

Math 557 August 29

Key Concepts

- **\mathcal{L} -structures:** Provide a realm to interpret \mathcal{L} -formulas over.
 - non-empty universe M ,
 - interpretation of all constant, function, and relation symbols over M .
- **structure isomorphism:** bijection π between the universes of two \mathcal{L} -structures that preserves interpretations of symbols. General version of the well-known idea of isomorphism of mathematical structures.
- **assignment:** function $\alpha : \mathcal{V} \rightarrow M$ that assigns all variables values in the universe of a structure. Recursively extends to terms.
- **satisfaction relation:** $\mathcal{M} \models \varphi[\alpha]$ means the \mathcal{L} -formula φ is true when interpreted in the \mathcal{L} -structure \mathcal{M} and all variables are assigned values according to assignment α . This is defined recursively over the structure of a formula.

Problems

We warm up with a quote often attributed to Lincoln.

Exercise 0.1.

> You can fool all the people some of the time, and some of the people all the time, but you cannot fool all the people all the time.

Define a suitable language and formalize this statement in first-order logic.

Next, a couple of problems to see the definition of structures and the satisfaction relation \models at work.

Exercise 0.2.

Let $\mathcal{L} = \{\square, \triangle, c\}$, where \square is a binary relation symbol, \triangle is a binary function symbol, and c is a constant symbol.

For each of the following formulas, find two \mathcal{L} -structures and (if necessary) assignments to the free variable(s) such that the formula holds in one structure but fails in the other.

$$\varphi \equiv \forall x \exists y \square \triangle xy c \quad (1)$$

$$\psi \equiv \exists x \forall y (x = c \wedge \square \triangle zc y) \quad (2)$$

Next, let's see some general properties of structures we can describe through first-order formulas.

Exercise 0.3.

Let n be a natural number and let \mathcal{L} be any language. Find formulas $\psi^{\geq}, \psi^{\leq}, \psi^=$ such that, for any \mathcal{L} -structure \mathcal{M} ,

$$\mathcal{M} \models \psi^{\geq} \iff |M| \geq n \quad (3)$$

$$\mathcal{M} \models \psi^{\leq} \iff |M| \leq n \quad (4)$$

$$\mathcal{M} \models \psi^= \iff |M| = n \quad (5)$$

$$(6)$$

i Question

Can we find a formula θ such that

$$\mathcal{M} \models \theta \iff |M| \text{ is finite?}$$

We will answer this question in a few weeks...

Exercise 0.4.

Let $\mathcal{L} = \{\cdot, e\}$ be the language of groups. Show that there is a sentence φ such that $\mathcal{M} \models \varphi$ if and only if

$$M \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

! Take-home Problems

1. Let \mathcal{L} be any finite language and let \mathcal{M} be a finite \mathcal{L} -structure. Show that there is an \mathcal{L} -sentence φ such that

$$\mathcal{N} \models \varphi \iff \mathcal{N} \cong \mathcal{M}.$$

2. Give an example of a language \mathcal{L} and an \mathcal{L} -sentence ψ such that
 - there is at least one \mathcal{L} -structure A such that $A \models \psi$,
 - for all L -structures A , if $A \models \psi$, then the universe A of A is infinite.