

Week 7 - Exercises

Key concepts

(Ultra)filters

A **filter** \mathcal{F} on a set I is a nonempty collection of subsets of I satisfying:

1. $\emptyset \notin \mathcal{F}$
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$
3. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$

An **ultrafilter** \mathcal{U} is a maximal filter, equivalently:

For all $A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

Reduced products

Given: filter \mathcal{F} on I and structures $(\mathcal{M}_i)_{i \in I}$. Let $M = \prod_{i \in I} M_i$ and define

$$a \sim_{\mathcal{F}} b \iff \{i \in I : a_i = b_i\} \in \mathcal{F}.$$

The universe of \mathcal{M}/\mathcal{F} is the quotient $M/\sim_{\mathcal{F}}$, with elements denoted $a_{\mathcal{F}}$ (alternatively, a/\mathcal{F}).

- **Relations:**

$$R^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) : \iff \{i \in I : \mathcal{M}_i \models R(\vec{a}_i)\} \in \mathcal{F}.$$

- **Functions:**

$$f^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) = [(f^{\mathcal{M}_i}(\vec{a}_i))_{i \in I}]_{\mathcal{F}}.$$

- **Constants:**

$$c^{\mathcal{M}/\mathcal{F}} = ((c^{\mathcal{M}_i})_{i \in I})_{\mathcal{F}}.$$

Łoś' Theorem

Let $\mathcal{M}/\mathcal{U} = \prod_{i \in I} \mathcal{M}_i/\mathcal{U}$ be an ultraproduct. For every \mathcal{L} -formula $\varphi(x_1, \dots, x_n)$ and tuples $\vec{a} \in \prod_{i \in I} M_i$,

$$\mathcal{M}/\mathcal{U} \models \varphi[\vec{a}_{\mathcal{U}}] \iff \{i \in I : \mathcal{M}_i \models \varphi[\vec{a}_i]\} \in \mathcal{U}.$$

Problems

Exercise 0.1 (Principal filters). Show that for every $A \subseteq I$,

$$\{X \subseteq I : A \subseteq X\}$$

is a filter on I . Show that it is an ultrafilter if and only if $|A| = 1$.

Exercise 0.2 (Finite sets). Show that every ultrafilter on a finite set is principal.

Exercise 0.3 (Free ultrafilters on infinite sets). Show that a free ultrafilter on an infinite set I cannot contain any finite subsets of I .

Exercise 0.4 (Ultrafilter existence). Use Zorn's Lemma to show that any family $\mathcal{A} \subseteq \mathcal{P}(I)$ with the FIP can be extended to an ultrafilter \mathcal{U} on I .

Use this to show that any infinite set has a free ultrafilter.

Exercise 0.5 (Ultraproducts with principal ultrafilters). Show that if \mathcal{U} is a principal ultrafilter on I , then every ultraproduct

$$\mathcal{M}/u = \prod_{i \in I} \mathcal{M}_i / \mathcal{U}$$

is isomorphic to some \mathcal{M}_j , $j \in I$.

For principal ultrafilters, ultraproducts do not lead to anything new.

Exercise 0.6 (Ultraproducts of fields). For any prime p let \mathbb{F}_p the field with p elements (of characteristic p), and let \mathcal{U} be a free ultrafilter on the set P of prime numbers.

Show that the ultraproduct

$$\prod_{p \in P} \mathbb{F}_p / \mathcal{U}$$

is a field of characteristic 0.

Ultrapowers

If we take an ultraproduct over the same structure \mathcal{M} along an index set I , we call this an **ultrapower**, denoted by

$$\mathcal{M}^I / \mathcal{U}$$

Let \mathcal{U} be a free ultrafilter on an (infinite) set I .

- Show that the map $j : b \mapsto (b)_{i \in I} / \mathcal{U}$ defines an **elementary embedding** of \mathcal{M} into $\mathcal{M}^I / \mathcal{U}$ (i.e. it is injective and the image is an elementary substructure).
- Show that if \mathcal{M} is infinite, j is not a surjection.

*If we apply this to \mathbb{N} in the language of arithmetic, this yields another way to obtain **non-standard models of arithmetic**.*

Further explorations

Lindenbaum-Tarski algebra

Given a language \mathcal{L} , define an equivalence relation on the set of all \mathcal{L} -sentences by

$$\sigma \sim \tau \iff \vdash \sigma \leftrightarrow \tau$$

The equivalence classes will then form a *Boolean algebra*, the **Lindenbaum-Tarski algebra** \mathcal{B} with the operations

$$\begin{aligned} [\sigma] \wedge [\tau] &:= [\sigma \wedge \tau] \\ [\sigma] \vee [\tau] &:= [\sigma \vee \tau] \\ \neg[\sigma] &:= [\neg\sigma] \end{aligned}$$

The *bottom* element is $0 := [\sigma \wedge \neg\sigma]$, the *top* element is $1 := [\sigma \vee \neg\sigma]$.

We can define an order by putting $[\sigma] \leq [\tau] : \iff \vdash \sigma \rightarrow \tau$. (This corresponds to the order that is defined on any Boolean algebra via $a \leq b : \iff a = a \wedge b$.)

A *filter* on a Boolean algebra is defined in the same way it is defined on a set I (in fact, a filter on a set I is simply a filter on the Boolean algebra induced on its power set by taking intersections, unions, and complements): $F \subseteq \mathcal{B}$ is a filter if it does not contain the bottom element, is closed under \wedge and closed upward under \leq .

- Show that if T is a deductively closed, consistent \mathcal{L} -theory, the set

$$\{[\sigma] : \sigma \in T\}$$

is a filter in \mathcal{B} .

- Show that if T is complete, the above filter is an ultrafilter.

Keisler-Shelah Theorem

From Łoś' Theorem, we know that if two ultrapowers are isomorphic, then the original structures are elementary equivalent. Remarkably, the converse holds, too, and hence gives an *algebraic characterization of elementary equivalence*.

Theorem 0.1 (Keisler-Shelah). *For any two structures \mathcal{M}, \mathcal{N} ,*

$$\mathcal{M} \equiv \mathcal{N} \iff \text{there exists an index set } I \text{ and an ultrafilter } \mathcal{U} \text{ on } I \text{ such that } \mathcal{M}^I/\mathcal{U} \cong \mathcal{N}^I/\mathcal{U}$$

Other interesting directions

- [Ramsey theory and ultrafilters](#)