第七章

微分方程

(习题课)

题组一、一阶微分方程

1.求下列方程的通解.

(1) $(x+2y-4)dx+(2x+y-5)dy=0$

解: 令 $x=X+h,\ y=Y+k,$则 $dx=dX,dy=dY$

代入原方程得 $$\frac{dY}{dX}=-\frac{X+2Y+h+2k-4}{2X+y+2h+k-5}$$

解方程组 $\begin{cases} h+2k-4=0 \\ 2h+k-5=0 \end{cases}$ 得: $h=2,k=1$

令 $x=X+2,\ y=Y+1,$原方程成为 $$\frac{dY}{dX}=-\frac{1+2\dfrac{Y}{X}}{2+\dfrac{Y}{X}}$$

令 $\frac{Y}{X}=u$,则$Y=uX$,$\frac{dY}{dX}=u+X\frac{du}{dX}$,于是方程变为:

$$X\frac{du}{dX}=-\frac{u^2+4u+1}{2+u}$$

1 (2). $y^2dx-(y^2+2xy-x)dy=0$

解: $y^2dx-(y^2+2xy-x)dy=0$

$$\downarrow$$

$$\frac{dx}{dy}-\frac{2y-1}{y^2}x=1$$ (一阶线性微分方程)

$$\downarrow$$

$$\boxed{P(y)=-\frac{2y-1}{y^2},Q(y)=1}$$

$$x=e^{-\int P(y)dy}[\int Q(y)e^{\int P(y)dy}dy+C]$$

$$=e^{\int\frac{2y-1}{y^2}dy}[\int e^{\int-\frac{2y-1}{y^2}dy}dy+C]=y^2(1+Ce^{\frac{1}{y}})$$

$$\downarrow$$

$$x=y^2(1+Ce^{\frac{1}{y}})$$

1 (3). $x(2x^3+y)y'-6y^2=0$

解: $x(2x^3+y)y'-6y^2=0$

$$\downarrow$$

$$\frac{dx}{dy}-\frac{1}{6y}x=\frac{1}{3y^2}x^4 \quad (\,n=4\text{ 的伯努利方程})$$

$$\downarrow \quad \boxed{\text{令} z=x^{1-4}=x^{-3}}$$

$$\frac{dz}{dy}+\frac{1}{2y}z=-\frac{1}{y^2}$$

$$\downarrow$$

$$z==e^{-\int\frac{1}{2y}dy}\left[\int\left(-\frac{1}{y^2}\right)e^{\int\frac{1}{2y}dy}dy+C\right]=\frac{2}{y}+\frac{C}{\sqrt{y}}$$

$$\downarrow$$

$$x^{-3}=\frac{2}{y}+\frac{C}{\sqrt{y}}$$

$$1(4).\quad \tan y \cdot y' - \ln\cos y = xe^x$$

解: $\tan y \cdot y' - \ln\cos y = xe^x \longrightarrow -\dfrac{d\ln\cos y}{dx} - \ln\cos y = xe^x$

$\boxed{\text{令} z = -\ln\cos y} \longrightarrow \dfrac{dz}{dx} + z = xe^x$ (一阶线性微分方程)

$$\downarrow$$

$$z = e^{-\int P(x)dx}[\int Q(x)e^{\int P(x)dx}dx + C]$$

$$= e^{-\int dx}[\int xe^x e^{\int dx}dx + C] = e^{-x}[x\frac{e^{2x}}{2} - \frac{e^{2x}}{4} + C]$$

$$\downarrow$$

$$-\ln\cos y = e^{-x}[x\frac{e^{2x}}{2} - \frac{e^{2x}}{4} + C]$$

1(5). $xdy = y(xy-1)dx$

解:

$$xdy = y(xy-1)dx \longrightarrow \frac{dy}{dx} + \frac{1}{x}y = y^2$$ (n=2 的贝努利方程

$$\xrightarrow{\boxed{令 z = y^{1-2} = y^{-1}}} \frac{dz}{dx} - \frac{1}{x}z = -1 \longrightarrow z = -(\ln|x| + c)x$$

$$\longrightarrow y^{-1} = -(\ln|x| + c)x$$

2.求下列方程的特解.

(1) $(1+y^2)dx - x(1+x)ydy = 0 \quad y\big|_{x=1} = 0.$

解: $(1+y^2)dx - x(1+x)ydy = 0 \xrightarrow{\boxed{\text{分离变量}}} \dfrac{dx}{x(1+x)} = \dfrac{y}{1+y^2}dy$

$$\longrightarrow \ln\frac{x}{1+x} = \frac{1}{2}\ln(1+y^2) + \ln C_1$$

$$\left.\begin{array}{r}\longrightarrow (\dfrac{x}{1+x})^2 = (1+y^2)C_1{}^2 \\ y\big|_{x=1} = 0\end{array}\right\} \longrightarrow C_1 = \frac{1}{2}$$

$$\longrightarrow (\frac{x}{1+x})^2 = \frac{1}{4}(1+y^2)$$

2(2). $x \cdot y' + x + \sin(x+y) = 0, \quad y\big|_{x=\frac{\pi}{2}} = 0.$

解: $x \cdot y' + x + \sin(x+y) = 0, \xrightarrow{\boxed{\text{令} z = x + y}} x\dfrac{dz}{dx} + \sin z = 0$

$$\xrightarrow{\boxed{\text{分离变量}}} \frac{dz}{\sin z} = -\frac{1}{x}dx \longrightarrow \left.\begin{array}{r} \ln|\csc z - \cot z| = -\ln x + \ln C \\ y\big|_{x=\frac{\pi}{2}} = 0. \end{array}\right\}$$

$$\left.\begin{array}{l} \longrightarrow C = \dfrac{\pi}{2} \\ \ln|\csc z - \cot z| = -\ln x + \ln C \end{array}\right\} \longrightarrow \frac{1-\cos(x+y)}{\sin(x+y)} = \frac{\pi}{2x}$$

2(3). $2yy' + 2xy^2 = xe^{-x^2}$, $y(0) = 1$.

解: $2yy' + 2xy^2 = xe^{-x^2}$, $\longrightarrow \dfrac{dy^2}{dx} + 2xy^2 = xe^{-x^2}$

$\boxed{令 z = y^2}$

$\longrightarrow \dfrac{dz}{dx} + 2xz = xe^{-x^2} \longrightarrow z = e^{-\int 2xdx}[\int xe^{-x^2} e^{\int 2xdx} + C]$

$$\left.\begin{array}{l} \longrightarrow y^2 = e^{-x^2}[\dfrac{x^2}{2} + C] \\ y(0) = 1. \end{array}\right\} \longrightarrow C = 1$$

$$\left.\begin{array}{l} C = 1 \\ y^2 = e^{-x^2}[\dfrac{x^2}{2} + C] \end{array}\right\}$$

$\longrightarrow y^2 = e^{-x^2}(\dfrac{x^2}{2} + 1) \longrightarrow y = \sqrt{e^{-x^2}(\dfrac{x^2}{2} + 1)}$

(2) 已知$\int_0^1 f(tx)dt = \frac{1}{2}f(x)-1$，试在 $x \neq 0$ 条件下求 $f(x)$

解:
$$\left.\begin{aligned} &\int_0^1 f(tx)dt = \frac{1}{x}\int_0^1 f(tx)d(tx) = \frac{1}{x}\int_0^x f(u)du \\ &\int_0^1 f(tx)dt = \frac{1}{2}f(x)-1 \end{aligned}\right\}$$

$$\longrightarrow \frac{1}{x}\int_0^x f(u)du = \frac{1}{2}f(x)-1 \longrightarrow \int_0^x f(u)du = \frac{x}{2}f(x)-x$$

$$\xrightarrow{\boxed{\text{两边关于}x\text{求导}}} f(x) = \frac{1}{2}f(x)+\frac{x}{2}f'(x)-1$$

$$\longrightarrow f'(x)-\frac{1}{x}f(x) = \frac{2}{x} \longrightarrow f(x) = -2+Cx$$

(一阶线性微分方程)

4. 设 $y_1(x), y_2(x)$ 是微分方程 $y' + p(x)y = q(x)$ 的两个不同的解，求证：对于该方程的任意一个解 $y(x)$ 都满足: $\dfrac{y(x)-y_1(x)}{y_2(x)-y_1(x)} = c$ (c为任意常数).

解: 由微分方程解的结构知 $y_2(x)-y_1(x)$ 是方程所对应齐次方程的解．于是非齐次方程的通解为

$$y(x) = C(y_2(x)-y_1(x)) + y_1(x) \longrightarrow \frac{y(x)-y_1(x)}{y_2(x)-y_1(x)} = C$$

题组二、高阶微分方程

1. 解初值问题 $y'' = 3\sqrt{y}$，$y(0)=1$，$y'(0)=2$.

解: 令$y' = p(y)$

$$\left.\begin{array}{l} y' = p(y) \\ \downarrow \\ y'' = p\dfrac{dp}{dy} \\ y'' = 3\sqrt{y} \end{array}\right\} \longrightarrow p\frac{dp}{dy} = 3\sqrt{y} \longrightarrow \left.\begin{array}{r} \dfrac{1}{2}p^2 = 2y^{\frac{3}{2}} + C_1 \\ y(0)=1 \\ y'(0)=2 \end{array}\right\}$$

$$\left.\begin{array}{r} \longrightarrow C_1 = 0 \\ \dfrac{1}{2}p^2 = 2y^{\frac{3}{2}} + C_1 \end{array}\right\} \longrightarrow \left.\begin{array}{r} y'(0)=2 \\ p = \pm 2y^{\frac{3}{4}} \end{array}\right\} \longrightarrow p = 2y^{\frac{3}{4}}$$

$$\longrightarrow \frac{dy}{dx} = 2y^{\frac{3}{4}} \longrightarrow \left.\begin{array}{r} y(0)=1 \\ 4y^{\frac{1}{4}} + C_2 = 2x \end{array}\right\} \longrightarrow x = 2y^{\frac{1}{4}} - 2$$

2. 解初值问题 $y''(x+y'^2)=y'$ 初始条件$y(1)=y'(1)=1.$

解: 令$y'=p(x)$

$$\left.\begin{array}{r} y'=p(x) \\ \downarrow \\ y''=p'(x) \\ y''(x+y'^2)=y' \end{array}\right\} \longrightarrow p'(x+p^2)=p \longrightarrow \frac{dx}{dp}-\frac{1}{p}x=p$$

$$\longrightarrow x=e^{\int\frac{1}{p}dp}[\int pe^{-\int\frac{1}{p}dp}dp+C_1] \longrightarrow \left.\begin{array}{r} y'(1)=p(1)=1 \\ x=C_1p+p^2 \end{array}\right\} \longrightarrow C_1=0$$

$$\longrightarrow x=p^2 \longrightarrow \left.\begin{array}{r} p=\pm\sqrt{x} \\ y'(1)=p(1)=1>0 \end{array}\right\} \longrightarrow p=\sqrt{x} \longrightarrow \frac{dy}{dx}=\sqrt{x}$$

$$\longrightarrow \left.\begin{array}{r} y=\frac{2}{3}x^{\frac{3}{2}}+C_2 \\ y(1)=1 \end{array}\right\} \longrightarrow C_2=\frac{1}{3} \longrightarrow y=\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{3}$$

3.写出下列方程的特解

(1) $y'' - (a+b)y' + aby = x^2 e^{3x}$ ，其中 a , b 为常数.

解: 方程的特征方程为 $r^2 - (a+b)r + ab = 0 \longrightarrow \begin{cases} r_1 = a \\ r_2 = b \end{cases}$

(1) 若$a \neq 3$,$b \neq 3$,则$\lambda = 3$ 不是特征根.

于是原方程特解为: $y^* = x^0(Ax^2 + Bx + C)e^{3x}$

将其代入方程比较系数得

$$A = \frac{1}{(3-a)(3-b)}, B = \frac{2(a+b)-12}{(3-a)^2(3-b)^2},$$

$$C = \frac{2(a^2 + ab - 9a + b^2 - 9b + 27)}{(3-a)^3(3-b)^3}.$$ 所以

$y^* =$

接3(1).

(2) 若a和b中有一个等于3, 则$\lambda = 3$ 是单根.

于是原方程特解为: $y^* = x(Ax^2 + Bx + C)e^{3x}$

将其代入方程比较系数得

$$A = \frac{1}{3(3-b)}, B = -\frac{1}{(b-3)^2}, C = -\frac{2}{(b-3)^3}$$

所以 $y^* = \cdots$

(3) 若$a = b = 3$则$\lambda = 3$ 是二重特征根.

于是原方程特解为: $y^* = x^2(Ax^2 + Bx + C)e^{3x}$

将其代入方程比较系数得

$$A = \frac{1}{12}, B = 0, C = 0$$

所以 $y^* = \cdots$

$3(2).\ \ y'' + 2y' + 5y = xe^{-x}\sin 2x + \sin^2 x + \sin\dfrac{x}{2}\cos\dfrac{x}{2}$

解: 方程的特征方程为 $r^2 + 2r + 5 = 0 \longrightarrow r_{1,2} = -1 \pm 2i.$

对于 $f_1(x) = xe^{-x}\sin 2x$，$\lambda \pm i\omega = -1 \pm 2i$ 是特征根. 所以

$$y_1^* = x[(A_1x + B_1)\cos 2x + (C_1x + D_1)\sin 2x]e^{-x}$$

对于 $f_2(x) = \sin^2 x = \dfrac{1-\cos 2x}{2} = \underline{\dfrac{1}{2}} - \underline{\dfrac{1}{2}\cos 2x}$

$\lambda \pm i\omega = 0 \pm 2i$ 不是特征根. $P_n(x) = -\dfrac{1}{2}$，所以

$$y_2^* = A_3 + (A_4\cos 2x + A_5\sin 2x)$$

接3(2).

对于 $f_3(x)=\sin\frac{x}{2}\cos\frac{x}{2}=\frac{1}{2}\sin x$,

$\lambda\pm i\omega=0\pm i$ 不是特征根. $P_n(x)=\frac{1}{2}$,

所以 $y_3^*=A_6\cos x+A_7\sin x$

因此 $y^*=y_1^*+y_2^*+y_3^*$

4. 求方程 $y'' - 2y' + y = e^x$ 的通解.

解: 方程的特征方程为 $r^2 - 2r + 1 = 0 \longrightarrow r_{1,2} = 1$

$\longrightarrow$ 齐次方程的通解 $Y = (C_1 + C_2 x)e^x$

设非齐次方程的特解 $y^* = x^2 A e^x$

代入原方程并比较系数得 $A = \dfrac{1}{2}$

$\longrightarrow y^* = \dfrac{1}{2}x^2 e^x$

$\longrightarrow$ 非齐次方程的通解 $y = (C_1 + C_2 x)e^x + \dfrac{1}{2}x^2 e^x$

5. 求方程 $y'' + y = \cos^2 x$ 在原点处与直线 $y = 2x$ 相切的特解。

解: 方程的特征方程为 $r^2 + 1 = 0 \longrightarrow r_{1,2} = \pm i$

$\longrightarrow$ 齐次方程的通解 $Y = C_1 \cos x + C_2 \sin x$

$$f(x) = \cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2} + \frac{1}{2}\cos 2x$$

所以设 $y^* = A + (B\cos 2x + C\sin 2x)$ 代入原方程得

$$A = \frac{1}{2}, B = -\frac{1}{6}, C = 0 \longrightarrow y^* = \frac{1}{2} - \frac{1}{6}\cos 2x$$

因此 $\left.\begin{array}{l} y = C_1 \cos x + C_2 \sin x + \frac{1}{2} - \frac{1}{6}\cos 2x \\ y(0) = 0,\ y'(0) = 2 \end{array}\right\} \longrightarrow \left\{\begin{array}{l} C_1 = -\frac{1}{3} \\ C_2 = 2 \end{array}\right.$

$$\longrightarrow y = -\frac{1}{3}\cos x + 2\sin x + \frac{1}{2} - \frac{1}{6}\cos 2x$$

6.设 $f(x)$ 为二阶可导函数,且

$$f(x)=\sin x+\int_0^x (x-t)f(t)dt$$ 试求 $f(x)$ 。

解: 原方程可化为 $f(x)=\sin x+x\int_0^x f(t)dt-\int_0^x tf(t)dt$

$\longrightarrow f'(x)=\cos x+\int_0^x f(t)dt \longrightarrow f''(x)-f(x)=-\sin x$

$\longrightarrow r^2-1=0 \longrightarrow r_{1,2}=\pm 1 \longrightarrow Y=C_1e^{-x}+C_2e^x$

设 $y^*=A\cos x+B\sin x$ 代入原方程得 $y^*=\dfrac{1}{2}\sin x$

$\longrightarrow y=C_1e^{-x}+C_2e^x+\dfrac{1}{2}\sin x$

$f(0)=0,\quad f'(0)=1$ $\longrightarrow C_1=-\dfrac{1}{4}, C_2=\dfrac{1}{4}$

$\longrightarrow y=-\dfrac{1}{4}e^{-x}+\dfrac{1}{4}e^x+\dfrac{1}{2}\sin x$

7.设函数 $y = f(x)$ 满足方程 $y'' + m^2 y = 0 \quad (m > 0)$

试证：$F(x) = [f''(x)]^2 + m^2[f'(x)]^2$ 与 x 无关。

解： 方程的特征方程为 $r^2 + m^2 = 0 \longrightarrow r_{1,2} = \pm mi$

$\longrightarrow$ 齐次方程的通解 $y = C_1 \cos mx + C_2 \sin mx$

$$\left.\begin{array}{l}\longrightarrow y' = -mC_1 \sin mx + mC_2 \cos mx \\ \longrightarrow y'' = -m^2 C_1 \cos mx - m^2 C_2 \sin mx \\ \quad F(x) = [f''(x)]^2 + m^2[f'(x)]^2\end{array}\right\}$$

$$\longrightarrow F(x) = m^4 (C_1^2 + C_2^2)$$

因此 $F(x) = [f''(x)]^2 + m^2[f'(x)]^2$ 与 x 无关。

8．已知函数 $y = e^{2x} + (x+1)e^x$ 是二阶常系数非齐次线性方程 $y'' + ay' + by = ce^x$ 的一个特解，试确定常数 a , b , c并求方程的通解.

解: 将 $y = e^{2x} + (x+1)e^x$ 代入 $y'' + ay' + by = ce^x$ 得:

$$\left.\begin{array}{l} a = -3, b = 2, c = -1 \\ y'' + ay' + by = ce^x \end{array}\right\} \longrightarrow y'' - 3y' + 2y = -e^x$$

特征方程为 $r^2 - 3r + 2 = 0 \longrightarrow r_1 = 2, r_2 = 1$

$$\longrightarrow \left.\begin{array}{l} Y = C_1 e^{2x} + C_2 e^x \\ y^* = e^{2x} + (x+1)e^x \end{array}\right\} \longrightarrow y = C_1 e^{2x} + C_2 e^x + e^{2x} + (x+1)e^x$$

$$\longrightarrow y = C_3 e^{2x} + C_4 e^x + xe^x$$

题组三、应用题

1. 曲线上点(x,y)在 x 轴上垂足为 $M(x,0)$,曲线上点(x,y)处的切线为T,已知 M 到 T 垂线之长等于1,试求曲线族的方程，并求一曲线使之与 y 轴正交。

(曲线在 $x=0$处的切线与y轴垂直)

解: 根据题意作图如右．设曲线方程为 $y=f(x)$,

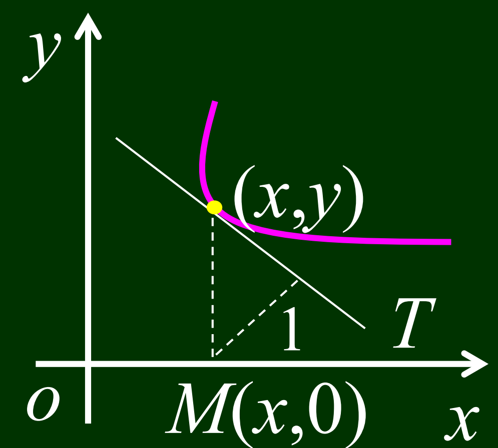
则过点(x,y)的切线方程为 $Y-y=y'(X-x)$．由题设可知

$$|y|\Big/\sqrt{1+(y')^2}=1 \longrightarrow y'=\pm\sqrt{y^2-1}$$

$$\longrightarrow \operatorname{arch}y=\pm(x+C) \longrightarrow y=\operatorname{ch}(x+C)$$

$$\longrightarrow y'=\operatorname{sh}(x+C) \longrightarrow y'(0)=0$$

$$\longrightarrow C=0 \longrightarrow y=\operatorname{ch}x$$ 与y轴正交.



2. 设一容器内有100升溶液，其中含有10升净盐，若每分钟向容器内以匀速注入3升净水，同时以每分钟2升的速率放出浓度均匀的溶液，问过程开始一小时后，溶液中还有多少净盐?

解: 设 t 时刻容器内含盐量为x kg，则 $x = x(t)$. 假设

t到$t+dt$ $(dt>0)$ 内容器中含盐量由x变到$x+dx$ $(dx<0)$.

又设$P(t)$为t时刻容器内盐的浓度，若dt很小， 则在

$[t, t+dt]$内$P(t)$的变化也很小. 因此可以认为

$P(t)$在$[t, t+dt]$内是不变化的. 且$P(t)$就等于t时刻的值.

于是在$[t, t+dt]$内流出的盐量是

接2.

$$2P(t)dt = 2\cdot\frac{x}{100+3t-2t}dt$$

又因在[t , t +d t]内盐减少了- d x，所以有

$$-dx = \frac{2x}{100+(3-2)t}dt$$

又知 $x(0) = 10$，解之得 $x = \dfrac{10^5}{(100+t)^2}$

所以 $x(60) \approx 3.9$

3. 设 $y = y(x)$ 是一条向上凸的连续曲线，其上任一点 (x, y)处的曲率为$1/\sqrt{1+y'^2}$,且此曲线上点(0 ,1)处的切线方程为$y = x + 1$ ，求该曲线的方程。

解: 已知 $y'' < 0$

$$\left.\begin{array}{l} y'' < 0 \\ k = |y''| / (\sqrt{1+y'^2})^3 \end{array}\right\} \longrightarrow \left.\begin{array}{r} y'' = -(1+y'^2) \\ \text{设} y' = p(x) \end{array}\right\} \longrightarrow p' + p^2 + 1 = 0$$

$$\longrightarrow \left.\begin{array}{r} \arctan p = -x + C_1 \\ y'(0) = 1 \end{array}\right\} \longrightarrow C_1 = \frac{\pi}{4} \longrightarrow y' = \tan(\frac{\pi}{4} - x)$$

$$\longrightarrow y = \int \tan(\frac{\pi}{4} - x)dx \longrightarrow \left.\begin{array}{r} y = \ln|\cos(\frac{\pi}{4} - x)| + C_2 \\ y(0) = 1 \end{array}\right\}$$

$$\longrightarrow C_2 = 1 - \ln\tfrac{\sqrt{2}}{2} \longrightarrow y = \ln\cos(\frac{\pi}{4} - x) + 1 - \ln\tfrac{\sqrt{2}}{2}$$