# Math 557 Sep 15

## The Completeness Theorem

### Key Concepts

- The **Completeness Theorem** states that

$$T \vdash_{\mathcal{L}} \varphi \iff T \vDash \varphi$$

- The $\Rightarrow$-direction is **Soundness**, which we already proved.
- The $\Leftarrow$-direction is usually proved in the following form:

  If $T$ is consistent, then $T$ has a model.

- To construct a model of a consistent theory, we consider the **constant terms**

$$K := \{t \colon t \ \mathcal{L}\text{-term without variables}\}$$

  and **identify provably equal terms**:

$$s \sim t \ :\iff \ T \vdash_{\mathcal{L}} s = t$$

- The **canonical term structure** $\mathcal{A}$ of $T$ has universe $A = K/\sim$.
- Moreover, we put
  - $c^{\mathcal{A}} := [c]$
  - $f^{\mathcal{A}}([t_1], \dots, [t_n]) := [ft_1 \dots t_n]$
  - $R^{\mathcal{A}}([t_1], \dots, [t_n]) \ :\iff \ T \vdash_{\mathcal{L}} Rt_1 \dots t_n$
- It holds that for any *atomic* sentence $\sigma$,

$$\mathcal{A} \vDash \sigma \quad \iff \quad T \vdash \sigma$$

### Problems

Discuss

We all believe (I think) that $(\mathbb{Z}, +, 0)$ is a model of the group axioms. To what extent does this prove that the group axioms are consistent?

**Exercise 0.1.**
Verify that $\sim$ is an equivalence relation.

**Exercise 0.2.**
Verify that the definition of $c^{\mathcal{A}}, f^{\mathcal{A}}, R^{\mathcal{A}}$ does not depend on the choice of representative for $[c]$ and $[t_i]$.

**Exercise 0.3.**
Verify the claim that for atomic sentences,

$$\mathcal{A}_T \vDash \sigma \quad \iff \quad T \vdash \sigma$$