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Category Theory

Chapter 1

Problem 9.1. The objects of **Rel** are sets, and an arrow $A \to B$ is a relation from A to B, that is, a subset $R \subseteq A \times B$. The equality relation $\{\langle a, a\rangle \in A \times A \mid a \in A\}$ is the identity arrow on a set A. Composition in Rel is to be given by

$$S \circ R = \{\langle a, c\rangle \in A \times C \mid \exists b(\langle a, b\rangle \in R \& \langle b, c\rangle \in S)\}$$

for $R \subseteq A \times B$ and $S \subseteq B \times C$.

Solution. (a) Show that Rel is a category.
Composability:
Shown above.

Identity:
Shown above.

Associativity:
Let $D \in \mathbf{Rel}$ such that $A \xrightarrow{R} B \xrightarrow{S} C \xrightarrow{T} D$. Consider $\langle a, d\rangle \in T \circ (S \circ R)$. I need to show that $\langle a, d\rangle \in (T \circ S) \circ R$. By composition, there exists a c such that $\langle a, c\rangle \in S \circ R$ and $\langle c, d\rangle \in T$. Again, by composition, there exists a b such that $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in S$. Since $\langle c, d\rangle \in T$, I have that $\langle b, d\rangle \in T \circ S$. Finally, since $\langle a, b\rangle \in R$, then $\langle a, d\rangle \in (T \circ S) \circ R$.
Showing $(T \circ S) \circ R \subset T \circ (S \circ R)$ is done symmetrically.

Unit:
Let $\langle a, b\rangle \in R$. By *identity*, $\langle a, a\rangle \in 1_A$ and $\langle b, b\rangle \in 1_B$.
Hence, $\langle a, b\rangle \in R \circ 1_A, 1_B \circ R$. If $\langle a, b\rangle \in R \circ 1_A, 1_B \circ R$; then definitionally $\langle a, b\rangle \in R$. This shows that $R \circ 1_A = R = 1_B \circ R$.

(b) Show also that there is a functor G : Sets $\to$ Rel taking objects to themselves and each function $f : A \to B$ to its graph,

$$G(f) = \{\langle a, f(a)\rangle \in A \times B \mid a \in A\}$$

I wish to show that

$$G(f : A \to B) = G(f) : G(A) \to G(B)$$

for any function f. By the definition of G

$$G(f) : G(A) \to G(B) = G(f) : A \to B.$$

By the definition of $G(f)$, it is clear that $G(f) \subseteq A \times B$. Thus, $G(f) : A \to B$ is an arrow in **Rel**.
Now I wish to show that $G(1_A) = 1_{G(A)}$:

$$\begin{aligned} G(1_A) &= \{\langle a, 1_A(a)\rangle \in A \times A \mid a \in A\} \\ &= \{\langle a, a\rangle \in A \times A \mid a \in A\} && (1_A(a) = a) \\ &= 1_A \\ &= 1_{G(A)} && (G(A) = A) \end{aligned}$$

Finally, I wish to show

$$G(fg) = \{\langle a, fg(a)\rangle \in A \times C \mid a \in A\} \text{ and}$$
$$GfGg = \{\langle a, c\rangle \in A \times C \mid \exists b(\langle a, b\rangle \in Gg \& \langle b, c\rangle \in Gf)\}$$

where $A \xrightarrow{g} B \xrightarrow{f} C$ are any two arrows in **Sets**. If either set is empty, then A or C is empty. In that case, both sets are empty and equal. Let $\langle a, fg(a)\rangle \in G(fg)$. By taking $g(a) \in B$, it's clear that $\langle a, fg(a)\rangle \in GfGg$. Now let $\langle a, c\rangle \in GfGg$. There then exists some $b \in B$ such that $\langle a, b\rangle \in Gg$. By definition, $b = g(a)$ and hence $c = fg(a)$. Thus, $\langle a, c\rangle = \langle a, fg(a)\rangle \in G(fg)$.
(c) Finally, show that there is a (contravariant)functor $K : \mathbf{Rel}^{\mathrm{Op}} \to \mathrm{Rel}$ taking each relation $R \subseteq A \times B$ to its converse $R^c \subseteq B \times A$, where,

$$\langle a, b\rangle \in R^c \Leftrightarrow \langle b, a\rangle \in R$$

Consider

$$\begin{aligned} K(SR) &= K(\{\langle a, c\rangle \in A \times C \mid \exists b(\langle a, b\rangle \in R \,\&\, \langle b, c\rangle \in S)\}) \\ &= \{\langle c, a\rangle \in C \times A \mid \exists b(\langle c, b\rangle \in S \,\&\, \langle b, a\rangle \in R)\} \\ &= \{\langle c, a\rangle \in C \times A \mid \exists b(\langle c, b\rangle \in KS \,\&\, \langle b, a\rangle \in KR)\} \\ &= KRKS. \end{aligned}$$

□

Problem 9.2. (a)$\mathbf{Rel^{op}} \ncong \mathbf{Rel}$
Since isos must be covariant by definition, exercise 1.9.1.c above shows this.
(b)$\mathbf{Sets^{op}} \ncong \mathbf{Sets}$
Consider $F : \mathbf{Sets^{op}} \to \mathbf{Sets}$ such that $F(S) = S$ for all sets S and $F(f : A \to B) = f^* : B \to A$. Given $A \xrightarrow{f} B \xrightarrow{g} C$, I have that $F(g \circ f) = f^* \circ g^* = F(f) \circ F(g)$. Hence F is contravariant and the opposite cateogories are not iso.

Solution. □

Problem 9.3. (a) Show that in Sets, the isomorphisms are exactly the bijections. (b) Show that in Monoids, the isomorphisms are exactly the bijective homomorphisms. (c) Show that in Posets, the isomorphisms are not the same as the bijective homomorphisms.

Solution. (c) You call functions homos when the sets are enhanced with some structure and the functions are structure preserving. For posets, monotonic functions are homomorphisms. □