Math 557 Nov 19

The First Incompleteness Theorem

Theorem 0.1 (Gödel-Rosser Theorem)**.**
Let T be a recursive set of (Gödel numbers of) $\mathcal{L}$-sentences such that:

1. *T is consistent, i.e., there is no L-sentence σ with $\ulcorner\sigma\urcorner \in T$ and at the same time $\ulcorner\neg\sigma\urcorner \in T$,*
2. *T contains all Σ_1 and Π_1-sentences that hold in PA^-:*

$$\{\ulcorner\sigma\urcorner : \sigma \in \Sigma_1 \cup \Pi_1, \sigma \textit{ sentence }, \mathsf{PA}^- \vdash \sigma\} \subseteq T.$$

Then T is Π_1-incomplete, i.e., there exists a Π_1-sentence τ with

$$\ulcorner\tau\urcorner \notin T \textit{ and } \ulcorner\neg\tau\urcorner \notin T.$$

Proof. Since T is recursive, there exists a Σ_1-formula $\theta(v)$ such that for all $n \in \mathbb{N}$

$$n \in T \Longrightarrow \mathsf{PA}^- \vdash \theta(\underline{n}),$$
$$n \notin T \Longrightarrow \mathsf{PA}^- \vdash \neg\theta(\underline{n}).$$

By the Diagonal Lemma, there exists a Π_1-sentence τ with

$$\mathsf{PA}^- \vdash \tau \leftrightarrow \neg\theta(\underline{\ulcorner\tau\urcorner}).$$

From assumption (2) it follows that

$$\ulcorner\tau\urcorner \in T \Rightarrow \mathsf{PA}^- \vdash \theta(\underline{\ulcorner\tau\urcorner}) \Rightarrow \mathsf{PA}^- \vdash \neg\tau \Rightarrow \ulcorner\neg\tau\urcorner \in T$$

and likewise, using (1),

$$\ulcorner\neg\tau\urcorner \in T \Rightarrow \ulcorner\tau\urcorner \notin T \Rightarrow \mathsf{PA}^- \vdash \neg\theta(\underline{\ulcorner\tau\urcorner}) \Rightarrow \mathsf{PA}^- \vdash \tau \Rightarrow \ulcorner\tau\urcorner \in T,$$

so that neither $\ulcorner\tau\urcorner$ nor $\ulcorner\neg\tau\urcorner$ can be in T. $\square$

Theorem 0.2 (First Gödel Incompleteness Theorem)**.**
Let T be a consistent[1] and recursively axiomatizable theory in language L which extends the theory PA^-. Then T is incomplete. In particular, there exists a Π_1-sentence τ with

$$T \nvdash \tau \textit{ and } T \nvdash \neg\tau.$$

[1] The original version of Gödel's theorem had the stronger assumption of ω-consistency of T (i.e., for all L-formulas $\theta(v)$ with $T \vdash \theta(\underline{n})$ for all $n \in \mathbb{N}$, the theory $T \cup \{\forall v\, \theta(v)\}$ is consistent). Rosser proved this stronger version in 1936.

Proof. We assume that T is recursively axiomatizable as well as complete (which implies that it is consistent, by definition), and derive a contradiction. As we proved in a previous lecture, the assumption implies that the set

$$S := \{\ulcorner\sigma\urcorner : \sigma \text{ sentence }, T \vdash \sigma\} = \ulcorner(T^{\vdash})\urcorner$$

is recursive. Thus the hypothesis of the Gödel-Rosser Theorem is met, which implies S is Π_1-incomplete—a contradiction! □

Every consistent extension T of PA^- has, by Lindenbaum's Lemma, a complete and consistent extension, none of which, by the above theorem, can be recursive.

The Paris-Harrington Theorem

The use of the Diagonal Lemma yields rather "artificial" witnesses of incompleteness. Are there "natural" mathematical theorems that PA cannot decide? In 1982, Paris and Harrington[2] found an example, based on *Ramsey theory*.

Pigeonhole Principles

In its simplest form, the **pigeonhole principle** states:

> If n elements are distributed into $m < n$ many pigeonholes, then one of the pigeonholes must contain at least 2 elements.

The generalization to infinite sets is:

> If an infinite set is partitioned into finitely many sets, then at least one of these sets must be infinite:
>
> $$A = A_0 \,\dot{\cup} \ldots \dot{\cup}\, A_k \text{ infinite } \Rightarrow \exists i \leq k\,(A_i \text{ infinite}).$$

For further generalization, we set

$$[A]^n := \{x \subseteq A \mid |x| = n\} \text{ the set of } n\text{-element subsets of } A.$$

A partition of the n-element subsets of A into k parts can also be represented by a function

$$f : [A]^n \to k$$

(where $A_i = \{x \in A \mid f(x) = i\}$ are then the partition sets), and such an f is more intuitively called a **coloring** (of $[A]^n$) with k colors. A subset $X \subseteq A$ with $[X]^n \subseteq A_i$ for some $i < k$ (whose n-tuples are thus monochromatic) is called **homogeneous** for the partition f.

Theorem 0.3 (Ramsey's Theorem)**.**
For every coloring $f : [\mathbb{N}]^n \to k$ of the n-element subsets of natural numbers with k colors, there exists an infinite subset $X \subseteq \mathbb{N}$ such that all n-element subsets of X have the same color:

$$f : [\mathbb{N}]^n \to k \Rightarrow \exists X \subseteq \mathbb{N}\,(X \textit{ infinite } \wedge \forall u, v \in [X]^n\, f(u) = f(v)).$$

This states that every coloring of the n-element subsets of natural numbers with finitely many colors has an infinite homogeneous subset. This theorem is provable in set theory with a weak form of the axiom of choice (see, e.g., D. Marker: *Model Theory: An Introduction*, § 5.1).

Definition 0.1 (Relatively Large Sets)**.**
A (finite) set H of ordinal numbers is called **relatively large** if and only if

$$\mathrm{card}(H) \geq \min(H).$$

[2] *A mathematical Incompleteness in Peano Arithmetic.* In: Handbook of Mathematical Logic, ed. Barwise, Elsevier 1982

If $n, k \in \mathbb{N}$, $\alpha, \gamma \in \mathrm{On}$, then let

$$\gamma \longrightarrow (\alpha)^n_k \quad \text{resp.} \quad \gamma \overset{\min}{\longrightarrow} (\alpha)^n_k$$

mean that for every coloring f of the n-element subsets of γ with k colors, there exists a (relatively large) subset $H \subseteq \gamma$ of order type α that is homogeneous for f.

Theorem 0.3 above thus states: $\forall n, k,\ \omega \to (\omega)^n_k$. From this follows the **finitary Ramsey theorem** (using a compactness argument):

$$\forall m, n, k \, \exists r \ \ r \to (m)^n_k.$$

This is a theorem about natural numbers and can be expressed in the language of Peano arithmetic and proven in PA.However, an (apparently) slight strengthening cannot:

Theorem 0.4 (Paris-Harrington Theorem)**.**
The statement

$$\forall m, n, k \, \exists r \, r \overset{\min}{\longrightarrow} (m)^n_k$$

is provable in set theory, but not in PA *(provided* PA *is consistent). The existence of these numbers r can no longer be shown generally in* PA*; in fact, they lead to very large numbers: If*

$$\sigma(n) = \textit{the smallest } r \textit{ with } r \overset{\min}{\longrightarrow} (n+1)^n_n,$$

then the function σ eventually dominates every recursive function f (i.e., for every recursive function f there exists a p such that $f(n) < \sigma(n)$ for all $n \geq p$).