Math 557 Oct 8

Ultraproducts

Direct Products

Let $(\mathcal{M}_i)_{i\in I}$ be a family of L-structures.

We define the **direct product**

$$\mathcal{M} = \prod_{i\in I} \mathcal{M}_i$$

as follows:

1. The universe is the Cartesian product $M = \prod_{i\in I} M_i$. If a is an element of M, we denote its i-th component (an element of M_i) by a_i and extend this notation to vectors: if $\vec{a}$ is a finite tuple in M^n, $\vec{a}_i$ denotes the n-tuple in M_i consisting of the M_i-entries of $\vec{a}$.
2. For each relation symbol $R \in \mathcal{L}$,
$$R^{\mathcal{M}}(\vec{a}) :\iff \forall i \in I,\, \vec{a}_i \in R^{\mathcal{M}_i}$$
3. For each function symbol $f \in \mathcal{L}$,
$$f^{\mathcal{M}}(\vec{a}) := (f^{\mathcal{M}_i}(\vec{a}_i))_{i\in I}.$$
4. For each constant $c \in \mathcal{L}$,
$$c^{\mathcal{M}} = (c^{\mathcal{M}_i})_{i\in I}.$$

Examples and Observations

- The direct product of groups is again a group (componentwise operation).
- The direct product of fields is **not** a field:
$$(1,0)\cdot(0,1) = (0,0).$$
- The direct product of linear orders is only a **partial order**.

We often want to preserve properties that hold in "most" component structures.
To formalize "most," we use **filters** on I.

Filters and Ultrafilters

A **filter** $\mathcal{F}$ on a set I is a nonempty collection of subsets of I satisfying:

1. $\emptyset \notin \mathcal{F}$
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$
3. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$

An **ultrafilter** $\mathcal{U}$ is a maximal filter, equivalently:

For all $A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

Ultrafilters interact nicely with logical operators:

- $A \notin \mathcal{U} \iff I\setminus A \in \mathcal{U}$,
- $A \in \mathcal{U} \wedge B \in \mathcal{U} \iff A \cap B \in \mathcal{U}$,
- $A \in \mathcal{U} \vee B \in \mathcal{U} \iff A \cup B \in \mathcal{U}$.

Examples

- A **principal filter** is of the form
$$\mathcal{F}_A = \{X \subseteq I : A \subseteq X\}$$
for some nonempty $A \subseteq I$.
If $A = \{a\}$, then $\mathcal{F}_A$ is a **principal ultrafilter**.
- A **free** (non-principal) ultrafilter exists on every infinite set I
(via Zorn's Lemma / Boolean prime ideal theorem).

Existence of Ultrafilters

A family of sets has the **finite intersection property (FIP)** if every finite subfamily has nonempty intersection.

Theorem 0.1.
If a family $\mathcal{A} \subseteq \mathcal{P}(I)$ has the FIP,
then there exists an ultrafilter $\mathcal{U}$ on I with $\mathcal{A} \subseteq \mathcal{U}$.

Reduced Products

Given a filter $\mathcal{F}$ on I and structures $(\mathcal{M}_i)_{i\in I}$, define the **reduced product**

$$\mathcal{M}/\mathcal{F}$$

as follows.

Let $M = \prod_{i\in I} M_i$. For $a, b \in M$, define

$$a \sim_{\mathcal{F}} b \iff \{\, i \in I : a_i = b_i \,\} \in \mathcal{F}.$$

The universe of $\mathcal{M}/\mathcal{F}$ is the quotient $M/{\sim_{\mathcal{F}}}$, with elements denoted $a_{\mathcal{F}}$ (alternatively, $a/\mathcal{F}$).

For symbols of $\mathcal{L}$:

- **Relations:**
$$R^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) :\iff \{\, i : \mathcal{M}_i \models R(\vec{a}_i) \,\} \in \mathcal{F}.$$
- **Functions:**
$$f^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) = [\,(f^{\mathcal{M}_i}(\vec{a}_i))_{i\in I}\,]_{\mathcal{F}}.$$
- **Constants:**
$$c^{\mathcal{M}/\mathcal{F}} = ((c^{\mathcal{M}_i})_{i\in I})_{\mathcal{F}}.$$

Exercise 0.1. Check that the above definition does not depend on the choice of representative for each equivalence class

Ultraproducts

If $\mathcal{U}$ is an **ultrafilter** on I, the reduced product

$$\prod_{i\in I} \mathcal{M}_i/\mathcal{U}$$

is called the **ultraproduct** of $(\mathcal{M}_i)_{i\in I}$ modulo $\mathcal{U}$.

When all $\mathcal{M}_i$ are the same structure $\mathcal{M}$, we get an **ultrapower**

$$\mathcal{M}^I/\mathcal{U}.$$

Łoś' Theorem

Let $\mathcal{M}/\mathcal{U} = \prod_{i\in I} \mathcal{M}_i/\mathcal{U}$ be an ultraproduct.

Theorem 0.2.
For every $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and tuples $\vec{a} \in \prod_{i\in I} M_i$,

$$\mathcal{M}/\mathcal{U} \vDash \varphi[\vec{a}_{\mathcal{U}}] \iff \{\, i \in I : \mathcal{M}_i \vDash \varphi[\vec{a}_i]\,\} \in \mathcal{U}.$$

For any $\mathcal{L}$-formula $\varphi(v_0, \dots, v_{n-1})$ a a tuple $\vec{a} \in \prod M_i$ we define the *Boolean extension* as

$$\|\varphi(\vec{a})\| := \{i \in I | \mathcal{M}_i \vDash \varphi[\vec{a}_i]\}$$

Lemma 0.1.

1. $\|\neg\varphi(\vec{a})\| = I \backslash \|\varphi(\vec{a})\|$,
2. $\|(\varphi \wedge \psi)(\vec{a})\| = \|\varphi(\vec{a})\| \cap \|\psi(\vec{a})\|$,
3. $\|(\varphi \vee \psi)(\vec{a})\| = \|\varphi(\vec{a})\| \cup \|\psi(\vec{a})\|$,
4. *For all tuples $\vec{a}$ and elements b in A:*
$$\|\varphi(\vec{a}, b)\| \subseteq \|(\exists v_n \varphi)(\vec{a})\|,$$
and there exists $b \in M$ such that
$$\|\varphi(\vec{a}, b)\| = \|(\exists v_n \varphi)(\vec{a})\|.$$

Proof. (1)-(3) and the first part of (4) follow directly from the definition of ultrafilters (and the definition of $\vDash$).

For the second part of (4), we observe that for every

$$i \in \|(\exists v_n \varphi)(\vec{a})\|$$

there exists $b_i \in A_i$ with $\mathcal{M}_i \vDash \varphi[\vec{a}_i, b_i]$. For all other $j \in I \setminus \|(\exists v_n \varphi)(\vec{a})\|$ we choose an aribitrary $b_j \in A_j$. This yields a sequence $b = (b_i)_{i\in I}$ with $b \in M$, for which

$$\|(\exists v_n \varphi)(\vec{a})\| \subseteq \|\varphi(\vec{a}, b)\|.$$

Together with the first part we obtain $=$. $\square$

Proof of Łoś's Theorem

We proceed by induction over the formula height. A straightforward argument shows that the interpretation of functions and symbols in $\mathcal{M}$ extends to terms in the following way:

$$t^{\mathcal{M}/\mathcal{U}}(\vec{a}/U) = (t^{\mathcal{M}_i}(\vec{a}_i))_{i\in I}/\mathcal{U} = t^{\mathcal{M}}(\vec{a})/\mathcal{U}$$

This easily implies the statement for atomic formulas.

For $\varphi \equiv \neg\psi$, we have

$$\begin{aligned}
\mathcal{M}/\mathcal{U} \vDash \neg\psi[\vec{a}_{\mathcal{U}}] &\iff \mathcal{M}/\mathcal{U} \nvDash \psi[\vec{a}_{\mathcal{U}}] \\
&\iff \{\, i : \mathcal{M}_i \vDash \psi[\vec{a}_i]\,\} \notin \mathcal{U} && \text{(I.H.)} \\
&\iff \|\psi(\vec{a})\| \notin \mathcal{U} \\
&\iff \neg\|\psi(\vec{a})\| \in \mathcal{U} && \text{($\mathcal{U}$ is an ultrafilter)} \\
&\iff \|\neg\psi(\vec{a})\| \in \mathcal{U} && \text{(Lemma (i))} \\
&\iff \{\, i : \mathcal{M}_i \vDash \neg\psi[\vec{a}_i]\,\} \in \mathcal{U}.
\end{aligned}$$

The case $\varphi \equiv (\psi \wedge \theta)$ is similar.

Finally, assume $\varphi \equiv \exists y\psi$. Then

$$\begin{aligned}
\mathcal{M}/\mathcal{U} \vDash \exists y\, \psi[\vec{a}_{\mathcal{U}}] &\iff \exists b_{\mathcal{U}} \in \mathcal{M}/\mathcal{U} \text{ such that } \mathcal{M}/\mathcal{U} \vDash \psi[\vec{a}_{\mathcal{U}}, b_{\mathcal{U}}] \\
&\iff \exists b \in M \text{ such that } \{\, i : \mathcal{M}_i \vDash \psi[\vec{a}_i, b_i] \,\} \in \mathcal{U} \quad \text{(I.H.)} \\
&\iff \exists b \in M \text{ such that } \|\psi(\vec{a}, b)\| \in \mathcal{U} \\
&\iff \|\exists y\, \psi(\vec{a})\| \in \mathcal{U} \qquad (\mathcal{U} \text{ ultrafilter, Lemma (4)}) \\
&\iff \{\, i : \mathcal{M}_i \vDash \exists y\, \psi[\vec{a}_i] \,\} \in \mathcal{U}.
\end{aligned}$$

This completes the proof.

Applications

- **Preservation of theories:**
  If each $\mathcal{M}_i \vDash T$, then $\prod_i \mathcal{M}_i/\mathcal{U} \vDash T$.
- **Ultrapowers:**
  $\mathcal{M} \equiv \mathcal{M}^I/\mathcal{U}$, i.e., a structure is elementarily equivalent to any of its ultrapowers.
- **Nonstandard models:**
  For example, the ultrapower $\mathbb{N}^{\mathbb{N}}/\mathcal{U}$ (with $\mathcal{U}$ non-principal) yields a countably saturated nonstandard model of arithmetic.

A new proof of the compactness theorem

Suppose T is an $\mathcal{L}$-theory for which every finite subset Δ has a model $\mathcal{M}_\Delta$.

Let $I = \{\Delta : \Delta \subseteq T \text{ finite}\}$.

For $\sigma \in T$, let

$$\hat{\sigma} = \{\Delta \in I : \sigma \in \Delta\} \subseteq I$$
$$E = \{\hat{\sigma} : \sigma \in T\} \subseteq \mathcal{P}(I)$$

E has FIP (finite intersection property), since for $\hat{S}_1, \dots, \hat{S}_n \in E$,

$$\{\hat{\sigma}_1, \dots, \hat{\sigma}_n\} \in \hat{\sigma}_1 \cap \dots \cap \hat{\sigma}_n.$$

By Theorem 0.1, E can be extended to an ultrafilter $\mathcal{U}$ on I.

Claim: $\mathcal{M}/\mathcal{U} = \prod_{\Delta \in I} \mathcal{M}_\Delta/\mathcal{U} \vDash T$.

Let $\sigma \in T$. Then for $\Delta \in I$,

$$\Delta \in \hat{\sigma} \;\Rightarrow\; \sigma \in \Delta \;\Rightarrow\; \mathcal{M}_\Delta \vDash \sigma$$

Hence

$$\hat{\sigma} \subseteq \{\Delta \in I : \mathcal{M}_\Delta \vDash \sigma\}$$

Since $\hat{\sigma} \in E$ and $E \subseteq \mathcal{U}$, we have $\hat{\sigma} \in \mathcal{U}$, which implies

$$\{\Delta \in I : \mathcal{M}_\Delta \vDash \sigma\} \in \mathcal{U}$$

By Łoś' Theorem, $\mathcal{M} \vDash \sigma$.