Math 557 Sep 17

Completing Theories

Key Concepts

- We previously defined the *term model* $\mathcal{A}$. It holds that for any *atomic* sentence σ,

$$\mathcal{A} \models \sigma \quad \Longleftrightarrow \quad T \vdash \sigma$$

- Trying to extend this to arbitrary sentence via induction, the *negation case* looks like

$$\mathcal{A} \models \neg\sigma \iff \mathcal{A} \not\models \sigma \overset{\text{ind hyp}}{\iff} T \nvdash \sigma$$

  We would like to show that this is equivalent to $T \vdash \neg\sigma$.

- One direction follows from T being consistent, but for the other direction, T may not be strong enough to prove this.
- We therefore need to *extend* T to a complete theory.

Problems

Exercise 0.1.
Verify that indeed for all atomic sentences σ,

$$\mathcal{A} \models \sigma \quad \Longleftrightarrow \quad T \vdash \sigma$$

Exercise 0.2.
Recall that a theory T is *maximally consistent* if it is consistent but does not have any consistent proper extensions. T is called *deductively closed* if the deductive closure of T,

$$T^{\vdash} = \{\sigma : T \vdash \sigma\}$$

is equal to T.

- Show that a maximally consistent theory is complete and deductively closed.
- Show that if T is complete, then $T^{\vdash}$ is maximally consistent.

Exercise 0.3.
Is every consistent, deductively closed theory complete?

Exercise 0.4.
Show that the union of an increasing sequence of consistent theories is consistent.

Exercise 0.5.
Extend Lindebaum's theorem on the existence of maximally consistent extension from countable to arbitrary languages.

Exercise 0.6.
Fix a language $\mathcal{L}$. Let X be the set of all maximally consistent $\mathcal{L}$-theories. For an $\mathcal{L}$-sentence σ, let

$$\langle\sigma\rangle = \{T \in X \colon \sigma \in T\}$$

Show that

- $\langle \sigma \wedge \tau \rangle = \langle \sigma \rangle \cap \langle \tau \rangle$
- $\langle \neg\sigma \rangle = X \setminus \langle \sigma \rangle$

Exercise 0.7.
Continuing the previous exercise, let $\mathcal{O}$ be the topology generated by the sets $\langle \sigma \rangle$. Show that

- each $\langle \sigma \rangle$ is clopen,
- the $\langle \sigma \rangle$ form a basis for $\mathcal{O}$,
- $\mathcal{O}$ is Hausdorff.