Principles of Mathematical Analysis Notes

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1 Chapter 1 Exercises

1.1

Proof. To prove (a) by contradiction, let $r + x = \frac{p}{q}$ for some $p, q \in \mathbb{Z}$. It follows that $x = \frac{p-rq}{q}$ which contradicts $x \in \mathbb{I}$. Thus, $r + x \in \mathbb{I}$.
Similarly, to prove (b) by contradiction, let $rx = \frac{p}{q}$. It follows that $x = \frac{p}{qr}$ which contradicts $x \in \mathbb{I}$. Thus, $rx \notin \mathbb{Q}$. $\square$

1.2

Proof. To prove this by contradiction, assume $\frac{p^2}{q^2} = 12$. It follows that

$$p^2 = 12q^2 = 2^2 \cdot 3^1 \cdot q^2.$$

By the fundamental theorem of arithmetic, p^2 must factor into a product of primes of even multiplicity. By the same argument, q^2 must factor into a product of an even multiplicity of 3, contradicting the unique factorization of p. Therefore, the assumption is false and $\sqrt{12} \in \mathbb{I}$. $\square$

1.3

Proof. To prove (a), by (M5)

$$\frac{1}{x}xy = \frac{1}{x}xz = y = z.$$

$\square$

Proof. To prove (b), by (M4)

$$x1 = x = xy.$$

By (a), $1 = y$. □

Proof. To prove (c), by (M5)

$$x\frac{1}{x} = 1 = xy.$$

By (a), $y = \frac{1}{x}$. □

Proof. To prove (d), by (M5)

$$\frac{1}{x}1/(1/x) = 1 = \frac{1}{x}x.$$

By (a), $1/(1/x) = x$. □

1.4

Proof. By the definition of lower bound, $\alpha \leq x$ for every $x \in E$. By the definition of upper bound, $x \leq \beta$. Combining inequalities, $\alpha \leq x \leq \beta$, which implies $\alpha \leq \beta$. □

1.5

Proof. Since A is bounded below, let $\alpha = \inf A$. By the definition of greatest lower bound, $x \geq \alpha$ for all $x \in A$. It follows that $-x \leq -\alpha$ for all $-x \in A$. Let $-x = y$ for some $y \in -A$. Therefore, $y \leq -\inf A = \sup -A$, for all $y \in -A$. It follows that

$$(-1) - \inf A = (-1) \sup -A = \inf A = -\sup -A.$$

□

2 Addition in the Real Number Field

Proof. Let α and β be cuts, such that $\alpha \subset \beta$. Let $r \in \alpha$ and $s \in \beta$. The cut defined by $\alpha + \beta$ is thus the set of all $r + s$. Since $\alpha \in \beta$, by (II), $r - s \in \alpha$. Since $r = r - s + s$, we can say $(r - s) + (s) \in \alpha + \beta$ and therefore $r \in \alpha + \beta$. $\square$

Proof. To verify that $\alpha + \beta$ satisfies (II), for some $r' \in \alpha$ such that $r < r'$ and $s' \in \beta$ such that $s < s'$. It follows that $r + s < r' + s' \in \alpha + \beta$. $\square$