

Math 557 Sep 3

Validities

Key Concepts

- **Validity:** An \mathcal{L} -formula ψ that is valid in *any* \mathcal{L} -structure under *any* assignment α , i.e.

$$\mathcal{M} \models \psi[\alpha] \quad \text{for any } \mathcal{M}, \alpha.$$

- **Sources of validities:**
 - *Propositional tautologies* – e.g. formulas of the form $\psi \rightarrow \psi$ or $\psi \vee \neg\psi$
 - *Equality* – the way $=$ is interpreted implies formulas like $\forall x \, x = x$ are validities.
 - *Quantifiers*
- **Fundamental questions:**
 - Can we characterize the set of validities *syntactically*?
 - In particular, is there an algorithm that, on input φ , determines whether φ is a validity or not?

Problems

Exercise 0.1 (Warmup).

Determine whether the following formulas are validities.

- $((\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_3)) \rightarrow ((\varphi_1 \rightarrow \varphi_2) \rightarrow (\varphi_1 \rightarrow \varphi_3)))$
- $\forall x(\varphi \wedge \psi) \leftrightarrow (\forall x\varphi \wedge \forall y\psi)$

Exercise 0.2.

Let $g : \{0, 1\}^n \rightarrow \{0, 1\}$ be a Boolean function. Show that there exists a propositional formula $F(p_1, \dots, p_n)$ such that, for any assignment δ to the variables p_1, \dots, p_n ,

$$\delta^*(F) = g(\delta(p_1), \dots, \delta(p_n)).$$

Exercise 0.3.

Formally verify that every propositional tautology is a validity.

Exercise 0.4.

Verify, using the definition of the satisfaction relation \models , that

$$\forall x, y (x = y \rightarrow f(x) = f(y)) \quad (f \text{ unary function symbol})$$

and

$$\forall x (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x \psi) \quad (x \text{ not free in } \varphi)$$

are validities.