

## The Completeness Theorem

### Key Concepts

- The **Completeness Theorem** states that

$$T \vdash_{\mathcal{L}} \varphi \iff T \models \varphi$$

- The  $\Rightarrow$ -direction is **Soundness**, which we already proved.
- The  $\Leftarrow$ -direction is usually proved in the following form:

If  $T$  is consistent, then  $T$  has a model.

- To construct a model of a consistent theory, we consider the **constant terms**

$$K := \{t : t \text{ } \mathcal{L}\text{-term without variables}\}$$

and **identify provably equal terms**:

$$s \sim t : \iff T \vdash_{\mathcal{L}} s = t$$

- The **canonical term structure**  $\mathcal{A}$  of  $T$  has universe  $A = K / \sim$ .
- Moreover, we put

$$\begin{aligned} - c^{\mathcal{A}} &:= [c] \\ - f^{\mathcal{A}}([t_1], \dots, [t_n]) &:= [ft_1 \dots t_n] \\ - R^{\mathcal{A}}([t_1], \dots, [t_n]) &: \iff T \vdash_{\mathcal{L}} Rt_1 \dots t_n \end{aligned}$$

- It holds that for any *atomic* sentence  $\sigma$ ,

$$\mathcal{A} \models \sigma \iff T \vdash \sigma$$

### Problems

#### Discuss

We all believe (I think) that  $(\mathbb{Z}, +, 0)$  is a model of the group axioms. To what extent does this prove that the group axioms are consistent?

#### Exercise 0.1.

Verify that  $\sim$  is an equivalence relation.

#### Exercise 0.2.

Verify that the definition of  $c^{\mathcal{A}}, f^{\mathcal{A}}, R^{\mathcal{A}}$  does not depend on the choice of representative for  $[c]$  and  $[t_i]$ .

#### Exercise 0.3.

Verify the claim that for atomic sentences,

$$\mathcal{A}_T \models \sigma \iff T \vdash \sigma$$