MATH 557 Midterm 2 Preparation

The second midterm will have two parts:

1. Reproduce a proof (in sufficient detail) of one the theorems below we covered in class.
2. Present a proof to one of the exercises (or a closely related problem) listed on this page.

Theorems

Theorem 1

Any two countable models of DLO are isomorphic.

Theorem 2

Suppose for $\mathcal{L}$-theories $\mathcal{M}, \mathcal{N}$, $\mathcal{M} \subseteq \mathcal{N}$ and for any formula $\psi(x, \vec{y})$ and any $\vec{a} \in M$, if there exists $b \in N$ such that $\mathcal{N} \models \psi[b, \vec{a}]$, then there also exists $c \in M$ such that $\mathcal{N} \models \psi[c, \vec{a}]$. Then $\mathcal{M} \preceq \mathcal{N}$.

Theorem 3

Suppose T is a consistent $\mathcal{L}$-theory with no finite models. If T is κ-categorical for some $\kappa \geq |\mathcal{L}|$, then T is complete.

Theorem 4

Let $\mathcal{A}$ be an $\mathcal{L}$-structure, κ an infinite cardinal with $\operatorname{card}(\mathcal{L}) \leq \kappa$ and $\kappa \leq \operatorname{card}(A)$. Then there exists a structure $\mathcal{B}$ with

$$\mathcal{B} \preceq \mathcal{A}\ ,\ \operatorname{card}(B) = \kappa.$$

Theorem 5

Let $\mathcal{A}$ be an infinite $\mathcal{L}$-structure, κ a cardinal with $\operatorname{card}(\mathcal{L}) \leq \kappa$ and $\operatorname{card}(A) \leq \kappa$. Then there exists a structure $\mathcal{B}$ with

$$\mathcal{A} \preceq \mathcal{B}\ ,\ \operatorname{card}(B) = \kappa.$$

Theorem 6

Let $\mathcal{M}/\mathcal{U} = \prod_{i \in I} \mathcal{M}_i/\mathcal{U}$ be an ultraproduct. For every $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and tuples $\vec{a} \in \prod_{i \in I} M_i$,

$$\mathcal{M}/\mathcal{U} \models \varphi[\vec{a}_{\mathcal{U}}] \iff \{\, i \in I : \mathcal{M}_i \models \varphi[\vec{a}_i] \,\} \in \mathcal{U}.$$

Theorem 7

Any two countable homogeneous structures with the same age are isomorphic.

Theorem 8

Let $\mathcal{L}$ be a finite language with only relation symbols. Let $\overline{K}$ be a class of finite $\mathcal{L}$-structures such that there are only countably many isomorphism types in $\overline{K}$.
Then $\overline{K}$ is an amalgamation class if and only if $\overline{K}$ is the age of a countable homogeneous $\mathcal{L}$-structure.

Problems

Problem 1

Let $\mathcal{L}$ be an arbitrary language. Show that for finite $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$,

$$\mathcal{M} \equiv \mathcal{N} \iff \mathcal{M} \cong \mathcal{N}$$

Problem 2

A model $\mathcal{M}$ of an $\mathcal{L}$-theory T is **prime** if for every $\mathcal{N} \vDash T$, there exists an elementary embedding of $\mathcal{M}$ into $\mathcal{N}$.

- Show that $\mathbb{N}$ is a prime model of $\mathrm{Th}(\mathbb{N}, +, \cdot, 0, 1)$.
- Show that if T is an ω-categorical theory over a countable language and has no finite models, T has a prime model.

Problem 3

If we take an ultraproduct over the same structure $\mathcal{M}$ along an index set I, we call this an **ultrapower**, denoted by

$$\mathcal{M}^I/\mathcal{U}$$

Let $\mathcal{U}$ be a free ultrafilter on an (infinite) set I. The map $j : b \mapsto (b)_{i\in I}/\mathcal{U}$ defines an **elementary embedding** of $\mathcal{M}$ into $\mathcal{M}^I/\mathcal{U}$ (i.e. it is injective and the image is an elementary substructure).
Show that if $\mathcal{M}$ is infinite, j is not a surjection.

Problem 4

Let G_N be the set of all simple graphs on the vertex set $\{1, \dots, N\}$. Let $\mathbb{P}_N(\sigma)$ be the probability that an $\mathcal{L}_G$-sentence σ holds for a random graph in G_N, i.e.

$$\mathbb{P}_N(\sigma) = \frac{|\{\mathcal{G} \in G_N : \mathcal{G} \vDash \sigma\}|}{|G_N|}.$$

Show that for any $\mathcal{L}_G$-sentence σ,

$$\text{either } \lim_{N\to\infty} \mathbb{P}_N(\sigma) = 0 \text{ or } \lim_{N\to\infty} \mathbb{P}_N(\sigma) = 1.$$