Math 557 Oct 8

Ultraproducts

Direct Products

Let $(\mathcal{M}_i)_{i\in I}$ be a family of L-structures.

We define the direct product

$$\mathcal{M} = \prod_{i \in I} \mathcal{M}_i$$

as follows:

- 1. The universe is the Cartesian product $M = \prod_{i \in I} M_i$. If a is an element of M, we denote its i-th component (an element of M_i) by a_i and extend this notation to vectors: if \vec{a} is a finite tuple in M^n , \vec{a}_i denotes the n-tuple in M_i consisting of the M_i -entries of \vec{a} .
- 2. For each relation symbol $R \in \mathcal{L}$,

$$R^{\mathcal{M}}(\vec{a}) : \iff \forall i \in I, \, \vec{a}_i \in R^{\mathcal{M}_i}$$

3. For each function symbol $f \in \mathcal{L}$,

$$f^{\mathcal{M}}(\vec{a}) := (f^{\mathcal{M}_i}(\vec{a}_i))_{i \in I}.$$

4. For each constant $c \in \mathcal{L}$,

$$c^{\mathcal{M}} = (c^{\mathcal{M}_i})_{i \in I}.$$

Examples and Observations

- The direct product of groups is again a group (componentwise operation).
- The direct product of fields is **not** a field:

$$(1,0) \cdot (0,1) = (0,0).$$

• The direct product of linear orders is only a **partial order**.

We often want to preserve properties that hold in "most" component structures. To formalize "most," we use **filters** on I.

Filters and Ultrafilters

A filter \mathcal{F} on a set I is a nonempty collection of subsets of I satisfying:

- 1. $\emptyset \notin \mathcal{F}$
- 2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$
- 3. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$

An ultrafilter \mathcal{U} is a maximal filter, equivalently:

For all
$$A \subseteq I$$
, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

Ultrafilters interact nicely with logical operators:

- $A \notin \mathcal{U} \iff I \backslash A \in \mathcal{U}$,
- $A \in \mathcal{U} \land B \in \mathcal{U} \iff A \cap B \in \mathcal{U}$,
- $A \in \mathcal{U} \lor B \in \mathcal{U} \iff A \cup B \in \mathcal{U}$.

Examples

• A principal filter is of the form

$$\mathcal{F}_A = \{X \subseteq I : A \subseteq X\}$$

for some nonempty $A \subseteq I$.

If $A = \{a\}$, then \mathcal{F}_A is a **principal ultrafilter**.

• A **free** (non-principal) ultrafilter exists on every infinite set *I* (via Zorn's Lemma / Boolean prime ideal theorem).

Existence of Ultrafilters

A family of sets has the finite intersection property (FIP) if every finite subfamily has nonempty intersection.

Theorem 0.1.

If a family $A \subseteq \mathcal{P}(I)$ has the FIP,

then there exists an ultrafilter \mathcal{U} on I with $\mathcal{A} \subseteq \mathcal{U}$.

Reduced Products

Given a filter \mathcal{F} on I and structures $(\mathcal{M}_i)_{i\in I}$, define the **reduced product**

$$\mathcal{M}/\mathcal{F}$$

as follows.

Let $M = \prod_{i \in I} M_i$. For $a, b \in M$, define

$$a \sim_{\mathcal{F}} b \iff \{i \in I : a_i = b_i\} \in \mathcal{F}.$$

The universe of \mathcal{M}/\mathcal{F} is the quotient $M/\sim_{\mathcal{F}}$, with elements denoted $a_{\mathcal{F}}$ (alternatively, a/\mathcal{F}).

For symbols of \mathcal{L} :

• Relations:

$$R^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) : \iff \{\,i: \mathcal{M}_i \models R(\vec{a}_i)\,\} \in \mathcal{F}.$$

• Functions:

$$f^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) = [\,(f^{\mathcal{M}_i}(\vec{a}_i))_{i \in I}\,]_{\mathcal{F}}.$$

• Constants:

$$c^{\mathcal{M}/\mathcal{F}} = ((c^{\mathcal{M}_i})_{i \in I})_{\mathcal{F}}.$$

Exercise 0.1. Check that the above definition does not depend on the choice of representative for each equivalence class

Ultraproducts

If \mathcal{U} is an ultrafilter on I, the reduced product

$$\prod_{i\in I}\mathcal{M}_i/\mathcal{U}$$

is called the **ultraproduct** of $(\mathcal{M}_i)_{i\in I}$ modulo \mathcal{U} .

When all \mathcal{M}_i are the same structure \mathcal{M} , we get an **ultrapower**

$$\mathcal{M}^I/\mathcal{U}$$
.

Łoś' Theorem

Let $\mathcal{M}/\mathcal{U} = \prod_{i \in I} \mathcal{M}_i/\mathcal{U}$ be an ultraproduct.

Theorem 0.2.

For every \mathcal{L} -formula $\varphi(x_1, \dots, x_n)$ and tuples $\vec{a} \in \prod_{i \in I} M_i$,

$$\mathcal{M}/\mathcal{U} \models \varphi[\vec{a}_{\mathcal{U}}] \iff \{ i \in I : \mathcal{M}_i \models \varphi[\vec{a}_i] \} \in \mathcal{U}.$$

For any \mathcal{L} -formula $\varphi(v_0,\ldots,v_{n-1})$ a a tuple $\vec{a}\in\prod M_i$ we define the Boolean extension as

$$\|\varphi(\vec{a})\| := \{i \in I | \mathcal{M}_i \models \varphi[\vec{a}_i]\}$$

Lemma 0.1.

- 1. $\|\neg \varphi(\vec{a})\| = I \setminus \|\varphi(\vec{a})\|$,
- 2. $\|(\varphi \wedge \psi)(\vec{a})\| = \|\varphi(\vec{a})\| \cap \|\psi(\vec{a})\|,$
- 3. $\|(\varphi \vee \psi)(\vec{a})\| = \|\varphi(\vec{a})\| \cup \|\psi(\vec{a})\|,$
- 4. For all tuples \vec{a} and elements b in A:

$$\|\varphi(\vec{a},b)\| \subseteq \|(\exists v_n\varphi)(\vec{a})\|,$$

and there exists $b \in M$ such that

$$\|\varphi(\vec{a}, b)\| = \|(\exists v_n \varphi)(\vec{a})\|.$$

Proof. (1)-(3) and the first part of (4) follow directly from the definition of ultrafilters (and the definition of \models).

For the second part of (4), we observe that for every

$$i \in \|(\exists v_n \varphi)(\vec{a})\|$$

there exists $b_i \in A_i$ with $\mathcal{M}_i \models \varphi[\vec{a}_i, b_i]$. For all other $j \in I \setminus \|(\exists v_n \varphi)(\vec{a})\|$ we choose an aribitrary $b_j \in A_j$. This yields a sequence $b = (b_i)_{i \in I}$ with $b \in M$, for which

$$\|(\exists v_n \varphi)(\vec{a})\| \subseteq \|\varphi(\vec{a}, b)\|.$$

Together with the first part we obtain =.

Proof of Łoś's Theorem

We proceed by induction over the formula height. A straightforward argument shows that the interpretation of functions and symbols in \mathcal{M} extends to terms in the following way:

$$t^{\mathcal{M}/\mathcal{U}}(\vec{a}/U) = (t^{\mathcal{M}_i}(\vec{a}_i))_{i \in I}/\mathcal{U} = t^{\mathcal{M}}(\vec{a})/\mathcal{U}$$

This easily implies the statement for atomic formulas.

For $\varphi \equiv \neg \psi$, we have

$$\begin{split} \mathcal{M}/\mathcal{U} \models \neg \psi[\vec{a}_{\mathcal{U}}] &\iff \mathcal{M}/\mathcal{U} \not\models \psi[\vec{a}_{\mathcal{U}}] \\ &\iff \left\{ i: \mathcal{M}_i \models \psi[\vec{a}_i] \right\} \notin \mathcal{U} \qquad \text{(I.H.)} \\ &\iff \|\psi(\vec{a})\| \notin \mathcal{U} \\ &\iff \neg \|\psi(\vec{a})\| \in \mathcal{U} \qquad \qquad (\mathcal{U} \text{ is an ultrafilter)} \\ &\iff \|\neg \psi(\vec{a})\| \in \mathcal{U} \qquad \qquad \text{(Lemma (i))} \\ &\iff \left\{ i: \mathcal{M}_i \models \neg \psi[\vec{a}_i] \right\} \in \mathcal{U}. \end{split}$$

The case $\varphi \equiv (\psi \wedge \theta)$ is similar.

Finally, assume $\varphi \equiv \exists y \psi$. Then

$$\begin{split} \mathcal{M}/\mathcal{U} \models \exists y \, \psi[\vec{a}_{\mathcal{U}}] &\iff \exists b_{\mathcal{U}} \in \mathcal{M}/\mathcal{U} \text{ such that } \mathcal{M}/\mathcal{U} \models \psi[\vec{a}_{\mathcal{U}}, b_{\mathcal{U}}] \\ &\iff \exists b \in M \text{ such that } \{\,i : \mathcal{M}_i \models \psi[\vec{a}_i, b_i] \,\} \in \mathcal{U} \quad \text{(I.H.)} \\ &\iff \exists b \in M \text{ such that } \|\psi(\vec{a}, b)\| \in \mathcal{U} \\ &\iff \|\exists y \, \psi(\vec{a})\| \in \mathcal{U} \qquad \qquad (\mathcal{U} \text{ ultrafilter, Lemma (4))} \\ &\iff \{\,i : \mathcal{M}_i \models \exists y \, \psi[\vec{a}_i] \,\} \in \mathcal{U}. \end{split}$$

This completes the proof.

Applications

• Preservation of theories:

If each $\mathcal{M}_i \models T$, then $\prod_i \mathcal{M}_i / \mathcal{U} \models T$.

• Ultrapowers:

 $\mathcal{M} \equiv \mathcal{M}^I/\mathcal{U}$, i.e., a structure is elementarily equivalent to any of its ultrapowers.

• Nonstandard models:

For example, the ultrapower $\mathbb{N}^{\mathbb{N}}/\mathcal{U}$ (with \mathcal{U} non-principal) yields a countably saturated nonstandard model of arithmetic.

A new proof of the compactness theorem

Suppose T is an \mathcal{L} -theory for which every finite subset Δ has a model \mathcal{M}_{Δ} .

Let
$$I = {\Delta : \Delta \subseteq T \text{ finite}}.$$

For $\sigma \in T$, let

$$\begin{split} \hat{\sigma} &= \{\Delta \in I : \sigma \in \Delta\} \subseteq I \\ E &= \{\hat{\sigma} : \sigma \in T\} \subseteq \mathcal{P}(I) \end{split}$$

E has FIP (finite intersection property), since for $\hat{S}_1,\dots,\hat{S}_n\in E,$

$$\{\hat{\sigma}_1,\dots,\hat{\sigma}_n\}\in\hat{\sigma}_1\cap\dots\cap\hat{\sigma}_n.$$

By Theorem 0.1, E can be extended to an ultrafilter \mathcal{U} on I.

Claim: $\mathcal{M}/\mathcal{U} = \prod_{\Delta \in I} \mathcal{M}_{\Delta}/\mathcal{U} \models T$.

Let $\sigma \in T$. Then for $\Delta \in I$,

$$\Delta \in \hat{\sigma} \ \Rightarrow \ \sigma \in \Delta \ \Rightarrow \ \mathcal{M}_{\Delta} \models \sigma$$

Hence

$$\hat{\sigma} \subseteq \{\Delta \in I: \mathcal{M}_{\Delta} \models \sigma\}$$

Since $\hat{\sigma} \in E$ and $E \subseteq \mathcal{U}$, we have $\hat{\sigma} \in \mathcal{U}$, which implies

$$\{\Delta \in I: \mathcal{M}_{\Delta} \models \sigma\} \in \mathcal{U}$$

By Łoś' Theorem, $\mathcal{M} \models \sigma$.