

## The First Incompleteness Theorem

**Theorem 0.1** (Gödel-Rosser Theorem).

Let  $T$  be a recursive set of (Gödel numbers of)  $\mathcal{L}$ -sentences such that:

1.  $T$  is consistent, i.e., there is no  $L$ -sentence  $\sigma$  with  $\ulcorner \sigma \urcorner \in T$  and at the same time  $\ulcorner \neg \sigma \urcorner \in T$ ,
2.  $T$  contains all  $\Sigma_1$  and  $\Pi_1$ -sentences that hold in  $\text{PA}^-$ :

$$\{\ulcorner \sigma \urcorner : \sigma \in \Sigma_1 \cup \Pi_1, \sigma \text{ sentence}, \text{PA}^- \vdash \sigma\} \subseteq T.$$

Then  $T$  is  $\Pi_1$ -incomplete, i.e., there exists a  $\Pi_1$ -sentence  $\tau$  with

$$\ulcorner \tau \urcorner \notin T \text{ and } \ulcorner \neg \tau \urcorner \notin T.$$

*Proof.* Since  $T$  is recursive, there exists a  $\Sigma_1$ -formula  $\theta(v)$  such that for all  $n \in \mathbb{N}$

$$\begin{aligned} n \in T &\implies \text{PA}^- \vdash \theta(\underline{n}), \\ n \notin T &\implies \text{PA}^- \vdash \neg \theta(\underline{n}). \end{aligned}$$

By the Diagonal Lemma, there exists a  $\Pi_1$ -sentence  $\tau$  with

$$\text{PA}^- \vdash \tau \leftrightarrow \neg \theta(\ulcorner \tau \urcorner).$$

From assumption (2) it follows that

$$\ulcorner \tau \urcorner \in T \implies \text{PA}^- \vdash \theta(\ulcorner \tau \urcorner) \implies \text{PA}^- \vdash \neg \tau \implies \ulcorner \neg \tau \urcorner \in T$$

and likewise, using (1),

$$\ulcorner \neg \tau \urcorner \in T \implies \ulcorner \tau \urcorner \notin T \implies \text{PA}^- \vdash \neg \theta(\ulcorner \tau \urcorner) \implies \text{PA}^- \vdash \tau \implies \ulcorner \tau \urcorner \in T,$$

so that neither  $\ulcorner \tau \urcorner$  nor  $\ulcorner \neg \tau \urcorner$  can be in  $T$ . □

**Theorem 0.2** (First Gödel Incompleteness Theorem).

Let  $T$  be a consistent<sup>1</sup> and recursively axiomatizable theory in language  $L$  which extends the theory  $\text{PA}^-$ . Then  $T$  is incomplete. In particular, there exists a  $\Pi_1$ -sentence  $\tau$  with

$$T \not\vdash \tau \text{ and } T \not\vdash \neg \tau.$$

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<sup>1</sup>The original version of Gödel's theorem had the stronger assumption of  $\omega$ -consistency of  $T$  (i.e., for all  $L$ -formulas  $\theta(v)$  with  $T \vdash \theta(\underline{n})$  for all  $n \in \mathbb{N}$ , the theory  $T \cup \{\forall v \theta(v)\}$  is consistent). Rosser proved this stronger version in 1936.

*Proof.* We assume that  $T$  is recursively axiomatizable as well as complete (which implies that it is consistent, by definition), and derive a contradiction. As we proved in a previous lecture, the assumption implies that the set

$$S := \{ \ulcorner \sigma \urcorner : \sigma \text{ sentence, } T \vdash \sigma \} = \ulcorner (T^+) \urcorner$$

is recursive. Thus the hypothesis of the Gödel-Rosser Theorem is met, which implies  $S$  is  $\Pi_1$ -incomplete—a contradiction!  $\square$

Every consistent extension  $T$  of  $\text{PA}^-$  has, by Lindenbaum's Lemma, a complete and consistent extension, none of which, by the above theorem, can be recursive.

## The Paris-Harrington Theorem

The use of the Diagonal Lemma yields rather “artificial” witnesses of incompleteness. Are there “natural” mathematical theorems that PA cannot decide? In 1982, Paris and Harrington<sup>2</sup> found an example, based on *Ramsey theory*.

### Pigeonhole Principles

In its simplest form, the **pigeonhole principle** states:

If  $n$  elements are distributed into  $m < n$  many pigeonholes, then one of the pigeonholes must contain at least 2 elements.

The generalization to infinite sets is:

If an infinite set is partitioned into finitely many sets, then at least one of these sets must be infinite:

$$A = A_0 \dot{\cup} \dots \dot{\cup} A_k \text{ infinite} \Rightarrow \exists i \leq k (A_i \text{ infinite}).$$

For further generalization, we set

$$[A]^n := \{x \subseteq A \mid |x| = n\} \text{ the set of } n\text{-element subsets of } A.$$

A partition of the  $n$ -element subsets of  $A$  into  $k$  parts can also be represented by a function

$$f : [A]^n \rightarrow k$$

(where  $A_i = \{x \in A \mid f(x) = i\}$  are then the partition sets), and such an  $f$  is more intuitively called a **coloring** (of  $[A]^n$ ) with  $k$  colors. A subset  $X \subseteq A$  with  $[X]^n \subseteq A_i$  for some  $i < k$  (whose  $n$ -tuples are thus monochromatic) is called **homogeneous** for the partition  $f$ .

**Theorem 0.3** (Ramsey's Theorem).

*For every coloring  $f : [\mathbb{N}]^n \rightarrow k$  of the  $n$ -element subsets of natural numbers with  $k$  colors, there exists an infinite subset  $X \subseteq \mathbb{N}$  such that all  $n$ -element subsets of  $X$  have the same color:*

$$f : [\mathbb{N}]^n \rightarrow k \Rightarrow \exists X \subseteq \mathbb{N} (X \text{ infinite} \wedge \forall u, v \in [X]^n f(u) = f(v)).$$

This states that every coloring of the  $n$ -element subsets of natural numbers with finitely many colors has an infinite homogeneous subset. This theorem is provable in set theory with a weak form of the axiom of choice (see, e.g., D. Marker: *Model Theory: An Introduction*, § 5.1).

**Definition 0.1** (Relatively Large Sets).

A (finite) set  $H$  of ordinal numbers is called **relatively large** if and only if

$$\text{card}(H) \geq \min(H).$$

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<sup>2</sup>A *mathematical Incompleteness in Peano Arithmetic*. In: Handbook of Mathematical Logic, ed. Barwise, Elsevier 1982

If  $n, k \in \mathbb{N}$ ,  $\alpha, \gamma \in \text{On}$ , then let

$$\gamma \longrightarrow (\alpha)_k^n \quad \text{resp.} \quad \gamma \xrightarrow{\min} (\alpha)_k^n$$

mean that for every coloring  $f$  of the  $n$ -element subsets of  $\gamma$  with  $k$  colors, there exists a (relatively large) subset  $H \subseteq \gamma$  of order type  $\alpha$  that is homogeneous for  $f$ .

Theorem 0.3 above thus states:  $\forall n, k, \omega \rightarrow (\omega)_k^n$ . From this follows the **finitary Ramsey theorem** (using a compactness argument):

$$\forall m, n, k \exists r \ r \rightarrow (m)_k^n.$$

This is a theorem about natural numbers and can be expressed in the language of Peano arithmetic and proven in PA. However, an (apparently) slight strengthening cannot:

**Theorem 0.4** (Paris-Harrington Theorem).

*The statement*

$$\forall m, n, k \exists r \ r \xrightarrow{\min} (m)_k^n$$

*is provable in set theory, but not in PA (provided PA is consistent). The existence of these numbers  $r$  can no longer be shown generally in PA; in fact, they lead to very large numbers: If*

$$\sigma(n) = \text{the smallest } r \text{ with } r \xrightarrow{\min} (n+1)_n^n,$$

*then the function  $\sigma$  eventually dominates every recursive function  $f$  (i.e., for every recursive function  $f$  there exists a  $p$  such that  $f(n) < \sigma(n)$  for all  $n \geq p$ ).*