

# Math 557 Sep 19

## Henkin Theories

### Key Concepts

- An  $\mathcal{L}$ -theory  $T$  is a **Henkin theory** if for every sentence of the form  $\exists x\psi$  there exists a constant  $\mathcal{L}$ -term  $t$  such that

$$T \vdash_{\mathcal{L}} \exists x\psi \rightarrow \psi_{t/x}$$

- **Questions:**
  - What are Henkin theories needed for?
  - Do they exist?
  - Can we extend a given consistent theory to a Henkin theory that is still consistent?

### Problems

#### Exercise 0.1.

As before, let  $\mathcal{A}_T$  be the term model of  $T$ . Pinpoint exactly where we might run into difficulties when we are trying to prove

$$\mathcal{A}_T \models \exists x\psi \iff T \vdash_{\mathcal{L}} \exists x\psi$$

#### Exercise 0.2 (Interlude).

What if in  $\exists x\psi$  the variable  $x$  is not free? Does that make a difference regarding being a Henkin theory?

#### Exercise 0.3.

Can you find a couple of examples of Henkin theories?

The basic construction step for extending an  $\mathcal{L}$ -theory  $T$  to a Henkin theory is for every sentence of the form  $\sigma \equiv \exists x\psi$ ,

- add a new constant symbol  $c_\sigma$  to  $\mathcal{L}$
- add the formula  $\exists x\psi \rightarrow \psi_{c_\sigma/x}$  to  $T$ .

#### Exercise 0.4.

Why could the addition of a constant theoretically lead to  $T$  being inconsistent, even though  $T$  itself remains unchanged?

And why does  $T$  actually remain consistent?

The Henkin extension  $T_H$  of  $T$  is obtained by an iterative process:

Single iteration step:

- $\mathcal{L}' = \mathcal{L} \cup \{c_\sigma : \sigma \text{ } \mathcal{L}\text{-sentence of the form } \exists x\psi\}$
- $\Gamma = \{\exists x\psi(x) \rightarrow \psi_{c_\sigma/x} : \sigma \text{ } \mathcal{L}\text{-sentence of the form } \exists x\psi\}$
- $T' = T \cup \Gamma$  (an  $\mathcal{L}'$ -theory)

#### Exercise 0.5.

Why is  $T'$  above not necessarily a Henkin theory?

Iteration process:

- $\mathcal{L}_0 = \mathcal{L}$ ,  $T_0 = T$

- $\mathcal{L}_{n+1} = \mathcal{L}'_n, T_{n+1} = T'_n$
- $\mathcal{L}_H = \bigcup_{n \in \mathbb{N}} \mathcal{L}_n, T_H = \bigcup_{n \in \mathbb{N}} T_n$

**Exercise 0.6.**

Why is  $T_H$  a Henkin theory?

$T_H$  is no longer a purely symbolic extension of  $T$  (in the sense that we simply extend the language), since we add new sentences to  $T$  (the sentences in  $\Gamma$  for each iteration step).

**! Take-home Problem**

Show that for every  $\mathcal{L}$ -formula  $\varphi$ ,

$$T_H \vdash_{\mathcal{L}_H} \varphi \iff T \vdash_{\mathcal{L}} \varphi$$