Arithmetical Formulas (Translated)

Bounded Quantifiers

For terms t and formulas φ in the language of PA^-, we write

$$\begin{aligned} \exists x < t\, \varphi \text{ for } \quad & \exists x(x < t \wedge \varphi) \\ \forall x < t\, \varphi \text{ for } \quad & \forall x(x < t \to \varphi) \end{aligned}$$

and call $\exists x < t$ and $\forall x < t$ **bounded quantifiers**.

Arithmetical Hierarchy

Definition

- φ is a Δ_0-formula: $\iff$ φ contains at most bounded quantifiers,
- φ is a Σ_1-formula: $\iff$ $\varphi = \exists \vec{x}\, \psi$ for a Δ_0-formula ψ,
- φ is a Π_1-formula: $\iff$ $\varphi = \forall \vec{x}\, \psi$ for a Δ_0-formula ψ.

This is the beginning of the **arithmetical hierarchy**. Setting

$$\Sigma_0 = \Pi_0 = \Delta_0,$$

we can continue:

- φ is a Σ_{n+1}-formula $\iff$ $\varphi = \exists \vec{x}\, \psi$ for a Π_n-formula ψ,
- φ is a Π_{n+1}-formula $\iff$ $\varphi = \forall \vec{x}\, \psi$ for a Σ_n-formula ψ.

Thus, a Σ_3-formula has the form $\exists \vec{x}\ \forall \vec{y}\ \exists \vec{z}\ \psi$, where ψ contains at most bounded quantifiers. This means that bounded quantifiers are not counted; Σ or Π indicates whether the formula begins with a (finite) sequence of $\exists$-quantifiers or $\forall$-quantifiers respectively, and the index counts the quantifier blocks. So it depends less on the number of quantifiers than on the number of quantifier alternations.

In this classification, we do not distinguish between logically equivalent formulas, so that every Π_n-formula for $n < m$ is also a Σ_m- and Π_m-formula (by simply prefixing the formula with additional quantifiers over variables that do not occur). Thus we can also define the formula sets

$$\Delta_n = \Sigma_n \cap \Pi_n.$$

This gives us the following picture of the **arithmetical hierarchy**:

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      Δ    Δ     Δ     Δ    ...


        Σ    Π    Σ   Π    Σ   Π
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Many useful properties of natural numbers can be expressed using Δ_0-formulas, e.g.:

$$x \text{ is irreducible } \iff 1 < x \wedge \forall u < x\ \forall v < x\ \neg(u \cdot v = x).$$

We will show that the recursive relations coincide with the sets that can be defined by Δ_1-formulas in the natural numbers, and that the graph of a recursive function can be defined by a Σ_1-formula in the natural numbers. We begin with the

Lemma on Δ_0-Formulas

Lemma

For every Δ_0-formula $\theta(\vec{v})$, the relation

$$R(\vec{a}) \iff \mathbb{N} \vDash \theta(\vec{a})$$

is primitive recursive.

Proof. We show by induction on $\mathrm{lz}(\theta)$ that the associated characteristic function

$$c_\theta(\vec{x}) = \begin{cases} 1 & \text{if } \mathbb{N} \vDash \theta(\vec{x}) \\ 0 & \text{otherwise} \end{cases}$$

is primitive recursive.

First, the functions $x+1, x+y, x \cdot y$ are p.r., and thus every term in $\mathbb{N}$ defines a primitive recursive function. Since the functions $\mathrm{eq}(x,y) = \overline{\mathrm{sg}}(|x-y|)$ and $\mathrm{sg}(y \dot{-} x)$ are also primitive recursive (and the p.r. functions are closed under composition), the claim holds for the atomic formulas $t = s, t < s$.

For the case of propositional operations, use

$$c_{\neg\theta}(\vec{x}) = 1 \dot{-} c_\theta(\vec{x}),\ c_{\theta\wedge\psi}(\vec{x}) = c_\theta(\vec{x}) \cdot c_\psi(\vec{x}),\ c_{\theta\vee\psi}(\vec{x}) = \min(c_\theta(\vec{x}), c_\psi(\vec{x})).$$

Finally, if θ is a Δ_0-formula, t is a term and $\psi(\vec{x}) = \forall y < t(\vec{x})\ \theta(\vec{x}, y)$, then the claim follows from

$$c_\psi(\vec{x}) = \mathrm{eq}(t(\vec{x}), (\mu y \leq t(\vec{x})(c_\theta(\vec{x}, y) = 0))).$$

One argues similarly in the case of the formula $\exists y < t(\vec{x})\ \theta(\vec{x}, y)$ (or reduces this case to the earlier one using negation). $\square$

The converse of the above lemma does not hold: there are primitive recursive sets that cannot be defined by any Δ_0-formula in the natural numbers.