LINEAR ALGEBRA – KEY TERMS & MAIN RESULTS

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<tr><th>Key terms</th><th>Problems with solutions</th><th>Exercises - Do yourself</th></tr>
<tr><td colspan="3">Chapter 1. Systems of Linear Equations</td></tr>
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<td>Reduced row-echelon form</td>
<td>Ex. Find x and y such that the matrix
$\begin{pmatrix} 1 & 1 & -1 & 3 & 5 \\ 0 & x & 1 & 0 & -2 \\ 0 & y & x & 2 & 1 \end{pmatrix}$ is a reduced row-echelon matrix.
Solution.
Consider row 2, two possible cases for x’s value: 0 or 1
• x = 0 ➔ y = 0, so the 3rd row becomes [0 0 0 2 1], which is impossible.
• x = 1 ➔ y = 0 and row 3 is [0 0 1 2 1], which is possible.
Conclusion: x = 1 and y = 0.</td>
<td>1/ Find x and y such that the matrix
$\begin{pmatrix} 1 & 1 & -1 & 3 & 5 \\ 0 & y & x & 0 & -2 \\ 0 & 0 & x & 1 & -1 \end{pmatrix}$ is a reduced row-echelon matrix.</td>
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<td>Consistent and inconsistent system</td>
<td>Ex1. Solve the system
x + 2y + 3z = 0
2x + 4y – z = 0
x + 2y – z = 0
Solution.
• Step 1. Carry augmented matrix to reduced row-echelon form:
$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 2 & 4 & -1 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right) \xrightarrow[-r_1+r_3]{-2r_1+r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & -7 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right)$
$\xrightarrow{-\frac{1}{7}r_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right) \xrightarrow{4r_2+r_3} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$
$\xrightarrow{-3r_2+r_1} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$
• Step 2. From the last matrix, the system has infinitely many solution described as below:
y = t (parameter = any number) // no leading one with respect to y
z = 0
x = -2t
• Step 3. Conclusion: solution set is {(-2t, t, 0) where t is arbitrary}

Ex2. Find all values of m such that the system</td>
<td>2/ a/ Solve the system
x - y + 2z = 0
-x + y – z = 0
b/ Solve the system corresponding to the augmented matrix
$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -2 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

3/ Find all values of m such that the system
$\begin{cases} x - y + 2z = -1 \\ -y + z = 1 \\ x - y + mz = 0 \end{cases}$
has unique solution.</td>
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	$\begin{cases} x - y + 2z = 2 \\ -2x + y - z = -1 \\ x + y + mz = 0 \end{cases}$ has ***unique solution***. **Solution.** $\left[\begin{array}{ccc\|c} 1 & -1 & 2 & 2 \\ -2 & 1 & -1 & 1 \\ 1 & 1 & m & 0 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & -1 & 2 & 2 \\ 0 & -1 & 3 & 5 \\ 0 & 2 & m-2 & -2 \end{array}\right]$ $\to \left[\begin{array}{ccc\|c} 1 & -1 & 2 & 2 \\ 0 & 1 & -3 & -5 \\ 0 & 2 & m-2 & -2 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & -1 & 2 & 2 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & m+4 & 8 \end{array}\right]$ From the last matrix, the system has unique solution when m + 4 ≠ 0 Conclusion: m ≠ -4.	
Rank of a matrix r(A)	**Ex.** Find the ***rank*** of the matrix. $A = \begin{bmatrix} 0 & -2 & 1 & 3 \\ 1 & -2 & -1 & 1 \\ -2 & 2 & 3 & 1 \end{bmatrix}.$ **Solution.** In general, carry A to a row echelon matrix, and rank(A) = number of leading ones. $\begin{bmatrix} 0 & -2 & 1 & 3 \\ 1 & -2 & -1 & 1 \\ -2 & 2 & 3 & 1 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & -2 & 1 & 3 \\ -2 & 2 & 3 & 1 \end{bmatrix}$ $\xrightarrow{2r_1 + r_3} \begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & -2 & 1 & 3 \\ 0 & -2 & 1 & 3 \end{bmatrix} \xrightarrow{-r_2 + r_3} \begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ We can see the latest matrix can be carried to a row echelon matrix with ***2 leading ones***. So, rank(A) = 2.	**4/** Find the ***rank*** of the matrix. $A = \begin{bmatrix} 1 & -2 & 1 & -3 \\ -2 & 0 & -1 & 1 \\ 2 & 2 & -2 & 3 \end{bmatrix}.$
the number of free parameters p = n – r of a homogeneous system	**Ex.** A homogeneous system has the coefficient matrix of **rank 8**. If there are 11 linear equations involving **13 variables (or unknowns)** in the system, then how many ***free parameters*** in the solution set of the system? **Solution.** p: number of parameters n: number of variables r = rank of the coefficient matrix p = n – r = 13 – 8 = 5.	**5/** A homogeneous system has the coefficient matrix of **rank 7**. If there are 13 linear equations involving **15 variables (or unknowns)** in the system, then how many ***free parameters*** in the solution set of the system?

Chapter 2-3. Matrix Algebra

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<td>Matrix addition A + B, scalar multiplication (k.A) and transpose A^T

Matrix multiplication A·B

Matrix inverse A^{-1}</td>
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Ex1. Given A = $A=\begin{bmatrix}-2 & 1/2 & 3\\ 3/2 & -1 & 0\end{bmatrix}, B=\begin{bmatrix}1 & 0\\ -3 & 2\\ 1 & 5\end{bmatrix}$

Find 2A – B^T.

Solution.

$2A=\begin{bmatrix}-4 & 1 & 6\\ 3 & -2 & 0\end{bmatrix}$

$B^T=\begin{bmatrix}1 & -3 & 1\\ 0 & 2 & 5\end{bmatrix}$

So,

$2A-B^T=\begin{bmatrix}-4 & 1 & 6\\ 3 & -2 & 0\end{bmatrix}-\begin{bmatrix}1 & -3 & 1\\ 0 & 2 & 5\end{bmatrix}$

$=\begin{bmatrix}-5 & 4 & 5\\ 3 & -4 & -5\end{bmatrix}$

Ex2. Find $(2A)^{-1}$ if $A=\begin{bmatrix}-1 & 3\\ 2 & 3\end{bmatrix}$

Solution.

$A=\begin{bmatrix}-1 & 3\\ 2 & 3\end{bmatrix}\Rightarrow 2A=\begin{bmatrix}-2 & 6\\ 4 & 6\end{bmatrix}$

$\Rightarrow (2A)^{-1}=\frac{1}{-36}\begin{bmatrix}6 & -6\\ -4 & -2\end{bmatrix}=\begin{bmatrix}-1/6 & 1/6\\ 1/9 & 1/18\end{bmatrix}$

Another way.

$A=\begin{bmatrix}-1 & 3\\ 2 & 3\end{bmatrix}\Rightarrow A^{-1}=\frac{1}{-9}\begin{bmatrix}3 & -3\\ -2 & -1\end{bmatrix}$

$\Rightarrow (2A)^{-1}=\frac{1}{2}A^{-1}=\begin{bmatrix}-1/6 & 1/6\\ 1/9 & 1/18\end{bmatrix}$

Ex3. Find A if $\left(A^T-2I\right)^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}$.

Solution.

$\left(A^T-2I\right)^{-1}=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}\Leftrightarrow A^T-2I=\begin{bmatrix}1 & 3\\ 1 & 4\end{bmatrix}^{-1}=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}$

$A^T=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}+2I=\begin{bmatrix}4 & -3\\ -1 & 1\end{bmatrix}+2\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

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6/ Find A^{-1} if

$A=\begin{bmatrix}-1 & -2\\ 3 & 3\end{bmatrix}$

7/ Find A if

$\left(A^T-2I\right)^{-1}=\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}$.

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	$A^T = \begin{bmatrix} 6 & -3 \\ -1 & 3 \end{bmatrix}$ $\Rightarrow A = \begin{bmatrix} 6 & -1 \\ -3 & 3 \end{bmatrix}$	
Invertible and determinant	**Ex.** Find all values of x such that the matrix $\begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & -3 \\ 1 & x & 1 \end{bmatrix}$ has **an inverse**. **Solution.** $\det\begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & -3 \\ 1 & x & 1 \end{bmatrix} = -x-10$ A has an inverse iff det(A) ≠ 0 ⇔ x ≠ -10.	**8/** Find all values of x such that the matrix $\begin{bmatrix} 1 & -2 & 1 \\ -2 & 0 & 3 \\ -1 & 2 & x \end{bmatrix}$
Linear transformations $T\left(a\vec{u}+b\vec{v}\right)$ $= aT\left(\vec{u}\right)+bT\left(\vec{v}\right)$	**Ex.** Let T: R^2 → R^2 be a **linear transformation** such that T(u) = (-1, 2) and T(v) = (-1, 1) Find T(2u – 3v). **Solution.** T(2u – 3v) = 2T(u) – 3T(v) = 2(-1, 2) -3(-1, 1) = (1, 1).	**9/** Let T: R^2 → R^2 be a **linear transformation** such that T(u) = (1, -2) and T(v) = (1, 2) Find T(3u – 2v).
Determinants of 2x2, 3x3, 4x4 matrices det(A)	**Ex.** Find $\begin{vmatrix} a & -2 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & -2 \end{vmatrix}$ **Solution.** $\begin{vmatrix} a & -2 & 0 \\ 1 & 1 & 0 \\ 2 & 3 & -2 \end{vmatrix} = -2(-1)^{3+3}\det\begin{bmatrix} a & -2 \\ 1 & 1 \end{bmatrix} = -2(a+2)$	**10/** Given $A = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & k \\ 0 & 1 & -3 \end{bmatrix}$ a/ Find det(A). b/ Find k such that A has an inverse.
Properties of determinants	**Ex.** Suppose A and B are 3x3 matrices such that |A| = 3, |B| = -6. a/ Find $\|2AB^{-1}\|$ b/ Find $\|3A^TBA^{-2}\|$ **Solution.** a/ $\|2AB^{-1}\| = 2^3\|A\|\frac{1}{\|B\|} = \frac{8\cdot 3}{-6} = -4$ b/ $\|3A^TBA^{-2}\|$ = $3^3\|A^T\|\|B\|\frac{1}{\|A\|^2} = 3^3\|A\|\|B\|\frac{1}{\|A\|^2} = \frac{3^3\cdot(-6)}{3} = -54$	**11/** Suppose A and B are 4x4 matrices such that |A| = -2, |B| = 3. a/ Find $\|2AB^T\|$ b/ Find $\|A^2B^{-1}A^{-1}\|$
(i, j)-cofactor and A^{-1}. **$(-1)^{i+j}$ det(delete row i, delete**	**Ex.** Find (2, 3)-cofactor and (3, 1)-cofactor of A if	**12/** Find (2, 3)-cofactor and (3, 1)-cofactor of A if

column j)	$A=\begin{bmatrix}-1&2&1\\2&0&-3\\1&2&1\end{bmatrix}$ **Solution.** (2, 3)-cofactor = $c_{23}=(-1)^{2+3}\det\begin{pmatrix}-1&2\\1&2\end{pmatrix}=4$ (3, 1)-cofactor = $c_{31}=(-1)^{3+1}\det\begin{pmatrix}2&1\\0&-3\end{pmatrix}=-6$	$A=\begin{bmatrix}-1&2&5\\2&3&-1\\-1&0&2\end{bmatrix}$
Adjugate matrix	**Ex.** Find the first row of the adjugate of $\begin{bmatrix}1&-1&2\\3&0&5\\0&1&-1\end{bmatrix}$ **Solution.** Find the $\boxed{\text{first row}}$ of the adjugate of $\begin{bmatrix}\boxed{1}&-1&2\\\boxed{3}&0&5\\\boxed{0}&1&-1\end{bmatrix}$ Solution. The first row of **adjugate matrix** of A is **cofactors** c_{11}, c_{21}, c_{31}: $\boxed{c_{11}}=(-1)^{1+1}\det\begin{pmatrix}0&5\\1&-1\end{pmatrix}=-5$ $\boxed{c_{21}}=(-1)^{2+1}\det\begin{pmatrix}-1&2\\1&-1\end{pmatrix}=1$ $\boxed{c_{31}}=(-1)^{3+1}\det\begin{pmatrix}-1&2\\0&5\end{pmatrix}=-5$ The first row of adj(A) is: [-5 1 -5].	13/ Find the second row of the adjugate matix of $A=\begin{bmatrix}-1&2&5\\2&3&-1\\-1&0&2\end{bmatrix}$
eigenvalues	**Ex.** Find all **eigenvalues** of the matrix of the matrix $\begin{bmatrix}1&1&-1\\0&0&-1\\0&2&-3\end{bmatrix}$ **Solution.** • det(xI – A) = $\begin{vmatrix}x-1&-1&1\\0&x&1\\0&-2&x+3\end{vmatrix}=(x-1)\left[x(x+3)+2\right]$ $=(x-1)(x^2+3x+2)$ • det(xI – A) = 0 ⇔ x = 1, x = -1, x = -2. • Eigenvalues: 1, -1, -2	14/ Find all **eigenvalues** of the matrix of the matrix $\begin{bmatrix}1&-1&4\\3&2&-1\\2&1&-1\end{bmatrix}$ Choose one from options below (i) -2, 1, 3 (ii) 2, -1, -3 (iii) -1, -2, 3 (iv) 2, 1, -1

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<td>* Note that for a multiple choice question: first we can find det(A) = 2.
Then, choose options which the product of values is 2
a) $2, 3, 4$ b) $-3, 3, 4$
d) $-3, 0, 4$ e) $-1, -2, 1$
e) is the possible option because det(A) = 2 = (-1)(-2).1</td>
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Chapter 5. The Vector Space R^n

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<td>Linear independence, Linear dependence</td>
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Ex1. Find all values of x such that the set {(1, 0, -2); (-2, 1, 1); (1, -3, x)} is linearly **independent**.

Solution.

- We solve the system for a, b, c

a(1, 0, -2) + b(-2, 1, 1) + c(1, -3, x) = (0, 0, 0)

Equivalently, in augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ -2 & 1 & x & 0 \end{array}\right]$$

- Carry the matrix to row-echelon form

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ -2 & 1 & x & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & x-2 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & x-11 & 0 \end{array}\right]$$

We want the set is linearly independent, so the system must have solution a = 0, b = 0, c = 0.

⇨ $x - 10 \neq 0$.

Ex2. Find all values of a such that the set {(1, -1, 1); (2, 1, 3); (-1, a, 2)} is linearly **dependent**.

Solution.

Similar to the previous example, solve the system

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & 3 & a & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & a+1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & a+1 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & a+1 & 0 \\ 0 & 0 & 1-3(a+1) & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & a+1 & 0 \\ 0 & 0 & -3a-2 & 0 \end{array}\right]$$

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15/ Find all values of x such that the set {(1, -1, 2); (-2, 0, 1); (-1, x, 3)} is linearly **independent**.

16/ Find all values of a such that the set {(1, 1, 0); (2, 1, 3); (-1, 0, a)} is linearly **dependent**.

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	We need values of a such that the set is linearly **dependent** ➔ -3a – 2 = 0 ⇔ a = -2/3.	
Spanning sets, span	**Ex.** Given U = span{(-1, 0, 1); (2, -1, 1)}. a/ Does the vector (1, -2, 3) belong to U? b/ Find all values of m such that (-2, 2, m)$\in$ U. **Solution.** a/ We want to find a, b such that (1, -2, 3) = a(-1, 0, 1) + b(2, -1, 1) Or equivalent, 1 = -a + 2b (1) -2 = 0a – b (2) 3 = a + b (3) Solve for a, b from (1), (2) ➔ a = 3, b = 2 ⇨ (3) becomes: 3 = 5 (!) Conclusion: vector (1, -2, 3) does not belong to U. b/ (-2, 2, m)$\in$ U if and only if the system (-2, 2, m) = a(-1, 0, 1) + b(2, -1, 1) has solution a, b. Or equivalent, -2 = -a + 2b (1) 2 = 0a – b (2) m = a + b (3) Solve for a, b from (1), (2) ➔ a = -2, b = -2 ⇨ (3) becomes: m = -4 Conclusion: m = -4	**17/** Given U = span{(1, -1, 0); (-2, 1, 1)}. Find all values of m such that (0, -1, m)$\in$ U.
Basis of a vector space, Dimension	**Ex1.** Given U = span{(1, 2, 1); (3, 2, 0); (-1, 2, 2)}. Find the **dimension** of U (find dim(U)). **Solution.** First, check for independence of the set {(1, 2, 1); (3, 2, 0); (-1, 2, 2)} $\left[\begin{array}{ccc\|c} 1 & 3 & -1 & 0 \\ 2 & 2 & 2 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & 3 & -1 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$ $\to \left[\begin{array}{ccc\|c} 1 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & 3 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ From the last matrix, the set is NOT INDEPENDENT. Only **two vectors** make an independent set ➔ Two vectors are chosen to form a basis of U ➔ dim(U) = 2. **Ex2.** Find all values of x such that dim(V) = 2 where V = span{(1, -1, 2); (-1, 0, 3); (2, -3, x)}. **Solution.**	**18/** Given U = span{(1, 2, 0); (-3, 1, 1); (1, 3, -1)}. Find the **dimension** of U (find dim(U)). **19/** Given U = span{(1, 2, 0, 1); (-3, 0, 1, -2); (1, 1, -1, 3)}. Find the **dimension** of U (find dim(U)).

	$\left[\begin{array}{ccc\|c} 1 & -1 & 2 & 0 \\ -1 & 0 & -3 & 0 \\ 2 & 3 & x & 0 \end{array}\right] \to \left[\begin{array}{ccc\|c} 1 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 5 & x-4 & 0 \end{array}\right]$ $\left[\begin{array}{ccc\|c} 1 & -1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & x-9 & 0 \end{array}\right]$ dim(V) = 2 if and only if x = 9.	
Column space Col(A) and row space row(A)	**Ex.** Find **dim(col(A))** if A = $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ -2 & 6 & -4 & 0 \end{bmatrix}$. **Solution.** $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ -2 & 6 & -4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 2 & -4 & 2 \end{bmatrix}$ $\to \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ⇨ Dim(col(A)) = rank(A) = 3.	**20/** Find **dim(col(A))** if A $= \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 2 & -3 & 4 & 2 \end{bmatrix}$.

END OF PART II – LINEAR ALGEBRA