

## Week 7 - Exercises

### Key concepts

#### (Ultra)filters

A **filter**  $\mathcal{F}$  on a set  $I$  is a nonempty collection of subsets of  $I$  satisfying:

1.  $\emptyset \notin \mathcal{F}$
2. If  $A, B \in \mathcal{F}$ , then  $A \cap B \in \mathcal{F}$
3. If  $A \in \mathcal{F}$  and  $A \subseteq B \subseteq I$ , then  $B \in \mathcal{F}$

An **ultrafilter**  $\mathcal{U}$  is a maximal filter, equivalently:

For all  $A \subseteq I$ , either  $A \in \mathcal{U}$  or  $I \setminus A \in \mathcal{U}$ .

#### Reduced products

Given: filter  $\mathcal{F}$  on  $I$  and structures  $(\mathcal{M}_i)_{i \in I}$ . Let  $M = \prod_{i \in I} M_i$  and define

$$a \sim_{\mathcal{F}} b \iff \{i \in I : a_i = b_i\} \in \mathcal{F}.$$

The universe of  $\mathcal{M}/\mathcal{F}$  is the quotient  $M/\sim_{\mathcal{F}}$ , with elements denoted  $a_{\mathcal{F}}$  (alternatively,  $a/\mathcal{F}$ ).

- **Relations:**

$$R^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) : \iff \{i \in I : \mathcal{M}_i \models R(\vec{a}_i)\} \in \mathcal{F}.$$

- **Functions:**

$$f^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) = [(f^{\mathcal{M}_i}(\vec{a}_i))_{i \in I}]_{\mathcal{F}}.$$

- **Constants:**

$$c^{\mathcal{M}/\mathcal{F}} = ((c^{\mathcal{M}_i})_{i \in I})_{\mathcal{F}}.$$

#### Łoś' Theorem

Let  $\mathcal{M}/\mathcal{U} = \prod_{i \in I} \mathcal{M}_i/\mathcal{U}$  be an ultraproduct. For every  $\mathcal{L}$ -formula  $\varphi(x_1, \dots, x_n)$  and tuples  $\vec{a} \in \prod_{i \in I} M_i$ ,

$$\mathcal{M}/\mathcal{U} \models \varphi[\vec{a}_{\mathcal{U}}] \iff \{i \in I : \mathcal{M}_i \models \varphi[\vec{a}_i]\} \in \mathcal{U}.$$

### Problems

**Exercise 0.1** (Principal filters). Show that for every  $A \subseteq I$ ,

$$\{X \subseteq I : A \subseteq X\}$$

is a filter on  $I$ . Show that it is an ultrafilter if and only if  $|A| = 1$ .

**Exercise 0.2** (Finite sets). Show that every ultrafilter on a finite set is principal.

**Exercise 0.3** (Free ultrafilters on infinite sets). Show that a free ultrafilter on an infinite set  $I$  cannot contain any finite subsets of  $I$ .

**Exercise 0.4** (Ultrafilter existence). Use Zorn's Lemma to show that any family  $\mathcal{A} \subseteq \mathcal{P}(I)$  with the FIP can be extended to an ultrafilter  $\mathcal{U}$  on  $I$ .

Use this to show that any infinite set has a free ultrafilter.

**Exercise 0.5** (Ultraproducts with principal ultrafilters). Show that if  $\mathcal{U}$  is a principal ultrafilter on  $I$ , then every ultraproduct

$$\mathcal{M}/\mathcal{U} = \prod_{i \in I} \mathcal{M}_i/\mathcal{U}$$

is isomorphic to some  $\mathcal{M}_j$ ,  $j \in I$ .

*For principal ultrafilters, ultraproducts do not lead to anything new.*

**Exercise 0.6** (Ultraproducts of fields). For any prime  $p$  let  $\mathbb{F}_p$  the field with  $p$  elements (of characteristic  $p$ ), and let  $\mathcal{U}$  be a free ultrafilter on the set  $P$  of prime numbers.

Show that the ultraproduct

$$\prod_{p \in P} \mathbb{F}_p/\mathcal{U}$$

is a field of characteristic 0.

## Ultrapowers

If we take an ultraproduct over the same structure  $\mathcal{M}$  along an index set  $I$ , we call this an **ultrapower**, denoted by

$$\mathcal{M}^I/\mathcal{U}$$

Let  $\mathcal{U}$  be a free ultrafilter on an (infinite) set  $I$ .

- Show that the map  $j : b \mapsto (b)_{i \in I}/\mathcal{U}$  defines an **elementary embedding** of  $\mathcal{M}$  into  $\mathcal{M}^I/\mathcal{U}$  (i.e. it is injective and the image is an elementary substructure).
- Show that if  $\mathcal{M}$  is infinite,  $j$  is not a surjection.

*If we apply this to  $\mathbb{N}$  in the language of arithmetic, this yields another way to obtain **non-standard models of arithmetic**.*

## Further explorations

### Lindenbaum-Tarski algebra

Given a language  $\mathcal{L}$ , define an equivalence relation on the set of all  $\mathcal{L}$ -sentences by

$$\sigma \sim \tau \iff \vdash \sigma \leftrightarrow \tau$$

The equivalence classes will then form a *Boolean algebra*, the **Lindenbaum-Tarski algebra**  $\mathcal{B}$  with the operations

$$\begin{aligned} [\sigma] \wedge [\tau] &:= [\sigma \wedge \tau] \\ [\sigma] \vee [\tau] &:= [\sigma \vee \tau] \\ \neg[\sigma] &:= [\neg\sigma] \end{aligned}$$

The *bottom* element is  $0 := [\sigma \wedge \neg\sigma]$ , the *top* element is  $1 := [\sigma \vee \neg\sigma]$ .

We can define an order by putting  $[\sigma] \leq [\tau] : \iff \vdash \sigma \rightarrow \tau$ . (This corresponds to the order that is defined on any Boolean algebra via  $a \leq b : \iff a = a \wedge b$ .)

A *filter* on a Boolean algebra is defined in the same way it is defined on a set  $I$  (in fact, a filter on a set  $I$  is simply a filter on the Boolean algebra induced on its power set by taking intersections, unions, and complements):  $F \subseteq \mathcal{B}$  is a filter if it does not contain the bottom element, is closed under  $\wedge$  and closed upward under  $\leq$ .

- Show that if  $T$  is a deductively closed, consistent  $\mathcal{L}$ -theory, the set

$$\{[\sigma] : \sigma \in T\}$$

is a filter in  $\mathcal{B}$ .

- Show that if  $T$  is complete, the above filter is an ultrafilter.

### Keisler-Shelah Theorem

From Łoś' Theorem, we know that if two ultrapowers are isomorphic, then the original structures are elementary equivalent. Remarkably, the converse holds, too, and hence gives an *algebraic characterization of elementary equivalence*.

**Theorem 0.1** (Keisler-Shelah). *For any two structures  $\mathcal{M}, \mathcal{N}$ ,*

$$\mathcal{M} \equiv \mathcal{N} \iff \text{there exists an index set } I \text{ and an ultrafilter } \mathcal{U} \text{ on } I \text{ such that } \mathcal{M}^I/\mathcal{U} \cong \mathcal{N}^I/\mathcal{U}$$

### Other interesting directions

- [Ramsey theory and ultrafilters](#)