§4 分块矩阵

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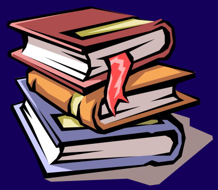

一. 矩阵的分块

例如，$A=\left(\begin{array}{cc|ccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{array}\right) \overset{记}{=} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

A_{ij} 称为A 的子块

$$A=\left(\begin{array}{c|c|c|c|c} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{array}\right)$$

$\overset{记}{=} \begin{pmatrix} A_1 & A_2 & A_3 & A_4 & A_5 \end{pmatrix}$ A_i称为A 的子块

以子块为元素的矩阵称为**分块矩阵**.

二. 分块矩阵的运算法则

(1) **加法.** $A=\begin{pmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & & \vdots \\ A_{s1} & \cdots & A_{sr} \end{pmatrix}$, $B=\begin{pmatrix} B_{11} & \cdots & B_{1r} \\ \vdots & & \vdots \\ B_{s1} & \cdots & B_{sr} \end{pmatrix}$

子块 A_{ij} 与 B_{ij} 有相同的行列数

则 $A+B=\begin{pmatrix} A_{11}+B_{11} & \cdots & A_{1r}+B_{1r} \\ \vdots & & \vdots \\ A_{s1}+B_{s1} & \cdots & A_{sr}+B_{sr} \end{pmatrix}$

(2) **数乘.** A **同上**, λ **为数, 则**

$$\lambda A=\begin{pmatrix} \lambda A_{11} & \cdots & \lambda A_{1r} \\ \vdots & & \vdots \\ \lambda A_{s1} & \cdots & \lambda A_{sr} \end{pmatrix}$$

(3) 矩阵乘法

将 $A_{m\times l}, B_{l\times n}$ 按如下形式分块：

$$A=\begin{pmatrix} A_{11} & \cdots & A_{1t} \\ \vdots & & \vdots \\ A_{s1} & \cdots & A_{st} \end{pmatrix},\quad B=\begin{pmatrix} B_{11} & \cdots & B_{1r} \\ \vdots & & \vdots \\ B_{t1} & \cdots & B_{tr} \end{pmatrix}$$

$A_{i1},\cdots,A_{it}$ 的列数分别等于 $B_{1j},\cdots,A_{tj}$ 的行数

则 $$AB=\begin{pmatrix} C_{11} & \cdots & C_{1r} \\ \vdots & & \vdots \\ C_{s1} & \cdots & C_{sr} \end{pmatrix}$$

其中 $$C_{ij}=\sum_{k=1}^{t} A_{ik}B_{kj}\ (i=1,\cdots s;\ j=1,\cdots,t)$$

例1. 设 $A=\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline -1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)$, $B=\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ \hline 1 & 0 & 4 & 1 \\ -1 & -1 & 2 & 0 \end{array}\right)$, 求$AB$, A^3.

(P48例 17)

解: 将A, B 分块,

$$A=\begin{pmatrix} E & O \\ A_1 & E \end{pmatrix},\quad B=\begin{pmatrix} B_{11} & E \\ B_{21} & B_{22} \end{pmatrix},$$

其中

$$A_1=\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix},\ B_{11}=\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix},\ B_{21}=\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix},\quad B_{22}=\begin{pmatrix} 4 & 1 \\ 2 & 0 \end{pmatrix}$$

$$AB=\begin{pmatrix} B_{11} & E \\ A_1B_{11}+B_{21} & A_1+B_{22} \end{pmatrix}=\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ \hline -2 & 4 & 3 & 3 \\ -1 & 1 & 3 & 1 \end{array}\right)$$

返回 上页 下页 结束

$$A^2 = \begin{pmatrix} E & O \\ A_1 & E \end{pmatrix}\begin{pmatrix} E & O \\ A_1 & E \end{pmatrix} = \begin{pmatrix} E & O \\ 2A_1 & E \end{pmatrix}$$

$$A^3 = A^2 A = \begin{pmatrix} E & O \\ 2A_1 & E \end{pmatrix}\begin{pmatrix} E & O \\ A_1 & E \end{pmatrix} = \begin{pmatrix} E & O \\ 3A_1 & E \end{pmatrix}$$

$$= \left(\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline -3 & 6 & 1 & 0 \\ 3 & 3 & 0 & 1 \end{array}\right)$$

问: $A^n = ?$

$$A = \left(\begin{array}{cc:cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline -1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right)$$ **其中** $A_1 = \begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix}$

(4) **转置** $A^{\mathrm{T}}=\begin{pmatrix} A_{11} & \cdots & A_{1r} \\ \vdots & & \vdots \\ A_{s1} & \cdots & A_{sr} \end{pmatrix}^{\mathrm{T}}=\begin{pmatrix} A_{11}^{\mathrm{T}} & \cdots & A_{s1}^{\mathrm{T}} \\ \vdots & & \vdots \\ A_{1r}^{\mathrm{T}} & \cdots & A_{sr}^{\mathrm{T}} \end{pmatrix}$

(5) **分块对角阵** $A=\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix}$, A_k 均为方阵

性质: ① $|A|=|A_1||A_2|\cdots|A_r|$

② $|A_i|\neq 0\ (i=1,\cdots,r) \longrightarrow$ A **可逆, 且**

$$A^{-1}=\begin{pmatrix} A_1^{-1} & & & \\ & A_2^{-1} & & \\ & & \ddots & \\ & & & A_r^{-1} \end{pmatrix}$$

$$\begin{pmatrix} & & A_1 \\ & ⋰ & \\ A_r & & \end{pmatrix}^{-1}=\begin{pmatrix} & & A_r^{-1} \\ & ⋰ & \\ A_1^{-1} & & \end{pmatrix}$$

例2. $A=\left(\begin{array}{c|cc}5&0&0\\ \hline 0&3&1\\0&2&1\end{array}\right)$，求 A^{-1} (P49 例 18)

解: $A=\begin{pmatrix}A_1&\\&A_2\end{pmatrix}$

$A_1=(5)$, $A_1^{-1}=\frac{1}{5}$

$A_2=\begin{pmatrix}3&1\\2&1\end{pmatrix}$, $A_2^{-1}=\frac{1}{|A_2|}\begin{pmatrix}1&-1\\-2&3\end{pmatrix}=\begin{pmatrix}1&-1\\-2&3\end{pmatrix}$

$$\therefore\quad A^{-1}=\begin{pmatrix}A_1^{-1}&\\&A_2^{-1}\end{pmatrix}=\begin{pmatrix}\frac{1}{5}&0&0\\0&1&-1\\0&-2&3\end{pmatrix}$$

三. 两种重要的分块法

约定：列向量表为 $\vec{\alpha}, \vec{\beta}, \vec{x}, \vec{a}, \cdots$

行向量表为 $\vec{\alpha}^{\mathrm{T}}, \vec{\beta}^{\mathrm{T}}, \vec{x}^{\mathrm{T}}, \vec{a}^{\mathrm{T}}, \cdots$

(1) 按行分块

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \vec{\alpha}_1^{\mathrm{T}} \\ \vec{\alpha}_2^{\mathrm{T}} \\ \vdots \\ \vec{\alpha}_m^{\mathrm{T}} \end{pmatrix}$$

记 $\vec{\alpha}_i^{\mathrm{T}} = (a_{i1}, a_{i2}, \cdots, a_{in})$

(1) **按行分块** $A=\begin{pmatrix}\vec{\alpha}_1^{\mathrm{T}}\\ \vec{\alpha}_2^{\mathrm{T}}\\ \vdots\\ \vec{\alpha}_m^{\mathrm{T}}\end{pmatrix}$

(2) 按列分块

$$A=\left(\begin{array}{c|c|cc|c} a_{11} & a_{12} & \cdots & \cdots & a_{1n}\\ a_{12} & a_{22} & \cdots & \cdots & a_{2n}\\ \vdots & \vdots & & & \vdots\\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn}\end{array}\right)=(\vec{\beta}_1,\ \vec{\beta}_2,\ \cdots,\ \vec{\beta}_n)$$

记 $\vec{\beta}_j=\begin{pmatrix}a_{1j}\\ a_{2j}\\ \vdots\\ a_{mj}\end{pmatrix}$

- 矩阵乘法的向量表示

$$AB=\begin{pmatrix}\vec{a}_1^{\mathrm{T}}\\ \vec{a}_2^{\mathrm{T}}\\ \vdots\\ \vec{a}_m^{\mathrm{T}}\end{pmatrix}(\vec{b}_1,\ \vec{b}_2,\ \cdots,\ \vec{b}_n)=\begin{pmatrix}\vec{a}_1^{\mathrm{T}}\vec{b}_1 & \vec{a}_1^{\mathrm{T}}\vec{b}_2 & \cdots & \vec{a}_1^{\mathrm{T}}\vec{b}_n\\ \vec{a}_2^{\mathrm{T}}\vec{b}_1 & \vec{a}_2^{\mathrm{T}}\vec{b}_2 & \cdots & \vec{a}_2^{\mathrm{T}}\vec{b}_n\\ \vdots & \vdots & & \vdots\\ \vec{a}_m^{\mathrm{T}}\vec{b}_1 & \vec{a}_m^{\mathrm{T}}\vec{b}_2 & \cdots & \vec{a}_m^{\mathrm{T}}\vec{b}_n\end{pmatrix}$$

$$=(c_{ij})_{m\times n}$$

$$c_{ij}=\vec{a}_i^{\mathrm{T}}\vec{b}_j=(a_{i1},a_{i2},\cdots,a_{in})\begin{pmatrix}a_{1j}\\ a_{2j}\\ \vdots\\ a_{mj}\end{pmatrix}$$

$$=\sum_{k=1}^{n}a_{ik}b_{kj}$$

• 矩阵与对角阵之积的特点

$\Lambda_m = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$ 左乘 $A_{m\times n}$

$$\Lambda_m A_{m\times n} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_m \end{pmatrix} \begin{pmatrix} \vec{a}_1^{\mathrm{T}} \\ \vec{a}_2^{\mathrm{T}} \\ \vdots \\ \vec{a}_m^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \lambda_1 \vec{a}_1^{\mathrm{T}} \\ \lambda_2 \vec{a}_2^{\mathrm{T}} \\ \vdots \\ \lambda_m \vec{a}_m^{\mathrm{T}} \end{pmatrix}$$

$\Lambda_n = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$ 右乘 $A_{m\times n}$

$$A_{m\times n}\Lambda_n = (\vec{a}_1,\ \vec{a}_2,\ \cdots,\ \vec{a}_n) \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$= (\lambda_1 \vec{a}_1,\ \lambda_2 \vec{a}_2,\ \cdots,\ \lambda_n \vec{a}_n)$$

例3. 设 $A^{\mathrm{T}}A = O$, 证明 $A = O$.

证: **设** $A = (a_{ij})_{m\times n}$, **按列分块:** $A = (\vec{a}_1,\ \vec{a}_2,\ \cdots,\ \vec{a}_n)$

$$A^{\mathrm{T}}A = \begin{pmatrix} \vec{a}_1^{\mathrm{T}} \\ \vec{a}_2^{\mathrm{T}} \\ \vdots \\ \vec{a}_n^{\mathrm{T}} \end{pmatrix} (\vec{a}_1,\ \vec{a}_2,\ \cdots,\ \vec{a}_n) = \begin{pmatrix} \vec{a}_1^{\mathrm{T}}\vec{a}_1 & \vec{a}_1^{\mathrm{T}}\vec{a}_2 & \cdots & \vec{a}_1^{\mathrm{T}}\vec{a}_n \\ \vec{a}_2^{\mathrm{T}}\vec{a}_1 & \vec{a}_2^{\mathrm{T}}\vec{a}_2 & \cdots & \vec{a}_2^{\mathrm{T}}\vec{a}_n \\ \vdots & \vdots & & \vdots \\ \vec{a}_n^{\mathrm{T}}\vec{a}_1 & \vec{a}_n^{\mathrm{T}}\vec{a}_2 & \cdots & \vec{a}_n^{\mathrm{T}}\vec{a}_n \end{pmatrix}$$

$\because A^{\mathrm{T}}A = O$, **所以对角元均为** 0, **即**

$$a_j^{\mathrm{T}}\vec{a}_j = (a_{1j}, a_{2j}, \cdots, a_{mj}) \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} = \sum_{k=1}^{m} a_{kj}^2 = 0 \ \ (j = 1, \cdots, n)$$

$$\therefore \quad a_{kj} = 0 \quad (k = 1, \cdots, m;\ j = 1, \cdots, n)$$

四. 线性方程组的几种表示法

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{cases}
\qquad (1)
$$

记 $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

$$
B = \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots & a_{1n} & b_1 \\ a_{12} & a_{22} & \cdots\cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots\cdots & a_{mn} & b_m \end{pmatrix}
$$

A 称为系数矩阵,

B 称为增广矩阵.

$$
B = (A \vdots \vec{b}) = (A, \vec{b}) = (\vec{a}_1,\ \vec{a}_2,\ \cdots,\ \vec{a}_n, \vec{b})
$$

表示法1. $A\vec{x}=\vec{b}$

表示法2. $\begin{pmatrix} \vec{\alpha}_1^{\mathrm{T}} \\ \vec{\alpha}_2^{\mathrm{T}} \\ \vdots \\ \vec{\alpha}_m^{\mathrm{T}} \end{pmatrix} \vec{x} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

相当于

$$\vec{\alpha}_i^T \vec{x} = b_i$$
$$(i=1, 2, \cdots, m)$$

表示法3. $(\vec{a}_1, \ \vec{a}_2, \ \cdots, \ \vec{a}_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \vec{b}$

即 $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$

五. 克拉默法则的证明

克拉默法则:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n \end{cases} \qquad (1)$$

若其系数行列式 $D \neq 0$, 则它有唯一解:

$$x_j = \frac{D_j}{D} \qquad (j = 1, 2, \cdots, n)$$

$$D_j = \begin{array}{c} \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix} \\ j \end{array}$$

证: 将 (1) 写成 $A\vec{x} = \vec{b}$

因 $D \neq 0$, 所以 A 可逆, 故得

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{|A|}A^*\vec{b} = \frac{1}{D}A^*\vec{b}$$

即
$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \frac{1}{D}\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

故
$$x_j = \frac{1}{D}(b_1 A_{1j} + b_2 A_{2j} + \cdots + b_n A_{nj})$$

$$= \frac{D_j}{D} \qquad (j = 1, 2, \cdots, n)$$

矩阵分块在行列式计算中的应用举例

设有3 阶方阵$A=(\vec{\alpha},\vec{\beta}_1,\vec{\beta}_2), B=(\vec{\beta},\vec{\beta}_1,\vec{\beta}_2)$,若 $|A|=2$, $|B|=\frac{1}{2}$, 求 $|A+B|$.

解: $|A+B|=\left|\vec{\alpha}+\vec{\beta},2\vec{\beta}_1,2\vec{\beta}_2\right|$

$$=\left|\vec{\alpha},2\vec{\beta}_1,2\vec{\beta}_2\right|+\left|\vec{\beta},2\vec{\beta}_1,2\vec{\beta}_2\right|$$

$$=2^2\left|\vec{\alpha},\vec{\beta}_1,\vec{\beta}_2\right|+2^2\left|\vec{\beta},\vec{\beta}_1,\vec{\beta}_2\right|$$

$$=8+2=10$$

作业

P55

25（2）； 26; 27; 28(1)