Homework 1 for
MATH 561, Set Theory

Due: Thursday Jan 26

Problem 1

Let $\mathcal{L}$ be a finite language and let $\mathcal{M}$ be a finite $\mathcal{L}$-structure. Show that there is an $\mathcal{L}$-sentence σ such that $\mathcal{N} \models \sigma$ if and only if $\mathcal{N} \cong \mathcal{M}$.

Problem 2

Let $\mathcal{M}$ be an $\mathcal{L}$-structure and suppose $X \subseteq M$. We say $b \in M$ is *algebraic* over X if there exist an $\mathcal{L}$-formula φ and $a_1, \ldots, a_m \in X$ such that $\mathcal{M} \models \varphi[b, a_1, \ldots, a_n]$ and $\{y \in M : \mathcal{M} \models \varphi[y, a_1, \ldots, a_m]\}$ is finite.

Let $\mathrm{acl}(X) = \{b \in M : b \text{ is algebraic over } X\}$ be the *algebraic closure* of X.

(a) Suppose that $x \in \mathrm{acl}(A)$. Show that there are $x_1, \ldots, x_m$ such that if σ is an automorphism of $\mathcal{M}$ with $\sigma(a) = a$ for all $a \in A$, then $\sigma(x) = x_i$ for some i. In other words, there are only finitely many conjugates for x under automprphisms of $\mathcal{M}$ fixing A.

(b) Show that $\mathrm{acl}(\mathrm{acl}(X)) = \mathrm{acl}(X)$.

(c) Show that if $x \in \mathrm{acl}(A)$, then $x \in \mathrm{acl}(A_0)$ for some finite $A_0 \subseteq A$.

(d) Show that if $A \subseteq B$, then $\mathrm{acl}(A) \subseteq \mathrm{acl}(B)$.

Problem 3

Suppose that $\mathcal{M} \preceq \mathcal{N}$ and $A \subseteq M$.

(a) Show that $\mathrm{acl}(A)$ in $\mathcal{M}$ is equal to $\mathrm{acl}(A)$ in $\mathcal{N}$.

(b) Give examples showing that this is not true if we only have $\mathcal{M} \equiv \mathcal{N}$ and $\mathcal{M} \subseteq \mathcal{N}$.

Problem 4

An ultrafilter $\mathcal{U}$ over $\mathbb{N}$ is *Ramsey* if for any infinite partition $(A_n : n \in \mathbb{N})$ of $\mathbb{N}$ so that $A_n \notin \mathcal{U}$ for all n, there exists $X \in \mathcal{U}$ with $|X \cap A_n| = 1$ for all n.

Show that an ultrafilter $\mathcal{U}$ over $\mathbb{N}$ is Ramsey if and only if every function $f : \mathbb{N} \to \mathbb{N}$ is either one-one on a set in $\mathcal{U}$ or constant on a set in $\mathcal{U}$.