Effect of bulk viscosity on acoustic wave equation

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November, 2022

Project report at the end of 7^{th} Semester

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Abstract

A fundamental acoustic equation is governed by bi-directional wave equation. It is obtained by considering dynamic coefficient of viscosity. Acoustic waves transport momentum and energy by consecutive compression, dilation through the intervening medium. Therefore, it is important to consider and study the effects of Second coefficient of visocity. Here, we attempt to study the effect of Second coefficient of viscosity. Acoustic equation has been obtained without considering the Stoke's hypothesis. Numerical techniques are needed to solve this partial differential equation. Numerical analysis has been performed to quantify and minimize the role of error while computing. Here, we use Global Spectral Analysis (GSA) with appropriate error metrics such as the numerical group velocity, numerical phase speed and the numerical amplification factor. Runge–Kutta–Nyström (RKN) method is used to discretize the temporal term. Spatial discretization is done by OUCS3, LELE and CD2 Scheme. Results of this analysis are reported here before performing actual computations.

0.1 Introduction

Acoustic noise control is a major engineering application. Noise induced in the interior of aerospace structural components are due to structural vibration and aerodynamics in origin. Sounds propagates in fluids as longitudinal waves, by undergoing compression and dilatation of the intervening medium which can be moving or stationary in the form of small perturbation in density and pressure as waves. While there is no net mass transfer due to sound propagation, momentum and energy are transported. A fundamental acoustic equation is governed by a bi-directional wave equation (1).it has been derived by considering just the first coefficient of viscosity μ (Dynamic viscosity). However, second coefficient of viscosity λ (Dilatational viscosity coefficient) is important for physical events where dilation and compression plays important role in transport of the property (8). The utility of the Stokes' hypothesis (5) has often been questioned in fluid dynamic and thermodynamic communities (4). For monoatomic gases, Stokes' hypothesis may hold, however, for polyatomic systems such as air, there are major concerns in assuming that the bulk viscosity is zero. Therefore, in acoustic wave propagation it is necessary to study the effect of Bulk viscosity.

0.2 Governing Equation

The fundamental partial differential governing equations of fluid flows are given by

1) Continuity Equation:

$$\frac{\partial \rho}{\partial t} + \rho_o \frac{\partial u}{\partial x} = 0 \tag{1}$$

2) Momentum Equation:

$$\rho_o \frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = (\lambda + 2\mu)\nabla^2 u \tag{2}$$

3) Equation of State:

$$\partial p = c^2 \partial \rho \tag{3}$$

Where:

ρ: Density of fluid

ρ_o: Density of fluid at a point

p: Excess Pressure of fluid

u: Velocity component of fluid in X direction

v: Velocity component of fluid in y direction

c: Phase speed

λ: Dilatational viscosity coefficient

μ: Shear viscosity coefficient

Acoustic equation by retaining both the coefficients of viscosity has been derived by using the above governing equations. For One Dimensional case, acoustic wave equation is given by:

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} + (\frac{\lambda+2\mu}{c^2 \rho_o})\frac{\partial^3 p}{\partial t \partial x^2} = 0 \tag{4}$$

The hydrodynamic and acoustic events occur on a completely different scales. That's why it becomes challenging to simultaneously simulate flow and sound field information. Computation of acoustic signal from the first principle is difficult, as the signal amplitude is often of the same order of numerical error. It is significantly smaller than associated amplitudes in routine computations. It requires high fidelity computations (6). Simulation of acoustic wave demands finer mesh as well as smaller time step (7). Most of the traditional methods of discretization are dispersive and dissipative (9). There is an intimate relation between space and time scales in such computing called as Dispersion relation and retaining this physical dispersion relation of an event is the task of Dispersion relation preservation (DRP) schemes.

0.3 Global Spectral Analysis

Dispersion relation for our equation and Physical amplification factor are given below:

For physical amplification factor inserting $p = \int\int Pe^{i(k_x x - \omega t)} dk_x d\omega$ into the above equation ,we will have::

$$(a)\ \frac{\partial^2 p}{\partial x^2} = -\int\int k_x^2 Pe^{i(k_x x - \omega t)} dk_x d\omega$$

$$(b)\ \frac{\partial^2 p}{\partial t^2} = -\int\int \omega^2 Pe^{i(k_x x - \omega t)} dk_x d\omega$$

$$(d)\ \frac{\partial}{\partial t}[\frac{\partial^2 p}{\partial x^2}] = i\int\int\int \omega[k_x^2] Pe^{i(k_x x - \omega t)} dk_x d\omega$$

substituting all the derivatives in above equation we have:

$$\omega^2 + i\frac{(\lambda+2\mu)}{\rho_o}(k_x^2)\omega - (k_x^2)c^2 = 0$$

Taking $\gamma = \frac{(\lambda+2\mu)}{\rho_o}$

$$\omega^2 + i\gamma(k_x^2)\omega - (k_x^2)c^2 = 0$$

solving above quadratic equation we will have

$$\omega_1 = i\frac{\gamma}{2}(k_x^2)[\sqrt{1 - \frac{4c^2}{(k_x^2)\gamma^2}} - 1]$$

$$\omega_2 = i\frac{\gamma}{2}(k_x^2)[-\sqrt{1 - \frac{4c^2}{(k_x^2)\gamma^2}} - 1]$$

so physical amplification factors are:

$G_{ph1} = e^{-i\omega_1 \Delta t}$

$G_{ph2} = e^{-i\omega_2 \Delta t}$

Numerical techniques are often employed for solving governing equations. Numerical simulations are an approximation to the exact solution and the accuracy depends on fidelity of the method. Numerical analysis plays an important role in quantification of errors of the numerical method. Traditionally finite difference based numerical schemes have been analyzed either using Von Neumann analysis or GK stability theory or time stability theory (?). These approaches have limitations like their inability to analyze in the spectral plane by full domain analysis with actual time-discretization method. On the other hand, a full domain spectral analysis with appropriate error metrics reveals more insight about stability/instability, dispersion, dissipation errors for all length and time scales (2). This information is important in designing and implementing dispersion relation preserving scheme.

In this work, we chose 1-D acoustic equation with bulk viscosity. Numerical analysis of this equation using Global spectral analysis has been performed with the appropriate error parameteres like numerical amplification factor, numerical group velocity, numerical phase velocity. It would be beneficial to perform such analysis before solving this equation.

We choose different spatial discretization schemes like CD2, LELE and OUCS3 along with Runge–Kutta–Nyström (RKN) time integration method. The stability/instability regions are identified and simulation parameters for achieving good accuracy are presented. Now looking at Numerical Global amplification factor

Here we are using implicit scheme for space discretization and RKN for time discretization. (RKN-OUCS3)

we have

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} + \left(\frac{\lambda+2\mu}{c^2\rho_o}\right)\frac{\partial}{\partial t}\left[\frac{\partial^2 p}{\partial x^2}\right] = 0$$

Now inserting a trial solution

$p = \int \int \hat{P} e^{i(k)} dk$

We seek converting equation in the form which will be later converted to k-plane using trial solution

$\frac{\partial^2 p}{\partial t^2} = f(P, P')$

so,

$f(p, p') = (\frac{\partial^2}{\partial x^2})[c^2 p + \gamma p']$

we are using Runge–Kutta–Nyström (RKN) method

$p^{(n+1)} = p^{(n)} + \Delta t p'^{(n)} + \frac{1}{96}(23K_1 + 75K_2 - 27K_3 + 25K_4)$

$p'^{(n+1)} = p'^{(n)} + \frac{1}{96}\frac{1}{\Delta t}(23K_1 + 125K_2 - 81K_3 + 125K_4)$

intend Using compact implicit scheme

$\frac{\partial^2 p}{\partial x^2} = \frac{[C^{(x)}]^{(2)}[p]}{\Delta x^2}$

$\frac{\partial^2 p}{\partial x^2}|_j = \int \frac{1}{\Delta x^2} \sum_{l=1}^{N} (C_{jl}^{x})^{(2)} \hat{P} e^{i(k_x x_j)} e^{i(k_x x_l - k_x x_j)} dk_x$

$\frac{\partial^2 p}{\partial x^2}|_j = \int \frac{1}{\Delta x^2} \sum_{l=1}^{N} (C_{jl}^{x})^{(2)} \hat{P} e^{i(k_x x_j)} \zeta_{lj}^{x} dk_x$

where , $\zeta_{lj}^{x} = e^{i(k_x x_l - k_x x_j)}$

$[-K_{eq}^{(2)}]_x = \frac{1}{\Delta x^2} \sum_{l=1}^{N} (C_{jl}^{x})^{(2)} \zeta_{lj}^{x}$

Now,

$\frac{\partial^2 p}{\partial x^2} = \int \sum_{l=1}^{N} [\frac{(C_{jl}^{x})^{(2)} \zeta_{lj}^{x}}{\Delta x^2}] \hat{P} e^{i(k_x x_j)} dk_x$

And

$\frac{\partial}{\partial t}(\frac{\partial^2 p}{\partial x^2}) = \int \sum_{l=1}^{N} [\frac{(C_{jl}^{x})^{(2)} \zeta_{lj}^{x}}{\Delta x^2}] \hat{P}' e^{i(k_x x_j)} dk_x$

Therefore,

$L(\hat{P}, \hat{P}') = \int (\frac{\gamma}{c^2}\hat{P}' + \hat{P}) B_j e^{i(k_x x_j)} dk_x$

Where:

$B_j = c^2 \sum_{l=1}^{N} [\frac{(C_{jl}^{x})^{(2)} \zeta_{lj}^{x}}{\Delta x^2}]$

Now we will calculate K_1, K_2, K_3, K_4 in spectral plane

$\hat{K}_1 = A_j \int S e^{k_x x_j} dk_x$

$\hat{K}_2 = A_j \int [S + \frac{2}{5}\Delta t S' + \frac{4}{25}A_j S] e^{ik_x x_j} dx$

$\hat{K}_3 = A_j \int [S + \frac{2}{3}\Delta t S' + \frac{4}{9}A_j S] e^{ik_x x_j} dx$

$\hat{K}_4 = A_j \int [S + \frac{4}{5}\Delta t S' + \frac{16}{25}A_j S + \frac{16}{125}\Delta t A_j S' + \frac{32}{625}A_j^2 S] e^{ik_x x_j} dx$

Where $A_j = \frac{1}{2}\Delta t^2 B_j$

$S = (\frac{\gamma}{c^2}P' + P)$

$S' = (\frac{\gamma}{c^2}P'' + P')$

Now, $23K_1 + 75K_2 - 27K_3 + 25K_4 = A_j \int [S(96 + 16A_j + \frac{32}{25}A_j^2) + S'\Delta t(32 + \frac{16}{25}A_j)] e^{ik_x x_j} dk_x$

$23K_1 + 75K_2 - 27K_3 + 25K_4 = A_j \int [(\frac{\gamma}{c^2}P' + P)(96 + 16A_j + \frac{32}{25}A_j^2) + (\frac{\gamma}{c^2}P'' + P')\Delta t(32 + \frac{16}{25}A_j)] e^{ik_x x_j} dk_x$

now using $\int \hat{P}'' e^{i(k_x x_j)} dk_x = \int (\frac{\gamma}{c^2}B_j \hat{P}' + B_j \hat{P}) e^{i(k_x x_j)} dk_x$

$=> \hat{P}'' = (\frac{\gamma}{c^2}B_j \hat{P}' + B_j \hat{P})$

substituting in above equation:

$23K_1 + 75K_2 - 27K_3 + 25K_4 = A_j \int [(\frac{\gamma}{c^2}P' + P)(96 + 16A_j + \frac{32}{25}A_j^2) + ((\frac{\gamma^2}{c^4}B_j + 1)P' + \frac{\gamma}{c^2}B_j P)\Delta t(32 + \frac{16}{25}A_j)] e^{ik_x x_j} dk_x$

now using expression of p for (RKN) :

$p^{(n+1)} = p^{(n)} + \Delta t p'^{(n)} + \frac{1}{96}A_j \int [(\frac{\gamma}{c^2}P' + P)(96 + 16A_j + \frac{32}{25}A_j^2) + ((\frac{\gamma^2}{c^4}B_j + 1)P' + \frac{\gamma}{c^2}B_j P)\Delta t(32 + \frac{16}{25}A_j)] e^{ik_x x_j} dk_x$

so,

$p^{(n+1)} = p^{(n)} + \Delta t p'^{(n)} + A_j \int [(\frac{\gamma}{c^2}P' + P)(1 + \frac{1}{6}A_j + \frac{1}{75}A_j^2) + ((\frac{\gamma^2}{c^4}B_j + 1)P' + \frac{\gamma}{c^2}B_j P)\Delta t(\frac{1}{3} + \frac{1}{150}A_j)] e^{ik_x x_j} dk_x$

Dividing whole expression by $P^{(n)}$ and using

$G_P = \frac{P_j^{(n+1)}}{P_j^{(n)}}$

$G_{P'} = \frac{P_j'^{(n+1)}}{P_j'^{(n)}}$

$H = \frac{P_j'^{(n)}}{P_j^{(n)}}$

$G_P = [1 + \Delta t H + A_j[(\frac{\gamma}{c^2 \Delta t} H\Delta t + 1)(1 + \frac{1}{6}A_j + \frac{1}{75}A_j^2) + ((\frac{\gamma^2}{c^4 \Delta t^2}\Delta t^2 B_j + 1)H\Delta t + \frac{\gamma}{c^2\Delta t}\Delta t^2 B_j)(\frac{1}{3} + \frac{1}{150}A_j)]]$

$G_P = [1 + \Delta t H + A_j[(\frac{P_e}{N_c} H\Delta t + 1)(1 + \frac{1}{6}A_j + \frac{1}{75}A_j^2) + ((2(\frac{P_e}{N_c})^2 A_j + 1)H\Delta t + 2\frac{P_e}{N_c}A_j)(\frac{1}{3} + \frac{1}{150}A_j)]]$

Now RKN for p'

$p'^{(n+1)} = p'^{(n)} + \frac{1}{96}\frac{1}{\Delta t}(23K_1 + 125K_2 - 81K_3 + 125K_4)$

$23K_1+125K_2-81K_3+125K_4 = A_j \int [S(192+64A_j+\frac{32}{5}A_j^2)+S'\Delta t(96+16A_j)]e^{ik_x x_j}dx$

$23K_1 + 125K_2 - 81K_3 + 125K_4 = A_j \int [(\frac{\gamma}{c^2}P' + P)(192 + 64A_j + \frac{32}{5}A_j^2) + ((\frac{\gamma^2}{c^4}B_j + 1)P' + \frac{\gamma}{c^2}B_j P)\Delta t(96 + 16A_j)]e^{ik_x x_j}dx$

placing the above equation in expression for p' RKN we have

therefore::

$G_{P'} = 1+\frac{1}{\Delta t}A_j[(\frac{\gamma}{c^2}+\frac{1}{H})(2+\frac{2}{3}A_j+\frac{1}{15}A_j^2)+((2\frac{\gamma^2}{c^4\Delta t^2}A_j+1)+2\frac{\gamma}{c^2\Delta t^2}A_j\frac{1}{H})\Delta t(1+\frac{1}{6}A_j)]$

$G_{P'} = 1+A_j[(\frac{P_e}{N_c}+\frac{1}{H\Delta t})(2+\frac{2}{3}A_j+\frac{1}{15}A_j^2)+((2(\frac{P_e}{N_c^2})^2 A_j+1+2\frac{P_e}{N_c}A_j\frac{1}{H\Delta t})(1+\frac{1}{6}A_j)]$

writing expression for both $G_{P'}$ and G_P

$G_P = [1 + \Delta t H + A_j[(\frac{P_e}{N_c} H\Delta t + 1)(1 + \frac{1}{6}A_j + \frac{1}{75}A_j^2) + ((2(\frac{P_e}{N_c})^2 A_j + 1)H\Delta t + 2\frac{P_e}{N_c}A_j)(\frac{1}{3} + \frac{1}{150}A_j)]]$

$G_{P'} = 1+A_j[(\frac{P_e}{N_c}+\frac{1}{H\Delta t})(2+\frac{2}{3}A_j+\frac{1}{15}A_j^2)+((2(\frac{P_e}{N_c^2})^2 A_j+1+2\frac{P_e}{N_c}A_j\frac{1}{H\Delta t})(1+\frac{1}{6}A_j)]$

where,

$P_e = \gamma\frac{\Delta t}{h^2}$

$N_c = c^2\frac{\Delta t^2}{h^2}$

$H\Delta t = Ln(G_P)$

0.4 Results and discussions (Numerical properties)

The spectral properties of the numerical scheme for solving this equation using RKN method for time integration and OUCS3 method for space discretisation are reported here. Subscripts 1 and 2 are, correspondingly, for the numerical amplification factor of signal and it's gradient. The unstable region is highlighted by red color in the plots. Numerical amplification factor should identically be equal to Physical amplification factor. Ratio of these terms is plotted for left and right running wave at diffusion number = 0.02 and 2 in Figure 1 to Figure 4. Unstable region can be observed to be extended with increasing diffusion number, implying range of Nc with desirable neutral stability shrinks with increase in diffusion number.

The normalised numerical group velocity, $Vgn/V1$ and $Vgn/V2$ contours shown in Figure 5 to 6. It can be seen that that for lower Nc values, both the methods are free of spurious upstream moving q-waves (group velocity of opposite sign) for all resolved kh range.

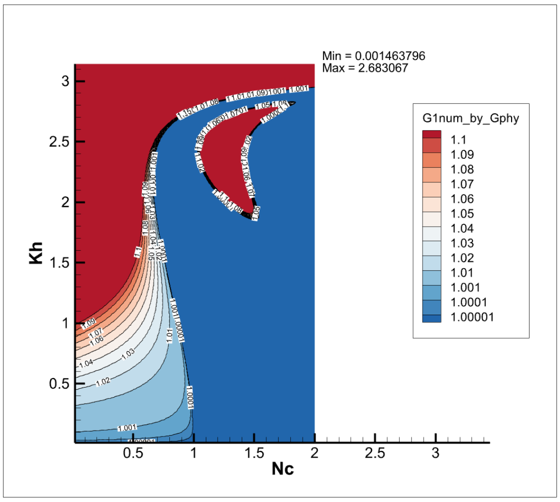

Figure 1: $G1/GPhy$ contours for $Pe = 0.02$.

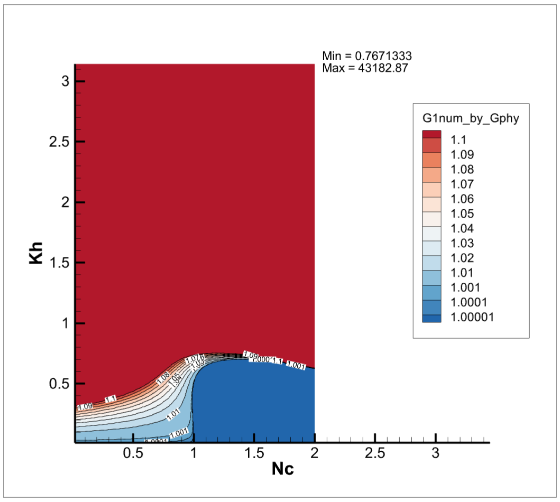

Figure 2: $G1/GPhy$ contours for $Pe = 2$.

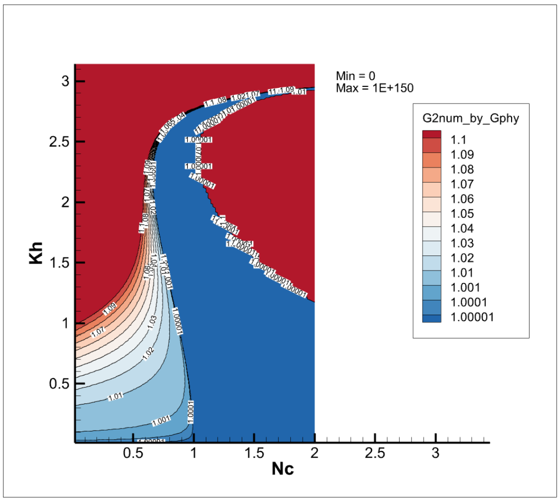

Figure 3: $G2/GPhy$ contours for $Pe = 0.02$.

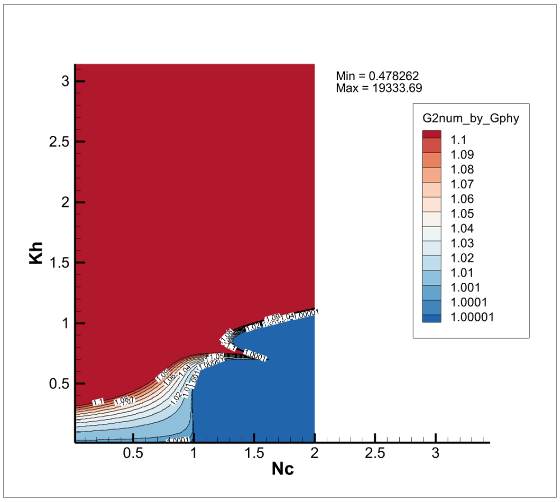

Figure 4: $G2/GPhy$ contours for $Pe = 2$.

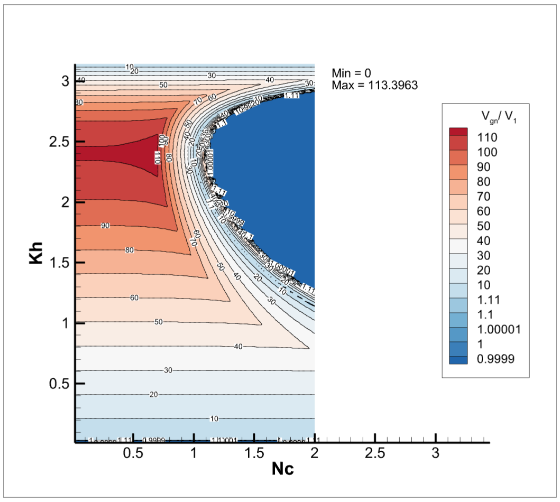

Figure 5: $Vg/V1$ contours for $Pe = 0.02$.

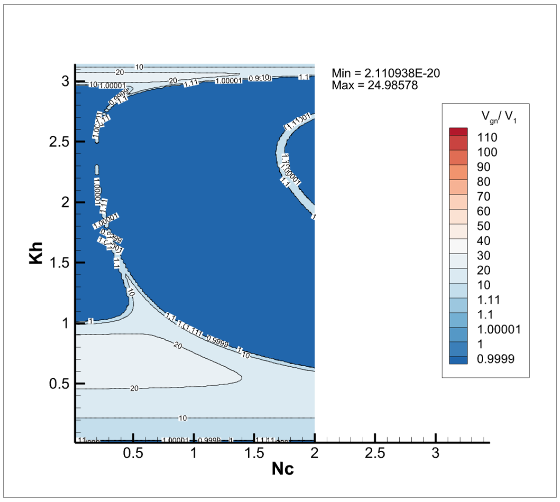

Figure 6: $Vg/V1$ contours for $Pe = 2$.

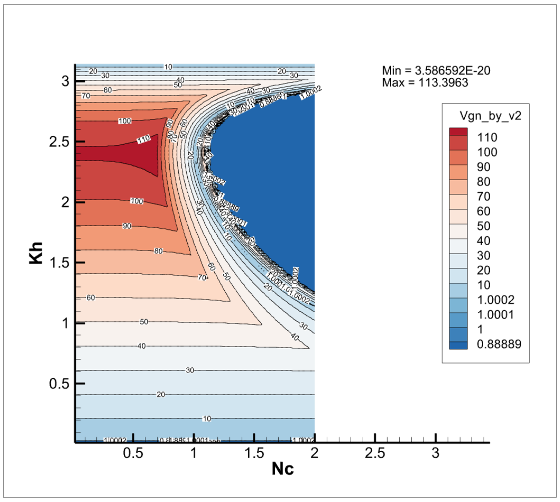

Figure 7: $Vg/V2$ contours for $Pe = 0.02$.

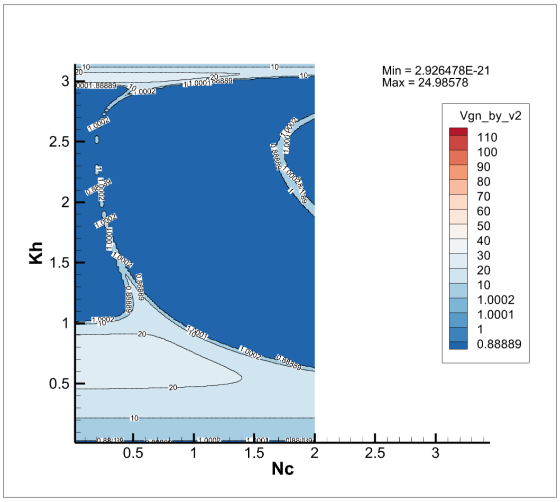

Figure 8: $Vg/V2$ contours for $Pe = 2$.

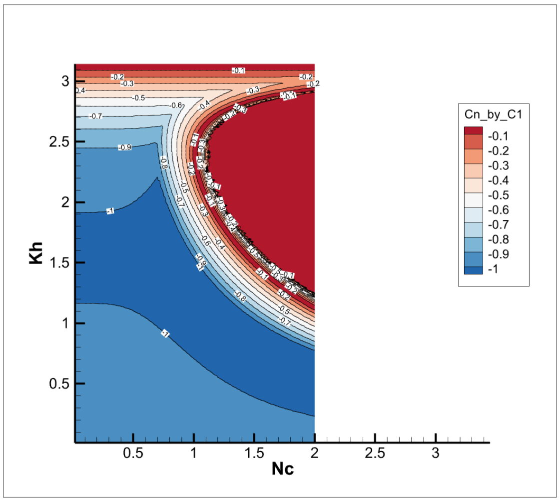

Figure 9: $Cn/C1$ contours for $Pe = 0.02$.

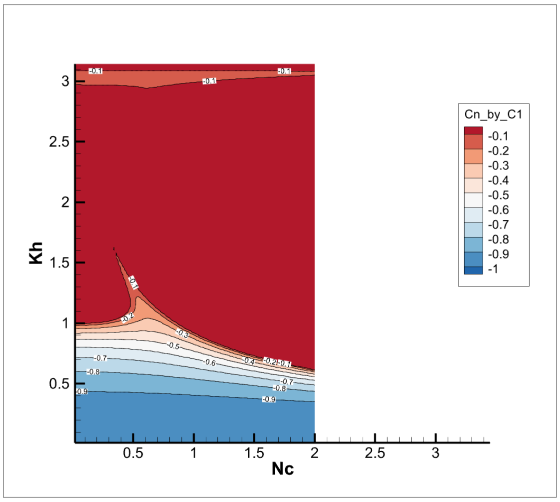

Figure 10: $Cn/C1$ contours for $Pe = 2$.

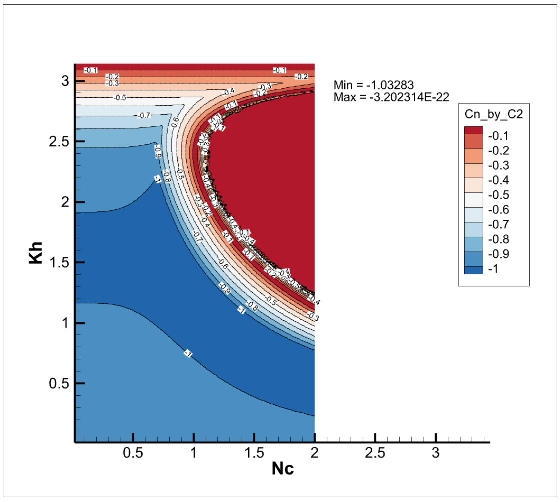

Figure 11: $Cn/C2$ contours for $Pe = 0.02$.

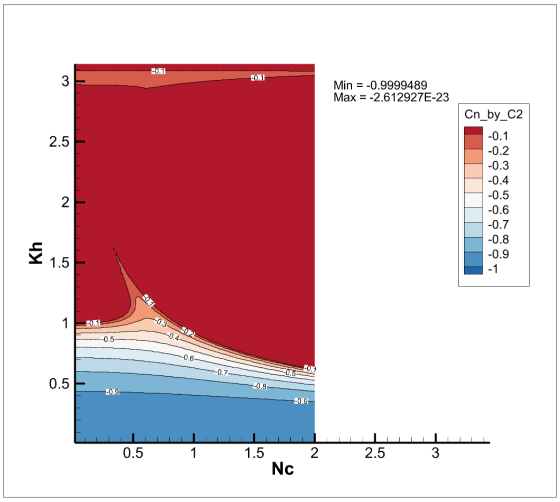

Figure 12: $Cn/C2$ contours for $Pe = 2$.

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