MATH 557 Midterm 3 Preparation

The third midterm will again have two parts:

1. Reproduce a proof (in sufficient detail) of one the theorems below we covered in class.
2. Present a proof to one of the exercises (or a closely related problem) listed on this page.

Notations, Axioms

In the following, $\mathcal{L}_A = \{0, 1, +, \cdot, <\}$ denotes the language of PA^-.

The axioms of PA^- are:

- **A1:** $(x + y) + z = x + (y + z)$
- **A2:** $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- **A3:** $x + y = y + x$
- **A4:** $x \cdot y = y \cdot x$
- **A5:** $x \cdot (y + z) = x \cdot y + x \cdot z$
- **A6:** $x + 0 = x \wedge x \cdot 0 = 0$
- **A7:** $x \cdot 1 = x$
- **A8:** $\neg\, x < x$
- **A9:** $x < y \wedge y < z \rightarrow x < z$
- **A10:** $x < y \vee x = y \vee y < x$
- **A11:** $x < y \rightarrow x + z < y + z$
- **A12:** $0 < z \wedge x < y \rightarrow x \cdot z < y \cdot z$
- **A13:** $x < y \rightarrow \exists z\, (x + z = y)$
- **A14:** $0 < 1 \wedge \forall x\, (0 < x \rightarrow 1 \leq x)$
- **A15:** $\forall x\, (0 \leq x)$

If we add the **induction scheme**

- **Ind:** $(\varphi(0, \vec{y}) \,\wedge\, \forall x(\varphi(x, \vec{y}) \rightarrow \varphi(x + 1, \vec{y})) \rightarrow \forall x\, \varphi(x, \vec{y})$

we obtain (a theory equivalent to) **full** PA.

Theorems

Theorem 1

For every Δ_0-formula $\theta(\vec{v})$, the relation

$$R(\vec{a}) :\iff \mathbb{N} \models \theta(\vec{a})$$

is primitive recursive.

Theorem 2

Let $\mathcal{N}, \mathcal{M}$ be $\mathcal{L}_A$-structures with $\mathcal{N} \subseteq_{end} \mathcal{M}$, and let $\vec{a} \in N$. Then for every Δ_0-formula $\varphi(\vec{v})$,

$$\mathcal{N} \models \varphi[\vec{a}] \iff \mathcal{M} \models \varphi[\vec{a}],$$

Theorem 3

If $f : \mathbb{N} \to \mathbb{N}$ is recursive, then there exists a Σ_1-formula $\theta(x, y)$ such that for all $m, n \in \mathbb{N}$,

$$f(n) = m \quad \Rightarrow \quad \mathsf{PA}^- \vdash \theta(\underline{n}, \underline{m})$$

(*This is one part of the Representability Theorem.*)

Theorem 4

Let T be a recursive set of (Gödel numbers of) $\mathcal{L}_A$-sentences such that:

1. T is consistent, i.e., there is no L-sentence σ with $\ulcorner \sigma \urcorner \in T$ and at the same time $\ulcorner \neg\sigma \urcorner \in T$,
2. T contains the deductive closure of PA^-, $\ulcorner (\mathsf{PA}^-)^{\vdash} \urcorner \subseteq T$.

Then T is incomplete, i.e., there exists a sentence τ with

$$\ulcorner \tau \urcorner \notin T \text{ and } \ulcorner \neg\tau \urcorner \notin T.$$

(*You can use the* Representability Theorem *as well as the* Diagonal Lemma.)

Theorem 5

If T is a consistent theory in the language $\mathcal{L}_A$, then not both the diagonal function d and the set $\ulcorner T^{\vdash} \urcorner$ are representable in T.

Problems

Problem 1

Show that the functions

$$\mathrm{rem}(x, y) = \text{remainder when } y \text{ is divided by } x$$

(put $\mathrm{rem}(0, y) = y$ to make it total) and

$$\mathrm{qt}(x, y) = \text{quotient when } y \text{ is divided by } x$$

(put $\mathrm{qt}(0, y) = 0$) are primitive recursive.
In other words, show that the uniquely determined functions (for $x \geq 1$) satisfying

$$y = \mathrm{qt}(x, y) \cdot x + \mathrm{rem}(x, y) \qquad 0 \leq \mathrm{rem}(x, y) < x$$

are primitive recursive.
You can use that the functions $x+y$, $x \cdot y$, $\max(x, y)$, $\min(x, y)$, $|x-y|$, $x \dot{-} y$, $\mathrm{sg}(x)$, $\overline{\mathrm{sg}}(x)$ are primitive recursive.

Problem 2

Show that for all $k \in \mathbb{N}$,

$$\mathsf{PA}^- \vdash \forall x \, (x \leq \underline{k} \to (x = \underline{0} \ \vee \ldots \vee \ x = \underline{k})$$

(*Hint: use (meta-)induction on k. For the inductive step, show first that*

$$\mathsf{PA}^- \vdash \forall x, y \, (y > x \to y \geq x + 1)$$

Problem 3

Let $\mathcal{M} \models \mathsf{PA}$ be non-standard. A *proper cut* in $\mathcal{M}$ is a set $I \subsetneq M$ that is closed downward under $<$ (the order of $\mathcal{M}$) and closed under successor. For example, the (copy of the) standard model $\mathbb{N}$ in $\mathcal{M}$ is a proper cut.

1. Show that if $\vec{a} \in M$ and $\mathcal{M} \models \varphi(b, \vec{a})$ for all $b \in I$, then there is $c > I$ in M such that $\mathcal{M} \models \forall x \leq c\, \varphi(x, \vec{a})$.
2. Suppose $\mathcal{M} \models \mathsf{PA}^-$ is non-standard and has the property that for any proper cut I, if $\varphi(x, y)$ is a formula and $\vec{a} \in M$ such that
$$\mathcal{M} \models \varphi(b, \vec{a}) \quad \text{for all } b \in I,$$
then there is $c > I$ in M such that $\mathcal{M} \models \forall x \leq c\, \varphi(x, \vec{a})$ (in other words, $\mathcal{M}$ satisfies the conclusion of 1.). Show that then $\mathcal{M} \models \mathsf{PA}$.

(*Note: For (2.), it suffices to show $\mathcal{M}$ satisfies the induction scheme.*)

Problem 4

Show that, for all functions $f : \mathbb{N}^k \to \mathbb{N}$ (where $k \geq 1$), if f is representable in PA^- then f is computable.
(*You can argue by Church-Turing Thesis.*)

Problem 5

(a) We say an $\mathcal{L}_A$-theory T' is a *finite extension* of an $\mathcal{L}_A$-theory T if $T' \supseteq T$ and $T' \setminus T$ is finite. Show that if T is decidable and T' is a finite extension of T, then T' is decidable.
(b) An $\mathcal{L}_A$-structure $\mathcal{A}$ is *strongly undecidable* if every $\mathcal{L}_A$-theory T with $\mathcal{A} \models T$ is undecidable. Show that the standard model $\mathbb{N}$ is strongly undecidable.