

# Math 557 Sep 10

## Logical Implication and Proof

### Key Concepts

- **Logical consequence:**
  - This is the semantical implication we are often working with in mathematical practice. We say  $T$  **logically implies**  $\varphi$ ,  $T \models \varphi$ , if for structure  $\mathcal{M}$ ,  $\mathcal{M} \models T$  implies  $\mathcal{M} \models \varphi$ .
- **Formal proof:**
  - $T \vdash \varphi$  means there is a formal (i.e. *syntactical*) derivation of  $\varphi$  from  $T$  using the formulas of  $T$ , the three kinds of *logical axioms* (propositional tautologies, equality and quantifier axioms), and the *inference rules* Modus Ponens and Generalization.

### Problems

**Exercise 0.1** (Warmup - Logical Implication).

Let  $T$  be an  $\mathcal{L}$ -theory. We say a theory  $T'$  is an **axiomatization** of  $T$  if for any  $\mathcal{L}$ -structure  $\mathcal{M}$ ,

$$\mathcal{M} \models T \iff \mathcal{M} \models T'$$

Show that for any axiomatization  $T'$  of  $T$ , for any  $\mathcal{L}$ -sentence  $\sigma$ ,

$$T \models \sigma \iff T' \models \sigma$$

**Exercise 0.2** (Warmup 2).

Recall that a *model* of a theory  $T$  is a structure  $\mathcal{M}$  such that for any sentence  $\sigma \in T$ ,  $\mathcal{M} \models \sigma$ . In this case we write  $\mathcal{M} \models T$ .

Argue that if  $T$  does not have a model, every sentence is a logical implication of  $T$ .

**Exercise 0.3** (Formal notion of proof – Warmup).

Verify that

$$\{\varphi, \neg\psi\} \vdash \neg(\varphi \rightarrow \psi)$$

**Exercise 0.4.**

Argue (semantically) that if  $x$  is not free in  $\psi$ ,

$$\{\varphi \rightarrow \psi\} \models \exists x\varphi \rightarrow \psi$$

Then prove this *syntactically*, i.e. show (under the same assumption) that

$$\{\varphi \rightarrow \psi\} \vdash \exists x\varphi \rightarrow \psi$$

**Exercise 0.5.**

Prove the *Soundness Theorem*, i.e. show that

$$T \vdash \varphi \Rightarrow T \models \varphi$$

**! Take-home problem**

Show that

$$\{\varphi \rightarrow \psi\} \vdash \exists x\varphi \rightarrow \exists x\psi$$

$$\{\varphi \rightarrow \psi\} \vdash \forall x\varphi \rightarrow \forall x\psi$$