Math 557 Sep 10

Logical Implication and Proof

Key Concepts

- **Logical consequence**:
  - This is the semantical implication we are often working with in mathematical practice. We say T **logically implies** φ, $T \models \varphi$, if for structure $\mathcal{M}$, $\mathcal{M} \models T$ implies $\mathcal{M} \models \varphi$.
- **Formal proof**:
  - $T \vdash \varphi$ means there is a formal (i.e. *syntactical*) derivation of φ from T using the formulas of T, the three kinds of *logical axioms* (propositional tautologies, equality and quantifier axioms), and the *inference rules* Modus Ponens and Generalization.

Problems

Exercise 0.1 (Warmup - Logical Implication)**.**
Let T be an $\mathcal{L}$-theory. We say a theory T' is an **axiomatization** of T if for any $\mathcal{L}$-structure $\mathcal{M}$,

$$\mathcal{M} \models T \iff \mathcal{M} \models T'$$

Show that for any axiomatization T' of T, for any $\mathcal{L}$-sentence σ,

$$T \models \sigma \iff T' \models \sigma$$

Exercise 0.2 (Warmup 2)**.**
Recall that a *model* of a theory T is a structure $\mathcal{M}$ such that for any sentence $\sigma \in T$, $\mathcal{M} \models \sigma$. In this case we write $\mathcal{M} \models T$.

Argue that if T does not have a model, every sentence is a logical implication of T.

Exercise 0.3 (Formal notion of proof – Warmup)**.**
Verify that

$$\{\varphi, \neg\psi\} \vdash \neg(\varphi \to \psi)$$

Exercise 0.4.
Argue (semantically) that if x is not free in ψ,

$$\{\varphi \to \psi\} \models \exists x\varphi \;\to\; \psi$$

Then prove this *syntactically*, i.e. show (under the same assumption) that

$$\{\varphi \to \psi\} \vdash \exists x\varphi \;\to\; \psi$$

Exercise 0.5.
Prove the *Soundness Theorem*, i.e. show that

$$T \vdash \varphi \;\Rightarrow\; T \vDash \varphi$$

❗ Take-home problem

Show that

$$\{\varphi \to \psi\} \vdash \exists x\varphi \;\to\; \exists x\psi$$
$$\{\varphi \to \psi\} \vdash \forall x\varphi \;\to\; \forall x\psi$$