Math 557 Sep 26

Midterm 1 Review

! Take-home Problem 1

Prove unique readability for the set of $\mathcal{L}$-formulas.

Solution 0.1. **First** prove **readibility**, i.e. the statement that for any $\mathcal{L}$-formula φ, exactly one of the following cases holds:

(1) There exist terms s, t such that $\varphi \equiv s = t$.
(2) There exists a relation symbol R and, if n is the arity of R, terms $t_1, \dots, t_n$ sucht that $\varphi \equiv Rt_1 \dots t_n$.
(3) There exists a formula ψ such that $\varphi \equiv \neg\psi$.
(4) There exist formulas ψ, θ such that $\varphi \equiv (\psi \wedge \theta)$.
(5) There exists a variable x and a formula ψ such that $\varphi \equiv \exists x\psi$.

To see this, let F be the subset of $\mathcal{L}^*$ such that every $\theta \in F$ satisfies exactly one of (1)-(5). Then F contains all atomic formulas and is closed under $\neg, \wedge, \exists x$. Since the set of $\mathcal{L}$-formulas is the smallest such set, it follows that every $\mathcal{L}$-formula is contained in F, and therefore has the desired property.

Second we argue that a proper initial segment of a formula cannot be a formula. We proceed by induction on the length l of φ. For $l = 1$ there is no formula of that length. Now suppose φ is a formula of length $l + 1$, and β is a proper initial segment. We may assume β is not empty. We consider theses (1)-(5) from readability.

Case $\varphi \equiv s = t$ Either $\beta \equiv s = t'$ with $t \subset t'$, but this would contradict that no proper initial segment of a term is a term. Or $\beta \equiv s =$: An easy induction over the length of a formula shows that no formula ends with $=$. Or $\beta \subseteq s$. Another easy induction shows that no formula can be a term or an initial segment of a term.

Case $\varphi \equiv Rt_1 \dots t_n$ If β were a formula, it has to be of the form $Rs_1 \dots s_n$. Comparing terms, we would see that one of the s_i is a proper initial segment of some t_j, which is impossible.

Case $\varphi \equiv \neg\theta$ Then $\beta \equiv \neg\beta'$ with β' a proper initial segment of θ. By inductive hypothesis, β' is not a formula, hence $\neg\beta'$ is not a formula either.

Case $\varphi \equiv (\psi \wedge \theta)$ Then β starts with (but does not end with). Another induction shows that for any formula, the number of ('s always equals the number of)'s.

Case $\varphi \equiv \exists x\varphi$ Neither '$\exists$' nor '$\exists x$' are formulas. If β is longer than that, it is of the form $\exists x\beta'$. By inductive hypothesis, β' is not a formula, and hence β is not a formula (by readability). varphi

Finally, we prove **unique readability**: The choices in (1)-(5) are unique.

Case (1) Suppose $\varphi \equiv s = t \equiv s' = t'$. Since no proper initial segment of a term is a term, it must hold that $s \equiv s'$ and $t \equiv t'$.

Case (2) Suppose $\varphi \equiv Rt_1 \dots t_n \equiv Ss_1 \dots s_k$. Then $R \equiv S$ and $n = k$. Comparing terms inductively, using again the fact that no proper initial segment of a term is a term, we get $s_i \equiv t_i$ for all i.

Case (3) Suppose $\varphi \equiv \neg\psi \equiv \neg\theta$. Immediately, we infer $\psi \equiv \theta$.

Case (4) Suppose $\varphi \equiv (\psi_1 \wedge \theta_1) \equiv (\psi_2 \wedge \theta_2)$. Assume $\psi_1 \not\equiv \psi_2$. Then ψ_1 is a proper initial segment of ψ_2 or vice versa (since both of them are part of φ). Either case is impossible due to the fact that no proper initial segment of a formula is a formula. Hence $\psi_1 \equiv \psi_2$ and thus also $\theta_1 \equiv \theta_2$.

Case (5) Suppose $\varphi \equiv \exists x\psi \equiv \exists y\theta$. By comparing entries, we get $x \equiv y$ and thus $, as desired.

! Take-home Problem 2

Let $\mathcal{L}$ be any finite language and let $\mathcal{M}$ be a finite $\mathcal{L}$-structure. Show that there is an $\mathcal{L}$-sentence φ such that

$$\mathcal{N} \vDash \varphi \iff \mathcal{N} \cong \mathcal{M}.$$

Solution 0.2. Suppose $\mathcal{L} = \{c_1, \cdots, c_k, f_1^{(a_1)}, \dots f_l^{(a_l)}, R_1^{(b_1)}, \dots, R_j^{(b_j)}\}$ where the $(a_i), (b_m)$ denote the arities of the respective symbols.

Since $\mathcal{M}$ is finite, we may assume $M = \{1, \dots, n\}$ for some $n \in \mathbb{N}$.

The basic idea is to collect all "elementary facts" about $\mathcal{M}$ in a single formula (think of a *group multiplication table*, just for all functions and relations).

Define

$$\begin{aligned}
\varphi \equiv\ \exists x_1, \dots, x_n \Big(& \bigwedge_{i \neq m} x_i \neq x_l \ \wedge\ \forall y \bigvee_{i \leq n} y = x_i && (1)\\
& \wedge \bigwedge_{i \leq k} c_i = x_{c_i^{\mathcal{M}}} && (2)\\
& \wedge \bigwedge_{i \leq l} \bigwedge_{\pi \in \{1,\dots,n\}^{a_i}} f_i x_{\pi(1)} \dots x_{\pi(a_i)} = x_{f_i^{\mathcal{M}}(\pi(1),\dots,\pi(a_i))} && (3)\\
& \wedge \bigwedge_{i \leq j} \bigwedge_{\pi \in \{1,\dots,n\}^{b_i}} \delta_\pi R_i x_{\pi(1)} \dots x_{\pi(b_i)} \Big) && (4)
\end{aligned}$$

where δ_π is empty if $R(\pi(1), \dots, \pi(b_i))$ holds in $\mathcal{M}$, and $\neg$ if not.

Assme $\mathcal{N} \vDash \varphi$. Due to the first line of the equation, N has exactly n elements, and let $r_i \in N$ be the witness to $\exists x_i$. Define a mapping $\tau : M \to N$ by letting $\tau(i) = r_i$. We claim that τ is an isomorphism.

By definition we have $\tau(c_i^{\mathcal{M}}) = r_{c_i^{\mathcal{M}}}$. Moreover, by the second line of the formula, $r_{c_i^{\mathcal{M}}}$ is the unique element of N that makes the formula $c_i = x_{c_i^{\mathcal{M}}}$ true. It follows that $\tau(c_i^{\mathcal{M}}) = c_i^{\mathcal{N}}$ for all $1 \leq i \leq k$.

Using a similar argument with the third line of the formula, we obtain

$$\tau(f_i^{\mathcal{M}}(s_1, \dots, s_{a_i})) = r_{f_i^{\mathcal{M}}(s_1,\dots,s_{a_i})} = f^{\mathcal{N}}(r_{s_1}, \dots, r_{s_{a_i}}) = f^{\mathcal{N}}(\tau(s_1), \dots, \tau(s_{a_i}))$$

The argument for $R^{\mathcal{M}}(s_1, \dots, s_{b_i}) \iff R^{`\mathcal{N}}(\tau(s_1), \dots, \tau(s_{b_i}))$ is similar.

On the other hand, if $\mathcal{M} \cong \mathcal{N}$ via τ, then $\mathcal{N} \vDash \psi[\tau(1), \dots, \tau(n)]$, where ψ is such that $\varphi \equiv \exists x_1, \dots, x_n\, \psi$, due to the fact that τ is an isomorphism, and thus $\mathcal{N} \vDash \varphi$.

! Take-home Problem 3

Give an example of a language $\mathcal{L}$ and an $\mathcal{L}$-sentence ψ such that

- there is at least one $\mathcal{L}$-structure A such that $A \vDash \psi$,
- for all L-structures A, if $A \vDash \psi$, then the universe A of A is infinite.

Solution 0.3. Let $\mathcal{L} = \{<\}$, where $<$ is a binary relation symbol. Define

$$\begin{aligned}
\psi \equiv\ & \forall x\ x \not< x && (5)\\
& \wedge \forall x, y\ x < y \vee x = y \vee y < x && (6)\\
& \wedge \forall x, y, z\ (x < y \wedge y < z) \to x < z && (7)\\
& \wedge \forall x \exists y\ x < y && (8)
\end{aligned}$$

The formula says that $<$ is a linear order with no maximal element.

Clearly, $(\mathbb{Z}, <) \vDash \psi$.

Now suppose $\mathcal{M} \vDash \psi$. Due to the last line of ψ, there exists a function $f : M \to M$ such that $x < f(x)$ for all $x \in M$. We claim that for any x and for any $n \neq m$,

$$f^{(n)}(x) \neq f^{(m)}(x)$$

This follows from antireflexivity (line one) and transitivity (line three).

Therefore, the set

$$\{x, f^{(1)}(x), f^{(2)}(x), \dots\}$$

is an infinite subset of N.

! Take-home problem 4

Show that

$$\{\varphi \to \psi\} \vdash \exists x\varphi \ \to \ \exists x\psi$$
$$\{\varphi \to \psi\} \vdash \forall x\varphi \ \to \ \forall x\psi$$

Solution 0.4. We give derivations below (with brief justifications). We collect simple substeps into a single one.

For $\{\varphi \to \psi\} \vdash \exists x\varphi \ \to \ \exists x\psi$:

Formula	Justification
$\varphi \to \psi$	given
$\psi \to \exists x\psi$	(Q2)
$\varphi \to \exists x\psi$	tautology
$\neg\exists x\psi \to \neg\varphi$	tautotolgy
$\forall x(\neg\exists x\psi \to \neg\varphi)$	$\forall$-intro
$\neg\exists x\psi \to \forall x\varphi$	(Q1), x not free in $\neg\exists x\psi$
$\neg\forall x\neg\varphi \to \exists x\psi$	tautology
$\exists x\varphi \to \exists x\psi$	(Q3)

For $\{\varphi \to \psi\} \vdash \forall x\varphi \ \to \ \forall x\psi$:

Formula	Justification
$\varphi \to \psi$	given
$\forall x\varphi \to \varphi$	Example 2.6.1 (d)
$\forall x\varphi \to \psi$	Tautology
$\forall x(\forall x\varphi \to \psi)$	$\forall$-intro
$\forall x\varphi \to \forall x\psi$	(Q1), x not free in $\forall x\varphi$

! Take-home Problem 6

Use the compactness theorem to show (without using the Axiom of Choice) that every set can be linearly ordered.
Try to strengthen this to:

Every partial order can be extended to a linear order.

Solution 0.5. We first argue that any finite partial order $(F, <)$ can be extended to a linear order.

We proceed by induction on the cardinality of F.

For $|F| = 0$ there is nothing to prove.

Now assume $|F| = n + 1$. Pick an arbitrary $a \in F$ and consider $F \setminus \{a\}$. By inductive hypothesis, $F \setminus \{a\}$ can be extended to a linear order $a_1 < a_2 < \cdots < a_n$.

If there does not exist i such that $a_i <_F a$, we can extend to a linear order on F by putting $a < a_i$ for all i. Otherwise let k be maximal such that $a_k < a$. Then

$$a_1 < \cdots < a_k < a < a_{k+1} < \cdots < a_n$$

defines a linear order that extends $(F, <)$.

Now let $(P, <)$ be an arbitrary partial order. We extend the language $\mathcal{L}_<$ of orders to $\mathcal{L}_P$, where we add a new constant symcol c_p for every $p \in P$.

Let T be the theory of linear orders (see first three lines of sentence φ in Problem 3) together with

$$\{c_p < c_q : p <_P q\}$$

In other words, if p is less than q according to P, we add a corresponding axiom to our theory.

Any model of T is a liner order that extends P.

Moreover, any finite subset $T_0 \subseteq T$ induces a partial order on a finite subset of P (induced by those p for which c_p is part of a formula in T_0). By our argument in the first part, this finite partial order extends to a linear order, thereby giving a model of T_0.

By compactness, T has a model $\mathcal{M} = (M, <)$. By choice of T, $(M, <)$ is a linear order. Furthermore, the mapping

$$c_p \mapsto c_p^{\mathcal{M}}$$

is one-to-one since $\mathcal{M} \models \forall x \; x \not< x$.

We can "pull back" the order on $\mathcal{M}$ to P by letting

$$p <' q \iff c_p^{\mathcal{M}} <^{\mathcal{M}} c_q^{\mathcal{M}}$$

By definition of T, $<'$ is a linear order that extends $<_P$.