Math 557 Sep 24

The Compactness Theorem

Key Concepts

- **Theorem:** A theory T has a model if and only if every finite subtheory $T_0 \subseteq T$ has a model.

Problems

Exercise 0.1.
Let X be the set of all maximally consistent $\mathcal{L}$-theories. Recall that the sets

$$\langle \sigma \rangle = \{T \in X : \sigma \in T\} \quad (\sigma \ \mathcal{L}\text{-sentence})$$

generate a Hausdorff topology on X.

Show that the topology is compact.

Exercise 0.2.
One can use the compactness theorem to construct non-standard models of arithmetic, i.e., models of $\mathrm{Th}(\mathbb{N}, 0, 1, +, \cdot, <)$ not isomorphic to $\mathbb{N}$.

Use the same technique for $\mathrm{Th}(\mathbb{R}, \{c_a : a \in \mathbb{R}\}, +, \cdot, <)$, where for every $a \in \mathbb{R}$ we add a constant symbol c_a to the language? What kind of structure do we obtain? Discuss.

> **!** Take-home Problem
>
> Use the compactness theorem to show (without using the Axiom of Choice) that every set can be linearly ordered.
> Try to trengthen this to:
> Every partial order can be extended to a linear order.