Math 557 Oct 6

Löwenheim-Skolem Theorems

Exercise 0.1. For finite structures $\mathcal{M}, \mathcal{N}$, $\mathcal{M} \preceq \mathcal{N}$ implies $M = N$

Hence, for finite structures, proper elementary substructures cannot exist. In contrast, infinite structures have an elementary substructure in every smaller infinite cardinality κ (as long as $\kappa \geq \operatorname{card}(\mathcal{L})$):

Down

Theorem 0.1 (Löwenheim-Skolem downward)**.** *Let $\mathcal{A}$ be an $\mathcal{L}$-structure, κ an infinite cardinal with* $\operatorname{card}(\mathcal{L}) \leq \kappa$ *and* $\kappa \leq \operatorname{card}(A)$. *Then there exists a structure $\mathcal{B}$ with*

$$\mathcal{B} \preceq \mathcal{A}\ ,\ \operatorname{card}(B) = \kappa.$$

Addition: If $A_0 \subseteq A$ is arbitrary with $\operatorname{card}(A_0) \leq \kappa$, then one can additionally require $A_0 \subseteq B$.

Proof. Let $A_0 \subseteq A$ be given with $\operatorname{card}(A_0) \leq \kappa \leq \operatorname{card}(A)$. By enlarging A_0 if necessary, we can assume that $\operatorname{card}(A_0) = \kappa$.

If for elements $a_1, \dots, a_n \in A_0$

$$(*) \quad \mathcal{A} \vDash \exists v_0 \varphi[a_1, \dots, a_n]$$

holds, we have to add a witness b for this existential quantifier to A_0; let A_1 be the set that arises from A_0 by adding such witnesses (for all possible existential formulas and all possible assignments with elements from A_0). Standard cardinal arithmetic yields $\operatorname{card}(A_1) = \operatorname{card}(A_0) = \kappa$.

Now, however, $(*)$ may possibly hold for a formula φ and new elements $a_1, \dots, a_n \in A_1$ that are not contained in A_0; thus, we have to iterate the procedure.

With A_i defined, add suitable elements from A, resulting in a set A_{i+1}, such that:

$$\begin{aligned}&\text{If for } \ a_1, \dots, a_n \in A_i : \mathcal{A} \vDash \exists v_0 \varphi[a_1, \dots, a_n],\\ &\text{then there exists an } a \in A_{i+1} \text{ with } \mathcal{A} \vDash \varphi[a, a_1, \dots, a_n].\end{aligned}$$

As in the first step, A_{i+1} can be obtained from A_i without changing the cardinality. Finally, we set

$$B := \bigcup_{i \in \mathbb{N}} A_i$$

and obtain a set B with $A_0 \subseteq B \subseteq A$ and $\operatorname{card}(B) = \kappa$.

In the first step, we already add all constants of $\mathcal{A}$ (using $(*)$ with the formula $\exists v_0(v_0 = c)$) and in all further steps we are closing under the functions of $\mathcal{A}$ (using the formula $\exists v_0(v_0 = f(v_1, \dots, v_n))$). It follows that B is the universe of a substructure $\mathcal{B}$ of $\mathcal{A}$. We can then conclude that $\mathcal{B} \preceq \mathcal{A}$ using the Tarski-Vaught test, as our construction is arranged precisely so that the Tarski-Vaught criterion is applicable. □

Up

The proof of the upward version is simpler and is based on the Compactness Theorem.

Theorem 0.2 (Löwenheim-Skolem upward)**.** *Let $\mathcal{A}$ be an infinite $\mathcal{L}$-structure, κ a cardinal with* $\operatorname{card}(\mathcal{L}) \leq \kappa$ *and* $\operatorname{card}(A) \leq \kappa$*. Then there exists a structure $\mathcal{B}$ with**

$$\mathcal{A} \preceq \mathcal{B}\ ,\ \operatorname{card}(B) = \kappa.$$

Proof. We first pick a set C with $A \subseteq C$ and $\operatorname{card}(C) = \kappa$ and extend the theory of $\mathcal{A}$ (with the help of new constants) so that every model has at least as many elements as C:

$$T' = \operatorname{Th}(\mathcal{A}) \cup \{\underline{c} \neq \underline{d} \mid c, d \in C, c \neq d\}.$$

By the Compactness Theorem, T' has a model, say $\mathcal{B}$, in which the new constants $\underline{c}$ are interpreted by elements of B – different constants by different elements of B. By passing to an isomorphic structure, we can assume that $\underline{c}^{\mathcal{B}} = c$ and thus $C \subseteq B$, so $\operatorname{card}(B) \geq \kappa$.

The language of T' has cardinality κ because $\kappa \geq \operatorname{card}(\mathcal{L})$, so we can also assume that $\operatorname{card}(B) = \kappa$ (by using the downward theorem).

Finally, because $\mathcal{B} \vDash \operatorname{Th}(\mathcal{A})$, we have $\mathcal{A} \equiv \mathcal{B}$. The stronger statement $\mathcal{A} \preceq \mathcal{B}$ is obtained by using the same argument, but using the elementary diagram $D(\mathcal{A}) = \operatorname{Th}(\mathcal{A}_A)$ instead of $\operatorname{Th}(\mathcal{A})$. $\square$

Some consequences

1. If $\mathcal{A}$ is an infinite $\mathcal{L}$-structure and $\kappa \geq \operatorname{card}(\mathcal{L})$ is a infinite cardinal, then there exists a structure $\mathcal{B}$ with $\operatorname{card}(B) = \kappa$ and
   - $\mathcal{B} \preceq \mathcal{A}$ in the case $\kappa \leq \operatorname{card}(A)$,
   - $\mathcal{A} \preceq \mathcal{B}$ in the case $\operatorname{card}(A) \leq \kappa$.
2. In particular, every theory T that has an infinite model has a model of cardinality κ for every cardinal $\kappa \geq \operatorname{card}(\mathcal{L})$.
3. More specifically: A theory T in a countable language $\mathcal{L}$ that has a model at all also has a countable model. This theorem of Löwenheim (1915) is one of the earliest results of mathematical logic.

The reals as a complete ordered field

Consider the structure $(\mathbb{R}, 0, 1, +, \cdot, <)$ over the language of ordered rings. By Löwenheim-Skolem downward, this has a countable elementary substructure $\mathcal{R}'$. $\mathcal{R}'$ is a field, so it has to contain $\mathbb{Q}$, and since it inherits the order from $\mathbb{R}$, it has to be dense in $\mathbb{R}$. Since $\mathcal{R}'$ is countable, the exists $r_0 \in \mathbb{R} \setminus R'$. The set

$$\{r \in R' : r < r_0\}$$

is bounded in $\mathcal{R}'$ but cannot have a least upper bound in $\mathcal{R}'$.

As $\mathcal{R}' \vDash \operatorname{Th}(\mathbb{R}, 0, 1, +, \cdot, <)$, it follows that the theory of complete *ordered fields is not first-order axiomatizable in the language of ordered rings.*

It can be shown that the *algebraic numbers* $\mathbb{R}_{\text{alg}}$ form such a countable elementary substructure of $\mathbb{R}$.