Math 557 Nov 5

Definability of Computable Functions

We now extend the connection between arithmetical formulas and computable functions significantly. Our goal is to show that the recursive relations on $\mathbb{N}$ are precisely those that are Δ_1-definable. At the core of this connection is the **Definability Theorem**. For a partial function f, let Γ_f denote the **graph** of f, i.e., the relation

$$\Gamma_f(\vec{x}, y) :\iff f(\vec{x}) = y.$$

The Definability Theorem

Theorem 0.1.
A partial function $f :\subseteq \mathbb{N} \to \mathbb{N}$ is computable if and only if its graph Γ_f over the natural numbers is definable by a Σ_1-formula, i.e., if there exists a Σ_1-formula θ such that for all $x, y \in \mathbb{N}$:

$$\Gamma_f(x, y) \iff \mathbb{N} \models \theta[x, y].$$

The direction ($\Leftarrow$) is relatively straightforward: Let $\theta = \exists z\, \psi$ be a Σ_1-formula that defines the graph Γ_f of a partial function f, where $\psi(x, y, z)$ is a Δ_0-formula. As Δ_0-formulas define primitive recursive relations, we obtain a computable function g by letting

$$g(x) = \mu z\, \psi(x, (z)_0, (z)_1).$$

Then $(g(x))_0$ is the smallest y with $\mathbb{N} \models \theta[x, y]$, if such a y exists, and undefined otherwise, since for each $x \in \mathbb{N}$ there is at most one y with $\mathbb{N} \models \theta[x, y]$. Thus $(g(x))_0 \cong f(x)$, and therefore f is partial computable.

For the proof of ($\Rightarrow$), we call partial functions f whose graph can be defined by a Σ_1-formula over the natural numbers *functions with a Σ_1-graph*. The obvious strategy is to show that the set of these functions contains the initial functions and is closed under composition, primitive recursion, and the μ-operator.

Most cases are straightforward (just by writing out the definitions), but when we get to closure under primitive recursion, we run into difficulties: Suppose the function f arises from functions g, h by primitive recursion:

$$f(x, 0) \cong g(x)$$
$$f(x, y + 1) \cong h(x, y, f(x, y))$$

We may assume that the graphs of g and h are definable by Σ_1-formulas. To describe the graph of f with this, we note that the value of $f(x, y)$ is computed through the preceding values

$$u_0 = f(x, 0),\ u_1 = f(x, 1),\ \dots,\ u_y = f(x, y),$$

We somehow have to *eliminate the function* f from this sequence to obtain an *explicit* definition (in the language of arithmetic).

The idea is to express the following through a formula:

> There exists a code u such that u codes a sequence of numbers $(u_0, \dots, u_y)$ in which the u_{i+1} is obtained from u_i using a valid application of the primitive recursion scheme with respect to g and h.

The problem is that we do not know a priori how long the sequence coded by u is. We therefore cannot use the preferred coding scheme

$$\langle u_0, \dots, u_y \rangle$$

which is Δ_0-definable. The coding scheme

$$(n_0, n_1, \dots, n_k) \quad \text{by} \quad p_0^{n_0+1} \cdot p_1^{n_1+1} \cdot \dots \cdot p_k^{n_k+1}.$$

on the other hand is uniform independent of the length of the sequence, but it uses exponentiation, which is not present in PA^- (and would have to be defined via recursion).

This is where Gödel, according to Mostowski, "had a phone call with God".

Gödel's Lemma

Lemma 0.1.
There exists a primitive recursive function β such that for every k and for every finite sequence $x_0, x_1, \dots x_{k-1} \in \mathbb{N}$, there exists a natural number c with

$$\textit{for all } i < k: \ \beta(c, i) = x_i.$$

In fact, there exists a Δ_0-formula $\theta(x, y, z)$ such that

$$\mathbb{N} \vDash \forall x, y \, \exists! z \, \theta(x, y, z),$$

and the formula $\theta(x, y, z)$ defines the function β over the natural numbers.

Proof. Let $x_0, x_1, \dots x_{k-1}$ be a sequence of natural numbers.

Set $m := b!$ where $b := \max(k, x_0, x_1, \dots x_{k-1})$. Then the sequence of numbers

$$m + 1, 2m + 1, \dots k \cdot m + 1$$

are pairwise coprime. By the theorem on simultaneous congruences (*Chinese Remainder Theorem*), there exists a natural number a with

$$a \equiv x_i \pmod{(i + 1)m + 1} \quad \text{for all } i < k.$$

Now we can choose $\langle a, m \rangle$ as a code for the sequence $x_0, x_1, \dots x_{k-1}$, because from this we can recover each x_i for every $i < k$ as the remainder of the division of a by the number $(i+1)m+1$. If $rem(x : y) = z$ denotes the remainder of the division of x by y (when $y \neq 0$ and $rem(x : 0) = 0$), then this is a p.r. function with a Δ_0-definition. We obtain another p.r. function by

$$\alpha(a, m, i) = rem(a : (1 + i)m + 1).$$

Let p_1, p_2 denote the inverse functions of the pairing function $\langle x, y \rangle$ above, for which

$$\langle p_1(x), p_2(x) \rangle = x, \quad \text{where } p_1(x), p_2(x) \leq x,$$

these are also p.r., and we finally obtain the **Gödel beta function** as

$$\beta(c, i) = \alpha(p_1(c), p_2(c), i).$$

□

Completing the proof of the Definability Theorem

Returning to the sequence

$$u_0 = f(x, 0), \, u_1 = f(x, 1), \, \dots, \, u_y = f(x, y),$$

we can now use the β-function to encode this by a single number u. The first value u_0 is determined by the graph of g, and the further values u_{i+1} are determined from u_i according to the recursion conditions via the graph of h:

$$\forall i < y \, \exists r, s \, [\beta(u,i) = r \wedge \beta(u,i+1) = s \wedge \Gamma_h(\vec{x}, i, r, s)]$$

This formula can be transformed (by choosing a sufficiently large w) into

$$\exists w \, \forall i < y \, \exists r, s < w \, [\beta(u,i) = r \wedge \beta(u,i+1) = s \wedge \Gamma_h(\vec{x}, i, r, s)].$$

The last value of the sequence encoded by u is the function value of f at the point (x, y). Thus we obtain the following description of the graph of f:

$$\begin{aligned} \Gamma_f(\vec{x}, y, z) \iff & \exists u, v, w \, (\Gamma_g(\vec{x}, v) \wedge \beta(u, 0) = v \wedge \beta(u, y) = z \wedge \\ & \forall i < y \, \exists r, s < w \, [\beta(u,i) = r \wedge \beta(u,i+1) = s \wedge \Gamma_h(\vec{x}, i, r, s)]), \end{aligned}$$

It is not hard to transform this into an equivalent Σ_1-formula.

Characterization of the Arithmetic Hierarchy

If we call a set $A \subseteq \mathbb{N}^k$ that is definable by a formula in Γ over the natural numbers a Γ-*set*, we obtain a new characterization of the first levels of the arithmetic hierarchy:

Corollary 0.1.

1. The Σ_1-sets are precisely the r.e. (recursively enumerable) relations $R \subseteq \mathbb{N}^k$.

2. The Π_1-sets are the complements of r.e. relations.

3. The Δ_1-sets are precisely the recursive relations $R \subseteq \mathbb{N}^k$.

We leave the short proof of (1) as an exercise. (2) follows from (1) by using negation, and for (3) use the fact that a set is computable if and only if the st and its complement are r.e.