

## Homogeneous Structures

**Recall:** We have seen that DLO is  $\omega$ -categorical, meaning up to isomorphism there is only one countable model. Since there are no finite models, by Vaught's test, DLO is complete.

**Question:** Are there other examples like this?

Let  $\mathcal{L}$  be a language and  $\mathcal{M}$  be an  $\mathcal{L}$ -structure.

**Definition 0.1.**

For  $A \subseteq M$ , we denote by  $\langle A \rangle^{\mathcal{M}}$  the smallest substructure of  $\mathcal{M}$  whose domain contains  $A$ .

We say  $\mathcal{N} \subseteq \mathcal{M}$  is finitely generated if  $\mathcal{N} = \langle A \rangle^{\mathcal{M}}$  for some finite  $A \subseteq M$ .

**Definition 0.2.**

We say  $\mathcal{M}$  is *homogeneous* if any isomorphism between finitely generated substructures of  $\mathcal{M}$  can be extended to an automorphism of  $\mathcal{M}$ .

Let  $\mathcal{L} = \{<\}$  and  $\mathcal{M} = (\mathbb{Q}, <)$ . The finitely generated substructures coincide with the finite substructures.

### The age of a structure

In the back-and-forth proof of  $\omega$ -categoricity of DLO, we used a homogeneity property, similarly for the proof that  $(\mathbb{Q}, <) \cong (\mathbb{R}, <)$ .

**Definition 0.3.**

The *age* of  $\mathcal{M}$ , denoted  $\text{age}(\mathcal{M})$ , is the class of all finitely generated  $\mathcal{L}$ -structures isomorphic to a substructure of  $\mathcal{M}$ .

$\text{age}(\mathbb{Q}, <)$  is the class of all finite linear orders.

**Lemma 0.1.**

*Any two countable homogeneous structures with the same age are isomorphic.*

*Sketch.* Let  $\mathcal{A} \subseteq \mathcal{M}$  and  $\mathcal{B} \subseteq \mathcal{N}$  be finitely generated, and let  $\pi : \mathcal{A} \rightarrow \mathcal{B}$  be an isomorphism.

Let  $a \in M \setminus A$ . We need to show that  $\pi$  can be extended to an isomorphism  $\mathcal{A} \cup \{a\} \rightarrow \mathcal{B}'$ . This suffices, since we can then use the back-and-forth argument (remember our structures are countable) to extend everything to an automorphism of  $\mathcal{M}$ .

Suppose  $\mathcal{A} = \langle E \rangle^{\mathcal{M}}$  and  $\mathcal{B} = \langle F \rangle^{\mathcal{N}}$ .

Let  $\mathcal{A}' = \langle E \cup \{a\} \rangle^{\mathcal{M}} \subseteq \mathcal{M}$ . Since  $\mathcal{M}$  and  $\mathcal{N}$  have the same age, there exists  $\mathcal{C} \subseteq \mathcal{N}$  finitely generated such that  $\mathcal{A}' \cong \mathcal{C}$  via some isomorphism  $g$ .

The map  $g$  is uniquely determined by its values on  $E \cup \{a\}$ . The restriction  $g|_E$  induces an isomorphism  $\langle E \rangle^{\mathcal{M}} \xrightarrow{\cong} \langle g(E) \rangle^{\mathcal{N}}$ .

Therefore,  $\pi \circ (g|_E)^{-1}$  is an isomorphism  $\langle g(E) \rangle^{\mathcal{N}} \xrightarrow{\cong} \mathcal{B}$ . Call this map  $\tau$ . Note that  $\langle g(E) \rangle^{\mathcal{N}}$  is a finitely generated subset of  $\mathcal{N}$ . Since  $\mathcal{N}$  is homogeneous,  $\tau$  extends to an automorphism  $\bar{\tau} : \mathcal{N} \rightarrow \mathcal{N}$ . Let  $\mathcal{B}'$  be the image of  $\mathcal{C}$  under  $\bar{\tau}$ . By definition of  $\bar{\tau}$ ,  $\mathcal{B} \subset \mathcal{B}'$  and  $\mathcal{A}'$  is isomorphic to  $\mathcal{B}'$  via the map  $\bar{\tau} \circ g$ .  $\square$

 Question

What do ages of countable homogeneous structures look like?