Math 557 Oct 13

Homogeneous Structures

Recall: We have seen that DLO is ω-categorical, meaning up to isomorphism there is only one countable model. Since there are no finite models, by Vaught's test, DLO is complete.

Question: Are there other examples like this?

Let $\mathcal{L}$ be a language and $\mathcal{M}$ be an $\mathcal{L}$-structure.

Definition 0.1.
For $A \subseteq M$, we denote by $\langle A \rangle^{\mathcal{M}}$ the smallest substructure of $\mathcal{M}$ whose domain contains A.

We say $\mathcal{N} \subseteq \mathcal{M}$ is finitely generated if $\mathcal{N} = \langle A \rangle^{\mathcal{M}}$ for some finite $A \subseteq M$.

Definition 0.2.
We say $\mathcal{M}$ is *homogeneous* if any isomorphism between finitely generated substructures of $\mathcal{M}$ can be extended to an automorphism of $\mathcal{M}$.

Let $\mathcal{L} = \{<\}$ and $\mathcal{M} = (\mathbb{Q}, <)$. The finitely generated substructures coincide with the finite substructures.

The age of a structure

In the back-and-forth proof of ω-categoricity of DLO, we used a homogeneity property, similarly for the proof that $(\mathbb{Q}, <) \cong (\mathbb{R}, <)$.

Definition 0.3.
The *age* of $\mathcal{M}$, denoted $\operatorname{age}(\mathcal{M})$, is the class of all finitely generated $\mathcal{L}$-structures isomorphic to a substructure of $\mathcal{M}$.

$\operatorname{age}(\mathbb{Q}, <)$ is the class of all finite linear orders.

Lemma 0.1.
Any two countable homogeneous structures with the same age are isomorphic.

Sketch. Let $\mathcal{A} \subseteq \mathcal{M}$ and $\mathcal{B} \subseteq \mathcal{N}$ be finitely generated, and let $\pi : \mathcal{A} \to \mathcal{B}$ be an isomorphism.

Let $a \in M \setminus A$. We need to show that π can be extended to an isomorphism $\mathcal{A} \cup \{a\} \to \mathcal{B}'$. This suffices, since we can then use the back-and-forth argument (remember our structures are countable) to extend everything to an automorphism of $\mathcal{M}$.

Suppose $\mathcal{A} = \langle E \rangle^{\mathcal{M}}$ and $\mathcal{B} = \langle F \rangle^{\mathcal{N}}$.

Let $\mathcal{A}' = \langle E \cup \{a\} \rangle^{\mathcal{M}} \subseteq \mathcal{M}$. Since $\mathcal{M}$ and $\mathcal{N}$ have the same age, there exists $\mathcal{C} \subseteq \mathcal{N}$ finitely generated such that $\mathcal{A}' \cong \mathcal{C}$ via some isomorphism g.

The map g is uniquely determined by its values on $E \cup \{a\}$. The restriction $g|_E$ induces an isomorphism $\langle E \rangle^{\mathcal{M}} \xrightarrow{\cong} \langle g(E) \rangle^{\mathcal{N}}$.

Therefore, $\pi \circ (g|_E)^{-1}$ is an isomorphism $\langle g(E) \rangle^{\mathcal{N}} \xrightarrow{\cong} \mathcal{B}$. Call this map τ. Note that $\langle g(E) \rangle^{\mathcal{N}}$ is a finitely generated subset of $\mathcal{N}$. Since $\mathcal{N}$ is homogeneous, τ extends to an automorphism $\overline{\tau} : \mathcal{N} \to \mathcal{N}$. Let $\mathcal{B}'$ be the image of $\mathcal{C}$ under $\overline{\tau}$. By definition of $\overline{\tau}$, $\mathcal{B} \subset \mathcal{B}'$ and $\mathcal{A}'$ is isomorphic to $\mathcal{B}'$ via the map $\overline{\tau} \circ g$. $\square$

Question

What do ages of countable homogeneous structures look like?