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Convolutional Neural Networks

1.1 Variational problems

Lemma 1. *Assume that u is continuous in $(0,1)$, then the following statements are equivalent*

(1) $u(x) = 0$.

(2) $\int_0^1 u(x)v(x)dx = 0$ *for any smooth (compactly supported) function v in* $(0,1)$.

Define function $v : [0,1] \to R$ and define space

$$V = \{v : \ v \text{ is continuous and } v(0) = v(1) = 0\}.$$

Given any $f : [0,1] \to R$, consider

$$J(v) = \frac{1}{2}\int_0^1 |v'|^2 dx - \int_0^1 f v dx.$$

Find $u \in V$ such that

$$u = \arg\min_{v \in V} J(v) \tag{1.1}$$

which is equivalent to: Find $u \in V$ such that

$$\begin{cases} -u'' = f, \ \ 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{1.2}$$

Proof. For any $v \in V, t \in R$, let $g(t) = J(u + tv)$. Since $u = \arg\min_{v \in V} J(v)$ means $g(t) \geq g(0)$. Hence, for any $v \in V$, 0 is the global minimum of the function $g(t)$. Therefore $g'(0) = 0$ implies

$$\int_0^1 u'v'dx = \int_0^1 f v dx \quad \forall v \in V.$$

By integration by parts, which is equivalent to

$$\int_0^1 (-u'' - f)v dx = 0 \quad \forall v \in V.$$

By variational principal Lemma 1, we obtain

$$\begin{cases} -u'' = f, \; 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases} \tag{1.3}$$

□

Let V_h be finite element space and $\{\varphi_1, \varphi_2, \cdots \varphi_n\}$ be a nodal basis of the V_h. Let $\{\psi_1, \psi_2, \cdots, \psi_n\}$ be a dual basis of $\{\varphi_1, \varphi_2, \cdots \varphi_n\}$, namely $(\varphi_i, \psi_j) = \delta_{ij}$.

$$J(v_h) = \frac{1}{2}\int_0^1 |v_h'|^2 dx - \int_0^1 f v_h dx. \tag{1.4}$$

Let

$$u_h = \sum_{i=1}^n v_i \varphi,$$

then

$$u_h = \arg\min_{v_h \in V_h} J(v_h) \tag{1.5}$$

is equivalent to: Find $u_h \in V_h$

$$a(u_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V_h. \tag{1.6}$$

where

$$a(u_h, v_h) = \int_0^1 u_h' v_h' dx.$$

Which is equivalent to: Find $u_h \in V_h$

$$a(u_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V_h, \tag{1.7}$$

which is equivalent to solving $\underset{\sim}{A}\mu = b$, where $\underset{\sim}{A} = (a_{ij})_{ij}^n$ and $a_{ij} = a(\varphi_j, \varphi_i)$ and $b_i = \int_0^1 f\varphi_i dx$. Namely

$$\frac{1}{h}\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \\ \mu_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \\ b_n \end{pmatrix}. \tag{1.8}$$

Which can be rewritten as

$$\frac{-\mu_{i-1} + 2\mu_i - \mu_{i+1}}{h} = b_i, \quad 1 \le i \le n, \quad \mu_0 = \mu_{n+1} = 0. \tag{1.9}$$

Using the convolution notation, (1.9) can be written as

$$A * \mu = b, \tag{1.10}$$

where $A = \frac{1}{h}[-1, 2, -1]$.

1.2 Introduction

Let us first briefly describe finite difference methods and finite element methods for the numerical solution of the following boundary value problem

$$-\Delta u = f, \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad \Omega = (0,1)^2. \tag{1.11}$$

For the x direction and the y direction, we consider the partition:

$$0 = x_0 < x_1 < \cdots < x_{n+1} = 1, \quad x_i = \frac{j}{n+1}, \quad (i = 0, \cdots, n+1); \tag{1.12}$$

$$0 = y_0 < y_1 < \cdots < y_{n+1} = 1, \quad y_j = \frac{j}{n+1}, \quad (j = 0, \cdots, n+1). \tag{1.13}$$

Such a uniform partition in the x and y directions leads us to a special example in two dimensions, a uniform square mesh $\mathbb{R}_h^2 = \{(ih, jh); i, j \in \mathbb{Z}\}$ (Figure 1.2). Let $\Omega_h = \Omega \cap \mathbb{R}_h^2$, the set of interior mesh points and $\partial\Omega_h = \partial\Omega \cap \mathbb{R}_h^2$, the set of boundary mesh points.

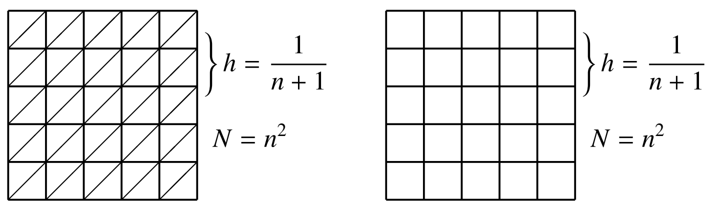

Fig. 1.1. Two-dimensional uniform grid for finite element and finite difference

1.3 Finite element methods

We consider two finite elements: continuous linear element and bilinear element. These two finite element methods find $u_h \in V_h$ such that

$$(\nabla u_h, \nabla v_h) = (f, v_h), \ \forall v_h \in V_h.$$

The above formulation can be written as

$$\underset{\sim}{A}\underset{\sim}{u} = \underset{\sim}{f},$$

with $\underset{\sim}{A}_{(j-1)n+i,(l-1)n+k} = (\nabla\phi_{kl}, \nabla\phi_{ij})$, $\underset{\sim}{f}_{(j-1)n+i,(l-1)n+k} = (f, \phi_{ij})$.

Basis functions ϕ_{ij} satisfy

$$\phi_{ij}(x_k, y_l) = \delta_{(i,j),(k,l)}. \tag{1.14}$$

1.3.1 Linear finite element

Continuous linear finite element discretization of (1.11) on the left triangulation in Fig 1.2. The discrete space for linear finite element is

$$\mathcal{V}_h = \{v_h : v_h|_K \in P_1(K) \text{ and } v_h \text{ is globally continuous}\}.$$

Denote $E_{i,j} = [x_i, x_{i+1}] \times [y_i, y_{i+1}] = K^U_{i,j} \cup K^D_{i,j}$. For linear element case,

(1.15)

$$\begin{aligned}
(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) &= \sum_{i,j=1}^{n} \int_{E_{i,j}} \nabla \mathbf{u}_h \cdot \nabla \mathbf{v}_h dxdy = \sum_{i,j=1}^{n} \int_{K^U_{i,j}} \nabla \mathbf{u}_h \cdot \nabla \mathbf{v}_h dxdy + \sum_{i,j=1}^{n} \int_{K^D_{i,j}} \nabla \mathbf{u}_h \cdot \nabla \mathbf{v}_h dxdy \\
&= \sum_{i,j=1}^{n} \int_{K^U_{i,j}} \Big(\frac{u_{i,j+1} - u_{i+1,j+1}}{h} \frac{v_{i,j+1} - v_{i+1,j+1}}{h} + \frac{u_{i,j+1} - u_{i,j}}{h} \frac{v_{i,j+1} - v_{i,j}}{h}\Big) dxdy \\
&+ \sum_{i,j=1}^{n} \int_{K^D_{i,j}} \Big(\frac{u_{i+1,j} - u_{i,j}}{h} \frac{v_{i+1,j} - v_{i,j}}{h} + \frac{u_{i+1,j} - u_{i+1,j+1}}{h} \frac{v_{i+1,j} - v_{i+1,j+1}}{h}\Big) dxdy \\
&= \sum_{i,j=1}^{n} \int_{K^U_{i,j}} \Big(\frac{u_{i,j+1} - u_{i+1,j+1}}{h} \frac{v_{i,j+1} - v_{i+1,j+1}}{h} + \frac{u_{i,j+1} - u_{i,j}}{h} \frac{v_{i,j+1} - v_{i,j}}{h}\Big) dxdy \\
&+ \sum_{i,j=1}^{n} \int_{K^D_{i,j+1}} \Big(\frac{u_{i+1,j+1} - u_{i,j+1}}{h} \frac{v_{i+1,j+1} - v_{i,j+1}}{h} + \frac{u_{i+1,j+1} - u_{i+1,j+2}}{h} \frac{v_{i+1,j+1} - v_{i+1,j+2}}{h}\Big) dxdy \\
&= \sum_{i,j=1}^{n} \frac{h^2}{2} \Big(\frac{u_{i,j+1} - u_{i+1,j+1}}{h} \frac{v_{i,j+1} - v_{i+1,j+1}}{h} + \frac{u_{i,j+1} - u_{i,j}}{h} \frac{v_{i,j+1} - v_{i,j}}{h}\Big) \\
&+ \sum_{i,j=1}^{n} \frac{h^2}{2} \Big(\frac{u_{i+1,j+1} - u_{i,j+1}}{h} \frac{v_{i+1,j+1} - v_{i,j+1}}{h} + \frac{u_{i+1,j+1} - u_{i+1,j+2}}{h} \frac{v_{i+1,j+1} - v_{i+1,j+2}}{h}\Big) \\
&= \sum_{i,j=1}^{n} h^2 \Big(\frac{u_{i+1,j} - u_{i,j}}{h} \frac{v_{i+1,j} - v_{i,j}}{h} + \frac{u_{i,j+1} - u_{i,j}}{h} \frac{v_{i,j+1} - v_{i,j}}{h}\Big) \\
&= \sum_{i,j=1}^{n} \Big[(u_{i+1,j} - u_{i,j})(v_{i+1,j} - v_{i,j}) + (u_{i,j+1} - u_{i,j})(v_{i,j+1} - v_{i,j})\Big] \\
&= (A * u, v)_{l^2}.
\end{aligned}$$

where $A = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix}$ and $A * u$ is given by (1.16).

It is easy to verify that the formulation for the linear element method is

$$4u_{i,j} - (u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}) = f_{i,j}, \quad u_{i,j} = 0 \text{ if } i \text{ or } j \in \{0, n+1\}, \tag{1.16}$$

where

$$f_{i,j} = \int_{\Omega} f(x,y)\phi_{i,j}(x,y) \mathrm{d}x\mathrm{d}y \approx h^2 f(x_i, y_j). \tag{1.17}$$

Proposition 1. *The mapping $A*$ has following properties*

1. *A is symmetric, namely*

$$(A * u, v)_{l^2} = (u, A * v)_{l^2}.$$

2. $(A * v, v)_F > 0, \quad if \quad v \neq 0.$
3. $A * u = f$ *if and only if*

$$u \in \arg\min_{v\in\mathcal{V}_h} J(v) = \frac{1}{2}(A * v, v) - (f, v). \tag{1.18}$$

4. *The eigenvalues λ_{kl} and eigenvectors u^{kl} of A are given by*

$$\lambda_{kl} = 4(\sin^2 \frac{k\pi}{2(n+1)} + \sin^2 \frac{l\pi}{2(n+1)}),$$

$$u^{kl}_{ij} = \sin \frac{ki\pi}{n+1} \sin \frac{lj\pi}{n+1}, \ 1 \leq i \leq n, \ 1 \leq j \leq n,$$

and $\rho(A) < 8$. Furthermore,

$$\lambda_{n,n} = 8\cos^2 \frac{\pi}{2(n+1)} \approx 8(1 - (\frac{\pi}{2(n+1)})^2) \approx 8 - \frac{2\pi^2}{(n+1)^2}$$

1.3.2 Bilinear element

Continuous bilinear finite element discretization of (1.11) on the right mesh in Fig. 1.2. The discrete space for linear finite element is

$$\mathcal{V}_h = \{v_h : v_h|_K \in \{1, \ x, \ y, \ xy\} \text{ and } v_h \text{ is globally continuous}\}.$$

For bilinear element case, we have

(1.19)

$$\begin{aligned}
(\nabla \mathbf{u}_h, \nabla \mathbf{v}_h) = & \sum_{i,j=1}^{n} \int_{E_{i,j}} \nabla \mathbf{u}_h, \nabla \mathbf{v}_h dxdy \\
= & \sum_{i,j=1}^{n} \int_{E_{i,j}} \left(\frac{(u_{i+1,j} - u_{i,j})(y_{j+1} - y)}{h^2} + \frac{(u_{i,j+1} - u_{i+1,j+1})(y - y_j)}{h^2} \right) \\
& \left(\frac{(v_{i+1,j} - v_{i,j})(y_{j+1} - y)}{h^2} + \frac{(v_{i,j+1} - v_{i+1,j+1})(y - y_j)}{h^2} \right) \\
& + \left(\frac{(u_{i,j+1} - u_{i,j})(x_{i+1} - x)}{h^2} + \frac{(u_{i+1,j} - u_{i+1,j+1})(x - x_i)}{h^2} \right) \\
& \left(\frac{(v_{i,j+1} - v_{i,j})(x_{i+1} - x)}{h^2} + \frac{(v_{i+1,j} - v_{i+1,j+1})(x - x_i)}{h^2} \right) dxdy \\
= & (A * u, v)_{l^2}.
\end{aligned}$$

where $A = \begin{pmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{pmatrix}$ and $A * u$ is given by (1.20).

And we have

$$8u_{ij} - (u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} + u_{i+1,j+1} + u_{i-1,j-1} + u_{i-1,j+1} + u_{i+1,j-1}) = f_{i,j}, \tag{1.20}$$

and $u_{i,j} = 0$ if i or $j \in \{0, n+1\}$.

References