Math 557 Nov 21

Consistency and Undecidability

The Second Incompleteness Theorem

As we saw previously, for a recursivley axiomatizable theory T, the relation

$$\mathrm{Prf}(x,y) :\iff x = \ulcorner\psi\urcorner, y = \langle \ulcorner\varphi_1\urcorner, \dots, \ulcorner\varphi_n\urcorner\rangle, \varphi_1, \dots, \varphi_n \text{ is a } T\text{-proof of } \psi$$

is recursive. It follows that for such T, the relation

$$\begin{aligned} \mathrm{prov}_T(v) &:\Leftrightarrow \text{"v is the Gödel number of a formula } \varphi \text{ and } \varphi \text{ is provable in } T\text{"} \\ &\Leftrightarrow \exists y \ \mathrm{Prf}(x,y) \end{aligned}$$

is r.e. and thus definable by a Σ_1-formula $\theta(v)$.

In PA[1] one can verify the following **derivability conditions**:

$$\begin{aligned} &(D1) \quad T \vdash \sigma \ \Rightarrow \ T \vdash \theta(\ulcorner\sigma\urcorner) \\ &(D2) \quad T \vdash \ \theta(\ulcorner\sigma \to \tau\urcorner) \to (\theta(\ulcorner\sigma\urcorner) \to \theta(\ulcorner\tau\urcorner)) \\ &(D3) \quad T \vdash \ \theta(\ulcorner\sigma\urcorner) \to \theta(\ulcorner\theta(\ulcorner\sigma\urcorner)\urcorner) \end{aligned}$$

Here σ, τ are arbitrary sentences of the language $\mathcal{L}_A$, and the Gödel numbers n should be replaced in these formulas by the corresponding terms $\underline{n}$.

Using such a proof predicate, one can also formalize the **consistency** of a theory, for example:

$$\mathrm{Con}_T \quad :\Leftrightarrow \quad \neg\theta(\ulcorner 0 = 1\urcorner).$$

Theorem 0.1 (Second Gödel Incompleteness Theorem)**.**
Let T be a theory in the language L for which there exists a formal proof predicate θ as defined above (e.g., $T =$ PA). Then:

$$T \ \textit{consistent} \quad \Longrightarrow \quad T \nvdash \mathrm{Con}_T.$$

Sketch. By the Diagonalization Lemma, there exists a sentence τ such that

$$T \vdash \tau \ \leftrightarrow \ \neg\theta(\underline{\ulcorner\tau\urcorner}) \tag{1}$$

If $T \vdash \tau$, it follows from $(D1)$ that

$$T \vdash \theta(\underline{\ulcorner\tau\urcorner})$$

By (1), this implies T is inconsistent. This yields

[1] PA$^-$ is not strong enough for this

$$T \text{ consistent} \quad \Rightarrow \quad T \nvdash \tau. \tag{2}$$

One can show that this proof can be formalized and represented in T using the proof predicate θ, thus

$$T \vdash \mathrm{Con}_T \to \neg\theta(\underline{\ulcorner \tau \urcorner}). \tag{3}$$

If T is consitent, by (2), $T \nvdash \tau$. Using (1), this implies $\neg\theta(\underline{\ulcorner \tau \urcorner})$. Thus, by (3), $T \nvdash \mathrm{Con}_T$. $\square$

Truth is Not Definable

While the usual syntactic concepts formed for a formal language $\mathcal{L}$ can be defined in the language of number theory and their essential properties can be proven in PA, this is not possible for the semantic notion of truth:

Theorem 0.2 (Tarski's Undefinability Theorem)**.**
Let $\mathcal{M} \vDash \mathsf{PA}^-$. Then there is no $\mathcal{L}_A$-formula $\theta(v)$ such that for all natural numbers n

$$\mathcal{M} \vDash \theta(\underline{n}) \iff n = \ulcorner \sigma \urcorner \textit{ for an sentence } \sigma \textit{ with } \mathcal{M} \vDash \sigma.$$

Proof. If there were such a truth definition θ, then by the Diagonal Lemma we could find a sentence G such that

$$\mathsf{PA}^- \vdash G \leftrightarrow \neg\theta(\underline{\ulcorner G \urcorner}),$$

in particular,

$$\mathcal{M} \vDash G \quad \iff \quad \mathcal{M} \vDash \neg\theta(\underline{\ulcorner G \urcorner}).$$

On the other hand, for the truth definition we would need:

$$\mathcal{M} \vDash G \iff \mathcal{M} \vDash \theta(\underline{\ulcorner G \urcorner}),$$

which yields a contradiction. $\square$

Undecidability

Theorem 0.3 (Theorem)**.**
If T is a consistent theory in the language $\mathcal{L}_A$, then not both the diagonal function d and the set $\ulcorner T^{\vdash} \urcorner$ are representable in T.

Proof. We assume that both d is represented by a formula $\delta(x, y)$ and the set $\ulcorner T \urcorner$ is represented by a formula $\varphi_T(v_0)$ in T, and we choose $\theta = \neg\varphi_T$, so that from the Diagonal Lemma we obtain the existence of a formula G such that

$$T \vdash G \leftrightarrow \neg\varphi_T(\underline{\ulcorner G \urcorner}). \tag{4}$$

Case 1: $T \vdash G$, thus $\ulcorner G \urcorner \in \ulcorner T^{\vdash} \urcorner$. Then by representability, $T \vdash \varphi_T(\underline{\ulcorner G \urcorner})$ and with (4) also $T \vdash \neg G$, i.e., T would be inconsistent.

Case 2: $T \nvdash G$, thus $\ulcorner G \urcorner \notin \ulcorner T^{\vdash} \urcorner$. Then again by representability, $T \vdash \neg\varphi_T(\underline{\ulcorner G \urcorner})$, and with (4) $T \vdash G$, contradiction! $\square$

Corollary 0.1 (Church)**.**
If T is a consistent theory in the language $\mathcal{L}_A$ which extends PA^-, then T is undecidable.

Proof. If T extends PA^-, every recursive function, including d, is representable in T. Hence, by Theorem 0.3, $\ulcorner T^{\vdash} \urcorner$ is not representable. It follows that $\ulcorner T^{\vdash} \urcorner$ is not recursive (since all recursive sets are representable in any consistent extension of PA^-. $\square$

Corollary 0.2 (Church)**.**
The set

$$\mathrm{VAL} := \{\ulcorner\sigma\urcorner : \sigma \;\; \mathcal{L}_A\text{-sentence with } \vdash \sigma\}$$

(i.e. the set of all sentences that are validities) is not recursive.

Exercise 0.1. We say an $\mathcal{L}_A$-theory T' is a *finite extension* of an $\mathcal{L}_A$-theory T if $T' \supseteq T$ and $T' \setminus T$ is finite. Show that if T is decidable and T' is a finite extension of T, then T' is decidable.

Corollary 0.2. Since PA^- is finite, it is a finite extension of the empty theory $T = \emptyset$. Since $\mathrm{VAL} = \emptyset^{\vdash}$, by Exercise 0.1, if VAL were recursive so would be $(\mathsf{PA}^-)^{\vdash}$, contradicting Corollary 0.1. ☐