Sample Midterm 1 for MATH 185 Solutions

Problem 1

If the followings statements are true, answer "TRUE". If not, give a brief explanation why.

(1) If $f : \mathbb{C} \to \mathbb{C}$ is complex differentiable at a, so is the function $g(z) := \overline{f(\overline{z})}$.

Solution. FALSE, consider e.g. the function

$$f(z) = \begin{cases} z & \text{if } z \in \mathbb{H}, \\ \overline{z} & \text{otherwise.} \end{cases}$$

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(2) The set $\{z \in \mathbb{C} : |z^2 - 3| < 1\}$ is a domain.

Solution. FALSE. The set contains $\sqrt{3}$ and $\sqrt{-3}$, but it is easy to see that for all $t \in \mathbb{R}$, ti is not in the set. So there is no path in the set connecting $\sqrt{3}$ and $\sqrt{-3}$. ■

(3) The set $\{z \in \mathbb{C} : |z| < 1 \wedge |z + 1| < \sqrt{2}\}$ is a star-shaped domain.

Solution. TRUE. ■

(4) The function $f(z) = -1/z^4$ has an anti-derivative in $D = \mathbb{C}^\bullet$.

Solution. TRUE. ■

(5) $\int_\alpha \frac{2z-1}{z^2-8z+15} = 0$, where α is the unit circle around 0.

Solution. TRUE. ■

Problem 2

Compute the real and imaginary part of $((1+i)/\sqrt{2})^k$ for $k \in \mathbb{Z}$.

Solution. $z = ((1+i)/\sqrt{2})^k$ is an 8th root of unity. Therefore, $z^{8m+j} = z^j$. Hence there are only 8 possible values. These are, for $\mathrm{Re}(z^k)$,

$$1, 1/\sqrt{2}, 0, -1/\sqrt{2}, -1, -1/\sqrt{2}, 0, 1/\sqrt{2}.$$

For $\mathrm{Im}(z^k)$ we get

$$0, 1/\sqrt{2}, 1, 1/\sqrt{2}, 0, -1/\sqrt{2}, -1, -1/\sqrt{2}.$$

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Problem 3

Determine in which points the function $f(x+iy) = sin^2(x+y) + icos^2(x+y)$ is complex differentiable and compute the derivative at those points.

Solution. Obviously, the function is totally differentiable in the sense of real analysis. So we only have to check the C-R-equations. The partial derivatives are

$$\partial_i u = 2\sin(x+y)\cos(x+y) \quad \text{and} \quad \partial_i v = -2\sin(x+y)\cos(x+y), \quad \text{for } i = 1,2.$$

Hence, the C-R-equations hold precisely for those $z = x + iy$ for which

$$\sin(x+y) = 0 \quad \text{or} \quad \cos(x+y) = 0.$$

This is precisely the set $\{z : x + y = k\pi/2 \text{ for some } k \in \mathbb{Z}\}$, which is a set of parallel lines of distance $\pi/2$ in the complex plane. In particular, the function is nowhere analytic. ■

Problem 4

Compute

$$\int_\alpha \frac{1}{|z|^2} dz,$$

where α is the circle of radius 3 around 0.

Solution. The circle is parametrized by $\alpha(t) = 3e^{it}$, $t \in [0, 2\pi]$. So

$$\int_\alpha \frac{1}{|z|^2} dz = \int_0^{2\pi} \frac{1}{|3e^{it}|^2} 3ie^{it} dt = \frac{i}{3} \int_0^{2\pi} e^{it} dt = 0.$$

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