

## Representability

We have established a closed connection between computability and definability over  $\mathbb{N}$ , but how much of that can  $\text{PA}^-$  actually prove? We need to make sure it can *represent* sufficiently simple (i.e., computable) functions and sets faithfully.

**Definition 0.1.** Let  $T$  be a theory in the language of arithmetic  $L$  that extends  $\text{PA}^-$ . A (total) function  $f : \mathbb{N}^k \rightarrow \mathbb{N}$  is **representable** in  $T$  iff there exists an  $L$ -formula  $\theta(x_1, \dots, x_k, y)$  such that for all  $n_1, \dots, n_k, m \in \mathbb{N}$ :

- (a)  $T \vdash \exists! y \theta(\underline{n_1}, \dots, \underline{n_k}, y)$ , and
- (b)  $f(n_1, \dots, n_k) = m \Rightarrow T \vdash \theta(\underline{n_1}, \dots, \underline{n_k}, \underline{m})$ .

Similarly, a set  $S \subseteq \mathbb{N}^k$  is **representable** in the theory  $T$  iff there exists an  $L$ -formula  $\theta(x_1, \dots, x_k)$  such that for all  $n_1, \dots, n_k \in \mathbb{N}$ :

- (c)  $(n_1, \dots, n_k) \in S \Rightarrow T \vdash \theta(\underline{n_1}, \dots, \underline{n_k})$ , and
- (d)  $(n_1, \dots, n_k) \notin S \Rightarrow T \vdash \neg \theta(\underline{n_1}, \dots, \underline{n_k})$ .

If the function  $f$  (or the set  $S$ ) is representable by a  $\Sigma_1$ -formula, then  $f$  (or  $S$ , respectively) is called  **$\Sigma_1$ -representable**.

Note that by definition, representability is preserved when we pass to a theory  $T' \supseteq T$ .

**Theorem 0.1** (Representation Theorem).

- (i) Every recursive function  $f : \mathbb{N}^k \rightarrow \mathbb{N}$  is  $\Sigma_1$ -representable in  $\text{PA}^-$ .
- (ii) Every recursive set  $S \subseteq \mathbb{N}^k$  is  $\Sigma_1$ -representable in  $\text{PA}^-$ .

*Proof.* (i) Let  $f : \mathbb{N}^k \rightarrow \mathbb{N}$  be a recursive function, so its graph  $\Gamma_f$  is definable over the natural numbers by a  $\Sigma_1$ -formula  $\exists z \varphi(\vec{x}, y, z)$ , where  $\varphi$  has only bounded quantifiers. Since every formula of the form  $\exists z \varphi$  is equivalent in  $\text{PA}^-$  to  $\exists u \exists z (z < u \wedge \varphi)$ , we may assume that  $z$  is just a single variable  $z$ . We now form the  $\Delta_0$ -formula  $\psi(\vec{x}, y, z)$ :

$$\varphi(\vec{x}, y, z) \wedge \forall u, v \leq y + z (u + v < y + z \rightarrow \neg \varphi(\vec{x}, u, v)).$$

We now claim that the  $\Sigma_1$ -formula  $\exists z \psi(\vec{x}, y, z)$  represents the function  $f$  in  $\text{PA}^-$ :

First we show (b). Assume that  $f(n_1, \dots, n_k) = m$  holds, thus  $\mathbb{N} \models \exists z \varphi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, z)$ . The number  $m$  is uniquely determined since  $f$  is a function. Choose  $l$  as the smallest number such that  $\mathbb{N} \models \varphi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, l)$ . Then clearly  $\mathbb{N} \models \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, l)$  also holds, and thus  $\mathbb{N} \models \exists z \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, z)$ . As a  $\Sigma_1$ -sentence, this sentence is preserved under all end extensions of the standard model to a model of  $\text{PA}^-$ , thus it holds in all models of  $\text{PA}^-$ , and therefore  $\text{PA}^- \vdash \exists z \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, z)$  by the Completeness Theorem.

The proof of (a) uses a similar argument: Let  $f(n_1, \dots, n_k) = m$  and let  $l$  again be the smallest number such that  $\mathbb{N} \models \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, l)$  holds. Let  $\mathcal{M} \models \text{PA}^-$ . We claim that  $m$  is the only element of  $M$  that satisfies the formula  $\psi(\underline{n_1}, \dots, \underline{n_k}, x, l)$  in  $\mathcal{M}$ .  $\mathcal{M} \models \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, l)$  due to the absoluteness of  $\Delta_0$ -formulas. If  $a, b \in M$  are two elements such that  $\mathcal{M} \models \psi(\underline{n_1}, \dots, \underline{n_k}, a, b)$ , then we must have  $a, b \leq m + l$ . For if one of them were greater than  $m + l$ , we would also have  $a + b > m + l$ . We assumed

$$\mathcal{M} \models \psi(\underline{n_1}, \dots, \underline{n_k}, a, b),$$

so by definition of  $\psi$  we have

$$\mathcal{M} \models \varphi(\underline{n_1}, \dots, \underline{n_k}, a, b) \wedge \forall u, v \leq a + b (u + v < a + b \rightarrow \neg \varphi(\underline{n_1}, \dots, \underline{n_k}, u, v)).$$

But since  $m + l < a + b$ , this would imply  $\neg \varphi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, \underline{l})$  in  $\mathcal{M}$ , contradicting the choice of  $m$  and  $l$ .

As  $\mathcal{M}$  is an end-extension of  $\mathbb{N}$ ,  $a, b \leq m + l$  implies  $a, b \in \mathbb{N}$ . Thus, again by the absoluteness of  $\Delta_0$ -formulas, we have  $\mathbb{N} \models \psi(\underline{n_1}, \dots, \underline{n_k}, a, b)$ , from which  $\mathbb{N} \models \exists z \psi(\underline{n_1}, \dots, \underline{n_k}, a, z)$  follows, and therefore  $a = m$ .

**(ii)** now follows easily: If  $S$  is recursive, then the characteristic function  $c_S$  is computable, thus by (i) represented by a  $\Sigma_1$ -formula  $\theta(\vec{x}, y)$  in  $\text{PA}^-$ . Then  $S$  is represented by the formula  $\theta(\vec{x}, 1)$ , since we have:

$$\text{PA}^- \vdash \neg \theta(\vec{x}, 1) \iff \text{PA}^- \vdash \theta(\vec{x}, 0).$$

□