Sample Final for MATH 104

Problem 1

[Review all important definition and results of the relevant material.]

Problem 2

If the followings statements are true, answer "TRUE". If not, give a brief explanation why.

(1) If F is a field and $x, y \in F$, then $x \cdot y = 0$ implies $x = 0$ or $y = 0$.

Solution. TRUE (Theorem 3.1 (vi) in Ross) ■

(2) If $S, T \subseteq \mathbb{R}$ are bounded and $\sup S = \inf T$, then $S \cap T \neq \emptyset$.

Solution. FALSE – Consider $S = (-1, 0)$ and $T = (0, 1)$. ■

(3) If (s_n) is a sequence of real numbers, and for some $k \geqslant 1$, $\lim_n (s_{n+k} - s_n) \to 0$, then (s_n) is a Cauchy sequence.

Solution. FALSE – Consider $s_n = (-1)^n$, $k = 2$. ■

(4) The series $\sum \frac{n!}{n^n}$ converges absolutely.

Solution. TRUE (Homework 4.1 (b)) ■

(5) If a set contains no interior points, it is closed.

Solution. FALSE – Consider $\mathbb{Q}$, the set of all rationals. ■

(6) The set of nondecreasing functions from $\mathbb{Q}$ into $\{0, 1\}$ is countable.

Solution. TRUE ($\mathbb{Q}$ is countable, use Problem 5 of first midterm.) ■

(7) If f is differentiable and $f(-x) = f(x)$, then $f'(-x) = -f'(x)$.

Solution. TRUE (use the chain rule) ■

(8) If $f : [0, 1] \to [0, 1]$ is bijective, and $f(0) = 0$ and $f(1) = 1$, then f is continuous on $[0, 1]$.

Solution. FALSE – consider for example

$$f(x) = \begin{cases} 0 & x = 0, \\ -x + 1 & 0 < x < 1, \\ 1 & x = 1. \end{cases}$$

■

(9) If $0 \leqslant f(x) \leqslant g(x)$ for all $x \in [a, b]$, and g is Riemann integrable on $[a, b]$, then f is Riemann integrable on $[a, b]$.

Solution. FALSE – consider $g(x) = 1$ for all $x \in [0, 1]$, and

$$f(x) = \begin{cases} 1 & x \in [0, 1] \cap \mathbb{Q}, \\ 0 & \text{otherwise}. \end{cases}$$

■

(10) If f and g are not differentiable at $x = 0$, then $f \cdot g$ is not differentiable at 0.

Solution. FALSE – consider $f(x) = g(x) = |x|$. Then $fg(x) = x^2$. ■

Problem 3

Show that if E is a compact subset of $\mathbb{R}$, then $\sup E$ and $\inf E$ belong to E.

Solution. Ross, solution to problem 13.13 ■

Problem 4

Show that if f is differentiable on (a, b) and $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on (a, b).

Solution. Ross, Corollary 29.7 ■

Problem 5

Show that there does not exist a sequence (p_n) of polynomials that converges uniformly to e^x on $\mathbb{R}$.

Solution. Ross, solution to problem 27.3 (b) ■

Problem 6

Suppose $0 < t < 1$. Let $s_1 = 1$ and $s_{n+1} = t(s_n + 1)$. Show that (s_n) converges (hint: bounded and monotone) and calculate $\lim_n s_n$.

Solution. The usual argument for recursively defined sequences shows that if the limit s of (s_n) exists, it must satisfy the equation $s = t(s + 1)$, hence $s = t/(1 - t)$. If $t \leqslant 1/2$, then an easy induction shows that s_n is nonincreasing. Furthermore, s_n is obviously > 0 for all n, so (s_n) converges. If $t > 1/2$, one can use induction to show that s_n is nondecreasing, and that $s_n \leqslant t/(1 - t)$ for all n (use inqeuality $[t/(1 - t) + 1]t \leqslant t/(1 - t)$). ■

Problem 7

Let $f : (a, b) \to \mathbb{R}$ be differentiable. Suppose $\lim_{x \to b} f(x) = \infty$. Show that $\lim_{x \to b} f'(x) = \infty$, provided that the limit exists.

Solution. Since $\lim_{x \to b} f(x) = \infty$, we can choose a sequence $x_n \nearrow b$ such that $f(x_{n+1}) - f(x_n) = 1$. By the mean value theorem, we can find z_n between x_n and x_{n+1} such that

$$f'(z_n) = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n} = \frac{1}{x_{n+1} - x_n}.$$

Obviously, $z_n \nearrow b$, so if $\lim_{x \to b} f'(x)$ exists, it follows that $f'(z_n) \to \lim_{x \to b} f'(x)$. But (x_n) is a Cauchy sequence, so $\lim_n f'(z_n) = \lim_n 1/(x_{n+1} - x_n) \to \infty$. ■

Problem 8

Suppose that f is a continuous function on $[a, b]$ and that $f(x) \geqslant 0$ for all $x \in [a, b]$. Show that if $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

Solution. See Ross, soltuion to problem 34.11 ■