

## Fraïssé Theory of Random Graphs

### Key facts from Fraïssé theory

To simplify things, we state everything in the context of a **finite language that contains only constants and relations**. In this case *finitely generated* is equivalent to *finite*.

- $\mathcal{M}$  is **homogeneous** if any isomorphism between finite substructures of  $\mathcal{M}$  can be extended to an automorphism of  $\mathcal{M}$ .
- $\text{age}(\mathcal{M})$  is the class of all finite  $\mathcal{L}$ -structures isomorphic to a substructure of  $\mathcal{M}$ .
- Any two countable homogeneous structures with the same age are isomorphic.
- A class of finite structures  $\bar{K}$  is an **amalgamation class** if it satisfies (HP), (JEP), and (AP). In words, every substructure of a structure in  $\bar{K}$  is in  $\bar{K}$ , any two structures in  $\bar{K}$  embed into another structure from  $\bar{K}$ , and any two structures with a common substructure (up to isomorphism) amalgamate to another structure that preserves the common substructure.
- **Fraïssé's theorem:** Suppose  $\bar{K}$  is a class of finite  $\mathcal{L}$ -structures such that there are only countably many isomorphism types in  $\bar{K}$ . Then  $\bar{K}$  is an amalgamation class iff  $\bar{K}$  is the age of a countable homogeneous  $\mathcal{L}$ -structure.

### Simple graphs

We consider the theory of *undirected graphs without self-loops*, also called **simple graphs**. Let  $\mathcal{L}_G = \{E\}$  be the language of graphs with one binary relation symbol  $E$ .

Simple graphs are formalized by the axioms

- $\forall x \neg E(x, x)$
- $\forall x, y (E(x, y) \rightarrow E(y, x))$

We denote the theory of simple graphs by  $T_G$

**Exercise 0.1.** Verify that

$$\bar{K} = \{\mathcal{G} : \mathcal{G} \models T_G, \mathcal{G} \text{ finite}\}$$

is an amalgamation class with countably many isomorphism types.

It follows that  $\bar{K}$  has a Fraïssé limit. What does this limit look like?

### Random graphs

We add the following **extension axioms**:

$$\sigma_{n,m} \equiv \forall x_1, \dots, x_n \forall y_1, \dots, y_m \left( \bigwedge_{i=1}^n \bigwedge_{j=1}^m x_i \neq y_j \rightarrow \exists z \left( \bigwedge_{i=1}^n (z \neq x_i \wedge E(z, x_i)) \wedge \bigwedge_{j=1}^m (z \neq y_j \wedge \neg E(z, y_j)) \right) \right)$$

Let

$$T_{RG} = T_G \cup \{\exists x, y (x \neq y)\} \cup \{\sigma_{n,m} : n, m \geq 1\}$$

**Exercise 0.2.** Show that every countable model of  $T_{RG}$  is homogeneous.

**Exercise 0.3.** Show that if  $\mathcal{R}$  is a model of  $T_{RG}$ ,  $\text{age}(\mathcal{R}) = \overline{K}$ .

It follows from Fraïssé's Theorem that  $T_{RG}$  has a countable model  $\mathcal{R}$ , and that this model is unique up to isomorphism.

**Exercise 0.4.** Show that  $T_{RG}$  is complete.

$\mathcal{R}$  is called the **random graph** and  $T_{RG}$  the *theory of the random graph*. Why? Let  $G_N$  be the set of all simple graphs on the vertex set  $\{1, \dots, N\}$ . Put a probability distribution on  $G_N$  by giving every graph the same probability. (Alternatively, we could independently assign edges to any vertex pair with probability  $1/2$ . This is called the **Erdős-Rényi model**.) Given an  $\mathcal{L}_G$ -sentence  $\sigma$ , let  $\mathbb{P}_N(\sigma)$  be the probability that  $\sigma$  holds for a random graph in  $G_N$ , i.e.

$$\mathbb{P}_N(\sigma) = \frac{|\{\mathcal{G} \in G_N : \mathcal{G} \models \sigma\}|}{|G_N|}.$$

It turns out, for large  $N$ , graphs will satisfy the extension axioms with high probability.

**Exercise 0.5.** Show that for any  $n, m \geq 1$ ,

$$\lim_{N \rightarrow \infty} \mathbb{P}_N(\sigma_{n,m}) = 1.$$

This justifies referring to  $\mathcal{R}$  as *the random graph*, since it can be seen as the “limit” of finite random graphs.

## Further explorations

### 0-1 law for graphs

**Theorem 0.1.** For any  $\mathcal{L}_G$ -sentence  $\sigma$ ,

$$\text{either } \lim_{N \rightarrow \infty} \mathbb{P}_N(\sigma) = 0 \text{ or } \lim_{N \rightarrow \infty} \mathbb{P}_N(\sigma) = 1.$$

Moreover,  $T_{RG}$  axiomatizes the theory

$$\{\sigma : \lim_{N \rightarrow \infty} \mathbb{P}_N(\sigma) = 1\}$$

and this theory is complete.

To prove the theorem, use that  $T_{RG}$  is complete together with Exercise 0.5. Give it a try!

### Other Fraïssé classes

There are other classes of structures that are amalgamation classes, for example:

- finite fields of characteristic  $p$
- finite groups
- (non-trivial) finite Boolean algebras

What kind of object is the Fraïssé limit in each case?