Math 557 Oct 31

Problems on PA and PA^-

Problem 1

In PA, we can define $1 := S(0)$.
Show that

$$\mathsf{PA} \vdash \forall x\,(x \cdot 1 = x)$$

Problem 2: Non-standard models of PA^-

The set $\mathbb{Z}[X]$ of polynomials in one variable X with integer coefficients is a commutative ring with the usual operations. One can order this ring by setting for a polynomial $p = a_n X^n + \dots a_1 X + a_0$ with leading coefficient $a_n \neq 0$:

$$a_n X^n + \dots a_1 X + a_0 > 0 :\iff a_n > 0$$

and thus ordering polynomials $p, q \in \mathbb{Z}[X]$ by $p < q :\iff q - p > 0$.

1. Verify that the subset $\mathbb{Z}[X]^+$ of polynomials $p \in \mathbb{Z}[X]$ with $p \geq 0$ is a model of PA^-.
2. What does the natural number n correspond to in $\mathbb{Z}[X]^+$? In other words, what is the interpretation of the constant term $\underline{n}$?
3. Identify an element of $\mathbb{Z}[X]^+$ that is larger than any "natural number" and thus "infinitely large".
4. Is $\mathbb{Z}[X]^+$ a model of PA?

Problem 3: Overspill

Let $\mathcal{M} \models \mathsf{PA}$ be non-standard. A **proper cut** in $\mathcal{M}$ is a set $I \subsetneq M$ that is an initial segment of M and closed under successor, e.g. the standard model $\mathbb{N}$.
Show that if $\vec{a} \in M$ and $\mathcal{M} \models \varphi(b, \vec{a})$ for all $b \in I$, then there is $c > I$ in M such that $\mathcal{M} \models \forall x \leq c \varphi(x, \vec{a})$.

Take-Home 1

Show that if $\mathcal{M} \models \mathsf{PA}^-$, is non-standard, and satisfies the conclusion of the previous problem, then $\mathcal{M} \models \mathsf{PA}$.

Arithmetical Formulas

Bounded Quantifiers

For terms t and formulas φ in the language of PA^-, we write

$$\begin{aligned} \exists x < t\ \varphi \text{ for } &\quad \exists x(x < t \wedge \varphi) \\ \forall x < t\ \varphi \text{ for } &\quad \forall x(x < t \to \varphi) \end{aligned}$$

and call $\exists x < t$ and $\forall x < t$ **bounded quantifiers**.

Arithmetical Hierarchy

Definition 1

- φ is a Δ_0**-formula**: $:\iff$ φ contains at most bounded quantifiers,
- φ is a Σ_1**-formula**: $:\iff$ $\varphi = \exists\vec{x}\ \psi$ for a Δ_0-formula ψ,
- φ is a Π_1**-formula**: $:\iff$ $\varphi = \forall\vec{x}\ \psi$ for a Δ_0-formula ψ.

This is the beginning of the **arithmetical hierarchy**. Setting

$$\Sigma_0 = \Pi_0 = \Delta_0,$$

we can continue:

- φ is a Σ_{n+1}**-formula** $\iff$ $\varphi = \exists\vec{x}\ \psi$ for a Π_n-formula ψ,
- φ is a Π_{n+1}**-formula** $\iff$ $\varphi = \forall\vec{x}\ \psi$ for a Σ_n-formula ψ.

Thus, a Σ_3-formula has the form $\exists\vec{x}\ \forall\vec{y}\ \exists\vec{z}\ \psi$, where ψ contains at most bounded quantifiers. This means that bounded quantifiers are not counted; Σ or Π indicates whether the formula begins with a (finite) sequence of $\exists$-quantifiers or $\forall$-quantifiers respectively, and the index counts the quantifier blocks. So it depends less on the number of quantifiers than on the number of quantifier alternations.

In this classification, we do not distinguish between logically equivalent formulas, so that every Π_n-formula for $n < m$ is also a Σ_m- and Π_m-formula (by simply prefixing the formula with additional quantifiers over variables that do not occur). Thus we can also define the formula classes

$$\Delta_n = \Sigma_n \cap \Pi_n.$$

This gives us the following picture of the **arithmetical hierarchy**:

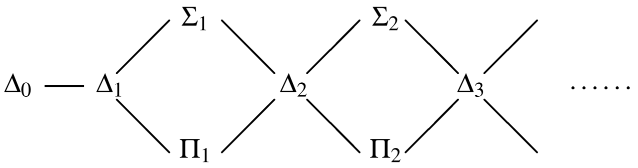

Figure 1: Arithmetical Hierarchy

Many fundamental properties of natural numbers can be expressed using Δ_0-formulas, e.g.:

$$x \text{ is irreducible } \iff 1 < x \wedge \forall u < x\ \forall v < x\ \neg(u \cdot v = x).$$

Computability of arithmetical predicates

We will show that the recursive relations coincide with the sets that can be defined by Δ_1-formulas, and that the graph of a recursive function can be defined by a Σ_1-formula. We start by showing that Δ_0-definable functions are primitive recursive.

Lemma 2

For every Δ_0-formula $\theta(\vec{v})$, the relation

$$R(\vec{a}) :\iff \mathbb{N} \vDash \theta(\vec{a})$$

is primitive recursive.

Proof. We show by induction on the height of θ that the associated characteristic function

$$c_\theta(\vec{x}) = \begin{cases} 1 & \text{if } \mathbb{N} \vDash \theta(\vec{x}) \\ 0 & \text{otherwise} \end{cases}$$

is primitive recursive.

First, the functions $x+1, x+y, x\cdot y$ are p.r., and thus every term in $\mathbb{N}$ defines a primitive recursive function. Since the functions $\mathrm{eq}(x,y) = \overline{\mathrm{sg}}(|x-y|)$ and $\mathrm{sg}(y \dot{-} x)$ are primitive recursive and the primitive recursive functions are closed under composition, the claim holds for the atomic formulas $t = s, t < s$.

For the case of propositional operations, use

$$\begin{aligned} c_{\neg\theta}(\vec{x}) &= 1 \dot{-} c_\theta(\vec{x}) \\ c_{\theta\wedge\psi}(\vec{x}) &= c_\theta(\vec{x}) \cdot c_\psi(\vec{x}) \\ c_{\theta\vee\psi}(\vec{x}) &= \min(c_\theta(\vec{x}), c_\psi(\vec{x})) \end{aligned}$$

Finally, if ψ is a Δ_0-formula, t is a term and $\theta(\vec{x}) = \forall y < t(\vec{x})\ \psi(\vec{x}, y)$, then the claim follows from

$$c_\theta(\vec{x}) = \mathrm{eq}(t(\vec{x}), (\mu y \leq t(\vec{x})(c_\psi(\vec{x}, y) = 0)).$$

One argues similarly in the case of the formula $\exists y < t(\vec{x})\ \psi(\vec{x}, y)$ (or reduces this case to the earlier one using negation). □

Remark: The converse of the above lemma does not hold: there are primitive recursive sets that cannot be defined by any Δ_0-formula in the natural numbers.

More Exercises

Problem 4

Show that the following relations and functions are primitive recursive.

x divides y

$\mathrm{rem}(x, y)$ (remainder when y is divided by x)

x is prime

$n \mapsto p_n$, where p_n is the nth prime

Take-Home 2

Show that the *Euler totient function*

$$\phi(x) = \text{ \# of } m \leq x \text{ relatively prime to } x$$

is primitive recursive.

Take-Home 3

Show that the set of *generalized Mersenne primes*,

$$\left\{ N \text{ prime} : N = \frac{p^n - 1}{p - 1} \text{ for some prime } p \text{ and } n \geq 2 \right\}$$

is primitive recursive.