

# Math 557 Sep 12

## Consistency and Completeness

### Key Concepts

- **Consistency:**  
A theory  $T$  is **consistent** if there does not exist a formula  $\psi$  such that  $T \vdash \psi$  and  $T \vdash \neg\psi$ .
- **Completeness:**  
A theory  $T$  is **complete** if it is consistent and for every sentence  $\sigma$ ,  $T \vdash \sigma$  or  $T \vdash \neg\sigma$ .
- In a complete theory, every statement is determined, in the sense that is either true or false in the theory.
- An important example of a complete theory is the *theory of a fixed structure*  $\mathcal{M}$ ,

$$\text{Th}(\mathcal{M}) = \{\sigma : \mathcal{M} \models \sigma\}$$

- If, on the other hand, we are given a complete theory  $T$ , does  $T$  have a model? This is the subject of the **Completeness Theorem**. It actually shows that any *consistent* theory has a model.

### Problems

**Exercise 0.1** (Carry-over from Sep 10).

Prove the *Soundness Theorem*, i.e. show that

$$T \vdash \varphi \Rightarrow T \models \varphi$$

**Exercise 0.2.**

Why is the theory of a structure,  $\text{Th}(\mathcal{M})$ , indeed a complete theory?

**Exercise 0.3.**

Show that a theory  $T$  is inconsistent iff for every formula  $\psi$ ,  $T \vdash \psi$

**Exercise 0.4.**

For each of the following classes  $\mathbf{K}$  of structures, determine whether they are *axiomatizable*, i.e. whether there exists a theory  $T$  (in the language of the structures) such that

$$\mathbf{K} = \{\mathcal{M} : \mathcal{M} \models T\}$$

(In this case  $K$  is also called *elementary*.)

If  $\mathbf{K}$  is axiomatizable, discuss whether it is actually *finitely axiomatizable* (i.e. can  $T$  be chosen finite). And is  $T$  complete?

- $\mathbf{K}$  = Abelian groups
- $\mathbf{K}$  = infinite sets
- $\mathbf{K}$  = fields of characteristic  $p$  ( $p$  prime)
- $\mathbf{K}$  = bipartite graphs
- $\mathbf{K}$  = torsion groups (*all elements have finite order*)
- $\mathbf{K}$  = algebraically closed fields of characteristic 0

*(Note: We do not yet have the tools to rigorously answer all these questions, but try an “educated” guess.)*

**Exercise 0.5.**

Suppose that  $T \cup \{\neg\varphi\}$  is inconsistent. Show that  $T \vdash \varphi$ .

*(This is a technical formulation of the legitimacy of proofs by contradiction.)*