第十章　重积分

题组一：二重积分

一. 计算二重积分

1. $I=\iint\limits_{D} xy\cos(xy^2)dxdy$，其中

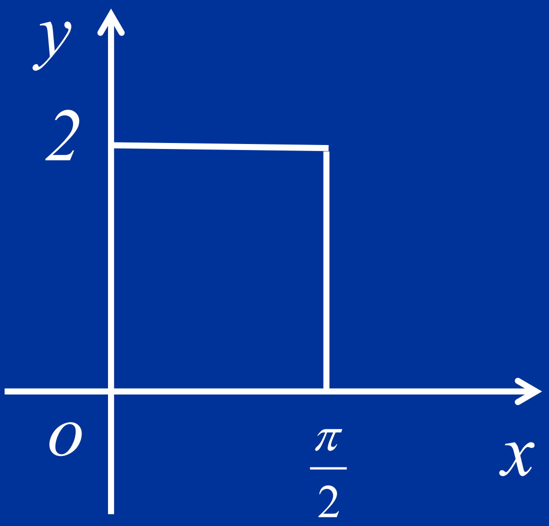
$$D=\{(x,y)\,|\,0\le x\le\frac{\pi}{2},\ 0\le y\le 2\}.$$

解:

$$I=\int_0^{\frac{\pi}{2}}dx\int_0^2 xy\cos(xy^2)dy$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}dx\int_0^2\cos(xy^2)d(xy^2)$$

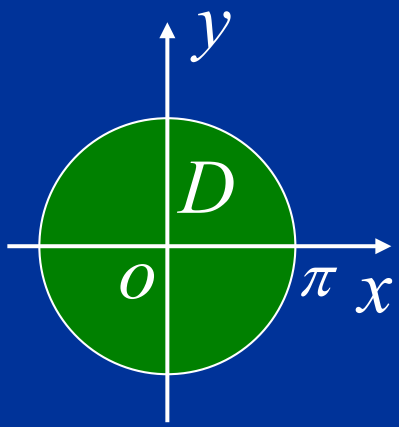
$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin(xy^2)\Big|_0^2 dx\ =0$$



2.求 $$I=\iint\limits_D e^{-(x^2+y^2-\pi)}\sin(x^2+y^2)dxdy,$$

其中 $D=\{(x,y)\,|\,x^2+y^2\le\pi\}.$

解: $$I=\iint\limits_D e^{-r^2+\pi}\sin r^2 rdrd\theta$$



$$=e^{\pi}\int_0^{2\pi}d\theta\int_0^{\sqrt{\pi}}re^{-r^2}\sin r^2dr$$

$$\underline{\underline{t=r^2}}\ \ \pi e^{\pi}\int_0^{\pi}e^{-t}\sin tdt$$

$$=\frac{\pi(1+e^{\pi})}{2}$$

3.求 $I=\iint\limits_D (x^3y^5+\sin y\cos x)dxdy$，其中D是由三点A(1,1),B(-1,1),C(-1,-1)围成的三角形区域.

解: 积分区域如图. 作辅助线*OB*将积分区域分为两部分.

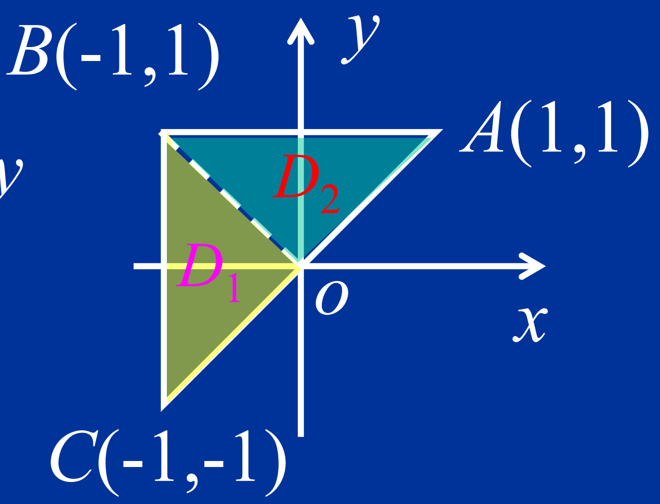
则 $$I=\iint\limits_D x^3y^5dxdy+\iint\limits_D \sin y\cos xdxdy$$

$$=\iint\limits_{D_1} x^3y^5dxdy+\iint\limits_{D_2} x^3y^5dxdy$$

$$+\iint\limits_{D_1} \sin y\cos xdxdy+\iint\limits_{D_2} \sin y\cos xdxdy$$

$$=0+0+0+\iint\limits_{D_2} \sin y\cos xdxdy$$

$$=2\int_0^1 \sin ydy\int_0^y \cos xdx=1-\frac{1}{2}\sin 2$$



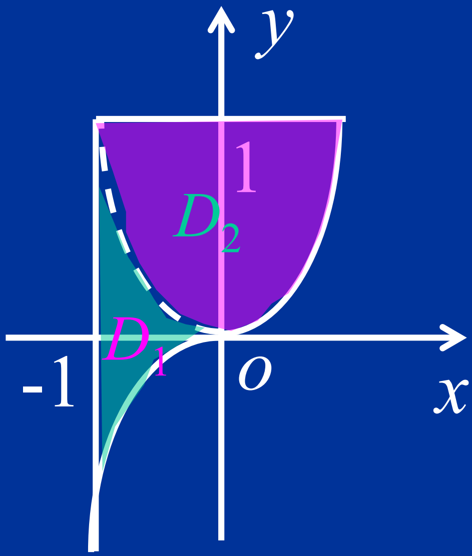
4.求 $I=\iint\limits_D x^5[\sin^7 y+y^3 f(x^2+y^2)]dxdy$ 其中D是由 $y=x^3$, $x=-1$, $y=1$ 围成.

解: 积分区域如图. 作辅助线 $y=-x^3\ (x<0)$ 积分区域被分为两部分. 于是

$$I=\iint\limits_{D_1} x^5[\sin^7 y+y^3 f(x^2+y^2)]dxdy$$

$$+\iint\limits_{D_2} x^5[\sin^7 y+y^3 f(x^2+y^2)]dxdy$$

$$=0$$



5.求 $I=\iint\limits_D (x+y)dxdy$，其中

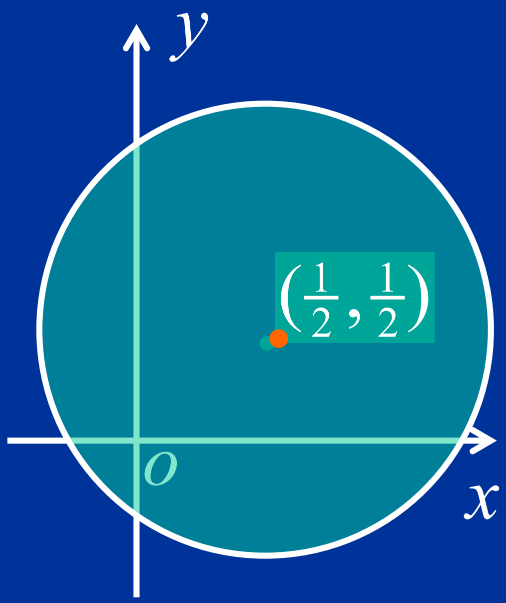
$$D=\{(x,y)\mid x^2+y^2\le x+y+1\}.$$

解: $x^2+y^2=x+y+1 \longrightarrow (x-\frac{1}{2})^2+(y-\frac{1}{2})^2=(\sqrt{\frac{3}{2}})^2$

积分区域如图．则均匀圆形薄片的形心为 $(\frac{1}{2},\frac{1}{2})$．

圆的面积为 $A=\frac{3}{2}\pi$ 利用形心的坐标公式有:

$$\frac{1}{2}=\frac{\iint\limits_D xdxdy}{\frac{3}{2}\pi},\quad \frac{1}{2}=\frac{\iint\limits_D ydxdy}{\frac{3}{2}\pi}$$



所以 $I=\frac{1}{2}\frac{3}{2}\pi+\frac{1}{2}\frac{3}{2}\pi=\frac{3}{2}\pi$

6. 求 $I=\iint\limits_D (|x|+y)dxdy$, 其中 $D=\{(x,y)\,\big|\, |x|+|y|\le 1\}$.

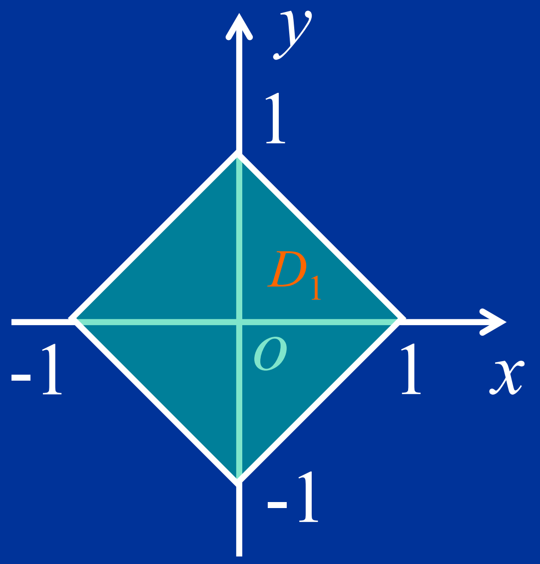
解: 积分区域如图. 利用对称性有

$$I = \iint\limits_D |x|dxdy + \iint\limits_D ydxdy$$

$$= 4\iint\limits_{D_1} x\, dxdy + 0$$

$$= 4\int_0^1 xdx\int_0^{1-x} dy$$

$$= \frac{2}{3}$$



7. 求 $I=\iint\limits_D |\cos(x+y)|dxdy$, 其中D是由 $y=x$,

$y=0,\ x=\dfrac{\pi}{2}$ 围成.

解: 积分区域如图. 作辅助线 $x+y=\frac{\pi}{2}$ 积分区域

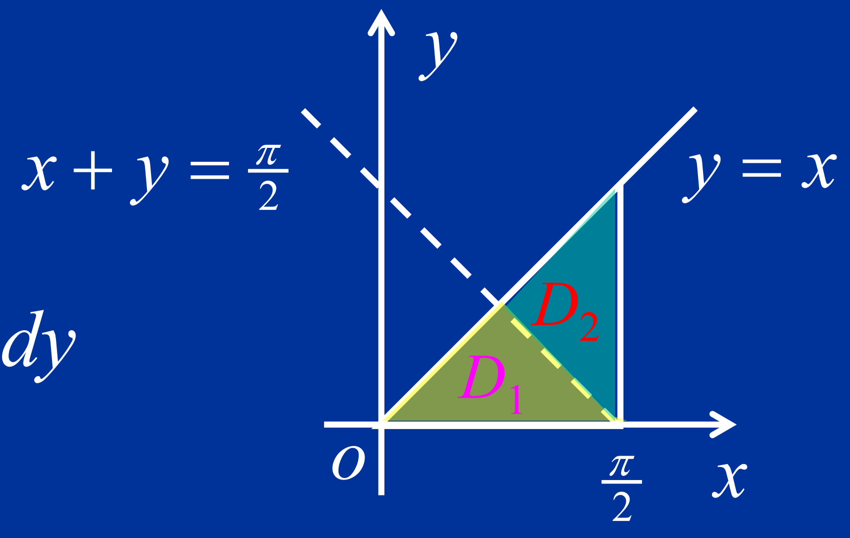
分为两部分. 于是

$$I=\iint\limits_{D_1}\cos(x+y)dxdy-\iint\limits_{D_2}\cos(x+y)dxdy$$

$$=\int_0^{\frac{\pi}{4}}dy\int_y^{\frac{\pi}{2}-y}\cos(x+y)dx$$

$$-\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}dx\int_{\frac{\pi}{2}-x}^{x}\cos(x+y)dy$$

$$=\frac{\pi}{2}+1$$



8. 设 $f(x,y)=\begin{cases} x+y\ ,\ x^2\le y\le 2x^2 \\ 0\ ,\ \text{其他} \end{cases}$，求 $I=\iint\limits_D f(x,y)dxdy$，

其中 $D=\{(x,y)\big|\ 0\le x\le 1\ ,\ 0\le y\le 1\}$.

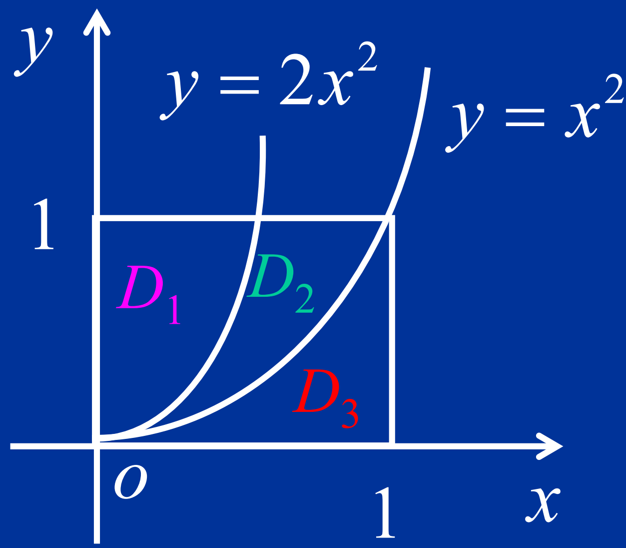
解: 积分区域如图.

$$I=\iint\limits_{D_1} f(x,y)dxdy+\iint\limits_{D_2} f(x,y)dxdy+\iint\limits_{D_3} f(x,y)dxdy$$

$$=\iint\limits_{D_1} 0\,dxdy+\iint\limits_{D_2}(x+y)dxdy+\iint\limits_{D_3} 0\,dxdy$$

$$=\iint\limits_{D_2}(x+y)dxdy$$

$$=\int_0^1 dy\int_{\sqrt{\frac{y}{2}}}^{\sqrt{y}}(x+y)dx=\tfrac{1}{5}\left(\tfrac{21}{8}-\sqrt{2}\right)$$

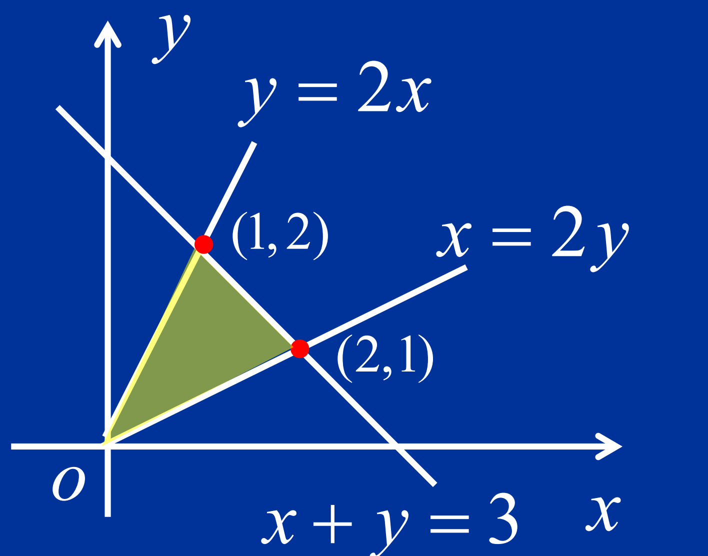


二. 二次积分

1. 将 $I=\iint\limits_D f(x,y)dxdy$ 化为直角坐标系下的累次积分,

其中 D: $y\le 2x,\ x\le 2y,\ x+y\le 3.$

解: 积分区域如图. 所以



$$I=\iint\limits_D f(x,y)dxdy$$

$$=\int_0^1 dx\int_{\frac{x}{2}}^{2x} f(x,y)dy$$

$$+\int_1^2 dx\int_{\frac{x}{2}}^{3-x} f(x,y)dy$$

或

$$=\int_0^1 dy\int_{\frac{y}{2}}^{2y} f(x,y)dx+\int_1^2 dy\int_{\frac{y}{2}}^{3-y} f(x,y)dx$$

2. 将 $I=\int_0^1 dx\int_0^1 f(x,y)dy$ 化为极坐标系下的累次积分.

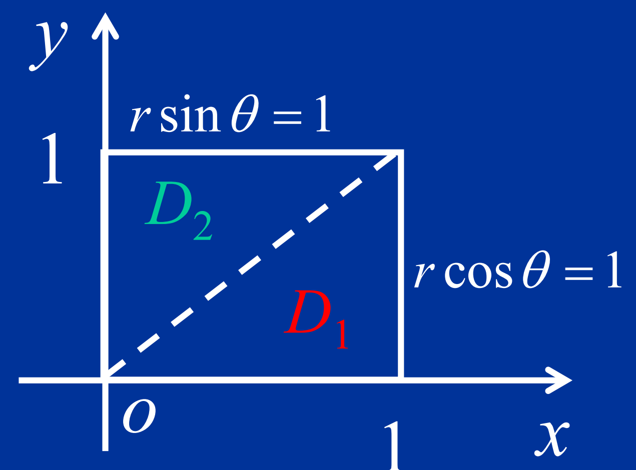
解: 积分区域如图. 作辅助线将积分区域分为两部分.

则 $$I=\iint\limits_{D_1} f(r\cos\theta, r\sin\theta)rdrd\theta$$

$$+\iint\limits_{D_2} f(r\cos\theta, r\sin\theta)rdrd\theta$$

$$=\int_0^{\frac{\pi}{4}} d\theta\int_0^{\frac{1}{\cos\theta}} f(r\cos\theta, r\sin\theta)rdr$$

$$+\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta\int_0^{\frac{1}{\sin\theta}} f(r\cos\theta, r\sin\theta)rdr$$



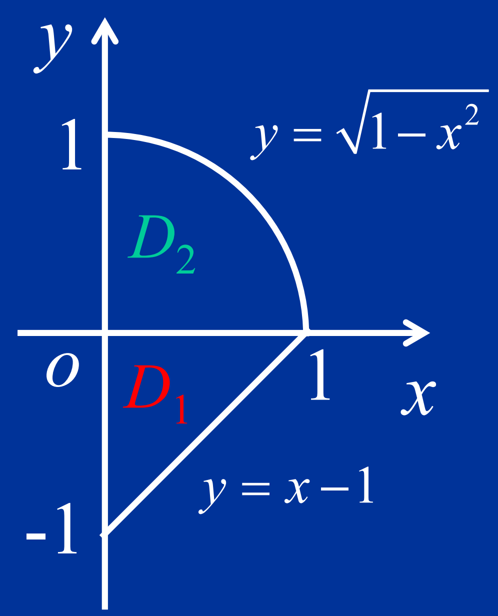
3. 交换下列二次积分的顺序

(1) $I=\int_0^1 dx\int_{x-1}^{\sqrt{1-x^2}} f(x,y)dy$

解: 积分区域如图. 所以

$$I=\int_{-1}^{0} dy\int_0^{y+1} f(x,y)dx$$

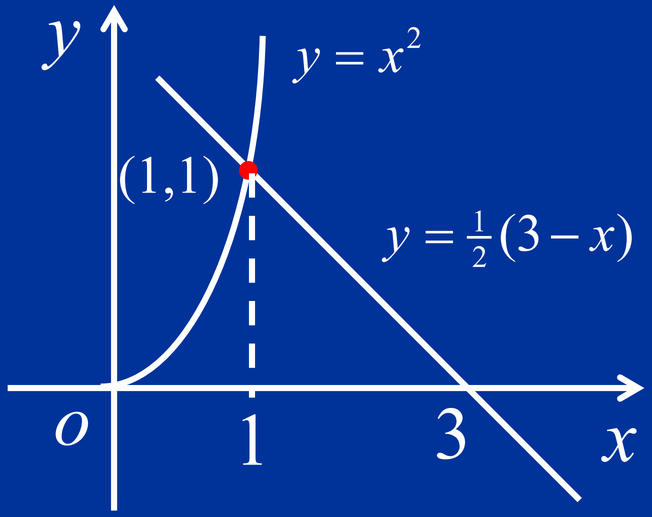
$$+\int_0^1 dy\int_0^{\sqrt{1-y^2}} f(x,y)dx$$



(2) $$I=\int_0^1 dx\int_0^{x^2} f(x,y)dy+\int_1^3 dx\int_0^{\frac{1}{2}(3-x)} f(x,y)dy$$

解: 积分区域如图.

$$I=\int_0^1 dy\int_{\sqrt{y}}^{3-2y} f(x,y)dx$$



(3) $\int_{\frac{1}{2}}^{1} dx \int_{\frac{1}{x}}^{x} f(x,y)dy$

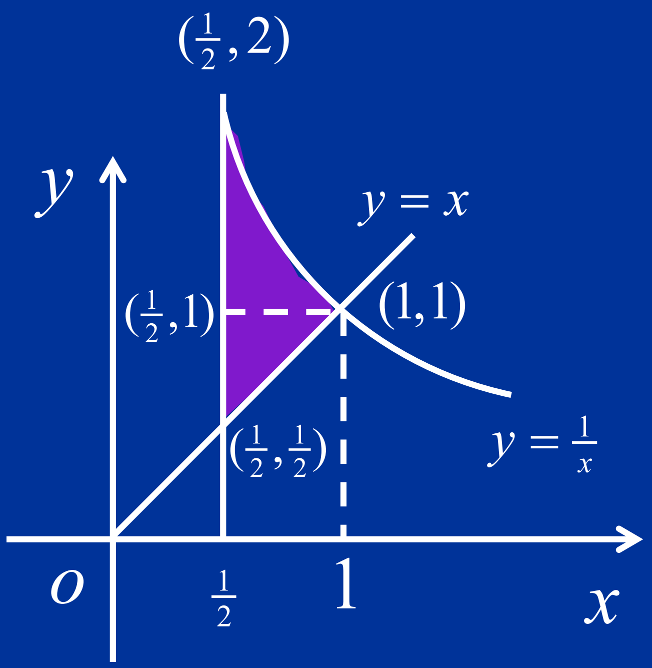
解: 积分区域如图.

$$\int_{\frac{1}{2}}^{1} dx \int_{\frac{1}{x}}^{x} f(x,y)dy = -\int_{\frac{1}{2}}^{1} dx \int_{x}^{\frac{1}{x}} f(x,y)dy$$

$$= -\iint_{D} f(x,y)dxdy$$

$$= -\int_{\frac{1}{2}}^{1} dy \int_{\frac{1}{2}}^{y} f(x,y)dx$$

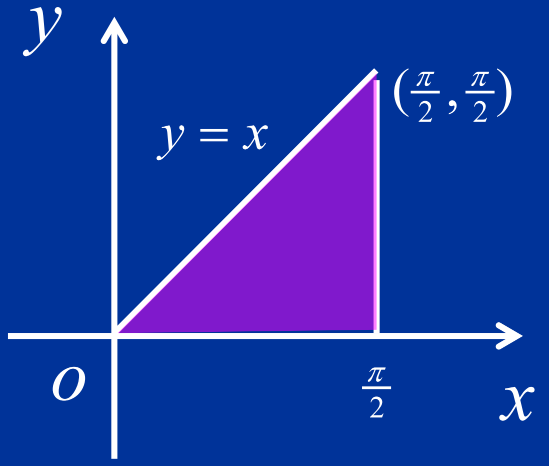
$$-\int_{1}^{2} dy \int_{\frac{1}{2}}^{\frac{1}{y}} f(x,y)dx$$



4. 计算 $I=\int_0^{\frac{\pi}{2}}dy\int_y^{\frac{\pi}{2}}\frac{\sin x}{x}dx.$

解: 积分区域如图. 交换积分顺序得:

$$I=\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}dx\int_0^x dy=1$$



5. 设 $f(x)=\int_{x^3}^{x} e^{-y^2}dy$, 计算 $I=\int_0^1 x^2 f(x)dx.$

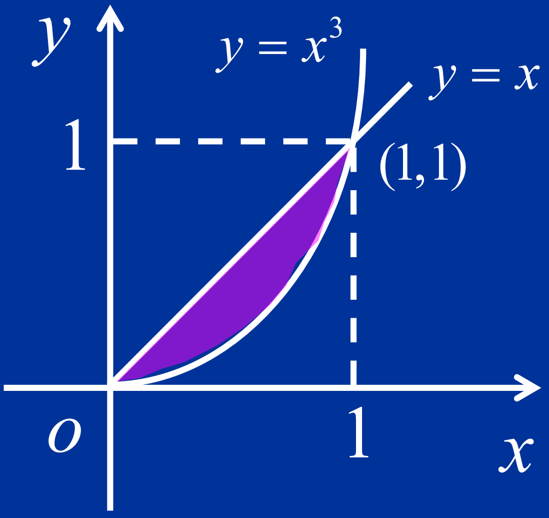
解: 积分区域如图. 根据题意知

$$I=\int_0^1 x^2\left[\int_{x^3}^{x} e^{-y^2}dy\right]dx$$

交换积分顺序

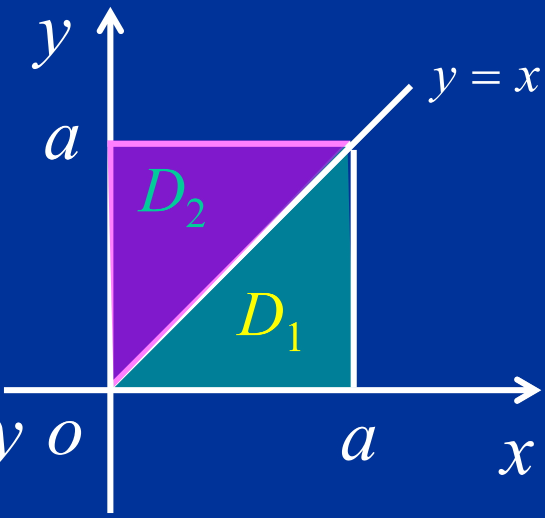
$$=\int_0^1 e^{-y^2}dy\int_y^{\sqrt[3]{y}} x^2dx$$

$$=\frac{1}{6e}$$



6. 设 $f(x)$ 在 $[0,a]$ $(a>0)$ 上连续, 证明

$$\int_0^a dx\int_0^x f(x)f(y)dy=\frac{1}{2}[\int_0^a f(x)dx]^2.$$



解: 积分区域如图.

$$\int_0^a dx\int_0^x f(x)f(y)dy=\iint_{D_1} f(x)f(y)dxdy$$

而 $$\iint_{D_1} f(x)f(y)dxdy=\iint_{D_2} f(x)f(y)dxdy$$

$$\therefore \int_0^a f(x)dx\int_0^x f(y)dy$$

$$=\frac{1}{2}(\iint_{D_1} f(x)f(y)dxdy+\iint_{D_2} f(x)f(y)dxdy)$$

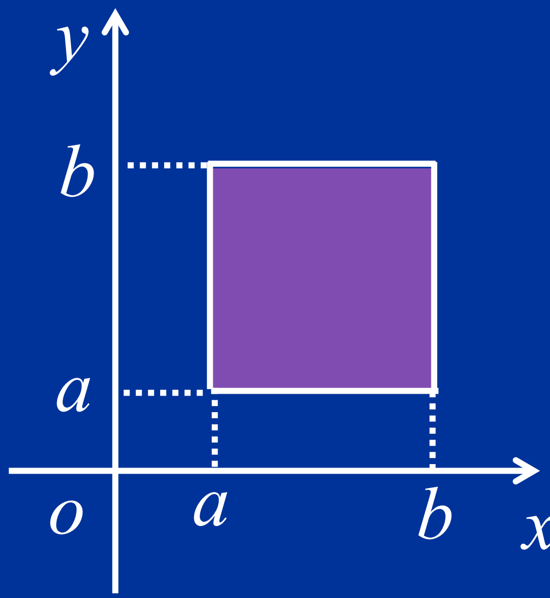
$$=\frac{1}{2}\iint_{D} f(x)f(y)dxdy=\frac{1}{2}\int_0^a f(x)dx\int_0^a f(y)dy=\frac{1}{2}[\int_0^a f(x)dx]^2$$

三. 二重积分的应用

1. 利用二重积分证明不等式

设 $f(x)$在$[a,b]$ $(a>0)$上连续, 且 $f(x)>0$, 则

$$\int_a^b f(x)dx\int_a^b \frac{1}{f(x)}dx \ge (b-a)^2.$$



解: 积分区域如图.

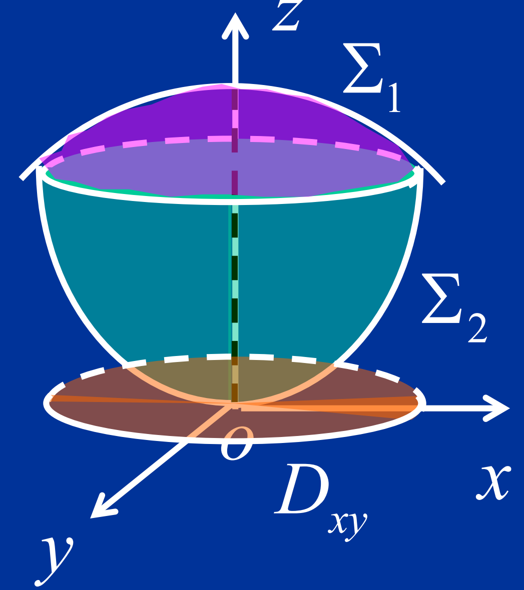
$$\int_a^b f(x)dx\int_a^b \frac{1}{f(x)}dx = \begin{cases} \iint\limits_D \dfrac{f(y)}{f(x)}dxdy \\ \iint\limits_D \dfrac{f(x)}{f(y)}dxdy \end{cases}$$

$$=\frac{1}{2}\iint\limits_D \left[\frac{f(y)}{f(x)}+\frac{f(x)}{f(y)}\right]dxdy = \frac{1}{2}\iint\limits_D \frac{f^2(x)+f^2(y)}{f(x)f(y)}dxdy$$

$$\ge \frac{1}{2}\iint\limits_D \frac{2f(x)f(y)}{f(x)f(y)}dxdy = (b-a)^2$$

2. 求抛物面 $z=x^2+y^2$ 与球面 $x^2+y^2+z^2=6$ 所围立体的体积和表面积.

解: 积分区域如图.



$$\left.\begin{aligned} x^2+y^2+z^2&=6 \\ z&=x^2+y^2 \end{aligned}\right\} \Rightarrow D_{xy}: x^2+y^2\le 2$$

$$\Rightarrow \begin{cases} \Sigma_2: z=x^2+y^2 \\ \Sigma_1: z=\sqrt{6-x^2-y^2} \end{cases}$$

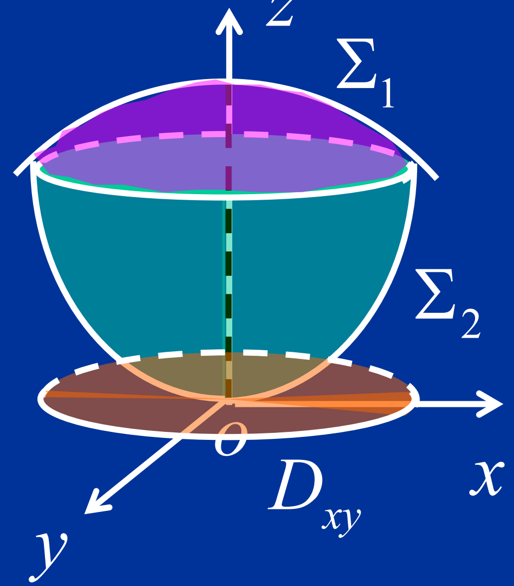
$$\Rightarrow \begin{cases} S_2=\iint\limits_{D_{xy}}\sqrt{1+4x^2+4y^2}\,dxdy=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\sqrt{1+4r^2}\,rdr=\cdots \\ S_1=\iint\limits_{D_{xy}}\sqrt{\dfrac{6}{6-x^2-y^2}}\,dxdy=\sqrt{6}\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}\dfrac{rdr}{\sqrt{6-r^2}}=\cdots \end{cases}$$

$$\Rightarrow S=S_1+S_2=\frac{13}{3}\pi+(12\pi-4\sqrt{6}\pi)=\pi(\frac{49}{3}-4\sqrt{6})$$

$$V = \iint\limits_{D} (z_1 - z_2)dxdy = \iint\limits_{D} [\sqrt{6 - x^2 - y^2} - (x^2 + y^2)]dxdy$$

$$= \int_0^{2\pi} d\theta \int_0^{\sqrt{2}} (\sqrt{6 - r^2} - r^2) r dr$$

$$= 2\pi(-\frac{11}{3} + 2\sqrt{6})$$



3. 证明曲面 $z=4+x^2+y^2$ 上任一点处的切平面与曲面 $z=x^2+y^2$ 所围立体体积为定值.

解: 设曲面上任一点的坐标为 (x_0, y_0, z_0), 则过该点的切平面方程为 $2x_0(x-x_0)+2y_0(y-y_0)-(z-z_0)=0$

$$\left.\begin{array}{l}\text{即}\quad 2x_0x+2y_0y-z-(2x_0{}^2+2y_0{}^2-z_0)=0\\ \text{又知}\qquad z_0=4+x_0{}^2+y_0{}^2\end{array}\right\}\Rightarrow$$

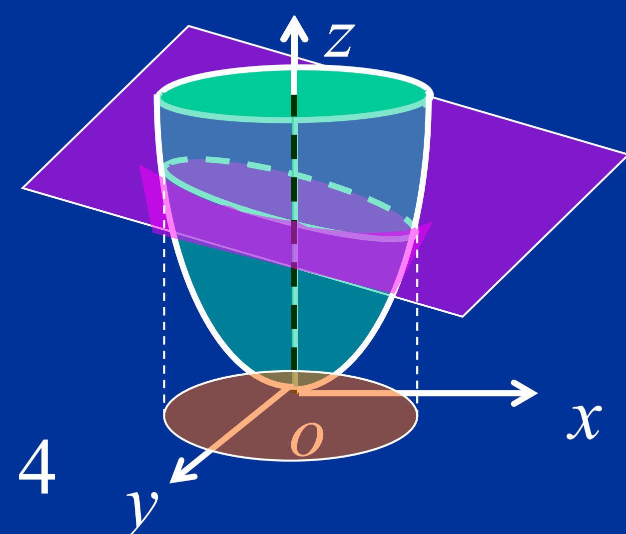
切平面方程为 $2x_0x+2y_0y-z-x_0{}^2-y_0{}^2+4=0$

切平面和曲面所围立体如图.

$$\left.\begin{array}{l}z=x^2+y^2\\ 2x_0x+2y_0y-z-x_0{}^2-y_0{}^2+4=0\end{array}\right\}$$

消去 z

$$\Longrightarrow\quad D_{xy}:(x-x_0)^2+(y-y_0)^2=4$$



$$V = \iint\limits_{D_{xy}} [(2x_0 x + 2y_0 y - x_0{}^2 - y_0{}^2 + 4) - (x^2 + y^2)]dxdy$$

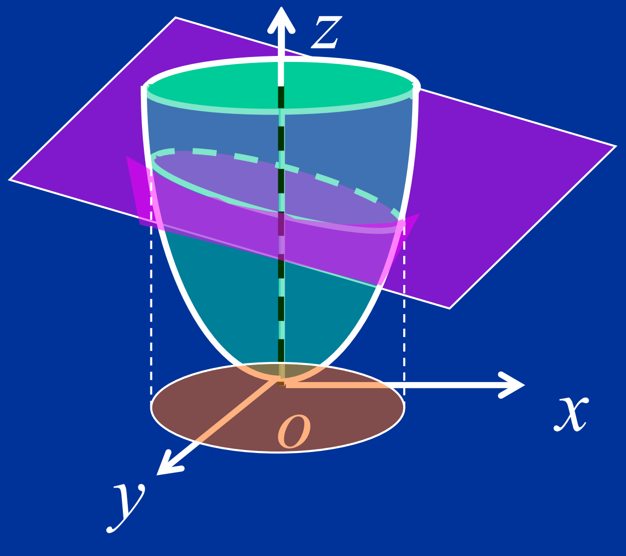
$$= \iint\limits_{D_{xy}} [4 - (x - x_0)^2 - (y - y_0)^2]dxdy$$

$$x - x_0 = r\cos\theta$$

$$y - y_0 = r\sin\theta$$

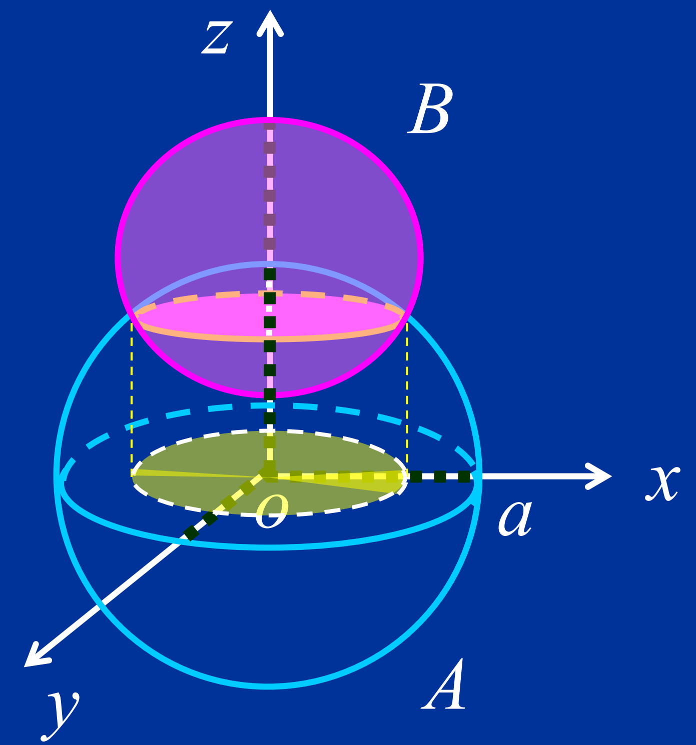
$$= \int_0^{2\pi} d\theta \int_0^2 (4 - r^2) r dr$$

$$= 8\pi$$



4. 已知球A的半径为a，另求一球B,球心在球A的球面上,问球B的半径R为多少时,球B位于球A内部的表面积为最大,并求出最大表面积.

解: 根据题意作图如右. 则



$$\left.\begin{aligned} A: &\ x^2+y^2+z^2=a^2 \\ B: &\ x^2+y^2+(z-a)^2=R^2 \end{aligned}\right\} \Rightarrow$$

$$\left.\begin{aligned} D_{xy}: &\ x^2+y^2\le\frac{R^2}{4a^2}(4a^2-R^2) \\ \Sigma: &\ z=a-\sqrt{R^2-x^2-y^2} \end{aligned}\right\}$$

$$\Rightarrow S=\iint_{D_{xy}}\sqrt{1+z_x{}^2+z_y{}^2}\,dxdy=\iint_{D_{xy}}\frac{R}{\sqrt{R^2-x^2-y^2}}dxdy$$

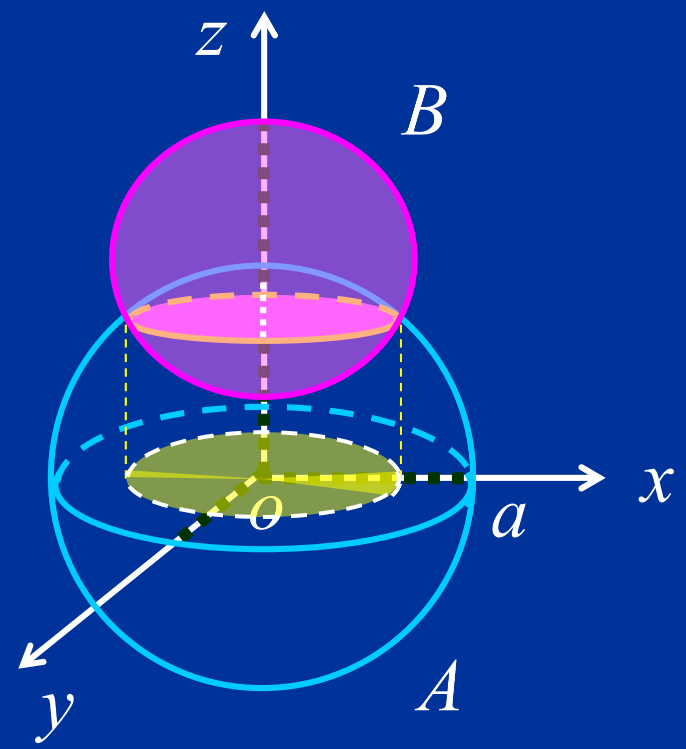
$$=\int_0^{2\pi}d\theta\int_0^{\frac{R}{2a}\sqrt{4a^2-R^2}}\frac{R}{\sqrt{R^2-r^2}}rdr=2\pi R^2-\frac{\pi}{a}R^3$$

$$S(R) = 2\pi R^2 - \frac{\pi}{a} R^3$$

$$\Rightarrow \quad S'(R) = 4\pi R - \frac{3\pi}{a} R^2 = 0$$

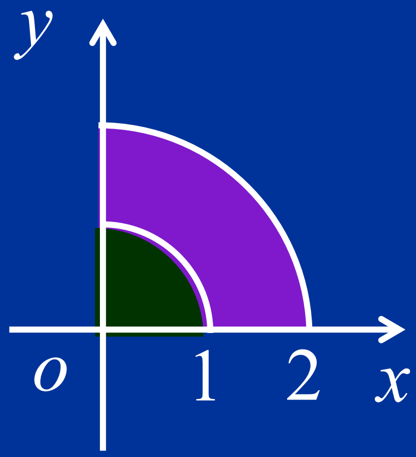
$$\Rightarrow \quad R = \frac{4}{3} a$$

$$\Rightarrow \quad \max S = S\left(\frac{4}{3} a\right) = \frac{32}{27} \pi a^2 .$$



5. 求由两同心圆 $x^2+y^2=1$ 和 $x^2+y^2=4$ 所围在第一象限内的四分之一圆环板的重心,其中面密度ρ为常数.

解: 根据题意作图如右. 由对称性知 $\bar{x}=\bar{y}$.



板面积 $A=\dfrac{3}{4}\pi$

$$\iint_D x\,dxdy=\int_0^{\frac{\pi}{2}}d\theta\int_1^2 r\cos\theta\, r\,dr=\frac{7}{3}$$

所以 $\bar{x}=\bar{y}=\dfrac{\iint_D x\,dxdy}{A}=\dfrac{7}{3}\Big/\dfrac{3}{4}\pi=\dfrac{28}{9\pi}$

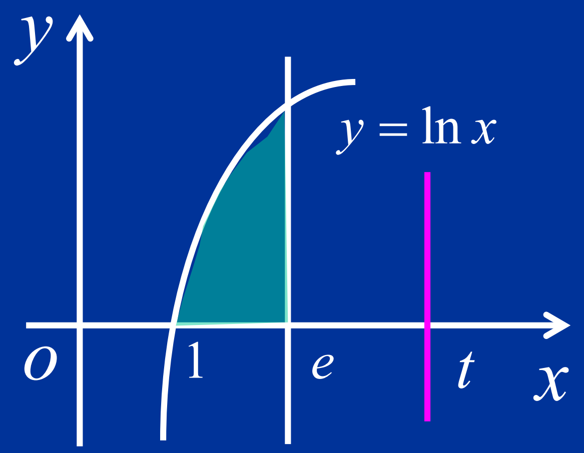
故重心坐标为 $\left(\dfrac{28}{9\pi},\dfrac{28}{9\pi}\right)$.

6. 设有一由 $y=\ln x$ ，x 轴及 $x=e$所围成的均匀薄片，密度ρ=1,求此薄片绕 $x=t$ 旋转的转动惯量 $I(t)$，并求 t 的值使 $I(t)$ 最小.

解: 根据题意作图如右.

$$I(t)=\iint_D \rho(x-t)^2\,dxdy=\int_1^e dx\int_0^{\ln x}(x-t)^2dy$$

$$=t^2-\frac{1}{2}(e^2+1)t+\frac{2}{9}e^3+\frac{1}{9}$$



令 $I'(t)=2t-\frac{1}{2}(e^2+1)=0$

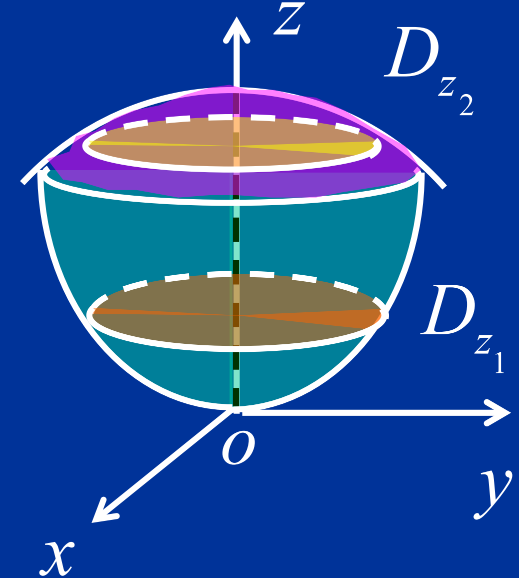
得 $t=\frac{1}{4}(e^2+1)$

故 $\min I(t)=I(\frac{1}{4}(e^2+1))=\frac{2}{9}e^3+\frac{1}{9}-\frac{1}{16}(e^2+1)^2$

题组二：三重积分

1. 已知 $I=\iiint\limits_{\Omega} z\,dxdydz$, 其中Ω由 $x^2+y^2+z^2\le 4$与 $z\ge\frac{1}{3}(x^2+y^2)$ 围成,试将 I 分别化为三种坐标系下的三次积分,并计算其值.

解: 积分区域如图.



直角坐标系下采用“先二后一”法.

$$I=\int_0^1 zdz\iint_{D_{z_1}}dxdy+\int_1^2 zdz\iint_{D_{z_2}}dxdy$$

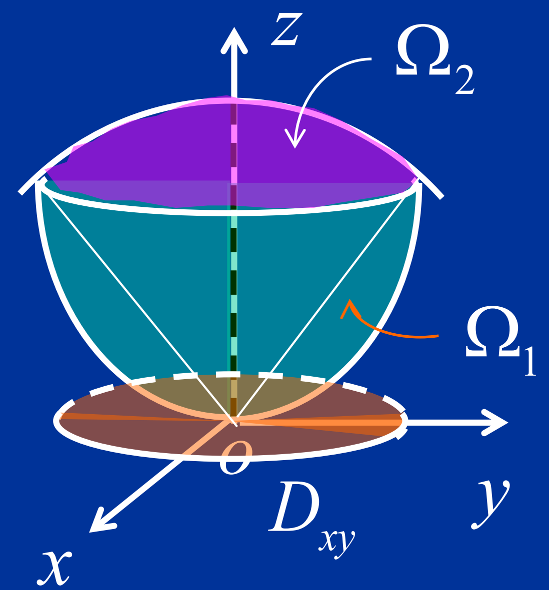
$$=\int_0^1 z\pi 3zdz+\int_1^2 z\pi(4-z^2)dz$$

$$=\frac{13}{4}\pi\,.$$

采用柱坐标.

$$I = \int_0^{2\pi} d\theta \int_0^{\sqrt{3}} r dr \int_{\frac{1}{3}r^2}^{\sqrt{4-r^2}} z dz \ = \cdots$$

采用球坐标. 作辅助线后积分区域



被分为如图两部分. 于是

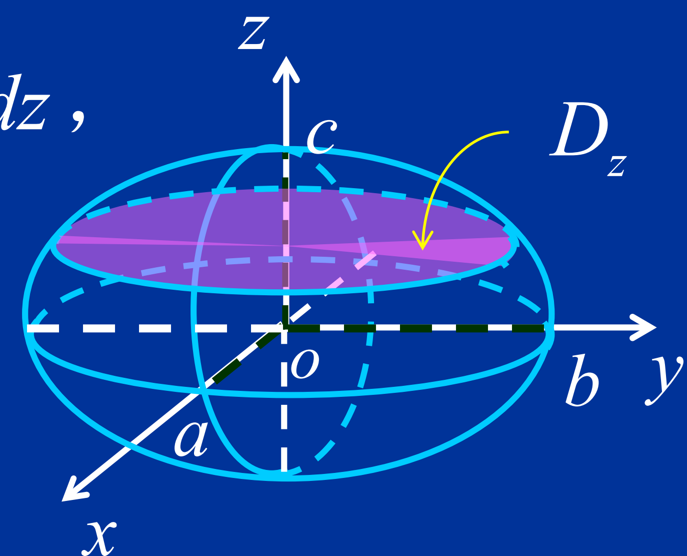
$$I = \int_0^{2\pi} d\theta \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} d\varphi \int_0^{\frac{3\cos\varphi}{\sin^2\varphi}} r\cos\varphi r^2 \sin\varphi dr$$

$$+\int_0^{2\pi} d\theta \int_0^{\frac{\pi}{3}} d\varphi \int_0^{2} r\cos\varphi r^2 \sin\varphi dr$$

$= \cdots$

2. 计算 $I=\iiint\limits_{\Omega}(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2})dxdydz$，

其中 $\Omega: \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1.$



解: 积分区域如图.

$$\iiint\limits_{\Omega} z^2 dv = 2\int_0^c z^2 dz \iint_{D_z} dxdy$$

$$=2\int_0^c z^2 \pi ab(1-\frac{z^2}{c^2})dz=\frac{4}{15}\pi abc^3$$

同理 $\iiint\limits_{\Omega} x^2 dv = 2\int_0^a x^2 dx \iint_{D_x} dydz = \frac{4}{15}\pi a^3 bc$

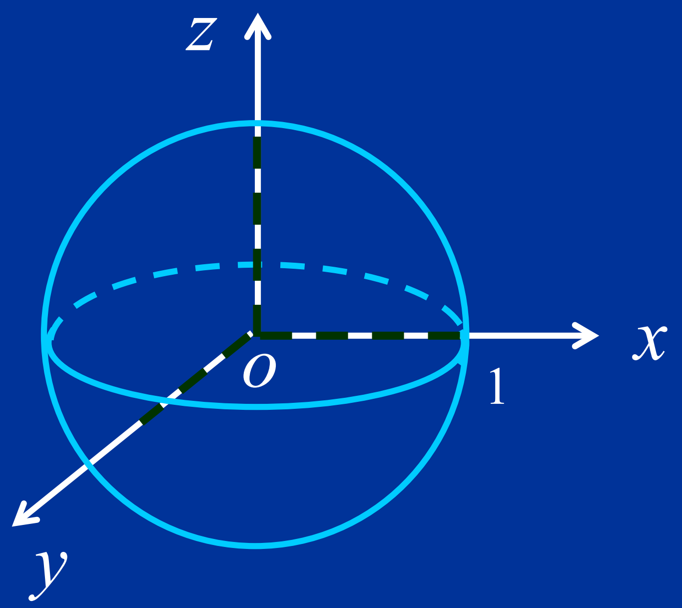
$$\iiint\limits_{\Omega} y^2 dv = \frac{4}{15}\pi ab^3 c$$

故 $I=\frac{4}{5}\pi abc$

3. 计算 $I=\iiint\limits_{\Omega}\dfrac{ze^{\sqrt{x^2+y^2+z^2}}}{1+x^2+y^2+z^2}dxdydz$, 其中Ω由

$x^2+y^2+z^2=1$ 围成.

解: 积分区域如图.



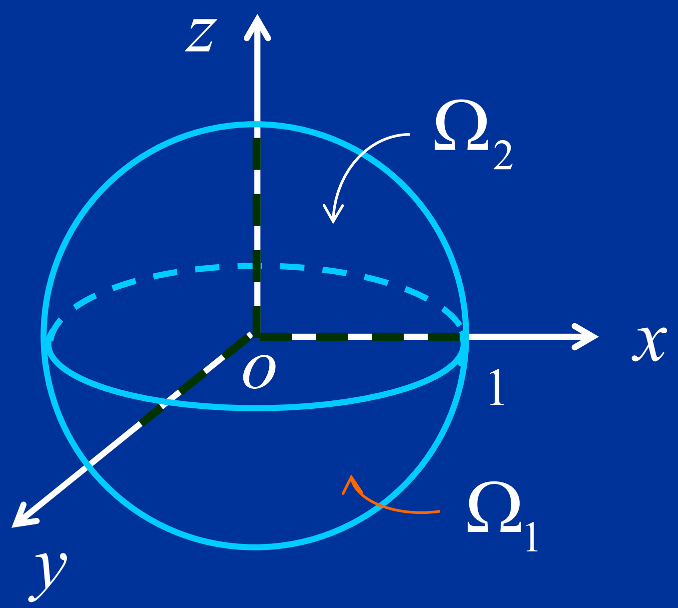
积分区域关于三坐标面对称,

被积函数关于 z 是奇函数,

故积分为零.

此题也可用“先一后二”法讨论.

4. 计算 $I=\iiint\limits_{\Omega} e^{|z|}dxdydz$，其中 $\Omega: x^2+y^2+z^2\le 1$.



解: 由区域的对称性及被积函数的特征，得

$$I=2\iiint\limits_{\Omega_2} e^{z}dxdydz$$

$$=2\int_0^1 e^z dz\iint_{D_z} dxdy$$

$$=2\int_0^1 \pi(1-z^2)e^z dz$$

$$=2\pi\ .$$

5. 计算 $I=\iiint\limits_{\Omega}(x+2y+3z)dxdydz$， 其中Ω由

$x^2+y^2+z^2-2x-2y-2z-6=0$ 围成.

解: $\Omega:\ (x-1)^2+(y-1)^2+(z-1)^2=9$

重心坐标为(1 , 1 , 1)， $V=\frac{4}{3}\pi 3^3=36\pi$

$$\iiint\limits_{\Omega} xdv=\overline{x}\cdot V=36\pi$$

同理 $\iiint\limits_{\Omega} ydv=36\pi$， $\iiint\limits_{\Omega} zdv=36\pi$

所以 $I=216\pi$.

6. 当 $f(x)$ 连续时,证明:

$$\iiint\limits_{x^2+y^2+z^2\le 1} f(z)dv = \pi\int_{-1}^{1} f(x)(1-x^2)dx\,.$$

证明:

$$\iiint\limits_{x^2+y^2+z^2\le 1} f(z)dv = \int_{-1}^{1} f(z)dz\iint_{D_z} dxdy$$

$$= \int_{-1}^{1} f(z)\pi(1-z^2)dz$$

$$= \pi\int_{-1}^{1} f(x)(1-x^2)dx\,.$$

7. 设 $f(x)$为连续函数,且 $\Phi(t)=\iiint\limits_{\Omega} f(x^2+y^2+z^2)dxdydz$

其中Ω由 $x^2+y^2+z^2=t^2\ (t>0)$ 围成, 求 $\Phi'(t)$.

解: 利用球坐标有

$$\Phi(t)=\int_0^{2\pi}d\theta\int_0^{\pi}d\varphi\int_0^{t}f(r^2)r^2\sin\varphi dr$$

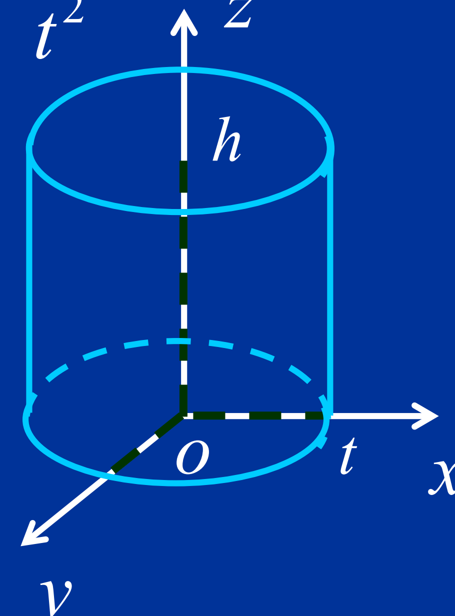
$$=\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi d\varphi\int_0^{t}f(r^2)r^2dr$$

$$=-4\pi\int_0^{t}f(r^2)r^2dr$$

$$\Phi'(t)=-4\pi f(t^2)t^2\ .$$

8. 设$f(x)$为连续函数,$F(t)=\iiint\limits_{\Omega}[z^2+f(x^2+y^2)]dxdydz$,

其中 $\Omega: 0\le z\le h, x^2+y^2\le t^2$, 求 $\lim\limits_{t\to 0^+}\dfrac{F(t)}{t^2}$.



解: 积分区域如图. 利用柱坐标有

$$F(t)=\int_0^{2\pi}d\theta\int_0^t dr\int_0^h[z^2+f(r^2)]rdz$$

$$=\frac{1}{3}\pi h^3t^2+2h\pi\int_0^t f(r^2)rdr$$

$$\lim_{t\to 0^+}\frac{F(t)}{t^2}=\lim_{t\to 0^+}\frac{\frac{1}{3}\pi h^3t^2+2h\pi\int_0^t f(r^2)rdr}{t^2}$$

$$=\frac{1}{3}\pi h^3+h\pi f(0)$$

9. 求曲面 $z=(x^2+y^2+z^2)^2$ 所围立体体积.

解: 由曲面方程可知, 立体位于xOy面上部, 且关于 xOz yOz面对称, 并与xOy面相切, 故在球坐标系下所围立体为

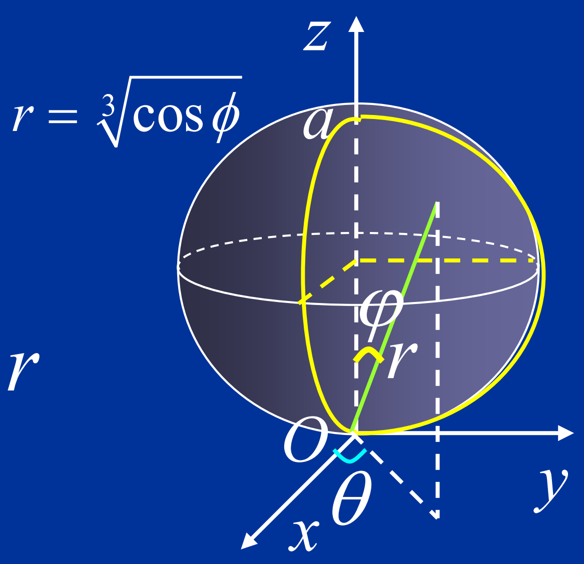
$$\Omega: 0 \le r \le \sqrt[3]{\cos\phi},\ 0 \le \varphi \le \pi/2,\ 0 \le \theta \le 2\pi$$

利用对称性, 所求立体体积为

$$V = \iiint_\Omega \mathrm{d}v$$

$$= 4\int_0^{\pi/2} \mathrm{d}\theta \int_0^{\pi/2} \sin\varphi \,\mathrm{d}\varphi \int_0^{\sqrt[3]{\cos\phi}} r^2 \,\mathrm{d}r$$

$$= \frac{2}{3}\pi \int_0^{\pi/2} \sin\phi \cos\phi \,\mathrm{d}\phi = \frac{1}{3}\pi$$



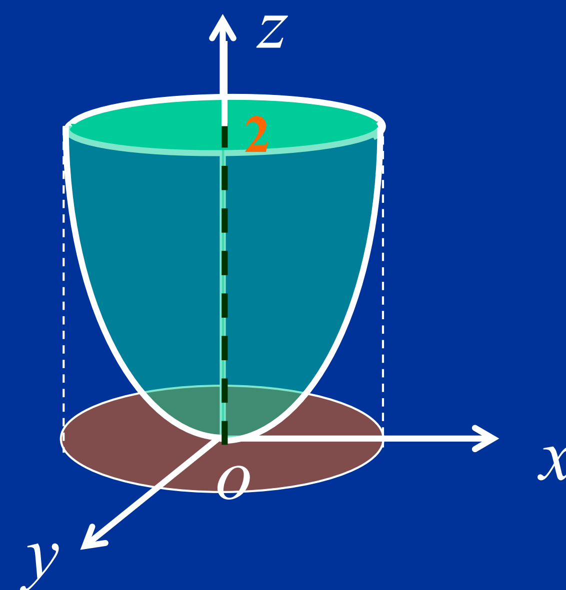
$$\boxed{\mathrm{d}v = r^2 \sin\varphi \mathrm{d}r \mathrm{d}\varphi \mathrm{d}\theta}$$

10. 已知 yoz 平面内一条曲线 $z=y^2$，将其绕 z 轴旋转得一旋转曲面,此曲面与 $z=2$ 所围立体在任一点的密度为 $\rho(x,y,z)=\sqrt{x^2+y^2}$, 求该立体对轴的转动惯量 .

解: 根据题意作图如右. 则

$$\Omega: \ x^2+y^2 \le z \le 2$$

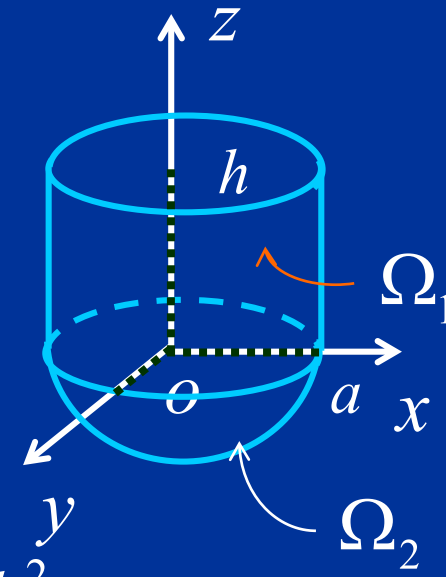
$$D_{xy}: \ x^2+y^2 \le 2$$



$$I_z=\iiint\limits_{\Omega}(x^2+y^2)\rho(x,y,z)dv$$

$$=\int_0^{2\pi}d\theta\int_0^{\sqrt{2}}r^4dr\int_{r^2}^2dz=\frac{32}{35}\sqrt{2}\pi$$

11. 在球心位于原点, 半径为 a 的均匀半球体靠圆形平面一侧接一个底半径与球半径相等且材料相同的圆柱体, 并使拼接后的整个物体的重心在球心,求圆柱体的高.



解: 根据题意作图如右. 则重心在原点.

设圆柱体高为 h, 且圆柱体和球体分别为 Ω_1,Ω_2. 则

$$0=\bar{z}=\frac{1}{V}\iiint_{\Omega} z\,dv=\frac{1}{V}\iiint_{\Omega_1} z\,dv+\frac{1}{V}\iiint_{\Omega_2} z\,dv$$

$$\iiint_{\Omega_1} z\,dv=\int_0^h z\,dz\iint_D dx\,dy=\int_0^h \pi a^2 z\,dz=\frac{1}{2}\pi a^2 h^2$$

$$\iiint_{\Omega_2} z\,dv=\int_{-a}^0 z\,dz\iint_{D_z} dx\,dy=\int_{-a}^0 \pi(a^2-z^2)z\,dz=-\frac{1}{4}\pi a^4$$

所以 $\frac{1}{2}\pi a^2 h^2-\frac{1}{4}\pi a^4=0$ 解得 $h=\frac{\sqrt{2}}{2}a$.