

# MATH 557 Midterm 2 Preparation

The second midterm will have two parts:

1. Reproduce a proof (in sufficient detail) of one the theorems below we covered in class.
2. Present a proof to one of the exercises (or a closely related problem) listed on this page.

## Theorems

### Theorem 1

Any two countable models of DLO are isomorphic.

### Theorem 2

Suppose for  $\mathcal{L}$ -theories  $\mathcal{M}, \mathcal{N}$ ,  $\mathcal{M} \subseteq \mathcal{N}$  and for any formula  $\psi(x, \vec{y})$  and any  $\vec{a} \in M$ , if there exists  $b \in N$  such that  $\mathcal{N} \models \psi[b, \vec{a}]$ , then there also exists  $c \in M$  such that  $\mathcal{N} \models \psi[c, \vec{a}]$ . Then  $\mathcal{M} \preceq \mathcal{N}$ .

### Theorem 3

Let  $\mathcal{A}$  be an  $\mathcal{L}$ -structure,  $\kappa$  an infinite cardinal with  $\text{card}(\mathcal{L}) \leq \kappa$  and  $\kappa \leq \text{card}(A)$ . Then there exists a structure  $\mathcal{B}$  with

$$\mathcal{B} \preceq \mathcal{A}, \text{card}(\mathcal{B}) = \kappa.$$

### Theorem 4

Let  $\mathcal{A}$  be an infinite  $\mathcal{L}$ -structure,  $\kappa$  a cardinal with  $\text{card}(\mathcal{L}) \leq \kappa$  and  $\text{card}(A) \leq \kappa$ . Then there exists a structure  $\mathcal{B}$  with

$$\mathcal{A} \preceq \mathcal{B}, \text{card}(\mathcal{B}) = \kappa.$$

### Theorem 5

Suppose  $T$  is a consistent  $\mathcal{L}$ -theory with no finite models. If  $T$  is  $\kappa$ -categorical for some  $\kappa \geq |\mathcal{L}|$ , then  $T$  is complete.

### Theorem 6

Let  $\mathcal{M}/\mathcal{U} = \prod_{i \in I} \mathcal{M}_i/\mathcal{U}$  be an ultraproduct. For every  $\mathcal{L}$ -formula  $\varphi(x_1, \dots, x_n)$  and tuples  $\vec{a} \in \prod_{i \in I} M_i$ ,

$$\mathcal{M}/\mathcal{U} \models \varphi[\vec{a}_{\mathcal{U}}] \iff \{i \in I : \mathcal{M}_i \models \varphi[\vec{a}_i]\} \in \mathcal{U}.$$

### Theorem 7

Any two countable homogeneous structures with the same age are isomorphic.

## Problems

### Problem 1

Let  $\mathcal{L}$  be an arbitrary language. Show that for finite  $\mathcal{L}$ -structures  $\mathcal{M}, \mathcal{N}$ ,

$$\mathcal{M} \equiv \mathcal{N} \iff \mathcal{M} \cong \mathcal{N}$$

### Problem 2

A model  $\mathcal{M}$  of an  $\mathcal{L}$ -theory  $T$  is **prime** if for every  $\mathcal{N} \models T$ , there exists an elementary embedding of  $\mathcal{M}$  into  $\mathcal{N}$ .

- Show that  $\mathbb{N}$  is a prime model of  $\text{Th}(\mathbb{N}, +, \cdot, 0, 1)$ .
- Show that if  $T$  is an  $\omega$ -categorical theory over a countable language and has no finite models,  $T$  has a prime model.

### Problem 3

If we take an ultraproduct over the same structure  $\mathcal{M}$  along an index set  $I$ , we call this an **ultrapower**, denoted by

$$\mathcal{M}^I/\mathcal{U}$$

Let  $\mathcal{U}$  be a free ultrafilter on an (infinite) set  $I$ . The map  $j : b \mapsto (b)_{i \in I}/\mathcal{U}$  defines an **elementary embedding** of  $\mathcal{M}$  into  $\mathcal{M}^I/\mathcal{U}$  (i.e. it is injective and the image is an elementary substructure).

Show that if  $\mathcal{M}$  is infinite and  $I = \mathbb{N}$ ,  $j$  is not a surjection.

### Problem 4

Let  $G_N$  be the set of all simple graphs on the vertex set  $\{1, \dots, N\}$ . Let  $\mathbb{P}_N(\sigma)$  be the probability that an  $\mathcal{L}_G$ -sentence  $\sigma$  holds for a random graph in  $G_N$ , i.e.

$$\mathbb{P}_N(\sigma) = \frac{|\{\mathcal{G} \in G_N : \mathcal{G} \models \sigma\}|}{|G_N|}.$$

Show that for any  $\mathcal{L}_G$ -sentence  $\sigma$ ,

$$\text{either } \lim_{N \rightarrow \infty} \mathbb{P}_N(\sigma) = 0 \text{ or } \lim_{N \rightarrow \infty} \mathbb{P}_N(\sigma) = 1.$$