Math 557 Sep 5

Substitution

Key Concepts

- **Substitution**:
  - Basic idea: $\varphi_{\bar{s}/\bar{x}}$ is obtained by replacing all occurrences of the variable x_i by the term s_i.
  - Uncontrolled substitution may cause issues with quantifiers. If we try to substitute a variable into the range of a quantifier, we rename the quantified variable to an unused variable ($\exists x \ldots$ becomes $\exists u \ldots$).
- **Substitution Lemma**:
  - Substitution behaves "as expected" with respect to evaluation and satisfaction.
  - Evaluating a substituted term yields the same value as evaluating the original term under the "substituted" assignment (i.e. the assignment in which we replace the assignment to x by the value of s under α).
  - A substituted formula holds in $\mathcal{M}$ under assignment α iff the original formula holds in $\mathcal{M}$ under the "substituted" assignment.

Problems

Exercise 0.1 (Carry-over from Sep 3)**.**
Show that if x is not free in φ, $\mathcal{M} \vDash \varphi[\alpha]$ implies $\mathcal{M} \vDash \forall x\, \varphi[\alpha]$.

Then verify that

$$\forall x(\varphi \to \psi) \ \to \ (\varphi \to \forall x \psi) \quad (x \text{ not free in } \varphi)$$

is a validity.

Exercise 0.2.

- Show that if t is a term, then $t_{\bar{s}/\bar{x}}$ is a term.
- Show that if φ is a formula, $\varphi_{\bar{s}/\bar{x}}$ is a formula of the same height.

Exercise 0.3.
Use the Substitution Lemma to verify that

$$\varphi_{t/x} \ \to \ \exists x\, \varphi$$

is a validity.

Exercise 0.4.
Show that if y does not occur in ψ,

$$[\psi_{y/x}]_{x/y} \equiv \psi$$

Find a counterexample that shows this no longer holds if y *does* occur in ψ.