

Definability of Computable Functions

We now extend the connection between arithmetical formulas and computable functions significantly. Our goal is to show that the recursive relations on \mathbb{N} are precisely those that are Δ_1 -definable. At the core of this connection is the **Definability Theorem**. For a partial function f , let Γ_f denote the **graph** of f , i.e., the relation

$$\Gamma_f(\vec{x}, y) : \Longleftrightarrow f(\vec{x}) = y.$$

The Definability Theorem

Theorem 0.1.

A partial function $f : \subseteq \mathbb{N} \rightarrow \mathbb{N}$ is computable if and only if its graph Γ_f over the natural numbers is definable by a Σ_1 -formula, i.e., if there exists a Σ_1 -formula θ such that for all $x, y \in \mathbb{N}$:

$$\Gamma_f(x, y) \Longleftrightarrow \mathbb{N} \models \theta[x, y].$$

The direction (\Leftarrow) is relatively straightforward: Let $\theta = \exists z \psi$ be a Σ_1 -formula that defines the graph Γ_f of a partial function f , where $\psi(x, y, z)$ is a Δ_0 -formula. As Δ_0 -formulas define primitive recursive relations, we obtain a computable function g by letting

$$g(x) = \mu z \psi(x, (z)_0, (z)_1).$$

Then $(g(x))_0$ is the smallest y with $\mathbb{N} \models \theta[x, y]$, if such a y exists, and undefined otherwise, since for each $x \in \mathbb{N}$ there is at most one y with $\mathbb{N} \models \theta[x, y]$. Thus $(g(x))_0 \cong f(x)$, and therefore f is partial computable.

For the proof of (\Rightarrow) , we call partial functions f whose graph can be defined by a Σ_1 -formula over the natural numbers *functions with a Σ_1 -graph*. The obvious strategy is to show that the set of these functions contains the initial functions and is closed under composition, primitive recursion, and the μ -operator.

Most cases are straightforward (just by writing out the definitions), but when we get to closure under primitive recursion, we run into difficulties: Suppose the function f arises from functions g, h by primitive recursion:

$$\begin{aligned} f(x, 0) &\cong g(x) \\ f(x, y + 1) &\cong h(x, y, f(x, y)) \end{aligned}$$

We may assume that the graphs of g and h are definable by Σ_1 -formulas. To describe the graph of f with this, we note that the value of $f(x, y)$ is computed through the preceding values

$$u_0 = f(x, 0), u_1 = f(x, 1), \dots, u_y = f(x, y),$$

We somehow have to *eliminate the function f* from this sequence to obtain an *explicit* definition (in the language of arithmetic).

The idea is to express the following through a formula:

There exists a code u such that u codes a sequence of numbers (u_0, \dots, u_y) in which the u_{i+1} is obtained from u_i using a valid application of the primitive recursion scheme with respect to g and h .

The problem is that we do not know a priori how long the sequence coded by u is. We therefore cannot use the preferred coding scheme

$$\langle u_0, \dots, u_y \rangle$$

which is Δ_0 -definable. The coding scheme

$$(n_0, n_1, \dots, n_k) \quad \text{by} \quad p_0^{n_0+1} \cdot p_1^{n_1+1} \cdot \dots \cdot p_k^{n_k+1}.$$

on the other hand is uniform independent of the length of the sequence, but it uses exponentiation, which is not present in PA^- (and would have to be defined via recursion).

This is where Gödel, according to Mostowski, “had a phone call with God”.

Gödel's Lemma

Lemma 0.1.

There exists a primitive recursive function β such that for every k and for every finite sequence $x_0, x_1, \dots, x_{k-1} \in \mathbb{N}$, there exists a natural number c with

$$\text{for all } i < k : \beta(c, i) = x_i.$$

In fact, there exists a Δ_0 -formula $\theta(x, y, z)$ such that

$$\mathbb{N} \models \forall x, y \exists! z \theta(x, y, z),$$

and the formula $\theta(x, y, z)$ defines the function β over the natural numbers.

Proof. Let x_0, x_1, \dots, x_{k-1} be a sequence of natural numbers.

Set $m := b!$ where $b := \max(k, x_0, x_1, \dots, x_{k-1})$. Then the sequence of numbers

$$m+1, 2m+1, \dots, k \cdot m+1$$

are pairwise coprime. By the theorem on simultaneous congruences (*Chinese Remainder Theorem*), there exists a natural number a with

$$a \equiv x_i \pmod{(i+1)m+1} \quad \text{for all } i < k.$$

Now we can choose $\langle a, m \rangle$ as a code for the sequence x_0, x_1, \dots, x_{k-1} , because from this we can recover each x_i for every $i < k$ as the remainder of the division of a by the number $(i+1)m+1$. If $\text{rem}(x : y) = z$ denotes the remainder of the division of x by y (when $y \neq 0$ and $\text{rem}(x : 0) = 0$), then this is a p.r. function with a Δ_0 -definition. We obtain another p.r. function by

$$\alpha(a, m, i) = \text{rem}(a : (1+i)m+1).$$

Let p_1, p_2 denote the inverse functions of the pairing function $\langle x, y \rangle$ above, for which

$$\langle p_1(x), p_2(x) \rangle = x, \quad \text{where } p_1(x), p_2(x) \leq x,$$

these are also p.r., and we finally obtain the **Gödel beta function** as

$$\beta(c, i) = \alpha(p_1(c), p_2(c), i).$$

□

Completing the proof of the Definability Theorem

Returning to the sequence

$$u_0 = f(x, 0), u_1 = f(x, 1), \dots, u_y = f(x, y),$$

we can now use the β -function to encode this by a single number u . The first value u_0 is determined by the graph of g , and the further values u_{i+1} are determined from u_i according to the recursion conditions via the graph of h :

$$\forall i < y \exists r, s [\beta(u, i) = r \wedge \beta(u, i + 1) = s \wedge \Gamma_h(\vec{x}, i, r, s)]$$

This formula can be transformed (by choosing a sufficiently large w) into

$$\exists w \forall i < y \exists r, s < w [\beta(u, i) = r \wedge \beta(u, i + 1) = s \wedge \Gamma_h(\vec{x}, i, r, s)].$$

The last value of the sequence encoded by u is the function value of f at the point (x, y) . Thus we obtain the following description of the graph of f :

$$\begin{aligned} \Gamma_f(\vec{x}, y, z) \iff \exists u, v, w (\Gamma_g(\vec{x}, v) \wedge \beta(u, 0) = v \wedge \beta(u, y) = z \wedge \\ \forall i < y \exists r, s < w [\beta(u, i) = r \wedge \beta(u, i + 1) = s \wedge \Gamma_h(\vec{x}, i, r, s)]), \end{aligned}$$

It is not hard to transform this into an equivalent Σ_1 -formula.

Characterization of the Arithmetic Hierarchy

If we call a set $A \subseteq \mathbb{N}^k$ that is definable by a formula in Γ over the natural numbers a Γ -set, we obtain a new characterization of the first levels of the arithmetic hierarchy:

Corollary 0.1.

1. The Σ_1 -sets are precisely the r.e. (recursively enumerable) relations $R \subseteq \mathbb{N}^k$.
2. The Π_1 -sets are the complements of r.e. relations.
3. The Δ_1 -sets are precisely the recursive relations $R \subseteq \mathbb{N}^k$.

We leave the short proof of (1) as an exercise. (2) follows from (1) by using negation, and for (3) use the fact that a set is computable if and only if the set and its complement are r.e.