第四章

不定定积分

(习题课)

题组一：　基本积分法

求下列不定积分

1. $\int \frac{dx}{\sqrt{x+1}-\sqrt{x-1}}$.

解: $\int \frac{dx}{\sqrt{x+1}-\sqrt{x-1}} = \frac{1}{2}\int(\sqrt{x+1}+\sqrt{x-1})dx$

$$=\frac{1}{2}\int\sqrt{x+1}dx+\frac{1}{2}\int\sqrt{x-1}dx$$

$$=\frac{1}{2}\int\sqrt{x+1}d(x+1)+\frac{1}{2}\int\sqrt{x-1}d(x-1)$$

$$=\frac{1}{3}(x+1)^{\frac{3}{2}}+\frac{1}{3}(x-1)^{\frac{3}{2}}+c$$

2. $\int \frac{4\sin^2 x + 5\cos^2 x}{1+\sin x} dx$.

解: $\int \frac{4\sin^2 x + 5\cos^2 x}{1+\sin x} dx$

$$= \int \frac{(4+\cos^2 x)\cdot(1-\sin x)}{(1+\sin x)\cdot(1-\sin x)} dx$$

$$= \int \frac{(4+\cos^2 x)\cdot(1-\sin x)}{\cos^2 x} dx$$

$$= 4\int \sec^2 x dx + 4\int \frac{d\cos x}{\cos^2 x} + \int 1 dx - \int \sin x dx$$

$$= 4\tan x - 4\sec x + x + \cos x + C$$

3. $\displaystyle\int \frac{dx}{\sin^3 x \cos^5 x}$.

解: $$\int \frac{dx}{\sin^3 x \cos^5 x} = \int \frac{(\sin^2 x + \cos^2 x)^2 dx}{\sin^3 x \cos^5 x}$$

$$= \int \frac{\sin x}{\cos^5 x} dx + 2\int \frac{1}{\sin x \cos^3 x} dx + \int \frac{1}{\sin^3 x \cos x} dx$$

$$= -\int \frac{1}{\cos^5 x} d\cos x + 2\int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^3 x} dx$$

$$+ \int \frac{\sin^2 x + \cos^2 x}{\sin^3 x \cos x} dx$$

$$= -\int \frac{1}{\cos^5 x} d\cos x - 2\int \frac{d\cos x}{\cos^3 x} + 2\int \frac{1}{\sin x \cos x} dx$$

$$+ \int \frac{1}{\sin x \cos x} dx + \int \frac{d\sin x}{\sin^3 x}$$

$$= \frac{1}{4\cos^4 x} + \frac{1}{\cos^2 x} + 3\ln\left|\tan x\right| - \frac{1}{2\sin^2 x} + C$$

5. $\int e^{5+\sin^2 x} \sin 2x dx$.

解: $\int e^{5+\sin^2 x} \sin 2x dx = \int e^{5+\sin^2 x} 2\sin x \cos x \cdot dx$.

$$= \int e^{5+\sin^2 x} d \sin^2 x$$

$$= e^{5+\sin^2 x} + C$$

6. $\int x^x(1+\ln x)dx$.

解: $\int x^x(1+\ln x)dx = \int e^{x\ln x}d(x\ln x)$

$= e^{x\ln x} + C$

7. $\displaystyle\int \frac{1-\ln x}{(x-\ln x)^2}\,dx$.

解: $$\int \frac{1-\ln x}{(x-\ln x)^2}\,dx = \int \frac{1-\ln x}{x^2}\cdot\frac{1}{\left(1-\frac{\ln x}{x}\right)^2}\,dx .$$

$$= \int \frac{1}{\left(1-\frac{\ln x}{x}\right)^2}\,d\frac{\ln x}{x}$$

$$= \frac{1}{1-\dfrac{\ln x}{x}} + C \quad = \frac{x}{x-\ln x} + C$$

8. $\int \frac{\sin x \cos x dx}{\sqrt[3]{a^2 \sin^2 x + b^2 \cos^2 x}}$.

解: $\int \frac{\sin x \cos x dx}{\sqrt[3]{a^2 \sin^2 x + b^2 \cos^2 x}} = \frac{1}{2} \int \frac{d \sin^2 x}{\sqrt[3]{a^2 \sin^2 x + b^2 (1 - \sin^2 x)}}$

$$= \frac{1}{2} \int \frac{d \sin^2 x}{\sqrt[3]{b^2 + (a^2 - b^2) \sin^2 x}}$$

$$= \frac{1}{2} \frac{1}{a^2 - b^2} \int \frac{d[b^2 + (a^2 - b^2) \sin^2 x]}{\sqrt[3]{b^2 + (a^2 - b^2) \sin^2 x}}$$

$$= \frac{3}{4} \frac{1}{a^2 - b^2} (b^2 + (a^2 - b^2) \sin^2 x)^{\frac{2}{3}} + C$$

9. $\int \frac{\arcsin x}{x^2} \cdot \frac{x^2+1}{\sqrt{1-x^2}} dx$.

解: $\int \frac{\arcsin x}{x^2} \cdot \frac{x^2+1}{\sqrt{1-x^2}} dx \xrightarrow[\boxed{-\frac{\pi}{2}<t<\frac{\pi}{2}}]{\boxed{\text{令} x=\sin t}}$

$$= \int \frac{t}{\sin^2 t} \cdot \frac{\sin^2 t+1}{\cos t} \cdot \cos t dt$$

$$= \int t(\csc^2 t+1) dt \quad = -t\cot t + \int \cot t dt + \frac{1}{2}t^2$$

$$= \frac{1}{2}t^2 - t\cot t + \ln|\sin t| + C$$

$$= \frac{1}{2}\arcsin^2 x - \frac{\sqrt{1-x^2}}{x}\arcsin x + \ln|x| + C$$

10. $\int \frac{\sqrt{x-1}\arctan\sqrt{x-1}}{x}dx$.

解: $\int \frac{\sqrt{x-1}\arctan\sqrt{x-1}}{x}dx$

$\boxed{令\ t=\sqrt{x-1}}$

$$=\int \frac{t\cdot\arctan t}{t^2+1}\cdot 2tdt=2\int \frac{(t^2+1-1)\arctan t}{t^2+1}dt$$

$$=2\int \arctan t dt-2\int \arctan t d\arctan t$$

$$=2t\arctan t-\ln(1+t^2)+(\arctan t)^2+C$$

$$=2\sqrt{x-1}\arctan\sqrt{x-1}-\ln x+(\arctan\sqrt{x-1})^2+C$$

11. $\int \frac{x^3}{\sqrt{1+x^2}} dx$.

解: $\int \frac{x^3}{\sqrt{1+x^2}} dx = \int x^2 \frac{x}{\sqrt{1+x^2}} dx = \int x^2 d\sqrt{1+x^2}$

$= x^2\sqrt{1+x^2} - \int \sqrt{1+x^2}\, dx^2$

$= x^2\sqrt{1+x^2} - \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C$

12. $\int \frac{\ln\cos x}{\cos^2 x} dx$.

解: $\int \frac{\ln\cos x}{\cos^2 x} dx = \int \ln\cos x \, d\tan x$

$= \tan x \ln\cos x + \int \tan^2 x dx$

$= \tan x \ln\cos x + \tan x - x + C$

13. $\int x\ln\left(x^4+x\right)dx$.

解: $\int x\ln\left(x^4+x\right)dx = \int \ln\left(x^4+x\right)d(\frac{x^2}{2})$

$$= \ln\left(x^4+x\right)\cdot\frac{x^2}{2} - \int\frac{4x^3+1}{x^4+x}\cdot\frac{x^2}{2}dx$$

$$= \ln\left(x^4+x\right)\cdot\frac{x^2}{2} - \frac{1}{2}\int[4x - \frac{3x}{(x+1)(x^2-x+1)}]dx$$

$$= \ln\left(x^4+x\right)\cdot\frac{x^2}{2} - x^2 + \frac{1}{2}\int[-\frac{1}{x+1} + \frac{x+1}{x^2-x+1}]dx$$

$$= \ln\left(x^4 + x\right) \cdot \frac{x^2}{2} - x^2 - \frac{1}{2}\ln|x+1|$$

$$+ \frac{1}{2}\int \frac{\frac{1}{2}[(2x-1)+3]}{x^2 - x + 1}dx$$

$$= \ln\left(x^4 + x\right) \cdot \frac{x^2}{2} - x^2 - \frac{1}{2}\ln|x+1|$$

$$+ \frac{1}{2}\int \frac{1}{2}\frac{d(x^2 - x + 1)}{x^2 - x + 1} + \frac{3}{4}\int \frac{d(x - \frac{1}{2})}{(x - \frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2}$$

$$= \ln\left(x^4 + x\right) \cdot \frac{x^2}{2} - x^2 - \frac{1}{2}\ln|x+1|$$

$$+ \frac{1}{4}\ln\left|x^2 - x + 1\right| + \frac{\sqrt{3}}{2}\arctan\frac{2x-1}{\sqrt{3}} + C$$

14. $\int \frac{\arctan e^x}{e^{2x}}dx$.

解: $\int \frac{\arctan e^x}{e^{2x}}dx = \int \arctan e^x d(-\frac{1}{2}e^{-2x})$

$$= -\frac{1}{2}e^{-2x} \cdot \arctan e^x - \int \frac{e^x}{1+e^{2x}} \cdot (-\frac{1}{2}e^{-2x})dx$$

$$= -\frac{1}{2}e^{-2x} \cdot \arctan e^x + \frac{1}{2}\int \frac{e^{-x}}{(1+e^{2x})}dx$$

$$= -\frac{1}{2}e^{-2x} \cdot \arctan e^x - \frac{1}{2}\int \frac{e^{-2x}}{e^{-2x}+1}de^{-x}$$

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$$= -\frac{1}{2}e^{-2x} \cdot \arctan e^{x} - \frac{1}{2}\int \frac{(e^{-2x}+1)-1}{e^{-2x}+1} de^{-x}$$

$$= -\frac{1}{2}e^{-2x} \cdot \arctan e^{x} - \frac{1}{2}e^{-x} + \frac{1}{2}\arctan e^{-x} + C$$

15. $\int \frac{xe^x}{(1+x)^2}dx$.

解: $\int \frac{xe^x}{(1+x)^2}dx = \int xe^x d(-\frac{1}{1+x})$

$$= -\frac{xe^x}{1+x} + \int (1+x)e^x \cdot \frac{1}{1+x}dx$$

$$= -\frac{xe^x}{1+x} + e^x + C$$

16. $\int e^x \dfrac{1+\sin x}{1+\cos x} dx$.

解: $\int e^x \dfrac{1+\sin x}{1+\cos x} dx = \int e^x \dfrac{(1+\sin x)\cdot(1-\cos x)}{\sin^2 x} dx$

$$= \int e^x (\csc^2 x - \csc x \cot x + \csc x - \cot x) dx$$

$$= \int e^x d(-\cot x) + \int e^x d \csc x$$

$$+ \int e^x \csc x dx - \int e^x \cot x dx$$

$$= -e^x \cot x + \cancel{\int e^x \cot x dx} + e^x \csc x - \cancel{\int e^x \csc x dx}$$

$$+ \cancel{\int e^x \csc x dx} - \cancel{\int e^x \cot x dx}$$

$$= -e^x \cot x + e^x \csc x + C$$

17. $I_n = \int x^n \ln x dx \quad (n \neq -1)$.

解: $$\int x^n \ln x dx = \int \ln x d(\frac{x^{n+1}}{n+1})$$

$$= \ln x \cdot \frac{x^{n+1}}{n+1} - \int \frac{1}{x} \cdot \frac{x^{n+1}}{n+1} dx$$

$$= \ln x \cdot \frac{x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C$$

18. $I_n = \int \tan^n x dx$.

解: $I_n = \int \tan^n x dx = \int \tan^{n-2} x(\sec^2 x - 1)dx$

$$= \int \tan^{n-2} x d\tan x - I_{n-2} = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

又知

$$I_0 = x + C \quad , I_1 = \int \tan x dx = -\ln|\cos x| + C$$

利用递推公式可求得 I_n .

题组二:　　特殊函数的不定积分

求下列不定积分

1. $\int \frac{x^2}{x^2-5x+6}dx$.

解: $\int \frac{x^2}{x^2-5x+6}dx = \int [1+\frac{5x-6}{(x-2)(x-3)}]dx$

$= \int [1+\frac{9}{x-3}-\frac{4}{x-2}]dx$

$= x+9\ln|x-3|-4\ln|x-2|+C$

2. $\int \frac{x+1}{x^2-2x+5}dx$.

解: $\int \frac{x+1}{x^2-2x+5}dx$

$$=\frac{1}{2}\int \frac{d(x^2-2x+5)}{x^2-2x+5}+2\int \frac{d(x-1)}{(x-1)^2+2^2}$$

$$=\frac{1}{2}\ln(x^2-2x+5)+\arctan\frac{x-1}{2}+C$$

3. $\int \frac{x^2+1}{1+x^4}dx$.

解: $\int \frac{x^2+1}{1+x^4}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \int \frac{d(x-\frac{1}{x})}{(x-\frac{1}{x})^2+2}$

$= \frac{1}{\sqrt{2}}\arctan\frac{1}{\sqrt{2}}(x-\frac{1}{x})+C$

5. $\int \frac{x^3}{\left(1+x^3\right)^2} dx$.

解: $\int \frac{x^3}{\left(1+x^3\right)^2} dx = \frac{1}{3}\int \frac{x}{\left(1+x^3\right)^2} d(1+x^3)$

$= \frac{1}{3}\int x d(-\frac{1}{1+x^3}) \quad = \frac{1}{3}x(-\frac{1}{1+x^3}) + \frac{1}{3}\int \frac{1}{1+x^3} dx$

$= -\frac{x}{3}\cdot\frac{1}{1+x^3} + \frac{1}{3}\int \frac{1}{(1+x)(1-x+x^2)} dx$

$= -\frac{x}{3}\cdot\frac{1}{1+x^3} + \frac{1}{9}\int \frac{dx}{1+x} - \frac{1}{9}\int \frac{x-2}{x^2-x+1} dx$

$$= -\frac{x}{3(1+x^3)} + \frac{1}{9}\ln|1+x| - \frac{1}{18}\int \frac{2x-1-3}{x^2-x+1}dx$$

$$= -\frac{x}{3(1+x^3)} + \frac{1}{9}\ln|1+x| - \frac{1}{18}\int \frac{d(x^2-x+1)}{x^2-x+1}$$

$$+\frac{1}{6}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{3}{4}}$$

$$= -\frac{x}{3(1+x^3)} + \frac{1}{9}\ln|1+x| - \frac{1}{18}\ln(x^2-x+1)$$

$$+\frac{1}{3\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}} + C$$

6. $\int \frac{dx}{x\left(1+x^{6}\right)}$.

解: $\int \frac{dx}{x\left(1+x^{6}\right)} = \int \frac{(1+x^{6})-x^{6}}{x\left(1+x^{6}\right)} dx$

$$= \int \frac{dx}{x} - \int \frac{x^{5}dx}{1+x^{6}}$$

$$= \ln|x| - \frac{1}{6}\ln(1+x^{6}) + C$$

$$= \frac{1}{6}\ln\left|\frac{x^{6}}{1+x^{6}}\right| + C$$

8. $\int \frac{x+\sin x}{1+\cos x}dx$.

解: $\int \frac{x+\sin x}{1+\cos x}dx = \int \frac{x}{1+\cos x}dx + \int \frac{\sin x}{1+\cos x}dx$

$$= \frac{1}{2}\int x\sec^2\frac{x}{2}dx + \int \tan\frac{x}{2}dx$$

$$= \int xd\tan\frac{x}{2} + \int \tan\frac{x}{2}dx$$

$$= x\tan\frac{x}{2} - \int \tan\frac{x}{2}dx + \int \tan\frac{x}{2}dx$$

$$= x\tan\frac{x}{2} + C$$

9. $\int \frac{\sin x}{2\sin x - \cos x} dx$.

解: 设 $\sin x = A(2\sin x - \cos x) + B(2\sin x - \cos x)'$

则 $A = \frac{2}{5}$, $B = \frac{1}{5}$

$$\therefore \text{原式} = \frac{1}{5}\int \frac{d(2\sin x - \cos x)}{2\sin x - \cos x} + \frac{2}{5}\int \frac{2\sin x - \cos x}{2\sin x - \cos x} dx$$

$$= \frac{1}{5}\ln\left|2\sin x - \cos x\right| + \frac{2}{5}x + c$$

10. $$\int \frac{dx}{\sin x\sqrt{1+\cos x}}.$$

解: $$\int \frac{dx}{\sin x\sqrt{1+\cos x}} = \int \frac{\sin x dx}{\sin^2 x\sqrt{1+\cos x}}$$

$$= -\int \frac{d\cos x}{(1-\cos^2 x)\sqrt{1+\cos x}}$$

$$\boxed{令 t = \sqrt{1+\cos x}}$$

$$= -\int \frac{d(t^2-1)}{[1-(t^2-1)^2]t} = 2\int \frac{dt}{t^2(t^2-2)} = \int \frac{dt}{t^2-2} - \int \frac{dt}{t^2}$$

$$= \frac{\sqrt{2}}{4} \ln \left| \frac{t - \sqrt{2}}{t + \sqrt{2}} \right| + \frac{1}{t} + C$$

$$= \frac{\sqrt{2}}{4} \ln \left| \frac{\sqrt{1 + \cos x} - \sqrt{2}}{\sqrt{1 + \cos x} + \sqrt{2}} \right| + \frac{1}{\sqrt{1 + \cos x}} + C$$

11. $\int \frac{dx}{x\sqrt{4-x^2}}$.

解: $$\int \frac{dx}{x\sqrt{4-x^2}} = \int \frac{xdx}{x^2\sqrt{4-x^2}} = \frac{1}{2}\int \frac{d(4-x^2)}{-x^2\sqrt{4-x^2}}$$

$$\Big\downarrow \quad \boxed{令 t=\sqrt{4-x^2}}$$

$$= \frac{1}{2}\int \frac{dt^2}{(t^2-4)t} \quad = \int \frac{dt}{t^2-4}$$

$$= \frac{1}{4}\ln\left|\frac{t-2}{t+2}\right| + C \quad = \frac{1}{4}\ln\left|\frac{\sqrt{4-x^2}-2}{\sqrt{4-x^2}+2}\right| + C$$

12. $$\int\sqrt{\frac{x}{1-x\sqrt{x}}}dx.$$

解: $$\int\sqrt{\frac{x}{1-x\sqrt{x}}}dx = \int\frac{\sqrt{x}}{\sqrt{1-x\sqrt{x}}}dx$$

$$=\frac{2}{3}\int\frac{d(x\sqrt{x})}{\sqrt{1-x\sqrt{x}}} = -\frac{2}{3}\int\frac{d(1-x\sqrt{x})}{\sqrt{1-x\sqrt{x}}}$$

$$=-\frac{4}{3}\sqrt{1-x\sqrt{x}}+C$$

13. $\displaystyle\int \frac{dx}{\sqrt{x(1-x)}}$.

解: $\displaystyle\int \frac{dx}{\sqrt{x(1-x)}} = 2\int \frac{d\sqrt{x}}{\sqrt{1-(\sqrt{x})^2}}$

$= 2\arcsin\sqrt{x} + C$

14. $\int \frac{xe^x}{\sqrt{e^x-2}}dx$.

$\boxed{令 t=\sqrt{e^x-2}}$

解:

则$dt=\frac{e^x}{2\sqrt{e^x-2}}dx, x=\ln(t^2+2)$

$$=\int 2\ln(t^2+2)dt \quad = 2t\ln(t^2+2)-2\int\frac{2t^2dt}{t^2+2}$$

$$=2t\ln(t^2+2)-4t+\frac{8}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}+C$$

$$=2x\sqrt{e^x-2}-4\sqrt{e^x-2}+\frac{8}{\sqrt{2}}\arctan\frac{\sqrt{e^x-2}}{\sqrt{2}}+C$$

15. $\int \arcsin\sqrt{\frac{x}{1+x}}dx$. $x>0$

解: $\int \arcsin\sqrt{\frac{x}{1+x}}dx$

$$\boxed{令 t=\arcsin\sqrt{\frac{x}{1+x}}\text{，则} x=\tan^2 t}$$

$$=\int t d(\tan^2 t)=t\tan^2 t-\int \tan^2 t dt$$

$$=t\tan^2 t-\tan t+t+C$$

$$=x\arcsin\sqrt{\frac{x}{1+x}}-\sqrt{x}+\arcsin\sqrt{\frac{x}{1+x}}+C$$

16. $\int x\sqrt{1-x^2}\arcsin x dx$.

解: $\int x\sqrt{1-x^2}\arcsin x dx$

$\boxed{\text{令} x=\sin t}$

$= \int \sin t \cdot \sqrt{1-\sin^2 t} \cdot t d(\sin t) \quad = \int t \cdot \sin t \cdot \cos^2 t dt$

$= -\frac{1}{3}\int t d(\cos^3 t) \quad = -\frac{1}{3} t\cos^3 t + \frac{1}{3}\int \cos^3 t dt$

$= -\frac{1}{3} t\cos^3 t + \frac{1}{3}\int (1-\sin^2 t) d\sin t$

$= -\frac{1}{3}(1-x^2)^{\frac{3}{2}}\arcsin x + \frac{1}{3}x - \frac{1}{9}x^3 + C$

题组三:　　其它

解下列各题

1. $\int \max\left(1, x^2\right) dx$.

解: $\int \max\left(1, x^2\right) dx = \begin{cases} \int x^2 dx, & x > 1 \\ \int x^2 dx, & x < -1 \\ \int 1 dx, & -1 \le x \le 1 \end{cases}$

$= \begin{cases} \dfrac{1}{3} x^3 + C_1, & x > 1 \\ \dfrac{1}{3} x^3 + C_2, & x < -1 \\ x + C_3, & -1 \le x \le 1 \end{cases}$

由连续性可得

$$C_2 = C_3 - \frac{2}{3}, C_1 = C_3 + \frac{2}{3}$$

所以

$$\int \max\left(1, x^2\right) dx = \begin{cases} \frac{1}{3}x^3 + C_3 + \frac{2}{3}, & x > 1 \\ \frac{1}{3}x^3 + C_3 - \frac{2}{3}, & x < -1 \\ x + C_3\ , & -1 \le x \le 1 \end{cases}$$

2. $\int x\left(\frac{\sin x}{x}\right)'' dx$.

解: $\int x\left(\frac{\sin x}{x}\right)'' dx = x(\frac{\sin x}{x})' - \int (\frac{\sin x}{x})' dx$

$$= \frac{x\cos x - \sin x}{x} - \frac{\sin x}{x} + C.$$

3. 已知 $f(x)$ 的一个原函数为 $(1+\sin x)\ln x$，

求 $\int xf'(x)dx$.

解: $\int xf'(x)dx = xf(x) - \int f(x)dx$

$$= x((1+\sin x)\ln x)' - (1+\sin x)\ln x + C$$

$$= x(\cos x \ln x + \frac{1+\sin x}{x}) - (1+\sin x)\ln x + C$$

4. 设 $f(\ln x)=\dfrac{\ln(1+x)}{x}$，计算 $\int f(x)dx$.

解:

$$\int f(x)dx=\int\frac{\ln(1+e^x)}{e^x}dx=\int\ln(1+e^x)d(-e^{-x})$$

$$=-e^{-x}\ln(1+e^x)-\int\frac{e^x}{1+e^x}(-e^{-x})dx$$

$$=-e^{-x}\ln(1+e^x)+\int\frac{1}{1+e^x}dx$$

$$=-e^{-x}\ln(1+e^x)+\int\frac{(e^x+1)-e^x}{1+e^x}dx$$

$$=-e^{-x}\ln(1+e^x)+x-\ln(e^x+1)+C$$

5. 设 $F(x)=\int\frac{x^3-a}{x-a}dx$ 为 x 的多项式,求常数 a 及 $F(x)$.

解: $$F(x)=\int(x^2+ax+a^2+\frac{a^3-a}{x-a})dx$$

$$=\frac{1}{3}x^3+\frac{a}{2}x^2+a^2x+(a^3-a)\ln|x-a|+C$$

由 $a^3-a=0 \Longrightarrow a=0, a=1, a=-1$.

接5.

$$F(x)=\begin{cases}\frac{1}{3}x^3+C, & a=0\\ \frac{1}{3}x^3+\frac{1}{2}x^2+x+C, & a=1\\ \frac{1}{3}x^3-\frac{1}{2}x^2+x+C, & a=-1\end{cases}$$

6. 设 $F'(x)=f(x), f'(x)$可微，且 $f^{-1}(x)$存在,

证明: $\int f^{-1}(x)dx = xf^{-1}(x)-F\left[f^{-1}(x)\right]+c$

(c 为常数)

证明:

$$(yf^{-1}(y)-F[f^{-1}(y)]+c)'$$

$$=f^{-1}(y)+y\frac{1}{f'(x)}-F'[f^{-1}(y)]\frac{1}{f'(x)}$$

$$=f^{-1}(y)+y\frac{1}{f'(x)}-f(x)\frac{1}{f'(x)}$$

$$=f^{-1}(y)+y\frac{1}{f'(x)}-y\frac{1}{f'(x)} = f^{-1}(y)$$