

MATH 557 Midterm 2 Preparation

The second midterm will have two parts:

1. Reproduce a proof (in sufficient detail) of one the theorems below we covered in class.
2. Present a proof to one of the exercises (or a closely related problem) listed on this page.

Theorems

Theorem 1

Any two countable models of DLO are isomorphic.

Theorem 2

Suppose for \mathcal{L} -theories \mathcal{M}, \mathcal{N} , $\mathcal{M} \subseteq \mathcal{N}$ and for any formula $\psi(x, \vec{y})$ and any $\vec{a} \in M$, if there exists $b \in N$ such that $\mathcal{N} \models \psi[b, \vec{a}]$, then there also exists $c \in M$ such that $\mathcal{N} \models \psi[c, \vec{a}]$. Then $\mathcal{M} \preceq \mathcal{N}$.

Theorem 3

Let \mathcal{A} be an \mathcal{L} -structure, κ an infinite cardinal with $\text{card}(\mathcal{L}) \leq \kappa$ and $\kappa \leq \text{card}(A)$. Then there exists a structure \mathcal{B} with

$$\mathcal{B} \preceq \mathcal{A}, \text{card}(\mathcal{B}) = \kappa.$$

Theorem 4

Let \mathcal{A} be an infinite \mathcal{L} -structure, κ a cardinal with $\text{card}(\mathcal{L}) \leq \kappa$ and $\text{card}(A) \leq \kappa$. Then there exists a structure \mathcal{B} with

$$\mathcal{A} \preceq \mathcal{B}, \text{card}(\mathcal{B}) = \kappa.$$

Theorem 5

Suppose T is a consistent \mathcal{L} -theory with no finite models. If T is κ -categorical for some $\kappa \geq |\mathcal{L}|$, then T is complete.

Theorem 6

Let $\mathcal{M}/\mathcal{U} = \prod_{i \in I} \mathcal{M}_i/\mathcal{U}$ be an ultraproduct. For every \mathcal{L} -formula $\varphi(x_1, \dots, x_n)$ and tuples $\vec{a} \in \prod_{i \in I} M_i$,

$$\mathcal{M}/\mathcal{U} \models \varphi[\vec{a}_{\mathcal{U}}] \iff \{i \in I : \mathcal{M}_i \models \varphi[\vec{a}_i]\} \in \mathcal{U}.$$

Theorem 7

Any two countable homogeneous structures with the same age are isomorphic.

Problems

Problem 1

Let \mathcal{L} be an arbitrary language. Show that for finite \mathcal{L} -structures \mathcal{M}, \mathcal{N} ,

$$\mathcal{M} \equiv \mathcal{N} \iff \mathcal{M} \cong \mathcal{N}$$

Problem 2

A model \mathcal{M} of an \mathcal{L} -theory T is **prime** if for every $\mathcal{N} \models T$, there exists an elementary embedding of \mathcal{M} into \mathcal{N} .

- Show that \mathbb{N} is a prime model of $\text{Th}(\mathbb{N}, +, \cdot, 0, 1)$.
- Show that if T is an ω -categorical theory over a countable language and has no finite models, T has a prime model.

Problem 3

If we take an ultraproduct over the same structure \mathcal{M} along an index set I , we call this an **ultrapower**, denoted by

$$\mathcal{M}^I/\mathcal{U}$$

Let \mathcal{U} be a free ultrafilter on an (infinite) set I . The map $j : b \mapsto (b)_{i \in I}/\mathcal{U}$ defines an **elementary embedding** of \mathcal{M} into $\mathcal{M}^I/\mathcal{U}$ (i.e. it is injective and the image is an elementary substructure).

Show that if \mathcal{M} is infinite and $I = \mathbb{N}$, j is not a surjection.

Problem 4

Let G_N be the set of all simple graphs on the vertex set $\{1, \dots, N\}$. Let $\mathbb{P}_N(\sigma)$ be the probability that an \mathcal{L}_G -sentence σ holds for a random graph in G_N , i.e.

$$\mathbb{P}_N(\sigma) = \frac{|\{\mathcal{G} \in G_N : \mathcal{G} \models \sigma\}|}{|G_N|}.$$

Show that for any \mathcal{L}_G -sentence σ ,

$$\text{either } \lim_{N \rightarrow \infty} \mathbb{P}_N(\sigma) = 0 \text{ or } \lim_{N \rightarrow \infty} \mathbb{P}_N(\sigma) = 1.$$