

Henkin Theories

Key Concepts

- An \mathcal{L} -theory T is a **Henkin theory** if for every sentence of the form $\exists x\psi$ there exists a constant \mathcal{L} -term t such that

$$T \vdash_{\mathcal{L}} \exists x\psi \rightarrow \psi_{t/x}$$

- **Questions:**
 - What are Henkin theories needed for?
 - Do they exist?
 - Can we extend a given consistent theory to a Henkin theory that is still consistent?

Problems

Exercise 0.1.

As before, let \mathcal{A}_T be the term model of T . Pinpoint exactly where we might run into difficulties when we are trying to prove

$$\mathcal{A}_T \models \exists x\psi \iff T \vdash_{\mathcal{L}} \exists x\psi$$

Exercise 0.2 (Interlude).

What if in $\exists x\psi$ the variable x is not free? Does that make a difference regarding being a Henkin theory?

Exercise 0.3.

Can you find a couple of examples of Henkin theories?

The basic construction step for extending an \mathcal{L} -theory T to a Henkin theory is for every sentence of the form $\sigma \equiv \exists x\psi$,

- add a new constant symbol c_σ to \mathcal{L}
- add the formula $\exists x\psi \rightarrow \psi_{c_\sigma/x}$ to T .

Exercise 0.4.

Why could the addition of a constant theoretically lead to T being inconsistent, even though T itself remains unchanged?

And why does T actually remain consistent?

The Henkin extension T_H of T is obtained by an iterative process:

Single iteration step:

- $\mathcal{L}' = \mathcal{L} \cup \{c_\sigma : \sigma \text{ } \mathcal{L}\text{-sentence of the form } \exists x\psi\}$
- $\Gamma = \{\exists x\psi(x) \rightarrow \psi_{c_\sigma/x} : \sigma \text{ } \mathcal{L}\text{-sentence of the form } \exists x\psi\}$
- $T' = T \cup \Gamma$ (an \mathcal{L}' -theory)

Exercise 0.5.

Why is T' above not necessarily a Henkin theory?

Iteration process:

- $\mathcal{L}_0 = \mathcal{L}$, $T_0 = T$

- $\mathcal{L}_{n+1} = \mathcal{L}'_n, T_{n+1} = T'_n$
- $\mathcal{L}_H = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n, T_H = \bigcup_{n \in \mathbb{N}} T_n$

Exercise 0.6.

Why is T_H a Henkin theory?

T_H is no longer a purely symbolic extension of T (in the sense that we simply extend the language), since we add new sentences to T (the sentences in Γ for each iteration step).

! Take-home Problem

Show that for every \mathcal{L} -formula φ ,

$$T_H \vdash_{\mathcal{L}_H} \varphi \iff T \vdash_{\mathcal{L}} \varphi$$