

## Amalgamation Classes

Let  $\overline{K}$  be a class of finitely generated structures.  $\overline{K}$  is called an **amalgamation class** if it has the following three properties:

(HP) Hereditary Property

If  $A \in \overline{K}$ ,  $\mathcal{B} \cong \mathcal{C} \in A$ , and  $\mathcal{C}$  is finitely generated, then  $\mathcal{B} \in \overline{K}$ .

(JEP) Joint Embedding Property

If  $A, \mathcal{B} \in \overline{K}$ , then there exists  $\mathcal{C} \in \overline{K}$  and embeddings

$$f_0 : A \rightarrow \mathcal{C}, \quad f_1 : \mathcal{B} \rightarrow \mathcal{C}$$

(AP) Amalgamation Property

If  $A, \mathcal{B}, \mathcal{C} \in \overline{K}$  with embeddings  $f_0 : A \rightarrow \mathcal{B}$  and  $f_1 : A \rightarrow \mathcal{C}$ , then there exists  $\mathcal{D} \in \overline{K}$  and embeddings

$$g_0 : \mathcal{B} \rightarrow \mathcal{D}, \quad g_1 : \mathcal{C} \rightarrow \mathcal{D}$$

such that  $g_0 \circ f_0 = g_1 \circ f_1$ .

### Exercise 0.1.

Show that (AP) does not imply (JEP).

(Hint: finite fields)

### Example 0.1.

$\overline{K} = \{(Z, <) : (Z, <) \text{ finite linear order}\}$  forms an amalgamation class.

## Fraïssé's Theorem

**Theorem 0.1.** Let  $\overline{K}$  be a class of finitely generated  $\mathcal{L}$ -structures such that there are only countably many isomorphism types in  $\overline{K}$ .

Then:  $\overline{K}$  is an amalgamation class  $\Leftrightarrow \overline{K}$  is the age of a countable homogeneous  $\mathcal{L}$ -structure.

*Proof.* ( $\Leftarrow$ ): Suppose  $\overline{K} = \text{age}(\mathcal{M})$  where  $\mathcal{M}$  is countable and homogeneous.

(HP): Holds by definition of age.

(JEP): Let  $\mathcal{A}, \mathcal{B} \in \text{age}(\mathcal{M})$ . Then  $A \cong \langle A \rangle^{\mathcal{M}}$  and  $\mathcal{B} \cong \langle B \rangle^{\mathcal{M}}$ , with  $A, B \subseteq M$  finite. Then  $\mathcal{A}$  and  $\mathcal{B}$  embed into  $\langle A \cup B \rangle^{\mathcal{M}}$ .

(AP): Suppose  $f_0 : \mathcal{A} \rightarrow \mathcal{B}$  and  $f_1 : \mathcal{A} \rightarrow \mathcal{C}$  are embeddings, where  $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \text{age}(\mathcal{M})$ .

Without loss of generality,  $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathcal{M}$  and  $f_0 = \text{id}_A$ .

Since  $\mathcal{M}$  is homogeneous, there exists an automorphism  $\pi : \mathcal{M} \xrightarrow{\sim} \mathcal{M}$  such that  $\pi^{-1}|_A = f_1$  (i.e.,  $\pi$  extends  $f_1^{-1}$ ).

Consider  $\mathcal{D} = \langle B \cup \pi^{-1}(C) \rangle^{\mathcal{M}}$ .  $\mathcal{D}$  amalgamates  $\mathcal{B}$  and  $\mathcal{C}$  via  $\text{id}_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{D}$  and  $\pi^{-1}|_C : \mathcal{C} \rightarrow \mathcal{D}$ .  $\square$

( $\Rightarrow$ ): Assume  $\overline{K}$  is an amalgamation class. We construct a structure  $\mathcal{N}$  with  $\text{age}(\mathcal{N}) = \overline{K}$  where  $\mathcal{N}$  is homogeneous and has domain  $N \subset \mathbb{N}$ .

Since  $\overline{K}$  has countably many isomorphism types, we enumerate the elements of  $\overline{K}$  (up to isomorphism) as  $(\mathcal{K}_e : e \in \mathbb{N})$  where  $K_e \subseteq \mathbb{N}$ .

Consider tuples  $(\bar{a}, \bar{b}, f)$  where  $\bar{a}_i, \bar{b}_i$  are finite subsets of  $\mathbb{N}$  and  $f$  is a partial function with  $\text{dom}(f) = \bar{a}$  and  $\text{ran}(f) \subseteq \bar{b}$ . Enumerate all such triples as  $(\bar{a}_e, \bar{b}_e, f_e)$  so that every  $(\bar{a}, \bar{b}, f)$  occurs infinitely often.

$\mathcal{N}$  will be the union of an increasing sequence  $(\mathcal{C}_n)_{n \in \mathbb{N}}$  where  $\mathcal{C}_n \in \overline{K}$ .

**Initialize:**  $\mathcal{C}_0 = \mathcal{K}_0$

Now assume we have defined  $\mathcal{C}_n$ .

**Case  $n = 2\ell$  (even):** Apply (JEP) to  $\mathcal{C}_n, \mathcal{K}_\ell$  to obtain  $\mathcal{C}_{n+1}$ .

**Case  $n = 2\ell + 1$  (odd):** If  $\bar{a}_\ell$  or  $\bar{b}_\ell \not\subseteq C_n$ , put  $\mathcal{C}_{n+1} = \mathcal{C}_n$ . If  $\bar{a}_\ell, \bar{b}_\ell \subseteq C_n$ , let  $\mathcal{A}_\ell = \langle \bar{a}_\ell \rangle^{\mathcal{C}_n}$  and  $\mathcal{B}_\ell = \langle \bar{b}_\ell \rangle^{\mathcal{C}_n} \subseteq \mathcal{C}_n$ . Apply (AP) to the embeddings  $\text{id} : \mathcal{A}_\ell \rightarrow \mathcal{C}_n$  and  $f : \mathcal{A}_\ell \rightarrow \mathcal{B}_\ell$  (induced by  $f_\ell : \bar{a}_\ell \rightarrow \bar{b}_\ell$ ). This yields a structure  $\mathcal{C}_{n+1} \in \overline{K}$ . Renaming if necessary, we can assume  $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$ .

Define  $\mathcal{N} = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$ .

**Claim 1:**  $\text{age}(\mathcal{N}) = \overline{K}$ .

In the even steps  $n = 2\ell$ , we ensure  $K_\ell \subseteq \mathcal{N}$ , so all structures isomorphic to  $K_\ell$  also enter  $\text{age}(\mathcal{N})$ . Since the  $(K_e)_{e \in \mathbb{N}}$  enumerates all isomorphism types of  $\overline{K}$ , we get  $\text{age}(\mathcal{N}) = \overline{K}$ .

**Claim 2:**  $\mathcal{N}$  is homogeneous.

Let  $\tau : A \rightarrow B$  be an isomorphism between finitely generated substructures  $\mathcal{A} = \langle \bar{a} \rangle^{\mathcal{N}}$  and  $\mathcal{B} = \langle \bar{b} \rangle^{\mathcal{N}}$  where  $\bar{a}, \bar{b} \in N$  are finite. We show that for any  $c_0 \in N \setminus B$ , there exists a partial isomorphism  $\tau' : \mathcal{A}' \rightarrow \mathcal{B}'$  where  $\mathcal{A}', \mathcal{B}'$  are finitely generated,  $\tau' \supseteq \tau$ , and  $c_0 \in \text{dom}(\tau')$ . (This suffices since we can continue via back-and-forth.)

There exists  $\ell$  such that  $f_\ell$  induces an embedding  $f : A \rightarrow \langle \mathcal{B} \cup \{c_0\} \rangle^{\mathcal{N}}$  (since every triple occurs infinitely often in the enumeration, in particular the triple  $(\bar{a}, \bar{b} \cup \{c_0\}, f)$ ).

In step  $n = 2\ell + 1$ ,  $\mathcal{B}_\ell = \langle \mathcal{B} \cup \{c_0\} \rangle^{\mathcal{N}}$  and  $\mathcal{C}_n$  are amalgamated over  $\mathcal{A}_\ell$ . This yields embeddings:

$$\begin{aligned} \text{id} : \mathcal{A} &\hookrightarrow \mathcal{C}_{n+1} \\ g : \mathcal{B} &\hookrightarrow \mathcal{C}_{n+1} \text{ where } g \circ f_\ell = \text{id}_{\mathcal{A}} \end{aligned}$$

Let  $a_0 = g(c_0)$ . Then  $(g|_{\mathcal{B}})^{-1} : \langle \bar{a} \cup \{a_0\} \rangle^{\mathcal{N}} \xrightarrow{\sim} \langle \bar{b} \cup \{c_0\} \rangle^{\mathcal{N}}$  is the desired isomorphism.