

## Ultraproducts

### Direct Products

Let  $(\mathcal{M}_i)_{i \in I}$  be a family of  $L$ -structures.

We define the **direct product**

$$\mathcal{M} = \prod_{i \in I} \mathcal{M}_i$$

as follows:

1. The universe is the Cartesian product  $M = \prod_{i \in I} M_i$ . If  $a$  is an element of  $M$ , we denote its  $i$ -th component (an element of  $M_i$ ) by  $a_i$  and extend this notation to vectors: if  $\vec{a}$  is a finite tuple in  $M^n$ ,  $\vec{a}_i$  denotes the  $n$ -tuple in  $M_i$  consisting of the  $M_i$ -entries of  $\vec{a}$ .
2. For each relation symbol  $R \in \mathcal{L}$ ,

$$R^{\mathcal{M}}(\vec{a}) : \iff \forall i \in I, \vec{a}_i \in R^{\mathcal{M}_i}$$

3. For each function symbol  $f \in \mathcal{L}$ ,

$$f^{\mathcal{M}}(\vec{a}) := (f^{\mathcal{M}_i}(\vec{a}_i))_{i \in I}.$$

4. For each constant  $c \in \mathcal{L}$ ,

$$c^{\mathcal{M}} = (c^{\mathcal{M}_i})_{i \in I}.$$

### Examples and Observations

- The direct product of groups is again a group (componentwise operation).
- The direct product of fields is **not** a field:

$$(1, 0) \cdot (0, 1) = (0, 0).$$

- The direct product of linear orders is only a **partial order**.

We often want to preserve properties that hold in “most” component structures. To formalize “most,” we use **filters** on  $I$ .

### Filters and Ultrafilters

A **filter**  $\mathcal{F}$  on a set  $I$  is a nonempty collection of subsets of  $I$  satisfying:

1.  $\emptyset \notin \mathcal{F}$
2. If  $A, B \in \mathcal{F}$ , then  $A \cap B \in \mathcal{F}$
3. If  $A \in \mathcal{F}$  and  $A \subseteq B \subseteq I$ , then  $B \in \mathcal{F}$

An **ultrafilter**  $\mathcal{U}$  is a maximal filter, equivalently:

For all  $A \subseteq I$ , either  $A \in \mathcal{U}$  or  $I \setminus A \in \mathcal{U}$ .

Ultrafilters interact nicely with logical operators:

- $A \notin \mathcal{U} \iff I \setminus A \in \mathcal{U}$ ,
- $A \in \mathcal{U} \wedge B \in \mathcal{U} \iff A \cap B \in \mathcal{U}$ ,
- $A \in \mathcal{U} \vee B \in \mathcal{U} \iff A \cup B \in \mathcal{U}$ .

## Examples

- A **principal filter** is of the form

$$\mathcal{F}_A = \{X \subseteq I : A \subseteq X\}$$

for some nonempty  $A \subseteq I$ .

If  $A = \{a\}$ , then  $\mathcal{F}_A$  is a **principal ultrafilter**.

- A **free** (non-principal) ultrafilter exists on every infinite set  $I$  (via Zorn's Lemma / Boolean prime ideal theorem).

## Existence of Ultrafilters

A family of sets has the **finite intersection property (FIP)** if every finite subfamily has nonempty intersection.

### Theorem 0.1.

*If a family  $\mathcal{A} \subseteq \mathcal{P}(I)$  has the FIP, then there exists an ultrafilter  $\mathcal{U}$  on  $I$  with  $\mathcal{A} \subseteq \mathcal{U}$ .*

## Reduced Products

Given a filter  $\mathcal{F}$  on  $I$  and structures  $(\mathcal{M}_i)_{i \in I}$ , define the **reduced product**

$$\mathcal{M}/\mathcal{F}$$

as follows.

Let  $M = \prod_{i \in I} M_i$ . For  $a, b \in M$ , define

$$a \sim_{\mathcal{F}} b \iff \{i \in I : a_i = b_i\} \in \mathcal{F}.$$

The universe of  $\mathcal{M}/\mathcal{F}$  is the quotient  $M/\sim_{\mathcal{F}}$ , with elements denoted  $a_{\mathcal{F}}$  (alternatively,  $a/\mathcal{F}$ ).

For symbols of  $\mathcal{L}$ :

- **Relations:**

$$R^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) : \iff \{i : \mathcal{M}_i \models R(\vec{a}_i)\} \in \mathcal{F}.$$

- **Functions:**

$$f^{\mathcal{M}/\mathcal{F}}(\vec{a}_{\mathcal{F}}) = [(f^{\mathcal{M}_i}(\vec{a}_i))_{i \in I}]_{\mathcal{F}}.$$

- **Constants:**

$$c^{\mathcal{M}/\mathcal{F}} = ((c^{\mathcal{M}_i})_{i \in I})_{\mathcal{F}}.$$

**Exercise 0.1.** Check that the above definition does not depend on the choice of representative for each equivalence class

## Ultraproducts

If  $\mathcal{U}$  is an **ultrafilter** on  $I$ , the reduced product

$$\prod_{i \in I} \mathcal{M}_i / \mathcal{U}$$

is called the **ultraproduct** of  $(\mathcal{M}_i)_{i \in I}$  modulo  $\mathcal{U}$ .

When all  $\mathcal{M}_i$  are the same structure  $\mathcal{M}$ , we get an **ultrapower**

$$\mathcal{M}^I / \mathcal{U}.$$

## Łoś' Theorem

Let  $\mathcal{M}/\mathcal{U} = \prod_{i \in I} \mathcal{M}_i/\mathcal{U}$  be an ultraproduct.

### Theorem 0.2.

For every  $\mathcal{L}$ -formula  $\varphi(x_1, \dots, x_n)$  and tuples

$$\vec{a} \in \prod_{i \in I} M_i,$$

$$\mathcal{M}/\mathcal{U} \models \varphi[\vec{a}_{\mathcal{U}}] \iff \{i \in I : \mathcal{M}_i \models \varphi[\vec{a}_i]\} \in \mathcal{U}.$$

For any  $\mathcal{L}$ -formula  $\varphi(v_0, \dots, v_{n-1})$  and a tuple  $\vec{a} \in \prod M_i$  we define the *Boolean extension* as

$$\|\varphi(\vec{a})\| := \{i \in I | \mathcal{M}_i \models \varphi[\vec{a}_i]\}$$

### Lemma 0.1.

1.  $\|\neg\varphi(\vec{a})\| = I \setminus \|\varphi(\vec{a})\|$ ,
2.  $\|(\varphi \wedge \psi)(\vec{a})\| = \|\varphi(\vec{a})\| \cap \|\psi(\vec{a})\|$ ,
3.  $\|(\varphi \vee \psi)(\vec{a})\| = \|\varphi(\vec{a})\| \cup \|\psi(\vec{a})\|$ ,
4. For all tuples  $\vec{a}$  and elements  $b$  in  $A$ :

$$\|\varphi(\vec{a}, b)\| \subseteq \|(\exists v_n \varphi)(\vec{a})\|,$$

and there exists  $b \in M$  such that

$$\|\varphi(\vec{a}, b)\| = \|(\exists v_n \varphi)(\vec{a})\|.$$

*Proof.* (1)-(3) and the first part of (4) follow directly from the definition of ultrafilters (and the definition of  $\models$ ).

For the second part of (4), we observe that for every

$$i \in \|(\exists v_n \varphi)(\vec{a})\|$$

there exists  $b_i \in A_i$  with  $\mathcal{M}_i \models \varphi[\vec{a}_i, b_i]$ . For all other  $j \in I \setminus \|(\exists v_n \varphi)(\vec{a})\|$  we choose an arbitrary  $b_j \in A_j$ . This yields a sequence  $b = (b_i)_{i \in I}$  with  $b \in M$ , for which

$$\|(\exists v_n \varphi)(\vec{a})\| \subseteq \|\varphi(\vec{a}, b)\|.$$

Together with the first part we obtain  $=$ . □

### Proof of Łoś's Theorem

We proceed by induction over the formula height. A straightforward argument shows that the interpretation of functions and symbols in  $\mathcal{M}$  extends to terms in the following way:

$$t^{\mathcal{M}/\mathcal{U}}(\vec{a}/U) = (t^{\mathcal{M}_i}(\vec{a}_i))_{i \in I}/\mathcal{U} = t^{\mathcal{M}}(\vec{a})/\mathcal{U}$$

This easily implies the statement for atomic formulas.

For  $\varphi \equiv \neg\psi$ , we have

$$\begin{aligned} \mathcal{M}/\mathcal{U} \models \neg\psi[\vec{a}_{\mathcal{U}}] &\iff \mathcal{M}/\mathcal{U} \not\models \psi[\vec{a}_{\mathcal{U}}] \\ &\iff \{i : \mathcal{M}_i \models \psi[\vec{a}_i]\} \notin \mathcal{U} \quad (\text{I.H.}) \\ &\iff \|\psi(\vec{a})\| \notin \mathcal{U} \\ &\iff \neg\|\psi(\vec{a})\| \in \mathcal{U} \quad (\mathcal{U} \text{ is an ultrafilter}) \\ &\iff \|\neg\psi(\vec{a})\| \in \mathcal{U} \quad (\text{Lemma (i)}) \\ &\iff \{i : \mathcal{M}_i \models \neg\psi[\vec{a}_i]\} \in \mathcal{U}. \end{aligned}$$

The case  $\varphi \equiv (\psi \wedge \theta)$  is similar.

Finally, assume  $\varphi \equiv \exists y\psi$ . Then

$$\begin{aligned}
\mathcal{M}/\mathcal{U} \models \exists y\psi[\vec{a}_\mathcal{U}] &\iff \exists b_\mathcal{U} \in \mathcal{M}/\mathcal{U} \text{ such that } \mathcal{M}/\mathcal{U} \models \psi[\vec{a}_\mathcal{U}, b_\mathcal{U}] \\
&\iff \exists b \in M \text{ such that } \{i : \mathcal{M}_i \models \psi[\vec{a}_i, b_i]\} \in \mathcal{U} \quad (\text{I.H.}) \\
&\iff \exists b \in M \text{ such that } \|\psi(\vec{a}, b)\| \in \mathcal{U} \\
&\iff \|\exists y\psi(\vec{a})\| \in \mathcal{U} \quad (\mathcal{U} \text{ ultrafilter, Lemma (4)}) \\
&\iff \{i : \mathcal{M}_i \models \exists y\psi[\vec{a}_i]\} \in \mathcal{U}.
\end{aligned}$$

This completes the proof.

## Applications

- **Preservation of theories:**  
If each  $\mathcal{M}_i \models T$ , then  $\prod_i \mathcal{M}_i/\mathcal{U} \models T$ .
- **Ultrapowers:**  
 $\mathcal{M} \equiv \mathcal{M}^I/\mathcal{U}$ , i.e., a structure is elementarily equivalent to any of its ultrapowers.
- **Nonstandard models:**  
For example, the ultrapower  $\mathbb{N}^\mathbb{N}/\mathcal{U}$  (with  $\mathcal{U}$  non-principal) yields a countably saturated nonstandard model of arithmetic.

## A new proof of the compactness theorem

Suppose  $T$  is an  $\mathcal{L}$ -theory for which every finite subset  $\Delta$  has a model  $\mathcal{M}_\Delta$ .

Let  $I = \{\Delta : \Delta \subseteq T \text{ finite}\}$ .

For  $\sigma \in T$ , let

$$\begin{aligned}
\hat{\sigma} &= \{\Delta \in I : \sigma \in \Delta\} \subseteq I \\
E &= \{\hat{\sigma} : \sigma \in T\} \subseteq \mathcal{P}(I)
\end{aligned}$$

$E$  has FIP (finite intersection property), since for  $\hat{S}_1, \dots, \hat{S}_n \in E$ ,

$$\{\hat{\sigma}_1, \dots, \hat{\sigma}_n\} \in \hat{\sigma}_1 \cap \dots \cap \hat{\sigma}_n.$$

By Theorem 0.1,  $E$  can be extended to an ultrafilter  $\mathcal{U}$  on  $I$ .

**Claim:**  $\mathcal{M}/\mathcal{U} = \prod_{\Delta \in I} \mathcal{M}_\Delta/\mathcal{U} \models T$ .

Let  $\sigma \in T$ . Then for  $\Delta \in I$ ,

$$\Delta \in \hat{\sigma} \Rightarrow \sigma \in \Delta \Rightarrow \mathcal{M}_\Delta \models \sigma$$

Hence

$$\hat{\sigma} \subseteq \{\Delta \in I : \mathcal{M}_\Delta \models \sigma\}$$

Since  $\hat{\sigma} \in E$  and  $E \subseteq \mathcal{U}$ , we have  $\hat{\sigma} \in \mathcal{U}$ , which implies

$$\{\Delta \in I : \mathcal{M}_\Delta \models \sigma\} \in \mathcal{U}$$

By Łoś' Theorem,  $\mathcal{M} \models \sigma$ .