Inverse Problems Assignment 1

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Note: I wrote 80% of this report before the required format was released. I know this report does not follow these requirements, but I hope this one time it is accepted.

1 Anomalies

The time anomaly is the increase in the time it takes a ray to hit a detector when introducing a new medium for the rays to pass through. Thus if a ray only passes through the $v = 5m/s$ medium, $t_\gamma = 0$. The time anomaly is found using:

$$t_\gamma = \int_\gamma s(u)du \tag{1}$$

Where u is the distance travelled in the new medium and s is the reciprocal velocity difference, which is:

$$s(u) = \begin{cases} 0 & \text{outside box} \\ 5 & \text{inside box} \end{cases} \tag{2}$$

Where the 5 comes from the two medium's difference in velocity: $\Delta v = (5.2-5)\frac{\mathrm{m}}{\mathrm{s}} \Rightarrow s(\text{inside box}) = \frac{1}{\Delta v} = 5$. So t_γ is found by:

1. Make a right-side triangle with the part of the ray inside the box as the hypotenuse.
2. Measure the lower side of the triangle, l and calculate the hypotenuse as $\Delta u = \sqrt{2l^2}$
3. Then $t_\gamma = s\Delta u$

Using this method, I find three different Δu values: $\Delta u_1 = \sqrt{2}, \Delta u_2 = \sqrt{8}, \Delta u_3 = \sqrt{18}$, which yields the corresponding time values:

$$t_1 = 5\sqrt{2}\,\mathrm{s}, \quad t_2 = 5\sqrt{8}\,\mathrm{s}, \quad t_3 = 5\sqrt{18}\,\mathrm{s} \tag{3}$$

Each detector receives a ray from the left and right direction, which can be seen in table 1 expressed in terms of the time expressions in Eq. (3).

2 Discretization

With 1×1 squares, $\Delta u = 1$ and we can discretize Eq. (1) by:

$$t_\gamma = \sum_{i=0}^{13}\sum_{j=0}^{11} g_\gamma(u_{ij})s(u_{ij}) \tag{4}$$

For each ray γ, sum over all the squares u_{ij} in the x direction: $i = 0, \cdots, 13$ and y: $j = 0, \cdots, 11$. $g_\gamma(u_{ij})$ is the weight at the ij'th square for a ray γ and $s(u_{ij})$ is the reciprocal velocity difference. This means that g_γ's job is to pick out the squares where the ray passes through i.e. be 0 if the ray does not pass through u_{ij} and 1 if it does.

Detector	Left Ray	Right Ray
1	0	t_3
2	0	t_3
3	0	t_3
4	0	t_2
5	0	t_1
6	t_1	0
7	t_2	0
8	t_3	0
9	t_3	0
10	t_3	0
11	t_3	0
12	t_3	0

Table 1: Time anomalies t_γ in both directions for each detector.

3 Formulate inverse problem

We wish to express the problem on the form:

$$\mathbf{d^{obs}} = \mathbf{Gm} \tag{5}$$

$$\Rightarrow \mathbf{t^{obs}} = \mathbf{Gs} \tag{6}$$

In the inverse situation, $\mathbf{t^{obs}}$ is known and $\mathbf{s}$ is unknown, and we have to formulate $\mathbf{G}$. From Eq. (4) we know what $\mathbf{G}$ will be full of ones and zeroes to reflect a ray passing through a square. Because we need both the left and right rays, $\mathbf{s}$ is:

$$\mathbf{s} = \begin{bmatrix} s_{0,0} & s_{1,0} & \cdots & s_{13,0} & s_{0,1} & \cdots & s_{13,1} \cdots s_{13,11} \end{bmatrix}^\top \tag{7}$$

Meaning the first 14 values (0 to and including 13) will be the x values for $y = 0$, and the next 14 values are the x values for $y = 1$ etc. Doing so we can construct $\mathbf{G}$ with detectors as rows, meaning it will have $2 \cdot 12 = 24$ rows and $14 \cdot 12 = 168$ columns matching the structure of the $\mathbf{s}$ vector as explained in Eq. (7). To further understand this matrix, we first express Eq. (4) in terms of a matrix equation:

$$t_\gamma = \mathbf{G}^\gamma \mathbf{s} \tag{8}$$

Where $\mathbf{s}$ is as defined in Eq. (7) and $\mathbf{G}^\gamma$ is the matrix with $g_\gamma(u_{ij})$ as entries. As an example, for the left beam hitting the detector at $x = 1$ (”$L1$”) we see that it passes through $u_{1,0}$ and $u_{0,1}$ and so $\mathbf{G}^{L1}$ is:

$$\mathbf{G}^{L1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{9}$$

Then, one can think of each row in $\mathbf{G}$ as a flattened[1] $\mathbf{G}^\gamma$ matrix. With this notation, we can neatly write up the resulting $\mathbf{G}$ matrix:

$$\mathbf{G} = \begin{bmatrix} \text{flatten}(\mathbf{G}^{L1}) \\ \text{flatten}(\mathbf{G}^{L2}) \\ \vdots \\ \text{flatten}(\mathbf{G}^{L12}) \\ \text{flatten}(\mathbf{G}^{R1}) \\ \vdots \\ \text{flatten}(\mathbf{G}^{R12}) \end{bmatrix} \tag{10}$$

Where flatten() refers to flattening the matrix and the superscripts the detector and ray direction, so as before $L1$ means "ray hitting the detector at $x = \mathbf{1}$ from the **L**eft". Thus the discrete inverse problem is formulated by plugging in Eq. (10) into (6).

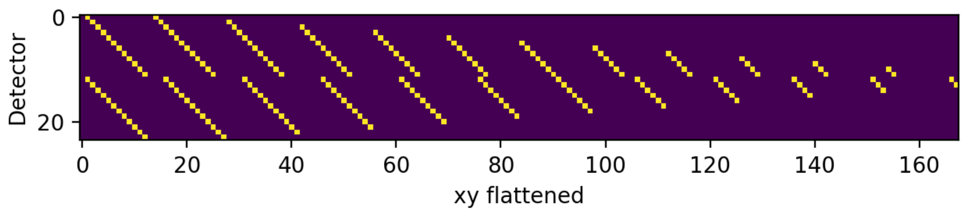

Figure 1: Illustration of the $\mathbf{G}$ matrix

4 Linearity

We have actually already shown that the problem is linear as we have successfully formulated the problem in terms of a matrix equation. This was possible because each time anomaly is a sum of the (linear) values in each square, and the sum of linear terms is still linear.

5 Uniqueness

From looking at the setup we quickly get a good feeling that the solution is most likely *not* unique, as a large part of the bottom of the box is not hit by any rays. However, in the inverse problem setup, we do not know where the box is, so the above argument only holds in our artificial setup. Instead, if we look at the shape of the $\mathbf{G}$ matrix from Eq. (10), which is 24 by 168 we see that we have many more equations (squares) than we have data ($\mathbf{t}^{obs}$). Thus the problem is underdetermined, which means we have an infinite number of best solutions i.e. the solution is **not unique**.

6 Tikhonov Regularization

Using Tikhonov regularization we have:

$$\mathbf{s} = (\mathbf{G}^\top \mathbf{G} + \varepsilon^2 \mathbf{I})^{-1} \mathbf{G}^\top \mathbf{t}^{\text{obs}} \tag{11}$$

I find the optimal $\hat{\varepsilon}$ by calculating the cost $C(\varepsilon) = ||\mathbf{t}^{\text{obs}} - \mathbf{G}\mathbf{s}||_2^2 - N\sigma^2$ for $\varepsilon \in [10^{-5}, 10]$ and find the value of ε where $C(\varepsilon)$ is closest to 0. This is illustrated in figure 2. And in figure 3 (see last page) the found $\mathbf{s}$ values can be seen.

[1]Flattened refers to transforming the matrix into a vector by "stacking" the matrix rows horizontally.

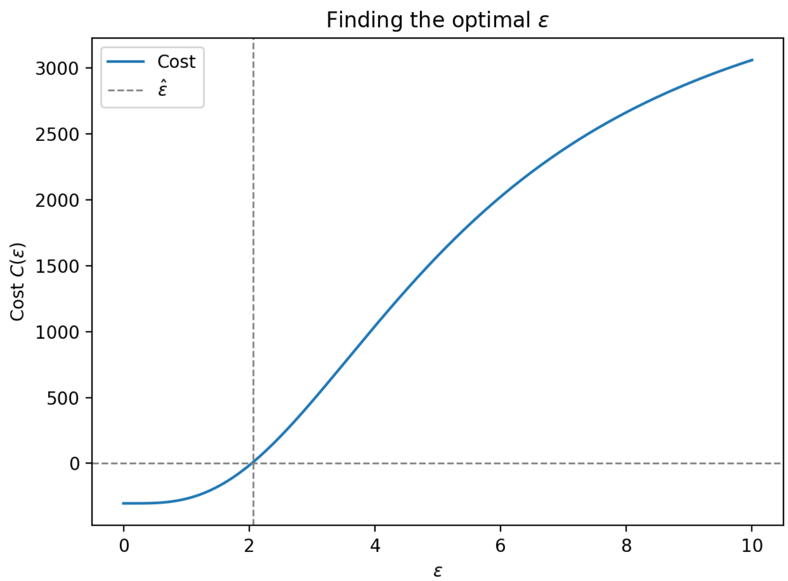

Figure 2: Finding the optimal ε value, which satisfies $C(\hat{\varepsilon}) = 0$

7 Model quality

From figure 3 it is obvious that our solution is not perfect, as parts where the rays do not hit are not explained well, such as the lower middle section. By creating a new, simplified scenario where the new medium only fills a single spot, say at $x = 5$ to $x = 6$ and $y = 1$ to $y = 2$, we can more easily see how the model works. This is illustrated in figure 4.

8 Discussion

From the true model, we know that the s values inside the box are 5 and outside is 0. In figure 3, only the top left part of the box has a high s value (around 2.4). This is because we only receive information through rays hitting the detectors, so any area with no rays will not be determined well, which is evident from figure 4 where the new medium is defined such that only two rays pass through. We also have areas with negative s values, which we know is incorrect, though it is not unphysical. It represents going faster than the non-box medium.

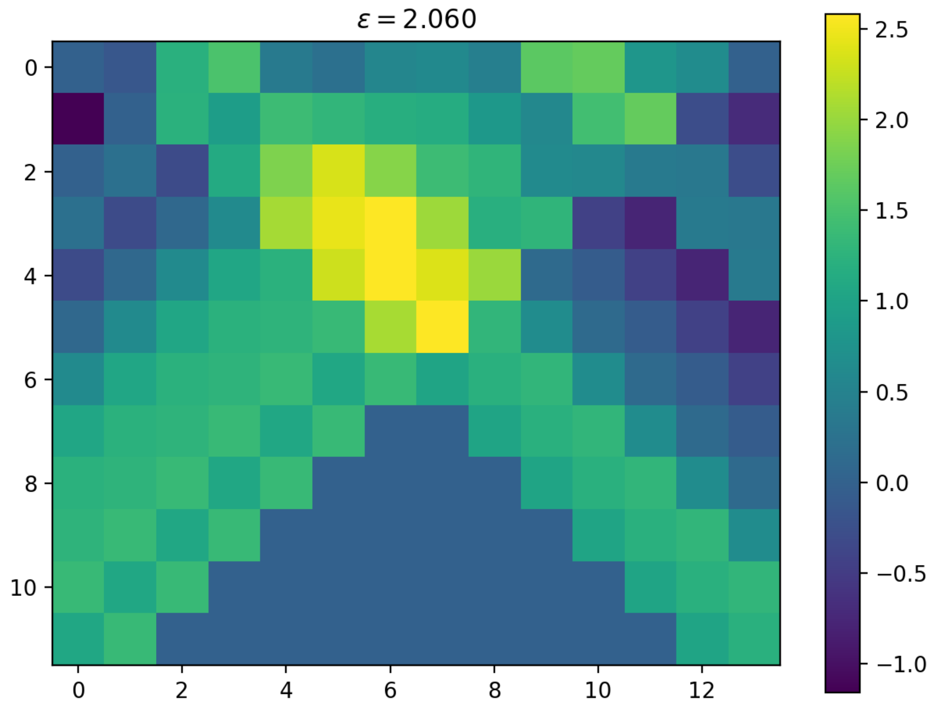

Figure 3: **s** solution values.

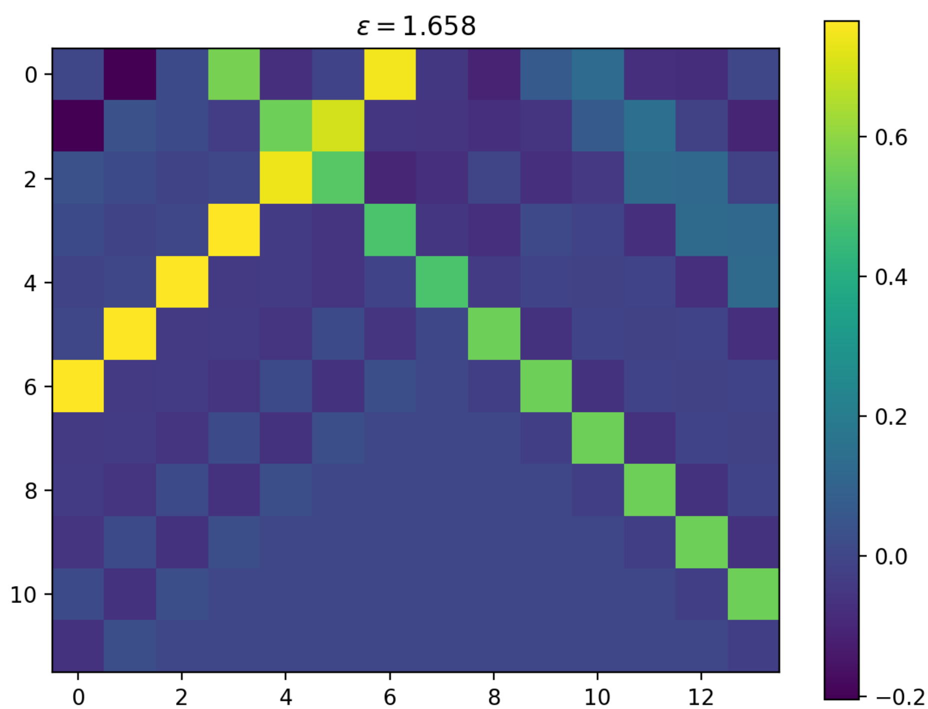

Figure 4: **s** solution values for a single square.