

## Consistency and Undecidability

### The Second Incompleteness Theorem

As we saw previously, for a recursively axiomatizable theory  $T$ , the relation

$$\text{Prf}(x, y) : \iff x = \ulcorner \psi \urcorner, y = \langle \ulcorner \varphi_1 \urcorner, \dots, \ulcorner \varphi_n \urcorner \rangle, \varphi_1, \dots, \varphi_n \text{ is a } T\text{-proof of } \psi$$

is recursive. It follows that for such  $T$ , the relation

$$\begin{aligned} \text{prov}_T(v) &:\Leftrightarrow "v \text{ is the Gödel number of a formula } \varphi \text{ and } \varphi \text{ is provable in } T" \\ &\Leftrightarrow \exists y \text{ Prf}(x, y) \end{aligned}$$

is r.e. and thus definable by a  $\Sigma_1$ -formula  $\theta(v)$ .

In  $\text{PA}^1$  one can verify the following **derivability conditions**:

- (D1)  $T \vdash \sigma \Rightarrow T \vdash \theta(\ulcorner \sigma \urcorner)$
- (D2)  $T \vdash \theta(\ulcorner \sigma \rightarrow \tau \urcorner) \rightarrow (\theta(\ulcorner \sigma \urcorner) \rightarrow \theta(\ulcorner \tau \urcorner))$
- (D3)  $T \vdash \theta(\ulcorner \sigma \urcorner) \rightarrow \theta(\ulcorner \theta(\ulcorner \sigma \urcorner) \urcorner)$

Here  $\sigma, \tau$  are arbitrary sentences of the language  $\mathcal{L}_A$ , and the Gödel numbers  $n$  should be replaced in these formulas by the corresponding terms  $\underline{n}$ .

Using such a proof predicate, one can also formalize the **consistency** of a theory, for example:

$$\text{Con}_T \iff \neg \theta(\ulcorner 0 = 1 \urcorner).$$

**Theorem 0.1** (Second Gödel Incompleteness Theorem).

Let  $T$  be a theory in the language  $L$  for which there exists a formal proof predicate  $\theta$  as defined above (e.g.,  $T = \text{PA}$ ). Then:

$$T \text{ consistent} \implies T \not\vdash \text{Con}_T.$$

*Sketch.* By the Diagonalization Lemma, there exists a sentence  $\tau$  such that

$$T \vdash \tau \Leftrightarrow \neg \theta(\ulcorner \tau \urcorner) \tag{1}$$

If  $T \vdash \tau$ , it follows from (D1) that

$$T \vdash \theta(\ulcorner \tau \urcorner)$$

By (1), this implies  $T$  is inconsistent. This yields

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<sup>1</sup> $\text{PA}^-$  is not strong enough for this

$$T \text{ consistent} \Rightarrow T \not\vdash \tau. \quad (2)$$

One can show that this proof can be formalized and represented in  $T$  using the proof predicate  $\theta$ , thus

$$T \vdash \text{Con}_T \rightarrow \neg\theta(\underline{\tau}). \quad (3)$$

If  $T$  is consistent, by (2),  $T \not\vdash \tau$ . Using (1), this implies  $\neg\theta(\underline{\tau})$ . Thus, by (3),  $T \not\vdash \text{Con}_T$ .  $\square$

## Truth is Not Definable

While the usual syntactic concepts formed for a formal language  $\mathcal{L}$  can be defined in the language of number theory and their essential properties can be proven in  $\text{PA}$ , this is not possible for the semantic notion of truth:

**Theorem 0.2** (Tarski's Undefinability Theorem).

*Let  $\mathcal{M} \models \text{PA}^-$ . Then there is no  $\mathcal{L}_A$ -formula  $\theta(v)$  such that for all natural numbers  $n$*

$$\mathcal{M} \models \theta(\underline{n}) \iff n = {}^\frown \sigma^\frown \text{ for an sentence } \sigma \text{ with } \mathcal{M} \models \sigma.$$

*Proof.* If there were such a truth definition  $\theta$ , then by the Diagonal Lemma we could find a sentence  $G$  such that

$$\text{PA}^- \vdash G \leftrightarrow \neg\theta(\underline{G}),$$

in particular,

$$\mathcal{M} \models G \iff \mathcal{M} \models \neg\theta(\underline{G}).$$

On the other hand, for the truth definition we would need:

$$\mathcal{M} \models G \iff \mathcal{M} \models \theta(\underline{G}),$$

which yields a contradiction.  $\square$

## Undecidability

**Theorem 0.3** (Theorem).

*If  $T$  is a consistent theory in the language  $\mathcal{L}_A$ , then not both the diagonal function  $d$  and the set  ${}^\frown T^{\perp\perp}$  are representable in  $T$ .*

*Proof.* We assume that both  $d$  is represented by a formula  $\delta(x, y)$  and the set  ${}^\frown T^{\perp\perp}$  is represented by a formula  $\varphi_T(v_0)$  in  $T$ , and we choose  $\theta = \neg\varphi_T$ , so that from the Diagonal Lemma we obtain the existence of a formula  $G$  such that

$$T \vdash G \leftrightarrow \neg\varphi_T(\underline{G}). \quad (4)$$

**Case 1:**  $T \vdash G$ , thus  ${}^\frown G^\frown \in {}^\frown T^{\perp\perp}$ . Then by representability,  $T \vdash \varphi_T(\underline{G})$  and with (4) also  $T \vdash \neg G$ , i.e.,  $T$  would be inconsistent.

**Case 2:**  $T \not\vdash G$ , thus  ${}^\frown G^\frown \notin {}^\frown T^{\perp\perp}$ . Then again by representability,  $T \vdash \neg\varphi_T(\underline{G})$ , and with (4)  $T \vdash G$ , contradiction!  $\square$

**Corollary 0.1** (Church).

*If  $T$  is a consistent theory in the language  $\mathcal{L}_A$  which extends  $\text{PA}^-$ , then  $T$  is undecidable.*

*Proof.* If  $T$  extends  $\text{PA}^-$ , every recursive function, including  $d$ , is representable in  $T$ . Hence, by Theorem 0.3,  ${}^\frown T^{\perp\perp}$  is not representable. It follows that  ${}^\frown T^{\perp\perp}$  is not recursive (since all recursive sets are representable in any consistent extension of  $\text{PA}^-$ ).  $\square$

**Corollary 0.2** (Church).

The set

$$\text{VAL} := \{\vdash \sigma : \sigma \text{ } \mathcal{L}_A\text{-sentence with } \vdash \sigma\}$$

(i.e. the set of all sentences that are validities) is not recursive.

**Exercise 0.1.** We say an  $\mathcal{L}_A$ -theory  $T'$  is a *finite extension* of an  $\mathcal{L}_A$ -theory  $T$  if  $T' \supseteq T$  and  $T' \setminus T$  is finite. Show that if  $T$  is decidable and  $T'$  is a finite extension of  $T$ , then  $T'$  is decidable.

**Corollary 0.2.** Since  $\text{PA}^-$  is finite, it is a finite extension of the empty theory  $T = \emptyset$ . Since  $\text{VAL} = \emptyset^+$ , by Exercise 0.1, if  $\text{VAL}$  were recursive so would be  $(\text{PA}^-)^+$ , contradicting Corollary 0.1.  $\square$