Math 557 Sep 26

Midterm 1 Review

> **!** Take-home Problem 1
>
> Prove unique readability for the set of $\mathcal{L}$-formulas.

Solution 0.1. **First** prove **readibility**, i.e. the statement that for any $\mathcal{L}$-formula φ, exactly one of the following cases holds:

(1) There exist terms s, t such that $\varphi \equiv s = t$.
(2) There exists a relation symbol R and, if n is the arity of R, terms $t_1, \dots, t_n$ sucht that $\varphi \equiv R t_1 \dots t_n$.
(3) There exists a formula ψ such that $\varphi \equiv \neg \psi$.
(4) There exist formulas ψ, θ such that $\varphi \equiv (\psi \wedge \theta)$.
(5) There exists a variable x and a formula ψ such that $\varphi \equiv \exists x \psi$.

To see this, let F be the subset of $\mathcal{L}^*$ such that every $\theta \in F$ satisfies exactly one of (1)-(5). Then F contains all atomic formulas and is closed under $\neg, \wedge, \exists x$. Since the set of $\mathcal{L}$-formulas is the smallest such set, it follows that every $\mathcal{L}$-formula is contained in F, and therefore has the desired property.

Second we argue that a proper initial segment of a formula cannot be a formula. We proceed by induction on the length l of φ. For $l = 1$ there is no formula of that length. Now suppose φ is a formula of length $l + 1$, and β is a proper initial segment. We may assume β is not empty. We consider theses (1)-(5) from readability.

Case $\varphi \equiv s = t$ Either $\beta \equiv s = t'$ with $t \subset t'$, but this would contradict that no proper initial segment of a term is a term. Or $\beta \equiv s =$: An easy induction over the length of a formula shows that no formula ends with $=$. Or $\beta \subseteq s$. Another easy induction shows that no formula can be a term or an initial segment of a term.

Case $\varphi \equiv R t_1 \dots t_n$ If β were a formula, it has to be of the form $R s_1 \dots s_n$. Comparing terms, we would see that one of the s_i is a proper initial segment of some t_j, which is impossible.

Case $\varphi \equiv \neg \theta$ Then $\beta \equiv \neg \beta'$ with β' a proper initial segment of θ. By inductive hypothesis, β' is not a formula, hence $\neg \beta'$ is not a formula either.

Case $\varphi \equiv (\psi \wedge \theta)$ Then β starts with (but does not end with). Another induction shows that for any formula, the number of ('s always equals the number of)'s.

Case $\varphi \equiv \exists x \varphi$ Neither '$\exists$' nor '$\exists x$' are formulas. If β is longer than that, it is of the form $\exists x \beta'$. By inductive hypothesis, β' is not a formula, and hence β is not a formula (by readability). varphi

Finally, we prove **unique readability**: The choices in (1)-(5) are unique.

Case (1) Suppose $\varphi \equiv s = t \equiv s' = t'$. Since no proper initial segment of a term is a term, it must hold that $s \equiv s'$ and $t \equiv t'$.

Case (2) Suppose $\varphi \equiv R t_1 \dots t_n \equiv S s_1 \dots s_k$. Then $R \equiv S$ and $n = k$. Comparing terms inductively, using again the fact that no proper initial segment of a term is a term, we get $s_i \equiv t_i$ for all i.

Case (3) Suppose $\varphi \equiv \neg \psi \equiv \neg \theta$. Immediately, we infer $\psi \equiv \theta$.

Case (4) Suppose $\varphi \equiv (\psi_1 \wedge \theta_1) \equiv (\psi_2 \wedge \theta_2)$. Assume $\psi_1 \not\equiv \psi_2$. Then ψ_1 is a proper initial segment of ψ_2 or vice versa (since both of them are part of φ). Either case is impossible due to the fact that no proper initial segment of a formula is a formula. Hence $\psi_1 \equiv \psi_2$ and thus also $\theta_1 \equiv \theta_2$.

Case (5) Suppose $\varphi \equiv \exists x \psi \equiv \exists y \theta$. By comparing entries, we get $x \equiv y$ and thus $, as desired.

> **!** Take-home Problem 2
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> Let $\mathcal{L}$ be any finite language and let $\mathcal{M}$ be a finite $\mathcal{L}$-structure. Show that there is an $\mathcal{L}$-sentence φ such that
>
> $$\mathcal{N} \vDash \varphi \iff \mathcal{N} \cong \mathcal{M}.$$

Solution 0.2. Suppose $\mathcal{L} = \{c_1, \cdots, c_k, f_1^{(a_1)}, \dots f_l^{(a_l)}, R_1^{(b_1)}, \dots, R_j^{(b_j)}\}$ where the $(a_i), (b_m)$ denote the arities of the respective symbols.

Since $\mathcal{M}$ is finite, we may assume $M = \{1, \dots, n\}$ for some $n \in \mathbb{N}$.

The basic idea is to collect all "elementary facts" about $\mathcal{M}$ in a single formula (think of a *group multiplication table*, just for all functions and relations).

Define

$$\varphi \equiv \exists x_1, \dots, x_n \left(\bigwedge_{i \neq m} x_i \neq x_l \ \wedge \ \forall y \bigvee_{i \leq n} y = x_i \right. \tag{1}$$

$$\wedge \bigwedge_{i \leq k} c_i = x_{c_i^{\mathcal{M}}} \tag{2}$$

$$\wedge \bigwedge_{i \leq l} \bigwedge_{\pi \in \{1, \dots, n\}^{a_i}} f_i x_{\pi(1)} \dots x_{\pi(a_i)} = x_{f_i^{\mathcal{M}}(\pi(1), \dots, \pi(a_i))} \tag{3}$$

$$\left. \wedge \bigwedge_{i \leq j} \bigwedge_{\pi \in \{1, \dots, n\}^{b_i}} \delta_{\pi} R_i x_{\pi(1)} \dots x_{\pi(b_i)} \right) \tag{4}$$

where δ_{π} is empty if $R(\pi(1), \dots, \pi(b_i))$ holds in $\mathcal{M}$, and $\neg$ if not.

Assme $\mathcal{N} \vDash \varphi$. Due to the first line of the equation, N has exactly n elements, and let $r_i \in N$ be the witness to $\exists x_i$. Define a mapping $\tau : M \to N$ by letting $\tau(i) = r_i$. We claim that τ is an isomorphism.

By definition we have $\tau(c_i^{\mathcal{M}}) = r_{c_i^{\mathcal{M}}}$. Moreover, by the second line of the formula, $r_{c_i^{\mathcal{M}}}$ is the unique element of N that makes the formula $c_i = x_{c_i^{\mathcal{M}}}$ true. It follows that $\tau(c_i^{\mathcal{M}}) = c_i^{\mathcal{N}}$ for all $1 \leq i \leq k$.

Using a similar argument with the third line of the formula, we obtain

$$\tau(f_i^{\mathcal{M}}(s_1, \dots, s_{a_i})) = r_{f_i^{\mathcal{M}}(s_1, \dots, s_{a_i})} = f^{\mathcal{N}}(r_{s_1}, \dots, r_{s_{a_i}}) = f^{\mathcal{N}}(\tau(s_1), \dots, \tau(s_{a_i}))$$

The argument for $R^{\mathcal{M}}(s_1, \dots, s_{b_i}) \iff R^{\grave{}\mathcal{N}}(\tau(s_1), \dots, \tau(s_{b_i}))$ is similar.

On the other hand, if $\mathcal{M} \cong \mathcal{N}$ via τ, then $\mathcal{N} \vDash \psi[\tau(1), \dots, \tau(n)]$, where ψ is such that $\varphi \equiv \exists x_1, \dots, x_n \, \psi$, due to the fact that τ is an isomorphism, and thus $\mathcal{N} \vDash \varphi$.

> **!** Take-home Problem 3
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> Give an example of a language $\mathcal{L}$ and an $\mathcal{L}$-sentence ψ such that
> - there is at least one $\mathcal{L}$-structure A such that $A \vDash \psi$,
> - for all L-structures A, if $A \vDash \psi$, then the universe A of A is infinite.

Solution 0.3. Let $\mathcal{L} = \{<\}$, where $<$ is a binary relation symbol. Define

$$\psi \equiv \forall x \ x \not< x \tag{5}$$
$$\wedge \ \forall x, y \ x < y \vee x = y \vee y < x \tag{6}$$
$$\wedge \ \forall x, y, z \ (x < y \wedge y < z) \to x < z \tag{7}$$
$$\wedge \ \forall x \exists y \ x < y \tag{8}$$

The formula says that $<$ is a linear order with no maximal element.

Clearly, $(\mathbb{Z}, <) \vDash \psi$.

Now suppose $\mathcal{M} \vDash \psi$. Due to the last line of ψ, there exists a function $f : M \to M$ such that $x < f(x)$ for all $x \in M$. We claim that for any x and for any $n \neq m$,

$$f^{(n)}(x) \neq f^{(m)}(x)$$

This follows from antireflexivity (line one) and transitivity (line three).

Therefore, the set

$$\{x, f^{(1)}(x), f^{(2)}(x), \dots \}$$

is an infinite subset of N.

! Take-home problem 4

Show that

$$\{\varphi \to \psi\} \vdash \exists x \varphi \ \to \ \exists x \psi$$
$$\{\varphi \to \psi\} \vdash \forall x \varphi \ \to \ \forall x \psi$$

Solution 0.4. We give derivations below (with brief justifications). We collect simple substeps into a single one.

For $\{\varphi \to \psi\} \vdash \exists x \varphi \ \to \ \exists x \psi$:

Formula	Justification
$\varphi \to \psi$	given
$\psi \to \exists x \psi$	(Q2)
$\varphi \to \exists x \psi$	tautology
$\neg \exists x \psi \to \neg \varphi$	tautotolgy
$\forall x(\neg \exists x \psi \to \neg \varphi)$	$\forall$-intro
$\neg \exists x \psi \to \forall x \varphi$	(Q1), x not free in $\neg \exists x \psi$
$\neg \forall x \neg \varphi \to \exists x \psi$	tautology
$\exists x \varphi \to \exists x \psi$	(Q3)

For $\{\varphi \to \psi\} \vdash \forall x \varphi \ \to \ \forall x \psi$:

Formula	Justification
$\varphi \to \psi$	given
$\forall x \varphi \to \varphi$	Example 2.6.1 (d)
$\forall x \varphi \to \psi$	Tautology
$\forall x(\forall x \varphi \to \psi)$	$\forall$-intro
$\forall x \varphi \to \forall x \psi$	(Q1), x not free in $\forall x \varphi$

! Take-home Problem

Show that for every $\mathcal{L}$-formula φ,
$$T_H \vdash_{\mathcal{L}_H} \varphi \ \iff \ T \vdash_{\mathcal{L}} \varphi$$

> **❗ Take-home Problem**
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> Use the compactness theorem to show (without using the Axiom of Choice) that every set can be linearly ordered.
> Try to strengthen this to:
>
> > Every partial order can be extended to a linear order.