MATH 557 Oct 15

Amalgamation Classes

Let $\overline{K}$ be a class of finitely generated structures. $\overline{K}$ is called an **amalgamation class** if it has the following three properties:

(HP) Hereditary Property

If $\mathcal{A} \in \overline{K}$, $\mathcal{B} \cong \mathcal{C} \in \mathcal{A}$, and $\mathcal{C}$ is finitely generated, then $\mathcal{B} \in \overline{K}$.

(JEP) Joint Embedding Property

If $\mathcal{A}, \mathcal{B} \in \overline{K}$, then there exists $\mathcal{C} \in \overline{K}$ and embeddings

$$f_0 : \mathcal{A} \to \mathcal{C}, \quad f_1 : \mathcal{B} \to \mathcal{C}$$

(AP) Amalgamation Property

If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \overline{K}$ with embeddings $f_0 : \mathcal{A} \to \mathcal{B}$ and $f_1 : \mathcal{A} \to \mathcal{C}$, then there exists $\mathcal{D} \in \overline{K}$ and embeddings

$$g_0 : \mathcal{B} \to \mathcal{D}, \quad g_1 : \mathcal{C} \to \mathcal{D}$$

such that $g_0 \circ f_0 = g_1 \circ f_1$.

Exercise 0.1.
Show that (AP) does not imply (JEP).

(*Hint: finite fields*)

Example 0.1.
$\overline{K} = \{(Z, <) : (Z, <) \text{ finite linear order}\}$ forms an amalgamation class.

Fraïssé's Theorem

Theorem 0.1. *Let $\overline{K}$ be a class of finitely generated $\mathcal{L}$-structures such that there are only countably many isomorphism types in $\overline{K}$.*

Then: $\overline{K}$ is an amalgamation class $\Leftrightarrow$ $\overline{K}$ is the age of a countable homogeneous $\mathcal{L}$-structure.

Proof. **($\Leftarrow$):** Suppose $\overline{K} = \text{age}(\mathcal{M})$ where $\mathcal{M}$ is countable and homogeneous.

(HP): Holds by definition of age.

(JEP): Let $\mathcal{A}, \mathcal{B} \in \text{age}(\mathcal{M})$. Then $\mathcal{A} \cong \langle A \rangle^{\mathcal{M}}$ and $\mathcal{B} \cong \langle B \rangle^{\mathcal{M}}$, with $A, B \subseteq M$ finite. Then $\mathcal{A}$ and $\mathcal{B}$ embed into $\langle A \cup B \rangle^{\mathcal{M}}$.

(AP): Suppose $f_0 : \mathcal{A} \to \mathcal{B}$ and $f_1 : \mathcal{A} \to \mathcal{C}$ are embeddings, where $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \text{age}(\mathcal{M})$.

Without loss of generality, $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathcal{M}$ and $f_0 = \text{id}_A$.

Since $\mathcal{M}$ is homogeneous, there exists an automorphism $\pi : \mathcal{M} \xrightarrow{\sim} \mathcal{M}$ such that $\pi^{-1}|_A = f_1$ (i.e., π extends f_1^{-1}).

Consider $\mathcal{D} = \langle B \cup \pi^{-1}(C)\rangle^{\mathcal{M}}$. $\mathcal{D}$ amalgamates $\mathcal{B}$ and $\mathcal{C}$ via $\mathrm{id}_{\mathcal{B}} : \mathcal{B} \to \mathcal{D}$ and $\pi^{-1}|_C : \mathcal{C} \to \mathcal{D}$. $\square$

($\Rightarrow$): Assume $\overline{K}$ is an amalgamation class. We construct a structure $\mathcal{N}$ with $\mathrm{age}(\mathcal{N}) = \overline{K}$ where $\mathcal{N}$ is homogeneous and has domain $N \subset \mathbb{N}$.

Since $\overline{K}$ has countably many isomorphism types, we enumerate the elements of $\overline{K}$ (up to isomorphism) as $(\mathcal{K}_e : e \in \mathbb{N})$ where $K_e \subseteq \mathbb{N}$.

Consider tuples $(\bar{a}, \bar{b}, f)$ where $\bar{a}_i, \bar{b}_i$ are finite subsets of $\mathbb{N}$ and f is a partial function with $\mathrm{dom}(f) = \bar{a}$ and $\mathrm{ran}(f) \subseteq \bar{b}$. Enumerate all such triples as $(\bar{a}_e, \bar{b}_e, f_e)$ so that every $(\bar{a}, \bar{b}, f)$ occurs infinitely often.

$\mathcal{N}$ will be the union of an increasing sequence $(\mathcal{C}_n)_{n\in\mathbb{N}}$ where $\mathcal{C}_n \in \overline{K}$.

Initialize: $\mathcal{C}_0 = \mathcal{K}_0$

Now assume we have defined $\mathcal{C}_n$.

Case $n = 2\ell$ (even): Apply (JEP) to $\mathcal{C}_n, \mathcal{K}_\ell$ to obtain $\mathcal{C}_{n+1}$.

Case $n = 2\ell + 1$ (odd): If $\bar{a}_\ell$ or $\bar{b}_\ell \nsubseteq C_n$, put $\mathcal{C}_{n+1} = \mathcal{C}_n$. If $\bar{a}_\ell, \bar{b}_\ell \subseteq C_n$, let $\mathcal{A}_\ell = \langle \bar{a}_\ell\rangle^{\mathcal{C}_n}$ and $\mathcal{B}_\ell = \langle \bar{b}_\ell\rangle^{\mathcal{C}_n} \subseteq \mathcal{C}_n$. Apply (AP) to the embeddings $\mathrm{id} : \mathcal{A}_\ell \to \mathcal{C}_n$ and $f : \mathcal{A}_\ell \to \mathcal{B}_\ell$ (induced by $f_\ell : \bar{a}_\ell \to \bar{b}_\ell$). This yields a structure $\mathcal{C}_{n+1} \in \overline{K}$. Renaming if necessary, we can assume $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$.

Define $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$.

Claim 1: $\mathrm{age}(\mathcal{N}) = \overline{K}$.
In the even steps $n = 2\ell$, we ensure $K_\ell \subseteq \mathcal{N}$, so all structures isomorphic to K_ℓ also enter $\mathrm{age}(\mathcal{N})$. Since the $(K_e)_{e\in\mathbb{N}}$ enumerates all isomorphism types of $\overline{K}$, we get $\mathrm{age}(\mathcal{N}) = \overline{K}$.

Claim 2: $\mathcal{N}$ is homogeneous.
Let $\tau : A \to \mathcal{B}$ be an isomorphism between finitely generated substructures $\mathcal{A} = \langle \bar{a}\rangle^{\mathcal{N}}$ and $\mathcal{B} = \langle \bar{b}\rangle^{\mathcal{N}}$ where $\bar{a}, \bar{b} \in N$ are finite. We show that for any $c_0 \in N \setminus B$, there exists a partial isomorphism $\tau' : \mathcal{A}' \to \mathcal{B}'$ where $\mathcal{A}', \mathcal{B}'$ are finitely generated, $\tau' \supseteq \tau$, and $c_0 \in \mathrm{dom}(\tau')$. (This suffices since we can continue via back-and-forth.)

There exists ℓ such that f_ℓ induces an embedding $f : A \to \langle \mathcal{B} \cup \{c_0\}\rangle^{\mathcal{N}}$ (since every triple occurs infinitely often in the enumeration, in particular the triple $(\bar{a}, \bar{b} \cup \{c_0\}, f)$).

In step $n = 2\ell + 1$, $\mathcal{B}_\ell = \in \langle \mathcal{B} \cup \{c_0\}\rangle^{\mathcal{N}}$ and $\mathcal{C}_n$ are amalgamated over A_ℓ. This yields embeddings:

$$\begin{aligned} \mathrm{id} &: \mathcal{A} \hookrightarrow \mathcal{C}_{n+1} \\ g &: \mathcal{B} \hookrightarrow \mathcal{C}_{n+1} \text{ where } g \circ f_\ell = \mathrm{id}_A \end{aligned}$$

Let $a_0 = g(c_0)$. Then $(g|_{\mathcal{B}})^{-1} : \langle \bar{a} \cup \{a_0\}\rangle^{\mathcal{N}} \xrightarrow{\sim} \langle \bar{b} \cup \{c_0\}\rangle^{\mathcal{N}}$ is the desired isomorphism.