

The First Incompleteness Theorem

Theorem 0.1 (Gödel-Rosser Theorem).

Let T be a recursive set of (Gödel numbers of) \mathcal{L} -sentences such that:

1. T is consistent, i.e., there is no L -sentence σ with $\ulcorner \sigma \urcorner \in T$ and at the same time $\ulcorner \neg \sigma \urcorner \in T$,
2. T contains all Σ_1 and Π_1 -sentences that hold in PA^- :

$$\{\ulcorner \sigma \urcorner : \sigma \in \Sigma_1 \cup \Pi_1, \sigma \text{ sentence}, \text{PA}^- \vdash \sigma\} \subseteq T.$$

Then T is Π_1 -incomplete, i.e., there exists a Π_1 -sentence τ with

$$\ulcorner \tau \urcorner \notin T \text{ and } \ulcorner \neg \tau \urcorner \notin T.$$

Proof. Since T is recursive, there exists a Σ_1 -formula $\theta(v)$ such that for all $n \in \mathbb{N}$

$$\begin{aligned} n \in T &\implies \text{PA}^- \vdash \theta(\underline{n}), \\ n \notin T &\implies \text{PA}^- \vdash \neg \theta(\underline{n}). \end{aligned}$$

By the Diagonal Lemma, there exists a Π_1 -sentence τ with

$$\text{PA}^- \vdash \tau \leftrightarrow \neg \theta(\ulcorner \tau \urcorner).$$

From assumption (2) it follows that

$$\ulcorner \tau \urcorner \in T \implies \text{PA}^- \vdash \theta(\ulcorner \tau \urcorner) \implies \text{PA}^- \vdash \neg \tau \implies \ulcorner \neg \tau \urcorner \in T$$

and likewise, using (1),

$$\ulcorner \neg \tau \urcorner \in T \implies \ulcorner \tau \urcorner \notin T \implies \text{PA}^- \vdash \neg \theta(\ulcorner \tau \urcorner) \implies \text{PA}^- \vdash \tau \implies \ulcorner \tau \urcorner \in T,$$

so that neither $\ulcorner \tau \urcorner$ nor $\ulcorner \neg \tau \urcorner$ can be in T . □

Theorem 0.2 (First Gödel Incompleteness Theorem).

Let T be a consistent¹ and recursively axiomatizable theory in language L which extends the theory PA^- . Then T is incomplete. In particular, there exists a Π_1 -sentence τ with

$$T \not\vdash \tau \text{ and } T \not\vdash \neg \tau.$$

¹The original version of Gödel's theorem had the stronger assumption of ω -consistency of T (i.e., for all L -formulas $\theta(v)$ with $T \vdash \theta(\underline{n})$ for all $n \in \mathbb{N}$, the theory $T \cup \{\forall v \theta(v)\}$ is consistent). Rosser proved this stronger version in 1936.

Proof. We assume that T is recursively axiomatizable as well as complete (which implies that it is consistent, by definition), and derive a contradiction. As we proved in a previous lecture, the assumption implies that the set

$$S := \{ \ulcorner \sigma \urcorner : \sigma \text{ sentence, } T \vdash \sigma \} = \ulcorner (T^+) \urcorner$$

is recursive. Thus the hypothesis of the Gödel-Rosser Theorem is met, which implies S is Π_1 -incomplete—a contradiction! \square

Every consistent extension T of PA^- has, by Lindenbaum's Lemma, a complete and consistent extension, none of which, by the above theorem, can be recursive.

The Paris-Harrington Theorem

The use of the Diagonal Lemma yields rather “artificial” witnesses of incompleteness. Are there “natural” mathematical theorems that PA cannot decide? In 1982, Paris and Harrington² found an example, based on *Ramsey theory*.

Pigeonhole Principles

In its simplest form, the **pigeonhole principle** states:

If n elements are distributed into $m < n$ many pigeonholes, then one of the pigeonholes must contain at least 2 elements.

The generalization to infinite sets is:

If an infinite set is partitioned into finitely many sets, then at least one of these sets must be infinite:

$$A = A_0 \dot{\cup} \dots \dot{\cup} A_k \text{ infinite} \Rightarrow \exists i \leq k (A_i \text{ infinite}).$$

For further generalization, we set

$$[A]^n := \{x \subseteq A \mid |x| = n\} \text{ the set of } n\text{-element subsets of } A.$$

A partition of the n -element subsets of A into k parts can also be represented by a function

$$f : [A]^n \rightarrow k$$

(where $A_i = \{x \in A \mid f(x) = i\}$ are then the partition sets), and such an f is more intuitively called a **coloring** (of $[A]^n$) with k colors. A subset $X \subseteq A$ with $[X]^n \subseteq A_i$ for some $i < k$ (whose n -tuples are thus monochromatic) is called **homogeneous** for the partition f .

Theorem 0.3 (Ramsey's Theorem).

For every coloring $f : [\mathbb{N}]^n \rightarrow k$ of the n -element subsets of natural numbers with k colors, there exists an infinite subset $X \subseteq \mathbb{N}$ such that all n -element subsets of X have the same color:

$$f : [\mathbb{N}]^n \rightarrow k \Rightarrow \exists X \subseteq \mathbb{N} (X \text{ infinite} \wedge \forall u, v \in [X]^n f(u) = f(v)).$$

This states that every coloring of the n -element subsets of natural numbers with finitely many colors has an infinite homogeneous subset. This theorem is provable in set theory with a weak form of the axiom of choice (see, e.g., D. Marker: *Model Theory: An Introduction*, § 5.1).

Definition 0.1 (Relatively Large Sets).

A (finite) set H of ordinal numbers is called **relatively large** if and only if

$$\text{card}(H) \geq \min(H).$$

²A *mathematical Incompleteness in Peano Arithmetic*. In: Handbook of Mathematical Logic, ed. Barwise, Elsevier 1982

If $n, k \in \mathbb{N}$, $\alpha, \gamma \in \text{On}$, then let

$$\gamma \longrightarrow (\alpha)_k^n \quad \text{resp.} \quad \gamma \xrightarrow{\min} (\alpha)_k^n$$

mean that for every coloring f of the n -element subsets of γ with k colors, there exists a (relatively large) subset $H \subseteq \gamma$ of order type α that is homogeneous for f .

Theorem 0.3 above thus states: $\forall n, k, \omega \rightarrow (\omega)_k^n$. From this follows the **finitary Ramsey theorem** (using a compactness argument):

$$\forall m, n, k \exists r \ r \rightarrow (m)_k^n.$$

This is a theorem about natural numbers and can be expressed in the language of Peano arithmetic and proven in PA. However, an (apparently) slight strengthening cannot:

Theorem 0.4 (Paris-Harrington Theorem).

The statement

$$\forall m, n, k \exists r \ r \xrightarrow{\min} (m)_k^n$$

is provable in set theory, but not in PA (provided PA is consistent). The existence of these numbers r can no longer be shown generally in PA; in fact, they lead to very large numbers: If

$$\sigma(n) = \text{the smallest } r \text{ with } r \xrightarrow{\min} (n+1)_n^n,$$

then the function σ eventually dominates every recursive function f (i.e., for every recursive function f there exists a p such that $f(n) < \sigma(n)$ for all $n \geq p$).