

# Math 557 Sep 3

## Validities

### Key Concepts

- **Validity:** An  $\mathcal{L}$ -formula  $\psi$  that is valid in *any*  $\mathcal{L}$ -structure under *any* assignment  $\alpha$ , i.e.

$$\mathcal{M} \models \psi[\alpha] \quad \text{for any } \mathcal{M}, \alpha.$$

- **Sources of validities:**

- *Propositional tautologies* – e.g. formulas of the form  $\psi \rightarrow \psi$  or  $\psi \vee \neg\psi$
- *Equality* – the way  $=$  is interpreted implies formulas like  $\forall x \, x = x$  are validities.
- *Quantifiers*

- **Fundamental questions:**

- Can we characterize the set of validities *syntactically*?
- In particular, is there an algorithm that, on input  $\varphi$ , determines whether  $\varphi$  is a validity or not?

### Problems

#### Exercise 0.1 (Warmup).

Determine whether the following formulas are validities.

- $((\varphi_1 \rightarrow (\varphi_2 \rightarrow \varphi_3)) \rightarrow ((\varphi_1 \rightarrow \varphi_2) \rightarrow (\varphi_1 \rightarrow \varphi_3)))$
- $\exists x(\varphi \wedge \psi) \leftrightarrow (\exists x\varphi \wedge \exists x\psi)$

#### Exercise 0.2.

Let  $g: \{0, 1\}^n \rightarrow \{0, 1\}$  be a Boolean function. Show that there exists a propositional formula  $F(p_1, \dots, p_n)$  such that, for any assignment  $\delta$  to the variables  $p_1, \dots, p_n$ ,

$$\delta^*(F) = g(\delta(p_1), \dots, \delta(p_n)).$$

#### Exercise 0.3.

Formally verify that every propositional tautology is a validity.

#### Exercise 0.4.

Verify, using the definition of the satisfaction relation  $\models$ , that

$$\forall x, y (x = y \rightarrow f(x) = f(y)) \quad (f \text{ unary function symbol})$$

and

$$\forall x (\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x \psi) \quad (x \text{ not free in } \varphi)$$

are validities.