

Representability

We have established a closed connection between computability and definability over \mathbb{N} , but how much of that can PA^- actually prove? We need to make sure it can *represent* sufficiently simple (i.e., computable) functions and sets faithfully.

Definition 0.1. Let T be a theory in the language of arithmetic L that extends PA^- . A (total) function $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is **representable** in T iff there exists an L -formula $\theta(x_1, \dots, x_k, y)$ such that for all $n_1, \dots, n_k, m \in \mathbb{N}$:

- (a) $T \vdash \exists! y \theta(\underline{n_1}, \dots, \underline{n_k}, y)$, and
- (b) $f(n_1, \dots, n_k) = m \Rightarrow T \vdash \theta(\underline{n_1}, \dots, \underline{n_k}, \underline{m})$.

Similarly, a set $S \subseteq \mathbb{N}^k$ is **representable** in the theory T iff there exists an L -formula $\theta(x_1, \dots, x_k)$ such that for all $n_1, \dots, n_k \in \mathbb{N}$:

- (c) $(n_1, \dots, n_k) \in S \Rightarrow T \vdash \theta(\underline{n_1}, \dots, \underline{n_k})$, and
- (d) $(n_1, \dots, n_k) \notin S \Rightarrow T \vdash \neg \theta(\underline{n_1}, \dots, \underline{n_k})$.

If the function f (or the set S) is representable by a Σ_1 -formula, then f (or S , respectively) is called **Σ_1 -representable**.

Note that by definition, representability is preserved when we pass to a theory $T' \supseteq T$.

Theorem 0.1 (Representation Theorem).

- (i) Every recursive function $f : \mathbb{N}^k \rightarrow \mathbb{N}$ is Σ_1 -representable in PA^- .
- (ii) Every recursive set $S \subseteq \mathbb{N}^k$ is Σ_1 -representable in PA^- .

Proof. (i) Let $f : \mathbb{N}^k \rightarrow \mathbb{N}$ be a recursive function, so its graph Γ_f is definable over the natural numbers by a Σ_1 -formula $\exists z \varphi(\vec{x}, y, z)$, where φ has only bounded quantifiers. Since every formula of the form $\exists z \varphi$ is equivalent in PA^- to $\exists u \exists z (z < u \wedge \varphi)$, we may assume that z is just a single variable z . We now form the Δ_0 -formula $\psi(\vec{x}, y, z)$:

$$\varphi(\vec{x}, y, z) \wedge \forall u, v \leq y + z (u + v < y + z \rightarrow \neg \varphi(\vec{x}, u, v)).$$

We now claim that the Σ_1 -formula $\exists z \psi(\vec{x}, y, z)$ represents the function f in PA^- :

First we show (b). Assume that $f(n_1, \dots, n_k) = m$ holds, thus $\mathbb{N} \models \exists z \varphi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, z)$. The number m is uniquely determined since f is a function. Choose l as the smallest number such that $\mathbb{N} \models \varphi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, l)$. Then clearly $\mathbb{N} \models \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, l)$ also holds, and thus $\mathbb{N} \models \exists z \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, z)$. As a Σ_1 -sentence, this sentence is preserved under all end extensions of the standard model to a model of PA^- , thus it holds in all models of PA^- , and therefore $\text{PA}^- \vdash \exists z \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, z)$ by the Completeness Theorem.

The proof of (a) uses a similar argument: Let $f(n_1, \dots, n_k) = m$ and let l again be the smallest number such that $\mathbb{N} \models \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, l)$ holds. Let $\mathcal{M} \models \text{PA}^-$. We claim that m is the only element of M that satisfies the formula $\psi(\underline{n_1}, \dots, \underline{n_k}, x, l)$ in \mathcal{M} . $\mathcal{M} \models \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, l)$ due to the absoluteness of Δ_0 -formulas. If $a, b \in M$ are two elements such that $\mathcal{M} \models \psi(\underline{n_1}, \dots, \underline{n_k}, a, b)$, then we must have $a, b \leq m + l$: For if one of them were greater than $m + l$, we would also have $a + b > m + l$. We assumed

$$\mathcal{M} \models \psi(\underline{n_1}, \dots, \underline{n_k}, a, b),$$

so by definition of ψ we have

$$\mathcal{M} \models \varphi(\underline{n_1}, \dots, \underline{n_k}, a, b) \wedge \forall u, v \leq a + b (u + v < a + b \rightarrow \neg \varphi(\underline{n_1}, \dots, \underline{n_k}, u, v)).$$

But since $m + l < a + b$, this would imply $\neg \varphi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, \underline{l})$ in \mathcal{M} , contradicting the choice of m and l .

As \mathcal{M} is an end-extension of \mathbb{N} , $a, b \leq m + l$ implies $a, b \in \mathbb{N}$. Thus, again by the absoluteness of Δ_0 -formulas, we have $\mathbb{N} \models \psi(\underline{n_1}, \dots, \underline{n_k}, a, b)$, from which $\mathbb{N} \models \exists z \psi(\underline{n_1}, \dots, \underline{n_k}, a, z)$ follows, and therefore $a = m$.

(ii) now follows easily: If S is recursive, then the characteristic function c_S is computable, thus by (i) represented by a Σ_1 -formula $\theta(\vec{x}, y)$ in PA^- . Then S is represented by the formula $\theta(\vec{x}, 1)$, since we have:

$$\text{PA}^- \vdash \neg \theta(\vec{x}, 1) \iff \text{PA}^- \vdash \theta(\vec{x}, 0).$$

□