

# MATH 557 Midterm 3 Preparation

The third midterm will again have two parts:

1. Reproduce a proof (in sufficient detail) of one the theorems below we covered in class.
2. Present a proof to one of the exercises (or a closely related problem) listed on this page.

## Notations, Axioms

In the following,  $\mathcal{L}_A = \{0, 1, +, \cdot, <\}$  denotes the language of  $\text{PA}^-$ .

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The axioms of  $\text{PA}^-$  are:

- **A1:**  $(x + y) + z = x + (y + z)$
- **A2:**  $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- **A3:**  $x + y = y + x$
- **A4:**  $x \cdot y = y \cdot x$
- **A5:**  $x \cdot (y + z) = x \cdot y + x \cdot z$
- **A6:**  $x + 0 = x \wedge x \cdot 0 = 0$
- **A7:**  $x \cdot 1 = x$
- **A8:**  $\neg x < x$
- **A9:**  $x < y \wedge y < z \rightarrow x < z$
- **A10:**  $x < y \vee x = y \vee y < x$
- **A11:**  $x < y \rightarrow x + z < y + z$
- **A12:**  $0 < z \wedge x < y \rightarrow x \cdot z < y \cdot z$
- **A13:**  $x < y \rightarrow \exists z (x + z = y)$
- **A14:**  $0 < 1 \wedge \forall x (0 < x \rightarrow 1 \leq x)$
- **A15:**  $\forall x (0 \leq x)$

If we add the **induction scheme**

- **Ind:**  $\forall \vec{y} (\varphi(0, \vec{y}) \wedge \forall x (\varphi(x, \vec{y}) \rightarrow \varphi(x + 1, \vec{y})) \rightarrow \forall x \varphi(x, \vec{y}))$

we obtain (a theory equivalent to) **full PA**.

## Theorems

### Theorem 1

For every  $\Delta_0$ -formula  $\theta(\vec{v})$ , the relation

$$R(\vec{a}) : \iff \mathbb{N} \models \theta(\vec{a})$$

is primitive recursive.

### Theorem 2

Let  $\mathcal{N}, \mathcal{M}$  be  $\mathcal{L}_A$ -structures with  $\mathcal{N} \subseteq_{\text{end}} \mathcal{M}$ , and let  $\vec{a} \in N$ . Then for every  $\Delta_0$ -formula  $\varphi(\vec{v})$ ,

$$\mathcal{N} \models \varphi[\vec{a}] \iff \mathcal{M} \models \varphi[\vec{a}],$$

(This is one part of the Representability Theorem.)

### Theorem 3

If  $f : \mathbb{N} \rightarrow \mathbb{N}$  is recursive, then there exists a  $\Sigma_1$ -formula  $\theta(x, y)$  such that for all  $m, n \in \mathbb{N}$ ,

$$f(n) = m \Rightarrow \text{PA}^- \vdash \theta(\underline{n}, \underline{m})$$

### Theorem 4

Let  $T$  be a recursive set of (Gödel numbers of)  $\mathcal{L}_A$ -sentences such that:

1.  $T$  is consistent, i.e., there is no  $L$ -sentence  $\sigma$  with  $\ulcorner \sigma \urcorner \in T$  and at the same time  $\ulcorner \neg \sigma \urcorner \in T$ ,
2.  $T$  contains the deductive closure of  $\text{PA}^-$ ,  $\ulcorner (\text{PA}^-)^{\perp\perp} \urcorner \subseteq T$ .

Then  $T$  is incomplete, i.e., there exists a sentence  $\tau$  with

$$\ulcorner \tau \urcorner \notin T \text{ and } \ulcorner \neg \tau \urcorner \notin T.$$

(You can use the Representability Theorem as well as the Diagonal Lemma.)

### Theorem 5

If  $T$  is a consistent theory in the language  $\mathcal{L}_A$ , then not both the diagonal function  $d$  and the set  $\ulcorner T^{\perp\perp} \urcorner$  are representable in  $T$ .

## Problems

### Problem 1

Show that the functions

$$\text{rem}(x, y) = \text{remainder when } y \text{ is divided by } x$$

(put  $\text{rem}(0, y) = y$  to make it total) and

$$\text{qt}(x, y) = \text{quotient when } y \text{ is divided by } x$$

(put  $\text{qt}(0, y) = 0$ ) are primitive recursive.

In other words, show that the uniquely determined functions (for  $x \geq 1$ ) satisfying

$$y = \text{qt}(x, y) \cdot x + \text{rem}(x, y) \quad 0 \leq \text{rem}(x, y) < x$$

are primitive recursive.

You can use that the functions  $x+y$ ,  $x \cdot y$ ,  $\max(x, y)$ ,  $\min(x, y)$ ,  $|x-y|$ ,  $x \dot{-} y$ ,  $\text{sg}(x)$ ,  $\bar{\text{sg}}(x)$  are primitive recursive.

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### Problem 2

Show that for all  $k \in \mathbb{N}$ ,

$$\text{PA}^- \vdash \forall x (x \leq \underline{k} \rightarrow (x = \underline{0} \vee \dots \vee x = \underline{k}))$$

(Hint: use (meta-)induction on  $k$ . For the inductive step, show first that

$$\text{PA}^- \vdash \forall x, y (y > x \rightarrow y \geq x + 1)$$

### Problem 3

Let  $\mathcal{M} \models \text{PA}$  be non-standard. A *proper cut* in  $\mathcal{M}$  is a set  $I \subsetneq M$  that is an initial segment of  $M$  and closed under successor, e.g. the standard model  $\mathbb{N}$ .

1. Show that if  $\vec{a} \in M$  and  $\mathcal{M} \models \varphi(b, \vec{a})$  for all  $b \in I$ , then there is  $c > I$  in  $M$  such that  $\mathcal{M} \models \forall x \leq c \varphi(x, \vec{a})$ .
2. Show that if  $\mathcal{M} \models \text{PA}^-$  is non-standard and has the property that for any proper cut  $I$  there is  $c > I$  in  $M$  such that  $\mathcal{M} \models \forall x \leq c \varphi(x, \vec{a})$  (i.e.,  $\mathcal{M}$  satisfies the conclusion of (1.)), then  $\mathcal{M} \models \text{PA}$ .  
(Hint: For (2.), it suffices to show  $\mathcal{M}$  satisfies the induction scheme.)

### Problem 4

Show that, for all functions  $f : \mathbb{N}^k \rightarrow \mathbb{N}$  (where  $k \geq 1$ ), if  $f$  is representable in  $\text{PA}^-$  then  $f$  is computable.  
(You can argue by Church-Turing Thesis.)

### Problem 5

- (a) We say an  $\mathcal{L}_A$ -theory  $T'$  is a *finite extension* of an  $\mathcal{L}_A$ -theory  $T$  if  $T' \supseteq T$  and  $T' \setminus T$  is finite. Show that if  $T$  is decidable and  $T'$  is a finite extension of  $T$ , then  $T'$  is decidable.
- (b) An  $\mathcal{L}_A$ -structure  $\mathcal{A}$  is *strongly undecidable* if every  $\mathcal{L}_A$ -theory  $T$  with  $\mathcal{A} \models T$  is undecidable. Show that the standard model  $\mathbb{N}$  is strongly undecidable.