

Consistency and Undecidability

The Second Incompleteness Theorem

As we saw previously, for a recursively axiomatizable theory T , the relation

$$\text{Prf}(x, y) : \iff x = \ulcorner \psi \urcorner, y = \langle \ulcorner \varphi_1 \urcorner, \dots, \ulcorner \varphi_n \urcorner \rangle, \varphi_1, \dots, \varphi_n \text{ is a } T\text{-proof of } \psi$$

is recursive. It follows that for such T , the relation

$$\begin{aligned} \text{prov}_T(v) &:\Leftrightarrow "v \text{ is the Gödel number of a formula } \varphi \text{ and } \varphi \text{ is provable in } T" \\ &\Leftrightarrow \exists y \text{ Prf}(x, y) \end{aligned}$$

is r.e. and thus definable by a Σ_1 -formula $\theta(v)$.

In PA^1 one can verify the following **derivability conditions**:

- (D1) $T \vdash \sigma \Rightarrow T \vdash \theta(\ulcorner \sigma \urcorner)$
- (D2) $T \vdash \theta(\ulcorner \sigma \rightarrow \tau \urcorner) \rightarrow (\theta(\ulcorner \sigma \urcorner) \rightarrow \theta(\ulcorner \tau \urcorner))$
- (D3) $T \vdash \theta(\ulcorner \sigma \urcorner) \rightarrow \theta(\ulcorner \theta(\ulcorner \sigma \urcorner) \urcorner)$

Here σ, τ are arbitrary sentences of the language \mathcal{L}_A , and the Gödel numbers n should be replaced in these formulas by the corresponding terms \underline{n} .

Using such a proof predicate, one can also formalize the **consistency** of a theory, for example:

$$\text{Con}_T : \Leftrightarrow \neg \theta(\ulcorner 0 = 1 \urcorner).$$

Theorem 0.1 (Second Gödel Incompleteness Theorem).

Let T be a theory in the language L for which there exists a formal proof predicate θ as defined above (e.g., $T = \text{PA}$). Then:

$$T \text{ consistent} \implies T \not\vdash \text{Con}_T.$$

Sketch. By the Diagonalization Lemma, there exists a sentence τ such that

$$T \vdash \tau \Leftrightarrow \neg \theta(\ulcorner \tau \urcorner) \tag{1}$$

If $T \vdash \tau$, it follows from (D1) that

$$T \vdash \theta(\ulcorner \tau \urcorner)$$

By (1), this implies $T \vdash \neg \tau$, hence T is inconsistent. This yields

¹ PA^- is not strong enough for this

$$T \text{ consistent} \Rightarrow T \not\vdash \tau. \quad (2)$$

One can show that this proof can be formalized and represented in T using the proof predicate θ , thus

$$T \vdash \text{Con}_T \rightarrow \neg\theta(\underline{\tau}). \quad (3)$$

If T is consistent, by (2), $T \not\vdash \tau$. Using (1), this implies $T \not\vdash \neg\theta(\underline{\tau})$. Thus, by (3), $T \not\vdash \text{Con}_T$. \square

Truth is Not Definable

While the usual syntactic concepts formed for a formal language \mathcal{L} can be defined in the language of number theory and their essential properties can be proven in PA , this is not possible for the semantic notion of truth:

Theorem 0.2 (Tarski's Undefinability Theorem).

Let $\mathcal{M} \models \text{PA}^-$. Then there is no \mathcal{L}_A -formula $\theta(v)$ such that for all natural numbers n

$$\mathcal{M} \models \theta(\underline{n}) \iff n = {}^\frown \sigma^\frown \text{ for an sentence } \sigma \text{ with } \mathcal{M} \models \sigma.$$

Proof. If there were such a truth definition θ , then by the Diagonal Lemma we could find a sentence G such that

$$\text{PA}^- \vdash G \leftrightarrow \neg\theta(\underline{G}),$$

in particular,

$$\mathcal{M} \models G \iff \mathcal{M} \models \neg\theta(\underline{G}).$$

On the other hand, for the truth definition we would need:

$$\mathcal{M} \models G \iff \mathcal{M} \models \theta(\underline{G}),$$

which yields a contradiction. \square

Undecidability

Theorem 0.3.

If T is a consistent theory in the language \mathcal{L}_A , then not both the diagonal function d and the set ${}^\frown T^{\perp\perp}$ are representable in T .

Proof. We assume that d is represented by a formula $\delta(x, y)$ and the set ${}^\frown T^{\perp\perp}$ is represented by a formula $\varphi_T(v_0)$ in T , and we choose $\theta = \neg\varphi_T$, so that from the Diagonal Lemma we obtain the existence of a formula G such that

$$T \vdash G \leftrightarrow \neg\varphi_T(\underline{G}). \quad (4)$$

Case 1: $T \vdash G$, thus ${}^\frown G^\frown \in {}^\frown T^{\perp\perp}$. Then by representability, $T \vdash \varphi_T(\underline{G})$ and with (4) also $T \vdash \neg G$, i.e., T would be inconsistent.

Case 2: $T \not\vdash G$, thus ${}^\frown G^\frown \notin {}^\frown T^{\perp\perp}$. Then again by representability, $T \vdash \neg\varphi_T(\underline{G})$, and with (4) $T \vdash G$, contradiction! \square

Corollary 0.1 (Church).

If T is a consistent theory in the language \mathcal{L}_A which extends PA^- , then T is undecidable.

Proof. If T extends PA^- , every recursive function, including d , is representable in T . Hence, by Theorem 0.3, ${}^\frown T^{\perp\perp}$ is not representable. It follows that ${}^\frown T^{\perp\perp}$ is not recursive (since all recursive sets are representable in any consistent extension of PA^-). \square

Corollary 0.2 (Church).

The set

$$\text{VAL} := \{\vdash \sigma : \sigma \text{ } \mathcal{L}_A\text{-sentence with } \vdash \sigma\}$$

(i.e. the set of all sentences that are validities) is not recursive.

Exercise 0.1. We say an \mathcal{L}_A -theory T' is a *finite extension* of an \mathcal{L}_A -theory T if $T' \supseteq T$ and $T' \setminus T$ is finite. Show that if T is decidable and T' is a finite extension of T , then T' is decidable.

Corollary 0.2. Since PA^- is finite, it is a finite extension of the empty theory $T = \emptyset$. Since $\text{VAL} = \emptyset^+$, by Exercise 0.1, if VAL were recursive so would be $(\text{PA}^-)^+$, contradicting Corollary 0.1. \square