VGU - CS Study Program
Algebra

Exercise Sheet: Relations

1. Let $\mathbb{Z}$ be the set of all integers.
   Describe the set $\mathbb{Z} \times \mathbb{Z}$.

2. Let $|A| = m$ and $|B| = n$.
   How many relations exist from A to B?

3. Is
   $$f(x) = \frac{x}{x+1}$$
   invertible?

4. For which real numbers a is the function
   $$f(x) = ax, \ x \in \mathbb{R}$$
   invertible? Determine the inverse function.

5. Let R_m be a relation of the integers $\mathbb{Z}$ defined by:
   $$(x, y) \in R_m \Leftrightarrow m|(x - y)$$
   Interpretation: $(x, y) \in R_m$ if and only if x and y have the same remainder when divided by m.

   We denote this by:
   $$x \equiv y \text{ mod } m$$
   i.e. x is congruent with y modulo m.

   Example: We have $1 \equiv 11$ mod 10, since both 1 and 11 have the remainder 1 when divided by 10.

   Your task: Investigate if R_5 is reflexive, symmetric, antisymmetric and transitive!

6. Is the equality relation, i.e. $=$, an equivalence relation?

7. Is the relation $>$ an equivalence relation?

8. Is the relation $\geq$ an equivalence relation?

9. Let
   $$A = \{ \text{ all positive integers divisible by 2 and not exceeding 30} \}.$$

(a) Find $|A|$

(b) Let

$$B = \{\text{ all positive integers divisible by 6 and not exceeding 30}\}.$$

$$C = \{\text{ all positive integers divisible by 8 and not exceeding 30}\}.$$

Find $B \cup C$, $B \cap C$, and binary string representations for B, C, $B \cup C$, $B \cap C$, $\bar{B}$.

10. Given A the set of all webpages. Let $R = \{(a, b) \in A \times A :$ there is at least 1 common link on we
Investigate following properties of R

   (a) reflexive
   (b) symmetric
   (c) transitive
   (d) antisymmetric
   (e) equivalent
   (f) partial order

11. Given $A = \{1, 2\}$. Let

$$R = \{(B, C) \in 2^A \times 2^A : B \subseteq C\}.$$

   (a) Find matrix representation M_R for R and $|R|$.
   (b) Investigate reflexive (symmetric, antisymmetric, transitive) properties of R.

12. Given a set $A = \{a, b, c\}$ and a relation R with matrix representation

$$M_R = \begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{matrix}$$

   (a) Find R^{-1} and its matrix representation
   (b) Find $\bar{R}$ and its matrix representation
   (c) Find R^2 and its matrix representation
   (d) Find $R \cup S$, $R \circ S$ and $S \circ R$ where

$$S = \{(a, b), (b, b), (b, c), (c, a), (c, b), (c, c)\}.$$

13. Given matrix representations of relations

$$M_{R_1} = \begin{matrix} 1 & 1 & 1 \\ 0 & 1 & 1, \\ 1 & 1 & 1 \end{matrix} \quad M_{R_2} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Determine whether R_1 and R_2 are equivalence relations or not.

14. Let R be the relation on the set of all cities in the world such that (a, b) in R if there is a direct non-stop airline flight from a to b. When is (a, b) in

    (a) R^2

    (b) R^3

    (c) R^{-1}

15. Let R be the relation $\{(a, b) : a \neq b\}$ on the set of integers. What is the reflexive closure of R?

16. Let R be the relation $\{(a, b) : a \text{ divides } b\}$ on the set of integers. What is the symmetric closure of R?

17. Let R be the relation on the set of all students containing the ordered pair (a, b) if a and b are in at least one common class and $a \neq b$. When (a, b) in

    (a) R^2

    (b) R^3

    (c) R^*

18. Given the matrix representation of the relation R on $\{a, b, c, d\}$ as following

$$M = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

    Find the transitive closure of R using naive and Warshall's algorithms.

19. Given the relation $R = \{(1, 2), (2, 1), (2, 3), (3, 4), (4, 1)\}$ on $\{1, 2, 3, 4\}$. Find

    (a) Reflexive closure of R

    (b) Symmetric closure of R

    (c) Transitive closure of R using naive algorithm and Warshall's algorithm

    (d) Reflexive transitive closure of R

    (e) Equivalent closure of R.

20. Given the relation $R = \{(1, 2), (1, 4), (3, 3), (4, 1)\}$ on $\{1, 2, 3, 4\}$. Find

    (a) reflexive and transitive closure of R

    (b) symmetric and transitive closure of R

(c) equivalent closure of R.

21. Given the relation R on the set of all bit strings such that $(s, t) \in R$ if and only if s and t contain the same number of 1s.

    (a) Prove that R is an equivalence class.

    (b) List all bit strings of length 4 equivalent to 01001.

    (c) How many bit strings of length n with exactly 2 occurrences of 1s are there?

22. Let R be the relation on the set ordered pairs of positive integers such that $((a, b), (c, d)) \in R$ if and only if $ad = bc$. Show that R is an equivalence relation.