

## Peano Arithmetic

While the algebraic theories of groups, rings, fields, ... have many models, (elementary) number theory studies properties of *one* model, which is called the **standard model** of number theory:

$$\mathcal{N} = (\mathbb{N}, +, \cdot, +1, 0).$$

Let  $\mathcal{L}_{\text{PA}}$  be the associated language, for which we also choose  $+$ ,  $\cdot$ ,  $0$  as (non-logical) symbols, but  $S$  for the unary successor operation  $+1$ . The theory of the standard model,  $\text{Th}(\mathcal{N})$ , is the set of  $L_{\text{PA}}$ -sentences that hold in this model. We have already seen as a consequence of the compactness theorem that there exist **non-standard models**, i.e., models that are not isomorphic to the standard model, because they have “infinitely large” numbers.

Moreover,  $\text{Th}(\mathcal{N})$  as a theory, while complete, is rather mysterious, since we do not know a priori which sentences exactly it comprises. The most important properties of the standard model are captured by the **Peano Axioms**.

### Definition 1: Peano Axioms

- P1.**  $Sx \neq 0$
- P2.**  $Sx = Sy \rightarrow x = y$
- P3.**  $x + 0 = x$
- P4.**  $x + Sy = S(x + y)$
- P5.**  $x \cdot 0 = 0$
- P6.**  $x \cdot Sy = x \cdot y + x$

To these we add the infinitely many **induction axioms**:

- Ind.**  $\varphi(0) \wedge \forall x(\varphi(x) \rightarrow \varphi(Sx)) \rightarrow \forall x \varphi(x)$

The theory comprising these axioms is called PA, **Peano Arithmetic**. As every model of  $\text{Th}(\mathcal{N})$  is also a model of PA, we have

### Theorem 2

There exists a (countable) model  $\mathcal{N}^*$  of PA which is not isomorphic to  $\mathcal{N}$ .

(In fact, we know by Loewenheim-Skolem that there exists a non-standard model in every infinite cardinality.)

These results can be interpreted as an expressive weakness of the language of first-order logic, because if one moves to a **language of second order**, in which one additionally has the possibility of using **second-order quantifiers**  $\exists X, \forall Y$  to quantify over subsets of the respective domain, then one can uniquely characterize (up to isomorphism) the standard model in this stronger theory:

### Definition 3: Second-Order Peano Axioms

The theory  $\text{PA}^{(2)}$  (Peano Axioms of second order) has the following axioms:

- P1.**  $\forall x \, 0 \neq Sx$
- P2.**  $\forall x \forall y (Sx = Sy \rightarrow x = y)$
- IND.**  $\forall X (0 \in X \wedge \forall x (x \in X \rightarrow Sx \in X) \rightarrow \forall x \, x \in X)$

#### Theorem 4

Every model of  $\text{PA}^{(2)}$  is isomorphic to the standard model  $(\mathbb{N}, S, 0)$ .

The second-order induction axiom (actually a *set-theoretic* axiom) is thus significantly more expressive than the corresponding induction schema, in which only first-order properties are allowed, which can only quantify over elements (instead of also over subsets) of natural numbers.

On the other hand, second-order logic has other disadvantages – most prominently, no completeness theorem holds.

### $\text{PA}^-$ : Peano Arithmetic without Induction

PA still has infinitely many (induction) axioms. We will introduce a finite subtheory that turns out is strong enough to capture many essential properties of arithmetic.

This theory is formalized in a language  $\mathcal{L}$  with the symbols  $0, 1, <, +, \cdot$ . The first axioms state that addition and multiplication are associative and commutative and satisfy the distributive law, and furthermore that 0 and 1 are neutral elements for the respective operations and 0 is a zero divisor:

#### Axioms A1-A7:

- **A1:**  $(x + y) + z = x + (y + z)$
- **A2:**  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- **A3:**  $x + y = y + x$
- **A4:**  $x \cdot y = y \cdot x$
- **A5:**  $x \cdot (y + z) = x \cdot y + x \cdot z$
- **A6:**  $x + 0 = x \wedge x \cdot 0 = 0$
- **A7:**  $x \cdot 1 = x$

Here we use the usual algebraic bracket conventions ( $\cdot$  binds more strongly than  $+$ ). For the  $<$ -relation, the laws of a linear order hold, which is compatible with addition and multiplication:

#### Axioms A8-A12:

- **A8:**  $\neg x < x$
- **A9:**  $x < y \wedge y < z \rightarrow x < z$
- **A10:**  $x < y \vee x = y \vee y < x$
- **A11:**  $x < y \rightarrow x + z < y + z$
- **A12:**  $0 < z \wedge x < y \rightarrow x \cdot z < y \cdot z$

A number can be subtracted from a larger one:

#### Axiom A13:

- **A13:**  $x < y \rightarrow \exists z (x + z = y)$

And finally, 1 is the successor of 0 and 0 is the smallest element (where as usual  $x \leq y : \iff x < y \vee x = y$ ):

#### Axioms A14-A15:

- **A14:**  $0 < 1 \wedge \forall x (0 < x \rightarrow 1 \leq x)$
- **A15:**  $\forall x (0 \leq x)$

From A14 it follows with A11 that more generally  $x + 1$  is the successor of  $x$ , and thus the order is *discrete*:

$$x < x + 1 \wedge \forall y (x < y \rightarrow x + 1 \leq y).$$

### Examples

1. In Peano Arithmetic PA, one can define the  $<$ -relation by  $x < y \iff \exists z (x + z + 1 = y)$  and thus obtain all axioms of  $\text{PA}^-$ . In particular, the standard model  $\mathbb{N}$  with the usual  $<$ -relation, the usual operations, and the natural numbers 0,1 is a model of  $\text{PA}^-$ .

2. The set  $\mathbb{Z}[X]$  of polynomials in one variable  $X$  with integer coefficients is a commutative ring with the usual operations. One can order this ring by setting for a polynomial  $p = a_n X^n + \dots a_1 X + a_0$  with leading coefficient  $a_n \neq 0$ :

$$a_n X^n + \dots a_1 X + a_0 > 0 : \iff a_n > 0$$

and thus ordering polynomials  $p, q \in \mathbb{Z}[X]$  by  $p < q : \iff q - p > 0$ . The subset  $\mathbb{Z}[X]^+$  of polynomials  $p \in \mathbb{Z}[X]$  with  $p \geq 0$  then becomes a model of  $\text{PA}^-$ , in which the polynomial  $X$  is larger than all constant polynomials and thus “infinitely large”.

## Relation to Discretely Ordered Rings

In a ring, there is a group with respect to addition, while A13 only allows a restricted inverse formation. If one replaces axioms A13 and A15 in  $\text{PA}^-$  with the axiom

### Axiom A16:

- **A16:**  $\forall x \exists z (x + z = 0)$

one obtains the algebraic theory **DOR** of **discretely ordered rings**, whose models include, for example, the rings  $\mathbb{Z}$  and  $\mathbb{Z}[X]$ . Every model  $\mathcal{M}$  of  $\text{PA}^-$  can be extended to a model  $\mathcal{R}$  of the theory **DOR** (following the same pattern by which one extends the natural numbers to the ring of integers), such that the non-negative elements of  $\mathcal{R}$  coincide with the original model. Conversely, for every model  $\mathcal{R}$  of the theory **DOR**, the restriction to the non-negative elements is a model of  $\text{PA}^-$ , so that one can *describe*  $\text{PA}^-$  as the theory of (the non-negative part of) discretely ordered rings.

## End Extensions

The standard model can be embedded into every model  $\mathcal{M}$  of  $\text{PA}$  as an initial segment. It turns out this already holds for models of the theory  $\text{PA}^-$ .

### Definition 5

Let  $L$  be a language containing a 2-ary symbol  $<$ , and let  $\mathcal{M}$  and  $\mathcal{N}$  be  $L$ -structures with  $\mathcal{M} \subseteq \mathcal{N}$ . Then  $\mathcal{N}$  is called an **end extension** of  $\mathcal{M}$  (and correspondingly  $\mathcal{M}$  is an **initial segment** of  $\mathcal{N}$ ) if and only if the larger set  $N$  does not add any further elements below an element of  $M$ :

$$\mathcal{M} \subseteq_{\text{end}} \mathcal{N} : \iff \text{for all } x \in M, y \in N : (y <^N x \Rightarrow y \in M).$$

Each natural number  $n$  is represented in the standard model, which we also simply denote by  $\mathbb{N}$  here, by the constant term

$$\underline{n} = 1 + \dots + 1 \quad (n \text{ times})$$

where  $\underline{0}$  is the constant 0.

### Theorem 6

Let  $\mathcal{M} \models \text{PA}^-$ . Then the map

$$n \mapsto \underline{n}^{\mathcal{M}}$$

defines an embedding of the standard model  $\mathbb{N}$  onto an initial segment of  $\mathcal{M}$ .

In particular, every model of  $\text{PA}^-$  is isomorphic to an end extension of the standard model  $\mathbb{N}$ .\*

*Proof.* By simple induction (in the meta-theory), one shows for all natural numbers  $n, k, l$ :

$$\begin{aligned} n = k + l &\implies \text{PA}^- \vdash \underline{n} = \underline{k} + \underline{l} \\ n = k \cdot l &\implies \text{PA}^- \vdash \underline{n} = \underline{k} \cdot \underline{l} \\ n < k &\implies \text{PA}^- \vdash \underline{n} < \underline{k} \end{aligned}$$

and

$$\text{PA}^- \vdash \forall x (x \leq \underline{k} \rightarrow x = \underline{0} \vee \dots \vee x = \underline{k})$$

The first three statements will later be generalized to all recursive functions and relations; They imply that the map  $n \mapsto \underline{n}^{\mathcal{M}}$  is a homomorphism, and, due to the last statement, the map is also an embedding onto an initial segment of  $\mathcal{M}$ .  $\square$

**Remark**

The standard model has no proper initial segment, and  $\mathbb{Z}[X]^+$  has  $\mathbb{N}$  as its only proper initial segment. On the other hand, every model  $\mathcal{M} \models \mathbf{PA}^-$  has a proper end extension that is also a model of  $\mathbf{PA}^-$ , namely the non-negative part of the polynomial ring  $R[X]$ , where  $R$  is the discretely ordered ring associated with the model  $\mathcal{M}$ .