

End Extensions

The standard model can be embedded into every model \mathcal{M} of PA as an initial segment. It turns out this already holds for models of the theory PA^- .

Definition 1

Let L be a language containing a 2-ary symbol $<$, and let \mathcal{M} and \mathcal{N} be L -structures with $\mathcal{M} \subseteq \mathcal{N}$. Then \mathcal{N} is called an **end extension** of \mathcal{M} (and correspondingly \mathcal{M} is an **initial segment** of \mathcal{N}) if and only if the larger set N does not add any further elements below an element of M :

$$\mathcal{M} \subseteq_{\text{end}} \mathcal{N} : \iff \text{for all } x \in M, y \in N : (y <^N x \Rightarrow y \in M).$$

Each natural number n is represented in the standard model, which we also simply denote by \mathbb{N} here, by the constant term

$$\underline{n} = 1 + \dots + 1 \quad (\text{n times})$$

where $\underline{0}$ is the constant 0.

Theorem 2

Let $\mathcal{M} \models \text{PA}^-$. Then the map

$$n \mapsto \underline{n}^{\mathcal{M}}$$

defines an embedding of the standard model \mathbb{N} onto an initial segment of \mathcal{M} .

In particular, every model of PA^- is isomorphic to an end extension of the standard model \mathbb{N} .*

Proof. By simple induction (in the meta-theory), one shows for all natural numbers n, k, l :

$$\begin{aligned} n = k + l &\implies \text{PA}^- \vdash \underline{n} = \underline{k} + \underline{l} \\ n = k \cdot l &\implies \text{PA}^- \vdash \underline{n} = \underline{k} \cdot \underline{l} \\ n < k &\implies \text{PA}^- \vdash \underline{n} < \underline{k} \end{aligned}$$

and

$$\text{PA}^- \vdash x \leq \underline{k} \rightarrow (x = \underline{0} \vee \dots \vee x = \underline{k})$$

The first three statements will later be generalized to all recursive functions and relations; they imply that the map $n \mapsto \underline{n}^{\mathcal{M}}$ is a homomorphism, and, due to the last statement, the map is also an embedding onto an initial segment of \mathcal{M} . \square

Remark

The standard model has no proper initial segment, and $\mathbb{Z}[X]^+$ has \mathbb{N} as its only proper initial segment. On the other hand, every model $\mathcal{M} \models \text{PA}^-$ has a proper end extension that is also a model of PA^- , namely the non-negative part of the polynomial ring $R[X]$, where R is the discretely ordered ring associated with the model \mathcal{M} .

Preservation Properties of End Extensions

In the previous lecture, we introduced *arithmetical formulas*. Δ_0 -formulas are at the bottom of the *arithmetical hierarchy*. We already saw that relations defined by such formulas are primitive recursive. Next, we will show that those formulas *mean the same thing in a structure as in all end extensions*. This will be crucial later on.

Theorem 3

Let \mathcal{N}, \mathcal{M} be structures of the language L of PA^- , with $\mathcal{N} \subseteq_{\text{end}} \mathcal{M}$, and let $\vec{a} \in N$. Then:

1. Every Δ_0 -formula $\varphi(\vec{v})$ is **absolute**:

$$\mathcal{N} \models \varphi[\vec{a}] \iff \mathcal{M} \models \varphi[\vec{a}],$$

2. Every Σ_1 -formula $\varphi(\vec{v})$ is **upward-persistent**:

$$\mathcal{N} \models \varphi[\vec{a}] \implies \mathcal{M} \models \varphi[\vec{a}],$$

3. Every Π_1 -formula $\varphi(\vec{v})$ is **downward-persistent**:

$$\mathcal{M} \models \varphi[\vec{a}] \implies \mathcal{N} \models \varphi[\vec{a}],$$

4. Every Δ_1 -formula $\varphi(\vec{v})$ is **absolute**:

$$\mathcal{N} \models \varphi[\vec{a}] \iff \mathcal{M} \models \varphi[\vec{a}].$$

Proof. We prove (i) by induction on the formula structure of $\varphi(\vec{v})$. Only the case of a bounded quantifier needs to be addressed. Since \mathcal{M} is an end extension of \mathcal{N} , M does not insert any new elements below an element of N , so that a bounded quantifier means the same thing in both structures. \square

Let $\Sigma_1\text{-Th}(\mathbb{N}) := \{\sigma \mid \sigma \text{ is a } \Sigma_1\text{-sentence with } \mathbb{N} \models \sigma\}$. Then we have:

Corollary 4

$$\text{PA}^- \models \Sigma_1\text{-Th}(\mathbb{N})$$

Proof. Let $\mathcal{N} \models \text{PA}^-$. By the previous theorem, we may assume that $\mathbb{N} \subseteq_{\text{end}} \mathcal{N}$, and the claim then follows from part (ii) above. \square

Thus one can prove in the theory PA^- all Σ_1 -sentences that hold in the standard model. This is no longer true for Π_1 -sentences. For instance, the Π_1 -sentence stating that every number is even or odd:

$$(*) \quad \forall x \exists y \leq x (x = 2 \cdot y \vee x = 2 \cdot y + 1)$$

is true in the standard model, but not in the PA^- -model $\mathbb{Z}[X]^+$.

Even true \forall -sentences (i.e., sentences of the form $\forall \vec{x} \psi$ with *quantifier-free* ψ) that hold in the standard model need not be provable in PA^- . For example, the \forall -sentence

$$\forall x, y (x^2 \neq 2 \cdot y^2)$$

is true in the standard model, but not provable in PA^- ($\mathbb{Z}/(X^2 - 2Y^2)$ is a counterexample).

Thus the above sentence $(*)$ is an example of a Π_1 -sentence that is not a \forall -sentence, where one cannot omit the bounded quantifier! (By the way, one can show that $\mathbb{Z}[X]^+$ is at least a model of all \forall -sentences that hold in the standard model.)