

# Math 557 Sep 17

## Completing Theories

### Key Concepts

- We previously defined the *term model*  $\mathcal{A}$ . It holds that for any *atomic* sentence  $\sigma$ ,

$$\mathcal{A} \models \sigma \iff T \vdash \sigma$$

- Trying to extend this to arbitrary sentence via induction, the *negation case* looks like

$$\mathcal{A} \models \neg \sigma \iff \mathcal{A} \not\models \sigma \stackrel{\text{ind hyp}}{\iff} T \not\vdash \sigma$$

We would like to show that this is equivalent to  $T \vdash \neg \sigma$ .

- One direction follows from  $T$  being consistent, but for the other direction,  $T$  may not be strong enough to prove this.
- We therefore need to *extend*  $T$  to a complete theory.

### Problems

#### Exercise 0.1.

Verify that indeed for all atomic sentences  $\sigma$ ,

$$\mathcal{A} \models \sigma \iff T \vdash \sigma$$

#### Exercise 0.2.

Recall that a theory  $T$  is *maximally consistent* if it is consistent but does not have any consistent proper extensions.  $T$  is called *deductively closed* if the deductive closure of  $T$ ,

$$T^+ = \{\sigma : T \vdash \sigma\}$$

is equal to  $T$ .

- Show that a maximally consistent theory is complete and deductively closed.
- Show that if  $T$  is complete, then  $T^+$  is maximally consistent.

#### Exercise 0.3.

Is every consistent, deductively closed theory complete?

#### Exercise 0.4.

Show that the union of an increasing sequence of consistent theories is consistent.

#### Exercise 0.5.

Extend Lindenbaum's theorem on the existence of maximally consistent extension from countable to arbitrary languages.

#### Exercise 0.6.

Fix a language  $\mathcal{L}$ . Let  $X$  be the set of all maximally consistent  $\mathcal{L}$ -theories. For an  $\mathcal{L}$ -sentence  $\sigma$ , let

$$\langle \sigma \rangle = \{T \in X : \sigma \in T\}$$

Show that

- $\langle \sigma \wedge \tau \rangle = \langle \sigma \rangle \cap \langle \tau \rangle$
- $\langle \neg \sigma \rangle = X \setminus \langle \sigma \rangle$

**Exercise 0.7.**

Continuing the previous exercise, let  $\mathcal{O}$  be the topology generated by the sets  $\langle \sigma \rangle$ . Show that

- each  $\langle \sigma \rangle$  is clopen,
- the  $\langle \sigma \rangle$  form a basis for  $\mathcal{O}$ ,
- $\mathcal{O}$  is Hausdorff.