Math 557 Oct 8

Ultraproducts

Direct Products

Let $(\mathcal{M}_i)_{i\in I}$ be a family of L-structures.

We define the **direct product**

$$\mathcal{M} = \prod_{i\in I} \mathcal{M}_i$$

as follows:

1. The universe is the Cartesian product $M = \prod_{i\in I} M_i$. If a is an element of M, we denote its i-th component (an element of M_i) by a_i and extend this notation to vectors: if $\vec{a}$ is a finite tuple in M^n, $\vec{a}_i$ denotes the n-tuple in M_i consisting of the M_i-entries of $\vec{a}$.
2. For each relation symbol $R \in \mathcal{L}$,
$$R^{\mathcal{M}}(\vec{a}) :\iff \forall i \in I,\, \vec{a}_i \in R^{\mathcal{M}_i}$$
3. For each function symbol $f \in \mathcal{L}$,
$$f^{\mathcal{M}}(\vec{a}) := (f^{\mathcal{M}_i}(\vec{a}_i))_{i\in I}.$$
4. For each constant $c \in \mathcal{L}$,
$$c^{\mathcal{M}} = (c^{\mathcal{M}_i})_{i\in I}.$$

Examples and Observations

- The direct product of groups is again a group (componentwise operation).
- The direct product of fields is **not** a field:
$$(1,0)\cdot(0,1) = (0,0).$$
- The direct product of linear orders is only a **partial order**.

We often want to preserve properties that hold in “most” component structures.
To formalize “most,” we use **filters** on I.

Filters and Ultrafilters

A **filter** $\mathcal{F}$ on a set I is a nonempty collection of subsets of I satisfying:

1. $\emptyset \notin \mathcal{F}$
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$
3. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$

An **ultrafilter** $\mathcal{U}$ is a maximal filter, equivalently:

For all $A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

Ultrafilters interact nicely with logical operators:

- $A \notin \mathcal{U} \iff I\backslash A \in \mathcal{U}$,
- $A \in \mathcal{U} \wedge B \in \mathcal{U} \iff A \cap B \in \mathcal{U}$,
- $A \in \mathcal{U} \vee B \in \mathcal{U} \iff A \cup B \in \mathcal{U}$.

Examples

- A **principal filter** is of the form
$$\mathcal{F}_A = \{X \subseteq I : A \subseteq X\}$$
for some nonempty $A \subseteq I$.
If $A = \{a\}$, then $\mathcal{F}_A$ is a **principal ultrafilter**.
- A **free** (non-principal) ultrafilter exists on every infinite set I
(via Zorn's Lemma / Boolean prime ideal theorem).

Existence of Ultrafilters

A family of sets has the **finite intersection property (FIP)** if every finite subfamily has nonempty intersection.

Theorem 0.1.
If a family $\mathcal{A} \subseteq \mathcal{P}(I)$ has the FIP,
then there exists an ultrafilter $\mathcal{U}$ on I with $\mathcal{A} \subseteq \mathcal{U}$.

Reduced Products

Given a filter $\mathcal{F}$ on I and structures $(\mathcal{M}_i)_{i\in I}$, define the **reduced product**

$$\mathcal{M}/\mathcal{F}$$

as follows.

Let $M = \prod_{i\in I} M_i$. For $a, b \in M$, define

$$a \sim_{\mathcal{F}} b \iff \{\, i \in I : a_i = b_i \,\} \in \mathcal{F}.$$

The universe of $\mathcal{M}/\mathcal{F}$ is the quotient $M/{\sim_{\mathcal{F}}}$, with elements denoted $a_{\mathcal{F}}$ (alternatively, $a/\mathcal{F}$).

For symbols of $\mathcal{L}$:

- **Relations:**
$$R^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) :\iff \{\, i : \mathcal{M}_i \models R(\vec{a}_i) \,\} \in \mathcal{F}.$$
- **Functions:**
$$f^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) = [\,(f^{\mathcal{M}_i}(\vec{a}_i))_{i\in I}\,]_{\mathcal{F}}.$$
- **Constants:**
$$c^{\mathcal{M}/\mathcal{F}} = ((c^{\mathcal{M}_i})_{i\in I})_{\mathcal{F}}.$$

Exercise 0.1. Check that the above definition does not depend on the choice of representative for each equivalence class

Ultraproducts

If $\mathcal{U}$ is an **ultrafilter** on I, the reduced product

$$\prod_{i\in I} \mathcal{M}_i/\mathcal{U}$$

is called the **ultraproduct** of $(\mathcal{M}_i)_{i\in I}$ modulo $\mathcal{U}$.

When all $\mathcal{M}_i$ are the same structure $\mathcal{M}$, we get an **ultrapower**

$$\mathcal{M}^I/\mathcal{U}.$$