Math 557 Nov 12

Representability

We have established a closed connection between computability and definability over $\mathbb{N}$, but how much of that can PA^- actually prove? We need to make sure it can *represent* sufficiently simple (i.e., computable) functions and sets faithfully.

Definition 0.1. Let T be a theory in the language of arithmetic L that extends PA^-. A (total) function $f : \mathbb{N}^k \to \mathbb{N}$ is **representable** in T iff there exists an L-formula $\theta(x_1, \dots, x_k, y)$ such that for all $n_1, \dots, n_k, m \in \mathbb{N}$:

- **(a)** $T \vdash \exists! y\, \theta(\underline{n_1}, \dots, \underline{n_k}, y)$, and
- **(b)** $f(n_1, \dots, n_k) = m \Rightarrow T \vdash \theta(\underline{n_1}, \dots, \underline{n_k}, \underline{m})$.

Similarly, a set $S \subseteq \mathbb{N}^k$ is **representable** in the theory T iff there exists an L-formula $\theta(x_1, \dots, x_k)$ such that for all $n_1, \dots, n_k \in \mathbb{N}$:

- **(c)** $(n_1, \dots, n_k) \in S \Rightarrow T \vdash \theta(\underline{n_1}, \dots, \underline{n_k})$, and
- **(d)** $(n_1, \dots, n_k) \notin S \Rightarrow T \vdash \neg\theta(\underline{n_1}, \dots, \underline{n_k})$.

If the function f (or the set S) is representable by a Σ_1-formula, then f (or S, respectively) is called **Σ_1-representable**.

Note that by definition, representability is preserved when we pass to a theory $T' \supseteq T$.

Theorem 0.1 (Represetation Theorem)**.**

(i) Every recursive function $f : \mathbb{N}^k \to \mathbb{N}$ is Σ_1-representable in PA^-.

(ii) Every recursive set $S \subseteq \mathbb{N}^k$ is Σ_1-representable in PA^-.

Proof. **(i)** Let $f : \mathbb{N}^k \to \mathbb{N}$ be a recursive function, so its graph Γ_f is definable over the natural numbers by a Σ_1-formula $\exists \vec{z}\, \varphi(\vec{x}, y, \vec{z})$, where φ has only bounded quantifiers. Since every formula of the form $\exists \vec{z}\, \varphi$ is equivalent in PA^- to $\exists u\, \exists \vec{z}\, (\vec{z} < u \wedge \varphi)$, we may assume that $\vec{z}$ is just a single variable z. We now form the Δ_0-formula $\psi(\vec{x}, y, z)$:

$$\varphi(\vec{x}, y, z) \wedge \forall u, v \leq y + z\, (u + v < y + z \to \neg\varphi(\vec{x}, u, v)).$$

We now claim that the Σ_1-formula $\exists z\, \psi(\vec{x}, y, z)$ represents the function f in PA^-:

First we show (b). Assume that $f(n_1, \dots, n_k) = m$ holds, thus $\mathbb{N} \vDash \exists z\, \varphi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, z)$. The number m is uniquely determined since f is a function. Choose l as the smallest number such that $\mathbb{N} \vDash \varphi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, \underline{l})$. Then clearly $\mathbb{N} \vDash \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, \underline{l})$ also holds, and thus $\mathbb{N} \vDash \exists z\, \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, \underline{z})$. As a Σ_1-sentence, this sentence is preserved under all end extensions of the standard model to a model of PA^-, thus it holds in all models of PA^-, and therefore $\mathsf{PA}^- \vdash \exists z\, \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, \underline{z})$ by the Completeness Theorem.

The proof of (a) uses a similar argument: Let $f(n_1, \dots, n_k) = m$ and let l again be the smallest number such that $\mathbb{N} \vDash \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, \underline{l})$ holds. Let $\mathcal{M} \vDash \mathsf{PA}^-$. We claim that m is the only element of M that satisfies the formula $\psi(\underline{n_1}, \dots, \underline{n_k}, x, \underline{l})$ in $\mathcal{M}$. $\mathcal{M} \vDash \psi(\underline{n_1}, \dots, \underline{n_k}, \underline{m}, \underline{l})$ due to the absoluteness of Δ_0-formulas. If $a, b \in M$ are two elements such that $\mathcal{M} \vDash \psi(\underline{n_1}, \dots, \underline{n_k}, a, b)$, then we must have $a, b \leq m + l$: For if one of them were greater than $m + l$, we would also have $a + b > m + l$. We assumed

$$\mathcal{M} \vDash \psi(\underline{n_1}, \ldots, \underline{n_k}, a, b),$$

so by definition of ψ we have

$$\mathcal{M} \vDash \varphi(\underline{n_1}, \ldots, \underline{n_k}, a, b) \wedge \forall u, v \leq a + b \; (u + v < a + b \rightarrow \neg\varphi(\underline{n_1}, \ldots, \underline{n_k}, u, v)).$$

But since $m + l < a + b$, this would imply $\neg\varphi(\underline{n_1}, \ldots, \underline{n_k}, \underline{m}, \underline{l})$ in $\mathcal{M}$, contradicting the choice of m and l.

As $\mathcal{M}$ is an end-extension of $\mathbb{N}$, $a, b \leq m + l$ implies $a, b \in \mathbb{N}$. Thus, again by the absoluteness of Δ_0-formulas, we have $\mathbb{N} \vDash \psi(\underline{n_1}, \ldots, \underline{n_k}, a, b)$, from which $\mathbb{N} \vDash \exists z \; \psi(\underline{n_1}, \ldots, \underline{n_k}, a, z)$ follows, and therefore $a = m$.

(ii) now follows easily: If S is recursive, then the characteristic function c_S is computable, thus by (i) represented by a Σ_1-formula $\theta(\vec{x}, y)$ in PA^-. Then S is represented by the formula $\theta(\vec{x}, 1)$, since we have:

$$\mathsf{PA}^- \vdash \neg\theta(\vec{x}, 1) \iff \mathsf{PA}^- \vdash \theta(\vec{x}, 0).$$

$\square$