Math 557 August 27

Problems

Exercise 0.1.
The set of L-terms $\mathcal{T}^{\mathcal{L}}$ is defined as the *smallest* set (with respect to $\subseteq$) containing variables and constant symbols and that is closed under function symbol application.

- Why do this set exist?
- Does a similar argument work for the set of $\mathcal{L}$-formulas?
- Find an example of a (non-empty) family of sets and a property P (applicable to the sets in the family) such that a $\subseteq$-smallest set with property P does not exist.

Exercise 0.2.
Prove the **Readbility Lemma**: For any term t, exactly one of the following holds.

1. There exists a $v \in \mathcal{V}$ such that $t \equiv v$.
2. There exists a $c \in \mathcal{C}^{\mathcal{L}}$ such that $t \equiv c$.
3. There exists a number $n \in \mathbb{N}$, a function symbol $f \in F_n^{\mathcal{L}}$, and terms $t_1, \dots, t_n$ such that $t \equiv ft_1 \dots t_n$.

Exercise 0.3.
Show that no proper initial segment of a term is a term. (*Use induction on the length of a term.*)

Deduce **unique readability**: The decomposition in (3.) above is unique.

> **!** Take-home Problem
>
> Prove **unique readability** for the set of $\mathcal{L}$-formulas.

Exercise 0.4.
The set C is defined as the smallest set of finite strings over the alphabet $\{\times, \circ\}$ such that

- (C1) $\times \in C$,
- (C2) $\circ \in C$,
- (C3) if $s, t \in C$ and the last symbol in s differs from the the first symbol in t, then $st \in C$ (where st is the concatenation of s and t).

Formulate the property of unique readability for the system (C1)-(C3) and argue that it does not hold for this system.

Exercise 0.5.
Let $\mathcal{L} = (0, <)$ be a language with one constant symbol 0 and one binary relation symbol $<$. In the following, for sake of (human) readability, we write $0 < v_1$ instead of $< 0v_1$.

Consider the expression

$$\phi \equiv (\exists v_2 \; v_1 < v_2 \wedge \exists v_1 \; v_2 < 0).$$

- Is this a correct $\mathcal{L}$-formula?
- Characterize every variable occurrence in ϕ as free or bound.