Math 557 Nov 3

End Extensions

The standard model can be embedded into every model $\mathcal{M}$ of PA as an initial segment. It turns out this already holds for models of the theory PA^-.

Definition 1

Let L be a language containing a 2-ary symbol $<$, and let $\mathcal{M}$ and $\mathcal{N}$ be L-structures with $\mathcal{M} \subseteq \mathcal{N}$. Then $\mathcal{N}$ is called an **end extension** of $\mathcal{M}$ (and correspondingly $\mathcal{M}$ is an **initial segment** of $\mathcal{N}$) if and only if the larger set N does not add any further elements below an element of M:

$$\mathcal{M} \subseteq_{end} \mathcal{N} :\iff \text{ for all } x \in M, y \in N : (y <^N x \Rightarrow y \in M).$$

Each natural number n is represented in the standard model, which we also simply denote by $\mathbb{N}$ here, by the constant term

$$\underline{n} = 1 + ... + 1 \quad (n \text{ times})$$

where $\underline{0}$ is the constant 0.

Theorem 2

Let $\mathcal{M} \vDash \mathsf{PA}^-$. Then the map

$$n \mapsto \underline{n}^{\mathcal{M}}$$

defines an embedding of the standard model $\mathbb{N}$ onto an initial segment of $\mathcal{M}$.
In particular, every model of PA^- is isomorphic to an end extension of the standard model $\mathbb{N}$.*

Proof. By simple induction (in the meta-theory), one shows for all natural numbers n, k, l:

$$\begin{aligned} n = k + l \quad &\Longrightarrow \quad \mathsf{PA}^- \vdash \underline{n} = \underline{k} + \underline{l} \\ n = k \cdot l \quad &\Longrightarrow \quad \mathsf{PA}^- \vdash \underline{n} = \underline{k} \cdot \underline{l} \\ n < k \quad &\Longrightarrow \quad \mathsf{PA}^- \vdash \underline{n} < \underline{k} \end{aligned}$$

and

$$\mathsf{PA}^- \vdash \forall x \, (\, x \leq \underline{k} \to x = \underline{0} \, \vee ... \vee \, x = \underline{k})$$

The first three statements will later be generalized to all recursive functions and relations; they imply that the map $n \mapsto \underline{n}^{\mathcal{M}}$ is a homomorphism, and, due to the last statement, the map is also an embedding onto an initial segment of $\mathcal{M}$. $\square$

Remark

The standard model has no proper initial segment, and $\mathbb{Z}[X]^+$ has $\mathbb{N}$ as its only proper initial segment. On the other hand, every model $\mathcal{M} \vDash \mathsf{PA}^-$ has a proper end extension that is also a model of PA^-, namely the non-negative part of the polynomial ring $R[X]$, where R is the discretely ordered ring associated with the model $\mathcal{M}$.