Math 557 Nov 3

End Extensions

The standard model can be embedded into every model $\mathcal{M}$ of PA as an initial segment. It turns out this already holds for models of the theory PA^-.

Definition 0.1.
Let L be a language containing a 2-ary symbol $<$, and let $\mathcal{M}$ and $\mathcal{N}$ be L-structures with $\mathcal{M} \subseteq \mathcal{N}$. Then $\mathcal{N}$ is called an **end extension** of $\mathcal{M}$ (and correspondingly $\mathcal{M}$ is an **initial segment** of $\mathcal{N}$) if and only if the larger set N does not add any further elements below an element of M:

$$\mathcal{M} \subseteq_{end} \mathcal{N} :\iff \text{ for all } x \in M, y \in N : (y <^N x \Rightarrow y \in M).$$

Each natural number n is represented in the standard model, which we also simply denote by $\mathbb{N}$ here, by the constant term

$$\underline{n} = 1 + ... + 1 \quad (n \text{ times})$$

where $\underline{0}$ is the constant 0.

Theorem 0.1.
Let $\mathcal{M} \models \mathsf{PA}^-$. Then the map

$$n \mapsto \underline{n}^{\mathcal{M}}$$

defines an embedding of the standard model $\mathbb{N}$ onto an initial segment of $\mathcal{M}$.

*In particular, every model of PA^- is isomorphic to an end extension of the standard model $\mathbb{N}$.**

Proof. By simple induction (in the meta-theory), one shows for all natural numbers n, k, l:

$$\begin{aligned} n = k + l \quad &\Longrightarrow \quad \mathsf{PA}^- \vdash \underline{n} = \underline{k} + \underline{l} \\ n = k \cdot l \quad &\Longrightarrow \quad \mathsf{PA}^- \vdash \underline{n} = \underline{k} \cdot \underline{l} \\ n < k \quad &\Longrightarrow \quad \mathsf{PA}^- \vdash \underline{n} < \underline{k} \end{aligned}$$

and

$$\mathsf{PA}^- \vdash \forall x \, (x \leq \underline{k} \to (x = \underline{0} \, \vee ... \vee \, x = \underline{k})$$

The first three statements will later be generalized to all recursive functions and relations; they imply that the map $n \mapsto \underline{n}^{\mathcal{M}}$ is a homomorphism, and, due to the last statement, the map is also an embedding onto an initial segment of $\mathcal{M}$. □

Remark

The standard model has no proper initial segment, and $\mathbb{Z}[X]^+$ has $\mathbb{N}$ as its only proper initial segment. On the other hand, every model $\mathcal{M} \models \mathsf{PA}^-$ has a proper end extension that is also a model of PA^-, namely the non-negative part of the polynomial ring $R[X]$, where R is the discretely ordered ring associated with the model $\mathcal{M}$.

Preservation Properties of End Extensions

In the previous lecture, we introduced *arithmetical formulas*. Δ_0-formulas are at the bottom of the *arithmetical hierarchy*. We already saw that relations defined by such formulas are primitive recursive. Next, we will show that those formulas *mean the same thing in a structure as in all end extensions*. This will be crucial later on.

Theorem 0.2.
Let $\mathcal{N}, \mathcal{M}$ be structures of the language L of PA^-, with $\mathcal{N} \subseteq_{end} \mathcal{M}$, and let $\vec{a} \in N$. Then:

1. *Every Δ_0-formula $\varphi(\vec{v})$ is **absolute**:*
$$\mathcal{N} \models \varphi[\vec{a}] \iff \mathcal{M} \models \varphi[\vec{a}],$$
2. *Every Σ_1-formula $\varphi(\vec{v})$ is **upward-persistent**:*
$$\mathcal{N} \models \varphi[\vec{a}] \Longrightarrow \mathcal{M} \models \varphi[\vec{a}],$$
3. *Every Π_1-formula $\varphi(\vec{v})$ is **downward-persistent**:*
$$\mathcal{M} \models \varphi[\vec{a}] \Longrightarrow \mathcal{N} \models \varphi[\vec{a}],$$
4. *Every Δ_1-formula $\varphi(\vec{v})$ is **absolute**:*
$$\mathcal{N} \models \varphi[\vec{a}] \iff \mathcal{M} \models \varphi[\vec{a}].$$

Proof. Part (i) is proved by induction on the formula structure of $\varphi(\vec{v})$. Most cases are straightforward (using that $\mathcal{N}$ is a substructure of $\mathcal{M}$). In the case of a bounded quantifier, one argues that, since $\mathcal{M}$ is an end extension of $\mathcal{N}$, M does not insert any new elements below an element of N, so that a bounded quantifier means the same thing in both structures.

The other parts follow easily from the definition of the satisfaction relation $\models$ and the Δ_0 case. ☐

Let $\Sigma_1\text{-Th}(\mathbb{N}) := \{\sigma \mid \sigma \text{ is a } \Sigma_1\text{-sentence with } \mathbb{N} \models \sigma\}$. Then we have:

Corollary 0.1.
$$\mathsf{PA}^- \models \Sigma_1\text{-}\mathrm{Th}(\mathbb{N})$$

Proof. Let $\mathcal{N} \models \mathsf{PA}^-$. By Theorem 0.1, we may assume that $\mathbb{N} \subseteq_{end} \mathcal{N}$, and the claim then follows from part (ii) above. ☐

Thus one can prove in the theory PA^- all Σ_1-sentences that hold in the standard model. This is no longer true for Π_1-sentences. For instance, the Π_1-sentence stating that every number is even or odd:

$$(*) \quad \forall x \, \exists y \leq x \, (x = 2 \cdot y \lor x = 2 \cdot y + 1)$$

is true in the standard model, but not in the PA^--model $\mathbb{Z}[X]^+$.

Even true $\forall$-sentences (i.e., sentences of the form $\forall \vec{x} \; \psi$ with *quantifier-free* ψ) that hold in the standard model need not be provable in PA^-. For example, the $\forall$-sentence

$$\forall x, y (x^2 \neq 2 \cdot y^2)$$

is true in the standard model, but not provable in PA^- ($\mathbb{Z}/(X^2 - 2Y^2)$ is a counterexample).

Thus the above sentence (*) is an example of a Π_1-sentence that is not a $\forall$-sentence, where one cannot omit the bounded quantifier! (By the way, one can show that $\mathbb{Z}[X]^+$ is at least a model of all $\forall$-sentences that hold in the standard model.)