Algebra

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Lecture 1: Introduction and vector operation review

Contents I

1. Linear algebra: $\sim$ 10-11 lectures
   1.1 Review: Vector operations in 2 and 3 dimensional spaces
   1.2 System of linear equations (LES)
      - Solution existence of a LES
      - LES and associated augmented matrices
      - Gauss and Gauss-Jordan elimination method
   1.3 Matrix Algebra
      - Terminologies
      - Operations with matrices
      - The inverse of a matrix
      - Partitioned matrix and matrix factorization (self-study)
   1.4 Determinants
   1.5 Vector spaces
   1.6 Linear transformations
   1.7 Coordinate systems and basic transition matrix

Contents II

2. General algebra: $\sim$ 9-10 lectures
   2.1 Propositional logic: Lehman et.al.'s (Chapter 3), or Rosen's (Chapter 1).
   2.2 Set theory review: Rosen's textbook
   2.3 Relations: Rosen's textbook (chapter 9)
      - Binary relations
      - Popular properties on relations: reflexivity, symmetricity, transitivity, anti-symmetricity.
      - Closures of a relation
   2.4 Group theory
      - Basic definitions
      - Generators for a group and cyclic groups
      - Cosets and Lagrange's theorem
      - Homomorphism and isomorphism
      - Quotient groups

References

- Linear algebra
  1. D. Lay, Linear Algebra and Its Applications, Pearson New International Edition, Pearson, 2014 (primary).
  2. Gilbert Strang, Linear Algebra and Its Applications, Fourth edition, Brooks/Cole Cengage Learning, 2006.
  3. Serge Lang, Introduction to Linear Algebra, Second edition, Springer.
- General algebra
  1. Kenneth Rosen, Discrete Mathematics and its applications, Mc Graw Hill education, 2013 (Logic, relations, group).
  2. Eric Lehman, F.T. Leighton, and A. R. Meyer, Mathematics for Computer Science, 2017 (Logic).
  3. Joseph A. Gallian, Contemporary Abstract Algebra, Cengage learning, 2017 (Groups)

Learning outcomes

1. Acquired with essential concepts, structures and methods of propositional algebra, general algebra and linear algebra. Particularly, well-acquired with basic algebraic structures necessary for the comprehension of formal structures in CS;
2. Have the ability to independently develop abstract concepts and to acquire basis techniques of algebra;
3. Acquired for analytical thinking, development of methodological expertise, handling abstract methods, structures and models.

Attendance and exam

1. To enable to pass the final exam, you are recommended to
   - attend the class at least 70% number of contact hours
   - do exercises/homework as much as possible
2. To get further point of views as well as to understand applications of the subject in the computer science field, please spend your time for reading textbooks.
3. Exam duration: 90 minutes with 7-12 questions. You are allowed to bring a two-sided A4 written with any contents. Pocket calculator is Not allowed.

Linear Algebra

Lecture 1: Vector operations in 2,3-dimensional spaces

Outline

- Geometric representation of vectors
- Coordinate representation of vectors
- Operations on vectors
- Inner/dotted product
- Length/norm of vectors
- Orthogonal vectors
- Angle between two vectors

Graphic representation of vectors

Definition (Vector)

A, B are points in the plane (or in the space). The vector $\vec{AB}$ is the directed segment from A to B.

Remark

Some variables, such as length, temperature, time are totally determined by just one number. These variables are so-called scalars. Other variables, such as speed, force need both a magnitude and a direction to be determined completely. These variables are vectors.

Examples: On the whiteboard

Definition (Equality of vectors)

Two vectors are equal if they have the same direction and the same length.

Notations

- Vectors are parallel: $\vec{AB} \uparrow\uparrow \vec{CD}$
- Vectors are antiparallel: $\vec{AB} \uparrow\downarrow \vec{CD}$
- Vectors: $\vec{a}, \vec{b}, \dots$
- Scalars: $\alpha, \beta, \gamma, \dots, \lambda, \mu, \dots$

Definition (Sum of the two vectors $\vec{a}$ and $\vec{b}$)

If we shift $\vec{b}$ in a parallel manner, such that its origin corresponds to the end point of $\vec{a}$, then $\vec{a}+\vec{b}$ starts at the origin of $\vec{a}$ and ends at the end of $\vec{b}$.

$$\vec{a}+\vec{b}=\vec{AB}+\vec{BC}=\vec{AC}$$

Theorem

1. *Commutative law:*
   $\vec{a}+\vec{b}=\vec{b}+\vec{a}$
2. *Associative law:*
   $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$

Proof of 1: On the whiteboard
Proof of 2: On the whiteboard
$\vec{a} + \vec{b} = \vec{AC}$
$(\vec{a} + \vec{b}) + \vec{c} = \vec{AC} + \vec{CD} = \vec{AD}$
$\vec{b} + \vec{c} = \vec{BD}$
$\vec{a} + (\vec{b} + \vec{c}) = \vec{AB} + \vec{BD} = \vec{AD} = (\vec{a} + \vec{b}) + \vec{c}$
Thus:
$(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

Definition (Special vectors)

1. Zero vector:
   $\vec{0} = \vec{AA}$
2. Opposite vector to $\vec{a} = \vec{AB}$:
   $-\vec{a} = \vec{BA}$

Remark
$\vec{a} + (\vec{-a}) = \vec{AB} + \vec{BA} = \vec{0}$

Definition
Let $\vec{a}, \vec{b}$ be vectors. Then

$$\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$$

Definition (Multiplication with a scalar)

Given a vector $\vec{a}$ and a scalar λ. Then, the multiplication $\vec{a}$ with a scalar λ is a vector with the following properties:

1. Length $(\lambda \cdot \vec{a}) = |\lambda| \cdot$ length $(\vec{a})$
2. For $\lambda > 0$ is $\lambda \cdot \vec{a} \uparrow\uparrow \vec{a}$
3. For $\lambda < 0$ is $\lambda \cdot \vec{a} \uparrow\downarrow \vec{a}$
4. For $\lambda = 0$ is $\lambda \cdot \vec{a} = \vec{0}$

Example: On the whiteboard.

Theorem

Let $\vec{a}, \vec{b}$ be vectors and $\lambda, \mu \in \mathbb{R}$. Then $\lambda \cdot \vec{a}$ has the following properties:

1. $(\lambda + \mu) \cdot \vec{a} = \lambda \cdot \vec{a} + \mu \cdot \vec{a}$
2. $(\lambda \cdot \mu) \cdot \vec{a} = \lambda \cdot (\mu \cdot \vec{a})$
3. $\lambda \cdot (\vec{a} + \vec{b}) = \lambda \cdot \vec{a} + \lambda \cdot \vec{b}$

Coordinate representation of vectors

Remark

Even if no direction is required to describe something, it can still be useful to summarize numbers in a vector as the next example will show.

Definition (Vector)

An n-dimensional vector $\vec{x}$ is given by

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

with $x_1 \in \mathbb{R}, \dots, x_n \in \mathbb{R}$.

Example: Coordinate representation of vectors in $\mathbb{R}^3$

To describe vectors, we consider a Cartesian coordinate system in the space through the origin $O = (0, 0, 0)$ and the unit points $E_1 = (1, 0, 0)$, $E_2 = (0, 1, 0)$ and $E_3 = (0, 0, 1)$ on the axes.
The vectors determined by the unit points are called unit vectors:

$$\begin{aligned} \vec{e_1} &= \vec{OE_1} \\ \vec{e_2} &= \vec{OE_2} \\ \vec{e_3} &= \vec{OE_3} \end{aligned}$$

Remark

The vector $\vec{a} = \vec{OA}$ to the point $A = (a_1, a_2, a_3)$ can we then express as a unique linear combination of the unit vectors:

$$\vec{a} = a_1 \cdot \vec{e_1} + a_2 \cdot \vec{e_2} + a_3 \cdot \vec{e_3}$$

Remark (Column notation)

$$\vec{a} = a_1 \cdot \vec{e_1} + a_2 \cdot \vec{e_2} + a_3 \cdot \vec{e_3} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

Definition (Arithmetic operations with vectors)

Let

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \text{ and } \vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

- **Addition and subtraction**

  We have

$$\vec{x} \pm \vec{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \pm \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} x_1 \pm y_1 \\ x_2 \pm y_2 \\ \vdots \\ x_n \pm y_n \end{pmatrix}$$

- Multiplication with a scalar
  For $\lambda \in \mathbb{R}$ we have

$$\lambda \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \lambda \cdot x_1 \\ \lambda \cdot x_2 \\ \vdots \\ \lambda \cdot x_n \end{pmatrix}$$

- Examples (3)

Remark

As we combine different vectors with each other, often linear combinations are created.

Definition (Linear combination)

Let $\vec{x}_1, \ldots, \vec{x}_m$ be $n-$dimensional vectors and $c_1 \in \mathbb{R}, \ldots, c_m \in \mathbb{R}$. Then

$$c_1 \cdot \vec{x}_1 + c_2 \cdot \vec{x}_2 + \ldots + c_m \cdot \vec{x}_m = \sum_{i=1}^{m} c_i \cdot \vec{x}_i$$

is a linear combination of the vectors $\vec{x}_1, \ldots, \vec{x}_m$.

Dot/Inner product

Definition

1. In 2-dimensional space, let $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$. We define their dot product of $\vec{a}$ and $\vec{b}$ to be

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1 b_1 + a_2 b_2$$

2. In 3-dimensional space, their dot or inner product is defined to be

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

3. Generally, in n-dimensional space, their dot or inner product is defined to be:

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Examples: On the whiteboard.

Remark

We can only compute the dot product of two vectors if they have the same dimensions. For example, it is not possible to compute the dot product as follows:

$$\vec{x} \cdot \vec{y} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

Theorem

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors and $\lambda \in \mathbb{R}$.

1. $(\lambda \cdot \vec{a}) \cdot \vec{b} = \lambda \cdot (\vec{a} \cdot \vec{b})$
2. $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
3. $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

- Examples (2)

Definition (Orthogonal vectors)

Two vectors $\vec{a}$ and $\vec{b}$ are called orthogonal if $\vec{a} \cdot \vec{b} = 0$.

Examples: On the whiteboard.

The norm or magnitude of a vector

Definition

Let $\vec{a}$ be a vector in $\mathbb{R}^n$. We define the norm or magnitude of $\vec{a}$, and denote by $|\vec{a}|$, the number

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}.$$

Examples:

1. For
$$\vec{a} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix},$$
the norm of $\vec{a}$ is
$$|\vec{a}| = \sqrt{\lambda_1^2 + \lambda_2^2}$$

2. For $\vec{a} = (\lambda_1, \lambda_2, \lambda_3),$ the norm of $\vec{a}$ is
$$|\vec{a}| = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$$

The norm and geometric length

Notes: When $n = 2$ and $n = 3$, then the definition of norm is compatible with the geometric length by using Pythagoras theorem.
Proof 1: Applying the Theorem of Pythagoras

$$a^2 = \lambda_1^2 + \lambda_2^2$$

Proof 2:

$$|\vec{a}|^2 = |\vec{OQ}|^2 + |\vec{QP}|^2 = |\vec{OR}|^2 + |\vec{RQ}|^2 + |\vec{QP}|^2 =$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2$$

Definition (Distance between two points in the space)

The distance between two points, $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, in the space is given by:

$$d = |\vec{A} - \vec{B}| = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}$$

- Example

General Pythagoras theorem

Theorem

Let $\vec{a}$ *and* $\vec{b}$ *be vectors in* $\mathbb{R}^n$*. Then, vectors* $\vec{a}$ *and* $\vec{b}$ *are perpendicular if and only if*

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2.$$

Angle between two vectors

Remark

How can we calculate the angle φ between two vectors $\vec{a}$ and $\vec{b}$ in term of their coordinates?

- Example

Theorem

Let $\vec{a}$ and $\vec{b}$ be vectors and φ the angle between these vectors. Then,

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos(\varphi)$$

- Example

Remark

Let $\vec{a}, \vec{b}$ be vectors and φ the angle between them. Then

$$\cos(\varphi) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$$

Linear Algebra

Lectures 2: Linear equation systems

Contents

- Linear equation systems (LES)
- Matrix representation of a LES
- Row-echelon form and Gaussian elimination
- Reduced row-echelon form and Gauss-Jordan elimination

Linear equation systems

In general, if a linear relation exists between the inputs and outputs of a system, we call it a linear system.

Remark (Applications of linear equation systems)

- Rotations in space
- Demand calculations for stock holding
- Modeling of customer streams
- Flow network
- Electrical circuits
- Quantum mechanics

Definition (Linear equation)

A linear equation in n variables:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where

$$\begin{aligned} a_1, \ldots, a_n, b &: \text{real numbers} \\ x_1, \ldots, x_n &: \text{unknown or variables} \\ a_1, \ldots, a_n &: \text{are called coefficients of the equation.} \\ a_1 &: \text{leading coefficient or pivot if non-zero} \end{aligned}$$

Examples

- $2x + y - 4z = 0$
- $2x + y - 4z = 3$

Definition (Linear equation system (LES))

A system of m linear equations is called a linear equation system:

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}\right. \tag{1}$$

where

$a_{ij}, b_i,\ i = 1, \dots, m,\ j = 1, \dots, n$: real numbers;

$x_1, \dots, x_n$ unknowns or variables.

a_{ij} coefficients.

Examples

1. $\begin{cases} x + y &= 3 \\ x - y &= -1 \end{cases}$
2. $\begin{cases} x + y &= 3 \\ 2x + 2y &= 6 \end{cases}$
3. $\begin{cases} x + y &= 3 \\ x + y &= 1 \end{cases}$

Solution set of a LES

- A solution of the linear equation system (1) in n variables is a set of numbers $s_1, s_2, \ldots, s_n$ such that for all $i = 1, 2, \ldots, n$, then
$$a_{i1}s_1 + a_{i2}s_2 + \cdots + a_{in}s_n = b_i.$$
- Solution set is the set of all solutions of (1).

Consistency of a LES

A system of m linear equations in n variables:

$$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ & & & & \vdots & & & & \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{cases}$$

- consistent if it has at least one solution;
- inconsistent if it has no solution;
- If $b_i = 0$ for all $i = 1, \dots, m$, then the LES is homogeneous, otherwise inhomogeneous.

Notes: Every system of linear equations has either

1. exactly one solution,
2. infinitely many solution, or
3. no solution.

Equivalent systems

- Two systems of linear equations are called equivalent if they have precisely the same solution set.
- **Notes:**
  Eliminations: We use following operations on a system of linear equations to produce an equivalent system:
  1. Interchange two equations
  2. Multiply an equation by a nonzero constant
  3. Add a multiple of an equation to another equation.

Example

Solve a system of linear equations

$$\begin{cases} x - 2y + 3z &= 9 \\ -x + 3y &= -4 \\ 2x - 5y + 5z &= 17 \end{cases} \tag{2}$$

Solution: Details on the whiteboard.

Coefficient matrix of a LES I

- The coefficient matrix of the LES 1 is defined by

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \dots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix},$$

- Notes:
  1. Every entry a_{ij} in a matrix is a real number;
  2. A matrix with m rows and n columns is said to be of size $m \times n$;
  3. If $m = n$, then the matrix is called square of order n;
  4. For a square matrix, the entries $a_{11}, a_{22}, \dots a_{nn}$ are called the main diagonal entries.

Coefficient matrix of a LES II

- the right-hand side with the vector

$$\vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$$

- Matrix form of the LES 1: $Ax = b$, where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ldots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Augmented matrix

- Augmented matrix for the LES (1)

$$\left[A|\vec{b}\right] = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & \ldots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

- The n numbers $x_1, \ldots, x_n$ of the solution of the LES is called solution vector:

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

- Example: The augmented matrix of the LES (2) is

$$[A|b] = \left(\begin{array}{cccc|c} 1 & -2 & 3 & 9 & 9 \\ -1 & 3 & 0 & -4 & -4 \\ 2 & -5 & 5 & 17 & 12 \end{array}\right).$$

Gaussian elimination: Row echelon form

- It is especially easy to solve a LES, if the coefficient matrix is given in row echelon form.
- A matrix is in row echelon form if the following three conditions are fulfilled:
  1. All nonzero rows are above any rows of all zeros.
  2. The leading entry (the pivot or the leftmost non-zero entry) in a nonzero row is 1.
  3. Each leading 1 of a row is in a column to the right of the leading 1 of the row above it.
- Note: All entries in a column below a leading entry are zeros.

Examples I

1. By elimination, the augmented matrix of the LES (2) could be transformed to the row-echelon form:

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

2. The augmented matrix

$$\left(\begin{array}{ccccc|c} 0 & 2 & -3 & 4 & 1 & 7 \\ 0 & 0 & 0 & 5 & 2 & 4 \\ 0 & 0 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

represents the system of linear equations

Examples II

$$\begin{cases} 2y - 3z + 4w + t & = 7 \\ 5w + 2t & = -4 \\ -3t & = 1 \end{cases}$$

Back-substitution for solving a LES with a coefficient matrix in row echelon form

1. We solve the equations for the leading variables (i.e. the variables corresponding to leading entries, also called basic variables).
2. Starting with the last (= bottom) we substitute the calculated variables into the equations above.
3. Free variables (i.e. variables not corresponding to leading entries) are assigned arbitrary values.

Remark

If the rows without leading ones are contradictory, then the solution is the empty set. Hence, the LES is inconsistent.

- Examples (3)

Remark

An arbitrary LES is usually not in row echelon form. We must then first transform the system to the row echelon form. The transformation must be done in such way that the solution set remains the same as for the original system.

Remark

Two LES with the same set of solutions are called equivalent.

Remark (Elementary row operations)

- Interchange: Interchange two rows: $r_{ij} : R_i \leftrightarrow R_j$
- Scaling: Multiply all entries in a row by a nonzero constant: $r_i^{(k)} : kR_i \rightarrow R_i$
- Replacement: Replace one row by the sum of itself and a multiple of another row: $r_{ij}^{(k)} : kR_i + R_j \rightarrow R_j$.

Theorem

If the augmented matrix $\left(A^*|\vec{b}^*\right)$ *is transformed from* $\left(A|\vec{b}\right)$ *by the use of a finite number of elementary row operations, then the solution sets to these two systems are equivalent. We say that the systems are row equivalent.*

Remark (Gauss elimination)

A given augmented matrix is used as a starting point: $\left(A|\vec{b}\right)$.
Using elementary row operations, this augmented matrix is transformed into an equivalent augmented matrix: $\left(A^*|\vec{b}^*\right)$ with A^* in row echelon form.
We proceed as follows:

1. We look for the leftmost column with at least one nonzero element.
2. In this column, we look for the element with the largest absolute value. We interchange this row and the first row. (Pivotsearch to obtain numerical stability).
3. If the uppermost number $a \neq 0$ is, then we multiply the first row by $\frac{1}{a}$ to create a leading 1.
4. We add a multiple of the first row two the other rows, to create zeros below the leading 1.
5. These steps are repeated for a submatrix. The submatrix is obtained, by covering the first row (or the actual first row and all rows above it). The steps are repeated until the whole matrix is in row echelon form.

Examples:

1. $\begin{bmatrix} 0 & 0 & -2 & 0 & 8 & 12 \\ 2 & 8 & -6 & 4 & 12 & 28 \\ 2 & 4 & -5 & 6 & -5 & 4 \end{bmatrix}$

2. $\begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$

Gauss-Jordan elimination: The produce for reducing a matrix to a reduced row-echelon form which satisfies the following:

1. it is a row-echelon form;
2. for each non-zero row, the first non-zero entry from the left is equal to 1 and it is called pivot or leading 1.
3. Every column that has a pivot/leading 1 has zeros in every position above and below its pivot.

Example: On the whiteboard.

Definition (Rank)

Given a matrix A. The number of the nonzero rows of A^*, is called the rank of matrix A and denoted by $r(A)$ or simply r in case A is shown.

Remark (Solutions to a LES)

Using the augmented matrix $\left(A^* | \vec{b}^*\right)$ in row echelon form, we can identify the kind of solutions that the corresponding LES has (no solution, exactly one solution, infinitely many solutions).
Back-substitution can be used to determine the set of solutions.

Theorem (The existence of solutions)

The LES with the augmented matrix $\left(A|\vec{b}\right)$ *is consistent if after applying the Gaussian elimination algorithm we arrive at the augmented matrix*
$\left(A^*|\vec{b}^*\right)$ *with:*

$$b^*_{r+1} = \ldots = b^*_m = 0$$

where A *is an* $m \times n$ *matrix.*
Then we have:

- $r = n \Rightarrow$ *the LES has exactly one solution*
- $r < n \Rightarrow$ *the LES has infinitely many solutions (and* $n - r$ *of the* x_j*:s can be taken arbitrary)*

- Examples (5)

Remark (Homogeneous linear equation systems)

A LES with augmented matrix $\left(A|\vec{0}\right)$ is called a homogeneous LES. The right-hand side of a homogeneous LES is always the null vector and has at least the trivial solution $\vec{x} = \vec{0}$.

Theorem

For a homogeneous LES, we have:

- $r = n \Leftrightarrow$ *the LES has only the trivial solution* $\vec{0}$
- $r < n \Leftrightarrow$ *the LES has infinitely many solutions and* $n - r$ *of the* x_j*:s can be taken arbitrary.*

Remark

A homogeneous LES with $n > m$ (with more unknowns than equations) has infinitely many solutions.

Remark

For a homogeneous LES is it sufficient to transform the coefficient matrix A with elementary row operations, since the null vector remains unchanged.

Definition (Square linear equation systems)

A LES with $m = n$ (Number of equations $=$ Number of unknowns) is called a square LES.

Theorem

A square LES has a unique solution if and only if $r = n$.

Remark (Solution of a square LES with a unique solution by Gauss-Jordan elimination)

We can obtain the solution in the following way from the augmented matrix $\left(A^* | \vec{b}^*\right)$:

1. We add a multiple of the last row to the other rows, to create zeros above the leading 1.
2. We repeat the first step for the submatrix obtained by covering the last row and repeat this until the first row.

The result is an augmented matrix of the form

$$\left(A^{**}|\vec{b}^{**}\right) = \left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1^{**} \\ 0 & 1 & \cdots & 0 & b_2^{**} \\ \vdots & & & \vdots & \vdots \\ 0 & \cdots & & 1 & b_n^{**} \end{array}\right)$$

i.e. we can immediately see the solution:

$$\vec{x} = \begin{pmatrix} b_1^{**} \\ b_2^{**} \\ \vdots \\ b_n^{**} \end{pmatrix}$$

Remark

The Gaussian algorithm together with back-substitution is also called Gauss-Jordan elimination.

- Example (2)

Linear Algebra

Lectures 3+4: Matrices

Outline

- Introduction
- Matrix operations
- Special matrices
- The inverse of a matrix
- Matrix and linear equation system

Introduction

1. Linear transformations like transition, reflection, rotation, can be considered as matrix actions.
2. Our ability to analyze and solve equations will be greatly enhanced when we can perform algebraic operations with matrices.
3. Transformations on equations of a linear system can be written as transformations on rows of a matrix.
4. Matrix algebra provides tools for manipulating matrix equations and creating various useful formulas in ways similar to doing ordinary algebra with real numbers.
5. The inverse of a matrix, if it exists, allows us to treat the matrix as a number which helps to find out the solution of the linear system explicitly.
6. Matrix algebra can be applied in many fields like economics, computer graphics, image processing, etc.

Introduction

Remark

A matrix is a rectangular array, filled with numbers.

- Example

Definition (Matrix)

A rectangular array A of $m \cdot n$ numbers in m rows and n columns

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called a matrix of size $m \times n$.
m is the number of rows of A;
n is the number of columns of A;
a_{ij}, for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, are called entries of A.
$M_{m \times n}$: the set of all matrices of size $m \times n$.

- Example:

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 3 \end{pmatrix} \in M_{2\times 3}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & -1 \\ -1 & 3 \end{pmatrix} \in M_{3\times 2}$$

Remark

A vector is only a special case of a matrix.

Definition

A matrix consisting of just one column is called a column vector:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

Definition

A matrix consisting of just one row is called a row vector:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

Definition (Equality of matrices)

Two matrices A and B are equal, if they have the same size $m \times n$ and if all elements are equal:

$$a_{ij} = b_{ij} \text{ for all } \; i = 1, \ldots, m, \; j = 1, \ldots, n$$

Matrix Operations: Addition and Subtraction of Matrices

Definition (Addition and Subtraction of Matrices)

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

and

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

be $m \times n$ matrices.

Then

$$A \pm B = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & \cdots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & \cdots & a_{2n} + b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & \cdots & a_{mn} + b_{mn} \end{pmatrix},$$

i.e. the matrices are added and subtracted element-wise.

- Example: On the whiteboard

Matrix Operations: Multiplication with a scalar

- Example: On the whiteboard

Definition (Multiplication with a Scalar)

Matrices are element-wise multiplied by a scalar. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

be an $m \times n$ matrix and let $\lambda \in \mathbb{R}$.

Then

$$\lambda \cdot A = \begin{pmatrix} \lambda \cdot a_{11} & \lambda \cdot a_{12} & \cdots & \lambda \cdot a_{1n} \\ \lambda \cdot a_{21} & \lambda \cdot a_{22} & \cdots & \lambda \cdot a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \lambda \cdot a_{m1} & \lambda \cdot a_{m2} & \cdots & \lambda \cdot a_{mn} \end{pmatrix}$$

- Example: On the whiteboard

Theorem (Calculation Rules)

Let A, B, C be $m \times n$ matrices and $\lambda, \mu \in \mathbb{R}$

1. $A + B = B + A$ *(Commutative)*
2. $(A + B) + C = A + (B + C)$ *(Associative)*
3. $\lambda \cdot (A + B) = \lambda \cdot A + \lambda \cdot B$ *(Distributive)*
4. $(\lambda + \mu) \cdot A = \lambda \cdot A + \mu \cdot A$ *(Distributive)*

- Example: On the whiteboard

Matrix Operations: Matrix Multiplication

- Example

Definition (Matrix multiplication)

For an $m \times n$ matrix A and an $n \times p$ matrix B. The matrix $m \times p$ matrix C with the elements

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

called the matrix product of A and B, denoted by $C = AB$.

Remark

- The matrix product AB is only defined if the number of columns of A is equal to the number of rows of B.
- Write the sizes of A and B beside each other:

$$m \times n \text{ and } q \times p$$

  The matrix product is only defined if the inner numbers are equal, i.e. $n = q$. Then the outer numbers tell the size of the matrix product: $m \times p$.

Remark

To determine the element in the i-th row and the j-th column of AB, we multiply the elements of the i-th row of A element-wise with the elements of the j-th column of B and add the generated products:

$$\begin{pmatrix} & \vdots & \\ a_{i1} & \cdots & a_{in} \\ & \vdots & \end{pmatrix} \cdot \begin{pmatrix} & b_{1j} & \\ \ldots & \vdots & \ldots \\ & b_{nj} & \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \cdots & c_{ij} & \cdots \\ & \vdots & \end{pmatrix}$$

- Examples (2): On the whiteboard.

Remark

The product of an $m \times 1$ column vector with a $1 \times m$ row vector is an $m \times m$ matrix:

$$\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} \cdot \begin{pmatrix} b_1 & \cdots & b_m \end{pmatrix} = \begin{pmatrix} a_1 b_1 & \cdots & a_1 b_m \\ \vdots & & \vdots \\ a_m b_1 & \cdots & a_m b_m \end{pmatrix}$$

Remark

The product of a $1 \times m$ row vector with an $m \times 1$ column vector is a 1×1 matrix, i.e. a scalar (only a number):

$$\begin{pmatrix} b_1 & \cdots & b_m \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix} = a_1 b_1 + \cdots + a_m b_m$$

This is a dot product written with matrix notation!

- Example

Remark (Missing Calculation Rules)

Many calculation rules for scalars hold also for matrices. But for the multiplication of matrices the commutative law does not hold. In general, is the following NOT fulfilled:

- Commutative law: $AB = BA$
- Zero law for products: $AB = 0 \Rightarrow A = 0$ or $B = 0$.
- Cancellation law: $AC = BC$ and $C \neq O \Rightarrow A = B$

Remark (Missing commutative law)

There are some situations, where the commutative law is missing for matrix multiplications:

- AB is defined, but BA is not defined.
- AB and BA are of different sizes.
- It is actually also possible that $AB \neq BA$ is not fulfilled, even if AB and BA have the same size:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}$$

but

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Matrix Operations: Transposing Matrices

Definition (Transposed matrix)

The $n \times m$ matrix that we obtain if we write the rows of the $m \times n$ matrix A as columns, is called the transposed matrix of A,

$$A^T = \begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \cdots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{pmatrix}$$

- Example: On the white board.

Properties of transposes

Theorem

Let A and B be matrices. Then

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(cA)^T = cA^T$
4. $(AB)^T = B^T A^T$

Some Special Matrices

Definition (Zero matrix)

$a_{ij} = 0$ for all i, j, i.e.

- Example:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Properties of zero matrices

Theorem

Let A be an $m \times n$ matrix and let c be a scalar. Then

1. $A + 0_{m \times n} = A$
2. $A + (-A) = 0_{m \times n}$
3. *if* $cA = 0_{m \times n}$, *then either* $c = 0$ *or* $A = 0_{m \times n}$

Definition (Square matrix)

An $n \times n$ matrix and the elements $a_{ii}, \ i = 1, \dots, n$ constitute the main diagonal.

- Example

Definition (Diagonal matrix)

A square matrix

$$d_{ij} = 0 \text{ for } i \neq j,$$

i.e.

$$D = \begin{pmatrix} d_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & d_{nn} \end{pmatrix}$$

- Example

Definition (Identity matrix)

A diagonal matrix with main diagonal elements $= 1$

$$d_{ij} = 0 \text{ for } i \neq j,$$

and

$$d_{ij} = 1 \text{ for } i = j,$$

is called the identity matrix and denoted by $I_{n \times n}$ or I in case n is shown.

- Example: $I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $I_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,

Definition (Upper triangular matrix)

A square matrix with

$$d_{ij} = 0 \text{ for } i > j,$$

i.e.

$$D = \begin{pmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & & \vdots \\ 0 & \cdots & d_{nn} \end{pmatrix}$$

- Example: On the whiteboard

Definition (Lower triangular matrix)

A square matrix with

$$d_{ij} = 0 \text{ for } i < j,$$

i.e.

$$D = \begin{pmatrix} d_{11} & \cdots & 0 \\ \vdots & & \vdots \\ d_{n1} & \cdots & d_{nn} \end{pmatrix}$$

- Example: On the whiteboard

Definition (Symmetric matrix)

A square matrix A with

$$a_{ij} = a_{ji} \text{ for all } i, j,$$

is called symmetric. In other words, A is symmetric if and only if $A = A^T$.

- Examples: Transportation matrices, social network matrices, image processing matrices...
- $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & -2 \\ -1 & -2 & 3 \end{pmatrix}$ is symmetric and $\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & \text{-2} \\ -1 & 0 & 3 \end{pmatrix}$ is not symmetric

The inverse of a matrix

Definition (Regular matrices)

An $n \times n$ matrix A is called regular if $rank(A) = n$ and singular if $rank(A) < n$.

Remark

$$A \text{ regular } \Leftrightarrow$$

$$rank(A) = n \Leftrightarrow$$

$$\text{Every LES } A\vec{x} = \vec{b} \text{ has exactly one solution} \Leftrightarrow$$

$$\text{The homogeneous LES } A\vec{x} = \vec{0} \text{ has only the trivial solution}$$

Theorem

If the $n \times n$ matrix A is regular, then a uniquely determined matrix $n \times n$ matrix X with $AX = I$ exists.

Definition (Invertible matrix)

An $n \times n$ matrix A is called invertible if an $n \times n$ matrix X exists with $AX = XA = I$, where I is the identity matrix of size $n \times n$.

Remark

The matrix X with $AX = I$ is called the inverse to A, and is written $X = A^{-1}$.

Remark

1. A regular $\Leftrightarrow$ A invertible.
2. If A is invertible, then is X uniquely determined.
3. For the inverse A^{-1}, the following holds: A^{-1} is regular and

$$AA^{-1} = A^{-1}A = I$$

   with I the identity matrix.

Power of a square matrix

Given a square matrix $A \in M_{n \times n}$. We define

1. $A^0 = I$
2. $A^k = \underbrace{AA \cdots A}_{k \text{ times}}$ $(k \in \mathbb{N})$.
3. $A^r.A^s = A^{r+s}$
4. $(A^r)^s = A^{rs}$
5. If $D = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}$, then $D^k = \begin{pmatrix} d_1^k & 0 & \dots & 0 \\ 0 & d_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n^k \end{pmatrix}$

Theorem (Properties of inverse matrices)

Let A be an invertible matrix, let k be a positive integer, and c be a scalar not equal to zero. Then

1. A^{-1} *is invertible and* $(A^{-1})^{-1} = A$
2. A^k *is invertible and* $(A^k)^{-1} = (A^{-1})^k$
3. cA *is invertible and* $(cA)^{-1} = \frac{1}{c}A^{-1}$
4. A^T *is invertible and* $(A^T)^{-1} = (A^{-1})^T$
5. *If* A *and* B *are invertible, then* AB *is invertible and* $(AB)^{-1} = B^{-1}A^{-1}$.

Computation of the inverse

1. Augment A to the right with the identity matrix
2. Perform a Gauss-Jordan elimination

Then we obtain the inverse matrix A^{-1} on the right-hand side:

$$(A|I) = \left(I|A^{-1}\right)$$

- Notes:
  1. If A cannot be row reduced to I, then A is singular.
  2. At first is it usually not obvious if the matrix A is invertible at all. If this is not the case, then we would arrive at a zero row at the left-hand side of the augmented matrix. We would then conclude that A is singular and stop the process.

Examples

- Find the inverse of the following matrix $\begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -6 & 2 & 3 \end{pmatrix}$
- Solution: On the whiteboard.

Notes

From the definitions for matrix operations, a linear system of m equations and n variables $x_1, \ldots, x_n$ can be written as matrix operation:

$$Ax = b,$$

where A is the coefficient matrix of the system, x (resp. b) represents column vector of variables (the right hand side) of the system.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Theorem

Let A be an invertible $n \times n$ matrix.
Then the LES $A\vec{x} = \vec{b}$ has the unique solution

$$\vec{x} = A^{-1}\vec{b}$$

- Example

Remark

We can use this theorem if we have to solve many LES with the same coefficient matrix A but different right-hand sides $\vec{b}$.

Linear Algebra

Lectures 5+6: Determinants and applications

Outlines

- Determinant of a matrix
- Properties of determinants
- Gaussian elimination for calculating determinants
- Applications in inverse finding and solution solving

Determinants

Determinant is a number assigned to each square matrix. A single number, determinant, can tell only so much about a matrix. Still, it is amazing how much this number can do. We can use determinant for

1. the invertibility existence of a matrix;
2. the solution existence of a linear system;
3. finding the inverse of a matrix using cofactors;
4. solving explicitly a linear system by Cramer's rule;
5. measuring the dependence of $A^{-1}b$ on each element of b. If one parameter is changed in an experiment, or one observation is corrected, the "influence coefficient" in A^{-1} is a ratio of determinants.
6. measuring the amount by which a linear transformation changes the area of a figure. Finding the volume of a box in n-dimensional space.

Determinant of a 2×2 matrix

The number

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

is called the determinant of the 2×2 matrix

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Determinant of a 3×3 matrix

The number

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

$$= a_{11}.M_{11} - a_{12}.M_{12} + a_{13}M_{13}$$

is called the determinant of the 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where M_{ij} is **the determinant of the matrix obtained from A by removing its row i and column j**, and called minor of the entry a_{ij}.

Remarks

- For the computation of the determinant of a 3×3 matrix, we can use Sarrus' rule. Accordingly, its determinant is equal to the difference of the down-diagonal sum and up-diagonal sum of the following matrix:

$$\begin{array}{ccc|cc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

- We could also compute the determinant by expanding one row or one column and then compute the determinant for the remaining 2×2 matrices. This can also generalized and then applied to the determinant for an $n \times n$ matrix.

Cofactor expansion

In general, the determinant of a square matrix of size $n \times n$ is a number defined inductively as follow:

1. Pick any one row (column) of A;
2. For each entry in the row chosen, find its co-factor;
3. Multiply each entry in the row (column) chosen by its co-factor and take the sum that results the determinant of A.

Determinant of an $n \times n$ matrix

- Minor M_{ij} of the entry a_{ij}: the determinant of the $(n-1) \times (n-1)$ matrix obtained from A by removing its row i and column j.
- Cofactor of a_{ij}
$$C_{ij} = (-1)^{i+j} M_{ij}.$$
- The determinant of A is given by
  1. Cofactor expansion along the row i:
  $$\det A = |A| = \sum_{j=1}^{n} a_{ij} C_{ij} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}.$$
  2. Cofactor expansion along the column j:
  $$\det A = |A| = \sum_{i=1}^{n} a_{ij} C_{ij} = a_{1j} C_{i1} + a_{2j} C_{i2} + \cdots + a_{in} C_{nj}.$$

Examples

- **Notes:** The row (or column) containing the most zeros often if the best choice for expansion by cofactors.
- **Examples:** If A is an triangular matrix, then its determinant is the product of the entries on the main diagonal, where
  1. upper triangular matrix: All the entries below the main diagonal are zeros;
  2. lower triangular matrix: All the entries above the main diagonal are zeros;
  3. diagonal matrix: All the entries above and below the main diagonal are zeros.
- **Example:** Find the determinant

$$\begin{vmatrix} 2 & 1 & 0 \\ 1 & 1 & 4 \\ -3 & 2 & 5 \end{vmatrix}$$

Solution: On the white board.

Numerical note

By today's standard, a 25×25 matrix is small. Yet it would be impossible to calculate a 25×25 determinant by cofactor expansion. In general, a cofactor expansion requires over $n!$ multiplications, and $25!$ is approximately 1.5×10^{25}, it would spend approximately $500,000$ years to compute a 25×25 determinant by this method.

Basic properties

Theorem

- *If A contains a zero row or a zero column, then we have $det(A) = 0$.*
- $\det(A^T) = \det(A)$
- *If A is a triangular matrix, then we have:*

$$det(A) = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nm}$$

- $\det(I) = 1$, *where I is the identity matrix.*
- $\det(AB) = \det(A)\det(B)$. *Consequently,* $\det(A^{-1}) = \frac{1}{\det(A)}$.

Remark: In general, $\det(A + B) \neq \det(A) + \det(B)$.

Row operation rules for determinants

Theorem (Determinants and elementary row operations)

Let A be a square matrix

1. *If a multiple of one row (column) of A is added to another row (column) to produce a matrix B, then $\det B = \det A$.*
2. *If two rows of A are interchanged to produce B, then $\det B = -\det A$*
3. *If one row of A is multiplied by k to produce B, then $\det B = k \det A$.*

Proof.

On the white board. □

Remark: Most computer programs that compute $\det A$ for a general matrix A use the elementary row operations to reduce A to triangular matrix.

Examples

Compute $\det(A)$, where $A = \begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix}$

Solution: On the whiteboard!

Zero determinants

If A is a square matrix and any of the following conditions is true, then $\det(A) = 0$.

1. An entire row (column) consists of zeros;
2. Two rows (columns) are equal;
3. One row (resp. column) is a multiple of another row (resp. column).
4. One row (resp. column) is a linear combination of other rows (resp. column).

Examples

1. $\begin{vmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{vmatrix} = 0$

2. $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{vmatrix} = 0$

3. $\begin{vmatrix} a_1 & b_1 & a_1 + b_1 \\ a_2 & b_2 & a_2 + b_2 \\ a_3 & b_3 & a_3 + b_3 \end{vmatrix} = 0$

Theorem

For a square matrix A we have:

$$A \text{ is regular } \Leftrightarrow$$

$$A \text{ is invertible } \Leftrightarrow$$

$$det(A) \neq 0$$

Remark

If A invertible, then

$$det(A^{-1}) = \frac{1}{det(A)}$$

Theorem (Cramer's rule)

If A is an invertible square matrix (i.e. $det(A) \neq 0$), then the LES $A\vec{x} = \vec{b}$ has the unique solution

$$\vec{x} = \frac{1}{det(A)} \begin{pmatrix} D_1 \\ \vdots \\ D_n \end{pmatrix}$$

Here D_k is the determinant we would obtain if in $det(A)$ the k-th column is replaced by $\vec{b}$.

Proof.

On the whiteboard. $\square$

Remarks:

- Using determinants we can give a very nice and explicit formula for the solution to a linear system with n variables and n equations in case it has unique solution.
- Cramer's rule is needed in a variety of theoretical calculations. However, the formula is inefficient for hand calculations, except for 2×2 or perhaps 3×3 matrices because of the determinant computing complexity.
- In general, for solving a linear system, Gaussian and Gaussian-Jordan eliminations are used more often.

Examples

Determine the value of s for which the system has a unique solution, and use Cramer's rule to describe the solution

$$\begin{cases} 3cx - 2y = 4 \\ -6x + cy = 1 \end{cases}$$

Solution:

- View the system as $Ax = b$. Then

$$A = \begin{bmatrix} 3c & -2 \\ -6 & c \end{bmatrix}, \quad D_1 = \begin{bmatrix} 4 & -2 \\ 1 & c \end{bmatrix}, \quad D_2 = \begin{bmatrix} 3c & 4 \\ -6 & 1 \end{bmatrix},$$

- The system has unique solution if and only if $\det(A) \neq 0$:

$$\det(A) = 3c^2 - 12 = 3(c-2)(c+2) \neq 0 \Leftrightarrow c \neq \pm 2.$$

- When $c \neq \pm 2$, the solution to system is

$$x = \frac{\det(D_1)}{\det A} = \frac{4c+2}{3(c-2)(c+2)}.$$

$$y = \frac{\det(D_2)}{\det A} = \frac{c+8}{(c-2)(c+2)}.$$

Inverse matrix finding: Case of 2×2 matrix

Recall:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

A formula for the inverse matrix

Theorem

Let A be an invertible matrix and let C be cofactor matrix of A, i.e. C_{ij} are cofactors of entries a_{ij}. Then,

$$A^{-1} = \frac{1}{\det A} C^T.$$

Remark: The matrix C^T is called the adjoint matrix of A, denoted by $adj(A)$.

Proof.

On the white board. $\square$

Examples

Using cofactors for finding the inverse of

1. $$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

2. $$A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 1 & 2 \\ 1 & -2 & 1 \end{pmatrix}.$$

   Solution: On the white board.

Linear Algebra

Lectures 7+8+9+10+11: Vector spaces

Introduction

- Actually, a study of vector spaces is not much different from a study of $\mathbb{R}^n$ itself. We can use our geometric experience with $\mathbb{R}^2$, $\mathbb{R}^3$ to visualize many general concepts.
- We can use vector space terminology to tie together important facts about rectangular matrices like rank concept.
- In applications of linear algebra, subspaces of $\mathbb{R}^n$ usually arise in one of two ways: as the set of solutions of a homogeneous linear system, or as the set of all linear combinations of certain specified vectors. These lead to considering the null and column spaces of a matrix/linear transformation.

Outline

- Vector Spaces
- Subspaces of Vector Spaces
- Spanning Sets and Linear Independence
- Basis and Dimension
- Rank of a Matrix and Systems of Linear Equations
- Four fundamental subspaces
- Linear transformations
- Coordinate systems

Vector space I

Let V be a set on which two operations (**vector addition:** $+ : V \times V \to V$ and **scalar multiplication:** $\cdot : \mathbb{R} \times V \to V$) are defined. If the following axioms are satisfied for every u, v, and w in V and every scalar (real number) λ and μ, then V is called a *vector space*.

Vector space II

1. $u + v \in V$ (closed with the addition)
2. $u + v = v + u$ (commutative addition)
3. $u + v + w = u + (v + w)$ (associative addition)
4. V has a zero vector 0 s.t. $u + 0 = u$ for all $u \in V$
5. For each u in V, there is an opposite vector in V, denoted by $-u$, s.t. $u + (-u) = 0$
6. $\lambda u \in V$ (closed with scalar multiplication)
7. $\lambda(u + v) = \lambda u + \lambda v$
8. $(\lambda + \mu)u = \lambda u + \mu u$
9. $\lambda \mu u = \lambda(\mu u)$
10. $1.u = u$.

Examples I

1. n-tuple space: $\mathbb{R}^n$, with the vector addition and the multiplication with scalar.
   - Vector addition

     $$(u_1, u_2, \ldots, u_n) + (v_1, v_2, \ldots, v_n) = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).$$

   - Scalar multiplication

     $$k(u_1, u_2, \ldots, u_n) = (ku_1, ku_2, \ldots, ku_n).$$

2. Matrix space: $V = M_{m \times n}$ with the matrix addition and scalar multiplication. Example: $m = n = 2$
   - Matrix addition

     $$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} u_{11} + v_{11} & u_{12} + v_{12} \\ u_{21} + v_{21} & u_{22} + v_{22} \end{pmatrix}$$

   - Scalar multiplication

     $$k \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{pmatrix}$$

Examples II

3. The set of all real polynomials of degree not exceeding n: $V = P_n(x)$ together with the polynomial addition and polynomial scalar multiplication forms a vector space. (verify this please!)
4. The set of all solutions of a homogeneous linear equation system together with vector addition and scalar multiplication forms a vector space. For instance, the solutions of the equation $x + 2y - 4z = 0$. (verify this please!)
5. The set of all continuous functions on a given domain with the scalar multiplication and function addition.

Notes: To show that a set is not a vector space, you need only find one axiom that is not satisfied.

Examples:

- The set of all integers is not a vector space since let $\frac{1}{2} \in \mathbb{R}$ and $1 \in V$ then $(\frac{1}{2})(1) = \frac{1}{2} \notin V$.
- The set of all second-degree polynomials is not a vector space, since let $p(x) = x^2 - 2x - 1$ and $q(x) = -x^2 - 1$, then $p(x) + q(x) = -2x - 2$ which is not a second-degree polynomial.
- The set of all solutions of a non-homogeneous linear equation system (verify this please).

Subspaces

Definition

V: vector space

$W \subseteq V$ and $W \neq \emptyset$.

W is called a subspace of V if W together with the addition and the multiplication with a scalar inherited from V is a vector space.

Notes:

1. Two trivial subspaces of V: Zero vector space $\{\vec{0}\}$, and V.
2. Test for a subspace: a non-empty set W is a subspace of V if and only if it is closed with the addition and the multiplication with a scalar, *i.e.*,
   - If u and v are in W, then $u + v$ is in W
   - If u is in W and λ is any scalar, then λu is in W.

Examples

- A set of points on a line through the origin in the plane is a subspace of $\mathbb{R}^2$.
- Let W be the set of all 2×2 symmetric matrices. Then W is a subspace of $M_{2\times 2}$.
- The set of singular matrices of size 2×2 is not a subspace of $M_{2\times 2}$.
- The set of invertible matrices of size 2×2 is not a subspace of $M_{2\times 2}$.
- The set of solutions of a homogeneous system with n variables is a subspace of $\mathbb{R}^n$.

Lemma

The intersection of two subspaces is a subspace.

Linear combinations I

Definition (Linear combinations)

Let V be a vector space and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in V. Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be numbers. An expression of type

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \cdots + \alpha_k \mathbf{v}_k$$

is called a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$. The numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ are called the coefficients of the linear combination.

Lemma

The set of all linear combinations of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ in a vector space V is a subspace of V. This subspace is denoted by $span(S)$ and called **the span** *of S.*

Examples

1. All linear combinations of vector $(1, 2)$ in the plane $\mathbb{R}^2$ are on the line through two points $(1, 2)$ and the origin.
2. All linear combinations of vectors $(1, 1, 0)$ and $(1, 0, 0)$ in $\mathbb{R}^3$ are on the plane through 3 points $(0, 1, 0)$, $(1, 0, 0)$ and the origin.

Spanning set

1. The set of all linear combinations of k vectors $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ is called **the span of S** and denoted by $span(S)$. Hence,

$$span(S) = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k : \forall a_1, a_2, \ldots, a_k \in \mathbb{R}\}.$$

2. A **spanning set of a vector space**: Given a vector space V and a set $S \subseteq V$. If every vector of V can be written as a linear combination of vectors in S, *i.e.*, $V = span(S)$, then S is called a **spanning set of V**.

1. Some terminologies: Given a set of vectors S. If $span(S) = V$, then we can say that
   - S **spans (generates)** V;
   - V **is spanned by** S;
   - S is a **spanning set** of V.
2. Notes:

   i) $span(\emptyset) = \{\vec{0}\}$;

   ii) $S \subseteq span(S)$

   iii) $S_1, S_2 \subseteq V$, and $S_1 \subseteq S_2$, then $span(S_1) \subseteq span(S_2)$.

Lemma

Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ *be vectors in* V*. Then,*

i) $span(S)$ *is a subspace of* V*.*

ii) $span(S)$ *is the smallest subspace of* V *that contains* S*.*

Examples

1. $e_1 = (1,0,0), e_2 = (0,1,0), e_3 = (0,0,1)$ is a spanning set for $\mathbb{R}^3$;
2. $e_1 = 1, e_2 = x, e_3 = x^2$ is a spanning set for $P_2(x)$;
3. Let H be the set of all vectors of the form $(a-3b, b-a, a, b)$, where a and b are arbitrary scalars. We can write down column vector

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

which shows that $H = span\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = (1,-1,1,0)$ and $\mathbf{v}_2 = (-3,1,0,1)$.

Details are on the whiteboard!

Linear independence and linear dependence

Definition

Let $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a set of vectors in a vector space V. Consider the equation

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k = \vec{0}.$$

i) If the equation has only the trivial solution $c_1 = c_2 = \cdots = c_k = 0$, then S is called linearly independent.

ii) If the equation has a nontrivial solution (i.e., not all zeros), then S is called linearly dependent.

Notes

1. $\emptyset$ is linearly independent
2. If $\vec{0} \in S$, then S is linearly dependent.
3. If $\mathbf{v} \neq \vec{0}$, then $\{\mathbf{v}\}$ is linearly independent
4. If $S_1 \subseteq S_2$, then
    - S_1 is linearly dependent $\Rightarrow$ S_2 is linearly dependent;
    - S_2 is linearly independent $\Rightarrow$ S_1 is linearly independent;

Examples: Testing for linearly independent

1. Vectors $(1,1)$ and $(-3,2)$ are linearly independent in $\mathbb{R}^2$.
2. Determine whether the following set of vectors in P_2 is linearly independent or dependent.

$$S = \{1 + x - 2x^2, 2 + 5x - x^2, x + x^2\}.$$

3. Determine whether the following set of vectors in 2×2 matrix space is linearly independent or dependent.

$$S = \left\{ \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \right\}$$

4. Let $p_1(t) = 1$, $p_2(t) = t$ and $p_3(t) = 4 - t$ be polynomials. Then the set $\{p_1, p_2, p_3\}$ are linearly dependent since $p_3 + p_2 - 4p_1 = 0$.

Solution: On the whiteboard!

Remarks

The main difference between linear dependence in $\mathbb{R}^n$ and in a general vector space is that when the vectors are not n-tuples, the homogeneous equation usually cannot be written as a system of n linear equations. That is, the vectors cannot be made into the columns of a matrix A in order to study the equation $Ax = 0$. We must rely instead on the definition of linear dependence.

Theorem

Let V be a vector space. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be linearly independent elements in V. Let $\alpha_1, \ldots, \alpha_k$ and $\beta_1, \ldots, \beta_k$ be numbers such that

$$\alpha_1 \mathbf{v}_1 + \cdots + \alpha_k \mathbf{v}_k = \beta_1 \mathbf{v}_1 + \cdots + \beta_k \mathbf{v}_k.$$

Then we must have

$$\alpha_1 = \beta_1,\ \alpha_2 = \beta_2, \ldots,\ \alpha_k = \beta_k.$$

Proof.

On the whiteboard! □

Linear dependence and spanning set

Theorem (Spanning set theorem)

Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set in a vector space V, and let $H = span(S)$.

- *If one of the vectors in S, say v_k, is a linear combination of the remaining vectors in S, then the set formed from S by removing v_k still spans H.*
- *If $H \neq \{0\}$, then some subset of S is a basis for H.*

Proof.

On the whiteboard! $\square$

Basis and dimension I

- Definition of a basis: Let V be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} \subseteq V$. If
  (i) S spans V (i.e., $span(S) = V$)
  (ii) S is linearly independent,

  then S is called a **basis** of V.
- Examples:
  i) $\emptyset$ is a basis for $\{\vec{0}\}$
  ii) the standard basis for R^3: $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where

$$\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1).$$

  iii) Another basis for $\mathbb{R}^3$:
  $\mathbf{v}_1 = (1,1,1), \mathbf{v}_2 = (1,1,0), \mathbf{v}_3 = (1,0,0)$.

Basis and dimension II

iv) the standard basis for $M_{2\times 2}$:

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

v) The standard basis for polynomials of degree $\leq n$:

$$\{1, x, x^2, \ldots, x^n\}.$$

vi) Another basis for polynomials of degree $\leq n$:

$$\{1, (1+x), (1+x)^2, \ldots, (1+x)^n\}.$$

Properties of basis

Theorem

Let V be a vector space.

1. *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for V, then every vector in V can be written in one and only one way as a linear combination of vectors in S.*
2. *Let S be a basis of V. If $|S| = n$, then every set containing more than n vectors in V is linearly dependent.*

Maximal subset of linearly independence

A set of vectors $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ in V is called a **maximal** subset of linearly independent elements if S is linearly independent, and if $S \cup \{v\}$ is linearly dependent for any $v \notin S$.

Theorem

1. *If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a spanning set of V and if S is a maximal subset of linearly independent elements, then S is a basis of V.*
2. *Let V be a vector space and suppose that one basis has n elements, and another basis has m elements. Then $m = n$, i.e., all bases for a finite-dimensional vector space has the same number of vectors.*

Remarks

1. When a vector space V has a basis of finite elements, then the number of vectors in a basis for V is called the **dimension** of V, denoted by $\dim(V)$. In other words, if S is a finite set and S is a basis of V, then

$$\dim(V) = |S|.$$

2. A vector space V is called finite dimensional, if it has a basis consisting of a finite number of elements.
3. If a vector space V is not finite dimensional, then it is called **infinite dimensional**.

Remarks

1. $\dim(\{\vec{0}\}) = 0$.
2. $\dim(V) = n$ and $S \subseteq V$.
   - If S is a generating set, then $|S| \geq n$.
   - If S is a linear independent set, then $|S| \leq n$.
   - If S is a basis, then $|S| = n$.
3. If $\dim(V) = n$ and if W is a subspace of V, then $\dim(V) \leq n$.

Examples

1. $\dim(\mathbb{R}^b) = n$
2. $\dim(\mathbb{M}_{m\times n}) = mn$
3. $\dim(P_n(x)) = n + 1$, where $P_n(x)$ the vector space of all polynomials with degree $\leq n$.
4. $\dim(P(x)) = \infty$, where $P(x)$ the vector space of all polynomials.
5. Find the dimension of the subspace

   $$H = \{(a - 3b + 6c, 5a + 4d, b - 2c - d, 5d) : a, b, c, d \in \mathbb{R}\}.$$

   Answer: On the whiteboard.

Null and column spaces

In applications of linear algebra, subspaces of $\mathbb{R}^n$ usually arise in one two ways:

- as the set of all solutions to a homogeneous linear system
- as the set of all linear combinations of certain vectors.

These two subspaces are called null and column spaces. In the sense of linear transformation, they also called kernel and range of the linear transformation respectively.

Null and column spaces

Given a matrix A of size $m \times n$.

1. **Null space** of A is the set of all solutions of the homogeneous equation $Ax = 0$. In set notation,

$$Nul(A) = \{x \in \mathbb{R}^n : Ax = 0.\}$$

2. **Column space** of A is the set of all linear combinations of the columns of A. In the set notation,

$$Col(A) = span\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\} = \{\mathbf{b} : b = Ax \text{ for some } \mathbf{x} \in \mathbb{R}^n\},$$

where $\mathbf{a}_i$ are columns of A.

Theorem

Given a matrix A *of size* $m \times n$. *The null space* $Null(A)$ *of* A *is a subspace of* $\mathbb{R}^n$. *Equivalently, the set of all solutions to a homogeneous linear system with* m *equations,* n *unknown is a subspace of* $\mathbb{R}^n$.

Remarks

- There is no obvious relation between vectors in $Null(A)$ and the entries in A.
- The $Null(A)$ is defined implicitly, because it is defined by a condition that must be checked.
- No explicit list or description of the elements in $Null(A)$ is given. However, solving the equation $Ax = 0$ produces an explicit description of $Null(A)$.

Finding a basis for $Null(A)$

1. Find the reduced echelon-form $(A^{**}|0)$ of the augmented matrix $(A|0)$.
2. Write dependent variables in terms of free variables (free variables are variables corresponding to non-pivot columns of A^{**})
3. Find general solution to $Ax = 0$ in term of free variables.
4. Decompose the general solution vector into a linear combination of vectors where the coefficients are the free variables.
5. Vectors in the above step form a basis for $Null(A)$.
6. Moreover, $\dim(Null(A))$ is equal to the number of free variables.

Examples I

Find a spanning set and a basis for $Null(A)$.

1. $A = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}$.

   Solution: We reduce the augumented matrix $(A|0)$ to reduced-row echelon form:

$$A = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix} \left| \begin{matrix} 0 \\ 0 \end{matrix} \right. \to \left(\begin{matrix} 1 & -3 & -2 \\ 0 & -6 & -9 \end{matrix} \left| \begin{matrix} 0 \\ 0 \end{matrix} \right. \right) \to \left(\begin{matrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & -6 & -9 \end{matrix} \left| \begin{matrix} 0 \\ 0 \end{matrix} \right. \right)$$
$$\to \left(\begin{matrix} 1 & 0 & 1 \\ 0 & 1 & \frac{3}{2} \end{matrix} \left| \begin{matrix} 0 \\ 0 \end{matrix} \right. \right)$$

   Hence, x_1, x_2 are dependent variables and x_3 is free variable. We get

$$x = (x_1, x_2, x_3) = (-x_3, -\frac{3}{2}x_3, x_3) = x_3(-1, -\frac{3}{2}, 1)$$

   and $(-1, -\frac{3}{2}, 1)$ is a basis for $Null(A)$.

Examples II

2. $A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$.

Solution: On the whiteboard.

Column space

Theorem

Given a matrix A of size $m \times n$. Assume that A^* is the row-echelon form of A. Then, the column space of A is the space spanned by column of A corresponding to pivot columns in A^*.

Examples: $A = \begin{pmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{pmatrix}$. We have

$$A^* = \begin{pmatrix} 1 & -3 & -2 \\ 0 & -6 & -9 \end{pmatrix},$$

and

$$\{(1, -5), (-3, 9)\}$$

is a basis for $Col(A)$.

Examples

Find a basis and the dimension for $Col(A)$ where

$$A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}.$$

Solution: On the whiteboard.

Row space

- Similarly, we can define the row space of a matrix, denoted by $row(A)$, which is a subspace generated by its rows. Note that $\dim(row(A))$ is equal to the number of non-zero rows in the row-echelon form A^*. Furthermore, non-zero rows of A^* forms a basis for $row(A)$.
- Given $A \in M_{m \times n}$. It is straightforward that

$$\dim(Col(A)) = \dim(row(A)).$$

- We define the **rank** of a matrix A, denoted by $r(A)$, as the dimension of row (column) space of A.

The following is straightforward from the fact that

number of pivot columns+number of non-pivot columns = number of col

- $r(A) + \dim(null(A)) = n$
- $r(A) + \dim(null(A^T)) = m$

Theorem

Let $A \in M_{n\times n}$ *and* A *be invertible. The following are equivalent*

- *Columns of* A *form a basis of* $\mathbb{R}^n$*.*
- $col(A) = \mathbb{R}^n$
- $\dim(col(A)) = n$
- $r(A) = n$
- $null(A) = \{\vec{0}\}$
- $\dim(null(A)) = 0$

Linear algebra: Linear mappings

Lectures 7+8+9+10+11: Vector spaces

Introduction

- Among mappings, the linear mappings are the most important.
- A good deal of mathematics is devoted to reducing questions concerning arbitrary mappings to linear mappings.
- On the other hand, it is often possible to approximate an arbitrary mapping by a linear one, whose study is much easier than the study of the original mapping. This is done in the calculus of several variables.

Linear mappings

Let V, W be vector spaces on $\mathbb{R}$. A **linear mapping**

$$T : V \to W$$

is a mapping which satisfies the following two properties

- $T(u + v) = T(u) + T(v)$ for any $u, v \in V$
- $T(cu) = cT(u)$ for any $c \in \mathbb{R}$ and $u \in V$.

Examples I

- (Projection):

$$\begin{aligned} T : \mathbb{R}^3 &\to \mathbb{R}^2 \\ (x, y, z) &\mapsto (x, y) \end{aligned}$$

- (Inner product):

$$\begin{aligned} T : \mathbb{R}^3 &\to \mathbb{R} \\ (x, y, z) &\mapsto ax + by + cz, \end{aligned}$$

  where a, b, c are given numbers in $\mathbb{R}$.

- (Linear mapping given by a matrix):

$$\begin{aligned} T : \mathbb{R}^n &\to \mathbb{R}^m \\ \mathbf{v} &\mapsto A\mathbf{v}, \end{aligned}$$

  where A is a given matrix.

Examples II

- (A linear transformation from $M_{m\times n}$ to $M_{n\times n}$)

$$T : M_{m\times n} \to M_{n\times m}$$
$$A \mapsto A^T$$

- (Differentiation): T transforms a differentiable function f to its derivative.

Properties of linear mappings

Given a linear mapping

$$T : V \to W.$$

Then

- $T(\vec{0}) = \vec{0}$;
- $T(-\mathbf{v}) = -T(\mathbf{v})$
- If $\mathbf{v} = c_1 v_1 + \cdots + c_k v_k$, then

$$T(\mathbf{v}) = c_1 T(v_1 + c_2 T(v_2)) + \cdots + c_k T(v_k).$$

Examples

Consider a map T from $M_{m\times n}$ to $M_{n\times m}$:

$$T : M_{m\times n} \to M_{n\times m}$$
$$A \mapsto A^T$$

Then T is a linear transformation since

$$T(A+B) = (A+B)^T = A^T + B^T = T(A) + T(B),$$

and

$$T(cA) = (cA)^T = cA^T = cT(A).$$

Theorem

Let V and W be vector spaces. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V. Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n$ be arbitrary elements of W. Then there exists a unique linear mapping $T : V \to W$ such that

$$T(v_1) = w_1, \ldots, T(v_n) = w_n.$$

In the other words, a linear mapping is well defined by its images of a basis.

The kernel and image of a linear mapping

- **Kernel of a linear map T:** Let $T : V \to W$ be a linear map. Then the set of all vectors v in V that satisfies $T(v) = 0$ is called the **kernel** of T and is denoted by $ker(T)$.

$$ker(T) = \{v : T(v) = 0\}.$$

- **Examples**: Given

$$T(x) = Ax = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Then, $\ker(T)$ coincides with solution set $null(A)$ of $Ax = 0$. To calculate $\ker(T)$, we use Gauss-Jordan elimination as presented in the previous lecture.

Observation

The kernel of a linear transformation $T : V \to W$ is a n essentially solution set of a homogeneous system of linear equations, and there fore is a subspace of the domain V. For this reason, sometimes T is called the **nullspace** of T.

Some popular linear transformations

On the whiteboard.

- Composition of linear transformation
- Inverse linear transformation
- Matrix of a transformation in any bases:

Coordinate systems

On the white board.