

## End Extensions

The standard model can be embedded into every model  $\mathcal{M}$  of  $\text{PA}$  as an initial segment. It turns out this already holds for models of the theory  $\text{PA}^-$ .

### Definition 1

Let  $L$  be a language containing a 2-ary symbol  $<$ , and let  $\mathcal{M}$  and  $\mathcal{N}$  be  $L$ -structures with  $\mathcal{M} \subseteq \mathcal{N}$ . Then  $\mathcal{N}$  is called an **end extension** of  $\mathcal{M}$  (and correspondingly  $\mathcal{M}$  is an **initial segment** of  $\mathcal{N}$ ) if and only if the larger set  $N$  does not add any further elements below an element of  $M$ :

$$\mathcal{M} \subseteq_{\text{end}} \mathcal{N} : \iff \text{for all } x \in M, y \in N : (y <^N x \Rightarrow y \in M).$$

Each natural number  $n$  is represented in the standard model, which we also simply denote by  $\mathbb{N}$  here, by the constant term

$$\underline{n} = 1 + \dots + 1 \quad (n \text{ times})$$

where  $\underline{0}$  is the constant 0.

### Theorem 2

Let  $\mathcal{M} \models \text{PA}^-$ . Then the map

$$n \mapsto \underline{n}^{\mathcal{M}}$$

defines an embedding of the standard model  $\mathbb{N}$  onto an initial segment of  $\mathcal{M}$ .

In particular, every model of  $\text{PA}^-$  is isomorphic to an end extension of the standard model  $\mathbb{N}$ .\*

*Proof.* By simple induction (in the meta-theory), one shows for all natural numbers  $n, k, l$ :

$$n = k + l \implies \text{PA}^- \vdash \underline{n} = \underline{k} + \underline{l}$$

$$n = k \cdot l \implies \text{PA}^- \vdash \underline{n} = \underline{k} \cdot \underline{l}$$

$$n < k \implies \text{PA}^- \vdash \underline{n} < \underline{k}$$

and

$$\text{PA}^- \vdash x \leq \underline{k} \rightarrow (x = \underline{0} \vee \dots \vee x = \underline{k})$$

The first three statements will later be generalized to all recursive functions and relations; they imply that the map  $n \mapsto \underline{n}^{\mathcal{M}}$  is a homomorphism, and, due to the last statement, the map is also an embedding onto an initial segment of  $\mathcal{M}$ .  $\square$

### Remark

The standard model has no proper initial segment, and  $\mathbb{Z}[X]^+$  has  $\mathbb{N}$  as its only proper initial segment. On the other hand, every model  $\mathcal{M} \models \text{PA}^-$  has a proper end extension that is also a model of  $\text{PA}^-$ , namely the non-negative part of the polynomial ring  $R[X]$ , where  $R$  is the discretely ordered ring associated with the model  $\mathcal{M}$ .

## Preservation Properties of End Extensions

In the previous lecture, we introduced *arithmetical formulas*.  $\Delta_0$ -formulas are at the bottom of the *arithmetical hierarchy*. We already saw that relations defined by such formulas are primitive recursive. Next, we will show that those formulas *mean the same thing in a structure as in all end extensions*. This will be crucial later on.

### Theorem 3

Let  $\mathcal{N}, \mathcal{M}$  be structures of the language  $L$  of  $\text{PA}^-$ , with  $\mathcal{N} \subseteq_{\text{end}} \mathcal{M}$ , and let  $\vec{a} \in N$ . Then:

1. Every  $\Delta_0$ -formula  $\varphi(\vec{v})$  is **absolute**:

$$\mathcal{N} \models \varphi[\vec{a}] \iff \mathcal{M} \models \varphi[\vec{a}],$$

2. Every  $\Sigma_1$ -formula  $\varphi(\vec{v})$  is **upward-persistent**:

$$\mathcal{N} \models \varphi[\vec{a}] \implies \mathcal{M} \models \varphi[\vec{a}],$$

3. Every  $\Pi_1$ -formula  $\varphi(\vec{v})$  is **downward-persistent**:

$$\mathcal{M} \models \varphi[\vec{a}] \implies \mathcal{N} \models \varphi[\vec{a}],$$

4. Every  $\Delta_1$ -formula  $\varphi(\vec{v})$  is **absolute**:

$$\mathcal{N} \models \varphi[\vec{a}] \iff \mathcal{M} \models \varphi[\vec{a}].$$

*Proof.* We prove (i) by induction on the formula structure of  $\varphi(\vec{v})$ . Only the case of a bounded quantifier needs to be addressed. Since  $\mathcal{M}$  is an end extension of  $\mathcal{N}$ ,  $\mathcal{M}$  does not insert any new elements below an element of  $N$ , so that a bounded quantifier means the same thing in both structures.  $\square$

Let  $\Sigma_1\text{-Th}(\mathbb{N}) := \{\sigma \mid \sigma \text{ is a } \Sigma_1\text{-sentence with } \mathbb{N} \models \sigma\}$ . Then we have:

### Corollary 4

$$\text{PA}^- \vdash \Sigma_1\text{-Th}(\mathbb{N})$$

*Proof.* Let  $\mathcal{N} \models \text{PA}^-$ . By the previous theorem, we may assume that  $\mathbb{N} \subseteq_{\text{end}} \mathcal{N}$ , and the claim then follows from part (ii) above.  $\square$

Thus one can prove in the theory  $\text{PA}^-$  all  $\Sigma_1$ -sentences that hold in the standard model. This is no longer true for  $\Pi_1$ -sentences. For instance, the  $\Pi_1$ -sentence stating that every number is even or odd:

$$(*) \quad \forall x \exists y \leq x (x = 2 \cdot y \vee x = 2 \cdot y + 1)$$

is true in the standard model, but not in the  $\text{PA}^-$ -model  $\mathbb{Z}[X]^+$ .

Even true  $\forall$ -sentences (i.e., sentences of the form  $\forall \vec{x} \psi$  with *quantifier-free*  $\psi$ ) that hold in the standard model need not be provable in  $\text{PA}^-$ . For example, the  $\forall$ -sentence

$$\forall x, y (x^2 \neq 2 \cdot y^2)$$

is true in the standard model, but not provable in  $\text{PA}^-$  ( $\mathbb{Z}/(X^2 - 2Y^2)$  is a counterexample).

Thus the above sentence  $(*)$  is an example of a  $\Pi_1$ -sentence that is not a  $\forall$ -sentence, where one cannot omit the bounded quantifier! (By the way, one can show that  $\mathbb{Z}[X]^+$  is at least a model of all  $\forall$ -sentences that hold in the standard model.)