

# Math 557 August 29

## Key Concepts

- **$\mathcal{L}$ -structures:** Provide a realm to interpret  $\mathcal{L}$ -formulas over.
  - non-empty universe  $M$ ,
  - interpretation of all constant, function, and relation symbols over  $M$ .
- **structure isomorphism:** bijection  $\pi$  between the universes of two  $\mathcal{L}$ -structures that preserves interpretations of symbols. General version of the well-known idea of isomorphism of mathematical structures.
- **assignment:** function  $\alpha : \mathcal{V} \rightarrow M$  that assigns all variables values in the universe of a structure. Recursively extends to terms.
- **satisfaction relation:**  $\mathcal{M} \models \varphi[\alpha]$  means the  $\mathcal{L}$ -formula  $\varphi$  is true when interpreted in the  $\mathcal{L}$ -structure  $\mathcal{M}$  and all variables are assigned values according to assignment  $\alpha$ . This is defined recursively over the structure of a formula.

## Problems

We warm up with a quote often attributed to Lincoln.

### Exercise 0.1.

You can fool all the people some of the time, and some of the people all the time, but you cannot fool all the people all the time.

Define a suitable language and formalize this statement in first-order logic.

Next, a couple of problems to see the definition of structures and the satisfaction relation  $\models$  at work.

### Exercise 0.2.

Let  $\mathcal{L} = \{\square, \triangle, c\}$ , where  $\square$  is a binary relation symbol,  $\triangle$  is a binary function symbol, and  $c$  is a constant symbol.

For each of the following formulas, find two  $\mathcal{L}$ -structures and (if necessary) assignments to the free variable(s) such that the formula holds in one structure but fails in the other.

$$\varphi \equiv \forall x \exists y \square \triangle xy c \tag{1}$$

$$\psi \equiv \exists x \forall y (x = c \wedge \square \triangle zc y) \tag{2}$$

Next, let's see some general properties of structures we can describe through first-order formulas.

### Exercise 0.3.

Let  $n$  be a natural number and let  $\mathcal{L}$  be any language. Find formulas  $\psi^{\geq}, \psi^{\leq}, \psi^=$  such that, for any  $\mathcal{L}$ -structure  $\mathcal{M}$ ,

$$\mathcal{M} \models \psi^{\geq} \iff |M| \geq n \tag{3}$$

$$\mathcal{M} \models \psi^{\leq} \iff |M| \leq n \tag{4}$$

$$\mathcal{M} \models \psi^= \iff |M| = n \tag{5}$$

$$\tag{6}$$

### **i** Question

Can we find a formula  $\theta$  such that

$$\mathcal{M} \models \theta \iff |M| \text{ is finite?}$$

We will answer this question in a few weeks...

### **Exercise 0.4.**

Let  $\mathcal{L} = \{\cdot, e\}$  be the language of groups. Show that there is a sentence  $\varphi$  such that  $\mathcal{M} \models \varphi$  if and only if

$$M \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

### **! Take-home Problems**

1. Let  $\mathcal{L}$  be any finite language and let  $\mathcal{M}$  be a finite  $\mathcal{L}$ -structure. Show that there is an  $\mathcal{L}$ -sentence  $\varphi$  such that

$$\mathcal{N} \models \varphi \iff \mathcal{N} \cong \mathcal{M}.$$

2. Give an example of a language  $\mathcal{L}$  and an  $\mathcal{L}$ -sentence  $\psi$  such that
  - there is at least one  $\mathcal{L}$ -structure  $A$  such that  $A \models \psi$ ,
  - for all  $L$ -structures  $A$ , if  $A \models \psi$ , then the universe  $A$  of  $A$  is infinite.