

## Midterm 1 Review

### ! Take-home Problem 1

Prove unique readability for the set of  $\mathcal{L}$ -formulas.

*Solution 0.1.* **First** prove **readability**, i.e. the statement that for any  $\mathcal{L}$ -formula  $\varphi$ , exactly one of the following cases holds:

- (1) There exist terms  $s, t$  such that  $\varphi \equiv s = t$ .
- (2) There exists a relation symbol  $R$  and, if  $n$  is the arity of  $R$ , terms  $t_1, \dots, t_n$  such that  $\varphi \equiv R t_1 \dots t_n$ .
- (3) There exists a formula  $\psi$  such that  $\varphi \equiv \neg \psi$ .
- (4) There exist formulas  $\psi, \theta$  such that  $\varphi \equiv (\psi \wedge \theta)$ .
- (5) There exists a variable  $x$  and a formula  $\psi$  such that  $\varphi \equiv \exists x \psi$ .

To see this, let  $F$  be the subset of  $\mathcal{L}^*$  such that every  $\theta \in F$  satisfies exactly one of (1)-(5). Then  $F$  contains all atomic formulas and is closed under  $\neg, \wedge, \exists x$ . Since the set of  $\mathcal{L}$ -formulas is the smallest such set, it follows that every  $\mathcal{L}$ -formula is contained in  $F$ , and therefore has the desired property.

**Second** we argue that a proper initial segment of a formula cannot be a formula. We proceed by induction on the length  $l$  of  $\varphi$ . For  $l = 1$  there is no formula of that length. Now suppose  $\varphi$  is a formula of length  $l + 1$ , and  $\beta$  is a proper initial segment. We may assume  $\beta$  is not empty. We consider theses (1)-(5) from readability.

- Case**  $\varphi \equiv s = t$  Either  $\beta \equiv s = t'$  with  $t \subset t'$ , but this would contradict that no proper initial segment of a term is a term. Or  $\beta \equiv s =$ : An easy induction over the length of a formula shows that no formula ends with  $=$ . Or  $\beta \subseteq s$ . Another easy induction shows that no formula can be a term or an initial segment of a term.
- Case**  $\varphi \equiv R t_1 \dots t_n$  If  $\beta$  were a formula, it has to be of the form  $R s_1 \dots s_n$ . Comparing terms, we would see that one of the  $s_i$  is a proper initial segment of some  $t_j$ , which is impossible.
- Case**  $\varphi \equiv \neg \theta$  Then  $\beta \equiv \neg \beta'$  with  $\beta'$  a proper initial segment of  $\theta$ . By inductive hypothesis,  $\beta'$  is not a formula, hence  $\neg \beta'$  is not a formula either.
- Case**  $\varphi \equiv (\psi \wedge \theta)$  Then  $\beta$  starts with ( but does not end with ). Another induction shows that for any formula, the number of ( 's always equals the number of ) 's.
- Case**  $\varphi \equiv \exists x \psi$  Neither ' $\exists$ ' nor ' $\exists x$ ' are formulas. If  $\beta$  is longer than that, it is of the form  $\exists x \beta'$ . By inductive hypothesis,  $\beta'$  is not a formula, and hence  $\beta$  is not a formula (by readability).  $\varphi$

**Finally**, we prove **unique readability**: The choices in (1)-(5) are unique.

- Case (1)** Suppose  $\varphi \equiv s = t \equiv s' = t'$ . Since no proper initial segment of a term is a term, it must hold that  $s \equiv s'$  and  $t \equiv t'$ .
- Case (2)** Suppose  $\varphi \equiv R t_1 \dots t_n \equiv S s_1 \dots s_k$ . Then  $R \equiv S$  and  $n = k$ . Comparing terms inductively, using again the fact that no proper initial segment of a term is a term, we get  $s_i \equiv t_i$  for all  $i$ .
- Case (3)** Suppose  $\varphi \equiv \neg \psi \equiv \neg \theta$ . Immediately, we infer  $\psi \equiv \theta$ .
- Case (4)** Suppose  $\varphi \equiv (\psi_1 \wedge \theta_1) \equiv (\psi_2 \wedge \theta_2)$ . Assume  $\psi_1 \not\equiv \psi_2$ . Then  $\psi_1$  is a proper initial segment of  $\psi_2$  or vice versa (since both of them are part of  $\varphi$ ). Either case is impossible due to the fact that no proper initial segment of a formula is a formula. Hence  $\psi_1 \equiv \psi_2$  and thus also  $\theta_1 \equiv \theta_2$ .
- Case (5)** Suppose  $\varphi \equiv \exists x \psi \equiv \exists y \theta$ . By comparing entries, we get  $x \equiv y$  and thus  $\psi \equiv \theta$ , as desired.

### ! Take-home Problem 2

Let  $\mathcal{L}$  be any finite language and let  $\mathcal{M}$  be a finite  $\mathcal{L}$ -structure. Show that there is an  $\mathcal{L}$ -sentence  $\varphi$  such that

$$\mathcal{N} \models \varphi \iff \mathcal{N} \cong \mathcal{M}.$$

*Solution 0.2.* Suppose  $\mathcal{L} = \{c_1, \dots, c_k, f_1^{(a_1)}, \dots, f_l^{(a_l)}, R_1^{(b_1)}, \dots, R_j^{(b_j)}\}$  where the  $(a_i), (b_m)$  denote the arities of the respective symbols.

Since  $\mathcal{M}$  is finite, we may assume  $M = \{1, \dots, n\}$  for some  $n \in \mathbb{N}$ .

The basic idea is to collect all “elementary facts” about  $\mathcal{M}$  in a single formula (think of a *group multiplication table*, just for all functions and relations).

Define

$$\varphi \equiv \exists x_1, \dots, x_n \left( \bigwedge_{i \neq m} x_i \neq x_m \wedge \forall y \bigvee_{i \leq n} y = x_i \right) \quad (1)$$

$$\wedge \bigwedge_{i \leq k} c_i = x_{c_i^{\mathcal{M}}} \quad (2)$$

$$\wedge \bigwedge_{i \leq l} \bigwedge_{\pi \in \{1, \dots, n\}^{a_i}} f_i x_{\pi(1)} \dots x_{\pi(a_i)} = x_{f_i^{\mathcal{M}}(\pi(1), \dots, \pi(a_i))} \quad (3)$$

$$\wedge \bigwedge_{i \leq j} \bigwedge_{\pi \in \{1, \dots, n\}^{b_i}} \delta_{\pi} R_i x_{\pi(1)} \dots x_{\pi(b_i)} \quad (4)$$

where  $\delta_{\pi}$  is empty if  $R(\pi(1), \dots, \pi(b_i))$  holds in  $\mathcal{M}$ , and  $\neg$  if not.

Assume  $\mathcal{N} \models \varphi$ . Due to the first line of the equation,  $N$  has exactly  $n$  elements, and let  $r_i \in N$  be the witness to  $\exists x_i$ . Define a mapping  $\tau : M \rightarrow N$  by letting  $\tau(i) = r_i$ . We claim that  $\tau$  is an isomorphism.

By definition we have  $\tau(c_i^{\mathcal{M}}) = r_{c_i^{\mathcal{M}}}$ . Moreover, by the second line of the formula,  $r_{c_i^{\mathcal{M}}}$  is the unique element of  $N$  that makes the formula  $c_i = x_{c_i^{\mathcal{M}}}$  true. It follows that  $\tau(c_i^{\mathcal{M}}) = c_i^{\mathcal{N}}$  for all  $1 \leq i \leq k$ .

Using a similar argument with the third line of the formula, we obtain

$$\tau(f_i^{\mathcal{M}}(s_1, \dots, s_{a_i})) = r_{f_i^{\mathcal{M}}(s_1, \dots, s_{a_i})} = f_i^{\mathcal{N}}(r_{s_1}, \dots, r_{s_{a_i}}) = f_i^{\mathcal{N}}(\tau(s_1), \dots, \tau(s_{a_i}))$$

The argument for  $R^{\mathcal{M}}(s_1, \dots, s_{b_i}) \iff R^{\mathcal{N}}(\tau(s_1), \dots, \tau(s_{b_i}))$  is similar.

On the other hand, if  $\mathcal{M} \cong \mathcal{N}$  via  $\tau$ , then  $\mathcal{N} \models \psi[\tau(1), \dots, \tau(n)]$ , where  $\psi$  is such that  $\varphi \equiv \exists x_1, \dots, x_n \psi$ , due to the fact that  $\tau$  is an isomorphism, and thus  $\mathcal{N} \models \varphi$ .

### ! Take-home Problem 3

Give an example of a language  $\mathcal{L}$  and an  $\mathcal{L}$ -sentence  $\psi$  such that

- there is at least one  $\mathcal{L}$ -structure  $A$  such that  $A \models \psi$ ,
- for all  $\mathcal{L}$ -structures  $A$ , if  $A \models \psi$ , then the universe  $A$  of  $A$  is infinite.

*Solution 0.3.* Let  $\mathcal{L} = \{<\}$ , where  $<$  is a binary relation symbol. Define

$$\psi \equiv \forall x \, x \not< x \quad (5)$$

$$\wedge \forall x, y \, x < y \vee x = y \vee y < x \quad (6)$$

$$\wedge \forall x, y, z \, (x < y \wedge y < z) \rightarrow x < z \quad (7)$$

$$\wedge \forall x \exists y \, x < y \quad (8)$$

The formula says that  $<$  is a linear order with no maximal element.

Clearly,  $(\mathbb{Z}, <) \models \psi$ .

Now suppose  $\mathcal{M} \models \psi$ . Due to the last line of  $\psi$ , there exists a function  $f : M \rightarrow M$  such that  $x < f(x)$  for all  $x \in M$ . We claim that for any  $x$  and for any  $n \neq m$ ,

$$f^{(n)}(x) \neq f^{(m)}(x)$$

This follows from antireflexivity (line one) and transitivity (line three).

Therefore, the set

$$\{x, f^{(1)}(x), f^{(2)}(x), \dots\}$$

is an infinite subset of  $N$ .

### ! Take-home problem 4

Show that

$$\begin{aligned} \{\varphi \rightarrow \psi\} &\vdash \exists x\varphi \rightarrow \exists x\psi \\ \{\varphi \rightarrow \psi\} &\vdash \forall x\varphi \rightarrow \forall x\psi \end{aligned}$$

*Solution 0.4.* We give derivations below (with brief justifications). We collect simple substeps into a single one.

For  $\{\varphi \rightarrow \psi\} \vdash \exists x\varphi \rightarrow \exists x\psi$ :

Formula	Justification
$\varphi \rightarrow \psi$	given
$\psi \rightarrow \exists x\psi$	(Q2)
$\varphi \rightarrow \exists x\psi$	tautology
$\neg\exists x\psi \rightarrow \neg\varphi$	tautology
$\forall x(\neg\exists x\psi \rightarrow \neg\varphi)$	$\forall$ -intro
$\neg\exists x\psi \rightarrow \forall x\neg\varphi$	(Q1), $x$ not free in $\neg\exists x\psi$
$\neg\forall x\neg\varphi \rightarrow \exists x\varphi$	tautology
$\exists x\varphi \rightarrow \exists x\psi$	(Q3)

For  $\{\varphi \rightarrow \psi\} \vdash \forall x\varphi \rightarrow \forall x\psi$ :

Formula	Justification
$\varphi \rightarrow \psi$	given
$\forall x\varphi \rightarrow \varphi$	Example 2.6.1 (d)
$\forall x\varphi \rightarrow \psi$	Tautology
$\forall x(\forall x\varphi \rightarrow \psi)$	$\forall$ -intro
$\forall x\varphi \rightarrow \forall x\psi$	(Q1), $x$ not free in $\forall x\varphi$

### ! Take-home Problem 6

Use the compactness theorem to show (without using the Axiom of Choice) that every set can be linearly ordered.

Try to strengthen this to:

Every partial order can be extended to a linear order.

*Solution 0.5.* We first argue that any finite partial order  $(F, <)$  can be extended to a linear order.

We proceed by induction on the cardinality of  $F$ .

For  $|F| = 0$  there is nothing to prove.

Now assume  $|F| = n + 1$ . Pick an arbitrary  $a \in F$  and consider  $F \setminus \{a\}$ . By inductive hypothesis,  $F \setminus \{a\}$  can be extended to a linear order  $a_1 < a_2 < \dots < a_n$ .

If there does not exist  $i$  such that  $a_i <_F a$ , we can extend to a linear order on  $F$  by putting  $a < a_i$  for all  $i$ . Otherwise let  $k$  be maximal such that  $a_k < a$ . Then

$$a_1 < \dots < a_k < a < a_{k+1} < \dots < a_n$$

defines a linear order that extends  $(F, <)$ .

Now let  $(P, <)$  be an arbitrary partial order. We extend the language  $\mathcal{L}_<$  of orders to  $\mathcal{L}_P$ , where we add a new constant symbol  $c_p$  for every  $p \in P$ .

Let  $T$  be the theory of linear orders (see first three lines of sentence  $\varphi$  in Problem 3) together with

$$\{c_p < c_q : p <_P q\}$$

In other words, if  $p$  is less than  $q$  according to  $P$ , we add a corresponding axiom to our theory.

Any model of  $T$  is a linear order that extends  $P$ .

Moreover, any finite subset  $T_0 \subseteq T$  induces a partial order on a finite subset of  $P$  (induced by those  $p$  for which  $c_p$  is part of a formula in  $T_0$ ). By our argument in the first part, this finite partial order extends to a linear order, thereby giving a model of  $T_0$ .

By compactness,  $T$  has a model  $\mathcal{M} = (M, <)$ . By choice of  $T$ ,  $(M, <)$  is a linear order. Furthermore, the mapping

$$c_p \mapsto c_p^{\mathcal{M}}$$

is one-to-one since  $\mathcal{M} \models \forall x x \not< x$ .

We can “pull back” the order on  $\mathcal{M}$  to  $P$  by letting

$$p <' q \iff c_p^{\mathcal{M}} <^{\mathcal{M}} c_q^{\mathcal{M}}$$

By definition of  $T$ ,  $<'$  is a linear order that extends  $<_P$ .