

Math 557 Sep 5

Substitution

Key Concepts

- **Substitution:**

- Basic idea: $\varphi_{\bar{s}/\bar{x}}$ is obtained by replacing all occurrences of the variable x_i by the term s_i .
- Uncontrolled substitution may cause issues with quantifiers. If we try to substitute a variable into the range of a quantifier, we rename the quantified variable to an unused variable ($\exists x \dots$ becomes $\exists u \dots$).

- **Substitution Lemma:**

- Substitution behaves “as expected” with respect to evaluation and satisfaction.
- Evaluating a substituted term yields the same value as evaluating the original term under the “substituted” assignment (i.e. the assignment in which we replace the assignment to x by the value of s under α).
- A substituted formula holds in \mathcal{M} under assignment α iff the original formula holds in \mathcal{M} under the “substituted” assignment.

Problems

Exercise 0.1 (Carry-over from Sep 3).

Show that if x is not free in φ , $\mathcal{M} \models \varphi[\alpha]$ implies $\mathcal{M} \models \forall x \varphi[\alpha]$.

Then verify that

$$\forall x(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x \psi) \quad (x \text{ not free in } \varphi)$$

is a validity.

Exercise 0.2.

- Show that if t is a term, then $t_{\bar{s}/\bar{x}}$ is a term.
- Show that if φ is a formula, $\varphi_{\bar{s}/\bar{x}}$ is a formula of the same height.

Exercise 0.3.

Use the Substitution Lemma to verify that

$$\varphi_{t/x} \rightarrow \exists x \varphi$$

is a validity.

Exercise 0.4.

Show that if y does not occur in ψ ,

$$[\psi_{y/x}]_{x/y} \equiv \psi$$

Find a counterexample that shows this no longer holds if y *does* occur in ψ .