Lecture 3: Stationarity in Time Series

- **Definition and Examples**:
  - Time series data differs from other statistical data due to its temporal component.
  - Examples include fields like finance, ecology, and climatology.
- **Notation**:
  - Time series is denoted as (Y_t), where (t) represents the time point.
  - Importance of choosing the correct time scale (daily, weekly, monthly, etc.) based on the experiment's goal.

Key Functions in Time Series Analysis

1. **Mean Function (μ_t)**:

   - Represents the expected value at a specific time point.
   - $\mu_t = \mathbb{E}[Y_{t}]$, where Y_t is the time series.

2. **Variance Function (σ_t^2) or (γ_0)**:

   - Variance of the time series process at time (t).
   - $(\gamma_0 = \mathbb{E}[(Y_t - \mu_t)^2])$.
   - Assumes variance is finite and non-negative.

3. **Autocovariance Function $(\gamma_{t,s})$**:

   - Measures dependency between (Y_t) and (Y_s) at different time points.
   - Formula: $(\gamma_{t,s} = \mathbb{E}[(Y_t - \mu_t)(Y_s - \mu_s)])$.
   - Simplified as $(\mathbb{E}[Y_t Y_s] - \mu_t \mu_s)$.

4. **Autocorrelation Function $(\rho_{t,s})$**:

   - Correlation between (Y_t) and (Y_s).
   - Formula: $(\rho_{t,s} = \frac{\gamma_{t,s}}{\sqrt{\gamma_0(t) \cdot \gamma_0(s)}})$.

Terminology:

- "Auto" refers to the same series at different time points.
- Unlike classical statistics, where random variables (X) and (Y) are independent, time series variables (Y_t) and (Y_s) are dependent.

Stationarity in Time Series

1. **Importance of Stationarity**:

   - Stationarity simplifies handling joint distributions and moments.
   - Non-stationary data makes inference complex due to changing distributions/moments over time.

2. **Probability Distributions**:

   - **Marginal Distribution**:
      - Marginal PDF: $f(Y_t)$ or $f(Y_s)$.
      - Marginal CDF: $F(Y_t)$ or $F(Y_s)$.
   - **Joint Distribution**:
      - Joint PDF: $f(Y_t, Y_s)$.
      - For independent variables: $f(Y_t, Y_s) = f(Y_t) \cdot f(Y_s)$.
      - Time series variables are dependent; hence, this equality does not hold.

3. **Example of Dependency**:

   - Stock prices: Today's price influences tomorrow's price due to trends or patterns.

4. **Challenges with Joint PDFs**:

   - Time series data typically has one observation per time point.
   - If distributions or moments vary over time, identifying a joint distribution becomes difficult.

Understanding the Behavior of Time Series

- **Example**:

  - A hypothetical time series Y_t exhibits behavior such as rising, falling, stabilizing, and fluctuating.
  - The x-axis represents the time frame, and the y-axis represents the observed values Y_t.
  - At each time point t, there is only one observation of Y_t (e.g., $Y_1, Y_2, Y_3, \dots$).

- **Challenge**:

  - The time series process may change rapidly over time, making it difficult to fit a single probability distribution across all time points.
  - Joint distribution analysis becomes complex due to the process's variability.

- **Solution**:

  - Simplification is required for effective analysis, leading to the concept of **stationarity**.

Stationarity in Time Series

- **Definition**:

  - Stationarity assumes that the probability laws governing the time series do not change over time.
  - A stationary process is in statistical equilibrium, characterized by smooth behavior without rapid fluctuations.

- **Key Characteristics**:

  - If the process changes rapidly, it is **non-stationary**.
  - A stationary process maintains consistent statistical properties (e.g., mean, variance) over time.

- **Types of Stationary Processes**:

    1. **Strong (or Strict) Stationarity**:
        - Considers the **joint distribution** of random variables.
        - A process is strong stationary if the joint cumulative distribution function (CDF) does not change when time points are shifted by a constant (k).

Strong Stationarity

Key Concepts:

1. **First-Order Stationarity**:

    - A process is **first-order stationary** if its **one-dimensional CDF** is time-invariant.
    - Mathematically: $$F(Y_{t_1}) = F(Y_{t_1 + k})$$
        - (F): CDF of the time series.
        - (t_1): Initial time point.
        - (k): Time shift constant.
    - Holds true for any (t_1) and (k).

2. **Second-Order Stationarity**:

    - Extends first-order stationarity to the **joint CDF** of two random variables.
    - Mathematically: $$F(Y_{t_1}, Y_{t_2}) = F(Y_{t_1 + k}, Y_{t_2 + k})$$
        - (t_1, t_2): Two time points.
        - (k): Time shift constant.
    - Applies to all combinations of (t_1), (t_2), and (k).

3. **Nth-Order Stationarity**:

    - Generalizes to (n)-random variables: $$F(Y_{t_1}, Y_{t_2}, \dots, Y_{t_n}) = F(Y_{t_1 + k}, Y_{t_2 + k}, \dots, Y_{t_n + k})$$
        - $(t_1, t_2, \dots, t_n)$: Time points.
        - (k): Time shift constant.
    - Applies to any (n)-dimensional joint distribution.

Summary of Strong Stationarity:

- A strong stationary process ensures that shifting time points by a constant (k) does not alter the joint distribution.
- More restrictive than other forms of stationarity, as it considers the entire joint distribution.

Weak Stationarity (Overview)

- **Definition**:
    - Less restrictive than strong stationarity.
    - Does not require assumptions about the joint distribution or CDF.
- **Details**:
    - To be covered in the next lecture.
- **Comparison**:

    - Strong stationarity focuses on joint distributions, while weak stationarity imposes fewer constraints.

Conclusion

- Strong stationarity is a fundamental concept in time series analysis, emphasizing time invariance in joint distributions.
- Weak stationarity, which involves fewer restrictions, will be discussed further in the next lecture.