Math 557 Oct 10

Week 7 - Exercises

Key concepts

(Ultra)filters

A **filter** $\mathcal{F}$ on a set I is a nonempty collection of subsets of I satisfying:

1. $\emptyset \notin \mathcal{F}$
2. If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$
3. If $A \in \mathcal{F}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{F}$

An **ultrafilter** $\mathcal{U}$ is a maximal filter, equivalently:

For all $A \subseteq I$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

Reduced products

Given: filter $\mathcal{F}$ on I and structures $(\mathcal{M}_i)_{i\in I}$. Let $M = \prod_{i\in I} M_i$ and define

$$a \sim_{\mathcal{F}} b \iff \{\, i \in I : a_i = b_i \,\} \in \mathcal{F}.$$

The universe of $\mathcal{M}/\mathcal{F}$ is the quotient $M/{\sim_{\mathcal{F}}}$, with elements denoted $a_{\mathcal{F}}$ (alternatively, $a/\mathcal{F}$).

- **Relations:**

$$R^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) :\iff \{\, i : \mathcal{M}_i \models R(\vec{a}_i) \,\} \in \mathcal{F}.$$

- **Functions:**

$$f^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) = [\,(f^{\mathcal{M}_i}(\vec{a}_i))_{i\in I}\,]_{\mathcal{F}}.$$

- **Constants:**

$$c^{\mathcal{M}/\mathcal{F}} = ((c^{\mathcal{M}_i})_{i\in I})_{\mathcal{F}}.$$

Łoś' Theorem

Let $\mathcal{M}/\mathcal{U} = \prod_{i\in I} \mathcal{M}_i/\mathcal{U}$ be an ultraproduct. For every $\mathcal{L}$-formula $\varphi(x_1, \dots, x_n)$ and tuples $\vec{a} \in \prod_{i\in I} M_i$,

$$\mathcal{M}/\mathcal{U} \models \varphi[\vec{a}_{\mathcal{U}}] \iff \{\, i \in I : \mathcal{M}_i \models \varphi[\vec{a}_i] \,\} \in \mathcal{U}.$$

Problems

Exercise 0.1 (Principal filters)**.** Show that for every $A \subseteq I$,

$$\{X \subseteq I : A \subseteq X\}$$

is a filter on I. Show that it is an ultrafilter if and only if $|A| = 1$.

Exercise 0.2 (Finite sets)**.** Show that every ultrafilter on a finite set is principal.

Exercise 0.3 (Free ultrafilters on infinite sets)**.** Show that a free ultrafilter on an infinite set I cannot contain any finite subsets of I.

Exercise 0.4 (Ultrafilter existence)**.** Use Zorn's Lemma to show that any family $\mathcal{A} \subseteq \mathcal{P}(I)$ with the FIP can be extended to an ultrafilter $\mathcal{U}$ on I.

Use this to show that any infinite set has a free ultrafilter.

Exercise 0.5 (Ultrapoducts with principal ultrafilters)**.** Show that if $\mathcal{U}$ is a principal ultrafilter on I, then every ultraproduct

$$\mathcal{M}/\mathcal{U} = \prod_{i \in I} \mathcal{M}_i/\mathcal{U}$$

is isomorphic to some $\mathcal{M}_j$, $j \in I$.

For principal ultrafilters, ultraproducts do not lead to anything new.

Exercise 0.6 (Ultraproducts of fields)**.** For any prime p let $\mathbb{F}_p$ the field with p elements (of characteristic p), and let $\mathcal{U}$ be a free ultrafilter on the set P of prime numbers.

Show that the ultraproduct

$$\prod_{p \in P} \mathbb{F}_p/\mathcal{U}$$

is a field of characteristic 0.

Ultrapowers

If we take an ultraproduct over the same structure $\mathcal{M}$ along an index set I, we call this an **ultrapower**, denoted by

$$\mathcal{M}^I/\mathcal{U}$$

Let $\mathcal{U}$ be a free ultrafilter on an (infinite) set I.

- Show that the map $j : b \mapsto (b)_{i \in I}/\mathcal{U}$ defines an **elementary embedding** of $\mathcal{M}$ into $\mathcal{M}^I/\mathcal{U}$ (i.e. it is injective and the image is an elementary substructure).
- Show that if $\mathcal{M}$ is infinite, j is not a surjection.

*If we apply this to $\mathbb{N}$ in the language of arithmetic, this yields another way to obtain **non-standard models** of arithmetic.*

Further explorations

Lindenbaum-Tarski algebra

Given a language $\mathcal{L}$, define an equivalence relation of the set of all $\mathcal{L}$-sentences by

$$\sigma \sim \tau \iff \vdash \sigma \leftrightarrow \tau$$

The equivalence classes will then form a *Boolean algebra*, the **Lindenbaum-Tarksi algebra** $\mathcal{B}$ with the operations

$$[\sigma] \wedge [\tau] := [\sigma \wedge \tau]$$
$$[\sigma] \vee [\tau] := [\sigma \vee \tau]$$
$$\neg[\sigma] := [\neg\sigma]$$

The *bottom* element is $0 := [\sigma \wedge \neg\sigma]$, the *top* element is $1 := [\sigma \vee \neg\sigma]$.

We can define an order by putting $[\sigma] \leq [\tau] :\iff \vdash \sigma \to \tau$. (This corresponds to the order that is defined on any Boolean algebra via $a \leq b :\Leftrightarrow a = a \wedge b$.)

A *filter* on a Boolean algebra is defined in the same way it is defined on a set I (in fact, a filter on a set I is simply a filter on the Boolean algebra induced on its power set by taking intersections, unions, and complements): $F \subseteq \mathcal{B}$ is a filter if it does not contain the bottom element, is closed under $\wedge$ and closed upward under $\leq$.

- Show that if T is a deductively closed, consistent $\mathcal{L}$-theory, the set

$$\{[\sigma] : \sigma \in T\}$$

  is a filter in $\mathcal{B}$.

- Show that if T is complete, the above filter is an ultrafilter.

Keisler-Shelah Theorem

From Łoś' Theorem, we know that if two ultrapowers are isomorphic, then the original structures are elementary equivalent. Remarkably, the converse holds, too, and hence gives an *algebraic characterization of elementary equivalence.*

Theorem 0.1 (Keisler-Shelah)**.** *For any two structures $\mathcal{M}, \mathcal{N}$,*

$$\mathcal{M} \equiv \mathcal{N} \iff \text{there exists an index set } I \text{ and an ultrafilter } \mathcal{U} \text{ on } I \text{ such that } \mathcal{M}^I/\mathcal{U} \cong \mathcal{N}^I/\mathcal{U}$$

Other interesting directions

- Ramsey theory and ultrafilters