Math 557 Oct 3

Elementary Substructures

Key Concepts

- **Substrucure:** $\mathcal{M} \subseteq \mathcal{N}$ if $M \subseteq N$, $c^{\mathcal{M}} = c^{\mathcal{N}}$ for all $c \in \mathcal{L}$, $f^{\mathcal{M}} = f^{\mathcal{N}} \restriction_{\mathcal{M}}$ for all $f \in \mathcal{L}$, and $R^{\mathcal{M}} = R^{\mathcal{N}} \restriction_{\mathcal{M}}$ for all $R \in \mathcal{L}$.
- **Elementary substructure:** $\mathcal{M} \preceq \mathcal{N}$ if for all $\mathcal{L}$-formulas $\psi(\vec{x})$ and all $\vec{a} \in M$,

$$\mathcal{M} \vDash \psi[\vec{a}] \iff \mathcal{N} \vDash \psi[\vec{a}]$$

Problems

Exercise 0.1.
We have the following relations between structures $\mathcal{M}, \mathcal{N}$:

$$\subseteq,\ \preceq,\ \equiv,\ \cong$$

Draw a diagram indicating implications between these relations, giving counterexamples if one relation does not imply another.

Tarski-Vaught test

Exercise 0.2.
THM: *Suppose $\mathcal{M} \subseteq \mathcal{N}$ and that for any formula $\psi(x, \vec{y})$ and any $\vec{a} \in M$, if there exists $b \in N$ such that $\mathcal{N} \vDash \psi[b, \vec{a}]$, then there also exists $c \in M$ such that $\mathcal{N} \vDash \psi[c, \vec{a}]$. Then we have $\mathcal{M} \preceq \mathcal{N}$.*

Prove this theorem by induction in $\mathrm{ht}(\psi)$.

Before you start, which inductive case do you think will require the most work?

As an application of the Tarski-Vaught test, we get another criterion for $\preceq$ using automorphisms of the bigger structure.

Exercise 0.3.
Suppose $\mathcal{M} \subseteq \mathcal{N}$ and that for any finite subset $A \subseteq M$ and $b \in N$, there exists an automorphism of $\mathcal{N}$ that fixes A pointwise and maps b into M. Show that $\mathcal{M} \preceq \mathcal{N}$.

Exercise 0.4.
Use the previous criterion to show that

$$(\mathbb{Q}, <) \preceq (\mathbb{R}, <)$$

More on DLOs

We have seen previously that the theory DLO is $\aleph_0$-*categorical*, i.e., there is only one countable model up to isomorphism.

We will now see that this actually implies DLO is complete.

We need the following theorem which we will be an easy consequence of the Löwenheim-SKolem Theorems we will prove next week.

Theorem 0.1. *Let T be an $\mathcal{L}$-theory that has an infinite model. If κ is an infinite cardinal and $\kappa \geq |\mathcal{L}|$, then there is a model of T of cardinality κ.*

Exercise 0.5 (Vaught's test)**.** Suppose T is a consistent $\mathcal{L}$-theory with no finite models. If T is κ-categorical for some $\kappa \geq |\mathcal{L}|$, then T is complete.