第九章　多元函数微分学

题组一：概念题

1\. 设 $f(x,y)=\begin{cases}0\ , & xy=0\\ 1\ , & xy\neq 0\end{cases}$ （1）求 $\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y}.$

（2）讨论 $f(x,y)$ 的连续性.

（3）讨论 $f(x,y)$ 在 (0, 0) 是否可微.

解(1): 由题目可知: 在两坐标轴上 $f(x,y)=0,$

在其它点 $f(x,y)=1.$ 显然

$$f_x(0,0)=0,\quad f_y(0,0)=0.$$

$$\frac{\partial f}{\partial x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}$$

$$=\begin{cases}\lim\limits_{\substack{\Delta x\to 0\\ y=0}}\dfrac{f(x+\Delta x,0)-f(x,0)}{\Delta x}=0\\ \lim\limits_{\substack{\Delta x\to 0\\ x=0,y\neq 0}}\dfrac{f(0+\Delta x,y)-f(0,y)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{1}{\Delta x}=\infty\\ \lim\limits_{\substack{\Delta x\to 0\\ x\neq 0,y\neq 0}}\dfrac{f(x+\Delta x,y)-f(x,y)}{\Delta x}=\lim\limits_{\Delta x\to 0}\dfrac{1-1}{\Delta x}=0\end{cases}$$

所以 $\dfrac{\partial f}{\partial x}=\begin{cases}\infty, & x=0, y\neq 0\text{时}.\\ 0, & \text{其它}.\end{cases}$ 同理 $\dfrac{\partial f}{\partial y}=\begin{cases}\infty, & x\neq 0, y=0\text{时}.\\ 0, & \text{其它}.\end{cases}$

解(2): 显然 $f(0,0)=0.$

沿 $y=0$， $\lim\limits_{\substack{x\to 0\\ y=0}} f(x,y)=\lim\limits_{x\to 0} f(x,0)=0$

沿 $y=x$， $\lim\limits_{\substack{x\to 0\\ y=x}} f(x,y)=\lim\limits_{x\to 0} f(x,x)=1$

所以 $\lim\limits_{\substack{x\to 0\\ y\to 0}} f(x,y)$ 不存在.

从而函数 $f(x,y)$ 在(0,0)点不连续， 故

函数在整个平面上不连续.

解(3): 因函数 $f(x,y)$ 在(0,0)点不连续，

故函数在(0,0)点不可微.

2. 设 $f(x,y)=\begin{cases} x+y+\dfrac{x^3y}{x^4+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$,

证明: 函数 $f(x,y)$在 (0, 0) 不可微但沿任意方向

$\vec{l}=(\cos\alpha,\cos\beta)$ 的方向导数存在.

证明:

$$f_x(0,0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x,0)-f(0,0)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\Delta x+0+\dfrac{\Delta x^3\cdot 0}{\Delta x^4+0^2}}{\Delta x}=1$$

所以 $f_x(0,0)=1$ 同理 $f_y(0,0)=1$

考察 $\Delta z = f(0+\Delta x, 0+\Delta y) - f(0,0)$

是否等于 $f_x(0,0)\Delta x + f_y(0,0)\Delta y + o(\rho)$

因为

$$\frac{f(0+\Delta x, 0+\Delta y) - f(0,0) - [f_x(0,0)\Delta x + f_y(0,0)\Delta y]}{\rho}$$

$$= \frac{f(\Delta x, \Delta y) - \Delta x - \Delta y}{\rho} = \frac{(\Delta x)^3 \Delta y}{(\Delta x)^4 + (\Delta y)^2} \cdot \frac{1}{\sqrt{(\Delta x)^2 + (\Delta y)^2}}$$

$$\underline{\underline{\text{沿}\Delta y = (\Delta x)^2}} \frac{(\Delta x)^5}{2(\Delta x)^4 |\Delta x| \sqrt{1+(\Delta x)^2}} \xrightarrow[\Delta x \to 0^+]{} \frac{1}{2} \neq 0.$$

所以函数$f(x,y)$在 (0, 0) 不可微.

$$\left.\frac{\partial f}{\partial l}\right|_{(0,0)} = \lim_{\rho\to 0^+}\frac{f(\Delta x,\Delta y)-f(0,0)}{\rho}$$

$$= \lim_{\rho\to 0^+}\frac{f(\rho\cos\alpha,\rho\cos\beta)}{\rho}$$

$$= \cos\alpha+\cos\beta$$

3. 解下列各题

(1) 设 f 有二阶连续偏导数,且 $f(x,2x)=x$,

$$f_1'(x,2x)=x^2,\ f_{12}''(x,2x)=x^3,\ \text{求}\ f_{22}''(x,2x).$$

解: $f(x,2x)=x$

$\Big\downarrow$ 两边同时关于x求导

$$\left.\begin{aligned} &f_1'+2f_2'=1 \\ &f_1'(x,2x)=x^2 \end{aligned}\right\} \longrightarrow f_2'=\frac{1-x^2}{2}$$

两边同时关于x求导

$$\longrightarrow \left.\begin{aligned} &f_{21}''+2f_{22}''=-x \\ &f_{21}''=f_{12}''(x,2x)=x^3 \end{aligned}\right\} \longrightarrow f_{22}''=\frac{-x-x^3}{2}$$

(2) 设$\dfrac{\partial z}{\partial x}=\dfrac{x^2+y^2}{x}$, $z(1,y)=\sin y$,

求 $z(x,y),(x>0)$.

解:

$$\frac{\partial z}{\partial x}=\frac{x^2+y^2}{x}\longrightarrow z=\int\frac{\partial z}{\partial x}dx \qquad =\int\frac{x^2+y^2}{x}dx$$

$$\longrightarrow \left.\begin{array}{l} z=\dfrac{1}{2}x^2+y^2\ln x+\varphi(y) \\ z(1,y)=\sin y \end{array}\right\}\longrightarrow \varphi(y)=\sin y-\frac{1}{2}$$

$$\longrightarrow \quad z(x,y)=\frac{1}{2}x^2+y^2\ln x+\sin y-\frac{1}{2}$$

题组二： 计算与证明

1. 计算下列各题

(1) 设 $z = e^{xy}\ln(x^2+y^2)$, 求 dz .

解: $dz = \dfrac{\partial z}{\partial x}dx + \dfrac{\partial z}{\partial y}dy$

$$= [ye^{xy}\ln(x^2+y^2) + e^{xy}\frac{2x}{x^2+y^2}]dx$$

$$+[xe^{xy}\ln(x^2+y^2) + e^{xy}\frac{2y}{x^2+y^2}]dy$$

(2) 设 $u = x^{yz} + x^{y^z}$， 求 du .

解: $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

$$= (yzx^{yz-1} + y^z x^{y^z-1})dx$$

$$+(zx^{yz}\ln x + zy^{z-1}x^{y^z}\ln x)dy$$

$$+(yx^{yz}\ln x + y^z x^{y^z}\ln x \ln y)dz$$

(3) 设 $z=x^2e^{-y}\sin\frac{y}{x}$, 求点 $(\frac{2}{\pi},0)$ 处的二阶混合偏导数.

解:

$$\frac{\partial z}{\partial y}=x^2[-e^{-y}\sin\frac{y}{x}+e^{-y}\cos\frac{y}{x}(\frac{1}{x})]$$

$$=-e^{-y}(x^2\sin\frac{y}{x}-x\cos\frac{y}{x})$$

$$\left.\frac{\partial z}{\partial y}\right|_{y=0}=x \qquad \therefore \left.\frac{\partial^2 z}{\partial y\partial x}\right|_{\substack{x=\frac{2}{\pi}\\ y=0}}=1$$

2. 设 $z = f(x^2 + y^2, x^2 - y^2), y = x + \varphi(x)$,

其中 f, φ 为可微函数, 求 dz .

解: 设 $u = x^2 + y^2, v = x^2 - y^2$, 则 $z = f(u, v)$

于是 $dz = \dfrac{\partial z}{\partial u} du + \dfrac{\partial z}{\partial v} dv$

$$= f_1' \cdot d(x^2 + y^2) + f_2' \cdot d(x^2 - y^2)$$

$$\left.\begin{array}{r} = 2x(f_1' + f_2')dx + 2y(f_1' - f_2')dy \\ y = x + \varphi(x) \end{array}\right\} \Longrightarrow$$

$$dz = 2[x(f_1' + f_2') + (f_1' - f_2')(x + \varphi(x))(1 + \varphi'(x))]dx$$

3. 设 $z=\frac{1}{y}f(xy)+xf(\frac{y}{x})$, f 为可微函数, 求 dz .

解: $$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

$$=[\frac{1}{y}f'(xy)y+f(\frac{y}{x})+xf'(\frac{y}{x})(-\frac{y}{x^2})]\quad dx$$

$$+[-\frac{1}{y^2}f(xy)+\frac{1}{y}f'(xy)x+xf'(\frac{y}{x})\frac{1}{x}]\ dy$$

$$=[f'(xy)+f(\frac{y}{x})-\frac{y}{x^2}f'(\frac{y}{x})]dx$$

$$+[\frac{x}{y}f'(xy)+f'(\frac{y}{x})-\frac{1}{y^2}f(xy)]dy$$

4. 设 $z=f(x^2, e^{xy})$, f 有二阶连续偏导数, 求 dz, $\dfrac{\partial^2 z}{\partial x\partial y}$

解:

$$dz = f_1'\cdot d(x^2) + f_2'\cdot d(e^{xy})$$

$$= 2xf_1'\cdot dx + f_2'\cdot e^{xy}\cdot d(xy)$$

$$= 2xf_1'\cdot dx + f_2'\cdot e^{xy}\cdot (xdy + ydx)$$

$$= (2xf_1' + yf_2'\cdot e^{xy})dx + xe^{xy} f_2' dy$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}(xe^{xy} f_2')$$

$$= (e^{xy} + xye^{xy}) f_2' + xe^{xy}(f_{21}'' 2x + f_{22}''\cdot ye^{xy})$$

5. 设 $z=f(x,y)$ 满足方程 $6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}=0,$

证明：变换 $\begin{cases}\xi = x-2y\\ \eta = x+3y\end{cases}$ 可将方程化简为 $\frac{\partial^2 z}{\partial\xi\partial\eta}=0.$

解:

$$\left.\begin{array}{l}\xi = x-2y\\ \eta = x+3y\end{array}\right\}\longrightarrow \left.\begin{array}{l}x=\frac{1}{5}(3\xi+2\eta)\\ y=\frac{1}{5}(\eta-\xi)\\ z=f(x,y)\end{array}\right\}\longrightarrow$$

$$\longrightarrow\ \frac{\partial z}{\partial\xi}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial\xi}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial\xi}=\frac{3}{5}f_1'-\frac{1}{5}f_2'$$

$$\longrightarrow\ \frac{\partial^2 z}{\partial\xi\partial\eta}=\frac{\partial(\frac{3}{5}f_1'-\frac{1}{5}f_2')}{\partial\eta}$$

$$\frac{\partial^2 z}{\partial\xi\partial\eta} = \frac{\partial(\frac{3}{5} f_1' - \frac{1}{5} f_2')}{\partial\eta}$$

$$= \frac{3}{5}(f_{11}'' \frac{\partial x}{\partial\eta} + f_{12}'' \frac{\partial y}{\partial\eta}) - \frac{1}{5}(f_{21}'' \frac{\partial x}{\partial\eta} + f_{22}'' \frac{\partial y}{\partial\eta})$$

$$\left.\begin{aligned} &= \frac{1}{5}(6 f_{11}'' + f_{12}'' - f_{22}'') \\ &6\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\partial y} - \frac{\partial^2 z}{\partial y^2} = 0 \end{aligned}\right\} \longrightarrow \frac{\partial^2 z}{\partial\xi\partial\eta} = 0$$

6. 设方程 $F(x+\frac{z}{y}, y+\frac{z}{x})=0$ 确定了隐函数 $z=z(x,y)$

求 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}$.

解: 利用公式法.

$$F_x=F_1'+F_2'(-\frac{z}{x^2}) \quad F_y=F_1'(-\frac{z}{y^2})+F_2' \quad F_z=F_1'\frac{1}{y}+F_2'\frac{1}{x}$$

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{F_1'+F_2'(-\frac{z}{x^2})}{F_1'\frac{1}{y}+F_2'\frac{1}{x}} \qquad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{F_1'(-\frac{z}{y^2})+F_2'}{F_1'\frac{1}{y}+F_2'\frac{1}{x}}$$

所以 $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z-xy.$

7. 设函数 F, u 均有二阶连续偏导数，且 F_1' , F_2' 不同时为零，$F(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0$ 证明：$\frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2$.

解:

$$F(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0 \xrightarrow{\boxed{\text{分别对}x,y\text{求偏导数}}} \left.\begin{aligned} F_1' \frac{\partial^2 u}{\partial x^2} + F_2' \frac{\partial^2 u}{\partial x \partial y} = 0 \\ F_1' \frac{\partial^2 u}{\partial x \partial y} + F_2' \frac{\partial^2 u}{\partial y^2} = 0 \end{aligned}\right\}$$

$$\longrightarrow \left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} \Big/ \frac{\partial^2 u}{\partial x \partial y} = -\frac{F_2'}{F_1'} \\ \frac{\partial^2 u}{\partial x \partial y} \Big/ \frac{\partial^2 u}{\partial y^2} = -\frac{F_2'}{F_1'} \end{aligned}\right\} \longrightarrow \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2$$

8. 已知 $x+y-z=e^z$， $xe^x=\tan t$， $y=\cos t$，

求 $\left.\dfrac{dz}{dt}\right|_{t=0}$ 及 $\left.\dfrac{d^2z}{dt^2}\right|_{t=0}$．

解:

$$\begin{cases} x+y-z=e^z \\ xe^x=\tan t \\ y=\cos t \end{cases} \xrightarrow{\boxed{\text{两边同时对t求导}}} \begin{cases} \dfrac{dx}{dt}+\dfrac{dy}{dt}-\dfrac{dz}{dt}=e^z\dfrac{dz}{dt} \\ e^x\dfrac{dx}{dt}+xe^x\dfrac{dx}{dt}=\sec^2 t \\ \dfrac{dy}{dt}=-\sin t \end{cases}$$

$$\longrightarrow \quad \frac{dz}{dt}=\frac{1}{1+e^z}\left(\frac{dx}{dt}+\frac{dy}{dt}\right)$$

$$=\frac{1}{1+e^{z}}\left(\frac{\sec^{2} t}{e^{x}+\tan t}-\sin t\right)$$

当 $t=0$ 时， $x=0, y=1, z=0$ $\therefore \left.\frac{dz}{dt}\right|_{t=0}=\frac{1}{2}$

$$\left.\frac{d^{2} z}{dt^{2}}\right|_{t=0}=\left[\frac{-e^{z} \frac{dz}{dt}}{(1+e^{z})^{2}}\left(\frac{\sec^{2} t}{e^{x}+\tan t}-\sin t\right)\right.$$

$$+\frac{1}{1+e^{z}}\left(\frac{2 \sec^{2} t \tan t(e^{x}+\tan t)-\sec^{2} t(e^{x} \frac{dx}{dt}+\sec^{2} t)}{(e^{x}+\tan t)^{2}}\right.$$

$$\left.\left.-\cos t)\right]\right|_{t=0}=-\frac{13}{8}$$

9. 求函数 $f(x,y,z)=\dfrac{\sqrt{x^2+y^2}}{xyz}$ 在点 M(-1,3,-3) 沿曲线 $x=-t^2$，$y=3t^2$，$z=-3t^3$ 在该点的切线方向上的方向导数和梯度.

解: 点 M 处曲线切线的方向向量为:

$$\vec{s}=(-2t,6t,-9t^2)\Big|_{t=1}=(-2,6,-9)$$

其方向余弦为: $\cos\alpha=-\dfrac{2}{11}$， $\cos\beta=\dfrac{6}{11}$， $\cos\gamma=\dfrac{-9}{11}$.

又知 $\left.\dfrac{\partial f}{\partial x}\right|_M=\left.\dfrac{d}{dx}\left(\dfrac{\sqrt{9+x^2}}{-9x}\right)\right|_{x=-1}=\dfrac{1}{\sqrt{10}}$,

$$\left.\frac{\partial f}{\partial y}\right|_M = -\frac{1}{27\sqrt{10}}, \qquad \left.\frac{\partial f}{\partial z}\right|_M = \frac{\sqrt{10}}{27}.$$

$$\text{所以}\left.\frac{\partial f}{\partial l}\right|_M = -\frac{5\sqrt{10}}{99}$$

$$grad f(-1,3,-3) = \left(\frac{1}{\sqrt{10}}, -\frac{1}{27\sqrt{10}}, \frac{10}{27\sqrt{10}}\right)$$

题组三：应用题

1. 过直线 $\begin{cases} 10x+2y-2z=27 \\ x+y-z=0 \end{cases}$ 作曲面 $3x^2+y^2-z^2=27$

的切平面，求此切平面的方程.

解: 设切点为 $M_0(x_0, y_0, z_0)$，则 $\vec{n}=(6x_0, 2y_0, -2z_0)$.

过直线的平面束方程为

$$(10x+2y-2z-27)+\lambda(x+y-z)=0$$

则 $\vec{n}_\lambda=(10+\lambda, 2+\lambda, -2-\lambda)$

由 $\vec{n} // \vec{n}_\lambda$ 可得 $\dfrac{10+\lambda}{3x_0}=\dfrac{2+\lambda}{y_0}=\dfrac{2+\lambda}{z_0}$ （1）

由于切点既在曲面上也在切平面上，

$$\therefore \quad 3x_0^2 + y_0^2 - z_0^2 = 27 \qquad （2）$$

$$(10+\lambda)x_0 + (2+\lambda)y_0 - (2+\lambda)z_0 - 27 = 0 \qquad （3）$$

联立（1）（2）（3）解之

$$\begin{cases} x_0 = 3 \\ y_0 = 1 \\ z_0 = 1 \\ \lambda = -1 \end{cases} \qquad \begin{cases} x_0 = -3 \\ y_0 = -17 \\ z_0 = 17 \\ \lambda = -19 \end{cases}$$

所求切平面的方程为

$$9x + y - z - 27 = 0$$

和

$$9x + 17y - 17z + 27 = 0$$

3. 求曲线 $\begin{cases} 2x^2+3y^2+z^2=47 \\ x^2+2y^2=z \end{cases}$ 在点(-2,1,6)处的切线方程.

解: $\begin{cases} 2x^2+3y^2+z^2=47 \\ x^2+2y^2=z \end{cases} \xrightarrow{\boxed{\text{关于}x\text{求导}}} \begin{cases} 4x+6yy'+2zz'=0 \\ 2x+4yy'=z' \end{cases}$

$\longrightarrow \begin{cases} y'|_{(-2,1,6)}=\frac{28}{27} \\ z'|_{(-2,1,6)}=\frac{4}{27} \end{cases} \longrightarrow \vec{s}=(1,\frac{28}{27},\frac{4}{27})$

$\longrightarrow \dfrac{x+2}{27}=\dfrac{y-1}{28}=\dfrac{z-6}{4}.$

4 证明曲线 $\Gamma: x = ae^t \cos t, y = ae^t \sin t, z = ae^t$

与锥面 $x^2 + y^2 = z^2$ 的各母线相交成定角。

证明：设母线 l 和曲线的交点为 (x,y,z)

$$l: \begin{cases} x^2 + y^2 - z^2 = 0 \\ cx - y = 0 \end{cases}$$

$$\vec{l} = (-2z, -2cz, -2x - 2yc) = -2\sqrt{1+c^2}\,x(1, c, \sqrt{1+c^2})$$

Γ在点 (x, y, z) 的切向量

$$\vec{a} = ae^t(\cos t - \sin t, \cos t + \sin t, 1) = (x - y, x + y, z)$$

$$= x(1-c, 1+c, \sqrt{1+c^2})$$

设 l 和 a 的夹角为 θ ，则

$$\cos\theta = \frac{\vec{a}\cdot\vec{l}}{|\vec{a}||\vec{l}|} = \frac{\sqrt{2}}{\sqrt{3}}$$

5. 在曲面 $a\sqrt{x}+b\sqrt{y}+c\sqrt{z}=1\,(a>0,b>0,c>0)$

上做切平面,使得切平面与三坐标平面所围成的体积

最大,求切点的坐标.

解: 设切点坐标为 $M_0(x_0,y_0,z_0)$. 则

$$\vec{n}=(\tfrac{a}{2}x^{-\frac{1}{2}},\tfrac{b}{2}y^{-\frac{1}{2}},\tfrac{c}{2}z^{-\frac{1}{2}})\big|_{(x_0,y_0,z_0)}=(\tfrac{a}{2}x_0^{-\frac{1}{2}},\tfrac{b}{2}y_0^{-\frac{1}{2}},\tfrac{c}{2}z_0^{-\frac{1}{2}})$$

所以切平面方程为

$$\tfrac{a}{2}x_0^{-\frac{1}{2}}(x-x_0)+\tfrac{b}{2}y_0^{-\frac{1}{2}}(y-y_0)+\tfrac{c}{2}z_0^{-\frac{1}{2}}(z-z_0)=0$$

即 $$\frac{x}{\frac{\sqrt{x_0}}{a}}+\frac{y}{\frac{\sqrt{y_0}}{b}}+\frac{z}{\frac{\sqrt{z_0}}{c}}=1$$

所以

$$V=\frac{1}{6}\frac{\sqrt{x_0}}{a}\frac{\sqrt{y_0}}{b}\frac{\sqrt{z_0}}{c}=\frac{1}{6}\frac{\sqrt{x_0y_0z_0}}{abc}$$

问题转化为:

求函数 $f(x,y,z)=xyz$ 在条件 $a\sqrt{x}+b\sqrt{y}+c\sqrt{z}=1$

下的最大值.

设 $F(x,y,z)=xyz+\lambda(a\sqrt{x}+b\sqrt{y}+c\sqrt{z}-1)$

解之得: $M_0(\frac{1}{9a^2},\frac{1}{9b^2},\frac{1}{9c^2})$

6. 求方程 $x^2+y^2+z^2-xz-yz+2x+2y+2z-2=0$

所确定的隐函数 $z=z(x,y)$ 的极值。

解：第一步：求驻点　方程分别对x,y求偏导数

$$\begin{cases} 2x+2z\dfrac{\partial z}{\partial x}-z-x\dfrac{\partial z}{\partial x}-y\dfrac{\partial z}{\partial x}+2+2\dfrac{\partial z}{\partial x}=0 \\ 2y+2z\dfrac{\partial z}{\partial y}-z-y\dfrac{\partial z}{\partial y}-x\dfrac{\partial z}{\partial y}+2+2\dfrac{\partial z}{\partial y}=0 \end{cases} \quad ①$$

$$\begin{cases} \dfrac{\partial z}{\partial x}=0 \\ \dfrac{\partial z}{\partial y}=0 \end{cases} \Rightarrow \begin{cases} x=\dfrac{z}{2}-1 \\ y=\dfrac{z}{2}-1 \end{cases} \quad ②$$

将②代入原方程得：

$$z_1=-4+2\sqrt{6},\ z_2=-4-2\sqrt{6}$$

即驻点为：$A(-3+\sqrt{6},-3+\sqrt{6}),\ B(-3-\sqrt{6},-3-\sqrt{6})$

在A点处，①两边分别对x，y再求导，并将以上结果代入得：

$$A=\left.\frac{\partial^2 z}{\partial x^2}\right|_{(-3+\sqrt{6},-3+\sqrt{6},-4+2\sqrt{6})}<0,\quad B=\frac{\partial^2 z}{\partial x\partial y}=0,$$

$$C=\left.\frac{\partial^2 z}{\partial y^2}\right|_{(-3+\sqrt{6},-3+\sqrt{6},-4+2\sqrt{6})}<0,$$

$\Rightarrow A$为极大值点，极大值为$Z=-4+2\sqrt{6},$

同理可得：

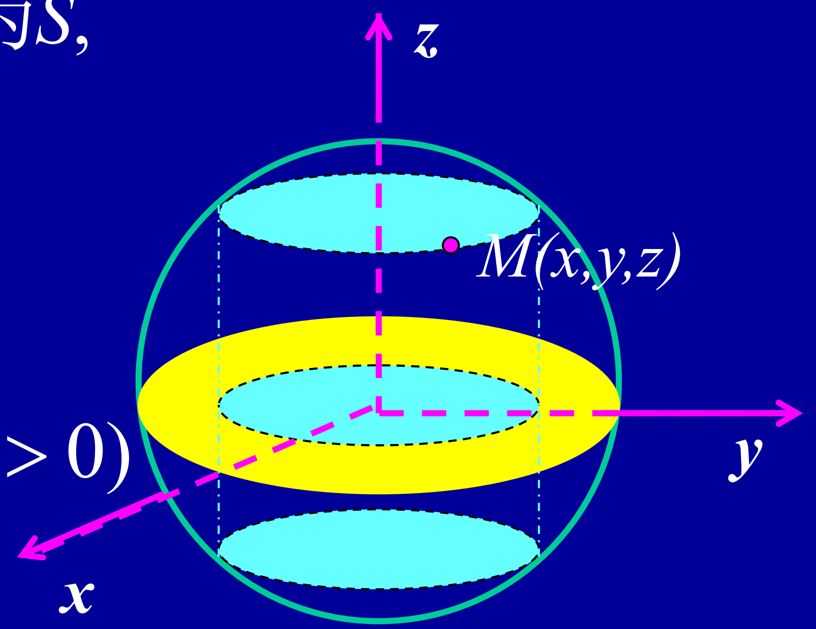
$\Rightarrow B$为极小值点，极小值为$Z=-4-2\sqrt{6}$,

8. 求 在半径为R的球内嵌入具有最大表面积的圆柱。

解: 设圆柱底半径为r,表面积为S,

$M(x,y,z)$为圆柱上底边界在

第一卦限内的任一点.



问题转化为:

$$S = 2\pi r \cdot 2z + 2\pi r^2 (r > 0, z > 0)$$

在条件 $x^2 + y^2 + z^2 = R^2$

即 $r^2 + z^2 = R^2$ 下的最大值.

设 $F(r,z) = 2\pi r \cdot 2z + 2\pi r^2 + \lambda(r^2 + z^2 - R^2)$

$\longrightarrow$ $r = \frac{\sqrt{R}}{2}\sqrt{2 + \frac{2}{\sqrt{5}}},\ h = 2z = R\sqrt{2 - \frac{2}{\sqrt{5}}}$ 时S最大.

9. 已知两平面曲线 $f(x,y)=0$及$\varphi(x,y)=0$，点(α,β)和(ξ,η) 分别为两曲线上的点，试证：如果这两点是两曲线上相距最近和相距最远的点，则

$$\frac{\alpha-\xi}{\beta-\eta}=\frac{f_x(\alpha,\beta)}{f_y(\alpha,\beta)}=\frac{\varphi_x(\xi,\eta)}{\varphi_y(\xi,\eta)}$$

证：设(x,y)和(X,Y) 分别为两曲线上的点，则两点间的距离为：$d=\sqrt{(X-x)^2+(Y-y)^2}$

问题转化为求函数 d在条件$f(x,y)=0,\varphi(X,Y)=0$下的最值问题。

设 $F=(X-x)^2+(Y-y)^2+\lambda_1 f(x,y)+\lambda_2\varphi(X,Y)$

$$\begin{cases} F_x=-2(X-x)+\lambda_1 f_x=0 \\ F_y=-2(Y-y)+\lambda_1 f_y=0 \\ F_X=2(X-x)+\lambda_2\varphi_x=0 \\ F_Y=2(Y-y)+\lambda_2\varphi_Y=0 \end{cases} \Rightarrow \begin{cases} \dfrac{X-x}{Y-y}=\dfrac{f_x}{f_y} \\ \dfrac{X-x}{Y-y}=\dfrac{\varphi_X}{\varphi_Y} \end{cases}$$

8. 求 $f(x,y,z)=\ln x+\ln y+3\ln z$ 在$x^2+y^2+z^2=5r^2$ $(x>0,y>0,z>0)$ 上的极大值，并以此结果证明：对任意 $a>0,b>0,c>0$ 有 $abc^3\le 27(\frac{a+b+c}{5})^5$ 。

解：令 $F(x,y,z)=\ln x+\ln y+3\ln z+\lambda(x^2+y^2+z^2-5r^2)$

$$\begin{cases} F_x=\dfrac{1}{x}+2\lambda x=0 \\ F_y=\dfrac{1}{y}+2\lambda y=0 \\ F_z=\dfrac{3}{z}+2\lambda z=0 \\ x^2+y^2+z^2-5r^2=0 \end{cases}$$

解之，得

$x = y = r, z = \sqrt{3}r$ 时 $f_{\max} = \ln 3\sqrt{3}r^5$

$\therefore f(x, y, z) = \ln xyz^3 \le \ln 3\sqrt{3}r^5$

因此 $xyz^3 \le 3\sqrt{3}(\frac{x^2 + y^2 + z^2}{5})^{\frac{5}{2}}$

取 $x = \sqrt{a}, y = \sqrt{b}, z = \sqrt{c}$

则有 $abc^3 \le 27(\frac{a + b + c}{5})^5$

作业

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15， 16， 17， 18