Math 557 Sep 3

Validities

Key Concepts

- **Validity**: An $\mathcal{L}$-formula ψ that is valid in *any* $\mathcal{L}$-structure under *any* assignment α, i.e.

$$\mathcal{M} \vDash \psi[\alpha] \quad \text{for any } \mathcal{M}, \alpha.$$

- **Sources of validities**:
  - *Propositional tautologies* – e.g. formulas of the form $\psi \to \psi$ or $\psi \vee \neg\psi$
  - *Equality* - the way $=$ is interpreted implies formulas like $\forall x\ x = x$ are validities.
  - *Quantifiers*
- **Fundamental questions**:
  - Can we characterize the set of validities *syntactically*?
  - In particular, is there an algorithm that, on input φ, determines whether φ is a validity or not?

Problems

Exercise 0.1 (Warmup)**.**
Determine whether the following formulas are validitites.

- $((\varphi_1 \to (\varphi_2 \to \varphi_3)) \to ((\varphi_1 \to \varphi_2) \to (\varphi_1 \to \varphi_3)))$
- $\exists x(\varphi \wedge \psi) \leftrightarrow (\exists x\varphi \wedge \exists x\psi)$

Exercise 0.2.
Let $g : \{0,1\}^n \to \{0,1\}$ be a Boolean function. Show that there exists a propositional formula $F(p_1, \dots, p_n)$ such that, for any assignment δ to the variables $p_1, \dots, p_n$,

$$\delta^*(F) = g(\delta(p_1), \dots, \delta(p_n)).$$

Exercise 0.3.
Formally verify that every propositional tautology is a validity.

Exercise 0.4.
Verify, using the definition of the satisfaction relation $\vDash$, that

$$\forall x, y(x = y \ \to\ f(x) = f(y)) \quad (f \text{ unary function symbol})$$

and

$$\forall x(\varphi \to \psi) \ \to\ (\varphi \to \forall x\psi) \quad (x \text{ not free in } \varphi)$$

are validities.