

Math 557 Sep 12

Consistency and Completeness

Key Concepts

- **Consistency:**

A theory T is **consistent** if there does not exist a formula ψ such that $T \vdash \psi$ and $T \vdash \neg\psi$.

- **Completeness:**

A theory T is **complete** if it is consistent and for every sentence σ , $T \vdash \sigma$ or $T \vdash \neg\sigma$.

- In a complete theory, every statement is determined, in the sense that is either true or false in the theory.
- An important example of a complete theory is the *theory of a fixed structure* \mathcal{M} ,

$$\text{Th}(\mathcal{M}) = \{\sigma : \mathcal{M} \models \sigma\}$$

- If, on the other hand, we are given a complete theory T , does T have a model? This is the subject of the **Completeness Theorem**. It actually shows that any *consistent* theory has a model.

Problems

Exercise 0.1 (Carry-over from Sep 10).

Prove the *Soundness Theorem*, i.e. show that

$$T \vdash \varphi \Rightarrow T \models \varphi$$

Exercise 0.2.

Why is the theory of a structure, $\text{Th}(\mathcal{M})$, indeed a complete theory?

Exercise 0.3.

Show that a theory T is inconsistent iff for every formula ψ , $T \vdash \psi$

Exercise 0.4.

For each of the following classes \mathbf{K} of structures, determine whether they are *axiomatizable*, i.e. whether there exists a theory T (in the language of the structures) such that

$$\mathbf{K} = \{\mathcal{M} : \mathcal{M} \models T\}$$

(In this case K is also called *elementary*.)

If \mathbf{K} is axiomatizable, discuss whether it is actually *finitely axiomatizable* (i.e. can T be chosen finite). And is T complete?

- \mathbf{K} = Abelian groups
- \mathbf{K} = infinite sets
- \mathbf{K} = fields of characteristic p (p prime)
- \mathbf{K} = bipartite graphs
- \mathbf{K} = torsion groups (*all elements have finite order*)
- \mathbf{K} = algebraically closed fields of characteristic 0

(Note: We do not yet have the tools to rigorously answer all these questions, but try an “educated” guess.)

Exercise 0.5.

Suppose that $T \cup \{\neg\varphi\}$ is inconsistent. Show that $T \vdash \varphi$.

(This is a technical formulation of the legitimacy of proofs by contradiction.)