Math 557 Sep 15

The Completeness Theorem

Key Concepts

- The **Completeness Theorem** states that

$$T \vdash_{\mathcal{L}} \varphi \iff T \models \varphi$$

- The $\Rightarrow$-direction is **Soundness**, which we already proved.
- The $\Leftarrow$-direction is usually proved in the following form:

  If T is consistent, then T has a model.

- To construct a model of a consistent theory, we consider the **constant terms**

$$K := \{t \colon t \ \mathcal{L}\text{-term without variables}\}$$

  and **identify provably equal terms**:

$$s \sim t \ :\iff \ T \vdash_{\mathcal{L}} s = t$$

- The **canonical term structure** $\mathcal{A}$ of T has universe $A = K/\sim$.
- Moreover, we put
  - $c^{\mathcal{A}} := [c]$
  - $f^{\mathcal{A}}([t_1], \dots, [t_n]) := [f t_1 \dots t_n]$
  - $R^{\mathcal{A}}([t_1], \dots, [t_n]) \ :\iff \ T \vdash_{\mathcal{L}} R t_1 \dots t_n$
- It holds that for any *atomic* sentence σ,

$$\mathcal{A} \models \sigma \quad \iff \quad T \vdash \sigma$$

Problems

> Discuss
>
> We all believe (I think) that $(\mathbb{Z}, +, 0)$ is a model of the group axioms. To what extent does this prove that the group axioms are consistent?

Exercise 0.1.
Verify that $\sim$ is an equivalence relation.

Exercise 0.2.
Verify that the definition of $c^{\mathcal{A}}, f^{\mathcal{A}}, R^{\mathcal{A}}$ does not depend on the choice of representative for $[c]$ and $[t_i]$.

Exercise 0.3.
Verify the claim that for atomic sentences,

$$\mathcal{A}_T \models \sigma \quad \iff \quad T \vdash \sigma$$