Math 557 Nov 10

Gödel Numbers

We can code formulas of the language of PA^- as natural numbers, the so-called *Gödel numbers*. They let us express syntactical properties of arithmetic formulas as number-theoretic properties in the language of arithmetic. This will be important for the self-referential argument at the heart of the first incompleteness theorem. **Below, "term" and "formula" always refer to the language** $\mathcal{L} = \{0, 1, +, *, <\}$.

Gödel numbers of terms

Using primitive recursion, we assign every term t a Gödel-Nummer $\ulcorner t \urcorner$:

$$\begin{aligned}
\ulcorner 0 \urcorner &= 1 \\
\ulcorner 1 \urcorner &= 3 \\
\ulcorner v_i \urcorner &= 3^2 \cdot 5^i \\
\ulcorner s + t \urcorner &= 3^3 \cdot 5^{\ulcorner s \urcorner} \cdot 7^{\ulcorner t \urcorner} \\
\ulcorner s \cdot t \urcorner &= 3^4 \cdot 5^{\ulcorner s \urcorner} \cdot 7^{\ulcorner t \urcorner}
\end{aligned}$$

Example 0.1.

$$\ulcorner v_3 + 1 \urcorner = 3^3 \cdot 5^{3^2 \cdot 5^3} \cdot 7^3$$

Gödel numbers of formulas

Using the Gödel numbers of terms, we can again recursively assign each formula φ a Gödel number $\ulcorner \varphi \urcorner$:

$$\begin{aligned}
\ulcorner s = t \urcorner &= 2 \cdot 5^{\ulcorner s \urcorner} \cdot 7^{\ulcorner t \urcorner} \\
\ulcorner s < t \urcorner &= 2 \cdot 3 \cdot 5^{\ulcorner s \urcorner} \cdot 7^{\ulcorner t \urcorner} \\
\ulcorner \neg \varphi \urcorner &= 2 \cdot 3^2 \cdot 5^{\ulcorner s \urcorner} \\
\ulcorner \varphi \wedge \psi \urcorner &= 2 \cdot 3^3 \cdot 5^{\ulcorner \varphi \urcorner} \cdot 7^{\ulcorner \psi \urcorner} \\
\ulcorner \exists v_i \varphi \urcorner &= 2 \cdot 3^4 \cdot 5^i \cdot 7^{\ulcorner \varphi \urcorner}
\end{aligned}$$

The following is straightforward based on the simple recursive definition of $\ulcorner . \urcorner$

Proposition 0.1.
The following sets are primitive recursive:

- $\{m : m \text{ is the Gödel number of a term } t\}$
- $\{n : n \text{ is the Gödel number of a formula } \varphi\}$

We can also effectively check for other syntactic properties. For example, the set

$$\{k : k \text{ is the Gödel number of a sentence } \sigma\}$$

is computable.

Definition 0.1.
A theory T is **decidable** if the set

$$\{\ulcorner \sigma \urcorner : T \vdash \sigma\}$$

is recursive.