Tommy O'Shaughnessy
Steve Awodey
Category Theory

Chapter 1

Problem 9.1. The objects of **Rel** are sets, and an arrow $A \to B$ is a relation from A to B, that is, a subset $R \subseteq A \times B$. The equality relation $\{\langle a, a\rangle \in A \times A \mid a \in A\}$ is the identity arrow on a set A. Composition in Rel is to be given by

$$S \circ R = \{\langle a, c\rangle \in A \times C \mid \exists b(\langle a, b\rangle \in R \& \langle b, c\rangle \in S)\}$$

for $R \subseteq A \times B$ and $S \subseteq B \times C$.

Solution. (a) Show that Rel is a category.
Composability:
Shown above.

Identity:
Shown above.

Associativity:
Let $D \in \mathbf{Rel}$ such that $A \xrightarrow{R} B \xrightarrow{S} C \xrightarrow{T} D$. Consider $\langle a, d\rangle \in T \circ (S \circ R)$. I need to show that $\langle a, d\rangle \in (T \circ S) \circ R$. By composition, there exists a c such that $\langle a, c\rangle \in S \circ R$ and $\langle c, d\rangle \in T$. Again, by composition, there exists a b such that $\langle a, b\rangle \in R$ and $\langle b, c\rangle \in S$. Since $\langle c, d\rangle \in T$, I have that $\langle b, d\rangle \in T \circ S$. Finally, since $\langle a, b\rangle \in R$, then $\langle a, d\rangle \in (T \circ S) \circ R$.
Showing $(T \circ S) \circ R \subset T \circ (S \circ R)$ is done symmetrically.

Unit:
Let $\langle a, b\rangle \in R$. By *identity*, $\langle a, a\rangle \in 1_A$ and $\langle b, b\rangle \in 1_B$.
Hence, $\langle a, b\rangle \in R \circ 1_A, 1_B \circ R$. If $\langle a, b\rangle \in R \circ 1_A, 1_B \circ R$; then definitionally $\langle a, b\rangle \in R$. This shows that $R \circ 1_A = R = 1_B \circ R$.

(b) Show also that there is a functor G : Sets $\to$ Rel taking objects to themselves and each function $f : A \to B$ to its graph,

$$G(f) = \{\langle a, f(a)\rangle \in A \times B \mid a \in A\}$$

I wish to show that

$$G(f : A \to B) = G(f) : G(A) \to G(B)$$

for any function f. By the definition of G

$$G(f) : G(A) \to G(B) = G(f) : A \to B.$$

By the definition of $G(f)$, it is clear that $G(f) \subseteq A \times B$. Thus, $G(f) : A \to B$ is an arrow in **Rel**.
Now I wish to show that $G(1_A) = 1_{G(A)}$:

$$\begin{aligned} G(1_A) &= \{\langle a, 1_A(a)\rangle \in A \times A \mid a \in A\} \\ &= \{\langle a, a\rangle \in A \times A \mid a \in A\} && (1_A(a) = a) \\ &= 1_A \\ &= 1_{G(A)} && (G(A) = A) \end{aligned}$$

Finally, I wish to show

$$G(fg) = \{\langle a, fg(a)\rangle \in A \times C \mid a \in A\} \text{ and}$$
$$GfGg = \{\langle a, c\rangle \in A \times C \mid \exists b(\langle a, b\rangle \in Gg \& \langle b, c\rangle \in Gf)\}$$

where $A \xrightarrow{g} B \xrightarrow{f} C$ are any two arrows in **Sets**. If either set is empty, then A or C is empty. In that case, both sets are empty and equal. Let $\langle a, fg(a)\rangle \in G(fg)$. By taking $g(a) \in B$, it's clear that $\langle a, fg(a)\rangle \in GfGg$. Now let $\langle a, c\rangle \in GfGg$. There then exists some $b \in B$ such that $\langle a, b\rangle \in Gg$. By definition, $b = g(a)$ and hence $c = fg(a)$. Thus, $\langle a, c\rangle = \langle a, fg(a)\rangle \in G(fg)$.
(c) Finally, show that there is a functor $K : \mathbf{Rel}^{\mathrm{Op}} \to \mathrm{Rel}$ taking each relation $R \subseteq A \times B$ to its converse $R^c \subseteq B \times A$, where,

$$\langle a, b\rangle \in R^c \Leftrightarrow \langle b, a\rangle \in R$$

Consider

$$\begin{aligned} K(SR) &= K(\{\langle a, c\rangle \in A \times C \mid \exists b(\langle a, b\rangle \in R \,\&\, \langle b, c\rangle \in S)\}) \\ &= \{\langle c, a\rangle \in C \times A \mid \exists b(\langle c, b\rangle \in S \,\&\, \langle b, a\rangle \in R)\} \\ &= \{\langle c, a\rangle \in C \times A \mid \exists b(\langle c, b\rangle \in KS \,\&\, \langle b, a\rangle \in KR)\} \\ &= KSKR. \end{aligned}$$

□

Problem 9.2.

Solution. □