. there is a path from each i to every other j, then the corresponding adjacency matrix is called irreducible, i.e.

for all i, j there exists n such that $(M^n)_{ij} > 0$.

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Irreducible Shifts

Irreducible shifts always have dense orbits, i.e. there exists an $x \in X_M$ such that the orbit of x under the shift map,

$$\mathcal{O}_x = \{x, T(x), T^2(x), \dots\}$$

is dense in the set X_M, that is, it gets arbitrarily close to every $y \in X_M$.

- Using irreducibility, we can an construct an x which contains every allowed word as a substring. The orbit of such an x must be dense in X_M.

On the other hand, if X_M has a dense orbit, then it must be irreducible.

- If the orbit of $x \in X_M$ is dense, then in particular every $i \in A$ must occur infinitely often in x.
- But this means we can get from any $i \in A$ to any other $j \in A$ in the underlying graph G (since x has only allowed words, i.e. substrings that correspond to paths in G).
- Hence G is strongly connected.

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Sofic Shifts

We can also consider edge shifts instead of vertex shifts.

We can interpret the graph of an edge shift as a non-deterministic finite automaton (NFA).

If we allow all states to be initial and accepting at the same time, then the set of allowed words for the edge shift corresponds to the language accepted by the NFA.

- Sofic shift = set of allowed words is a regular language.

Sofic Shifts

Sofic shifts do not have to be SFTs.

- Example: Even shift

$$X_E = \{x \in \{0,1\}^{\mathbb{N}} : \text{ between two 1s is even \# of 0s}\}.$$

- Not a SFT: forbidden words 101, 10001, 1000001, . . .
- But allowed words form regular language.