

Logical Implication and Proof

Key Concepts

- **Logical consequence:**
 - This is the semantical implication we are often working with in mathematical practice. We say T **logically implies** φ , $T \models \varphi$, if for structure \mathcal{M} , $\mathcal{M} \models T$ implies $\mathcal{M} \models \varphi$.
- **Formal proof:**
 - $T \vdash \varphi$ means there is a formal (i.e. *syntactical*) derivation of φ from T using the formulas of T , the three kinds of *logical axioms* (propositional tautologies, equality and quantifier axioms), and the *inference rules* Modus Ponens and Generalization.

Problems

Exercise 0.1 (Warmup - Logical Implication).

Let T be an \mathcal{L} -theory. We say a theory T' is an **axiomatization** of T if for any \mathcal{L} -structure \mathcal{M} ,

$$\mathcal{M} \models T \iff \mathcal{M} \models T'$$

Show that for any axiomatization T' of T , for any \mathcal{L} -sentence σ ,

$$T \models \sigma \iff T' \models \sigma$$

Exercise 0.2 (Warmup 2).

Recall that a *model* of a theory T is a structure \mathcal{M} such that for any sentence $\sigma \in T$, $\mathcal{M} \models \sigma$. In this case we write $\mathcal{M} \models T$.

Argue that if T does not have a model, every sentence is a logical implication of T .

Exercise 0.3 (Formal notion of proof – Warmup).

Verify that

$$\{\varphi, \neg\psi\} \vdash \neg(\varphi \rightarrow \psi)$$

Exercise 0.4.

Argue (semantically) that if x is not free in ψ ,

$$\{\varphi \rightarrow \psi\} \models \exists x \varphi \rightarrow \psi$$

Then prove this *syntactically*, i.e. show (under the same assumption) that

$$\{\varphi \rightarrow \psi\} \vdash \exists x \varphi \rightarrow \psi$$

Exercise 0.5.

Prove the *Soundness Theorem*, i.e. show that

$$T \vdash \varphi \Rightarrow T \models \varphi$$

! Take-home problem

Show that

$$\{\varphi \rightarrow \psi\} \vdash \exists x\varphi \rightarrow \exists x\psi$$

$$\{\varphi \rightarrow \psi\} \vdash \forall x\varphi \rightarrow \forall x\psi$$