Study program: Computer Science, VGU
Algebra

Exercise Sheet: Group

1. Let $(G.*)$ be a group. Prove that
   (a) G has only one identity element.
   (b) For each element $a \in G$, the inverse of a is unique.
   (c) If $a * b = a * c$, then $b = c$ (cancellation property)
   (d) $a^r * a^s = a^{r+s}$ for all $a \in G$ and $r, s \in \mathbb{Z}$.
   (e) $(a^r)^s = a^{rs}$ for all $a \in G$ and $r, s \in \mathbb{Z}$.
   (f) $(a * b)^{-1} = b^{-1} * a^{-1}$ for $a, b \in G$.
2. Consider the group $(\mathbb{Z}_{15}, +_{15})$. Find
   (a) $\langle 3 \rangle$ and the order of the element 3.
   (b) $\langle 10 \rangle$ and the order of the element 10.
3. Consider the group $(\mathbb{Z}_{12}^*, \cdot_{12})$. Find
   (a) its order
   (b) Draw the multiplication table for this group
   (c) Is it cyclic?
4. Given a group $(G, *)$ and $a \in G$. We define

   $$C(a) = \{b \in G : ab = ba\},$$

   and

   $$Cent(G) = \{a \in G : ab = ba \text{ for all } b \in G\}.$$

   Prove that $C(a)$ and $Cent(G)$ are subgroups of G.
5. How many different groups are there with
   (a) 2 elements
   (b) 3 elements
   (c) 4 elements

   Draw the multiplication table for each group.
6. Given a group $(\mathbb{Z}, +)$ and $H = \{0, \pm 3, \pm 6, \dots\}$.
   (a) Prove that H is a subgroup of $\mathbb{Z}$.

(b) Check whether the following pairs of cosets $11 + H$ and $7 + H$; and $5 + H$ and $-1 + H$ are the same or disjoint?

(c) Find $|\mathbb{Z} : H|$

7. Given a group $(\mathbb{Z}_{20}^*, \cdot_{20})$ and $H = \langle 3 \rangle$.

   (a) Find all left cosets of H in $\mathbb{Z}_{20}^*$.

   (b) Find $|H|$ and $|\mathbb{Z}_{20}^* : H|$.

8. Consider the group $(\mathbb{Z}_p^*, \cdot_p)$ where p is prime.

   (a) List all elements of $\mathbb{Z}_p^*$.

   (b) Prove that $\mathbb{Z}_p^*$ is cyclic

   (c) Show the little Fermat theorem that $a^{p-1} \equiv 1 \mod p$ for each

9. Prove that if $\phi(n)$ is the Euler's function of n, then

$$a^{\phi(n)} \equiv 1 \mod n.$$

10. Let $(G, *)$ be a group and $a \in G$ such that $o(a) = 15$. Find all the left cosets of $\langle a^5 \rangle$ in $\langle a \rangle$. What is $|\langle a \rangle : \langle a^5 \rangle|$?

11. Suppose that K is a proper subgroup of H and H is proper subgroup of G. If $|G| = 420$ and $|K| = 42$, what are the possible orders of H?

12. Prove that all subgroups of a cyclic group are cyclic groups.

13. Given a permutation $\sigma = 213546$ in S_6.

    (a) Represent σ in the cycle notation form

    (b) Find σ^{-1}, σ^2, σ^3.

    (c) Find the order of σ in S_6.

    (d) Find the index of $\langle \sigma \rangle$ in S_6.

14. Prove that $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_4$.

15. Given $\sigma = (2)(1, 3) \in S_3$. Let $H = \langle \sigma \rangle$.

    (a) Find all left cosets of H in S_3

    (b) Find all right cosets of H in S_3

    (c) What is $|H|$? what is $|S_3 : H|$?

16. How many non-isomorphic groups are there with

    (a) 2 elements

    (b) 3 elements

(c) 4 elements.

17. Prove that the direct product group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic if and only if $gcd(m, n) = 1$.

18. Prove the followings about group isomorphisms.

    (a) $(\mathbb{Z}_{10}, \cdot_{10}) \cong (\mathbb{Z}_4, +_4)$

    (b) $(\mathbb{Z}_{12}^*, \cdot_{12}) \ncong (\mathbb{Z}_4, +_4)$.

19. Consider the maps

$$\begin{aligned} \phi : \mathbb{Z} &\to \mathbb{Z}_3 \times \mathbb{Z}_4 \\ m &\mapsto (m \mod 3, m \mod 4) \end{aligned}$$

    (a) Prove that ϕ is a homomorphism

    (b) Find $\ker(\phi)$, $\phi(\mathbb{Z})$

    (c) Using the first isomorphism theorem to show that

$$(\mathbb{Z}_{12}, +_{12}) \cong \mathbb{Z}/12\mathbb{Z} \cong \mathbb{Z}_3 \times \mathbb{Z}_4.$$

    (d) How about are the results if $\phi : \mathbb{Z} \to \mathbb{Z}_6 \times \mathbb{Z}_4$?

20. Consider the group $(\mathbb{Z}_{24}, +_{24})$ and $H = \langle 8 \rangle$.

    (a) Describe all cosets of H in $\mathbb{Z}_{24}$

    (b) Find the order of the coset $14 + H$ in the quotient group $\mathbb{Z}_{24}/H$.

21. Prove that quotient groups of a cyclic group are cyclic.

22. Prove that every subgroup of index 2 is a normal subgroup.