

Amalgamation Classes

Let \overline{K} be a class of finitely generated structures. \overline{K} is called an **amalgamation class** if it has the following three properties:

(HP) Hereditary Property

If $A \in \overline{K}$, $B \cong C \in A$, and C is finitely generated, then $B \in \overline{K}$.

(JEP) Joint Embedding Property

If $A, B \in \overline{K}$, then there exists $C \in \overline{K}$ and embeddings

$$f_0 : A \rightarrow C, \quad f_1 : B \rightarrow C$$

(AP) Amalgamation Property

If $A, B, C \in \overline{K}$ with embeddings $f_0 : A \rightarrow B$ and $f_1 : A \rightarrow C$, then there exists $D \in \overline{K}$ and embeddings

$$g_0 : B \rightarrow D, \quad g_1 : C \rightarrow D$$

such that $g_0 \circ f_0 = g_1 \circ f_1$.

Exercise 0.1.

Show that (AP) does not imply (JEP).

(Hint: finite fields)

Example 0.1.

$\overline{K} = \{(Z, <) : (Z, <) \text{ finite linear order}\}$ forms an amalgamation class.

Fraïssé's Theorem

Theorem 0.1. Let \overline{K} be a class of finitely generated \mathcal{L} -structures such that there are only countably many isomorphism types in \overline{K} .

Then: \overline{K} is an amalgamation class $\Leftrightarrow \overline{K}$ is the age of a countable homogeneous \mathcal{L} -structure.

Proof. (\Leftarrow): Suppose $\overline{K} = \text{age}(\mathcal{M})$ where \mathcal{M} is countable and homogeneous.

(HP): Holds by definition of age.

(JEP): Let $A, B \in \text{age}(\mathcal{M})$. Then $A \cong \langle A \rangle^{\mathcal{M}}$ and $B \cong \langle B \rangle^{\mathcal{M}}$, with $A, B \subseteq M$ finite. Then A and B embed into $\langle A \cup B \rangle^{\mathcal{M}}$.

(AP): Suppose $f_0 : A \rightarrow B$ and $f_1 : A \rightarrow C$ are embeddings, where $A, B, C \in \text{age}(\mathcal{M})$.

Without loss of generality, $A, B, C \subseteq M$ and $f_0 = \text{id}_A$.

Since \mathcal{M} is homogeneous, there exists an automorphism $\pi : \mathcal{M} \xrightarrow{\sim} \mathcal{M}$ such that $\pi^{-1}|_A = f_1$ (i.e., π extends f_1^{-1}).

Consider $\mathcal{D} = \langle B \cup \pi^{-1}(C) \rangle^{\mathcal{M}}$. \mathcal{D} amalgamates \mathcal{B} and \mathcal{C} via $\text{id}_{\mathcal{B}} : \mathcal{B} \rightarrow \mathcal{D}$ and $\pi^{-1}|_C : \mathcal{C} \rightarrow \mathcal{D}$. \square

(\Rightarrow): Assume \overline{K} is an amalgamation class. We construct a structure \mathcal{N} with $\text{age}(\mathcal{N}) = \overline{K}$ where \mathcal{N} is homogeneous and has domain $N \subset \mathbb{N}$.

Since \overline{K} has countably many isomorphism types, we enumerate the elements of \overline{K} (up to isomorphism) as $(\mathcal{K}_e : e \in \mathbb{N})$ where $K_e \subseteq \mathbb{N}$.

Consider tuples (\bar{a}, \bar{b}, f) where \bar{a}_i, \bar{b}_i are finite subsets of \mathbb{N} and f is a partial function with $\text{dom}(f) = \bar{a}$ and $\text{ran}(f) \subseteq \bar{b}$. Enumerate all such triples as $(\bar{a}_e, \bar{b}_e, f_e)$ so that every (\bar{a}, \bar{b}, f) occurs infinitely often.

\mathcal{N} will be the union of an increasing sequence $(\mathcal{C}_n)_{n \in \mathbb{N}}$ where $\mathcal{C}_n \in \overline{K}$.

Initialize: $\mathcal{C}_0 = \mathcal{K}_0$

Now assume we have defined \mathcal{C}_n .

Case $n = 2\ell$ (even): Apply (JEP) to $\mathcal{C}_n, \mathcal{K}_\ell$ to obtain \mathcal{C}_{n+1} .

Case $n = 2\ell + 1$ (odd): If \bar{a}_ℓ or $\bar{b}_\ell \not\subseteq \mathcal{C}_n$, put $\mathcal{C}_{n+1} = \mathcal{C}_n$. If $\bar{a}_\ell, \bar{b}_\ell \subseteq \mathcal{C}_n$, let $\mathcal{A}_\ell = \langle \bar{a}_\ell \rangle^{\mathcal{C}_n}$ and $\mathcal{B}_\ell = \langle \bar{b}_\ell \rangle^{\mathcal{C}_n} \subseteq \mathcal{C}_n$. Apply (AP) to the embeddings $\text{id} : \mathcal{A}_\ell \rightarrow \mathcal{C}_n$ and $f : \mathcal{A}_\ell \rightarrow \mathcal{B}_\ell$ (induced by $f_\ell : \bar{a}_\ell \rightarrow \bar{b}_\ell$). This yields a structure $\mathcal{C}_{n+1} \in \overline{K}$. Renaming if necessary, we can assume $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$.

Define $\mathcal{N} = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n$.

Claim 1: $\text{age}(\mathcal{N}) = \overline{K}$.

In the even steps $n = 2\ell$, we ensure $K_\ell \subseteq \mathcal{N}$, so all structures isomorphic to K_ℓ also enter $\text{age}(\mathcal{N})$. Since the $(K_e)_{e \in \mathbb{N}}$ enumerates all isomorphism types of \overline{K} , we get $\text{age}(\mathcal{N}) = \overline{K}$.

Claim 2: \mathcal{N} is homogeneous.

Let $\tau : A \rightarrow B$ be an isomorphism between finitely generated substructures $\mathcal{A} = \langle \bar{a} \rangle^{\mathcal{N}}$ and $\mathcal{B} = \langle \bar{b} \rangle^{\mathcal{N}}$ where $\bar{a}, \bar{b} \in N$ are finite. We show that for any $c_0 \in N \setminus B$, there exists a partial isomorphism $\tau' : \mathcal{A}' \rightarrow \mathcal{B}'$ where $\mathcal{A}', \mathcal{B}'$ are finitely generated, $\tau' \supseteq \tau$, and $c_0 \in \text{dom}(\tau')$. (This suffices since we can continue via back-and-forth.)

There exists ℓ such that f_ℓ induces an embedding $f : A \rightarrow \langle \mathcal{B} \cup \{c_0\} \rangle^{\mathcal{N}}$ (since every triple occurs infinitely often in the enumeration, in particular the triple $(\bar{a}, \bar{b} \cup \{c_0\}, f)$).

In step $n = 2\ell + 1$, $\mathcal{B}_\ell = \langle \mathcal{B} \cup \{c_0\} \rangle^{\mathcal{N}}$ and \mathcal{C}_n are amalgamated over \mathcal{A}_ℓ . This yields embeddings:

$$\begin{aligned} \text{id} : \mathcal{A} &\hookrightarrow \mathcal{C}_{n+1} \\ g : \mathcal{B} &\hookrightarrow \mathcal{C}_{n+1} \text{ where } g \circ f_\ell = \text{id}_{\mathcal{A}} \end{aligned}$$

Let $a_0 = g(c_0)$. Then $(g|_{\mathcal{B}})^{-1} : \langle \bar{a} \cup \{a_0\} \rangle^{\mathcal{N}} \xrightarrow{\sim} \langle \bar{b} \cup \{c_0\} \rangle^{\mathcal{N}}$ is the desired isomorphism.