

## The Diagonal Lemma

In the following, we consider the **diagonal function**  $d$ , defined by

$$d(n) = \begin{cases} \ulcorner \forall y (y = \underline{n} \rightarrow \sigma(y)) \urcorner & \text{if } n = \ulcorner \sigma(v_0) \urcorner \text{ for an } L\text{-formula } \sigma(v_0) \\ 0 & \text{otherwise.} \end{cases}$$

### Lemma 0.1.

Let  $T$  be an  $L$ -theory,  $\theta(v_0)$  an  $L$ -formula with the single free variable  $v_0$ . If the diagonal function  $d$  is representable in  $T$ , then there exists an  $L$ -sentence  $G$  with the property

$$T \vdash G \leftrightarrow \theta(\ulcorner G \urcorner).$$

If  $\theta(v_0)$  is a  $\Pi_1$ -formula, then  $G$  can also be chosen as a  $\Pi_1$ -sentence.

*Proof.* If  $d$  is represented in  $T$  by the formula  $\delta(x, y)$ , then define

$$\psi(v_0) := \forall y (\delta(v_0, y) \rightarrow \theta(y)),$$

and set  $n := \ulcorner \psi(v_0) \urcorner$ .

**Note:**  $n$  is actually a number in which the variable  $v_0$  does not occur; however, the Gödel number of  $v_0$  does enter into the calculation of  $n$ .

For  $G$ , we now choose the sentence

$$G := \forall y (y = \underline{n} \rightarrow \psi(y)).$$

Then  $G$  has Gödel number  $d(\ulcorner \psi(v_0) \urcorner)$ . If  $\delta(x, y)$  is a  $\Sigma_1$ -formula and  $\theta(v)$  is a  $\Pi_1$ -formula, then  $\psi$  and  $G$  are (equivalent to)  $\Pi_1$ -formulas. Thus it remains to show that  $T \vdash G \leftrightarrow \theta(\ulcorner G \urcorner)$ :

It is clear that

$$\begin{aligned} T \vdash G &\leftrightarrow \psi(\underline{n}), & \text{i.e., by the definition of } \psi \\ (1) \quad T \vdash G &\leftrightarrow \forall y (\delta(\underline{n}, y) \rightarrow \theta(y)). \end{aligned}$$

But the fact that  $d$  is represented in  $T$  by  $\delta(x, y)$  means

$$d(n) = \ulcorner G \urcorner \Rightarrow T \vdash \delta(\underline{n}, \ulcorner G \urcorner) \quad \text{and} \quad T \vdash \exists! y \delta(\underline{n}, y).$$

Thus in particular

(2)  $T \vdash \forall y (\delta(\underline{n}, y) \leftrightarrow y = \underline{\ulcorner G \urcorner})$  and together with (1) it follows  
 $T \vdash G \leftrightarrow \forall y (y = \underline{\ulcorner G \urcorner} \rightarrow \theta(y))$ , hence  
 $T \vdash G \leftrightarrow \theta(\underline{\ulcorner G \urcorner})$ .

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