International Journal of Theoretical and Applied Finance
Vol. 3, No. 4 (2000) 675–702

VOLATILITY CLUSTERING IN FINANCIAL MARKETS: A MICROSIMULATION OF INTERACTING AGENTS

THOMAS LUX
Department of Economics, University of Kiel, Olshausenstr. 40, 24118 Kiel, Germany
E-mail: *lux@bwl.uni-kiel.de*

MICHELE MARCHESI
Department of Electrical and Electronic Engineering,
University of Cagliari, piazza D'Armi, 09123 Cagliari, Italy
E-mail: *michele@diee.unica.it*

Received 4 October 1999

The finding of clustered volatility and ARCH effects is ubiquitous in financial data. This paper presents a possible explanation for this phenomenon within a multi-agent framework of speculative activity. In the model, both chartist and fundamentalist strategies are considered with agents switching between both behavioural variants according to observed differences in pay-offs. Price changes are brought about by a market maker reacting to imbalances between demand and supply. Most of the time, a stable and efficient market results. However, its usual tranquil performance is interspersed by sudden transient phases of destabilisation. An outbreak of volatility occurs if the fraction of agents using chartist techniques surpasses a certain threshold value, but such phases are quickly brought to an end by stabilising tendencies. Formally, this pattern can be understood as an example of a new type of dynamic behaviour known as "on-off intermittency" in physics literature. Statistical analysis of simulated time series shows that the main stylised facts (unit roots in levels together with heteroscedasticity and leptokurtosis of returns) can be found in this "artificial" market.

Keywords: Volatility clustering; interacting agents; on-off intermittency.

1. Introduction

Both foreign exchange markets and national stock markets share a number of stylised facts for which a satisfactory explanation is still lacking in standard theories of financial markets. Pagan [35] provides an authoritative survey of those salient features that appear to be common characteristics of all financial markets together with the econometric techniques for dealing with them. As concerns foreign exchange markets, his description can be supplemented by recent reviews of their empirical regularities by de Vries [39] and Guillaume *et al.* [20]. Comparing these papers, the main difference seems to be that a remarkable number of facts presented by de Vries concern *negative* results, like the rejection of uncovered interest rate parity or purchasing power parity as well as other theoretically sensible but empirically doubtful relationships between exchange rates and other economic variables. However, his positive results are mostly striking uni-variate statistical features of the data which also play a prominent role in the other surveys and appear to be extremely uniform across various assets, nations and sampling horizons.[a] As it appears from the empirical literature, the features highlighted below are also routinely found in the prices and returns of financial and commodity futures as well as in prices for precious metals.

In this paper, we will concentrate on those three uni-variate properties which appear to be the most important and pervasive, and will try to provide an explanation using a multi-agent model of speculative activity. Beginning with the characteristics of share prices and foreign exchange rates themselves (or their logarithms), we encounter the following empirical regularity:

Fact 1. Unit root property of asset prices and spot exchange rates (or their logs).

More formally, denoting by p_t the price at time t, Fact 1 in its most elementary form implies that p_t follows an autoregressive process: $p_t = \rho p_{t-1} + \epsilon_t$ with stationary increments ϵ_t, and that one is usually unable to reject the hypothesis $\rho = 1$ using standard statistical procedures such as the Dickey-Fuller test. Expressed somewhat differently, one is unable to reject the hypothesis that financial prices follow a random walk or martingale.[b] While the implied non-stationarity and lack of predictability of spot rates appears to be at odds with traditional models of exchange rate determination, it squares well with the efficient market view of stock price determination and served as a starting point for models that propose a view of forex markets as arbitrage-free financial markets.

If levels (or logs) obey a unit root dynamics, returns or differences of logs should be stationary. In fact, this has been confirmed throughout the literature. However, certain distributional characteristics of returns also count as well-established facts which — in the words of de Vries — "have a sound statistical basis but for which no convincing economic explanation has been established". The first of these is:

Fact 2. Fat tail phenomenon.

Returns at weekly, daily and higher frequencies exhibit more probability mass in the tails and in the centre of the distribution than does the standard Normal. It is perhaps also remarkable that, besides this deviation from the Gaussian, the shape of the distribution usually appears well-behaved: namely, histograms of stock price or exchange rate returns mostly show a uni-modal bell shape with, in most cases, only modest levels of skewness. In the early literature, Fact 2 has been identified with excessive fourth moments (leptokurtosis). Kurtosis is, however, a very limited measure of deviations from Gaussian shape. Fortunately, recent literature provides

[a]Harrison [21] shows that 18th century financial data also share most of the characteristics of today's financial markets. In particular, they also exhibit leptokurtosis and volatility clustering.
[b]The former notion applies if increments ϵ_t are independently and identically distributed, while the latter only requires that $E[\epsilon_t] = 0$.

a sharper characterisation of the behaviour in the distribution's extreme parts: in particular, it could be established that the decline of probability mass in the outer parts follows a power law (whereas the Normal and a number of other often-used distributions have an exponential decline). As a consequence, the distribution in the tails can be approximated by a Pareto distribution:

$$F(x) = 1 - ax^{-\alpha} \tag{1.1}$$

Furthermore, the tail shape parameter α has mostly been found to hover between about 2 and 4 (with lower α values indicating fatter tails!) for both foreign exchange rates and stock returns.

While Fact 2 concerns the unconditional distribution of returns, the third item focuses on salient features of their conditional distribution. Using again the words of de Vries, it may be stated as follows:

Fact 3. *Volatility Clustering*: periods of quiescence and turbulence tend to cluster together.

Formally, this property can be identified with what is now known as ARCH effects: non-homogeneity of volatility together with highly significant autocorrelation in all measures of volatility despite insignificant autocorrelation in raw returns. There is some relationship between Facts 2 and 3 as persistence after shocks to volatility tends to generate a relatively high concentration of large returns. In fact, it has been shown in the econometrics literature that the popular (G)ARCH family of time series models generates unconditional distributions with a limiting behaviour conforming to the Pareto law (1.1). On the other hand, it has also been found that residuals from (G)ARCH specifications usually still exhibit fat tails so that the frequency of large returns can not solely be traced back to autoregressive behaviour of volatility. Also in this case, recent literature provides a somewhat sharper view on this stylised fact showing absolute returns rather than squared returns possess the highest degree of autocorrelation (cf. Ding *et al.* [12]).

Though properties 1 to 3 characterise the behaviour of almost all financial prices, they defy a straightforward explanation. It should be pointed out that the task of finding an explanation for these facts differs quite fundamentally from attempts to explain other economic phenomena. In particular, the economist's interest is often in questions of the type: how does one economic variable (e.g. the exchange rate) react to variations in some other variable (e.g. money supply). While linear dynamic models or comparative static analysis as a formal tool might appear perfectly adequate for the latter purpose (at least in order to gain first insights into the structure of the problem), this is not so when dealing with the above statistical findings. The reason is that an elementary requirement for any adequate analytical approach is that it must have the potential for bringing about the required behaviour in theoretical time series. Therefore, it seems rather obvious that one has to go beyond *linear* deterministic dynamics, which of course is insufficient to account for the phenomena under study. Furthermore, allowing for homogeneous (white) noise in some economic variables will also not achieve our goals simply because, in the view of Fact 3, we are dealing with *time-varying statistical behaviour*. In fact, the situation one faces is more often encountered by natural scientists. What one wishes to explain is a feature of the empirical time series *as a whole*. In the natural sciences, such characteristics of the data are often described by *scaling laws* and it is indeed possible to express Facts 2 and 3 in a similar fashion. First, Eq. (1.1) can be readily interpreted as a scaling law remaining probability in the tails, $1 - F(x)$, since the is scaling according to a power of x. Second, some formalisations of autoregressive heteroscedasticity concretised this feature as a hyperbolic decline of the autocorrelations of absolute or squared returns (cf. Brock and de Lima [8]).

Despite this somewhat unfamiliar research environment, some attempts have been made in recent economics literature towards an explanation of the above facts. Concerning the unit root property, both Kirman [24] and De Grauwe *et al.* [10] constructed complex structural models of speculative activity generating time series which are not distinguishable from a unit root process using standard tests. Interestingly, both contributions have as their starting point the model of Frankel and Froot [14] which focuses on the interaction of chartist and fundamentalist traders. Kirman extends this approach by adding a stochastic mechanism for the formation of majority opinion among chartists (see also Kirman [25]). De Grauwe *et al.* [10], on the other hand, investigate the chaotic dynamics resulting from an extended deterministic version of the original model set-up. In both models, the data-generating mechanism is surely not a (pure) unit root process and one finds the short-run dynamics dominated by speculative deviations from a fundamental-oriented path. As a consequence, the usual interpretation of non-rejection of a unit root as evidence *against* existence of speculative bubbles is called into question by these results. As concerns the approach taken by De Grauwe *et al.*, one may, however, object that the empirical search for chaos in financial time series has not been very successful — at least in the sense that no low-dimensional attractor could be identified.[c]

Turning to Fact 3, we find evidence of volatility clustering in complex simulation studies of financial markets by Grannan and Swindle [19] and in the artificial stock market of Arthur *et al.* [2]. Ramsey [36], on the other hand, shows how a statistical description of individual behaviour may quite generally give rise to dynamics with time varying second moments. Independently, Lux [29] made the same point when deriving the dynamics of second moments of a stochastic multi-agent model of speculation. However, while the model presented there seemed to yield results conforming with Fact 3, it appears to be too simple to confront it with other regularities. In another recent paper (Lux [30]), a chaotic model of speculative dynamics constructed along the lines of Day and Huang [9] and Lux [27] was

[c]However, such an approach may be justified by the recognition that an underlying chaotic dynamics may be concealed by additional noise which is surely present in economic data. It has indeed been shown, that rather small amounts of noise may strongly influence the results of certain tests for nonlinearity and chaotic dynamics.

shown to give rise to well-behaved, uni-modal and *leptokurtotic* distributions of returns. The model presented in this paper will be close in economic content to the one analysed in Lux [30]. However, in contrast to the earlier article, we will not be interested in the potential of cyclic or chaotic time paths but will concentrate on investigating the system's dynamics in the presence of *stable equilibria*. Though this restriction may at a first glance appear to be contradictory to the above outline of an appropriate research strategy and seems to constitute a refinement to a rather uninteresting case, it will turn out that this is not so. This seeming contradiction will be resolved by showing that an otherwise stable equilibrium can be subject to sudden transient phases of destabilisation. The characteristic features of these periods are bursts of severe fluctuations around the equilibrium which, however, quickly die out in the course of events with the system returning to a stable and calm state again. This type of punctuated equilibrium generates time series with clusters of excessive volatility interspersed among long tranquil periods. Statistical analysis shows that the resulting time paths for returns share the basic characteristics of real-life markets as captured in Facts 2 and 3 above. The time series of prices (or exchange rates) itself, on the other hand, resembles a random walk, thus conforming to Fact 1.

Similar dynamics have also been found in a somewhat different economic context recently by Youssefmir and Huberman [40] who dealt with the evolution of resource utilisation by adaptive agents.[d] In their paper, they conjectured that the same mechanism may serve as an explanation for volatility clustering in financial markets. The present paper confirms this conjecture.[e]

In natural science literature, a number of papers with qualitatively similar dynamic behaviour can be found which may, however, result from very different types of models (see e.g. Fujisaka and Yamada [15]; Heagy *et al.* [22]; Ott and Sommerer [34]). The phenomenon under study has been denoted *on-off intermittency*. Loosely speaking, the unifying feature of all examples of its occurrence is an attracting state (which may not always be a fixed point) becoming *temporarily* unstable due to a local bifurcation i.e. some key variable surpassing some stability threshold. This destabilisation may be generated in a deterministic manner (e.g. through weak coupling to another dynamics) or may occur stochastically. In any case, there will be no lasting deviation from the equilibrium as the system is driven back to stability by some endogenous mechanism.

In our model of chartist/fundamentalist interaction, the bifurcation parameter is the time-varying fraction of traders pursuing a chartist strategy. In general, agents are allowed to switch between a chartist and a fundamentalist trading strategy after comparing the respective profits. However, in the vicinity of the equilibrium the price (on average) equals the fundamental value and capital gains are

[d] A similar behaviour has also been observed by Glance [17].
[e] In personal communication, Michael Youssefmir in fact conjectured that this effect may be found in the type of model presented in Lux [27].

(on average) zero so that neither strategy is superior. As a consequence, switching between strategies occurs in an unsystematic manner and depends, so to say, on idiosyncratic motivation which is formalised using transition probabilities. Hence, the fraction of agents pursuing one or the other strategy follows a random walk and, sooner or later, leaves the region warranting a stable market. The ensuing destabilisation is characterised by an outbreak of severe fluctuations with a large fraction of traders switching to chartism and pursuing destabilising trend-following strategies. However, this situation does not last very long, as the temporary advantage of chartists disappears when the ensuing price bubble breaks down. After a reversal of the price trend, fundamentalists gain higher profits on average which leads to a conversion of chartists to the other strategy. This makes oscillations diminish and the state variables are pushed towards a stable market constellation again. However, every once in a while, this pattern will repeat. More picturesque, one may describe the market as being stable (and efficient) to a large extent, but inherently nervous with the potential of sudden, unforecastable eruptions.

The paper proceeds as follows: Sec. 2 presents the model, Sec. 3 gives some analytical results obtained through analysis of approximate mean value dynamics. In Sec. 4 details about the micro-simulations are presented, while Sec. 5 is devoted to statistical analyses of simulated data. Finally, Sec. 6, concludes the paper. The Appendix contains the formal proofs of two propositions.

2. A Model of Speculation

Recent survey studies on the behaviour and trading motives of participants in foreign exchange markets revealed a picture which seems to be largely at odds with earlier perceptions of efficient price formation.[f] First, a majority of foreign exchange dealers was found to neglect the fundamental determinants of exchange rates to a large extent. When asked about the sources of information they rely on they usually confess to having based their decisions on a variety of chartist techniques as well as on the observations of *flows*, i.e. orders of other market participants. The latter may be surveyed in an attempt to get a feel of the general tendency of the market. Alternatively and more conforming to standard theories, traders may try to extract private information from the activities of others.[g]

The recent literature on the economics of herding (e.g. Banerjee [3]; Orléan [33]) shows how the influence of the behaviour of others can generate majority opinions among an ensemble of individuals by a process of snow-ball like infection. As our

[f]Goodhart [18] constitutes the starting point for this strand of literature. Other contributions include Frankel and Froot [14], Allen and Taylor [1], Taylor and Allen [38], and Menkhoff [32].
[g]Although, in personal communication, a trader argued that unlike in stock markets there is practically no private information to be extracted from other traders' or customers' activities in foreign exchange markets. The reason for this was seen in the fact that virtually all news about macroeconomic data of major countries are made available at the same time to all traders in the market and therefore become public information immediately.

agents will consider "flows" explicitly as a source of information, herding and bandwagon effects are a common feature in the model. In addition to the prevalence of chartist techniques, Frankel and Froot [14] and Liu [26] both found that the weights placed on chartist or fundamentalist information were not constant over time. As a striking example, during the dollar bubble of the eighties, dealers became more and more prone to exclusive reliance on chartist device. According to Frankel and Froot, this can be explained by switching between strategies according to their past performance. Following these empirical observations, traders' switching between strategies will be a key element of the dynamics.

As the model closely follows the one presented in Lux [30], the exposition of the details will be kept as short as possible. The model considers the behaviour of an ensemble of N speculators. These traders may either adhere to chartist or fundamentalist practices. The number of chartists at any point in time will be denoted n_c, the number of fundamentalists is n_f, $n_c + n_f = N$. Furthermore, we distinguish between two subgroups of chartists: those with an optimistic disposition and those who are pessimistic about the market's development in the near future. The number of individuals in these groups is denoted by n_+ and n_-, respectively, and $n_+ + n_- = n_c$. The dynamics of the model encapsulates the endogenous switching of agents between the groups defined above and the price dynamics resulting from their market activities. Basically, we have three elements constituting the dynamics:

(1) Chartists switching between optimistic and pessimistic opinion: this element of the model is inspired by Kirman's analysis of opinion formation. However, our approach deviates from his set-up in two aspects. First, while he relies on pairwise communication, we will allow for interpersonal influence within larger groups, which may be thought of as a formalisation of the influence of "flows" on individual decisions. Second, the interpersonal factor is supplemented by a second one capturing the effects of "charts" in a very simple manner. The first type of influence is concretised by the individuals considering the majority opinion of others. The second (chart) component enters via the influence of the actual price trend. If both "signals" are in contradiction, the incentive to follow the crowd will be weakened, while if both are in harmony, there will be a strong pressure to believe in the majority's being right.

Formally, we use an *opinion index* x which is defined as the difference between optimistic (n_+) and pessimistic chartists (n_-) scaled by their total number ($n_c = n_+ + n_-$):

$$x = \frac{n_+ - n_-}{n_c}, x \in [-1, 1] . \tag{2.1}$$

Denoting the price change in continuous time by $\dot{p} = dp/dt$ and following a convenient formalisation for transition probabilities, the probability of a formerly pessimistic individual to switch to the optimistic group (π_{+-}) and *vice versa* (π_{-+})

within some small time interval Δt may be written as:

$$\begin{aligned} \pi_{+-} &= \nu_1 \left(\frac{n_c}{N} \exp(U_1)\right) \\ \pi_{-+} &= \nu_1 \left(\frac{n_c}{N} \exp(-U_1)\right) \end{aligned}, \qquad U_1 = \alpha_1 x + \alpha_2 \frac{\dot{p}}{\nu_1} \tag{2.2}$$

Here, ν_1 is a parameter for the frequency of revaluation of opinion, and α_1 and α_2 are parameters measuring the importance the individuals place on the majority opinion and actual price trend in forming expectations about future price changes.[h] The pressure to re-evaluate behaviour and possibly change to another strategy gets stronger when the actual development of the market points in a direction which is in contradiction to the individual's own expectation. In (2.2) the transitions between the optimistic and pessimistic subgroup are restricted to a fraction n_c/N of all chartists. To explain this, we have to forestall a bit of the next main ingredient of the model, that is, switches between chartist and fundamentalist strategy. Namely, it will be assumed that these behavioural changes are governed by comparison of profits between individuals of both types meeting each other. This introduces a dependence of transition probabilities on the number of individuals currently following a certain strategy. As the simplest way of formalising this dependence, the chance of meeting an agent from a certain subgroup was assumed to equal the fraction of the subgroup within the entire population. Hence, a chartist meets a fundamentalist with probability n_f/N [see Eq. (2.3)] and compares the difference in profits between both strategies. As a consequence, only the remaining fraction of chartists, $1 - n_f/N = n_c/N$, is not involved in this second type of behavioural changes and constitutes a pool of possible candidates for changes of opinion. Hence the inclusion of the term n_c/N in Eq. (2.2).

The formalisation (2.2) has a number of parallels in recent literature. In particular, it is close to the formalisation of the choice of different predictors adopted by Brock and LeBaron [7] and Brock and Hommes [5]. The key difference between our formalisation and theirs is that they are not concerned with a sluggish process of changes of fractions of agents within groups. Rather, in their discrete time model, the distribution of the population at any integer time step could be interpreted as the final result of such a process of revaluation of choices by the individuals. As in Youssefmir and Huberman [40] revaluations are assumed to occur *asynchronously* in our model. During an infinitesimal time increment Δt, the probability for an individual to switch from pessimistic to optimistic expectations is thus, $\pi_{+-}\Delta t$ with the first term being a time-varying function of both x and dp/dt. Similarly, a formerly optimistic individual will be found switching in the opposite direction with (time-varying) probability $\pi_{-+}\Delta t$.

[h]Note that α_1 and α_2 need not sum up to 1. Furthermore, the change in asset prices has to be divided by the parameter ν_1 for the frequency of agents' revision of expectations since one has to consider the mean price change over the average interval between successive revisions of opinion. The same applies to the transition probabilities in Eqs. (2.3) below.

Note that this formulation implies that switching towards expectations justified by the present state of the market will dominate. It will not, however, preclude a minority from switching in the opposite direction. Furthermore, even in the absence of external stimuli (i.e. if $x = dp/dt = 0$) changes of behaviour occur due to idiosyncratic factors not captured by the model.

(2) Switching between chartist and fundamentalist strategy: chartists are assumed to buy (sell) a fixed number of units, if they are optimistic (pessimistic). Fundamentalists on the other hand, are assumed to buy (sell) if the actual market price is below (above) the value derived from fundamental analysis. As outlined above, these behavioural changes are modelled in the following way: agents meet individuals from the other group, compare (myopic) excess profits from both strategies and with a probability depending on the pay-off differential, switch to the more successful strategy. Excess profits per unit (as compared to alternative investments) gained by chartists are given by $(r + dp/dt)/p - R$, with: r nominal dividends of the asset and R average real returns from other investments. While this holds for chartists who are extending the fraction of the asset under study in their portfolio (optimistic individuals), the behaviour of pessimistic traders is governed by another motive. Instead of gaining potential excess profits, their advantage consists in avoiding losses by searching for other investment opportunities. This advantage can be expressed as: $R - (r + dp/dt)/p$. We note that it will be assumed that $r/p_f = R$, i.e. evaluated at its fundamental value (p_f), in a state of stable prices ($dp/dt = 0$), the asset will yield the same return as other investments.

Fundamentalist traders entertain the hypothesis that the price will revert to the perceived fundamental value in the not too remote future. Therefore, they may use p_f instead of the actual price p to evaluate infrequently paid dividends. Hence, in their computation, real dividends conform to economy-wide average R. Excess profits, then, solely consist of the percentage deviation between p_f and p, as they intend to buy low and sell high. It follows that, irrespective of whether we are in a situation of undervaluation or overvaluation, fundamentalists' excess profits per unit of the asset can be written as: $s|(p - p_f)/p|$. The discount factor $s < 1$ is justified by the observation that these are *expected* gains and will be realised only after reversal to the fundamental value (chartists' excess profits, on the other hand, are immediately realised).

According to the above, transition probabilities for changes of strategies are formalised as follows:

$$
\begin{aligned}
&\left.\begin{aligned}
\pi_{+f} &= \nu_2 \left(\frac{n_+}{N}\exp(U_{2,1})\right) \\
\pi_{f+} &= \nu_2 \left(\frac{n_f}{N}\exp(-U_{2,1})\right)
\end{aligned}\right., \quad U_{2,1} = \alpha_3 \left(\left(r + \frac{\dot{p}}{\nu_2}\right)\Big/ p - R - s\left|\frac{p_f - p}{p}\right|\right) \\
&\left.\begin{aligned}
\pi_{-f} &= \nu_2 \left(\frac{n_-}{N}\exp(U_{2,2})\right) \\
\pi_{f-} &= \nu_2 \left(\frac{n_f}{N}\exp(-U_{2,2})\right)
\end{aligned}\right., \quad U_{2,2} = \alpha_3 \left(R - \left(r + \frac{\dot{p}}{\nu_2}\right)\Big/ p - s\left|\frac{p_f - p}{p}\right|\right)
\end{aligned}
\tag{2.3}
$$

The two transition probabilities in the first and second line depict switching between the fundamentalist group and the optimistic chartist subgroup, while in the third and fourth line, transition probabilities for switches towards and from the pessimistic group are formalised. In (2.3), ν_2 is again a parameter for the frequency of this type of transition, while α_3 is a measure of the pressure exerted by profit differentials (or, put the other way round, of the inertia of the reaction to profit differentials). As transitions are governed by some type of pair interaction, the probability for an individual to change strategy also depends on the number of individuals pursuing other strategies at that time. Interpreted somewhat differently, we may say that there is also a herd effect concerning choice of strategies.

(3) At last, we turn to the price formation process. It will be assumed that in the presence of non-zero net demand prices are adjusted by an auctioneer in the usual manner. However, in order to formulate this process in a way that fits into the present framework, the reaction of the market maker will also be formalised as a stochastic process. To achieve this, the auctioneer is assumed to adjust the price to the next higher (lower) possible value (one cent or pence) within the next time increment with a certain probability depending on the extent of the imbalance between demand and supply. Assuming furthermore that there are additional liquidity traders in the market whose excess demand is stochastic or that the value of excess demand (ED) is perceived with some imprecision by the auctioneer, we add a (small) noise term μ. Under these assumptions, we have the following transition probabilities for an increase or decrease of the market price by a fixed amount $\Delta p = \pm 0.01$:[i]

$$\begin{aligned} \pi_{\uparrow p} &= \max[0, \beta(ED + \mu)] \\ \pi_{\downarrow p} &= -\min[0, \beta(ED + \mu)] \end{aligned} \tag{2.4}$$

where β is a parameter for the reaction speed of the auctioneer. If, for example, the perceived excess demand is positive, an increase of the price towards the next elementary unit occurs with probability $\beta(ED + \mu)\Delta t$ within an infinitesimal time increment. Aggregate excess demand ED in (2.4) is composed of excess demand of chartists and fundamentalists, $ED = ED_c + ED_f$. The former is $ED_c = (n_+ - n_-)t_c$ since all chartists either buy or sell the same number t_c of units.[j] Fundamentalists' excess demand is given by $ED_f = n_f \gamma(p_f - p)$, depending on the deviation from the fundamental value, reaction strength γ and the number of individuals behaving this way at that time, n_f. Hence, similarly as with (2.2) and (2.3), the formalisation in (2.4) yields time-varying state-dependent transition probabilities.

[i]This restriction of the price change during an infinitesimal time increment does *not* mean that the market maker is insensitive to the level of excess demand. Large imbalances will give rise to a quick succession of adjustments until the market approaches an equilibrium between demand and supply.

[j]Note that our binary decision problem leaves no room for more complicated excess demand functions. However, it is not difficult to extend this binary problem to one covering different grades of optimism and pessimism.

3. Theoretical Results

As laid out in detail in earlier papers, a formal analysis of stochastic transition models like the above often proceeds by deriving approximate differential equations governing the time development of mean values of state variables. This can be justified by assuming a sharply peaked initial distribution of the endogenous variables. Formally, such mean value dynamics are usually found to be identical with differential equations derived for an infinite population. As shown in a similar model in Lux [30], the above framework can be studied deriving differential equations for the three variables x (opinion index), z (fraction of chartists in population, $z = n_c/N$) and p (market price). The results of a formal investigation into the number and stability of stationary states of this dynamics are summarised in the following propositions.[k]

Proposition 3.1. (a) *The mean-value dynamics of x, p and z possesses the following stationary solutions*:

(i) $x^* = 0$, $p^* = p_f$ *with arbitrary* z,
(ii) $x^* = 0$, $z^* = 1$ *with arbitrary* p,
(iii) $z^* = 0$, $p^* = p_f$ *with arbitrary* x;

(b) *no stationary states with both $x^* \neq 0$ and $p^* \neq p_f$ exist.*

The equilibria of major interest are those depicted in item (i) of the first part of the proposition. These stationary states are characterised by a balanced disposition among chartists and the price equal to the fundamental value. As can be deduced from transition probabilities (2.3), neither fundamentalists nor chartists have an advantage in this situation (note that $dp/dt = 0$ in any stationary equilibrium). This explains why the fraction of traders following chartist practices is arbitrary. Switching between subgroups, therefore occurs in a random fashion. One might argue that this leads to only minor fluctuations around the stationary state — a conjecture that turns out to be false as indicated in the introduction and shown in the next section.

Equilibria of types (ii) and (iii) are somewhat degenerate situations in which either the group of chartists or fundamentalists has declined to zero. Though they may not be very likely to be realised during limited time horizons with a sufficiently large population size, they will eventually come about in the course of events and will, then, act as absorbing states from which the dynamics does not return (in the same sense as any species will eventually become extinct in the course of time). As these states are not really characteristic for the system's development and the processes we are interested in, their occurrence by chance will be excluded through additional borderline conditions in the simulations.

[k] Proofs can be found in the Appendix.

Finally, the second part of Proposition 3.1 explicitly excludes existence of equilibria with an emerging majority along with lasting overvaluation or undervaluation. This result is mainly of interest in the light of earlier papers which focused on this type of equilibria (cf. Kirman [25], Lux [27]). Hence, there only remain the stationary states of type (i) which are characterised by efficient price formation. Proposition 3.2 is concerned with the stability of this continuum of steady states:

Proposition 3.2. *An equilibrium on the line* $(x^* = 0, p^* = p_f, z^*)$ *is unstable* (*repelling*) *if*

$$(\textit{cond } 1) \quad 2z^*\nu_1\left(\alpha_1 + \alpha_2\frac{\beta}{\nu_1}z^*T_c - 1\right) + 2(1-z^*)\alpha_3\beta z^*T_c/p_f - \beta(1-z^*)T_f > 0$$

or

$$(\textit{cond } 2) \quad \alpha_1 > 1 + \alpha_3\frac{\nu_2 T_c R}{\nu_1 T_f p_f} \ \textit{holds} \ (T_c \equiv Nt_c, T_f \equiv N\gamma)\,.$$

Related to the "degree of freedom" in the choice of z^*, one finds a zero root in the differential equation system governing mean-value dynamics (cf. the Appendix). Note that a zero root in a differential equation system is equivalent to a unit root in a difference equation. The zero root thus reflects the fact that after any stochastic disturbance the dynamics will move towards another member of the continuum of stable stationary states which implies that the fraction z will follow a path that is close to a random walk. Unfortunately, the discontinuity in the partial derivatives of $U_{2,1}$ and $U_{2,2}$ (due to the absolute terms in these expressions) precludes decisive statements about *stability*, but allows only to derive conditions for *instability* of steady states.

Proposition 3.2 can be summarised as follows: From (cond 2) one infers that an upper value for the reaction to "flows", say $\bar{\alpha}_1$, can be computed beyond which no stable equilibrium will exist at all. However, given that (cond 2) is not violated, (cond 1) may be used to calculate an interval of z^* values which are candidates for stable equilibria and to demarcate the region of unambiguous instability.

4. Simulation Details

The Poisson-type dynamics of asynchronous updating of strategies and opinions by the agents can only be approximated in simulations. In particular, one has to choose appropriately small time increments in order to avoid artificial synchronicity of decisions.[1] In the simulations below we considered an ensemble of $N = 500$

[1]It has been found in simulations of evolutionary games, that models which differ only in the assumption of synchronous vs. asynchronous decision-making can lead to fundamentally different results (see Glance [17], Appendix). This finding necessitates discussion about which of the two assumptions is likely to provide a better approximation to real-life situations. A casual look at the working of, for example, foreign exchange markets may convince one that decisions of economic agents are usually not synchronised. In this context, one may note the close similarity between the asynchronous structure of the model and recent work on the econometric modelling of intra-daily data using time-dependent Poisson processes (cf. Engle and Russell [13]).

agents speculating in our artificial financial market. After some experimentation, we designed a simulation program with some flexibility in the choice of the time increment. Namely, while we found $\Delta t = 0.01$ to yield satisfactory results for "normal" times,[m] the phenomenon of volatility bursts required a somewhat higher precision in order to not artificially restrict the maximum price changes between unit time steps. As a consequence, the precision of the simulations was automatically increased by a factor 5 (switching to $\Delta t = 0.002$) when the frequency of price changes became higher than average.

Our procedure requires that all the above Poisson rates be divided by 100 or 500 (depending on the precision of the simulation) in order to arrive at the probability for any single individual to change his behaviour during $[t, t + \Delta t)$. Similarly we assume that the auctioneer adjusts the prevailing price by one elementary unit (one cent or one pence) with probabilities $\pi_{p\uparrow}$ or $\pi_{\downarrow p}$ during one time increment. Since the time derivative, dp/dt, has been introduced in the determination of transition probabilities, a word on its interpretation in the simulation experiments is in order. Though one could simply use the price change during the last increment $[t - \Delta t, t)$ in place of dp/dt, this appears somewhat restrictive as, according to our simulation design, it only leaves the possibilities $\Delta p = -0.01$, 0 and $+0.01$. In order to get a broader set of possibilities, we considered the average of the price changes that took place during the interval $[t - 0.2, t)$.

As a slight modification of the theoretical model, occurrence of the 'absorbing states' $z = 0$ and $z = 1$ was excluded by setting a lower bound to the number of individuals in both the group of chartists and fundamentalists. In particular, it was assumed that agents cannot switch out of one strategy that has less than four members. The intention of this modification is solely to avoid the breakdown of simulations.

In order to initialise the simulations, agents are randomly distributed to the groups of optimistic and pessimistic chartist or fundamentalist traders, while the price is set equal to its fundamental value. Furthermore, the number of chartists is chosen sufficiently small at the outset to let the system settle down at the continuum of stable stationary states. The behaviour of agents is, then, updated after every time increment $\Delta t = 0.01$ according to the rules detailed above. At the same time, the auctioneer may also adjust the price in the presence of excess demand or supply.[n]

[m] The initial choice of $\Delta t = 0.01$ was motivated by comparing the results of micro-simulations with theoretical insights obtained by mean field approximations. In particular, we found good agreement between simulations and theory using 100 micro-intervals per unit time steps for simulations of the related models in Lux ([27, 29], and [30]).

[n] The practical implementation consists in drawing a number from a uniform distribution on [0,1] and comparing it with the actual value of the relevant transition probability divided by 100. If the random number was found to be smaller than the transition probability, the prescribed change of behaviour occurred, if it was greater, no change occurred. Note that, in the case of the auctioneer's adjusting of the market price division by 100 is already implemented by using cents (hundredths of dollars) instead of dollars as elementary units and no further adjustment of the transition probabilities is required in the simulations.

The resulting time paths of the state variables x, z and p were recorded at all integer time steps and returns calculated as the difference between the logs of successive price recordings: $ret_t = \ln(p_t) - \ln(p_{t-1})$.

In order to choose appropriate sets of parameter values, one would ideally like to calibrate the model by using relevant empirical observations on its various components. Unfortunately, we lack empirical estimates for all those parameters that are introduced in the "population dynamics" (e.g. frequencies of revaluation, ν_1 and ν_2, and weight factors, α_1, α_2 and α_3). On the other hand, most of the basic results on the statistical behaviour reported below turned out to be extremely robust and remained qualitatively unchanged for a wide range of parameter values. Given these observations, we confined ourselves to a very limited degree of "fine-tuning" of parameter values. The parameter sets exhibited below have been chosen using the criterion that the bandwidth for returns over unit time steps should roughly conform to what one usually observes with data from asset or foreign exchange markets at daily frequencies. Basically, this amounted to selecting parameter sets that did not generate maximum absolute returns exceeding 0.2 to 0.3 in the most extreme cases. In this way, our statistical results for returns over unit intervals are most easily comparable to the large body of applied literature on daily financial returns. Our choice of average real returns (per day) follows the same intention: $R = 0.0004$ yields yearly returns of about 15 per cent which is roughly consistent with empirical numbers. Even with this very limited attempt at calibrating our simulation design (which amounts more to a standardisation of the data), the time series characteristics turned out to be remarkably similar to empirical findings. Presumably, one could further improve the fit by elaborating on the complex relationship between the model's parameters and the emergent statistical features[o]

All simulations performed with this model confirmed our conjecture that volatility clustering and the on-off intermittency phenomenon should show up in this framework of speculative price formation in a financial market. The main feature of volatility bursts appears to be ubiquitous in this model and does not hinge on fine-tuning of the parameters. The only change brought about by progressing towards more extreme parameter values is that the bursts become much more severe and may then look like a dramatic exaggeration of those depicted in Fig. 1. In any case, however, the statistical characteristics do not undergo qualitative changes when varying the parameters of the model and, even with unrealistically violent fluctuations, the features of fat tails and volatility clustering are preserved.

The remainder of the paper reports the details of some simulation runs with parameters leading to a "realistic" range of fluctuations as described above.

[o]Bouchaud and Cont [4] also formulate a phenomenological model which bears some similarity to the present one. They make some efforts towards calibration of the parameters of the resulting Langevin equation using empirical numbers and describe various scenarios depending on market depth and reaction strength of individuals.

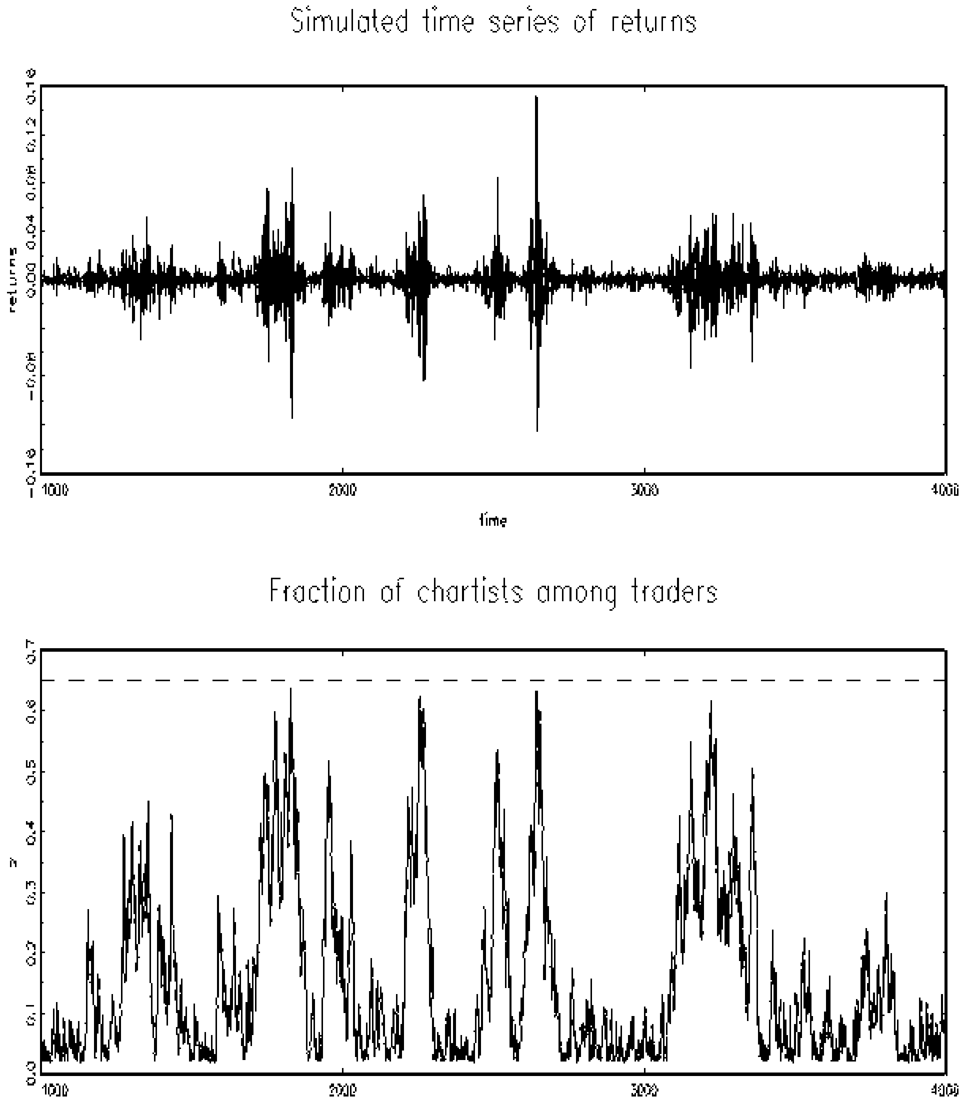

Fig. 1. Upper part: typical simulated time series of returns. Bottom part: simultaneous development of the fraction of chartists, $z = n_c/N$. The broken line indicates the critical value $\bar{z} = 0.65$ where a loss of stability is expected. Parameter values underlying this simulation are those of parameter set I, cf. main text.

Figure 1 shows the time development of returns and z, the fraction of chartist traders, from a simulation with the following parameters:

Parameter set I:
$N = 500$, $v_1 = 3$, $v_2 = 2$, $\beta = 6$, $T_c (\equiv Nt_c) = 10$, $T_f (\equiv N\gamma) = 5$,
$\alpha_1 = 0.6$, $\alpha_2 = 0.2$, $\alpha_3 = 0.5$, $p_f = 10$, $r = 0.004$, $R = 0.0004$, $s = 0.75$.

Furthermore, the noise in the transmission of the "signal" excess demand was assumed to follow a Normal distribution with mean equal to zero and standard deviation $\sigma = 0.05$.

The trajectory of returns, plotted over 3,000 time steps, clearly shows that this model behaves differently from other models with stochastic fluctuations around stationary solutions. In particular, we do *not* find the small homogenous disturbances with a constant variance that one would usually expect. Instead we see long calm periods punctuated with sudden bursts of clustered volatility in returns. It is worth noting that the behaviour of the simulated time series in the upper part of Fig. 1 conforms with empirical observations in quite a number of aspects. First, returns appear to be stationary and are also distributed rather symmetrically around zero. Second, they exhibit occasionally sudden, strong deviations which appear to come in clusters. Interestingly, volatility clustering seems to be a rather general feature of the dynamics and is not confined to the most severe fluctuations.

The bottom diagram confirms the viewpoint that "on-off intermittency" is responsible for the remarkable features of the time series of returns. Here we depict the trajectory of z together with the suspected bifurcation value $\bar{z}$ where the system loses stability. $\bar{z}$ can be calculated from (cond 1) and for the parameters underlying the simulation, is found to equal 0.65 which is demarcated by the broken line in Fig. 1. Looking at both the upper and lower part of the figure, the following interplay between both variables can be observed: as long as z is far from $\bar{z}$, its time development appears quite random. It is accompanied by small fluctuations of returns around zero whose magnitude seems to be correlated with the number of chartists, z. However, once z approaches $\bar{z}$, excessively large price changes set in. The reason is that, with a certain dominance of chartist practices, deviations from the fundamental equilibrium become self-reinforcing and the system cannot maintain its local stability any more.

Nevertheless, the dynamics does not diverge without bounds: deviations are checked after some time presumedly because of the superior performance of fundamentalists. Hence, sooner or later the market returns to its usual tranquil mode of operation after any outbreak of instability. The time needed until recovery (and with it the degree of volatility clustering) may depend on the parameters of the model. For example, variations of the parameters characterising the readiness of agents to quickly react on profit differentials, ν_2 and α_3, may result in differences concerning the frequency and extent of volatility bursts. Intensive study of the effects of parameter variations will be performed in future research.

By and large, Fig. 1 as well as similar results from other simulations performed so far are in agreement with the mechanism described by Youssefmir and Huberman and more generally, in a number of contributions in the natural science literature recently. We encountered a system (market) which can be characterised as being mostly stable and tranquil. This usual behaviour is however, disturbed by sudden bursts of surprisingly large fluctuations from time to time. In our example, bursts are brought about randomly due to the unsystematic development of the fraction of traders pursuing chartist strategies in the vicinity of the fundamental equilibrium.

5. Statistical Analysis of Simulated Data

Now we take a closer look at the statistical characteristics of some simulated data sets. In particular, it will be investigated whether or how far the data from our "artificial" market conform to the stylised Facts 1 to 3 of real-life markets outlined above.

Beginning with Fact 1 (unit roots), the standard tool to test for its presence is the well-known Dickey-Fuller test. Formally, in its most elementary form, this amounts to a test of the hypothesis:

$$H_0 : \rho = 1 \text{ vs. } H_1 : \rho < 1 \quad \text{in the regression} : p_t = \rho p_{t-1} + \epsilon_t \,.$$

Often the same test design is applied to the logarithms of the price series instead of the levels. The standard result when applying this test to foreign exchange rates or share prices is inability of rejection of a unit root process.

As an illustrative and representative example of the results obtained from simulations of the model, Table 1 depicts the outcome of the Dickey-Fuller test using four different parameter sets. In each case, a long record of 20,000 realisations has been divided into 40 subsamples with 500 entries, respectively. As can be seen in the third column of the table, from the total number of 160 one-sided tests, *none* led to a rejection of the presence of a unit root. Actually, the same holds for larger samples and the results are also not affected by using logs instead of levels. Furthermore, when applying the Augmented Dickey–Fuller test, i.e. regressing the price on p_{t-1} and $\Delta p = p_{t-1} - p_{t-2}$, the estimates of ρ are even closer to 1. One is, therefore, forced to conclude that the random components of the price dynamics conceal any systematic motion. This happens, although the time series itself is in fact bounded and stationary as is warranted by our constant fundamental value $p_f = 10$ in the simulations. However, as is visible in the example given in Fig. 2, for rather long

Table 1. Results of unit-root tests.

Parameters	range of $\hat{\rho}$	No. of rejections for one-sided test at 95% level	No. of rejections for two-sided test at 95% level
Parameter set I	0.999819–1.000022	0	0
Parameter set II	0.999977–1.000021	0	0
Parameter set III	0.999957–1.000030	0	3
Parameter set IV	0.999972–1.000014	0	2

Note: The test statistic of the Dickey–Fuller test is identical to the usual t-statistic of p_{t-1}. The difference is that, in the case of a unit root, this statistic has a non-standard distribution. Critical points can be looked up in, for example, Dickey *et al.* [11]. Parameter sets are given in footnote p.

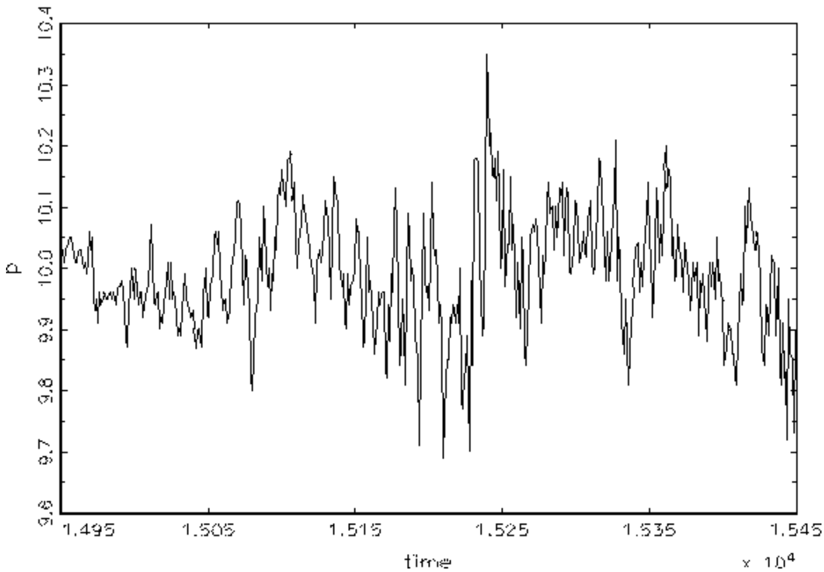

Fig. 2. Time series of prices. It is hard to decide by inspection whether the dynamics is stationary or non-stationary. This example is from a simulation with parameter set IV.

subperiods, it is quite hard to decide by visual inspection whether the motion is in fact stationary or non-stationary.

A somewhat surprising variation of the above results is obtained when performing two-sided tests of H_0 (see the last column of Table 1): Now one finds a number of cases where the unit-root hypothesis is rejected. However, these are all occurrences of $\hat{\rho} > 1$, i.e. H_0 is *rejected in favour of explosive roots* of the dynamics. It is interesting to note that an intrinsically stable and bounded dynamics like ours can even mimic such an extreme form of instability. Of course, the impression of explosive roots is connected with periods of emerging temporary instability. Note that the case $\rho > 1$ is usually identified with the theory of bubbles under rational expectations, whereas "fads" models like the present one are assumed to possess mean-reverting dynamics giving rise to $\rho < 1$. Obviously, though mean-reverting tendencies do exist in the model, they are concealed by the discontinuous nature of the dynamics. Hence, the above criterion may not enable one to unambiguously distinguish between RE bubbles and fads.

Now we turn to a characterisation of the dynamics of returns relating the computer-generated series to Facts 2 and 3. The first column of Table 2[p] gives the

[p]Parameter set I is as given above. Parameters in the remaining cases were: Parameter set II: $\nu_1 = 4$, $\nu_2 = 1$, $\beta = 4$, $T_c = 7.5$, $T_f = 5$, $\alpha_1 = 0.9$, $\alpha_2 = 0.25$, $\alpha_3 = 1$, $s = 0.75$, $\sigma = 0.1$; Parameter set III: $\nu_1 = \nu_2 = 0.5$, $\beta = 2$, $T_c = T_f = 10$, $\alpha_1 = 0.75$, $\alpha_2 = 0.25$, $\alpha_3 = 0.75$, $s = 0.8$, $\sigma = 0.1$; Parameter set IV: $\nu_1 = 2$, $\nu_2 = 0.6$, $\beta = 4$, $T_c = T_f = 5$, $\alpha_1 = 0.8$, $\alpha_2 = 0.2$, $\alpha_3 = 1$, $s = 0.75$, $\sigma = 0.05$; in all three cases fundamental price and returns are the same as with parameter set I: $p_f = 10$, $r = 0.004$, $R = 0.0004$.

Table 2. Fat tail property of the data: kurtosis and tail index estimates.

Parameters	kurtosis	median α_H from 10 samples of 2,000 observations (in parentheses: range of estimates) 2.5% tail	5% tail	10% tail
		2.04	2.11	1.93
Parameter set I	135.73	(1.61–4.50)	(1.51–2.64)	(1.26–2.44)
		2.82	2.52	2.18
Parameter set II	16.10	(2.28–3.73)	(2.00–3.17)	(1.55–2.36)
		4.63	3.48	2.86
Parameter set II	27.11	(2.41–6.82)	(2.33–8.60)	(1.80–4.84)
		3.08	2.46	1.97
Parameter set IV	37.74	(2.11–4.06)	(2.13–7.86)	(1.65–3.18)

Note: Parameter sets are those given in footnote p.

kurtosis statistics for the above four series of 20,000 time steps, respectively. As can be seen, we have excessive fourth moments in each case. As compared to empirical data at daily frequency, the results from parameter sets II to IV look very realistic while the first variant may seem to exhibit too high a degree of leptokurtosis.[q] However, note that kurtosis is a somewhat ambiguous concept and it is not entirely clear how to compare the statistics obtained for various time series. Furthermore, empirical power-law tails with exponents in the range 2 to 4 imply non-convergence of the fourth moment which also makes empirical estimates of the kurtosis statistics unreliable.

A more stringent characterisation of the fat tail property can be achieved using the Pareto approximation of Eq. (1). In particular, the tail index α gives a measure of the frequency of abnormally large returns. It has the advantage of (theoretical) invariance under time aggregation (provided, of course, the underlying data are truly stochastic), i.e. data from the same source but with different sampling frequency should tend to the same Pareto law.

In applied economics literature, an estimator proposed by Hill has become the standard work tool for estimation of the Pareto exponent of the tails. To compute the Hill tail index estimate, the sample elements of a series are put in descending order: $x_{(n)} \geq x_{(n-1)} \geq \cdots \geq x_{(n-k)} \geq \cdots \geq x_{(1)}$ with k the number of observations located in the "tail" of the distribution. The Hill estimate is then obtained as:

$$\alpha_H = \frac{1}{\frac{1}{k}\sum_{i=1}^{k}[\ln(x_{(n-l+1)}) - \ln(x_{(n-k)})]} \tag{5.1}$$

[q]However, such high numbers are not uncommon for thinly traded assets at daily frequencies and for more frequently traded ones at intra-daily frequencies.

Here, we apply this technique to the time series of returns. Since there is no indication of systematic differences between positive and negative price changes we merged both extremal regions by using absolute values of returns. As the estimate rests to some extent on an appropriate choice of the tail region (the cut-off value k in the above formula), Table 2 reports results for a range of tail sizes that are frequently used in empirical studies. Here we split the data records obtained with the above parameter sets into 10 subsamples of 2,000 observations, respectively, which is a sample size quite representative of many empirical studies. The results are, in fact, close to the usual empirical finding of tail indices somewhere between 2 and 5. There is also some indication of an increase of the estimate with decreasing tail size — a pattern which is also often encountered in empirical investigations.

Interestingly, the bursts are so strong in our examples, that tail index estimates even below 2 are found in some instances. It is well-known that for a stochastic process with a tail shape parameter < 2, the second moment does not converge, i.e. the sample variance tends to infinity with increasing sample size.[r] Here the situation is similar to the unit-root results: we *know* that, in our simulation design, the maximum increase and decrease of the price is bounded which means that all moments of this process do exist. Of course, with sample size $n \to \infty$, the estimator should be able to detect this finiteness (α_H would then, also tend to infinity). Again, the simulated series are able to "fool" the estimator behaving as if the third or even the second moment would not exist. Note again the similarity with the behaviour of empirical data. Here one might also argue that price changes are ultimately bounded. Nevertheless, all statistical analyses indicate moment condition failure with non-convergence of all moments of order higher than three or four.

Finally, we are interested in whether the impression of autocorrelated volatility is also confirmed using elementary statistical techniques. To this end, we first investigate the autocorrelation functions of raw returns as well as those of squared and absolute returns. Fig. 3 shows a representative graph. Evidently, autocorrelations of raw returns show only minor fluctuations around zero and are indicative of very restricted short memory in returns at best.[s] Squared and absolute returns, on the other hand, show much higher autocorrelation than the original data with absolute returns exhibiting the highest degree of autocorrelation. Again, the picture resembles that of empirical time series as outlined in the introduction. Note that no formal tests are reported because with 20,000 entries even the smallest degree of correlation will be detected by e.g. the Box–Ljung statistics. In fact, even the hypothesis of no serial correlation of raw returns is overwhelmingly rejected using the entire sample. However, one might easily select subsamples for which we get rejection for squared and absolute returns but not for raw returns.

[r]More generally, it can be shown that only moments of order smaller than the tail index exist.
[s]The negative spike at the first two lags appears somewhat too large when compared to empirical numbers. The reason for this negative short-run autocorrelation is probably that noise traders move too swifty from optimistic to pessimistic and *vice versa* during turbulent episodes. Further attempts at fine-tuning parameters may eliminate this feature.

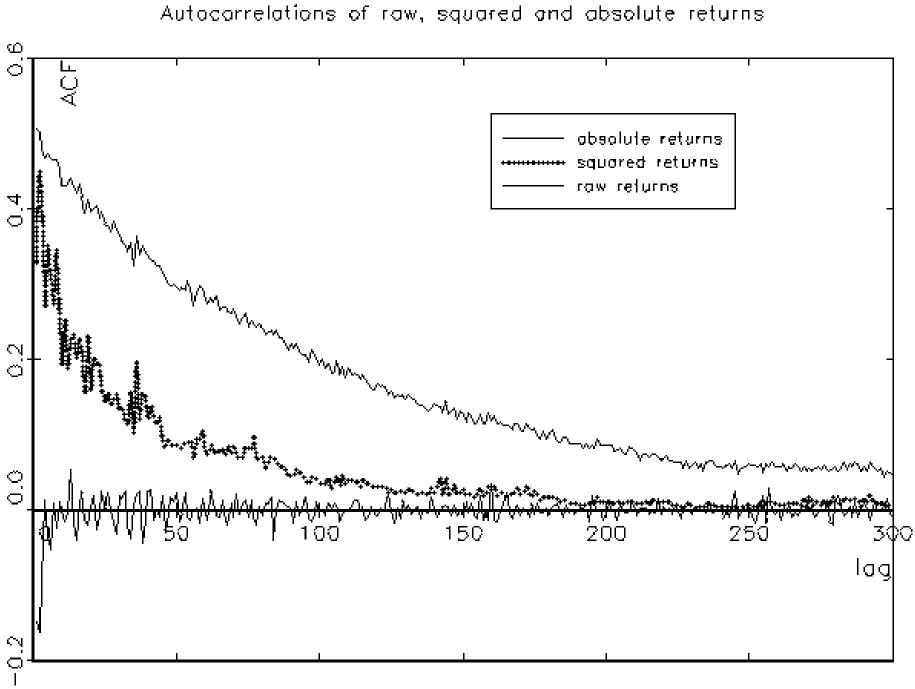

Fig. 3. Typical behaviour of the autocorrelations of raw, squared and absolute returns (bottom to top). The autocorrelation function here is computed from a simulation extending over 20,000 time steps. Underlying parameter values are those of parameter set IV.

As is evident from Fig. 3, dependence in volatility measures (squared and absolute returns) is significant over an extended time horizon with autocorrelation coefficients of absolute returns not even decaying to zero when considering three hundred lags. Such long-horizon effects in the level of volatility have also been found in real-life data. In recent literature, various stochastic processes have been introduced to deal with long-term dependence. Examples are fractional Gaussian noises and fractionally integrated ARMA processes both of which are characterised by a slow, hyperbolic decay of the autocorrelation coefficients. Of course, Fig. 3 is quite suggestive of the same slow decay of autocorrelation in the volatility measures from our simulated data. A simple and frequently used method of testing for long-memory *versus* only short-memory in a time series is the periodogram regression introduced by Geweke and Porter–Hudak [16]. Application of this technique again allows us to compare the performance of our artificial data with well-known results from real-life markets.

The Geweke/Porter–Hudak procedure consists in estimating the fractional differencing parameter d of a fractionally integrated ARMA model without specifying the short-memory part of the model. As the case $d = 0$ leads one back to the standard, short-memory ARMA processes, non-rejection of the hypothesis $d = 0$

Table 3. Long–term dependence in squared and absolute returns.

	squared returns		absolute returns	
Parameters	median d from 10 samples of 2,000 observations (in parentheses: range of estimates)	% rejections of $d = 0$ at 95% level	median d from 10 samples of 2,000 observations (in parentheses: range of estimates)	% rejections of $d = 0$ at 95% level
Parameter set I	0.17 (0.06–0.56)	40%	0.38 (0.21–0.64)	80%
Parameter set II	0.54 (0.37–0.86)	100%	0.63 (0.43–0.75)	100%
Parameter set II	0.50 (0.29–0.80)	100%	0.64 (0.26–0.81)	100%
Parameter set IV	0.52 (0.20–0.70)	90%	0.64 (0.17–0.88)	90%

Note: Parameter sets are those given in footnote p.

amounts to non-rejection of only short-term dependence in the data. Estimates significantly different from zero, on the other hand, can be interpreted as evidence in favour of long-term dependence. Concerning the autocorrelation function, $\rho(k)$, long memory shows up in an asymptotic scaling according to: $\rho(k) \sim k^{2d-1}$, i.e. hyperbolic decay. Empirical results usually point to (positive) long-term memory in various measures of volatility ($d > 0$) with raw returns (for what reason ever) usually exhibiting the strongest degree of dependence (cf. Ding *et al.* [12]).

Since the Geweke and Porter–Hudak technique is fairly straightforward and has been used in a large number of applied papers over the last years, we shall dispense with a detailed treatment here and proceed with the results given in Table 3. The estimates confirm the impression of Fig. 3: in most cases, long-term memory in volatility cannot be rejected using this standard procedure. Also, the degree of persistence appears to be higher in absolute returns than in squared returns judging by the median of the estimates.

One may also note, that some of the estimates fall into the region $d > 0.5$ indicating non-stationarity of the volatility process. As with the appearance of infinite moments, we know that the underlying process is, in fact, stationary but dependence in volatility is occasionally so strong that the Geweke/Porter–Hudak estimator delivers an indication of non-stationarity. It seems reassuring that similar results can be found in empirical applications.[t] Hence, with regard to the third stylised fact,

[t]For example, Lux [28] found point estimates ranging from 0.22 to 0.60 for absolute returns of German stocks. A similar finding of parameter estimates within the non-stationarity region occurs occasionally with GARCH models.

the computer-generated data again share reproduce the properties of empirical records in great detail.

6. Conclusion

In this paper, a model of financial markets has been developed in which clustered volatility was explained as the consequence of the market being subject to occasional temporary instability. The emerging picture of a vibrant market with a dominance of tranquil phases interspersed with volatile episodes seems to have some intuitive appeal. Statistical investigation of the simulated time paths showed that the main stylised facts (unit roots in levels together with fat tails and heteroscedasticity of returns) can be found in this "artificial" market.

Although some intuition and mathematical insight regarding the phenomenon of volatility bursts can be provided applying mean-field theory and local stability analysis, their appearance can only be demonstrated using numerical simulation. We believe that our findings extend beyond the stylised model of speculative behaviour analysed in this paper. Considering our results as well as those obtained in a different context by Youssefmir and Huberman the key increments for the emergence of volatility bursts (in models with many interacting agents) seem to be the following: (i) indeterminacy of the population composition in equilibrium (i.e. no strategy has an advantage within a stationary environment) and (ii) dependence of stability of the equilibrium itself on the composition of the population. We believe that these conditions are met by various economic models. As an example, we conjecture that many variants of Brock and Hommes' model of adaptive choice of predictors (Brock and Hommes [5]) may fulfil the above conditions and may, thus, also be likely candidates for the observation of volatility bursts.

Some future research avenues suggest themselves. First, it would be an interesting exercise to consider the effect of additional noise in fundamental data which have been assumed to be stationary throughout the paper. This line has been pursued in Lux and Marchesi [31] where we investigate the relationship between random changes in fundamental values (as an "input" to the market system) and price changes as the market's output. As it turns out, even if we assume that the news arrival process follows an innocent Gaussian distribution lacking both fat tails and volatility dependence, these features nevertheless show up again in the time series of returns. These results suggest that these statistical properties, thus, appear as "emergent phenomena" from the market process itself and do not stem from similar behaviour of fundamental values (see our above paper for a more elaborate discussion).

Second, one would like to know whether the process described in this paper is able to mimic empirical time series in more detail. As an example, it would be interesting to estimate ARCH models from our synthetic data and compare the results with empirical findings. Furthermore, it would also be interesting to see whether there is non-linear structure in the series that could be detected using, for example, the BDS test (cf. Brock *et al.* [6]). The mechanics of the process in fact suggest that there may be periods with more structure, namely those phases showing bursts and successive returns to normal performance, while most of the time the dynamics appears almost random. This would square well with recent findings of windows of recognisable structure in high-frequency forex markets (see Ramsey and Zhang [37]). However, this possible parallel remains to be confirmed by statistical analysis. From a more fundamental perspective the similarity between the phenomenon of on-off intermittency and the behaviour of the successful ARCH time series models developed in financial econometrics seems remarkable and may point to an explanation of the underlying phenomena.

Appendix A. Proof of Propositions

Proof of Proposition 3.1. ad (a): The reader may refer to the derivations detailed in Lux [30] in order to arrive at the following system of differential equations governing the time development of mean values of x, z, and p (without introducing explicit notation, x, z and p are understood as mean values of the original variables in the following):

$$\begin{aligned} \frac{dx}{dt} &= 2z\nu_1[\tanh(U_1) - x]\cosh(U_1) \\ &\quad + (1-z)(1-x^2)\nu_2[\sinh(U_{2,1}) - \sinh(U_{2,2})]\,, \\ \frac{dx}{dt} &= (1-z)z(1+x)\nu_2\sinh(U_{2,1}) + (1-z)z(1-x)\nu_2\sinh(U_{2,2})\,, \\ \frac{dp}{dt} &= \beta(xzT_c + (1-z)(p_f - p)T_f)\,. \end{aligned} \tag{A.1}$$

Here, the U-functions are defined as in the main text. It is straightforward to convince oneself that (i), (ii), and (iii) are all stationary states of the dynamics (A.1).

ad (b): Considering only the third equation of the triplet (A.1), one finds further solutions obeying $dp/dt = 0$, namely:

$$x = \frac{1-z}{z}\frac{T_f}{T_c}(p_f - p)\,. \tag{A.2}$$

If combinations (x', z', p') exist fulfilling (A.2), they would be characterised either by a positive value of x' together with $p' > p_f$ or $x' < 0$ and $p' < p_f$. However, it will be shown that no such equilibria exist for the entirety of the differential equation system (A.1): Inserting some $x' > 0$, $p' > p_f$ into $dz/dt = 0$, one obtains:

$$(1+x)\sinh(U_{2,1}(x', p')) + (1-x)\sinh(U_{2,2}(x', p')) = 0\,. \tag{A.3}$$

Solving for x' yields:

$$x' = \frac{\sinh(U_{2,2}(x', p')) + \sinh(U_{2,1}(x', p'))}{\sinh(U_{2,2}(x', p')) - \sinh(U_{2,1}(x', p'))}\,. \tag{A.4}$$

Note that: $U_{2,1}(x',p') = \alpha_3(\frac{r}{p'} - R - s|(p'-p_f)/p'|) < 0$ for any $p' > p_f$, while the sign of $U_{2,2}(x',p') = \alpha_3(R - \frac{r}{p'} - s|(p'-p_f)/p'|)$ cannot be unambiguously determined. However, one easily recovers that $U_{2,1}(x',p') < U_{2,2}(x',p') < -U_{2,1}(x',p')$, and consequently, $\sinh(U_{2,1}(x',p')) < \sinh(U_{2,2}(x',p')) < -\sinh(U_{2,1}(x',p'))$. It follows that the RHS of (A.4) should be negative which is in contradiction to the assumption $x' > 0$. A similar chain of arguments may be used to show that starting with $x' < 0$, $p' < p_f$ one arrives at a similar contradiction. As a consequence, there are no stationary states besides the ones given in part (a).

Proof of Proposition 3.2. In order to hopefully get some insights into the determinants of stability for equilibria along the line $(x^* = 0, z^*, p^* = p_f)$ one considers the entries of the three-by-three Jacobian ($\mathbf{A}$) of system (A.1), evaluated at an equilibrium of type (i):

$$a_{11} = \frac{d\dot{x}}{dx} = 2z^*\nu_1(\alpha_1 + \alpha_2(\beta/\nu_1)z^*T_c - 1) + 2(1-z^*)z^*\alpha_3\beta T_c/p_f$$

$$a_{13} = \frac{d\dot{x}}{dp} = -2z^*(1-z^*)\alpha_2\beta T_f - 2(1-z^*)\nu_2\alpha_3\left[\left(\frac{\beta}{\nu_2}(1-z^*)T_f p_f + r\right)/p_f^2\right],$$

$$a_{23} = \frac{d\dot{z}}{dp} = \begin{cases} 2(1-z^*)z^*\nu_2\alpha_3\dfrac{s}{p_f} & \text{if } p < p_f \\ -2(1-z^*)z^*\nu_2\alpha_3\dfrac{s}{p_f} & \text{if } p > p_f \end{cases}$$

$$a_{31} = \frac{d\dot{p}}{dx} = \beta z^* T_c\,, \quad a_{33} = \frac{d\dot{p}}{dp} = -\beta(1-z^*)T_f\,,$$

$$a_{12} = \frac{d\dot{x}}{dz} = 0\,, \qquad a_{21} = \frac{d\dot{z}}{dx} = 0\,, \qquad a_{22} = \frac{d\dot{z}}{dz} = 0\,, \qquad a_{32} = \frac{d\dot{p}}{dz} = 0\,.$$

The discontinuity in the entry a_{23} stems from the absolute term in the profit differential. In the case of continuous derivatives, stability hinges on the signs of the roots of the matrix A. Considering separately the cases $p < p_f$ and $p > p_f$, one finds that in both cases the roots $\lambda_i (i = 1,2,3)$ are determined by the same characteristic equation:

$$(a_{11} - \lambda)(-\lambda)(a_{33} - \lambda) + a_{13}a_{31}\lambda = 0\,. \tag{A.5}$$

It can readily be seen that one of the roots is always equal to zero. Furthermore, with $\lambda_1 = 0$, the conditions for the remaining roots to be negative reduce to:

$$\text{(i)} \quad a_{11} + a_{33} < 0 \qquad \text{and} \qquad \text{(ii)} \quad a_{11}a_{33} - a_{13}a_{31} > 0 \tag{A.6}$$

Obviously, the conditions for λ_2, $\lambda_3 < 0$ reduce to the stability conditions for a two-dimensional dynamics of the state-variables x and p with stationary z.

Appropriate manipulation of these conditions yields (cond 1) and (cond 2) in Proposition 3.2. Both the fact that with a zero root we are at the edge between stability and instability and the discontinuity in the partial derivatives preclude to draw inference concerning the stability of some point along the line ($x^* = 0, p^* = p_f, z^*$). However, in the case of instability of both adjacent subregimes (to the right and left of p_f) a point can be unambiguously classified as a local repeller (cf. Honkapohja and Ito [23]) leading to the statements given in Proposition 3.2.

Acknowledgments

The first version of this paper grew out of a keynote lecture given at the conference "Quantitative Methods in Finance '97" in Sydney. Later versions have benefited from the audiences of several university seminars, the Konstanz Seminar on Monetary Theory and Monetary Policy 1998, the 1998 Workshop on Economics with Heterogeneous Interacting Agents in Ancona, the 1998 Symposium of the Society of Computational Economics in Cambridge, the conference "Computational Finance '99" in New York and the 1999 Symposium of the Society of Nonlinear Dynamics and Econometrics at the same place. We are grateful to many participants of the above events for their helpful remarks and lively discussion and wish to thank in particular Carl Chiarella, Sergio Focardi, Reiner Franke, Gebhard Kirchgaessner, Kenichi Ueda and the editor-in-charge, Jean-Philippe Bouchaud, for their perceptive comments. Financial support by Deutsche Forschungsgemeinschaft, Sonderforschungsbereich 303 at the University of Bonn (the first author's prior affiliation) is also gratefully acknowledged.

References

[1] H. Allen and M. P. Taylor, *Charts, noise and fundamentals in the london foreign exchange market*, Economic Journal **100** (1990) 49–59.

[2] W. B. Arthur, J. H. Holland, B. LeBaron, R. Palmer and P. Taylor, *Asset Pricing under Endogenous Expectations in an Artificial Stock Market*, mimeo: Santa Fe Institute (1996).

[3] A. V. Banerjee, *The economics of roumors*, Review of Economic Studies **60** (1993) 309–327.

[4] J.-P. Bouchaud and R. Cont, *A Langevin approach to stock market fluctuations and market crashes*, European Physical J. **B6** (1998) 543–550.

[5] W. Brock and C. Hommes, *A rational route to randomness*, Econometrica **65** (1997) 1059–1095.

[6] W. Brock, D. Hsieh and B. LeBaron, *Nonlinear Dynamics, Chaos, and Instability: Statistical Theory and Economic Inference*, Cambridge: MIT-Press (1991).

[7] W. Brock and B. LeBaron, *A dynamical structural model for stock return volatility and trading volume*, Review of Economics and Statistics **78** (1996) 94–110.

[8] W. Brock and P. de Lima, *Nonlinear Time Series, Complexity Theory, and Finance*, in: eds. G. Maddala and C. Rao, *Handbook of Statistics* Vol. 14. Amsterdam: North–Holland (1995).

[9] R. Day and W. Huang, *Bulls, bears, and market sheep*, J. Economic Behavior and Organization **14** (1990) 299–329.

[10] P. De Grauwe, H. Dewachter and M. Embrechts, *Exchange Rate Theory: Chaotic Models of Foreign Exchange Markets*, Oxford: Blackwell Publ. (1993).

[11] D. A. Dickey, W. R. Bell and R. B. Miller, *Unit roots in time series models: tests and implications*, American Statistician **40** (1986) 12–26.

[12] Z. Ding, C. W. J. Granger and R. F. Engle, *A long memory property of stock market returns and a new model*, J. Empirical Finance **1** (1993) 83–106.

[13] R. F. Engle and J. R. Russell, *Forecasting the frequency of changes in quoted foreign exchange prices with the autoregressive conditional duration model*, J. Empirical Finance **4** (1997) 187–212.

[14] J. A. Frankel and K. A. Froot, *Understanding the US dollar in the eighties: The expectations of chartists and fundamentalists*, Economic Record, special issue (1986) 24–40.

[15] H. Fujisaka and T. Yamada, *Stability theory of synchronized motion in coupled-oscillator systems IV*, Progress of Theoretical Physics **75** (1986) 1087–1104.

[16] J. Geweke and S. Porter–Hudak, *The estimation and application of long-memory time series models*, J. Time Series Analysis **4** (1983) 221–238.

[17] N. Glance, *Dynamics with Expectations*, Ph. D. Thesis: Stanford Univ. (1993).

[18] C. Goodhart, *The foreign exchange market: A random walk with a dragging anchor*, Economica **55** (1988) 437–460.

[19] E. R. Grannan and G. H. Swindle, *Contrarians and Volatility Clustering*, mimeo: Santa Fe Institute (1994).

[20] D. M. Guillaume, M. M. Dacorogna, R. R. Davé, U. A. Müller, R. B. Olsen and O. V. Pictet, *From the bird's eye to the microscope: A survey of new stylized facts of the intra-daily foreign exchange markets*, Finance and Stochastics **1** (1997) 95–129.

[21] P. Harrison, *Similarities in the distribution of stock market price changes between the eighteenth and twentieth centuries*, J. Business **71** (1998) 55–79.

[22] J. F. Heagy, N. Platt and S. M. Hammel, *Characterization of on-off intermittency*, Physical Review E **49** (1994) 1140–1150.

[23] S. Honkapohja and T. Ito, *Stability with regime switching*, J. Economic Theory **29** (1985) 22–48.

[24] A. Kirman, *The Significance of Testing for Bubbles*, mimeo: European Univ. Institute (1992).

[25] A. Kirman, *Ants, rationality, and recruitment*, Quarterly J. Economics **108** (1993) 137–156.

[26] P. C. Liu, *The effects of the fundamentalists' and chartists' expectations on market survey*, Applied Financial Economics **6** (1996) 363–366.

[27] T. Lux, *Herd behaviour, bubbles and crashes*, J. Economic **105** (1995) 881–896.

[28] T. Lux, *Long-term stochastic dependence in financial markets: evidence from the German stock market*, Applied Economics Letters **3** (1996) 701–6.

[29] T. Lux, *Time variation of second moments from a noise trader/infection model*, J. Economic Dynamics and Control **22** (1997) 1–38.

[30] T. Lux, *The socio-economic dynamics of speculative markets: Interacting agents, chaos, and the fat tails of return distribution*, J. Economic Behavior and Organization **33** (1998) 143–165.

[31] T. Lux and M. Marchesi, *Scaling and criticality in a stochastic multi-agent model of a financial market*, Nature **397** (1999) 498–500.

[32] L. Menkhoff, *The noise trading approach: Questionnaire evidence from foreign exchange*, J. International Money and Finance **17** (1998) 547–564.

[33] A. Orléan, *Bayesian interactions and collective dynamics of opinion: Herd behavior and mimetic contagion*, J. Economic Behavior and Organization **28** (1995) 257–274.

[34] E. Ott and J. C. Sommerer, *Blowout bifurcations: the occurrence of riddled basins and on-off intermittency*, Physics Letters **A188** (1994) 39–47.

[35] A. Pagan, *The econometrics of financial markets*, J. Empirical Finance **3** (1996) 15–102.

[36] J. B. Ramsey, *On the existence of macro variables and of macro relationships*, J. Economic Behavior and Organization **30** (1996) 275–299.

[37] J. B. Ramsey and Z. Zhang, *The analysis of foreign exchange data using waveform dictionaries*, J. Empirical Finance **4** (1997) 341–372.

[38] M. P. Taylor and H. Allen, *The use of technical analysis in the foreign exchange market*, J. International Money and Finance **11** (1992) 304–314.

[39] C. G. de Vries, *Stylized facts of nominal exchange rate returns*, pp. 348–89, in ed. F. van der Ploeg, The Handbook of International Macroeconomics. Blackwell: Oxford (1994).

[40] M. Youssefmir and B. Huberman, *Clustered volatility in multiagent dynamics*, J. Economic Behavior and Organization **32** (1997) 101–118.