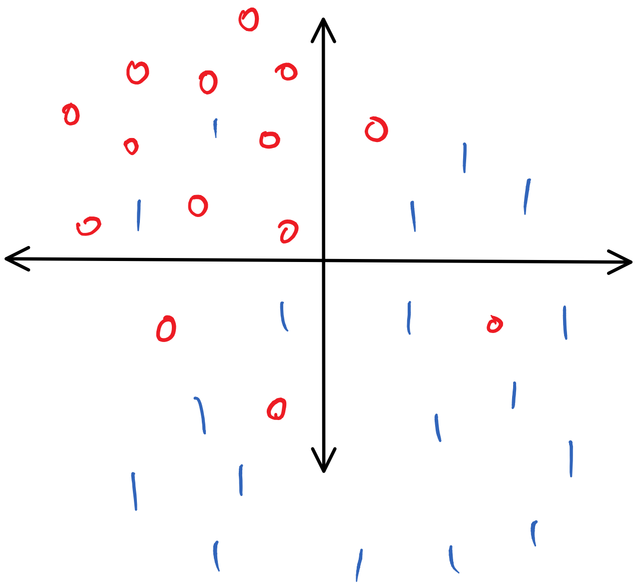
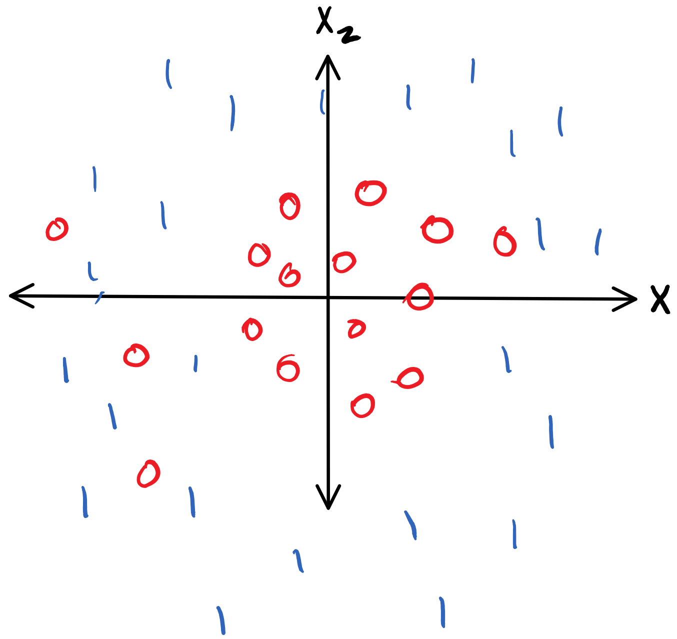
HiDi Embedding

Monday, May 8, 2017   8:11 PM



Linear decision boundary

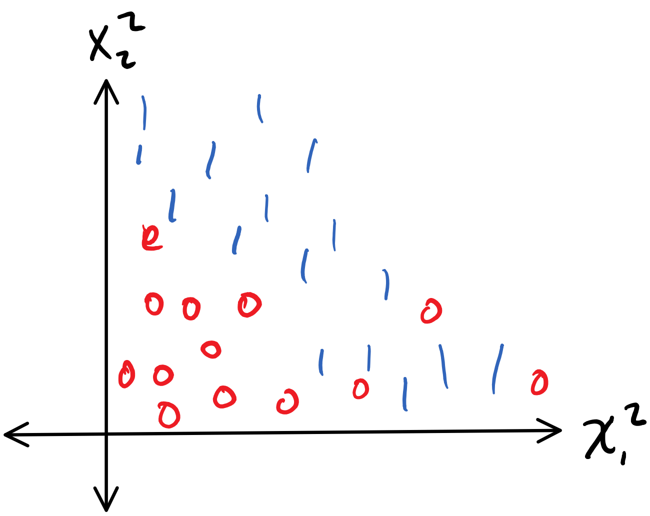


Non-linear decision boundary

<u>define</u> higher dimensional embedding $\Phi : \mathbb{R}^P \to \mathbb{R}^D$

$\Phi(x) \in \mathbb{R}^D$

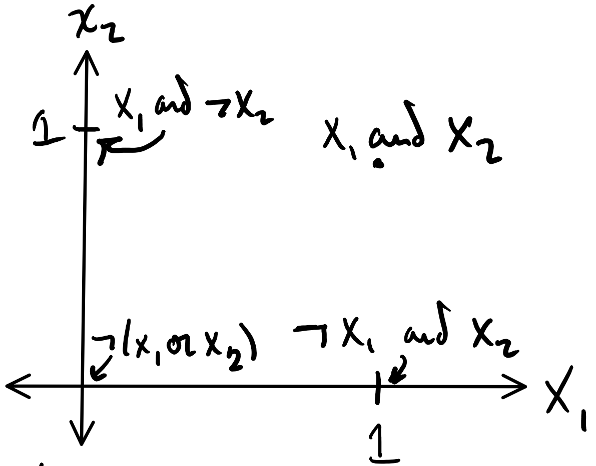
<u>ex</u> $\Phi(x_1, x_2) = (1, x_1, x_2, x_1^2, x_2^2)$



"Hi-di embedding can make linear methods non-linear"

— me, just now

ex Logic: $X_1, \dots, X_p$ are propositions encoded as $\{0,1\}$.

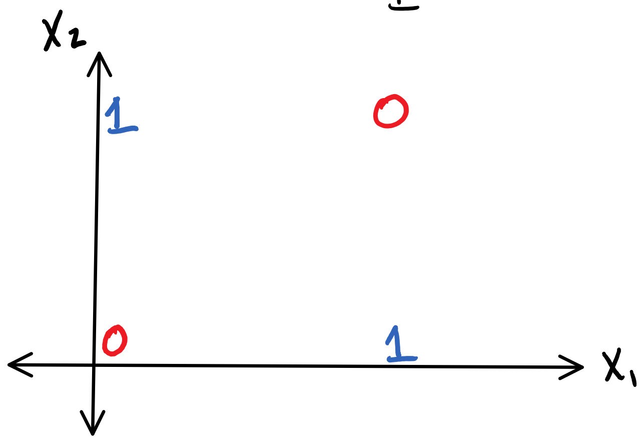


X_1 xor X_2 is

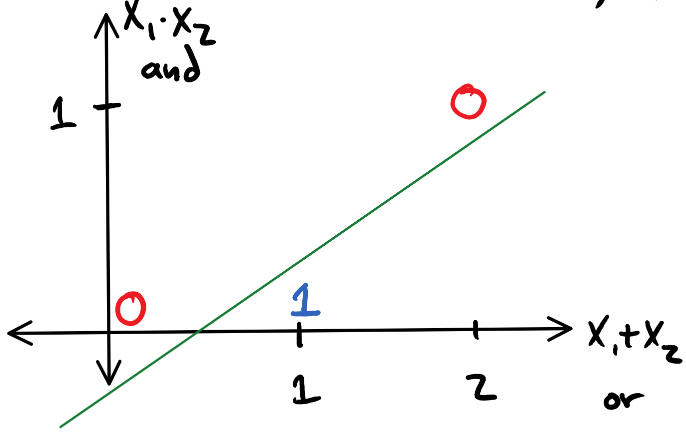
$(X_1 \text{ and } \neg X_2)$ or

$(X_2 \text{ and } \neg X_1)$

$\equiv (X_1 \text{ or } X_2) \text{ and } \neg(X_1 \text{ and } X_2)$

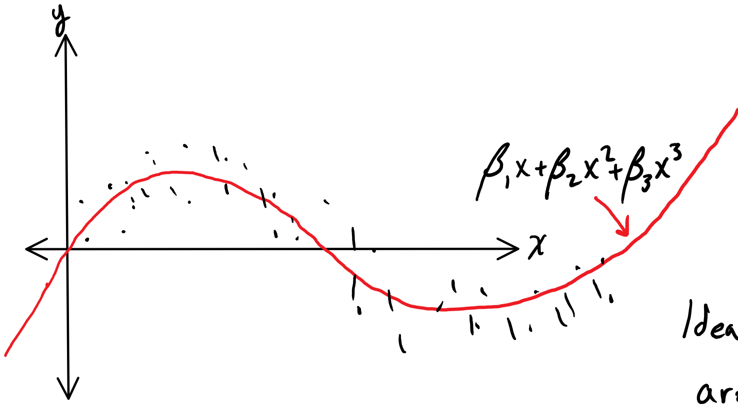


$\Phi(X_1, X_2) = \Phi(X_1, X_2, X_1 \cdot X_2)$



Basis Expansion

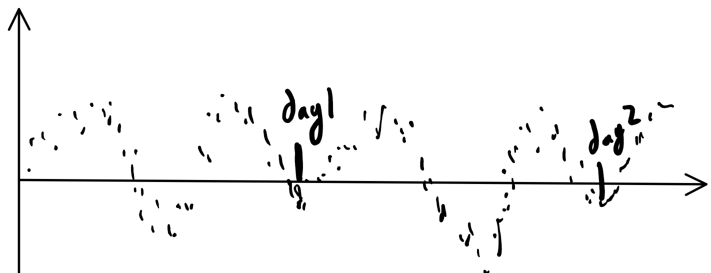
Monday, May 8, 2017 8:42 PM



Polynomial Basis

$\Phi(x) = (1, x, x^2, \ldots, x^d)$

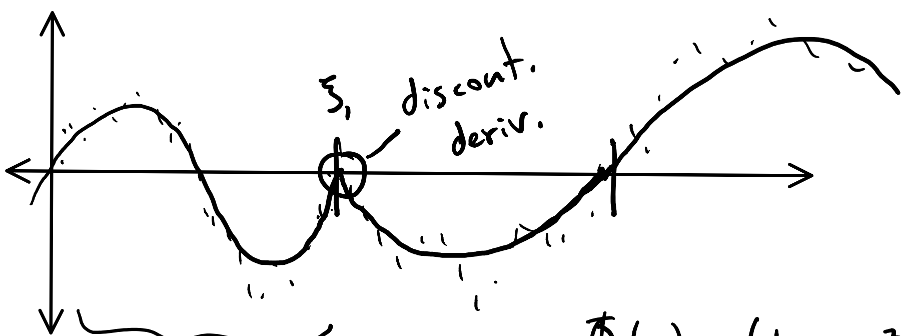
Idea: Taylor expansion around $x = 0$.



Periodic (1 day period)

$\sin(4\pi x) = \sin(2\pi(2x))$ "dilation"

Fourier basis: $\Phi(x) = (\sin(2\pi x), \cos(2\pi x), \sin(4\pi x), \cos(4\pi x), \ldots)$



cubic btw knots

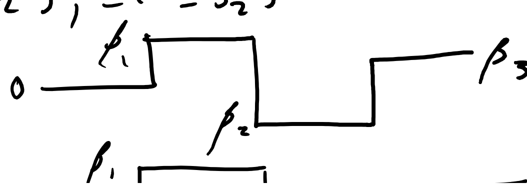
Piecewise polynomial

$\Phi(x) = (1, x, x^2, x^3, \underline{(x-\xi_1)_+}, \underline{(x-\xi_1)_+^2}, (x-\xi_1)_+^3, \ldots)$

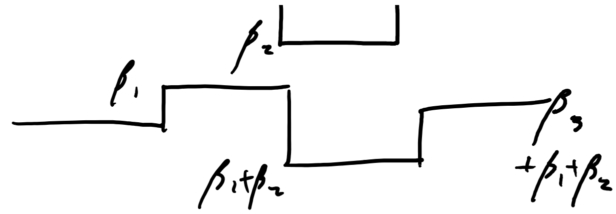
derivatives and constraints

$$\begin{array}{lccc} & \varphi_1 & \varphi_2 & \varphi_3 \\ 0^{th} \text{ order}: & \mathbb{1}\{0 \le x < \xi_1\}, & \mathbb{1}\{\xi_1 \le x < \xi_2\}, & \mathbb{1}\{x \ge \xi_2\} \end{array}$$

$f(x) = (\beta_1 \varphi_1 + \beta_2 \varphi_2 + \beta_3 \varphi_3)(x)$

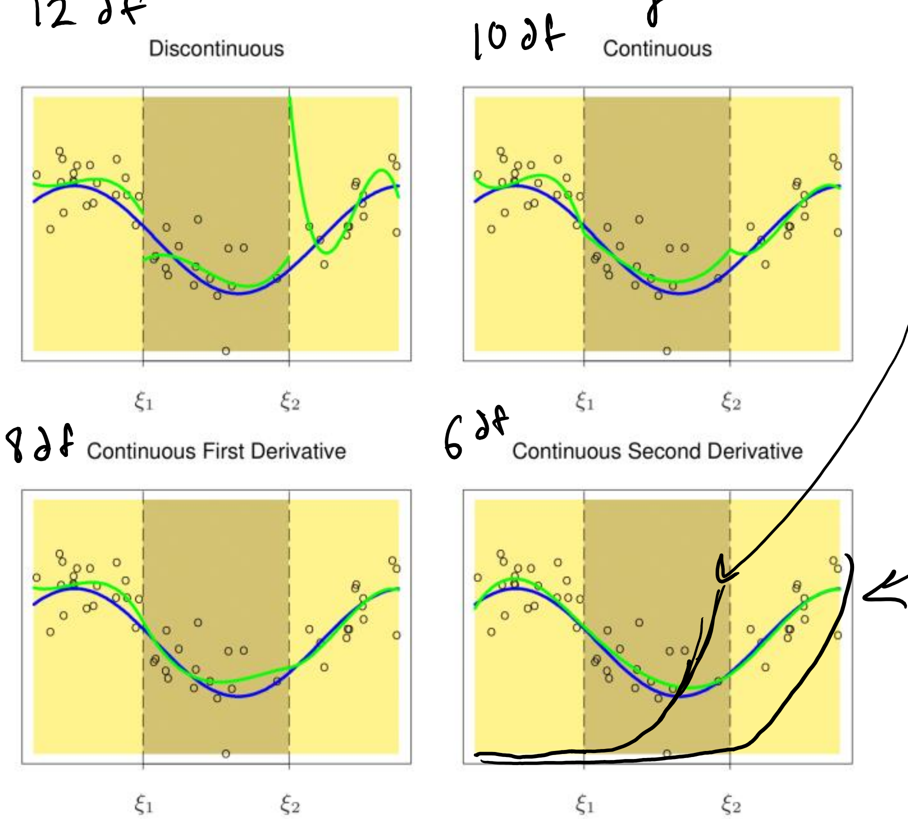


or $\mathbb{1}, \mathbb{1}\{x \ge \xi_1\}$ $\mathbb{1}\{x > \xi$ [illegible] or $1, 1\{x \geq \xi_1\}, 1\{x \geq \xi_2\}$



k^{th} order poly: $1\{x \geq \xi_j\}, (x-\xi_j)_+, (x-\xi_j)_+^2, \dots, (x-\xi_j)_+^k$

$k-1$ cont derivatives: removing

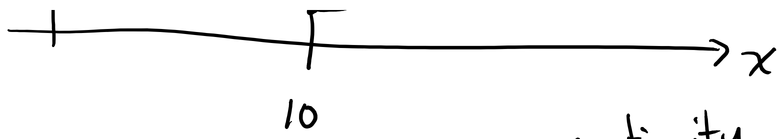


$\underline{1}, \underline{x}, \underline{x^2}, \underline{x^3},$
$\underline{(x-\xi_1)_+^3}, \underline{(x-\xi_2)_+^3}$

FIGURE 5.2. *A series of piecewise-cubic polynomials, with increasing orders of continuity.*

ESL 5.2

iii/ Change β changes fit everywhere (globally supported)



continuity constraint   cont. deriv. const.

$$\hat{f}(x) = \underbrace{\beta_0 + \beta_1 x + \beta_2 x^2}_{p(x)} + \beta_3 \cancel{\mathbb{1}\{x > 10\}} + \beta_4 \cancel{(x-10)_+} + \beta_5 (x-10)_+^2$$

left lim of $\hat{f}(x)$ at 10 : $p(10)$
right lim ... : $p(10) + \beta_3$
equal $\equiv \beta_3 = 0$

$$\hat{f}'(x) = \underbrace{\beta_1 + 2\beta_2 x}_{p'(x)} + \beta_4 \mathbb{1}\{x \geq 10\} + 2\beta_5 (x-10)_+$$

left lim of $\hat{f}'(x)$ at 10 : $p'(10)$
right lim ---- : $p'(10) + \beta_4$
equal $\equiv \beta_4 = 0$

$$\hat{f}(x) = \beta_0 + \beta_1 x + \beta_2 x^2 + \beta_5 (x-10)_+^2$$

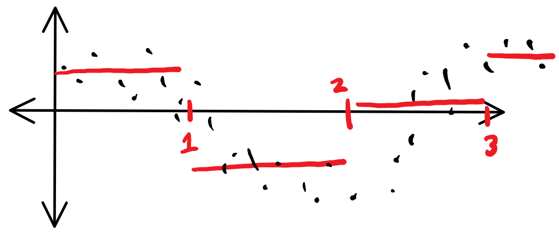
Localized Bases

Monday, May 8, 2017 9:55 PM

B-spline basis (cardinal)

knots at 1, 2, 3, …

0th order $\varphi_i(x) = \mathbb{1}\{i-1 \le x < i\}$



<u>translation</u>: $\varphi_{i+1}(x) = \varphi_1(x-i)$

<u>convolution</u>: $(g * h)(x) = \int_{-\infty}^{\infty} g(y)\, h(x-y)\, dy$

$(\varphi_1 * \varphi_1)(x) = \int \mathbb{1}\{0 \le y < 1\}\, \mathbb{1}\{0 \le x-y < 1\}\, dy$

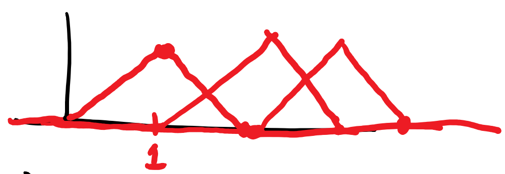
<u>derive</u>

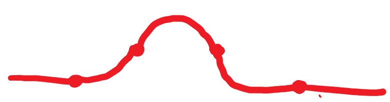
$x < 0 : 0$

$0 \le x < 1 : \int_0^x 1 \cdot dy = x$

$1 \le x < 2 : \int_{x-1}^{1} 1 \cdot dy = 1 - (x-1) = 2 - x$

$x \ge 2 : 0$





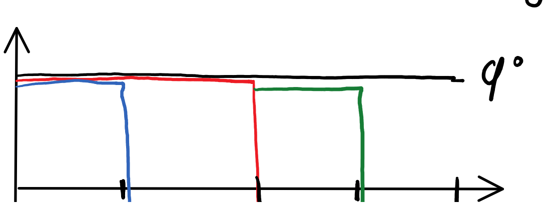
<u>def</u> $\varphi_1^{(k)} = \varphi_1^{(k-1)} * \varphi_1$ and $\varphi_{i+1}^{(k)} = \varphi_1(x-i)$

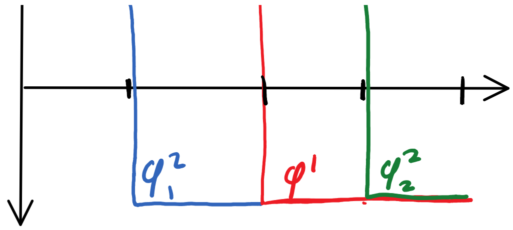
is the <u>k-th order (cardinal) B-spline basis</u>.

- localized, not orthogonal. $\left(\int \varphi_{i+1}^{(k)}(x)\, \varphi_i^{(k)}(x)\, dx \neq 0\right)$

<u>Haar Wavelets</u>

orthogonal!





- Wavelets are orthogonal
- Take "mother wavelet" φ^1 and translate and dilate $\varphi^{(k)}_{i+1}(x) = \varphi^1(2^k x - i)$
- Many other wavelets: Daubechies, Coiflets, etc.

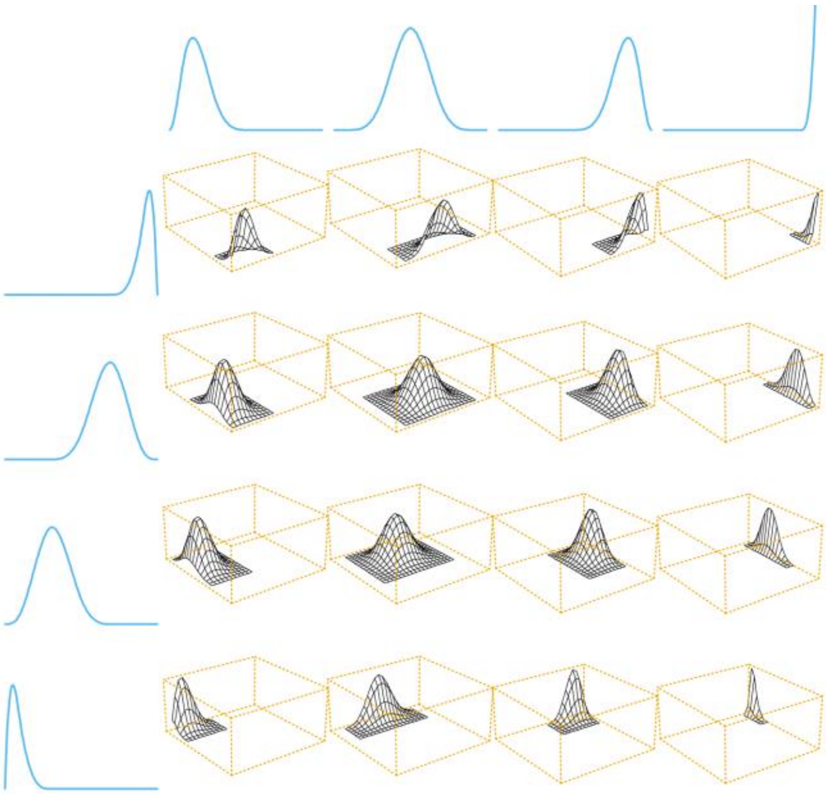
Define $Z_{jl} = \varphi_l(j)$ for l'th basis element,

lasso: $\min_\beta \|y - Z\beta\|_2^2 + \lambda\|\beta\|_1$

Z orthogonal $\longrightarrow$ soft-thresholding

Multidimensional Bases $x \in \mathbb{R}^2$ then

$\varphi_{ij}(x) = \varphi_i(x_1)\cdot\varphi_j(x_2)$ is Tensor product basis.



ESL 5.7