Math 557 Nov 17

The Diagonal Lemma

In the following, we consider the **diagonal function** d, defined by

$$d(n) = \begin{cases} \ulcorner \forall y\,(y = \underline{n} \rightarrow \sigma(y)) \urcorner & \text{if } n = \ulcorner \sigma(v_0) \urcorner \text{ for an } L\text{-formula } \sigma(v_0) \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 0.1.
Let T be an L-theory, $\theta(v_0)$ an L-formula with the single free variable v_0. If the diagonal function d is representable in T, then there exists an L-sentence G with the property

$$T \vdash G \leftrightarrow \theta(\underline{\ulcorner G \urcorner}).$$

If $\theta(v_0)$ is a Π_1-formula, then G can also be chosen as a Π_1-sentence.

Proof. If d is represented in T by the formula $\delta(x, y)$, then define

$$\psi(v_0) := \forall y\,(\delta(v_0, y) \rightarrow \theta(y)),$$

and set $n := \ulcorner \psi(v_0) \urcorner$.

Note: *n is actually a number in which the variable v_0 does not occur; however, the Gödel number of v_0 does enter into the calculation of n.*

For G, we now choose the sentence

$$G := \forall y\,(y = \underline{n} \rightarrow \psi(y)).$$

Then G has Gödel number $d(\ulcorner \psi(v_0) \urcorner)$. If $\delta(x, y)$ is a Σ_1-formula and $\theta(v)$ is a Π_1-formula, then ψ and G are (equivalent to) Π_1-formulas. Thus it remains to show that $T \vdash G \leftrightarrow \theta(\underline{\ulcorner G \urcorner})$:

It is clear that

$$T \vdash G \leftrightarrow \psi(\underline{n}), \qquad \text{i.e., by the definition of } \psi$$

$$(1) \quad T \vdash G \leftrightarrow \forall y(\delta(\underline{n}, y) \rightarrow \theta(y)).$$

But the fact that d is represented in T by $\delta(x, y)$ means

$$d(n) = \ulcorner G \urcorner \Rightarrow T \vdash \delta(\underline{n}, \underline{\ulcorner G \urcorner}) \quad \text{and} \quad T \vdash \exists! y\; \delta(\underline{n}, y).$$

Thus in particular

(2) $T \vdash \forall y\, (\delta(\underline{n}, y) \leftrightarrow y = \underline{\ulcorner G \urcorner})$ and together with (1) it follows

$T \vdash G \leftrightarrow \forall y\, (y = \underline{\ulcorner G \urcorner} \rightarrow \theta(y))$, hence

$T \vdash G \leftrightarrow \theta(\underline{\ulcorner G \urcorner})$.

$\square$