

## Definability of Computable Functions

We now extend the connection between arithmetical formulas and computable functions significantly. Our goal is to show that the recursive relations on  $\mathbb{N}$  are precisely those that are  $\Delta_1$ -definable. At the core of this connection is the **Definability Theorem**. For a partial function  $f$ , let  $\Gamma_f$  denote the **graph** of  $f$ , i.e., the relation

$$\Gamma_f(\vec{x}, y) : \Longleftrightarrow f(\vec{x}) = y.$$

### The Definability Theorem

#### Theorem 0.1.

A partial function  $f : \subseteq \mathbb{N} \rightarrow \mathbb{N}$  is computable if and only if its graph  $\Gamma_f$  over the natural numbers is definable by a  $\Sigma_1$ -formula, i.e., if there exists a  $\Sigma_1$ -formula  $\theta$  such that for all  $x, y \in \mathbb{N}$ :

$$\Gamma_f(x, y) \Longleftrightarrow \mathbb{N} \models \theta[x, y].$$

The direction  $(\Leftarrow)$  is relatively straightforward: Let  $\theta = \exists z \psi$  be a  $\Sigma_1$ -formula that defines the graph  $\Gamma_f$  of a partial function  $f$ , where  $\psi(x, y, z)$  is a  $\Delta_0$ -formula. As  $\Delta_0$ -formulas define primitive recursive relations, we obtain a computable function  $g$  by letting

$$g(x) = \mu z \psi(x, (z)_0, (z)_1).$$

Then  $(g(x))_0$  is the smallest  $y$  with  $\mathbb{N} \models \theta[x, y]$ , if such a  $y$  exists, and undefined otherwise, since for each  $x \in \mathbb{N}$  there is at most one  $y$  with  $\mathbb{N} \models \theta[x, y]$ . Thus  $(g(x))_0 \cong f(x)$ , and therefore  $f$  is partial computable.

For the proof of  $(\Rightarrow)$ , we call partial functions  $f$  whose graph can be defined by a  $\Sigma_1$ -formula over the natural numbers *functions with a  $\Sigma_1$ -graph*. The obvious strategy is to show that the set of these functions contains the initial functions and is closed under composition, primitive recursion, and the  $\mu$ -operator.

Most cases are straightforward (just by writing out the definitions), but when we get to closure under primitive recursion, we run into difficulties: Suppose the function  $f$  arises from functions  $g, h$  by primitive recursion:

$$\begin{aligned} f(x, 0) &\cong g(x) \\ f(x, y + 1) &\cong h(x, y, f(x, y)) \end{aligned}$$

We may assume that the graphs of  $g$  and  $h$  are definable by  $\Sigma_1$ -formulas. To describe the graph of  $f$  with this, we note that the value of  $f(x, y)$  is computed through the preceding values

$$u_0 = f(x, 0), u_1 = f(x, 1), \dots, u_y = f(x, y),$$

We somehow have to *eliminate the function  $f$*  from this sequence to obtain an *explicit* definition (in the language of arithmetic).

The idea is to express the following through a formula:

There exists a code  $u$  such that  $u$  codes a sequence of numbers  $(u_0, \dots, u_y)$  in which the  $u_{i+1}$  is obtained from  $u_i$  using a valid application of the primitive recursion scheme with respect to  $g$  and  $h$ .

The problem is that we do not know a priori how long the sequence coded by  $u$  is. We therefore cannot use the preferred coding scheme

$$\langle u_0, \dots, u_y \rangle$$

which is  $\Delta_0$ -definable. The coding scheme

$$(n_0, n_1, \dots, n_k) \quad \text{by} \quad p_0^{n_0+1} \cdot p_1^{n_1+1} \cdot \dots \cdot p_k^{n_k+1}.$$

on the other hand is uniform independent of the length of the sequence, but it uses exponentiation, which is not present in  $\text{PA}^-$  (and would have to be defined via recursion).

This is where Gödel, according to Mostowski, “had a phone call with God”.

### Gödel's Lemma

#### Lemma 0.1.

*There exists a primitive recursive function  $\beta$  such that for every  $k$  and for every finite sequence  $x_0, x_1, \dots, x_{k-1} \in \mathbb{N}$ , there exists a natural number  $c$  with*

$$\text{for all } i < k : \beta(c, i) = x_i.$$

*In fact, there exists a  $\Delta_0$ -formula  $\theta(x, y, z)$  such that*

$$\mathbb{N} \models \forall x, y \exists! z \theta(x, y, z),$$

*and the formula  $\theta(x, y, z)$  defines the function  $\beta$  over the natural numbers.*

*Proof.* Let  $x_0, x_1, \dots, x_{k-1}$  be a sequence of natural numbers.

Set  $m := b!$  where  $b := \max(k, x_0, x_1, \dots, x_{k-1})$ . Then the sequence of numbers

$$m + 1, 2m + 1, \dots, k \cdot m + 1$$

are pairwise coprime. By the theorem on simultaneous congruences (*Chinese Remainder Theorem*), there exists a natural number  $a$  with

$$a \equiv x_i \pmod{(i+1)m+1} \quad \text{for all } i < k.$$

Now we can choose  $\langle a, m \rangle$  as a code for the sequence  $x_0, x_1, \dots, x_{k-1}$ , because from this we can recover each  $x_i$  for every  $i < k$  as the remainder of the division of  $a$  by the number  $(i+1)m+1$ . If  $\text{rem}(x : y) = z$  denotes the remainder of the division of  $x$  by  $y$  (when  $y \neq 0$  and  $\text{rem}(x : 0) = 0$ ), then this is a p.r. function with a  $\Delta_0$ -definition. We obtain another p.r. function by

$$\alpha(a, m, i) = \text{rem}(a : (1+i)m+1).$$

Let  $p_1, p_2$  denote the inverse functions of the pairing function  $\langle x, y \rangle$  above, for which

$$\langle p_1(x), p_2(x) \rangle = x, \quad \text{where } p_1(x), p_2(x) \leq x,$$

these are also p.r., and we finally obtain the **Gödel beta function** as

$$\beta(c, i) = \alpha(p_1(c), p_2(c), i).$$

□

### Completing the proof of the Definability Theorem

Returning to the sequence

$$u_0 = f(x, 0), u_1 = f(x, 1), \dots, u_y = f(x, y),$$

we can now use the  $\beta$ -function to encode this by a single number  $u$ . The first value  $u_0$  is determined by the graph of  $g$ , and the further values  $u_{i+1}$  are determined from  $u_i$  according to the recursion conditions via the graph of  $h$ :

$$\forall i < y \exists r, s [\beta(u, i) = r \wedge \beta(u, i + 1) = s \wedge \Gamma_h(\vec{x}, i, r, s)]$$

This formula can be transformed (by choosing a sufficiently large  $w$ ) into

$$\exists w \forall i < y \exists r, s < w [\beta(u, i) = r \wedge \beta(u, i + 1) = s \wedge \Gamma_h(\vec{x}, i, r, s)].$$

The last value of the sequence encoded by  $u$  is the function value of  $f$  at the point  $(x, y)$ . Thus we obtain the following description of the graph of  $f$ :

$$\begin{aligned} \Gamma_f(\vec{x}, y, z) \iff \exists u, v, w (\Gamma_g(\vec{x}, v) \wedge \beta(u, 0) = v \wedge \beta(u, y) = z \wedge \\ \forall i < y \exists r, s < w [\beta(u, i) = r \wedge \beta(u, i + 1) = s \wedge \Gamma_h(\vec{x}, i, r, s)]), \end{aligned}$$

It is not hard to transform this into an equivalent  $\Sigma_1$ -formula.

### Characterization of the Arithmetic Hierarchy

If we call a set  $A \subseteq \mathbb{N}^k$  that is definable by a formula in  $\Gamma$  over the natural numbers a  $\Gamma$ -set, we obtain a new characterization of the first levels of the arithmetic hierarchy:

#### Corollary 0.1.

1. The  $\Sigma_1$ -sets are precisely the r.e. (recursively enumerable) relations  $R \subseteq \mathbb{N}^k$ .
2. The  $\Pi_1$ -sets are the complements of r.e. relations.
3. The  $\Delta_1$ -sets are precisely the recursive relations  $R \subseteq \mathbb{N}^k$ .