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Deep Learning Algorithms and Analysis

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Finite element spaces

1.1 Conforming linear finite element spaces

A conforming linear finite element function in a domain $\Omega \subset \mathbb{R}^d$ is a continuous function that is piecewise linear function with a grid or mesh consisting of a union of simplexes.

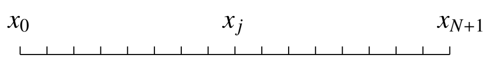

Fig. 1.1. 1D uniform grid

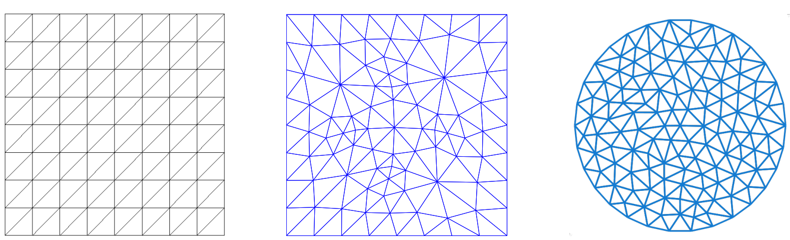

Fig. 1.2. 2D grids

1.1.1 Simplexes in $\mathbb{R}^d$

Let $\boldsymbol{x}_i = (x_{1,i}, \cdots, x_{d,i})^t, i = 1, \cdots, d+1$ be $d+1$ points in $\mathbb{R}^d$ which do not all lie in one hyper-plane. The *convex hull* of the $d+1$ points $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{d+1}$ (See Figure 1.3)

$$\tau := \{\boldsymbol{x} = \sum_{i=1}^{d+1} \lambda_i \boldsymbol{x}_i \,|\, 0 \le \lambda_i \le 1, i = 1 : d+1, \sum_{i=1}^{d+1} \lambda_i = 1\} \tag{1.1}$$

is defined as a *geometric d-simplex* generated (or spanned) by the vertices $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{d+1}$. For example, a triangle is a 2-simplex and a tetrahedron is a 3-simplex. For an integer $0 \leq m \leq d-1$, an m-dimensional face of τ is any m-simplex generated by $m+1$ of the vertices of τ. Zero-dimenisonal faces are vertices and one-dimensional faces are called edges of τ. The $(d-1)$-face opposite to the vertex $\boldsymbol{x}_i$ will be denoted by F_i.

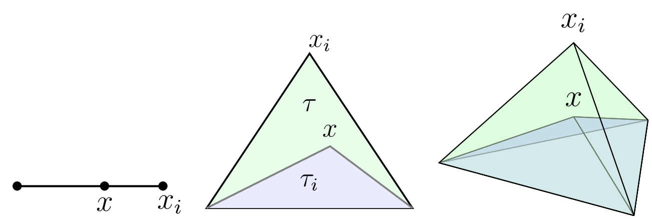

Fig. 1.3. Geometric explanation of barycentric coordinate

Barycentric coordinates

The numbers $\lambda_1(\boldsymbol{x}), \cdots, \lambda_{d+1}(\boldsymbol{x})$ are called *barycentric coordinates* of $\boldsymbol{x}$ with respect to the $d+1$ points $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_{d+1}$. There is a simple geometric meaning of the barycentric coordinates. Given a $\boldsymbol{x} \in \tau$, let $\tau_i(\boldsymbol{x})$ be the simplex with vertices $\boldsymbol{x}_i$ replaced by $\boldsymbol{x}$. Then it can be easily shown that

$$\lambda_i(\boldsymbol{x}) = |\tau_i(\boldsymbol{x})|/|\tau|, \tag{1.2}$$

where $|\cdot|$ is the Lebesgure measure in $\mathbb{R}^d$, namely area in two dimensions and volume in three dimensions. Note that $\lambda_i(\boldsymbol{x})$ is affine function of $\boldsymbol{x}$ and vanishes on the face F_i. We list the four basic properties of barycentric coordinate below:

1. $0 \leq \lambda_i(x) \leq 1$;
2. $\sum_{i=1}^{d+1} \lambda_i = 1$;
3. $\lambda_i \in P_1(\tau)$;
4. $\lambda_i(\boldsymbol{x}_j) = \delta_{ij}$.

1.1.2 Shape-regular and quai-uniform triangulations

Given a bounded polyhedral domain $\Omega \subset R^d$. A geometric triangulation (also called mesh or grid) $\mathcal{T}$ of Ω is a set of d-simplices such that

$$\cup\tau = \overline{\Omega}, \quad \text{and} \quad \mathring{\tau}_i \cap \mathring{\tau}_j = \emptyset.$$

Examples of triangulations for $\Omega = (0,1)$ $(d=1)$ are shown in Figure 1.1 for $\Omega = (0,1)$ $(d=1)$ and Figure 1.1 for $\Omega = (0,1)^2$ $(d=2)$.

Denote

$$h_\tau = \mathrm{diam}(\tau), h = \max_{\tau\in\mathcal{T}_h} h_\tau; \quad \underline{h} = \min_{\tau\in\mathcal{T}_h} h_\tau,$$

The first requirement is a topological property. A triangulation $\mathcal{T}$ is called *conforming* or *compatible* if the intersection of any two simplexes τ and τ' in $\mathcal{T}$ is either empty or a common lower dimensional simplex (nodes in two dimensions, nodes and edges in three dimensions).

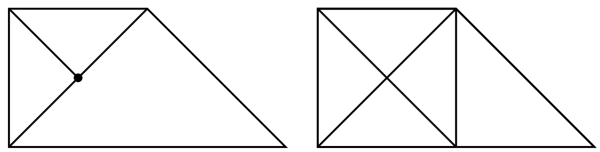

Fig. 1.4. Two triangulations. The left is non-conforming and the right is conforming.

The second important condition depends on the geometric structure. A set of triangulations $\mathscr{T}$ is called *shape regular* if there exists a constant c_0 such that

$$\max_{\tau\in\mathcal{T}} \frac{\mathrm{diam}(\tau)^d}{|\tau|} \le c_0, \quad \forall \mathcal{T} \in \mathscr{T}, \tag{1.3}$$

where $\mathrm{diam}(\tau)$ is the diameter of τ and $|\tau|$ is the measure of τ in $\mathbb{R}^d$. This assumption can also be represented as

$$\sup_{h\in\aleph} \max_{\tau\in\mathcal{T}_h} \frac{h_\tau}{\rho_\tau} \le \sigma_1 \tag{1.4}$$

where ρ_τ denotes the radius of the ball inscribed in τ. In two dimensions, it is equivalent to the minimal angle of each triangulation is bounded below uniformly in the shape regular class. We shall define $h_\tau = |\tau|^{1/n}$ for any $\tau \in \mathcal{T} \in \mathscr{T}$. By (1.3), $h_\tau \eqsim \mathrm{diam}(\tau)$ represents the size of an element $\tau \in \mathcal{T}$ for a shape regular triangulation $\mathcal{T} \in \mathscr{T}$.

In addition to (1.3), if

$$\frac{\max_{\tau\in\mathcal{T}} |\tau|}{\min_{\tau\in\mathcal{T}} |\tau|} \le \rho, \quad \forall \mathcal{T} \in \mathscr{T}, \tag{1.5}$$

$\mathscr{T}$ is called *quasi-uniform*. For quasi-uniform grids, $h_{\mathcal{T}} := \max_{\tau\in\mathcal{T}} h_\tau$, the mesh size of $\mathcal{T}$, is used to measure the approximation rate. In the FEM literature, we often write as $\mathcal{T}_h$.

The assumption (1.4) is a local assumption, as is meant by above definition, for $d = 2$ for example, it assures that each triangle will not degenerate into a segment in the limiting case. A triangulation satisfying this assumption is often called to be *shape regular*.

On the other hand, the assumption (1.5) is a global assumption, which says that the smallest mesh size is not too small compared with the largest mesh size of the same triangulation. By the definition, in a quasiuniform triangulation, all the elements are about the same size asymptotically.

Remark 1. In this course, unless otherwise noted, we restrict ourself to quasi-uniform simplicial triangulation. There are other type of meshes by partition the domain into quadrilateral (in 2-D), cubes (in 3-D), or other type of elements.

1.1.3 Finite element space

Given a shape regular triangulation $\mathcal{T}_h$ of Ω, we set

$$V_h := \{v \,|\, v \in C(\overline{\Omega}), \text{ and } v|_\tau \in P_1(\tau), \forall \tau \in \mathcal{T}_h\},$$

where $P_1(\tau)$ denotes the space of polynomials of degree 1 (linear) on $\tau \in \mathcal{T}_h$. Whenever we need to deal with boundary conditions, we further define $V_{h,0} = V_h \cap H_0^1(\Omega)$.

We note here that the global continuity is also necessary in the definition of V_h in the sense that if u has a square interable gradient, that is $u \in H^1(\Omega)$, and u is piecewise smooth, then u is continuous.

We always use n_h to denote the dimension of finite element spaces. For V_h, n_h is the number of vertices of the triangulation $\mathcal{T}_h$ and for $V_{h,0}$, n_h is the number of interior vertices.

Nodal basis functions and dual basis

For linear finite element spaces, we have the so called *a standard nodal basis functions* $\{\varphi_i, i = 1, \cdots n_h\}$ such that φ_i is piecewise linear (with respect to the triangulation) and $\varphi_i(x_j) = \delta_{i,j}$. Note that $\varphi_i|_\tau$ is the corresponding barycentrical coordinates of x_i. See Figure 1.5 for an illustration in 2-D.

Let $(\psi_i)_{i=1}^{n_h}$ be the dual basis of $(\varphi_i)_{i=1}^{n_h}$. Namely

$$(\psi_i, \varphi_j) = \delta_{i,j}, \quad i, j = 1, \ldots, n_h. \tag{1.6}$$

We notice that all the nodal basis functions $\{\varphi_i\}$ are locally supported, but their dual basis functions $\{\psi_i\}$ are in general not locally supported. The nodal basis functions $\{\varphi_i\}$ are easily constructed in terms of barycentric coordinate functions. The dual basis $\{\psi_i\}$ are only interesting for theoretical consideration and it is not necessary to know the actual constructions of these functions.

Therefore for any $v_h \in V_h$, we have the representation

$$v_h(x) = \sum_{i=1}^{n_h} v_h(x_i)\varphi_i(x).$$

Let us see how our construction looks like in one spatial dimension. Associated with the partition $\mathcal{T}_h = \{0 = x_0 < x_1 < \ldots < x_{n_h} < x_{n_h+1} = 1\}$, we define a linear finite element space

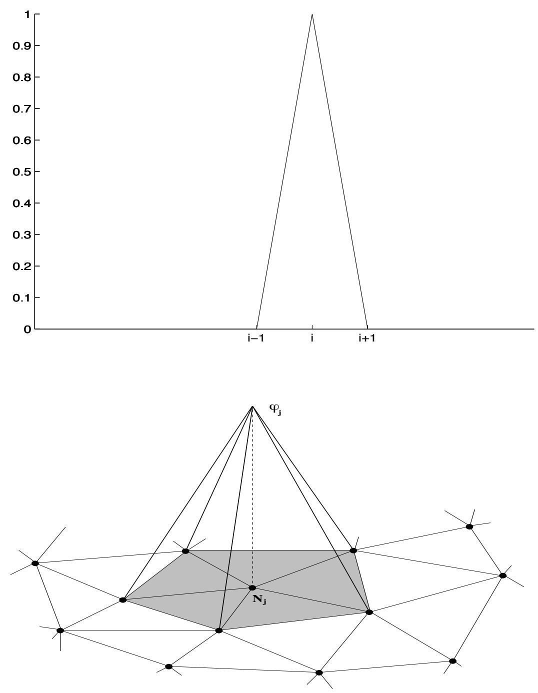

Fig. 1.5. Nodal basis functions in 1d and 2d

$$V_{h,0} = \{v: \ v \text{ is continuous and piecewise linear w. r. t. } \mathcal{T}_h, v(0) = v(1) = 0\}.$$

A plot of a typical element of $V_{h,0}$ is shown in Fig. 1.6.

It is easily calculated (as we already mentioned), that the dimension of V_h is equal to the number of internal vertices, and the nodal basis functions spanning $V_{h,0}$ (for $i = 1, 2, \cdots, n_h$) are (see also Fig. 1.5):

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h}, & x \in [x_{i-1}, x_i]; \\ \dfrac{x_{i+1} - x}{h}, & x \in [x_i, x_{i+1}]; \\ 0 & \text{elsewhere.} \end{cases}$$

1.2 Nodal value interpolant

Lemma 1. *For any $v \in V_h$,*

$$v(x) = \sum_{j=1}^{N_h} v(x_j)\phi_j(x).$$

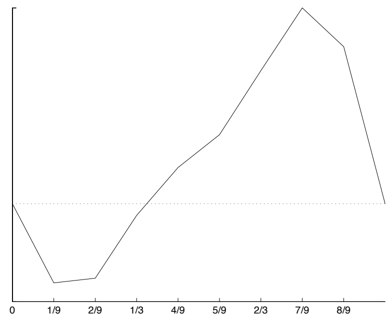

Fig. 1.6. Plot of a typical element from V_h.

Proof. Let $v_h(x) = \sum_{j=1}^{N_h} v(x_j)\phi_j(x)$. For an arbitrary simplex with $d+1$ ($\Omega \subseteq \mathbb{R}^d$) vertices $a_1, \cdots, a_{d+1}$:

$$v_h(x) = \sum_{j=1}^{d+1} v(a_j)\lambda_j(x)$$

Notice that

$$v_h(a_i) = \sum_{j=1}^{d+1} v(a_j)\lambda_j(a_i) = v(a_j),\ i = 1, \cdots, d+1$$

So the values of v_h and v are equal at $d+1$ points. Notice that both v_h and v are linear functions, so $v = v_h$. □

For any continuous function u, we define its finite element interpolation, $u_I \in V_{h,0}$, as follows:

$$u_I(x) = \sum_{i=1}^{N} u(x_i)\phi_i(x). \tag{1.7}$$

For any $v \in \mathcal{S}_0^h$, we can obviously write

$$v(x) = \sum_{i=1}^{N} v(x_i)\phi_i(x).$$

The nodal value interpolation operator $I_h : C(\bar{\Omega}) \mapsto V_h$ is defined as follows

$$(I_h u)(x_i) = u(x_i), \quad \forall\ x_i \in \mathcal{N}_h.$$

1.3 Error estimates

Theorem 1. *Assume that* $\{\mathcal{T}_h : h \in \aleph\}$ *is quasiuniform, then*

$$\inf_{v_h \in V_h} \|v - v_h\| + h|v - v_h|_1 \lesssim h^2 |v|_2 \quad \forall\ v \in H^2(\Omega) \tag{1.8}$$

Next we provide proofs of the above theorem for some special cases.

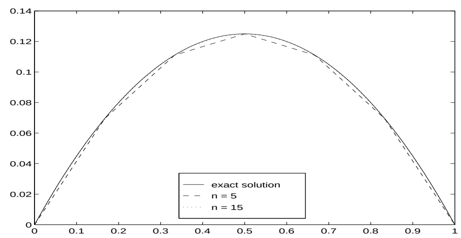

Fig. 1.7. Approximation of finite element space.

A proof of Theorem 1 for $d = 1$

Observe first that $e = (u - u_I)$ vanishes at the end points of each interval and e' is continuous, because e'' is square integrable. By the Rolle's theorem there exists $\xi_i \in (x_i, x_{i+1})$ such that $e'(\xi_i) = 0$. By the Fundamental Theorem of Calculus for $x \in (x_i, x_{i+1})$, we have that

$$e'(x) = \int_{\xi_i}^{x} e''(t)\, dt$$

Since u_I is linear on $[x_i, x_{i+1}]$ we have that $e''(t) = u''(t)$, and hence

$$[e'(x)]^2 = \left[\int_{\xi_i}^{x} u''(t)\, dt\right]^2.$$

Applying the Schwarz inequality to the right side then gives,

$$\begin{aligned}[e'(x)]^2 &\le \left|\int_{\xi_i}^{x} 1^2\, dt\right| \left|\int_{\xi_i}^{x} [u''(t)]^2\, dt\right| \\ &\le |\xi_i - x| \int_{x_i}^{x_{i+1}} [u''(t)]^2\, dt.\end{aligned}$$

Integrating from x_i to x_{i+1}, and observing that

$$\int_{x_i}^{x_{i+1}} |\xi_i - x|\, dx = \frac{1}{2}[(\xi_i - x_i)^2 + (x_{i+1} - \xi_i)^2] \le (x_i - x_{i+1})^2,$$

then gives

$$\int_{x_i}^{x_{i+1}} [e'(x)]^2 \, dx \le (x_{i+1} - x_i)^2 \int_{x_i}^{x_{i+1}} [u''(t)]^2 \, dt.$$

Finally, summing up on $(0, 1)$ then leads to:

$$\begin{aligned} \|e'\|_{0,\Omega} = \int_0^1 [e'(x)]^2 \, dx &= \sum_{i=1}^{n_h-1} \int_{x_i}^{x_{i+1}} [e'(x)]^2 \, dx \\ &\le \sum_{i=1}^{n_h-1} (x_{i+1} - x_i)^2 \int_{x_i}^{x_{i+1}} [u''(t)]^2 \, dt \\ &\le \max_i (x_{i+1} - x_i)^2 \int_0^1 [u''(t)]^2 \, dt. \end{aligned}$$

Since $e(x_i) = 0$, for any $x \in (x_i, x_{i+1})$,

$$|e(x)| = |\int_{x_i}^{x} e'(t) \, dt| \lesssim h|e|_{1,\tau} \lesssim h^2 |u|_{2,\tau}.$$

Summing up on $(0, 1)$ then leads to:

$$\|e\|_{0,\Omega} \lesssim h^2 |u|_{2,\Omega}.$$

This completes the proof of the estimate in one dimension.

A proof of Theorem 1 for $d = 1, 2, 3$

Let $x = (x^1, \dots, x^d)$ and $a_i = (a_i^1, \dots, a_i^d)$. Introducing the auxiliary functions

$$g_i(t) = v(a_i(t)), \text{ with } a_i(t) = a_i + t(x - a_i),$$

we have

$$g_i'(t) = (\nabla v)(a_i(t)) \cdot (x - a_i) = \sum_{l=1}^{d} (\partial_l v)(a_i(t))(x^l - a_i^l)$$

and

$$g_i''(t) = \sum_{k,l=1}^{d} \partial_{kl}^2 v)(a_i(t))(x^k - a_i^k)(x^l - a_i^l). \tag{1.9}$$

By Taylor expansion

$$g_i(0) = g_i(1) - g_i'(1) + \int_0^1 t g_i''(t) dt.$$

Namely

$$v(a_i) = v(x) - (\nabla v)(x) \cdot (x - a_i) + \int_0^1 t g_i''(t) dt. \tag{1.10}$$

Note that

$$(I_h v)(x) = \sum_{i=1}^{n+1} v(a_i)\lambda_i(x), \sum_{i=1}^{n+1} \lambda_i(x) = 1,$$

and

$$\sum_{i=1}^{n+1}(x - a_i)\lambda_i(x) = 0.$$

It follows that

$$(I_h v - v)(x) = \sum_{i=1}^{n+1} \lambda_i(x) \int_0^1 t g_i''(t)dt \tag{1.11}$$

Using (1.9) and the trivial fact that $|x^l - a_i^l| \le h$, we obtain

$$\|g_i''(t)\|_{L^2(\tau)} \le h^2 \sum_{k,l=1}^{d} \|(\partial_{kl}^2 v)(a_i(t))\|_{L^2(\tau_i^t)} \le h^2 t^{-n/2} \sum_{k,l=1}^{d} \|\partial_{kl}^2 v\|_{L^2(\tau)}$$

where we have used the following change of variable

$$y = a_i + t(x - a_i) : \tau \mapsto \tau_i^t \subset \tau \text{ with } dy = t^d dx.$$

Now taking the $L^2(\tau)$ norm on both hand of sides of (1.11), we get

$$\begin{aligned}
\|I_h v - v\|_{L^2(\tau)} &\le h^2 \sum_{i=1}^{n+1} \max_{x\in\tau} |\lambda_i(x)| \int_0^1 t\|g_i''(t)\|_{L^2(\tau)}\, dt \\
&\le (n+1) \int_0^1 t^{1/n/2} dt\, h^2 \sum_{k,l=1}^{d} \|\partial_{kl}^2 v\|_{L^2(\tau)} \\
&\le \frac{2(n+1)}{4-n} h^2 \sum_{k,l=1}^{d} \|\partial_{kl}^2 v\|_{L^2(\tau)} \\
&\le \frac{4n(n+1)}{4-n} h^2 |v|_{H^2(\tau)}
\end{aligned}$$

Now we prove the H^1 error estimate. Notice that

$$[\partial_j(I_h v - v)](x) = \sum_i (\partial_j \lambda_i)(x) \int_0^1 t g_i''(t)dt + \sum_i \lambda_i(x)\partial_j \int_0^1 t g_i''(t)dt$$

By (1.10),

$$\int_0^1 t g_i''(t)dt = v(a_i) - v(x) + (\nabla v)(x) \cdot (x - a_i)$$

therefore,

$$\begin{aligned}
&\partial_j \int_0^1 t g_i''(t)dt \\
&= -\partial_j v + (\nabla \partial_j v)(x)(x - a_i) + \nabla v \cdot e_j \quad (e_j \text{ is the } j\text{-th standard basis}) \\
&= (\nabla \partial_j v)(x)(x - a_i)
\end{aligned}$$

Notice that $\sum_i \lambda_i (\nabla \partial_j v)(x)(x - a_i) = 0$:

$$[\partial_j (I_h v - v)](x) = \sum_i (\partial_j \lambda_i)(x) \int_0^1 t g_i''(t)dt$$

Then the estimate for $|\nabla(I_h v - v)|_{L^2(\tau)}$ follows by a similar argument and the following obvious estimate

$$|(\nabla \lambda_i)(x)| \lesssim \frac{1}{h}.$$

On the proof of Theorem 3 for $d \geq 4$

The above proof using interpolation for Theorem 3 does not apply for $d \geq 4$. This is because when $d \geq 4$, the embedding relation between $H^2(\Omega)$ and $C(\bar{\Omega})$ is not true. Only continuous functions can have interpolations. In this case, one approach is to use the so-called Scott-Zhang interpolation **?**, the details can be found in **?**.

Theorem 2. *Let V_N be linear finite element space on a quasi-uniform simplicial triangulation consisting of N element. Then*

$$\inf_{v_h \in V_N} \|v - v_N\| + h|v - v_N|_1 \lesssim N^{-\frac{2}{d}} |v|_2 \quad \forall\, v \in H^2(\Omega) \tag{1.12}$$

We can refine and extend the above error estimate in many different ways.

Theorem 3. *Given any function v with certain regularity assumption (say $v \in H^2(\Omega)$), then there is a shape regular grid $\mathcal{T}_N$ consisting of N simplicial elements*

$$\inf_{v_h \in V_N} \|v - v_N\| + h|v - v_N|_1 \leq C(v) N^{-\frac{2}{d}}. \tag{1.13}$$

where V_N is linear finite element space on associated with $\mathcal{T}_N$.

Theorem 4. ? *For any function v that is not locally linear, we have*

$$\inf_{\dim V_N = N} \inf_{v_h \in V_N} \|v - v_N\| + h|v - v_N|_1 \geq c(v) N^{-\frac{2}{d}}. \tag{1.14}$$

where V_N be linear finite element space on associated with a shape regular mesh $\mathcal{T}_N$.