Math 557 Oct 10

Ultrapoducts - Exercises

Key concepts

(Ultra)filters

A **filter** $\mathcal{F}$ on a set I is a nonempty collection of subsets of I satisfying:

1. $\emptyset \notin \mathcal{F}$
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$
3. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$

An **ultrafilter** $\mathcal{U}$ is a maximal filter, equivalently:

For all $A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

Reduced products

Given: filter $\mathcal{F}$ on I and structures $(\mathcal{M}_i)_{i \in I}$. Let $M = \prod_{i \in I} M_i$ and define

$$a \sim_{\mathcal{F}} b \iff \{\, i \in I : a_i = b_i \,\} \in \mathcal{F}.$$

The universe of $\mathcal{M}/\mathcal{F}$ is the quotient $M/{\sim_{\mathcal{F}}}$, with elements denoted $a_{\mathcal{F}}$ (alternatively, $a/\mathcal{F}$).

- **Relations:**

$$R^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) :\iff \{\, i : \mathcal{M}_i \models R(\vec{a}_i) \,\} \in \mathcal{F}.$$

- **Functions:**

$$f^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) = [\,(f^{\mathcal{M}_i}(\vec{a}_i))_{i \in I}\,]_{\mathcal{F}}.$$

- **Constants:**

$$c^{\mathcal{M}/\mathcal{F}} = ((c^{\mathcal{M}_i})_{i \in I})_{\mathcal{F}}.$$

Łoś' Theorem

Let $\mathcal{M}/\mathcal{U} = \prod_{i \in I} \mathcal{M}_i/\mathcal{U}$ be an ultraproduct. For every $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and tuples $\vec{a} \in \prod_{i \in I} M_i$,

$$\mathcal{M}/\mathcal{U} \models \varphi[\vec{a}_{\mathcal{U}}] \iff \{\, i \in I : \mathcal{M}_i \models \varphi[\vec{a}_i] \,\} \in \mathcal{U}.$$