

Midterm 1 Review

! Take-home Problem 1

Prove unique readability for the set of \mathcal{L} -formulas.

Solution 0.1. **First** prove **readability**, i.e. the statement that for any \mathcal{L} -formula φ , exactly one of the following cases holds:

- (1) There exist terms s, t such that $\varphi \equiv s = t$.
- (2) There exists a relation symbol R and, if n is the arity of R , terms t_1, \dots, t_n such that $\varphi \equiv Rt_1 \dots t_n$.
- (3) There exists a formula ψ such that $\varphi \equiv \neg\psi$.
- (4) There exist formulas ψ, θ such that $\varphi \equiv (\psi \wedge \theta)$.
- (5) There exists a variable x and a formula ψ such that $\varphi \equiv \exists x\psi$.

To see this, let F be the subset of \mathcal{L}^* such that every $\theta \in F$ satisfies exactly one of (1)-(5). Then F contains all atomic formulas and is closed under $\neg, \wedge, \exists x$. Since the set of \mathcal{L} -formulas is the smallest such set, it follows that every \mathcal{L} -formula is contained in F , and therefore has the desired property.

Second we argue that a proper initial segment of a formula cannot be a formula. We proceed by induction on the length l of φ . For $l = 1$ there is no formula of that length. Now suppose φ is a formula of length $l + 1$, and β is a proper initial segment. We may assume β is not empty. We consider theses (1)-(5) from readability.

- Case** $\varphi \equiv s = t$ Either $\beta \equiv s = t'$ with $t \subset t'$, but this would contradict that no proper initial segment of a term is a term. Or $\beta \equiv s =$: An easy induction over the length of a formula shows that no formula ends with $=$. Or $\beta \subseteq s$. Another easy induction shows that no formula can be a term or an initial segment of a term.
- Case** $\varphi \equiv Rt_1 \dots t_n$ If β were a formula, it has to be of the form $Rs_1 \dots s_n$. Comparing terms, we would see that one of the s_i is a proper initial segment of some t_j , which is impossible.
- Case** $\varphi \equiv \neg\theta$ Then $\beta \equiv \neg\beta'$ with β' a proper initial segment of θ . By inductive hypothesis, β' is not a formula, hence $\neg\beta'$ is not a formula either.
- Case** $\varphi \equiv (\psi \wedge \theta)$ Then β starts with (but does not end with). Another induction shows that for any formula, the number of ('s always equals the number of) 's.
- Case** $\varphi \equiv \exists x\psi$ Neither ' \exists ' nor ' $\exists x$ ' are formulas. If β is longer than that, it is of the form $\exists x\beta'$. By inductive hypothesis, β' is not a formula, and hence β is not a formula (by readability). φ

Finally, we prove **unique readability**: The choices in (1)-(5) are unique.

- Case (1)** Suppose $\varphi \equiv s = t \equiv s' = t'$. Since no proper initial segment of a term is a term, it must hold that $s \equiv s'$ and $t \equiv t'$.
- Case (2)** Suppose $\varphi \equiv Rt_1 \dots t_n \equiv Ss_1 \dots s_k$. Then $R \equiv S$ and $n = k$. Comparing terms inductively, using again the fact that no proper initial segment of a term is a term, we get $s_i \equiv t_i$ for all i .
- Case (3)** Suppose $\varphi \equiv \neg\psi \equiv \neg\theta$. Immediately, we infer $\psi \equiv \theta$.
- Case (4)** Suppose $\varphi \equiv (\psi_1 \wedge \theta_1) \equiv (\psi_2 \wedge \theta_2)$. Assume $\psi_1 \not\equiv \psi_2$. Then ψ_1 is a proper initial segment of ψ_2 or vice versa (since both of them are part of φ). Either case is impossible due to the fact that no proper initial segment of a formula is a formula. Hence $\psi_1 \equiv \psi_2$ and thus also $\theta_1 \equiv \theta_2$.
- Case (5)** Suppose $\varphi \equiv \exists x\psi \equiv \exists y\theta$. By comparing entries, we get $x \equiv y$ and thus $\psi \equiv \theta$, as desired.

! Take-home Problem 2

Let \mathcal{L} be any finite language and let \mathcal{M} be a finite \mathcal{L} -structure. Show that there is an \mathcal{L} -sentence φ such that

$$\mathcal{N} \models \varphi \iff \mathcal{N} \cong \mathcal{M}.$$

Solution 0.2. Suppose $\mathcal{L} = \{c_1, \dots, c_k, f_1^{(a_1)}, \dots, f_l^{(a_l)}, R_1^{(b_1)}, \dots, R_j^{(b_j)}\}$ where the $(a_i), (b_m)$ denote the arities of the respective symbols.

Since \mathcal{M} is finite, we may assume $M = \{1, \dots, n\}$ for some $n \in \mathbb{N}$.

The basic idea is to collect all “elementary facts” about \mathcal{M} in a single formula (think of a *group multiplication table*, just for all functions and relations).

Define

$$\varphi \equiv \exists x_1, \dots, x_n \left(\bigwedge_{i \neq m} x_i \neq x_m \wedge \forall y \bigvee_{i \leq n} y = x_i \right) \quad (1)$$

$$\wedge \bigwedge_{i \leq k} c_i = x_{c_i^{\mathcal{M}}} \quad (2)$$

$$\wedge \bigwedge_{i \leq l} \bigwedge_{\pi \in \{1, \dots, n\}^{a_i}} f_i x_{\pi(1)} \dots x_{\pi(a_i)} = x_{f_i^{\mathcal{M}}(\pi(1), \dots, \pi(a_i))} \quad (3)$$

$$\wedge \bigwedge_{i \leq j} \bigwedge_{\pi \in \{1, \dots, n\}^{b_i}} \delta_{\pi} R_i x_{\pi(1)} \dots x_{\pi(b_i)} \quad (4)$$

where δ_{π} is empty if $R(\pi(1), \dots, \pi(b_i))$ holds in \mathcal{M} , and \neg if not.

Assume $\mathcal{N} \models \varphi$. Due to the first line of the equation, N has exactly n elements, and let $r_i \in N$ be the witness to $\exists x_i$. Define a mapping $\tau : M \rightarrow N$ by letting $\tau(i) = r_i$. We claim that τ is an isomorphism.

By definition we have $\tau(c_i^{\mathcal{M}}) = r_{c_i^{\mathcal{M}}}$. Moreover, by the second line of the formula, $r_{c_i^{\mathcal{M}}}$ is the unique element of N that makes the formula $c_i = x_{c_i^{\mathcal{M}}}$ true. It follows that $\tau(c_i^{\mathcal{M}}) = c_i^{\mathcal{N}}$ for all $1 \leq i \leq k$.

Using a similar argument with the third line of the formula, we obtain

$$\tau(f_i^{\mathcal{M}}(s_1, \dots, s_{a_i})) = r_{f_i^{\mathcal{M}}(s_1, \dots, s_{a_i})} = f^{\mathcal{N}}(r_{s_1}, \dots, r_{s_{a_i}}) = f^{\mathcal{N}}(\tau(s_1), \dots, \tau(s_{a_i}))$$

The argument for $R^{\mathcal{M}}(s_1, \dots, s_{b_i}) \iff R^{\mathcal{N}}(\tau(s_1), \dots, \tau(s_{b_i}))$ is similar.

On the other hand, if $\mathcal{M} \cong \mathcal{N}$ via τ , then $\mathcal{N} \models \psi[\tau(1), \dots, \tau(n)]$, where ψ is such that $\varphi \equiv \exists x_1, \dots, x_n \psi$, due to the fact that τ is an isomorphism, and thus $\mathcal{N} \models \varphi$.

! Take-home Problem 3

Give an example of a language \mathcal{L} and an \mathcal{L} -sentence ψ such that

- there is at least one \mathcal{L} -structure A such that $A \models \psi$,
- for all \mathcal{L} -structures A , if $A \models \psi$, then the universe A of A is infinite.

Solution 0.3. Let $\mathcal{L} = \{<\}$, where $<$ is a binary relation symbol. Define

$$\psi \equiv \forall x \, x \not< x \quad (5)$$

$$\wedge \forall x, y \, x < y \vee x = y \vee y < x \quad (6)$$

$$\wedge \forall x, y, z \, (x < y \wedge y < z) \rightarrow x < z \quad (7)$$

$$\wedge \forall x \exists y \, x < y \quad (8)$$

The formula says that $<$ is a linear order with no maximal element.

Clearly, $(\mathbb{Z}, <) \models \psi$.

Now suppose $\mathcal{M} \models \psi$. Due to the last line of ψ , there exists a function $f : M \rightarrow M$ such that $x < f(x)$ for all $x \in M$. We claim that for any x and for any $n \neq m$,

$$f^{(n)}(x) \neq f^{(m)}(x)$$

This follows from antireflexivity (line one) and transitivity (line three).

Therefore, the set

$$\{x, f^{(1)}(x), f^{(2)}(x), \dots\}$$

is an infinite subset of N .

! Take-home problem 4

Show that

$$\begin{aligned} \{\varphi \rightarrow \psi\} &\vdash \exists x\varphi \rightarrow \exists x\psi \\ \{\varphi \rightarrow \psi\} &\vdash \forall x\varphi \rightarrow \forall x\psi \end{aligned}$$

Solution 0.4. We give derivations below (with brief justifications). We collect simple substeps into a single one.

For $\{\varphi \rightarrow \psi\} \vdash \exists x\varphi \rightarrow \exists x\psi$:

Formula	Justification
$\varphi \rightarrow \psi$	given
$\psi \rightarrow \exists x\psi$	(Q2)
$\varphi \rightarrow \exists x\psi$	tautology
$\neg\exists x\psi \rightarrow \neg\varphi$	tautology
$\forall x(\neg\exists x\psi \rightarrow \neg\varphi)$	\forall -intro
$\neg\exists x\psi \rightarrow \forall x\neg\varphi$	(Q1), x not free in $\neg\exists x\psi$
$\neg\forall x\neg\varphi \rightarrow \exists x\varphi$	tautology
$\exists x\varphi \rightarrow \exists x\psi$	(Q3)

For $\{\varphi \rightarrow \psi\} \vdash \forall x\varphi \rightarrow \forall x\psi$:

Formula	Justification
$\varphi \rightarrow \psi$	given
$\forall x\varphi \rightarrow \varphi$	Example 2.6.1 (d)
$\forall x\varphi \rightarrow \psi$	Tautology
$\forall x(\forall x\varphi \rightarrow \psi)$	\forall -intro
$\forall x\varphi \rightarrow \forall x\psi$	(Q1), x not free in $\forall x\varphi$

! Take-home Problem 6

Use the compactness theorem to show (without using the Axiom of Choice) that every set can be linearly ordered.

Try to strengthen this to:

Every partial order can be extended to a linear order.

Solution 0.5. We first argue that any finite partial order $(F, <)$ can be extended to a linear order.

We proceed by induction on the cardinality of F .

For $|F| = 0$ there is nothing to prove.

Now assume $|F| = n + 1$. Pick an arbitrary $a \in F$ and consider $F \setminus \{a\}$. By inductive hypothesis, $F \setminus \{a\}$ can be extended to a linear order $a_1 < a_2 < \dots < a_n$.

If there does not exist i such that $a_i <_F a$, we can extend to a linear order on F by putting $a < a_i$ for all i . Otherwise let k be maximal such that $a_k < a$. Then

$$a_1 < \dots < a_k < a < a_{k+1} < \dots < a_n$$

defines a linear order that extends $(F, <)$.

Now let $(P, <)$ be an arbitrary partial order. We extend the language $\mathcal{L}_<$ of orders to \mathcal{L}_P , where we add a new constant symbol c_p for every $p \in P$.

Let T be the theory of linear orders (see first three lines of sentence φ in Problem 3) together with

$$\{c_p < c_q : p <_P q\}$$

In other words, if p is less than q according to P , we add a corresponding axiom to our theory.

Any model of T is a linear order that extends P .

Moreover, any finite subset $T_0 \subseteq T$ induces a partial order on a finite subset of P (induced by those p for which c_p is part of a formula in T_0). By our argument in the first part, this finite partial order extends to a linear order, thereby giving a model of T_0 .

By compactness, T has a model $\mathcal{M} = (M, <)$. By choice of T , $(M, <)$ is a linear order. Furthermore, the mapping

$$c_p \mapsto c_p^{\mathcal{M}}$$

is one-to-one since $\mathcal{M} \models \forall x \, x \not< x$.

We can “pull back” the order on \mathcal{M} to P by letting

$$p <' q \iff c_p^{\mathcal{M}} <^{\mathcal{M}} c_q^{\mathcal{M}}$$

By definition of T , $<'$ is a linear order that extends $<_P$.