

# Math 557 Sep 5

## Substitution

### Key Concepts

- **Substitution:**

- Basic idea:  $\varphi_{\bar{s}/\bar{x}}$  is obtained by replacing all occurrences of the variable  $x_i$  by the term  $s_i$ .
- Uncontrolled substitution may cause issues with quantifiers. If we try to substitute a variable into the range of a quantifier, we rename the quantified variable to an unused variable ( $\exists x \dots$  becomes  $\exists u \dots$ ).

- **Substitution Lemma:**

- Substitution behaves “as expected” with respect to evaluation and satisfaction.
- Evaluating a substituted term yields the same value as evaluating the original term under the “substituted” assignment (i.e. the assignment in which we replace the assignment to  $x$  by the value of  $s$  under  $\alpha$ ).
- A substituted formula holds in  $\mathcal{M}$  under assignment  $\alpha$  iff the original formula holds in  $\mathcal{M}$  under the “substituted” assignment.

### Problems

**Exercise 0.1** (Carry-over from Sep 3).

Show that if  $x$  is not free in  $\varphi$ ,  $\mathcal{M} \models \varphi[\alpha]$  implies  $\mathcal{M} \models \forall x \varphi[\alpha]$ .

Then verify that

$$\forall x(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x \psi) \quad (x \text{ not free in } \varphi)$$

is a validity.

**Exercise 0.2.**

- Show that if  $t$  is a term, then  $t_{\bar{s}/\bar{x}}$  is a term.
- Show that if  $\varphi$  is a formula,  $\varphi_{\bar{s}/\bar{x}}$  is a formula of the same height.

**Exercise 0.3.**

Use the Substitution Lemma to verify that

$$\varphi_{t/x} \rightarrow \exists x \varphi$$

is a validity.

**Exercise 0.4.**

Show that if  $y$  does not occur in  $\psi$ ,

$$[\psi_{y/x}]_{x/y} \equiv \psi$$

Find a counterexample that shows this no longer holds if  $y$  *does* occur in  $\psi$ .