Math 557 Sep 12

Consistency and Completeness

Key Concepts

- **Consistency**:
  A theory T is **consistent** if there does not exist a formula ψ such that $T \vdash \psi$ and $T \vdash \neg\psi$.
- **Completeness**:
  A theory T is **complete** if it is consistent and for every sentence σ, $T \vdash \sigma$ or $T \vdash \neg\sigma$.
- In a complete theory, every statement is determined, in the sense that is either true or false in the theory.
- An important example of a complete theory is the *theory of a fixed structure $\mathcal{M}$*,

$$\mathrm{Th}(\mathcal{M}) = \{\sigma : \mathcal{M} \models \sigma\}$$

- If, on the other hand, we are given a complete theory T, does T have a model? This is the subject of the **Completeness Theorem**. It actually shows that any *consistent* theory has a model.

Problems

Exercise 0.1 (Carry-over from Sep 10)**.**
Prove the *Soundness Theorem*, i.e. show that

$$T \vdash \varphi \;\Rightarrow\; T \models \varphi$$

Exercise 0.2.
Why is the theory of a structure, $\mathrm{Th}(\mathcal{M})$, indeed a complete theory?

Exercise 0.3.
Show that a theory T is inconsistent iff for every formula ψ, $T \vdash \psi$

Exercise 0.4.
For each of the following classes $\mathbf{K}$ of structures, determine whether they are *axiomatizable*, i.e. whether there exists a theory T (in the language of the structures) such that

$$\mathbf{K} = \{\mathcal{M} : \mathcal{M} \models T\}$$

(In this case K is also called *elementary*.)

If $\mathbf{K}$ is axiomatizable, discuss whether it is actually *finitely axiomatizable* (i.e. can T be chosen finite). And is T complete?

- $\mathbf{K}$ = Abelian groups
- $\mathbf{K}$ = infinite sets
- $\mathbf{K}$ = fields of characteristic p (p prime)
- $\mathbf{K}$ = bipartite graphs
- $\mathbf{K}$ = torsion groups (*all elements have finite order*)
- $\mathbf{K}$ = algebraically closed fields of characteristic 0

(*Note: We do not yet have the tools to rigorously answer all these questions, but try an "educated" guess.*)

Exercise 0.5.
Suppose that $T \cup \{\neg\varphi\}$ is inconsistent. Show that $T \vdash \varphi$.

(*This is a technical formulation of the legitimacy of proofs by contradiction.*)