Homework 10 for MATH 104

Solutions to selected exercises

Problem 1

Suppose $f : \mathbb{R} \to \mathbb{R}$. Call x a *fixed point* of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \neq 1$ for all t, prove that f has at most one fixed point.

Solution. Suppose f had two fixed points, x and y, where $x < y$. Then by the mean value theorem there exists z such that

$$f'(z) + \frac{f(y) - f(x)}{y - x} = \frac{y - x}{y - x} = 1,$$

which contradicts $f'(t) \neq 1$ for all t. ■

(b) Show that the function f defined by

$$f(t) = t + \frac{1}{1 + e^t}$$

satisfies $0 < f'(t) < 1$ for all t but has no fixed point.

Solution. We have

$$f'(t) = 1 - \frac{e^t}{(1 + e^t)^2}.$$

Clearly $e^t/(1 + e^t)^2 > 0$ for all t, so $f'(t) < 1$ for all t. On the other hand,

$$\frac{e^t}{(1 + e^t)^2} < \frac{1 + e^t}{(1 + e^t)^2} < \frac{1}{1 + e^t} < 1,$$

so $f'(t) > 0$ for all t.

A simple calculation shows that x is a fixed point of f if and only if $e^x = 0$, so f has no fixed point. ■

(c) Show that if $|f'(t)| \leqslant M$ for all t and some $M < 1$, then f has a fixed point.

Solution. Pick $s_0 \in \mathbb{R}$ arbitrary. Define $s_{n+1} = f(s_n)$. By Problem 9.4, (s_n) converges to some $s \in \mathbb{R}$. Since f is differentiable, it must be continuous. Hence $f(s_n) \to f(s)$. But since $s_{n+1} = f(s_n)$, we must also have $f(s_n) \to s$. By the uniqueness of the limit, $s = f(s)$. ■

Problem 2

[Newton's method, 10P]
Suppose $f : [a, b] \to \mathbb{R}$ is twice differentiable on $[a, b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geqslant \delta > 0$ and $0 \leqslant f''(x) \leqslant M$ for all $x \in [a, b]$.

(a) Show that there exists a unique point $\xi \in (a, b)$ such that $f(\xi) = 0$.

Solution. By the intermediate value theorem, there exists a $\xi \in (a, b)$ such that $f(\xi) = 0$. Suppose there were another $\zeta \in (a, b)$ with $f(\zeta) = 0$. Then, by Rolle's theorem there exists a z between ξ and ζ such that $f'(z) = 0$, contradicting the assumption $f'(x) > 0$ for all x. ■

(b) Let $x_1 \in (\xi, b)$ and define (x_n) by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Give a geometrical interpretation of this definition, by means of a tangent to the graph of f.

(c) Prove that $x_{n+1} < x_n$ and that $\lim_n x_n = \xi$.

Solution. We prove the following two assertions by simultaneous induction: (1) $x_n \geqslant \xi$ for all n and (2) $x_{n+1} < x_n$.

$x_1 > \xi$ holds by assumption. Since $x_1 \in (\xi, b)$, it follows that $f(x_1) > 0$. Furthermore, $f'(x_1) > 0$ by assumption. Hence $x_2 = x_1 - f(x_1)/f'(x_1) < x_1$. x_{n+1}

Given $\xi < x_n < x_1 < b$, it follows that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} < x_n,$$

since $f(x_n) > 0$ and $f'(x_n) > 0$. Also, (assume $x_{n+1} \neq \xi$)

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} > \xi \iff f'(x_n) > \frac{f(x_n)}{x_n - \xi}.$$

By the mean value theorem, $f(x_n)/(x_n - \xi)$ is equal to some $f'(z)$ for $\xi < z < x_n$. But $f'' \geqslant 0$, so f' is non-decreasing, which implies $f'(x_n) \geqslant f'(z)$, so $x_{n+1} < x_n$ follows.

It follows from these considerations that (x_n) is nonincreasing and bounded from below, hence it converges to some $x \in [\xi, b)$. Consider the function $g(t) = t - \frac{f(t)}{f'(t)}$. Clearly, g is continuous on $[\xi, b)$. An argument similar to Problem 10.1 (c) yields that $g(x) = x$. Since $f'(x) \neq 0$, $f(x) = 0$. But it has been shown in (a) that ξ is the unique zero of f, so $x = \xi$. ■

(d) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

Solution. We consider the Taylor series for $f(\xi)$ about x_n. Taylor's theorem yields

$$f(\xi) - [f(x_n) + f'(x_n)(\xi - x_n)] = \frac{f''(t_n)}{2}(x_n - \xi)^2$$

for some t_n between ξ and x_n. This in turn transforms into

$$x_n - \frac{f(x_n)}{f'(x_n)} - \xi = x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2.$$

■

(e) Set $A = M/2\delta$ and deduce that

$$0 \leqslant x_{n+1} - \xi \leqslant \tfrac{1}{A}[A(x_1 - \xi)]^{2^n}$$

Solution. $0 \leqslant x_{n+1} - \xi$ has been shown in (c). Obviously, $f''(t_n)/2f'(x_n) \leqslant M/2\delta = A$ by assumption. The claim now follows by a straightforward induction. ■

(f) Interpret Newton's method as a search for a fixed point of some function.

Solution. ξ is the fixed point of $g(t) = t - \frac{f(t)}{f'(t)}$, see (c). ■

Problem 3

Let F be the Cantor set (as defined for example in Ross, p.85). Let $f : [0, 1] \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1 & \text{if } x \in F, \\ 0 & \text{if } x \notin F. \end{cases}$$

Show that f is Riemann integrable.

Solution. Let P_n be the partition of $[0, 1]$ given by

$$P_n = \{a/3^n : a = 0, \ldots, 3^n\}.$$

By construction of F, $\sup(f, [t_{k-1}, t_k]) = 1$ for 2^n intervals $[t_{k-1}, t_k]$ of the form $[k - 1/3^n, k/3^n]$, whereas f is zero on the remaining intervals. Hence, $U(f, P_n) = 2^n 3^{-n} = (2/3)^n$. Hence $U(f, P_n) \to 0$ for $n \to \infty$, and therefore $U(f) \leqslant \lim U(f, P_n) = 0$. On the other hand, $f \geqslant 0$, so $L(f) \geqslant 0$. It follows that $0 \leqslant L(f) \leqslant U(f) \leqslant 0$, and f is integrable. ■