Math 497A: Introductory to Ramsey Theory

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1 Ramsey's Theorem on Graphs, continued

We can restate the Ramsey Theorem on Infinite Graph given at the end of the previous set of notes.

Theorem 1. (Ramsey's Theorem on Infinite Graphs) For every 2-coloring $c : [\mathbb{N}]^2 \to \{\text{red, blue}\}$, there exists and infinite set $H \subseteq \mathbb{N}$ and a color c_0 such that for every distinct pair of $x, y \in H$, we have that $c(\{x, y\}) = c_0$.

Infinite Ramsey-type theorems are important because one can often derive from them finite versions by using compactness arguments. However, these corollaries have the downside in that, while they still have the condition that your object or set of objects must be sufficiently large, there is nothing else that we can know about this threshold.

2 Other Ramsey-type Results

We have seen so far how Ramsey-type problems can be given in the realm of graph theory. However, an interesting point about Ramsey Theory is how ubiquitous it is. Many natural structures have properties that can be preserved under a partitioning of their sets.

Exercise: Use Ramseys theorem for graphs to show that for every positive integer k there exists a number $N(k)$ such that if $a_1, a_2, \dots, a_{N(k)}$ is a sequence of $N(k)$ integers, it has a non-increasing subsequence of length k or a non-decreasing subsequence of length k. Show that $N(k + 1) > k^2$.

As in the example above, Ramsey-type problems are many times answered by relating your objects to those for which a Ramsey-type result is already known.

The set of natural numbers provides a bevy of Ramsey-type results. Regularity in a set of natural numbers can come in many forms, for example, the property of additivity:

Exercise: Use Ramseys theorem for graphs to show that for every positive integer k there exists a number $M(k)$ such that if the set $\{1, 2, ..., M(k)\}$ is partitioned into two subsets, at least one of them contains a set of the form $\{x_1, x_2, \ldots, x_k, x_1 + \cdots + x_k\}$.

The existance of arithemtic progressions, sets of the form $m, m+r, m+2r, \ldots, m+(l-1)r$ for some m, r, and l, is another property which can be preserved. In this definition, m, r, and l are called the **base**, **modulus**, and **length** of the progression, respectively.

Theorem 2. (Van der Waerden's Theorem, 1927) For every $k \geq 1$, there exists an integer $W(k)$ such that if $\{1, \ldots, W(k)\}$ is partitioned into two sets, one set will neccesarily contain an arithmetic progression of length k.

We can also state an infinite version of Van der Waerden's Theorem. Be sure to note that the qualifier "infinite" refers to the size of the original set and not to the lengths of the arithmetic progressions.

Theorem 3. (Van der Waerden's Theorem, Infinite Version) If the positive integers are partitioned into two sets, one of the sets contains arbitrarily long arithmetic progressions.

Exercise: Show that Van der Waerdens theorem becomes false if we require that one of the two subsets contains infinite arithmetic progressions by giving a counterexample.

The last example of a Ramsey-type result we will give here is one that preserves geometric structure. We must first define our object and subobjects in question. Given a set $A = \{1, \cdots, a\}$, where $a \geq 1$, we can define the **n-dimensional cube over** A, or C_A^n, as

$$C_A^n := \{(x_1, \ldots, x_n) : x_i \in A\}.$$

For example, a tic-tac-toe board could be seen as $C_{1,2,3}^2$ whereas a Rubix Cube would be $C_{1,2,3}^3$.

As on a tic-tac-toe board, if we mark off certain coordinates, we can form lines. On $C_{1,2,3}^2$ the coordinates $(1,1)$, $(1,2)$, and $(1,3)$ form a vertical line whereas $(1,1)$, $(2,2)$, and $(3,3)$ form a diagonal line. You might have noticed that these lines have a pattern to them: the first can be written as $\{(1,*) : * \in 1,2,3\}$ and the second is $\{(*,*) : * \in 1,2,3\}$. These examples motivate the following definition: in the n-dimensional cube over A, a **combinatorial line** is a set of n-tuples having exactly one free variable (which is allowed to appear in any number of coordinates) that ranges over A; these are the regular subobjects we wish to consider.

Theorem 4. (Hales-Jewett Theorem, 1963) For every $a \geq 1$, there exists and integer $HJ(a)$ such that given any coloring $c : C_A^{HJ(a)} \to \{\text{blue, red}\}$ there will exist a monochomatic combinatorial line.

In all of these examples, we have been partitioning our object into exactly two sets. We will soon see that the number of sets does not actually matter; the only change will be that the threshold for the existance of these objects will grow substantially.