Course Title :- Mathematics - I for CSE stream

Course code :- 22MATS11

Module 1 :- Calculus :-

Polar Coordinates, polar curves, Angle b/w the radius vector and the tangent, angle b/w the two curves, pedal equations. Curvature and radius of curvature - Cartesian, Parametric, polar and pedal forms. problems.

Self - study :- Centre and circle of curvatures, Evolutes and Involutes.

Polar curves :-

Polar co-ordinates :-

Initially choosing a point 'O' in a plane is called pole (origin).

A line OX drawn through 'O' is called initial line (x-axis)

If P be any point in a XY plane then join the points OP. with the result an angle is formed at 'O', then the length of OP is denoted by 'r' which is called radius vector.

$P\hat{O}X = \theta$ is called Vectorial angle which is measured in anticlockwise direction.

the pair 'r and θ' is represented by (r, θ) are called polar co-ordinates.

Here 'r' is always +ve and θ lies b/w 0 & 2π.

Relationship b/w Cartesian and polar co-ordinates

Let 'O' be a pole, OX be the initial line and let P be any point in a xy-plane. Joining the points O and P an angle θ is formed at O.

From the fig,

$OP = r$, $OL = x$, $PL = y$ & $P\hat{O}L = \theta$, $P\hat{L}O = 90^\circ$

$\cos\theta = \frac{OL}{OP} = \frac{OX}{r}$ also, $\sin\theta = \frac{PL}{OP} = \frac{y}{r}$

$\boxed{x = r\cos\theta} \rightarrow (1)$ $\boxed{y = r\sin\theta} \rightarrow (2)$

squaring and adding the (1) and (2)

we get, $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$x^2 + y^2 = r^2$

$\boxed{r = \sqrt{x^2 + y^2}} \rightarrow (3)$

Dividing (2) and (1),

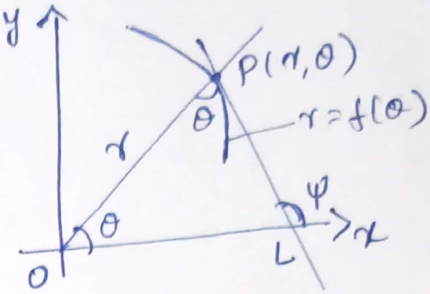
we get, $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta}$

$\frac{y}{x} = \tan\theta$

$\boxed{\theta = \tan^{-1}(y/x)} \rightarrow (4)$

Angle b/w Radius vector and tangent ②

Let $P(r,\theta)$ be any point on the curve $r=f(\theta)$



$\therefore\ x\hat{O}P=\theta$ and $OP=r$

Let PL be the tangent to the curve at 'P' subtending an angle φ with the and positive direction of the initial line and ϕ be the angle b/w the radius vector OP and the tangent PL, i,e $O\hat{P}L=\phi$.

From the fig, we have

$\varphi=\phi+\theta$ (an exterior angle = sum of interior opposite angles.)

$\tan\varphi=\tan(\phi+\theta)$

$$\tan\varphi=\frac{\tan\phi+\tan\theta}{1-\tan\phi\cdot\tan\theta} \longrightarrow (1)$$

Let (x,y) be the cartesian coordinates of P, so that we have,

$x=r\cos\theta$, $y=r\sin\theta$.

The geometrical meaning of the slope of tangent is

$$\tan\varphi=\frac{dy}{dx}$$

$$=\frac{dy}{d\theta}\times\frac{d\theta}{dx}$$

$$\tan\varphi=\frac{dy/d\theta}{dx/d\theta}$$

$$\tan\varphi=\frac{r\cos\theta+r'\sin\theta}{-r\sin\theta+r'\cos\theta}$$

But, $\frac{dy}{d\theta}=r\cos\theta+\sin\theta\frac{dr}{d\theta}$

$\frac{dx}{d\theta}=-r\sin\theta+\cos\theta\frac{dr}{d\theta}$

and $\frac{dr}{d\theta}=r'$

dividing throughout by $r' \cos\theta$ both donomenator and Numerator.

$$\tan\psi = \frac{\frac{r\cos\theta}{r'\cos\theta} + \frac{r'\sin\theta}{r'\cos\theta}}{\frac{-r\sin\theta}{r'\cos\theta} + \frac{r'\cos\theta}{r'\cos\theta}}$$

$$\tan\psi = \frac{\frac{r}{r'} + \tan\theta}{-\frac{r}{r'}\tan\theta + 1}$$

$$\tan\psi = \frac{\frac{r}{r'} + \tan\theta}{1 - \frac{r}{r'}\tan\theta} \longrightarrow (2)$$

comparing (1) and (2)

$\tan\phi = \frac{r}{r'}$ where $r' = \frac{dr}{d\theta}$

$$\boxed{\tan\phi = r\frac{d\theta}{dr}}$$

(or)

$$\frac{1}{\tan\phi} = \frac{1}{r}\frac{dr}{d\theta}$$

$$\boxed{\cot\phi = \frac{1}{r}\frac{dr}{d\theta}}$$

Hence the proof!

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Working Rule :-

Step 1:- Given the equation in the form $r = f(\theta)$, take log on both sides of the equation and then differentiate w.r.t θ which always gives $\frac{1}{r}\frac{dr}{d\theta}$ and use $\cot\phi$ for the same.

Step 2:- Simplify RHS and try to put it in terms of cotangent i.e 'cot' so that we obtain ϕ (or) ϕ_1 and ϕ_2.

Step 3:- $|\phi_2 - \phi_1|$ (or) $|\phi_1 - \phi_2|$ will give the angle of intersection

Step 4:- If the solution contains 'θ' then we have to find θ by solving the pair of equations to obtain the angle of intersection independent of θ.

Step 5:- Suppose we are not able to obtain ϕ_1 and ϕ_2 explicitly then we have to write the expressions for $\tan\phi_1$, $\tan\phi_2$ and use the formula $\tan(\phi_1 - \phi_2)$.

Step 6:- If $\tan(\phi_1 - \phi_2) = \alpha$ (say) then the angle of intersection is equal to $\tan^{-1}(\alpha)$.

Step 7:- Also if $\tan\phi_1 \cdot \tan\phi_2 = -1$ then $\tan(\phi_1 - \phi_2) = \infty$
$\Rightarrow \phi_1 - \phi_2 = \pi/2$.

Note :- The following allied and compound angles trignometric formulae will have frequent reference in problems.

1. $\sin(\pi/2 - \theta) = \cos\theta$
$\cos(\pi/2 - \theta) = \sin\theta$
$\tan(\pi/2 - \theta) = \cot\theta$
$\cot(\pi/2 - \theta) = \tan\theta$

2. $\sin(\pi/2 + \theta) = \cos\theta$
$\cos(\pi/2 + \theta) = -\sin\theta$
$\tan(\pi/2 + \theta) = -\cot\theta$
$\cot(\pi/2 + \theta) = -\tan\theta$

3. $\tan(\pi/4 + \theta) = \dfrac{1+\tan\theta}{1-\tan\theta}$

4. $\cot(\pi/4 + \theta) = \dfrac{1-\tan\theta}{1+\tan\theta}$

5. $1 + \cos\theta = 2\cos^2\theta/2$

6. $1 - \cos\theta = 2\sin^2\theta/2$

7. $\sin\theta = 2\sin\theta/2 \cdot \cos\theta/2$

8. $\cos\theta = \cos^2\theta/2 - \sin^2\theta/2$

Find the angle between the radius vector and the tangent for the following polar curves:-

1. $r = a(1-\cos\theta)$

taking log on b.s

$\log r = \log a + \log(1-\cos\theta)$

Diff. w.r.t θ,

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\sin\theta}{1-\cos\theta}$$

$$\cot\phi = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$\cot\phi = \cot\theta/2$$

$$\boxed{\phi = \theta/2}$$

2. $r^2\cos 2\theta = a^2$

taking log on b.s

$2\log r + \log(\cos 2\theta) = 2\log a$

Diff. w.r.t θ,

$$\frac{2}{r}\frac{dr}{d\theta} + \frac{-2\sin(2\theta)}{\cos 2\theta} = 0$$

$$\frac{1}{r}\frac{dr}{d\theta} = \tan 2\theta$$

$$\cot\phi = \cot(\pi/2 - 2\theta)$$

$$\boxed{\phi = \frac{\pi}{2} - 2\theta}$$

3. $r^m = a^m(\cos m\theta + \sin m\theta)$

Taking log on b.s

$m\log r = m\log a + \log(\cos m\theta + \sin m\theta)$

Diff. w.r.t θ, get

$$\frac{m}{r}\frac{dr}{d\theta} = 0 + \frac{(-m\sin m\theta + m\cos m\theta)}{(\cos m\theta + \sin m\theta)}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta}$$

$$\cot\phi = \frac{\cos m\theta(1-\tan m\theta)}{\cos m\theta(1+\tan m\theta)}$$

$$\cot\phi = \cot(\pi/4 + m\theta)$$

$$\boxed{\phi = \frac{\pi}{4} + m\theta}$$

4. $\frac{l}{r} = 1 + e\cos\theta$

Taking log on b.s

$\log l - \log r = \log(1+e\cos\theta)$

Diff. w.r.t θ,

$$0 - \frac{1}{r}\frac{dr}{d\theta} = \frac{-e\sin\theta}{1+e\cos\theta}$$

$$\cot\phi = \frac{e\sin\theta}{1+e\cos\theta}$$

$$\tan\phi = \frac{1+e\cos\theta}{e\sin\theta}$$

$$\boxed{\phi = \tan^{-1}\left(\frac{1+e\cos\theta}{e\sin\theta}\right)}$$

Find the angle b/w the radius vector and the tangent as indicated :- (4)

1. $r = a(1+\cos\theta)$ at $\theta = \pi/3$

$\log r = \log a + \log(1+\cos\theta)$

Diff. w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{\sin\theta}{1+\cos\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$\cot\phi = -\tan\theta/2$

$\cot\phi = \cot(\pi/2+\theta/2)$

$\phi = \frac{\pi}{2}+\frac{\theta}{2} \rightarrow (1)$

At $\theta = \pi/3$,

$(1) \Rightarrow \phi = \frac{\pi}{2}+\frac{\pi}{6}$

$$\boxed{\phi = \frac{2\pi}{3}}$$

2. $r\cos^2\theta/2 = a$ at $\theta = 2\pi/3$

$\log r + 2\log\cos(\theta/2) = \log a$

Diff. w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} + 2\frac{(-\frac{1}{2})\sin\theta/2}{\cos\theta/2} = 0$$

$\cot\phi = +\tan\theta/2$

$\cot\phi = \cot(\pi/2-\theta/2)$

$\phi = \frac{\pi}{2}-\theta/2 \rightarrow (1)$

At, $\theta = \frac{2\pi}{3}$,

$(1) \Rightarrow \phi = \frac{\pi}{2}-\frac{\pi}{3}$

$$\boxed{\phi = \frac{\pi}{6}}$$

3. $\frac{2a}{r} = 1-\cos\theta$ at $\theta = \frac{2\pi}{3}$

$\log 2a - \log r = \log(1-\cos\theta)$

Diff. w.r.t θ

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$$

$-\cot\phi = \cot\theta/2$

$\phi = -\theta/2 \rightarrow (1)$

At $\theta = \frac{2\pi}{3}$.

$$\boxed{\phi = -\frac{\pi}{3}}$$

4. $r = a(1+\sin\theta)$ at $\theta = \frac{\pi}{2}$

$\log r = \log a + \log(1+\sin\theta)$

Diff. w.r.t θ,

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta}{1+\sin\theta}$$

$\cot\phi = \frac{\cos\theta}{1+\sin\theta} \rightarrow (1)$

At $\theta = \pi/2$,

$\cot\phi = \frac{0}{1+1} = 0$

$$\boxed{\phi = \pi/2}$$

Show that the following pairs of curves intersect each other orthogonally :-

1. $r = a(1+\cos\theta)$ & $r = b(1-\cos\theta)$

$\log r = \log(1+\cos\theta)$

Diffn w.r.t θ

$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$

$\cot\phi_1 = -\tan\theta/2$

$\phi_1 = \pi/2 + \theta/2$

$\log r = \log(1-\cos\theta)$

Diffn w.r.t θ

$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$

$\cot\phi_2 = \cot\theta/2$

$\phi_2 = \theta/2$

Angle of intersection is

$|\phi_1 - \phi_2| = \left|\frac{\pi}{2} + \frac{\theta}{2} - \frac{\theta}{2}\right|$

$\boxed{|\phi_1 - \phi_2| = \pi/2}$

Thus the curves intersect each other orthogonally.

2) Similar way (Dec 2016)

$r = \frac{a}{1+\cos\theta}$ and $r = \frac{b}{1-\cos\theta}$

Proceeding the same lines

$\phi_1 = \frac{\pi}{2} - \frac{\theta}{2}$, $\phi_2 = -\theta/2$

$|\phi_1 - \phi_2| = \frac{\pi}{2}$

3) $r = a(1+\sin\theta)$ & $r = a(1-\sin\theta)$

taking log & Diffn w.r.t θ,

$\cot\phi_1 = \frac{\cos\theta}{1+\sin\theta}$, $\cot\phi_2 = \frac{-\cos\theta}{1-\sin\theta}$

$\tan\phi_1 = \frac{1+\sin\theta}{\cos\theta}$, $\tan\phi_2 = \frac{(1-\sin\theta)}{-\cos\theta}$

$\therefore \tan\phi_1 \cdot \tan\phi_2 = \frac{1-\sin^2\theta}{-\cos^2\theta}$

$= \frac{\cos^2\theta}{-\cos^2\theta}$

$\boxed{\tan\phi_1 \cdot \tan\phi_2 = -1}$

Thus the curves intersect each other orthogonally.

4. $r^n = a^n\cos n\theta$ & $r^n = b^n \sin n\theta$

taking log & Diffn w.r.t θ,

$\cot\phi_1 = -\tan n\theta$ & $\cot\phi_2 = \cot n\theta$

$\phi_1 = \frac{\pi}{2} + n\theta$ & $\phi_2 = n\theta$

$\therefore |\phi_1 - \phi_2| = \left|\frac{\pi}{2} + n\theta - n\theta\right|$

$\boxed{|\phi_1 - \phi_2| = \frac{\pi}{2}}$

$\therefore$ Thus, the curves intersect each other orthogonally.

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5. $r = ae^{\theta}$ and $re^{\theta} = b$

$\log r = \log e^{\theta}$

$\log r = \theta \log e$

Diff. w.r.t θ

$\frac{1}{r}\frac{dr}{d\theta} = 1$

$\cot\phi_1 = 1$

$\tan\phi_1 = 1$

$\log r + \theta \log e = \log b$

Diff. w.r.t

$\frac{1}{r}\frac{dr}{d\theta} + 1 = 0$

$\cot\phi_2 = -1$

$\tan\phi_2 = -1$

$\therefore \tan\phi_1 \cdot \tan\phi_2 = (1)(-1) = -1$

$\therefore$ The curves intersect each other orthogonally.

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6. $r = 4\sec^2\theta/2$ and $r = 9\,\mathrm{cosec}^2\,\theta/2$

$\log r = \log 4 + 2\log \sec\theta/2$

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{2}{\sec\theta/2} \times \sec\theta/2 \tan\theta/2 \cdot \frac{1}{2}$

$\cot\phi_1 = \tan\theta/2$

$\cot\phi_1 = \cot(\pi/2 - \theta/2)$

$\phi_1 = \pi/2 - \theta/2$

$\log r = \log 9 + 2\log \mathrm{cosec}\,\theta/2$

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-2\,\mathrm{cosec}\,\theta/2\,\cot\theta/2 \cdot \frac{1}{2}}{\mathrm{cosec}\,\theta/2}$

$\cot\phi_2 = -\cot\theta/2$

$\phi_2 = -\theta/2$

$\therefore |\phi_1 - \phi_2| = \left|\frac{\pi}{2} - \frac{\theta}{2} + \frac{\theta}{2}\right| = \pi/2$ //.

H.W
7. $r^2 \sin 2\theta = a^2$ and $r^2 \cos 2\theta = b^2$.

$\phi_1 = -2\theta$

$\phi_2 = \pi/2 - 2\theta$

$|\phi_1 - \phi_2| = \pi/2$ //.

Find the angle b/w the following pairs of curves:-

1. $r = a\log\theta$ and $r = \frac{a}{\log\theta}$

$\log r = \log a + \log(\log\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\theta(\log\theta)}$

$\cot\phi_1 = \frac{1}{\theta\log(\theta)}$

$\tan\phi_1 = \theta\log\theta$

$\log r = \log a - \log(\log\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{\theta(\log(\theta))}$

$\cot\phi_2 = -\frac{1}{\theta\log(\theta)}$

$\tan\phi_2 = -\theta\log\theta$

$\therefore \tan(\phi_1-\phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1 . \tan\phi_2}$

$\tan(\phi_1-\phi_2) = \frac{2\theta\log\theta}{1-(\theta\log\theta)^2} \rightarrow (1)$

To find θ,
Solving the pair of equations.
equaling the RHS. of $r = a\log\theta$ and $r = a/\log\theta$

$a\log\theta = \frac{a}{\log\theta}$

$(\log\theta)^2 = 1$

$\log\theta = 1$

$\boxed{\theta = e}$

$(1) \Rightarrow \tan(\phi_1-\phi_2) = \frac{2e}{1-e^2}$

$\phi_1-\phi_2 = \tan^{-1}\left(\frac{2e}{1-e^2}\right)$ //.

2. $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

$\log r = \log a + \log\theta - \log(1+\theta)$, $\log r = \log a - \log(1+\theta^2)$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta} - \frac{1}{1+\theta}$ $\qquad$ $\frac{1}{r}\frac{dr}{d\theta} = -\frac{2\theta}{1+\theta^2}$

$\cot\phi_1 = \frac{1}{\theta(1+\theta)}$ $\qquad$ $\cot\phi_2 = -\frac{2\theta}{1+\theta^2}$

$\tan\phi_1 = \theta(1+\theta)$ $\qquad$ $\tan\phi_2 = \frac{1+\theta^2}{-2\theta}$

$$\therefore \tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \cdot \tan\phi_2}$$

$$= \frac{\theta(1+\theta) + \frac{1+\theta^2}{2\theta}}{1 - \frac{\theta(1+\theta)(1+\theta^2)}{2\theta}} \rightarrow (1)$$

Comparing the given pairs of equation,

$$\frac{a\theta}{1+\theta} = \frac{a}{1+\theta^2}$$

$$\frac{\theta}{1+\theta} = \frac{1}{1+\theta^2}$$

$$\theta + \theta^3 = 1 + \theta$$

$$\theta^3 = 1$$

$$\boxed{\theta = 1}$$

(1) $\Rightarrow$ $\tan(\phi_1 - \phi_2) = \frac{2+1}{1-2} = -3$.

$\phi_1 - \phi_2 = \tan^{-1}(-3)$ //

3. $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$

$\log r = \log(\sin\theta + \cos\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$

$\cot\phi_1 = \frac{1-\tan\theta}{1+\tan\theta}$

$\cot\phi_1 = \cot(\pi/4 + \theta)$

$\phi_1 = \frac{\pi}{4} + \theta$

$\log r = \log 2 + \log\sin\theta$

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{\cos\theta}{\sin\theta}$

$\cot\phi_2 = \cot\theta$

$\phi_2 = \theta$

$\therefore (\phi_1 - \phi_2) = \frac{\pi}{4} + \theta - \theta = \pi/4$ //.

4. $r = a\theta$ and $r = a/\theta$

$\log r = \log a + \log\theta$

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{\theta}$

$\cot\phi_1 = \frac{1}{\theta}$

$\tan\phi_1 = \theta$

, $\log r = \log a - \log\theta$

$\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{\theta}$

$\cot\phi_2 = -\frac{1}{\theta}$

$\tan\phi_2 = -\theta$

$\therefore \tan(\phi_1 - \phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1 + \tan\phi_1 \tan\phi_2}$

$= \frac{\theta + \theta}{1 - \theta^2} \rightarrow (1)$

comparing,

$a\theta = \frac{a}{\theta}$

$\theta^2 = 1 \Rightarrow \boxed{\theta = \pm 1}$

(1) $\Rightarrow \tan(\phi_1 - \phi_2) = \infty \Rightarrow \tan^{-1}(\infty) = \pi/2$ //.

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5. $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$

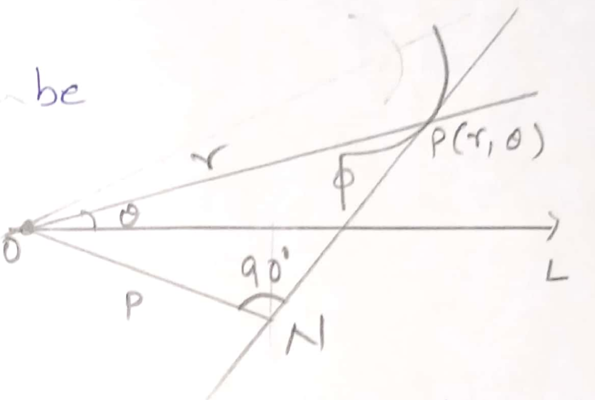
6. $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$

7. $r = 6\cos\theta$ and $r = 2(1+\cos\theta)$

8. $r^n = a^n \sec(n\theta + \alpha)$ and $r^n = b^n \sec(n\theta + \beta)$

9. $r = a(1+\cos\theta)$ and $r^2 = a^2 \cos 2\theta$

Length of the perpendicular from the pole to the tangent :-



Let 'O' be the pole and 'OL' be the initial line.

Let $P(r,\theta)$ be any point on the curve and hence we have

$OP = r$ and $L\hat{O}P = \theta$.

Draw $ON = p$ (say) $\perp r$ from the pole onto the tangent at P and let 'ϕ' be the angle made by the radius vector with the tangent.

From the fig, $O\hat{N}P = 90^\circ$ and $L\hat{O}P = \theta$

Now, from the right angled triangle ONP,

$$\sin\phi = \frac{ON}{OP}$$

$$\sin\phi = \frac{p}{r}$$

$$\boxed{p = r\sin\phi} \rightarrow (1)$$ → Basic expression for the length of the perpendicular p.

to present the expression for p terms of θ,

Now, Squaring and taking reciprocal of (1),

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left[1 + \cot^2\phi\right] \rightarrow (2)$$

WKT, the alternate expression for angle b/w radius vector and tangent is given by.

$$\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$$

$$\cot^2\phi = \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2$$

$$(2) \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

$$\boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2} //.$$

Find the pedal equation of the following curves

1. $\frac{2a}{r} = 1 + \cos\theta$

$\log 2a - \log r = \log(1+\cos\theta)$

$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1+\cos\theta}$

$\cot\phi = \tan\theta/2$

$\phi = \frac{\pi}{2} - \frac{\theta}{2}$

WKT, pedal eqn is given by

$p = r\sin\phi$

$p = r\sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$

$p = r\cos\frac{\theta}{2} \rightarrow (1)$

from the given curve

$\frac{2a}{r} = 1+\cos\theta$

$\frac{2a}{r} = 2\cos^2\theta/2$

$\cos\theta/2 = \sqrt{a/r}$

$(1) \Rightarrow p = r\sqrt{\frac{a}{r}} \Rightarrow \boxed{p = \sqrt{ar}} //.$

2. $r^2 = a^2\sec 2\theta$

$2\log r = 2\log a + \log\sec 2\theta$

$2\frac{1}{r}\frac{dr}{d\theta} = \frac{2(\sec 2\theta)\tan 2\theta)}{\sec 2\theta}$

$\cot\phi = \tan 2\theta$

$\phi = \frac{\pi}{2} - 2\theta$

WKT,

$p = r\sin\phi$

$p = r\sin(\pi/2 - 2\theta)$

$p = r\cos 2\theta \rightarrow (1)$

from the given curve

$r^2 = a^2\sec 2\theta$

$\frac{1}{r^2} = \frac{1}{a^2}\cos 2\theta$

$\frac{a^2}{r^2} = \cos 2\theta$

$(1) \Rightarrow p = r\frac{a^2}{r^2}$

$\boxed{p = \frac{a^2}{r}} //.$

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3. $r^m = a^m [\cos m\theta + \sin m\theta]$

$m \log r = m \log a + \log(\cos m\theta + \sin m\theta)$

$\frac{m}{r}\frac{dr}{d\theta} = \frac{(-\sin m\theta + \cos m\theta)(m)}{\cos m\theta + \sin m\theta}$

$\cot\phi = \frac{1 - \tan m\theta}{1 + \tan m\theta}$

$\phi = \frac{\pi}{4} + m\theta$

WKT,

$p = r\sin\phi$

$p = r\sin\left(\frac{\pi}{4} + m\theta\right)$

$p = r\left[\sin\frac{\pi}{4}\cos m\theta + \cos\frac{\pi}{4}\sin m\theta\right]$

But, $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$p = \frac{r}{\sqrt{2}}(\cos m\theta + \sin m\theta) \rightarrow (1)$

From the given curves,

$\frac{r^m}{a^m} = \cos m\theta + \sin m\theta$

$(1) \Rightarrow p = \frac{r}{\sqrt{2}}\left(\frac{r^m}{a^m}\right)$

$\boxed{p = \frac{r^{m+1}}{\sqrt{2}\, a^m}}$ //.

6. $r = 2a\cos\theta$

7. $r^n = a^n \cos n\theta$

8. $r(1 - \cos\theta) = 2a$

9. $r = a(1 + \cos\theta)$

4. $r = a e^{m\theta}$

$\log r = \log a + m\theta \log e$

(8)

$\frac{1}{r}\frac{dr}{d\theta} = m$

$\cot\phi = m$

WKT,

$\frac{1}{p^2} = \frac{1}{r^2}[1 + \cot^2\phi]$

$\frac{1}{p^2} = \frac{1}{r^2}[1 + m^2]$

$\boxed{p^2 = \frac{r^2}{1+m^2}}$ //.

5. $r^n = a^n \operatorname{sech} n\theta$

$n \log r = n \log a + \log \operatorname{sech} n\theta$

$n \frac{1}{r}\frac{dr}{d\theta} = \frac{\operatorname{sech} n\theta \tanh n\theta\,(n)}{\operatorname{sech} n\theta}$

$\cot\phi = \tanh n\theta$

WKT,

$\frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2\phi)$

$\frac{1}{p^2} = \frac{1}{r^2}(1 + \tan^2 h n\theta)$

$\frac{1}{p^2} = \frac{1}{r^2}(\sec^2 h n\theta) \rightarrow (1)$

From the given curve

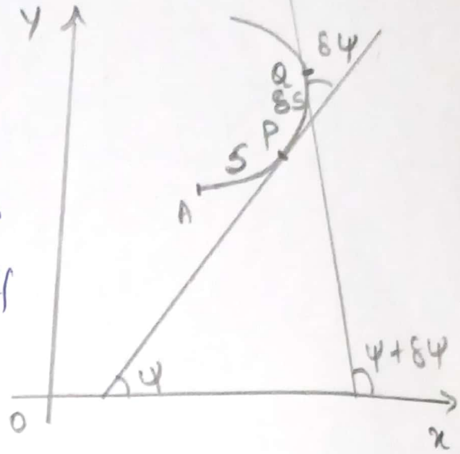
$\frac{r^n}{a^n} = \operatorname{sech} n\theta$

$(1) \Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(\frac{r^n}{a^n}\right)^2$

$p^2 = \frac{a^{2n} r^2}{r^{2n}}$

$\boxed{p^2 = \frac{a^{2n}}{r^{2n-2}}}$ //.

Curvature and Radius of Curvature :-



Consider a curve in x-y plane and Let A be the fixed point on it. P and Q are the neighbouring points on the curve such that the length of $AP = s$ and $PQ = \delta s$

so that, $AQ = s + \delta s$.

Let ψ and $\psi + \delta\psi$ are the angles made by the tangents at P and Q with the x-axis.

the angle $\delta\psi$ b/w the tangents is called bending of a curve which depends on δs.

The $\frac{\delta\psi}{\delta s}$ is the amount of bending of curve at P, which is called curvature of the curve.

Mathematically, it is defined as

$$\lim_{\delta s \to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$$, which is denoted by k.

i.e curvature, $k = \frac{d\psi}{ds}$

Further, if $k \neq 0$ then the reciprocal of the curvature is called radius of curvature which is denoted by 'ρ'.

i.e Radius of curvature, $\rho = \frac{1}{k} = \frac{ds}{d\psi}$.

Note :-

1) The amount of bending for a straight line at all the points on it is zero, i.e curvature of line is zero i.e $k = 0$.

2) Since the circle has a uniform bending at all the points the curvature of circle is always constant

Proof of Radius of curvature in Cartesian form

Suppose the cartesian equation of the curve C is given by $y = f(x)$ and 'A' be a fixed point on it.

Let $P(x, y)$ be a given point on C such that arc $AP = s$

then wKT, slope of tangent is given by

$$\frac{dy}{dx} = \tan\psi \longrightarrow (1)$$

where ψ is the angle made by the tangent to the curve C at P with x-axis and the arc length is given by

$$\frac{ds}{dx} = \sqrt{\left(1+\left(\frac{dy}{dx}\right)^2\right)} \longrightarrow (2)$$

Diff/ eq^n (1) w.r.t x, we get

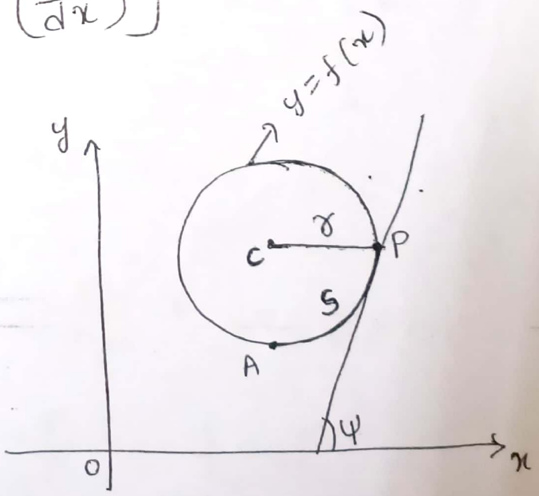
$$\frac{d^2y}{dx^2} = \sec^2\psi \cdot \frac{d\psi}{dx}$$

$$= (1+\tan^2\psi)\frac{d\psi}{ds}\cdot\frac{ds}{dx}$$

$$= \left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{1}{\rho}\sqrt{\left[1+\left(\frac{dy}{dx}\right)^2\right]}$$

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\boxed{\rho = \frac{(1+y_1^2)^{3/2}}{y_2}}$$



Expression for the radius of curvature in polar form

$\frac{d\phi}{d\theta}$

Let $r = f(\theta)$ be the equation of a polar curve

Let $OP = r$ be the radius vector and ϕ be the angle made by the radius vector with the tangent at $P(r, \theta)$.

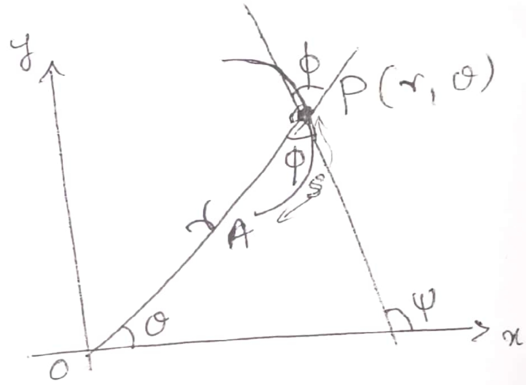
Let ψ be the angle made by the tangent at P with the initial line.

Let A be a fixed point on the curve and

Let $\widehat{AP} = s$

we have, (Exterior angle = sum of interior angles)

i.e $\psi = \theta + \phi$



Diff// w.r.t s

$$\frac{d\psi}{ds} = \frac{d\phi}{ds} + \frac{d\theta}{ds}$$

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{d\theta} \cdot \frac{d\theta}{ds}$$

$$\frac{1}{\rho} = \frac{d\theta}{ds}\left[1 + \frac{d\phi}{d\theta}\right]$$

$$\rho = \frac{\left(\frac{ds}{d\theta}\right)}{\left(1 + \frac{d\phi}{d\theta}\right)} \longrightarrow (1)$$

WKT, the angle b/w radius vector and ~~vector~~ tangent is given by

$$\tan\phi = r\frac{d\theta}{dr} = \frac{r}{\left(\frac{dr}{d\theta}\right)}$$

i.e $\tan\phi = \frac{r}{r_1}$ where $r_1 = \frac{dr}{d\theta} \longrightarrow (2)$

Diff// w.r.t θ, we get

$$\sec^2\phi \cdot \frac{d\phi}{d\theta} = \frac{r_1 \cdot r_1 - r r_2}{r_1^2} \quad \text{where } r_2 = \frac{d^2 r}{d\theta^2}$$

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$$\frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2 \sec^2\phi} = \frac{r_1^2 - rr_2}{r_1^2[1+\tan^2\phi]}$$ (10)

$$\frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2\left[1+\frac{r^2}{r_1^2}\right]}$$ [Using eq^n (2)]

$$\frac{d\phi}{d\theta} = \frac{r_1^2 - rr_2}{r_1^2 + r^2}$$

$$1+\frac{d\phi}{d\theta} = 1 + \frac{r_1^2 - rr_2}{r_1^2 + r^2}$$ [adding '1' on B.S]

$$1+\frac{d\phi}{d\theta} = \frac{r_1^2 + r^2 + r_1^2 - rr_2}{r_1^2 + r^2}$$

$$1+\frac{d\phi}{d\theta} = \frac{r^2 + 2r_1^2 - rr_2}{r_1^2 + r^2} \rightarrow (3)$$

Also, WKT, the arc length in polar form is given by $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2 + r_1^2} \rightarrow (4)$

Using (3) and (4) in (1) we get

$$\rho = \sqrt{r^2 + r_1^2} \cdot \frac{(r_1^2 + r^2)}{(r^2 + 2r_1^2 - rr_2)}$$

$$\boxed{\rho = \frac{(r_1^2 + r^2)^{3/2}}{r^2 + 2r_1^2 - rr_2}}$$

//.

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Formulae to be Remember :-

1. Radius of curvature in cartesian form.

$$\boxed{\rho = \frac{(1+y_1^2)^{3/2}}{y_2}}, \quad \text{where } y_1 = \frac{dy}{dx}, \quad y_2 = \frac{d^2y}{dx^2}.$$

Alternate form,

$$\boxed{\rho = \frac{(1+x_1^2)^{3/2}}{x_2}}, \quad \text{where } x_1 = \frac{dx}{dy}, \quad x_2 = \frac{d^2x}{dy^2}$$

2. Radius of curvature in polar form :

$$\boxed{\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}}, \quad \text{where } r_1 = \frac{dr}{d\theta}, \quad r_2 = \frac{d^2r}{d\theta^2}$$

3. Radius of curvature in parametric form :-

$$\boxed{\rho = \frac{(x_1^2+y_1^2)^{3/2}}{x_1y_2-y_1x_2}} \quad \text{where, } x_1 = \frac{dx}{dt}, \quad y_1 = \frac{dy}{dt}$$

$$x_2 = \frac{d^2x}{dt^2}, \quad y_2 = \frac{d^2y}{dt^2}$$

4. Radius of curvature in pedal form :-

$$\boxed{\rho = r\frac{dr}{dp}}.$$

Problems :-

1. Find the radius of curvature of the curve $x^3+y^3=3axy$ at $\left(\frac{3a}{2},\frac{3a}{2}\right)$.

Sol :-

$x^3+y^3=3axy$ Diff. w.r.t x

$3x^2+3y^2y_1=3a(xy_1+y)$

$3y^2y_1-3axy_1=3ay-3x^2$

$y_1(3y^2-3ax)=3ay-3x^2$

$$y_1=\frac{3ay-3x^2}{3y^2-3ax}$$

$$y_1=\frac{ay-x^2}{y^2-ax} \rightarrow (1)$$

Again Diff. w.r.t x

$$y_2=\frac{(y^2-ax)(ay_1-2x)-(ay-x^2)(2yy_1-a)}{(y^2-ax)^2} \rightarrow (2)$$

At $\left(\frac{3a}{2},\frac{3a}{2}\right)$

$$(1) \Rightarrow y_1=\frac{a\left(\frac{3a}{2}\right)-\left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2-a\left(\frac{3a}{2}\right)}$$

taking $\left(\frac{3a}{2}\right)$ common b.s Nr & Dr

$$y_1=\frac{\left(a-\frac{3a}{2}\right)}{\left(\frac{3a}{2}-a\right)}$$

$$y_1=\frac{2a-3a}{3a-2a}$$

$$y_1=\frac{-a}{a}$$

$$\boxed{y_1=-1}$$

At $\left(\frac{3a}{2},\frac{3a}{2}\right)$

$$(2) \Rightarrow y_2=\frac{\left[\left(\frac{3a}{2}\right)^2-a\left(\frac{3a}{2}\right)\right]\left[a(-1)-2\left(\frac{3a}{2}\right)\right]-\left[a\left(\frac{3a}{2}\right)-\left(\frac{3a}{2}\right)^2\right]\left[(2)\left(\frac{3a}{2}\right)(-1)-a\right]}{\left[\left(\frac{3a}{2}\right)^2-a\left(\frac{3a}{2}\right)\right]^2}$$

$$y_2=\frac{\left[\frac{9a^2}{4}-\frac{3a^2}{2}\right]\left[-4a\right]-\left[\frac{3a^2}{2}-\frac{9a^2}{4}\right]\left[-4a\right]}{\left[\frac{9a^2}{4}-\frac{3a^2}{2}\right]^2}$$

$$y_2=\frac{\frac{3a^2}{4}(-4a)+\left(\frac{3a^2}{4}\right)(-4a)}{\left(\frac{3a^2}{4}\right)^2}$$

$$=\frac{-6a^3(16)}{9a^4}$$

$$=\frac{-32a^3}{3a^4}$$

$$\boxed{y_2=\frac{-32}{3a}}$$

Now,
Radius of curvature is given by

$$\rho = \frac{\left[1+(y_1)^2\right]^{3/2}}{y_2}$$

$$= \frac{(1+1)^{3/2}}{-32/3a}$$

$$= \frac{3a\,(2)^{3/2}}{-32}$$

$$= \frac{-2\sqrt{2}\,(3a)}{32}$$

$$\boxed{\rho = -\frac{\sqrt{2}\,(3a)}{16}}$$ //.

2. Show that the radius of curvature for the catenary of uniform strength $y = a \log \sec(x/a)$ is $a \sec(x/a)$.

Sol:

$$y = a \log \sec(x/a)$$

$$y_1 = \frac{a \sec(x/a)\tan(x/a)\left(\frac{1}{a}\right)}{\sec(x/a)}$$

$$y_1 = \tan(x/a)$$

$$y_2 = \frac{\sec^2(x/a)}{a}$$

WKT,

$$\rho = \frac{\left[1+(y_1)^2\right]^{3/2}}{y_2} = \frac{a\left[1+\tan^2(x/a)\right]^{3/2}}{\sec^2(x/a)}$$

$$= \frac{a\left[\sec^{\cancel{2}}(x/a)\right]^{3/\cancel{2}}}{\sec^2(x/a)}$$

$$\boxed{\rho = a \sec(x/a)}$$

3. Find the radius of curvature for the curve $y^2 = \frac{4a^2(2a-x)}{x}$, where curve meets the x-axis. (12)

Sol:-

If the curve meets x-axis then $y = 0$

$$\therefore \quad 0 = \frac{4a^2(2a-x)}{x}$$

$$4a^2(2a-x) = 0$$

$$2a - x = 0$$

$$x = 2a$$

Thus, $(2a, 0)$ be the point on the curve at which we have to find ρ.

$$y^2 = \frac{4a^2(2a-x)}{x}$$

$$2yy_1 = \frac{x[-4a^2] - (8a^3 - 4a^2x)(1)}{x^2}$$

$$2yy_1 = \frac{-8a^3}{x^2}$$

At $(2a, 0)$

$$① \Rightarrow y_1 = \frac{4a^3}{0} = \infty$$

$$y_1 = \frac{-8a^3}{2yx^2}$$

$$y_1 = \frac{-4a^3}{yx^2} \rightarrow ①$$

~~Again, $y_2 = -4a^3$~~ [

At $(2a, 0)$, y_1 become ∞, hence consider $\frac{dx}{dy}$

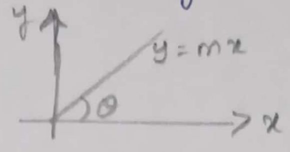
$$\frac{dx}{dy} = -\frac{x^2y}{4a^3} \rightarrow ① \qquad \frac{d^2x}{dy^2} = \frac{-1}{4a^3}\left[x^2 + y\,2x\,x_1\right]$$

at $(2a, 0)$, $x_1 = 0$ $\qquad$ at $(2a, 0)$, $x_2 = \frac{-1}{a}$

Now, $$\rho = \frac{[1+(x_1)^2]^{3/2}}{x_2} = \frac{(1+0)^{3/2}}{-1/a} = -a$$

$$|\rho| = |-a| = a \; //.$$

4. Find the radius of curvature of the curve $\sqrt{x} + \sqrt{y} = 4$ at the point where it cuts the line passing through origin making an angle 45° with x-axis.



$m = \tan\theta$

If $m = 1$ then $y = x,\ \theta = 45^\circ$

Sol:- The equation of the line passing through origin with an angle 45° is $y = x$

we shall find the point of intersection of this line with the given curve $\sqrt{x} + \sqrt{y} = 4 \rightarrow (1)$

Put $y = x$ in (1)

$$\sqrt{x} + \sqrt{x} = 4$$

$$2\sqrt{x} = 4$$

$$\sqrt{x} = 2$$

$$\boxed{x = 4} \quad \text{since } y = x, \quad \boxed{y = 4}$$

$\therefore (4, 4)$ is the point on the curve at which we have to find 'ρ'.

Now,

$$\cancel{\sqrt{x} + \sqrt{y} = 4}$$

$$\cancel{\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \times y_1 = 0}$$

$$\cancel{\frac{1}{2\sqrt{y}} \times y_1 = 0 - \frac{1}{2\sqrt{x}}}$$

$$\cancel{\frac{1}{\sqrt{y}} \times y_1 = \frac{1}{\sqrt{x}}}$$

$$\cancel{y_1 = \frac{2 - \sqrt{x}}{\sqrt{y}}}$$

$$\sqrt{x} + \sqrt{y} = 4$$

$$\frac{1}{2\sqrt{x}} + \frac{y_1}{2\sqrt{y}} = 0$$

$$\frac{y_1}{\sqrt{y}} = -\frac{1}{\sqrt{x}}$$

$$y_1 = -\sqrt{\frac{y}{x}} \rightarrow (1)$$

at $(4, 4)$ (1) $\Rightarrow$

$$\boxed{y_1 = -1}$$

Diff. (1) Again,

$$y_2 = -\left[\frac{(\sqrt{x})\frac{y_1}{2\sqrt{y}} - \sqrt{y}\,\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2}\right] \rightarrow (2)$$

at $(4, 4)$, (2) $\Rightarrow$

$$y_2 = -\left[\frac{\frac{-2}{4} - \frac{1}{2}}{4}\right]$$

$$\boxed{y_2 = 1/4}$$

Now,

$$\rho = \frac{\left[1 + (y_1)^2\right]^{3/2}}{y_2}$$

$$= \frac{(2)^{3/2}}{1/4}$$

$$\boxed{\rho = 8\sqrt{2}}$$

5. If ρ be the radius of curvature at any point $P(x, y)$ on the parabola $y^2 = 4ax$, then show that ρ^2 varies as $(SP)^3$ where S is a focus of the parabola

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Sol:-

$$y^2 = 4ax$$

$$2yy_1 = 4a$$

$$y_1 = \frac{2a}{y}$$

$$y_2 = -\frac{2a}{y^2}(y_1)$$

Now,

$$\rho = \frac{(1+(y_1)^2)^{3/2}}{y_2}$$

$$= \frac{\left[1 + \left(\frac{2a}{y}\right)^2\right]^{3/2}}{-2ay_1/y^2}$$

$$= \frac{\left(1 + \frac{4a^2}{y^2}\right)^{3/2} y^2}{-2ay_1}$$

$$\rho = \frac{(y^2+4a^2)^{3/2}}{-2ay_1y} \quad \text{But } y_1 = \frac{2a}{y}$$

$$= \frac{(y^2+4a^2)^{3/2}}{-4a^2} \quad \text{But } y^2 = 4ax$$

$$= \frac{(4ax+4a^2)^{3/2}}{-4a^2}$$

$$\rho = \frac{-1}{4a^2}(4a)^{3/2}(x+a)^{3/2}$$

SBS

$$\rho^2 = \frac{+1}{(4a^2)^2}(4a)^3(x+a)^3$$

$$\rho^2 = \frac{4}{a}(x+a)^3 \rightarrow (1)$$

To find $(x+a)$ in (1)

The co-ordinates of the focus of the parabola is $S = (a, 0)$ and $P(x, y)$ be any point on the parabola.

Now, $SP = \sqrt{(x-a)^2 + (y-0)^2}$

$$= \sqrt{x^2 - 2ax + a^2 + 4ax}$$

$$= \sqrt{x^2 + 2ax + a^2}$$

$$= \sqrt{(x+a)^2}$$

$$SP = x + a \rightarrow (2)$$

$(1) \Rightarrow$

$$\rho^2 = \frac{4}{a}(SP)^3$$

$$\rho^2 \propto (SP)^3 \text{ //.}$$

Note :-

6. Find the radius of curvature of the curve $r = ae^{\theta \cot\alpha}$ where, a & α are constants.

Sol:-

$$r = ae^{\theta \cot\alpha}$$

$$\log r = \log a + \theta \cot\alpha \log e$$

Diffu w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \cot\alpha$$

$$\cot\phi = \cot\alpha$$

$$\phi = \alpha$$

Now,

$$p = r\sin\phi$$

$$p = r\sin\alpha$$

Diffu w.r.t r

$$\frac{dp}{dr} = \sin\alpha$$

WKT,

$$\rho = r\frac{dr}{dp}$$

$$\rho = \frac{r}{\sin\alpha}$$

$$\boxed{\frac{\rho}{r} = \operatorname{cosec}\alpha \text{ (constant)}} //.$$

7. Show that the radius of curvature of the curve $r^n = a^n \cos n\theta$ varies inversly as r^{n-1}

Sol:-

$$r^n = a^n \cos n\theta$$

$$n\log r = n\log a + \log \cos n\theta$$

Diffu w.r.t θ

$$n\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{-n\sin n\theta}{\cos n\theta}$$

$$\cot\phi = -\tan n\theta$$

$$\cot\phi = \cot(\pi/2 + n\theta)$$

$$\phi = \frac{\pi}{2} + n\theta.$$

Now,

$$p = r\sin\phi$$

$$p = r\sin(\pi/2 + n\theta)$$

$$p = r\cos n\theta$$

But $\cos n\theta = \frac{r^n}{a^n}$

$$\therefore p = r\frac{r^n}{a^n}$$

$$p = \frac{r^{n+1}}{a^n}$$

Diff. w.r.t r

$$\frac{dp}{dr} = \frac{(n+1)r^n}{a^n}$$

WKT,

$$\rho = r\frac{dr}{dp}$$

$$\rho = \frac{r a^n}{(n+1)r^n} = \frac{a^n}{(n+1)r^{n-1}}$$

$$\therefore \boxed{\rho \propto \frac{1}{r^{n-1}}} //.$$

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8. Show that the curve $r(1-\cos\theta) = 2a$, the radius of curvature ρ^2 varies as r^3.

Sol:-

$$r(1-\cos\theta) = 2a \rightarrow (1)$$

$$\log r + \log(1-\cos\theta) = \log(2a)$$

Diffn w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0$$

$$\cot\phi = -\cot\theta/2$$

$$\phi = -\theta/2$$

Now,

$$p = r\sin\phi$$

$$p = r\sin(-\theta/2)$$

$$p = -r\sin\theta/2$$

But from (1),

$$r(1-\cos\theta) = 2a$$

$$r\,2\sin^2\theta/2 = 2a$$

$$\sin\theta/2 = \sqrt{\frac{a}{r}}$$

$$\therefore\ p = -r\sqrt{\frac{a}{r}}$$

$$p = -\sqrt{ar}$$

Diffn w.r.t r

$$\frac{dp}{dr} = \frac{-\sqrt{a}}{2\sqrt{r}}$$

WKT,

$$\rho = r\frac{dr}{dp}$$

$$\rho = -r\frac{2\sqrt{r}}{\sqrt{a}}$$

$$\rho^2 = \frac{4r^3}{a}$$

$$\boxed{\rho^2 \propto r^3}$$

//.

9. Show that the curve $r(1+\cos\theta) = 2a$, the radius of curvature ρ^2/r^3 is the constant.

Sol:-

$$r(1+\cos\theta) = 2a \rightarrow (1)$$

$$\log r + \log(1+\cos\theta) = \log 2a$$

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{-\sin\theta}{1+\cos\theta} = 0$$

$$\cot\phi = \tan\theta/2$$

$$\phi = \frac{\pi}{2} - \frac{\theta}{2}$$

Now,

$$p = r\sin\phi$$

$$p = r\sin(\pi/2 - \theta/2)$$

$$p = r\cos\theta/2$$

But from (1),

$$r(1+\cos\theta) = 2a$$

$$r\,2\cos^2\theta/2 = 2a$$

$$\cos\theta/2 = \sqrt{\frac{a}{r}}$$

$$\therefore p = \frac{r\sqrt{a}}{\sqrt{r}}$$

$$p = \sqrt{ar}$$

Diff w.r.t r

$$\frac{dp}{dr} = \frac{\sqrt{a}}{2\sqrt{r}}$$

WKT,

$$\rho = r\frac{dr}{dp}$$

$$= r\frac{2\sqrt{r}}{\sqrt{a}}$$

$$\rho = \frac{2r^{3/2}}{\sqrt{a}}$$

$$\rho^2 = \frac{4r^3}{a}$$

$$\boxed{\frac{\rho^2}{r^3} = \frac{4}{a} \text{ (constant)}}$$

10. Show that for the curve $r = a(1+\cos\theta)$ is ρ^2/r is a constant.

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Raising b.s to the power 1/3, we get

$$\left(\frac{2\rho}{a}\right)^{2/3} = \frac{x^2}{y^2}\left(1+\frac{y^4}{x^4}\right) \text{ (or) } \left(\frac{2\rho}{a}\right)^{2/3} = \frac{x^2}{y^2}+\frac{y^2}{x^2}$$

Radius of curvature in parametric form:

The Radius of curvature in parametric form is given by $\rho = \dfrac{(\dot{x}^2+\dot{y}^2)^{3/2}}{(\dot{x}\ddot{y}-\dot{y}\ddot{x})}$ taking $\dot{x} = f'(t)$, $\dot{y} = g'(t)$, $\ddot{x} = f''(t)$, $\ddot{y} = g''(t)$

Problems: Find the Radius of curvature at a point 't' on the curve for the following curves.

① $x = a(\theta+\sin\theta)$, $y = a(1-\cos\theta)$

Soln: $\dot{x} = a(1+\cos\theta)$ $\dot{y} = a\sin\theta$

$\ddot{x} = -a\sin\theta$ $\ddot{y} = a\cos\theta$

we have $\rho = \dfrac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \dfrac{(a^2(1+\cos\theta)^2+a^2\sin^2\theta)^{3/2}}{a(1+\cos\theta)a\cos\theta - a\sin\theta(-a\sin\theta)}$

$= \dfrac{a^3[2(1+\cos\theta)]^{3/2}}{a^2(1+\cos\theta)} = \dfrac{a[4\cos^2\theta/2]^{3/2}}{2\cos^2\theta/2}$

$\rho = 4a\cos\theta/2.$

② $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$

$\dot{x} = a[-\sin t + t\cos t + \sin t]$, $\dot{y} = a[\cos t + t\sin t - \cos t]$

$\dot{x} = a[t\cos t]$ $y = a[t\sin t]$

$\ddot{x} = a[-t\sin t + \cos t]$, $\ddot{y} = a[t\cos t + \sin t]$

$[\dot{x}^2+\dot{y}^2]^{3/2} = [a^2t^2\cos^2t + a^2t^2\sin^2t]^{3/2} = (a^2t^2)^{3/2} = a^3t^3$

$\dot{x}\ddot{y}-\dot{y}\ddot{x} = (at\cos t)[at\cos t + \sin t] - (at\sin t)[-at\sin t + a\cos t]$

$= a^2t^2\cos^2 t + a^2 t \sin t \cos t + a^2t^2\sin^2 t - a^2 t \sin t \cos$

$= a^2t^2$.

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{a^3t^3}{a^2t^2} = at.$$

③ P.T the radius of curvature at a point 't' of the ellipse $x = a\cos t$, $y = b\sin t$ is given by $\rho = \frac{a^2b^2}{p^3}$

Soln: The parametric equation of the ellipse are

$x = a\cos t$, $y = b\sin t$

$\dot{x} = -a\sin t$. $\dot{y} = b\cos t$

$\ddot{x} = -a\cos t$ $\ddot{y} = -b\sin t$

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}{ab} \rightarrow ①$$

Equation of the tangent to the ellipse at the point 't' is $\frac{x\cos t}{a} + \frac{y\sin t}{b} = 1$

The length of the $\perp^r$ from the centre of the ellipse on this tangent is, $\left[p = \frac{Ax_1+By_1+C}{\sqrt{A^2+B^2}}\right]$

$$p = \frac{1}{\left(\frac{\cos^2 t}{a^2}+\frac{\sin^2 t}{b^2}\right)^{1/2}} = \frac{ab}{\sqrt{a^2\sin^2 t + b^2\cos^2 t}} \rightarrow ②$$

using ② in ①

$$\rho = \frac{(ab)^3}{p^3}\cdot\frac{1}{ab} = \frac{a^2b^2}{p^3}$$

④ $x = a\cos t$ $y = a\sin t$

$\dot{x} = -a\sin t$ $\dot{y} = a\cos t$

$\ddot{x} = -a\cos t$ $\ddot{y} = -a\sin t$.

$$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y}-\dot{y}\ddot{x}} = \frac{(a^2\sin^2 t + a^2\cos^2 t)^{3/2}}{(-a\sin t)(-a\sin t)-a\cos t(-a\cos t)} = \frac{(a^2)^{3/2}}{a^2\sin^2 t + a^2\cos^2 t}$$

$$= \frac{a^3}{a^2} = a = \text{Const}.$$

$\therefore$ The radius of curvature of the given curve is constant.

(g)

$x = a(\cos t + \log \tan t/2)$, $y = a \sin t$

$\dot{x} = a[-\sin t + \frac{1}{\tan t/2} \sec^2 t/2 \cdot \frac{1}{2}]$, $\dot{y} = a\cos t$

$= a[-\sin t + \frac{1}{2\cos t/2 \sin t/2}]$ $\qquad \ddot{y} = -a\sin t$

$= a[-\sin t + \frac{1}{\sin t}] = a[\frac{1-\sin^2 t}{\sin t}]$

$\dot{x} = \frac{a\cos^2 t}{\sin t} = a\cot t \cdot \cos t.$

$\ddot{x} = a\{\cot t(-\sin t) + \cos t(-\operatorname{cosec}^2 t)\}$

$= a\{-\cot t \cdot \sin t - \cot t \operatorname{cosec} t\}$

$\ddot{x} = -a\cot t(\operatorname{cosec} t + \sin t)$

$\rho = \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} = \frac{(a^2\cot^2 t \cos^2 t + a^2\cos^2 t)^{3/2}}{a\cot t \cdot \cos t(-a\sin t) - a\cos t(-a\cot t(\operatorname{cosec} t + \sin t))}$

$= \frac{(a^2\cos^2 t)^{3/2} \cdot (1+\cot^2 t)^{3/2}}{-a^2\cos^2 t + a^2\cos^2 t(\operatorname{cosec}^2 t + 1)} = \frac{a^3\cos^3 t \operatorname{cosec}^3 t}{a^2\cos^2 t \operatorname{cosec}^2 t}.$

$= a\cos t \operatorname{cosec} t = a\cot t.$

Radius of curvature in pedal form:

Radius of curvature in pedal form is given by

$\boxed{\rho = r\frac{dr}{dp}}$ This formula is also known as $(p-r)$ form.

Problems: Find the radius of curvature of the following curves.

① $2ap^2 = r^3$

Diff w.r.t r we get $4ap\frac{dp}{dr} = 3r^2$ or $\frac{dr}{dp} = \frac{4ap}{3r^2}$

$$\rho = r\frac{dr}{dp} = r\cdot\frac{4ap}{3r^2} = \frac{4a\,r^{3/2}}{3r\sqrt{2a}} = \frac{2}{3}\sqrt{2ar}$$

(2) $p^2 = r^2 - a^2$

Diff w.r.t 'r', we get

$$2p\frac{dp}{dr} = 2r, \quad \frac{dp}{dr} = \frac{2r}{2p} \Rightarrow \frac{r}{p}, \quad \frac{dr}{dp} = p/r$$

$$\rho = r\frac{dr}{dp} = r\cdot\frac{p}{r} = p = \sqrt{r^2-a^2}$$

$$\rho = \sqrt{r^2-a^2}.$$

(3) $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$

Diff w.r.t 'r' we get

$$\frac{-2}{p^3}\frac{dp}{dr} = \frac{-2r}{a^2b^2} \quad (or) \quad \frac{dp}{dr} = \frac{rp^3}{a^2b^2} \quad (or) \quad \frac{dr}{dp} = \frac{a^2b^2}{p^3 r}$$

$$\rho = r\frac{dr}{dp} = \frac{ra^2b^2}{p^3 r} = \frac{a^2b^2}{p^3} = a^2b^2\left[\frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}\right]^{3/2}$$

$$\rho = \frac{(a^2+b^2-r^2)^{3/2}}{ab}$$

(4) Prove the radius of curvature formula in tangential form $\rho = p + \frac{d^2p}{d\psi^2}$.

Pf: w.k.t $\rho = \frac{ds}{d\psi}$ Also $\rho = r\frac{dr}{dp}$

$$\frac{ds}{d\psi} = r\frac{dr}{ds}\cdot\frac{ds}{dp} = r\cos\phi\frac{ds}{dp}$$

$$\Rightarrow \frac{dp}{ds}\cdot\frac{ds}{d\psi} = r\cos\phi$$

By Defn $\quad \frac{dp}{d\psi} = r\cos\phi \rightarrow (1)$

$$p = r\sin\phi \rightarrow (2)$$

Squaring & adding (1) & (2)

$$\left(\frac{dp}{d\psi}\right)^2 + p^2 = r^2$$

Diff w.r.t ψ, we get

$$2\frac{dp}{d\psi}\cdot\frac{d^2p}{d\psi^2} + 2p\frac{dp}{d\psi} = 2r\frac{dr}{d\psi}$$

$$\Rightarrow \frac{d^2p}{d\psi^2} + p = r\frac{dr}{d\psi}\cdot\frac{d\psi}{dp}$$

$$\Rightarrow \frac{d^2p}{d\psi^2} + p = r\frac{dr}{dp} \quad \text{or} \quad \frac{d^2p}{d\psi^2} + p = \rho.$$

$\therefore \rho = \dfrac{d^2p}{d\psi^2} + p \Rightarrow$ Is the formula for ρ in tangential polar form.

(5) Find the radius of curvature for the curve

$$pa^m = r^{m+1}$$

Soln:

$$pa^m = r^{m+1}$$

Diff w.r.t 'r' we get

$$a^m\frac{dp}{dr} = (m+1)r^m$$

$$\frac{dp}{dr} = \frac{(m+1)r^m}{a^m} \quad (\text{or}) \quad \frac{dr}{dp} = \frac{a^m}{(m+1)r^m}$$

$$r\frac{dr}{dp} = \frac{a^m}{(m+1)r^{m-1}}$$

$$\rho = \frac{a^m}{(m+1)r^{m-1}}.$$

Radius of curvature of polar form:

Radius of curvature in polar form is given by

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} \quad \text{where } r_1 = \frac{dr}{d\theta},\ r_2 = \frac{d^2r}{d\theta^2}$$

Problems:

① S.T the radius of curvature at any point on the cardioid $r = a(1-\cos\theta)$ is $\frac{2}{3}\sqrt{2ar}$ also S.T. $\frac{\rho^2}{r}$ is a constant.

Soln: $r = a(1-\cos\theta)$

$$r_1 = \frac{dr}{d\theta} = a\sin\theta \ \& \ r_2 = \frac{d^2r}{d\theta^2} = a\cos\theta$$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2} = \frac{[a^2(1-\cos\theta)^2+a^2\sin^2\theta]^{3/2}}{a^2(1-\cos\theta)^2+2a^2\sin^2\theta-a(1-\cos\theta)a\cos\theta}$$

$$= \frac{a(2-2\cos\theta)^{3/2}}{3(1-\cos\theta)} = \frac{2a\sqrt{2}}{3}(1-\cos\theta)^{1/2}$$

$$\rho = \frac{2a\sqrt{2}}{3}(2\sin^2\theta/2)^{1/2} = \frac{4}{3}a\sin\theta/2.$$

From the eqn of the curve $r = a(1-\cos\theta) = 2a\sin^2\theta/2$

$$\sin^2\theta/2 = \frac{r}{2a}, \quad \sin\theta/2 = \sqrt{\frac{r}{2a}}$$

$$\rho = \frac{4}{3}a\sqrt{\frac{r}{2a}} = \frac{2}{3}\sqrt{2ar}$$

Now $\rho^2 = \frac{8ar}{9} \Rightarrow \frac{\rho^2}{r} = \frac{8a}{9}$ = constant.

② Prove that for the curve $r^m = a^m\cos m\theta$, $\rho = \frac{a^m}{(m+1)r^{m-1}}$.

Soln: $r^m = a^m\cos m\theta$.

Take log on b.s.

$m\log r = m\log a + \log\cos m\theta$

Diff w.r.t θ

$$\frac{m}{r}\frac{dr}{d\theta} = \frac{-m\sin m\theta}{\cos m\theta} \quad \text{or} \quad \frac{dr}{d\theta} = -r\tan m\theta$$

$$\frac{d^2r}{d\theta^2} = -mr\sec^2 m\theta - \tan m\theta \frac{dr}{d\theta}$$

$$\rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1-rr_2} = \frac{(r^2+r^2\tan^2 m\theta)^{3/2}}{r^2+2r^2\tan^2 m\theta - r^2(\tan^2 m\theta - m\sec^2 m\theta)}$$

$$= \frac{[r^2(1+\tan^2 m\theta)]^{3/2}}{r^2+r^2\tan^2 m\theta + mr^2\sec^2 m\theta} = \frac{r^3\sec^3 m\theta}{r^2\sec^2 m\theta(1+m)}$$

$$= \frac{r\sec m\theta}{1+m} = \frac{r a^m}{(1+m)r^m} \quad \because \cos m\theta = \frac{r^m}{a^m}$$

$$\rho = \frac{a^m}{(1+m)r^{m-1}}$$

(3) S.T. for the curve $r(1-\cos\theta) = 2a$, ρ^2 varies as r^3

Soln: Given $r(1-\cos\theta) = 2a$

$$\log r + \log(1-\cos\theta) = \log 2a$$

Diff w.r.t θ

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{\sin\theta}{1-\cos\theta} = 0$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta} = \frac{-2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2} = -\cot\theta/2$$

$$\frac{dr}{d\theta} = -r\cot\theta/2 = r_1$$

Diff again w.r.t θ we get

$$r_2 = -\{r(-\tfrac{1}{2}\operatorname{cosec}^2\theta/2) + \cot\theta/2\, r_1\}$$

$$= -\{-\tfrac{r}{2}\operatorname{cosec}^2\theta/2 + \cot\theta/2(-r\cot\theta/2)\}$$

$$= \tfrac{r}{2}\operatorname{cosec}^2\theta/2 + r\cot^2\theta/2$$

we have $\rho = \dfrac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2-rr_2}$

$$= \frac{(r^2 + r^2\cot^2\theta/2)^{3/2}}{r^2 + 2r^2\cot^2\theta/2 - \frac{r^2}{2}\operatorname{cosec}^2\theta/2 - r^2\cot^2\theta/2}$$

$$\rho = \frac{[r^2(1+\cot^2\theta/2)]^{3/2}}{r^2[1+\cot^2\theta/2 - \frac{1}{2}\operatorname{cosec}^2\theta/2]}$$

$$\rho = \frac{r\operatorname{cosec}^3\theta/2}{\frac{1}{2}\operatorname{cosec}^2\theta/2} = 2r\operatorname{cosec}\theta/2$$

We have $r(1-\cos\theta) = 2a$

$$\Rightarrow r = \frac{2a}{1-\cos\theta} = \frac{2a}{2\sin^2\theta/2} = a\operatorname{cosec}^2\theta/2$$

$$\Rightarrow \frac{r}{a} = \operatorname{cosec}^2\theta/2 \text{ or } \operatorname{cosec}\theta/2 = \sqrt{r/a}$$

$$\rho = 2r\sqrt{r/a} \text{ (or) } \rho^2 = 4r^2 \cdot \frac{r}{a} = \frac{4r^3}{a}$$

$$\rho^2 = \left(\frac{4}{a}\right) r^3.$$

Thus ρ varies as r^3 which is the required result.

(4) S.T. for the curve $r = a\sin n\theta$ the radius of curvature at the pole is $\frac{na}{2}$.

Soln: Given, $r = a\sin n\theta$, $r_1 = an\cos n\theta$, $r_2 = -an^2\sin n\theta$

When $\theta = 0$, $r = 0$, $r_1 = an$, $r_2 = 0$

$$\rho = \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} = \frac{(0 + a^2n^2)^{3/2}}{0 + 2a^2n^2 - 0} = \frac{a^3n^3}{2a^2n^2}$$

$$\rho = \frac{an}{2} \text{ or } \rho = \frac{na}{2}.$$