Math 557 August 29

Key Concepts

- **$\mathcal{L}$-structures**: Provide a realm to interpret $\mathcal{L}$-formulas over.
  - non-empty universe M,
  - interpretation of all constant, function, and relation symbols over M.
- **structure isomorphism**: bijection π between the universes of two $\mathcal{L}$-structures that preserves interpretations of symbols. General version of the well-known idea of isomorphism of mathematical structures.
- **assignment**: function $\alpha : \mathcal{V} \to M$ that assigns all variables values in the universe of a structure. Recursively extends to terms.
- **satisfaction relation**: $\mathcal{M} \models \varphi[\alpha]$ means the $\mathcal{L}$-formula φ is true when interpreted in the $\mathcal{L}$-structure $\mathcal{M}$ and all variables are assigned values according to assignment α. This is defined recursively over the structure of a formula.

Problems

We warm up with a quote often attributed to Lincoln.

Exercise 0.1.
> You can fool all the people some of the time, and some of the people all the time, but you cannot fool all the people all the time.

Define a suitable language and formalize this statement in first-order logic.

Next, a couple of problems to see the definition of structures and the satisfaction relation $\models$ at work.

Exercise 0.2.
Let $\mathcal{L} = \{\square, \triangle, c\}$, where $\square$ is a binary relation symbol, $\triangle$ is a binary function symbol, and c is a constant symbol.

For each of the following formulas, find two $\mathcal{L}$-structures and (if necessary) assignments to the free variable(s) such that the formula holds in one structure but fails in the other.

$$\begin{align}
\varphi &\equiv \forall x \exists y \, \square \, \triangle xy \, c \tag{1} \\
\psi &\equiv \exists x \forall y (x = c \ \wedge \ \square \, \Delta zc \, y) \tag{2}
\end{align}$$

Next, let's see some general properties of structures we can describe through first-order formulas.

Exercise 0.3.
Let n be a natural number and let $\mathcal{L}$ be any language. Find formulas , $\psi^{\geq}, \psi^{\leq}, \psi^{=}$ such that, for any $\mathcal{L}$-structure $\mathcal{M}$,

$$\begin{align}
\mathcal{M} \vDash \psi^{\geq} \quad &\Longleftrightarrow \quad |M| \geq n \tag{3} \\
\mathcal{M} \vDash \psi^{\leq} \quad &\Longleftrightarrow \quad |M| \leq n \tag{4} \\
\mathcal{M} \vDash \psi^{=} \quad &\Longleftrightarrow \quad |M| = n \tag{5} \\
& \tag{6}
\end{align}$$

Question

Can we find a formula θ such that

$$\mathcal{M} \vDash \theta \quad \Longleftrightarrow \quad |M| \text{ is finite?}$$

We will answer this question in a few weeks...

Exercise 0.4.
Let $\mathcal{L} = \{\cdot, e\}$ be the language of groups. Show that there is a sentence φ such that $\mathcal{M} \vDash \varphi$ if and only if

$$M \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

Take-home Problems

1. Let $\mathcal{L}$ be any finite language and let $\mathcal{M}$ be a finite $\mathcal{L}$-structure. Show that there is an $\mathcal{L}$-sentence φ such that

$$\mathcal{N} \vDash \varphi \quad \Longleftrightarrow \quad \mathcal{N} \cong \mathcal{M}.$$

2. Give an example of a language $\mathcal{L}$ and an $\mathcal{L}$-sentence ψ such that
   - there is at least one $\mathcal{L}$-structure A such that $A \vDash \psi$,
   - for all L-structures A, if $A \vDash \psi$, then the universe A of A is infinite.