Math 557 Sep 24

The Compactness Theorem

Key Concepts

- **Theorem:** A theory T has a model if and only if every finite subtheory $T_0 \subseteq T$ has a model.

Problems

Exercise 0.1.
Let X be the set of all maximally consistent $\mathcal{L}$-theories. Recall that the sets

$$\langle \sigma \rangle = \{T \in X : \sigma \in T\} \quad (\sigma \ \mathcal{L}\text{-sentence})$$

generate a Hausdorff topology on X.

Show that the topology is compact.

Exercise 0.2.
One can use the compactness theorem to construct non-standard models of arithmetic, i.e., models of $\mathrm{Th}(\mathbb{N}, 0, 1, +, \cdot, <)$ not isomorphic to $\mathbb{N}$.

Use the same technique for $\mathrm{Th}(\mathbb{R}, \{c_a : a \in \mathbb{R}\}, +, \cdot, <)$, where for every $a \in \mathbb{R}$ we add a constant symbol c_a to the language? What kind of structure do we obtain? Discuss.

Exercise 0.3.
Use the compactness theorem to show (without using the Axiom of Choice) that every set can be linearly ordered.

Try to trengthen this to:

> Every partial order can be extended to a linear order.