Principles of Mathematical Analysis Notes

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1 Chapter 1 Exercises

1.1

Proof. To prove (a) by contradiction, let $r + x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$. It follows that $x = \frac{p-rq}{q}$ which contradicts $x \in \mathbb{I}$. Thus, $r + x \in \mathbb{I}$.
Similarly, to prove (b) by contradiction, let $rx = \frac{p}{q}$. It follows that $x = \frac{p}{qr}$ which contradicts $x \in \mathbb{I}$. Thus, $rx \notin \mathbb{Q}$. $\square$

1.2

Proof. To prove this by contradiction, assume $\frac{p^2}{q^2} = 12$. It follows that

$$p^2 = 12q^2 = 2^2 \cdot 3^1 \cdot q^2.$$

By the fundamental theorem of arithmetic, p^2 must factor into a product of primes of even multiplicity. By the same argument, q^2 must factor into a product of an even multiplicity of 3, contradicting the unique factorization of p. Therefore, the assumption is false and $\sqrt{12} \in \mathbb{I}$. $\square$

1.3

Proof. To prove (a), by (M5)

$$\frac{1}{x}xy = \frac{1}{x}xz = y = z.$$

$\square$

Proof. To prove (b), by (M4)

$$x1 = x = xy.$$

By (a), $1 = y$. $\square$

Proof. To prove (c), by (M5)

$$x\frac{1}{x} = 1 = xy.$$

By (a), $y = \frac{1}{x}$. $\square$

Proof. To prove (d), by (M5)

$$\frac{1}{x}1/(1/x) = 1 = \frac{1}{x}x.$$

By (a), $1/(1/x) = x$. $\square$

1.4

Proof. By the definition of lower bound, $\alpha \leq x$ for every $x \in E$. By the definition of upper bound, $x \leq \beta$. Combining inequalities, $\alpha \leq x \leq \beta$, which implies $\alpha \leq \beta$. $\square$

1.5

Proof. Since A is bounded below, let $\alpha = \inf A$. By the definition of greatest lower bound, $x \geq \alpha$ for all $x \in A$. It follows that $-x \leq -\alpha$ for all $-x \in A$. Let $-x = y$ for some $y \in -A$. Therefore, $y \leq -\inf A = \sup -A$, for all $y \in -A$. It follows that

$$(-1) - \inf A = (-1)\sup -A = \inf A = -\sup -A.$$

$\square$

1.6

Proof. To prove (a), first notice that $m = rn$. By Corollary 1.21, it follows that

$$(b^m)^{\frac{1}{n}} = (b^{rn})^{\frac{1}{n}} = b^r.$$

Theorem 1.21 shows that $b^r = (b^m)^{\frac{1}{n}}$. $\square$

Proof. To prove (b) by Corollary 1.21, it follows that

$$(b^r b^s)^{\frac{1}{r}} = b \cdot b^{\frac{s}{r}} = b^{\frac{s}{r}+1} = b^{\frac{s+r}{r}} = (b^{s+r})^{\frac{1}{r}}.$$

Thus, by Theorem 1.21, $b^r b^s = b^{r+s}$. $\square$

Proof. To prove (c), first we will prove that $B(x)$ has the least upper bound property. Let $\varepsilon > 0$, $b = 1 + \varepsilon$, and $t = x$. It follows that $b^t \leq b^x$, therefore $b^t \in B(x)$ and $B(x)$ is not empty. Next, notice that $x < x + \varepsilon$. Since $b > 1$,

$$b^{x-\varepsilon} < b^t \leq b^x < b^{x+\varepsilon}.$$

Notice that $b^{x+\varepsilon} \notin B(x)$ and $b^{x-\varepsilon}$ is not an upper bound, because $b^{x+\varepsilon} \nleq b^x$ and $b^{x-\varepsilon} < b^x \in B(x)$. By Definition 1.8, $b^x = \sup B(x)$ for any $x \in \mathbb{R}$. $\square$

Proof. To prove (d), ... $\square$

1.7

Proof. To prove (a) using the equality

$$b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + ... + a^{n-1})$$

where $a = 1$, yields

$$b^n - 1 = (b - 1)(b^{n-1} + b^{n-2} + ... + 1).$$

Notice that there are n terms in right side of the equality. Since $b > 1$, it follows by (D) that

$$b^n - 1 = b^{n-1}(b - 1) + b^{n-2}(b - 1) + ... + (b - 1) \geq n(b - 1).$$

$\square$

Proof. To prove (b) using the same equality, let $b = \beta^n$. Substituting β for b

$$b - 1 = \beta^n - 1 = (\beta - 1)(\beta^{n-1} + \beta^{n-2} + ... + 1),$$

Notice that $\beta = b^{\frac{1}{n}}$. Substituting

$$(b^{\frac{1}{n}} - 1)((b^{\frac{1}{n}})^{n-1} + (b^{\frac{1}{n}})^{n-2} + ... + 1) \geq n(b^{\frac{1}{n}} - 1),$$

is true because $b^x > 1$ for all $x \in \mathbb{R}$. □

Proof. To prove (c), first divide $n(b^{\frac{1}{n}} - 1) < b - 1$ by the assumed inequality $n(t-1) < b - 1$, giving $\frac{b^{\frac{1}{n}}-1}{t-1} < 1$. It follows that $b^{\frac{1}{n}} < t$. □

Proof. To prove (d), let $t = yb^{-w}$. Notice that $t > 1$ from the antecedent. It follows from (c) that $b^{\frac{1}{n}} < yb^{-w}$, and thus $b^{\frac{1}{n}}b^w < y$. By exercise 6(b), $b^{\frac{1}{n}+w} < y$. □

2 Addition in the Real Number Field

Proof. Let α and β be cuts, such that $\alpha \subset \beta$. Let $r \in \alpha$ and $s \in \beta$. The cut defined by $\alpha + \beta$ is thus the set of all $r + s$. Since $\alpha \in \beta$, by (II), $r - s \in \alpha$. Since $r = r - s + s$, we can say $(r - s) + (s) \in \alpha + \beta$ and therefore $r \in \alpha + \beta$. □

Proof. To verify that $\alpha + \beta$ satisfies (II), for some $r' \in \alpha$ such that $r < r'$ and $s' \in \beta$ such that $s < s'$. It follows that $r + s < r' + s' \in \alpha + \beta$. □