Math 557 Sep 19

Henkin Theories

Key Concepts

- An $\mathcal{L}$-theory T is a **Henkin theory** if for every sentence of the form $\exists x\psi$ there exists a constant $\mathcal{L}$-term t such that
$$T \vdash_{\mathcal{L}} \exists x\psi \;\rightarrow\; \psi_{t/x}$$
- **Questions:**
  - What are Henkin theories needed for?
  - Do they exist?
  - Can we extend a given consistent theory to a Henkin theory that is still consistent?

Problems

Exercise 0.1.
As before, let $\mathcal{A}_T$ be the term model of T. Pinpoint exactly where we might run into difficulties when we are trying to prove
$$\mathcal{A}_T \models \exists x\psi \iff T \vdash_{\mathcal{L}} \exists x\psi$$

Exercise 0.2 (Interlude)**.**
What if in $\exists x\psi$ the variable x is not free? Does that make a difference regarding being a Henkin theory?

Exercise 0.3.
Can you find a couple of examples of Henkin theories?

The basic construction step for extending an $\mathcal{L}$-theory T to a Henkin theory is for every sentence of the form $\sigma \equiv \exists x\psi$,

- add a new constant symbol c_σ to $\mathcal{L}$
- add the formula $\exists x\psi \;\rightarrow\; \psi_{c_\sigma/x}$ to T.

Exercise 0.4.
Why could the addition of a constant theoretically lead to T being inconsistent, even though T itself remains unchanged?

And why does T actually remain consistent?

The Henkin extension T_H of T is obtained by an iterative process:

Single iteration step:

- $\mathcal{L}' = \mathcal{L} \cup \{c_\sigma \colon \sigma \text{ } \mathcal{L}\text{-sentence of the form } \exists x\psi\}$
- $\Gamma = \{\exists x\psi(x) \rightarrow \psi_{c_\sigma/x} \colon \sigma \text{ } \mathcal{L}\text{-sentence of the form } \exists x\psi\}$
- $T' = T \cup \Gamma$ (an $\mathcal{L}'$-theory)

Exercise 0.5.
Why is T' above not necessarily a Henkin theory?

Iteration process:

- $\mathcal{L}_0 = \mathcal{L},\ T_0 = T$

- $\mathcal{L}_{n+1} = \mathcal{L}'_n$, $T_{n+1} = T'_n$
- $\mathcal{L}_H = \bigcup_{n\in\mathbb{N}} \mathcal{L}_n$, $T_H = \bigcup_{n\in\mathbb{N}} T_n$

Exercise 0.6.
Why is T_H a Henkin theory?

T_H is no longer a purely symbolic extension of T (in the sense that we simply extend the language), since we add new sentences to T (the sentences in Γ for each iteration step).

> **!** Take-home Problem
>
> Show that for every $\mathcal{L}$-formula φ,
> $$T_H \vdash_{\mathcal{L}_H} \varphi \iff T \vdash_{\mathcal{L}} \varphi$$