

## Löwenheim-Skolem Theorems

**Exercise 0.1.** For finite structures  $\mathcal{M}, \mathcal{N}$ ,  $\mathcal{M} \preceq \mathcal{N}$  implies  $\mathcal{M} = \mathcal{N}$

Hence, for finite structures, proper elementary substructures cannot exist. In contrast, infinite structures have an elementary substructure in every smaller infinite cardinality  $\kappa$  (as long as  $\kappa \geq \text{card}(\mathcal{L})$ ):

### Down

**Theorem 0.1** (Löwenheim-Skolem downward). *Let  $\mathcal{A}$  be an  $\mathcal{L}$ -structure,  $\kappa$  an infinite cardinal with  $\text{card}(\mathcal{L}) \leq \kappa$  and  $\kappa \leq \text{card}(A)$ . Then there exists a structure  $\mathcal{B}$  with*

$$\mathcal{B} \preceq \mathcal{A}, \text{card}(\mathcal{B}) = \kappa.$$

**Addition:** If  $A_0 \subseteq A$  is arbitrary with  $\text{card}(A_0) \leq \kappa$ , then one can additionally require  $A_0 \subseteq B$ .

*Proof.* Let  $A_0 \subseteq A$  be given with  $\text{card}(A_0) \leq \kappa \leq \text{card}(A)$ . By enlarging  $A_0$  if necessary, we can assume that  $\text{card}(A_0) = \kappa$ .

If for elements  $a_1, \dots, a_n \in A_0$

$$(*) \quad \mathcal{A} \models \exists v_0 \varphi[a_1, \dots, a_n]$$

holds, we have to add a witness  $b$  for this existential quantifier to  $A_0$ ; let  $A_1$  be the set that arises from  $A_0$  by adding such witnesses (for all possible existential formulas and all possible assignments with elements from  $A_0$ ). Standard cardinal arithmetic yields  $\text{card}(A_1) = \text{card}(A_0) = \kappa$ .

Now, however,  $(*)$  may possibly hold for a formula  $\varphi$  and new elements  $a_1, \dots, a_n \in A_1$  that are not contained in  $A_0$ ; thus, we have to iterate the procedure.

With  $A_i$  defined, add suitable elements from  $A$ , resulting in a set  $A_{i+1}$ , such that:

$$\begin{aligned} &\text{If for } a_1, \dots, a_n \in A_i : \mathcal{A} \models \exists v_0 \varphi[a_1, \dots, a_n], \\ &\text{then there exists an } a \in A_{i+1} \text{ with } \mathcal{A} \models \varphi[a, a_1, \dots, a_n]. \end{aligned}$$

As in the first step,  $A_{i+1}$  can be obtained from  $A_i$  without changing the cardinality. Finally, we set

$$B := \bigcup_{i \in \mathbb{N}} A_i$$

and obtain a set  $B$  with  $A_0 \subseteq B \subseteq A$  and  $\text{card}(B) = \kappa$ .

In the first step, we already add all constants of  $\mathcal{A}$  (using  $(*)$  with the formula  $\exists v_0 (v_0 = c)$ ) and in all further steps we are closing under the functions of  $\mathcal{A}$  (using the formula  $\exists v_0 (v_0 = f(v_1, \dots, v_n))$ ). It follows that  $B$  is the universe of a substructure  $\mathcal{B}$  of  $\mathcal{A}$ . We can then conclude that  $\mathcal{B} \preceq \mathcal{A}$  using the Tarski-Vaught test, as our construction is arranged precisely so that the Tarski-Vaught criterion is applicable.  $\square$

## Up

The proof of the upward version is simpler and is based on the Compactness Theorem.

**Theorem 0.2** (Löwenheim-Skolem upward). *Let  $\mathcal{A}$  be an infinite  $\mathcal{L}$ -structure,  $\kappa$  a cardinal with  $\text{card}(\mathcal{L}) \leq \kappa$  and  $\text{card}(A) \leq \kappa$ . Then there exists a structure  $\mathcal{B}$  with\**

$$\mathcal{A} \preceq \mathcal{B}, \text{card}(\mathcal{B}) = \kappa.$$

*Proof.* We first pick a set  $C$  with  $A \subseteq C$  and  $\text{card}(C) = \kappa$  and extend the theory of  $\mathcal{A}$  (with the help of new constants) so that every model has at least as many elements as  $C$ :

$$T' = \text{Th}(\mathcal{A}) \cup \{\underline{c} \neq \underline{d} \mid c, d \in C, c \neq d\}.$$

By the Compactness Theorem,  $T'$  has a model, say  $\mathcal{B}$ , in which the new constants  $\underline{c}$  are interpreted by elements of  $B$  – different constants by different elements of  $B$ . By passing to an isomorphic structure, we can assume that  $\underline{c}^{\mathcal{B}} = c$  and thus  $C \subseteq B$ , so  $\text{card}(B) \geq \kappa$ .

The language of  $T'$  has cardinality  $\kappa$  because  $\kappa \geq \text{card}(\mathcal{L})$ , so we can also assume that  $\text{card}(B) = \kappa$  (by using the downward theorem).

Finally, because  $\mathcal{B} \models \text{Th}(\mathcal{A})$ , we have  $\mathcal{A} \equiv \mathcal{B}$ . The stronger statement  $\mathcal{A} \preceq \mathcal{B}$  is obtained by using the same argument, but using the elementary diagram  $D(\mathcal{A}) = \text{Th}(\mathcal{A}_A)$  instead of  $\text{Th}(\mathcal{A})$ .  $\square$

## Some consequences

1. If  $\mathcal{A}$  is an infinite  $\mathcal{L}$ -structure and  $\kappa \geq \text{card}(\mathcal{L})$  is a infinite cardinal, then there exists a structure  $\mathcal{B}$  with  $\text{card}(B) = \kappa$  and
  - $\mathcal{B} \preceq \mathcal{A}$  in the case  $\kappa \leq \text{card}(A)$ ,
  - $\mathcal{A} \preceq \mathcal{B}$  in the case  $\text{card}(A) \leq \kappa$ .
2. In particular, every theory  $T$  that has an infinite model has a model of cardinality  $\kappa$  for every cardinal  $\kappa \geq \text{card}(\mathcal{L})$ .
3. More specifically: A theory  $T$  in a countable language  $\mathcal{L}$  that has a model at all also has a countable model. This theorem of Löwenheim (1915) is one of the earliest results of mathematical logic.

## The reals as a complete ordered field

Consider the structure  $(\mathbb{R}, 0, 1, +, \cdot, <)$  over the language of ordered rings. By Löwenheim-Skolem downward, this has a countable elementary substructure  $\mathcal{R}'$ .  $\mathcal{R}'$  is a field, so it has to contain  $\mathbb{Q}$ , and since it inherits the order from  $\mathbb{R}$ , it has to be dense in  $\mathbb{R}$ . Since  $\mathcal{R}'$  is countable, there exists  $r_0 \in \mathbb{R} \setminus \mathcal{R}'$ . The set

$$\{r \in \mathcal{R}' : r < r_0\}$$

is bounded in  $\mathcal{R}'$  but cannot have a least upper bound in  $\mathcal{R}'$ .

As  $\mathcal{R}' \models \text{Th}(\mathbb{R}, 0, 1, +, \cdot, <)$ , it follows that the theory of complete ordered fields is not first-order axiomatizable in the language of ordered rings.

It can be shown that the algebraic numbers  $\mathbb{R}_{\text{alg}}$  form such a countable elementary substructure of  $\mathbb{R}$ .