MATH 557 Oct 17

Fraïssé Theory of Random Graphs

Key facts from Fräissé theory

Simple graphs

We consider the theory of *undirected graphs without self-loops*, also called **simple graphs**. Let $\mathcal{L}_G = \{E\}$ be the language of graphs with one binary relation symbol E.

Simple graphs are formalized by the axioms

- $\forall x \neg E(x, x)$
- $\forall x, y\, (E(x, y) \to E(y, x))$

We denote the theory of simple graphs by T_G

Exercise 0.1. Verify that

$$\overline{K} = \{\mathcal{G} : \mathcal{G} \models T_G, \mathcal{G} \text{ finite}\}$$

is an amalgamation class with countably many isomorphism types.

It follows that $\overline{K}$ has a Fraïssé limit. What does this limit look like?

Random graphs

We now add the following **extension axioms**:

$$\sigma_{n,m} \equiv \forall x_1, \dots, x_n \forall y_1, \dots, y_m \, \left(\bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} x_i \neq y_j \; \to \; \exists z \, (\bigwedge_{i=1}^{n} (z \neq x_i \wedge E(z, x_i)) \wedge \bigwedge_{j=1}^{m} (z \neq y_j \wedge \neg E(z, y_j))) \right)$$

Let

$$T_{RG} = T_G \cup \{\exists x, y\, (x \neq y)\} \cup \{\sigma_{n,m} : n, m \geq 1\}$$

Exercise 0.2. Show that every countable model of T_{RG} is homogeneous.

Exercise 0.3. Show that if $\mathcal{R}$ is a model of T_{RG}, age$(\mathcal{R}) = \overline{K}$.

It follows from Fraïssé's Theorem that T_{RG} has a countable model $\mathcal{R}$, and that this model is unique up to isomorphism.

Exercise 0.4. Show that T_{RG} is complete.

$\mathcal{R}$ is called the **random graph** and T_{RG} the *theory of the random graph*. Why? Let G_N be the set of all simple graphs on the vertex set $\{1, \dots, N\}$. Put a probability distribution on G_N by giving every graph the same probability. (Alternatively, we could independently assign edges to any vertex pair with probability $1/2$. This is called the **Erdös-Renyi model**.) Given an $\mathcal{L}_G$-sentence σ, let $\mathbb{P}_N(\sigma)$ be the probability that σ holds for a random graph in G_N, i.e.

$$\mathbb{P}_N(\sigma) = \frac{|\{\mathcal{G} \in G_N : \mathcal{G} \models \sigma\}|}{|G_N|}.$$

It turns out, for large N, graphs will satisfy the extension axioms with high probability.

Exercise 0.5. Show that for any $n, m \geq 1$,

$$\lim_{N\to\infty} \mathbb{P}_N(\sigma_{n,m}) = 1.$$

This justifies referring to $\mathcal{R}$ as *the random graph*, since it can be seen as the "limit" of finite random graphs.

Further explorations

0-1 law for graphs

Theorem 0.1. *For any $\mathcal{L}_G$-sentence σ,*

$$\text{either } \lim_{N\to\infty} \mathbb{P}_N(\sigma) = 0 \text{ or } \lim_{N\to\infty} \mathbb{P}_N(\sigma_{n,m}) = 1.$$

It follows that T_{RG} axiomatizes the *almost sure theory$ of graphs

$$\{\sigma : \lim_{N\to\infty} \mathbb{P}_N(\sigma) = 1\}$$

(and we already know this theory is complete).

To prove the theorem, use that T_{RG} is complete together with Exercise 0.5. Give it a try!

Other Fraïssé classes

There are other classes of structures that are amalgamation classes, for example:

- finite fields of characteristic p
- finite groups
- (non-trivial) finite Boolean algebras

What could be the Fraïssé limit in each case?