Math 557 Oct 24

Primitive Recursive Functions

The following functions are certainly intuitively computable.

R1	$O(x) = 0$	*Zero function*
R2	$S(x) = x + 1$	*Successor function*
R3	$P_i^n(x_1, \ldots, x_n) = x_i$	*i-th Projection*

By **substituting** functions into already given functions we obtain a new function:

R4	$f(\vec{x}) \simeq h(g_1(\vec{x}), \ldots, g_k(\vec{x}))$	*Substitution*

Another important way to form new functions is by **primitive recursion**. According to this pattern the arithmetic operations addition, multiplication and exponentiation are defined.

R5	$f(\vec{x}, 0) \simeq g(\vec{x})$	*Primitive recursion*
	$f(\vec{x}, y+1) \simeq h(\vec{x}, y, f(\vec{x}, y))$	

Definition 1

The class **PRIM** of all **primitive recursive** (abbreviated: **p.r.**) functions is the smallest class of functions that contains the initial functions and is closed under substitution and primitive recursion. These functions are in particular total functions.

Examples of Primitive Recursive Functions

The following functions are primitive recursive:

$x + y$	*Sum*
$x \cdot y$	*Product*
x^y	*Exponentiation*
$\max(x, y)$	*Maximum*
$\min(x, y)$	*Minimum*
$\Vert x - y \Vert$	*Absolute difference*

and moreover,

$$x \dot{-} y = \begin{cases} x - y & \text{if } x \geq y, \\ 0 & \text{otherwise} \end{cases} \qquad \textit{Monus (difference on } \mathbb{N}\textit{)}$$

$$sg(x) = \begin{cases} 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0 \end{cases} \qquad \textit{Sign function}$$

$$\overline{sg}(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{if } x > 0 \end{cases} \qquad \textit{Negated sign function}$$

Proof

Addition, multiplication and exponentiation are obviously primitive recursive. For the remaining cases one first shows that the predecessor function $x \dot{-} 1$ is primitive recursive, namely because

$$\begin{aligned} 0 \dot{-} 1 &= 0 \\ (x+1) \dot{-} 1 &= x \end{aligned}$$

and then defines $\dot{-}$ by primitive recursion

$$\begin{aligned} x \dot{-} 0 &= x \\ x \dot{-} (y+1) &= (x \dot{-} y) \dot{-} 1 \end{aligned}$$

The remaining functions can be defined directly using $+$ and $\dot{-}$:

$$\begin{aligned} |x - y| &= (x \dot{-} y) + (y \dot{-} x) \\ \max(x, y) &= x + (y \dot{-} x) \\ \min(x, y) &= \max(x, y) \dot{-} |x - y| \\ \overline{sg}(x) &= 1 \dot{-} x \\ sg(x) &= 1 \dot{-} \overline{sg}(x) \end{aligned}$$

Closure Properties

Exercise 1

Show that primitive recursive functions are closed under
(i) Case distinction:
If $g, f_0, f_1, \ldots, f_k$ are primitive recursive, then so is f with

$$f(\vec{x}) = \begin{cases} f_0(\vec{x}) & \text{if} \quad g(\vec{x}) = 0, \\ f_1(\vec{x}) & \text{if} \quad g(\vec{x}) = 1, \\ \vdots & \vdots \\ f_k(\vec{x}) & \text{if} \quad g(\vec{x}) \geq k. \end{cases}$$

and
(ii) Bounded μ-operator:
If g is primitive recursive, then so is f with

$$f(\vec{x}, z) = \mu y < z \; (g(\vec{x}, y) = 0),$$

where

$$\mu y < z \; R(\vec{x}, y) = \begin{cases} \text{the smallest } y < z \text{ with } R(\vec{x}, y) & \textit{if such exists} \\ z & \text{otherwise.} \end{cases}$$