MONASH University

Information Technology

FIT1006
Business Information Analysis

Lecture 10
Probability (cont…)

Topics covered:

- Independent and conditional events.
- Probability trees.
- Bayes' Theorem.
- *Notes on background concepts for Bayes' Theorem.
- *Notes on background mathematics for probability distributions.

https://flux.qa (Feed code: SJ6KGV)

Question 1. Motivating problem

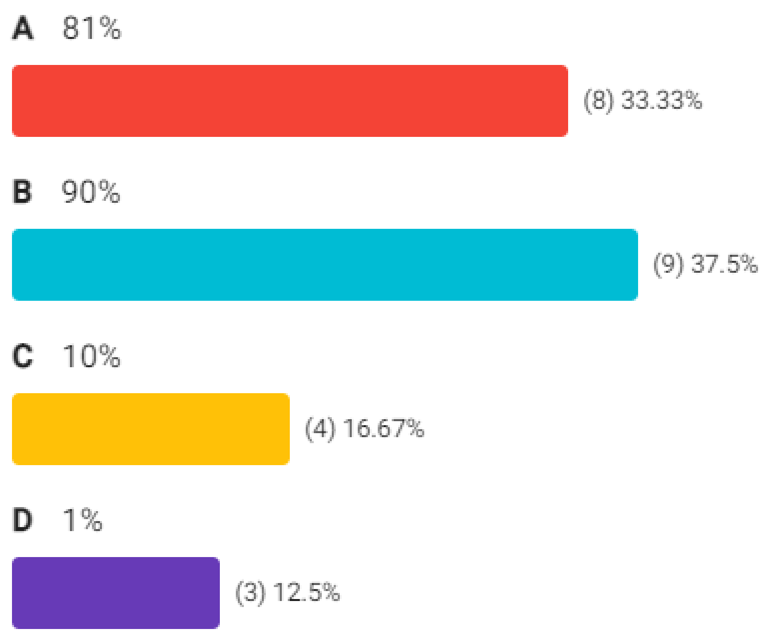
- The probability of disease X in the population is 1%
- If a person has disease X the probability they will test positive is 90%
- If a person doesn't have disease X the probability they will test positive is 9%

(Adapted from: Gigerenzer, G. et al, Knowing your chances. *Sci Am Mind*, April/May 2009)

- I have just been tested and the test is positive. What is the probability I have disease X?

Who is correct?

We will answer this later (Slide 13)



Today’s lecture

- How do we incorporate additional information in to our estimates of a probability?
- For the motivating problem, without knowing the test results there is a 0.01 chance that a person selected at random has disease X.
- If that person tested positive then we would expect the probability the person had the disease to increase. Similarly, it should decrease for a negative test.
- Bayes’ theorem gives us a tool for calculating these probabilities.

Bayes' Theorem

- Bayes' Theorem is a method for updating the probability of an event when the occurrence of that event is affected (conditional) on another event.
- The stages of a Bayesian problem:

1. Start with the *Prior* probability – this is the probability of an event in the absence of any other information. Sometimes called the *state of nature*.

   Probability of getting disease X is 1%

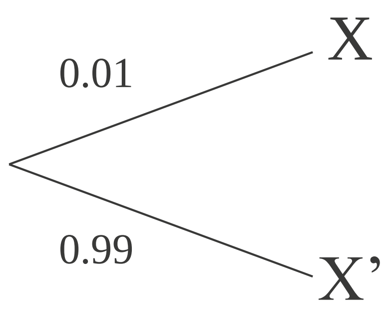
2. Receive additional information as *conditional probabilities*.

   doesn't have disease X the probability of being tested positive is 9%

3. Update the Prior probability using the additional information to determine the *Posterior* probability.

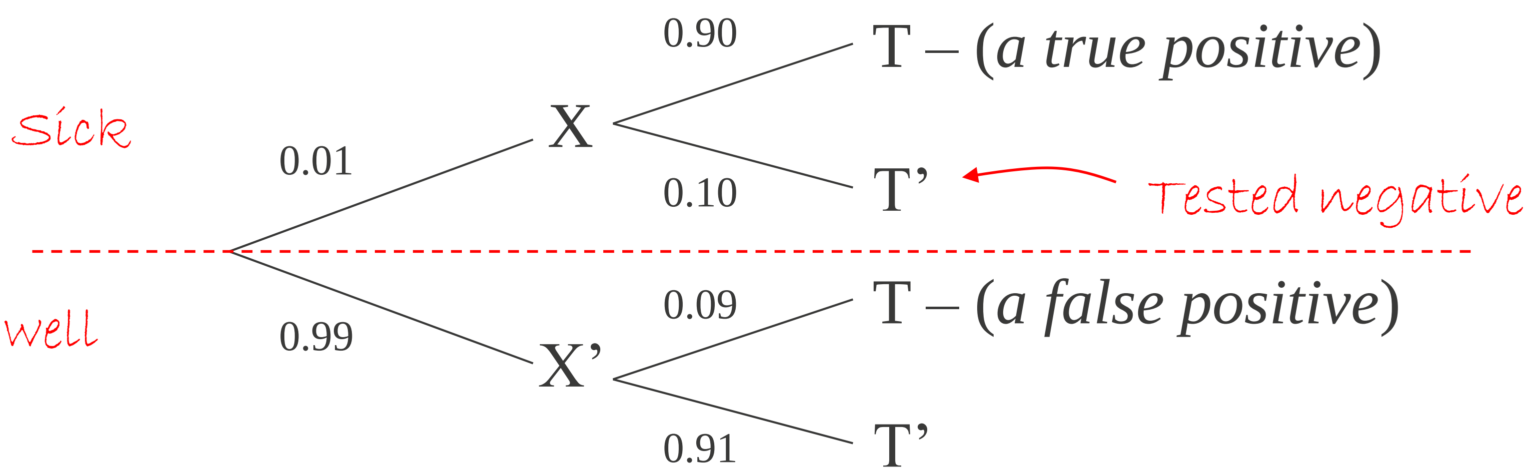
Prior probability

- Draw the motivating problem as a probability tree.
- Use X to indicate person has disease,
- ’ denotes complement.
- The first stage of the tree reflects the *state of nature*.



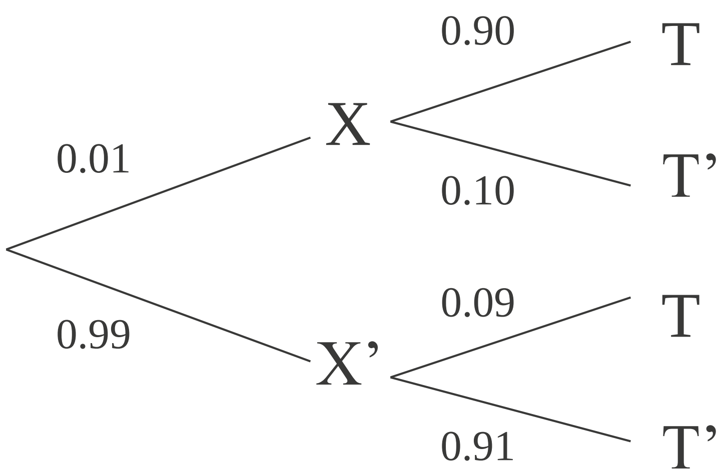
Conditional probabilities

- Use T to indicate a positive test result.
- Recall that: P(T|X) = 0.90 and P(T|X’) = 0.09.
- That is, the probability of positive result is conditional on a person’s disease state.



Joint probabilities

- Probabilities for each of the 4 situations corresponding to disease status *and* test outcome are evaluated. These are the *joint* probabilities.



$\rightarrow$ $\mathbf{X \cap T}$ **= 0.01 * 0.90 = 0.009**

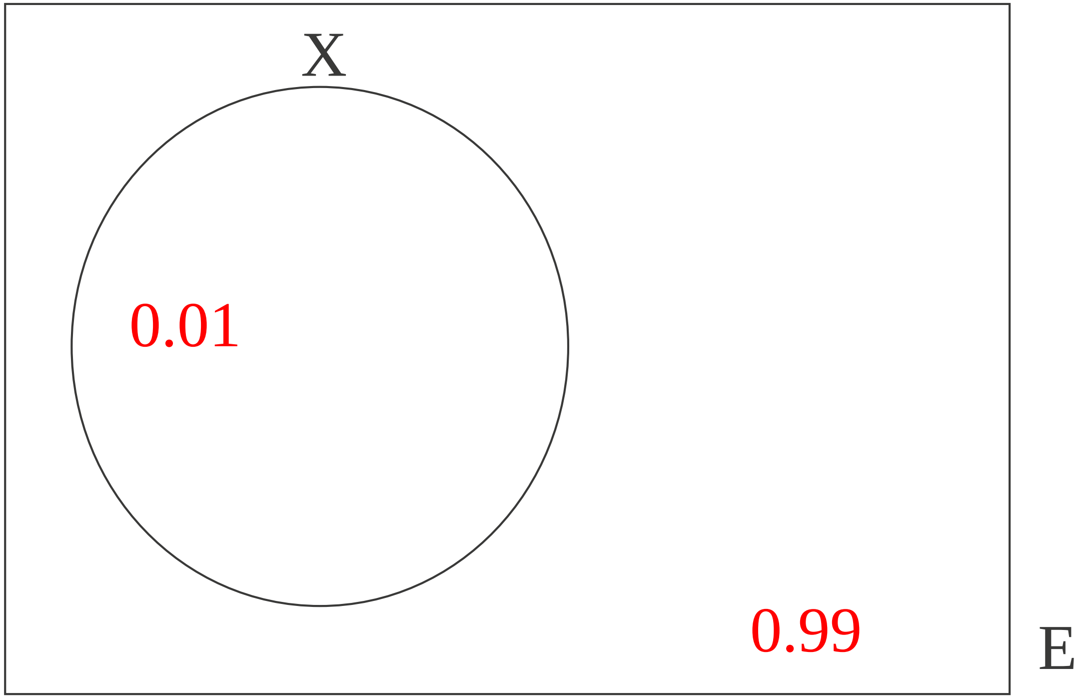
$\rightarrow$ $\mathbf{X \cap T'}$ **= 0.01 * 0.01 = 0.001**

$\rightarrow$ $\mathbf{X' \cap T}$ **= 0.99 * 0.09 = 0.0891**

$\rightarrow$ $\mathbf{X' \cap T'}$ **= 0.99 * 0.91 = 0.9009**

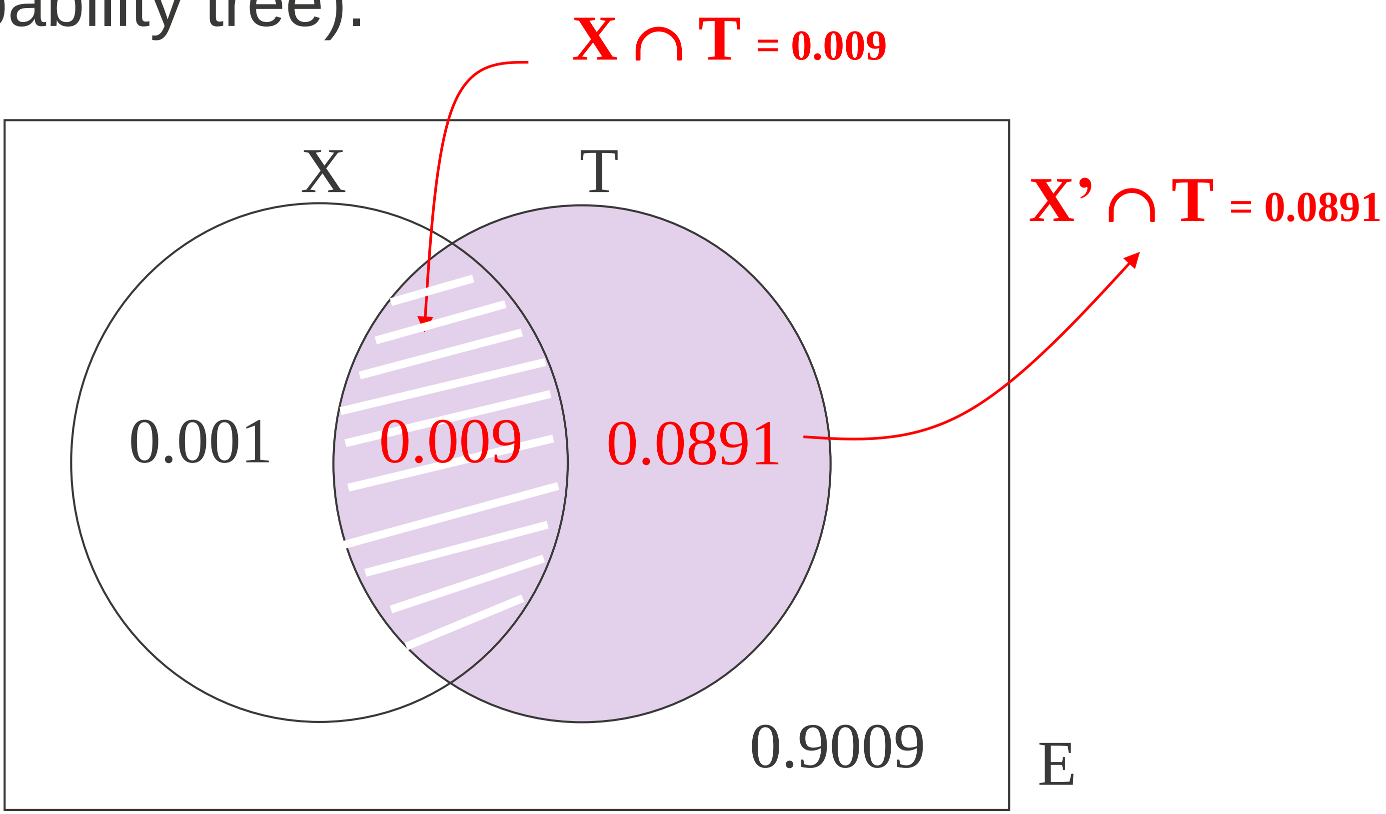
As a Venn Diagram

- Venn Diagram showing disease status. Without a test, the probability a person chosen at random has disease is 0.01 .
- This is the *prior* probability of X.



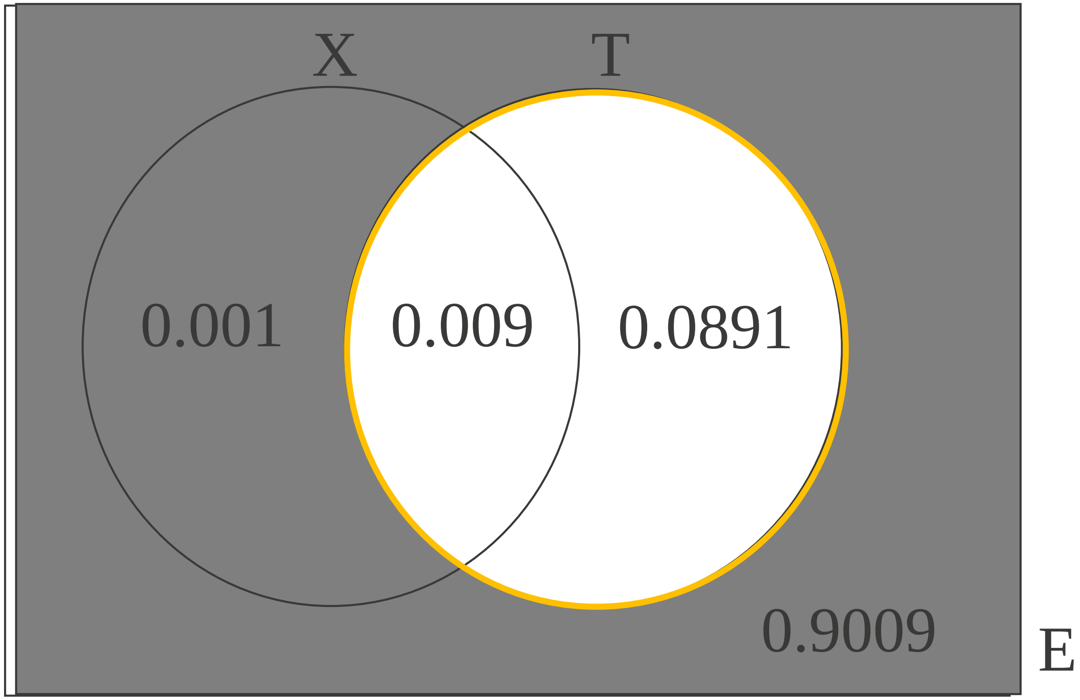
Redraw… showing test results

As a Venn Diagram showing disease status and test result after calculation of joint probabilities (using probability tree).



… positive test results only

- As a Venn Diagram showing disease status and test result.
- Conditioning on a positive test result means that only a subset of the original problem is relevant.



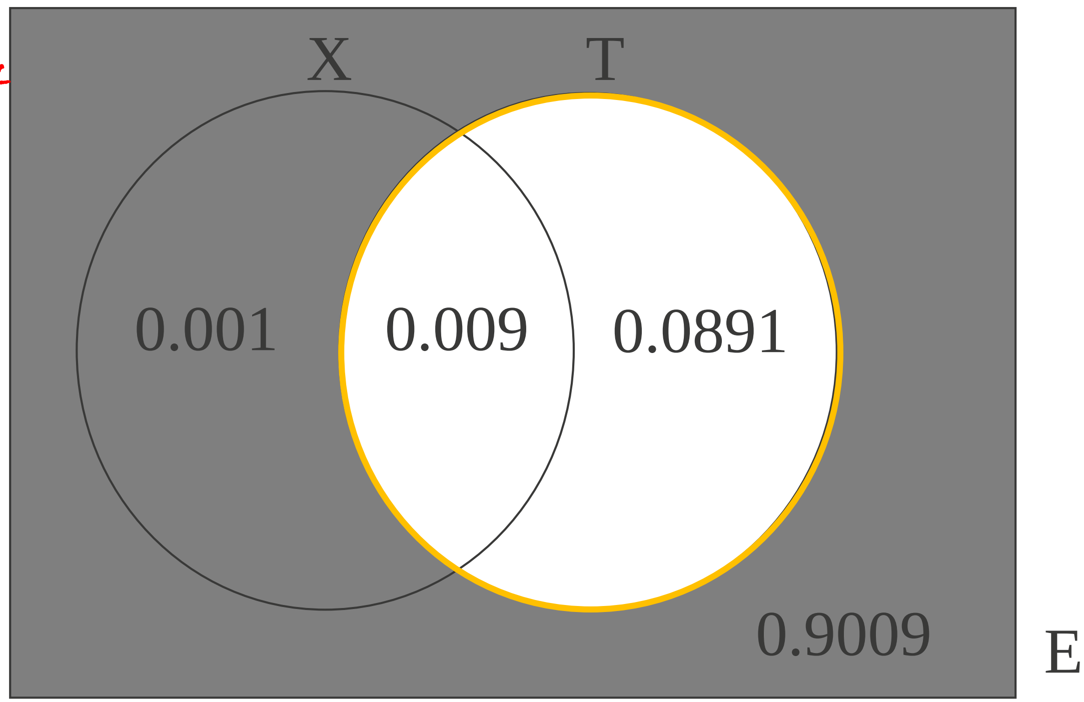
Posterior probabiliy

$0.009 + 0.0891$

The probability of a Positive test result (T) is $\underline{\;0.0981\;}$.

The probability of having disease after positive test is P(X|T) = $\underline{\;0.0917\;}$. The *posterior* probability of X.

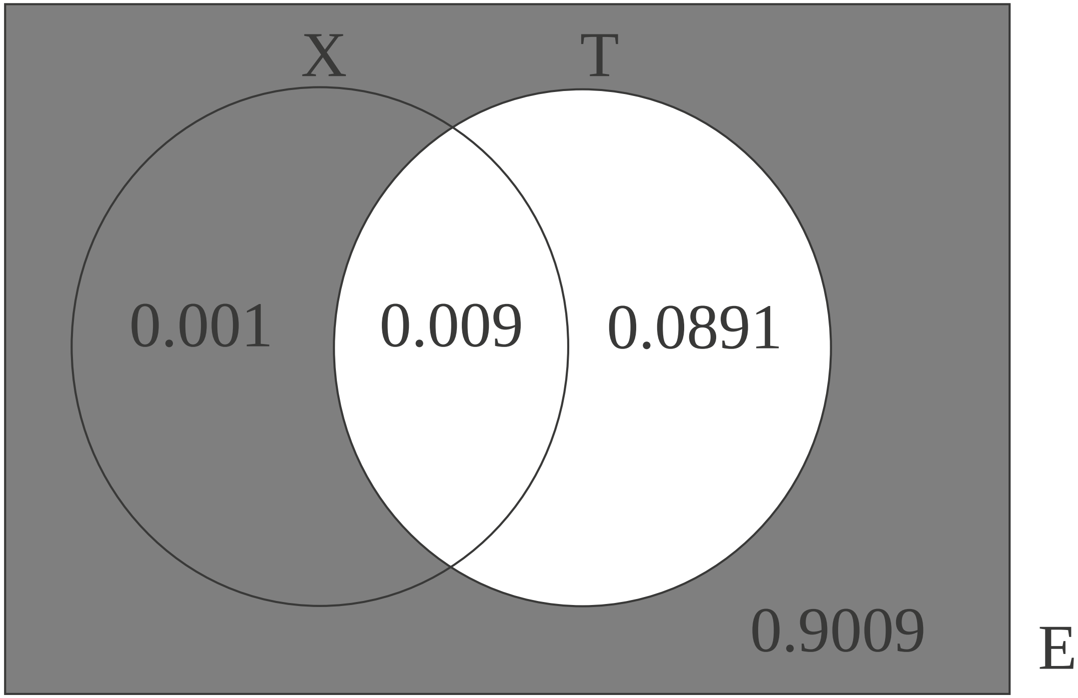
$$\frac{0.009}{0.0981} = 0.0917$$



… calculated

P(T) is 0.009 + 0.891 = 0.0981.

P(X|T) = 0.009 / 0.0981 = 0.0917. Approx 10% chance of having disease after a positive test result.



Summary

Begin with state of nature

$P(X) = 0.01$ *prior probability*

Incorporate additional information

$P(T|X) = 0.9$ *conditional probability*

$P(T|X') = 0.09$ *conditional (false positive)*

Calculate

$P(X \cap T)$ and $P(X' \cap T)$ *joint probabilities*

$P(T) = P(X \cap T) + P(X' \cap T)$

$P(X|T) = P(X \cap T)/P(T)$ *posterior probability*

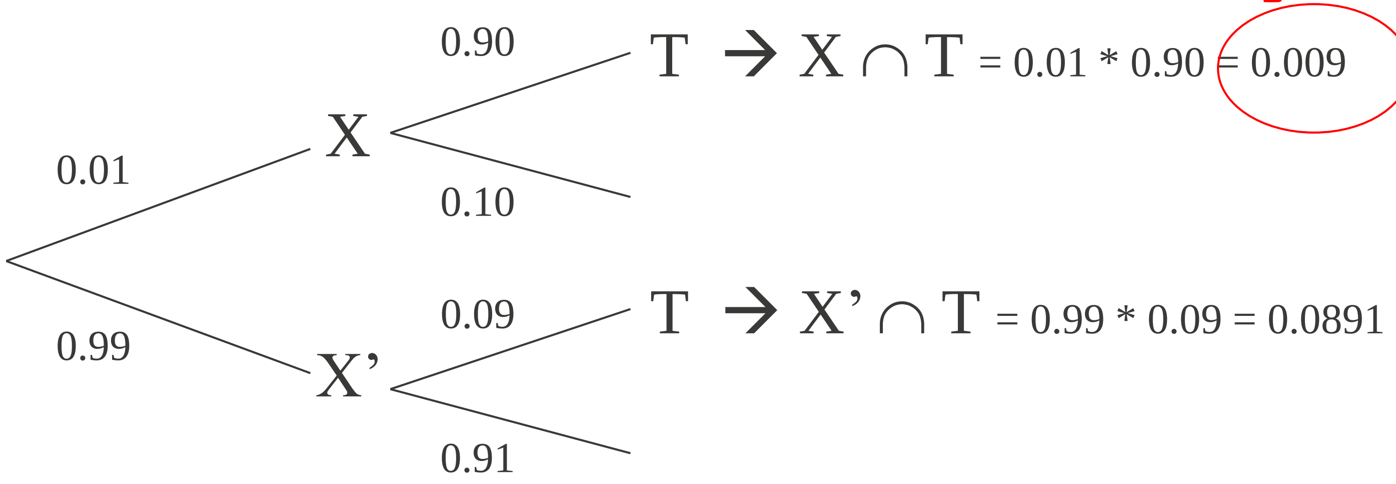
Summary

As a probability tree:

$P(X \cap T) + P(X' \cap T)$

P(T) is 0.009 + 0.891 = 0.0981.

P(X|T) = 0.009 / 0.0981 = 0.0917.



Summary

As a table:

P(T) is 0.009 + 0.891 = 0.0981.

P(X|T) = 0.009 / 0.0981 = 0.0917.

Disease	Prior	Test	Conditional	Joint	Posterior after +ve Test
Has	0.01	+ve	0.9	0.0090	0.0917
		-ve	~~0.1~~	~~0.0010~~	
Not	0.99	+ve*	0.09	0.0891	0.9083
		-ve	~~0.91~~	~~0.9009~~	
			Pr +ve =	0.0981	

Note: we could adapt this method to calculate posterior probability after a negative test.

https://flux.qa (Feed code: SJ6KGV)

Question 2

Using the Venn diagram below P(B) =

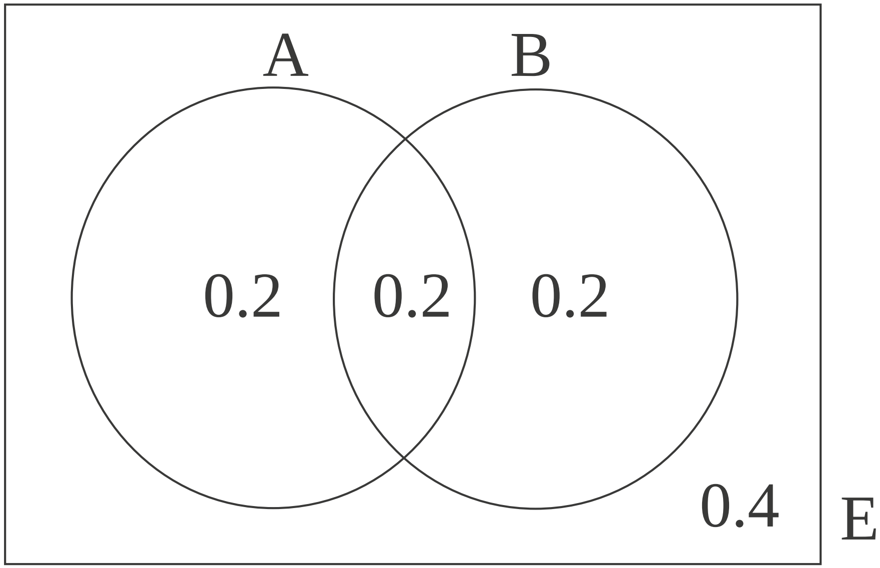
A. 0.2

✓ B. 0.4

C. 0.5

D. 0.6

E. None of these.

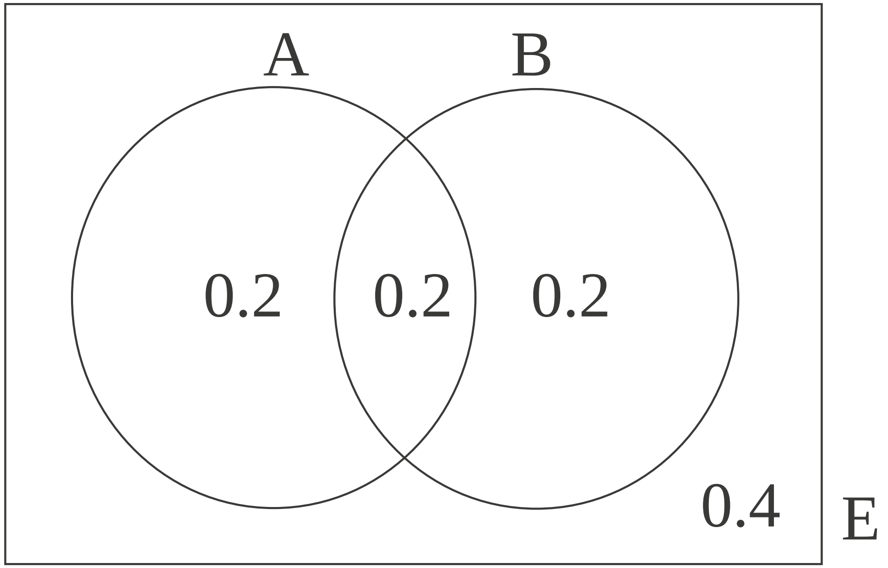


https://flux.qa (Feed code: SJ6KGV)

Question 3

Using the Venn diagram below P(A|B) =

A. 0.2
B. 0.4
✓ C. 0.5
D. 0.6
E. None of these.

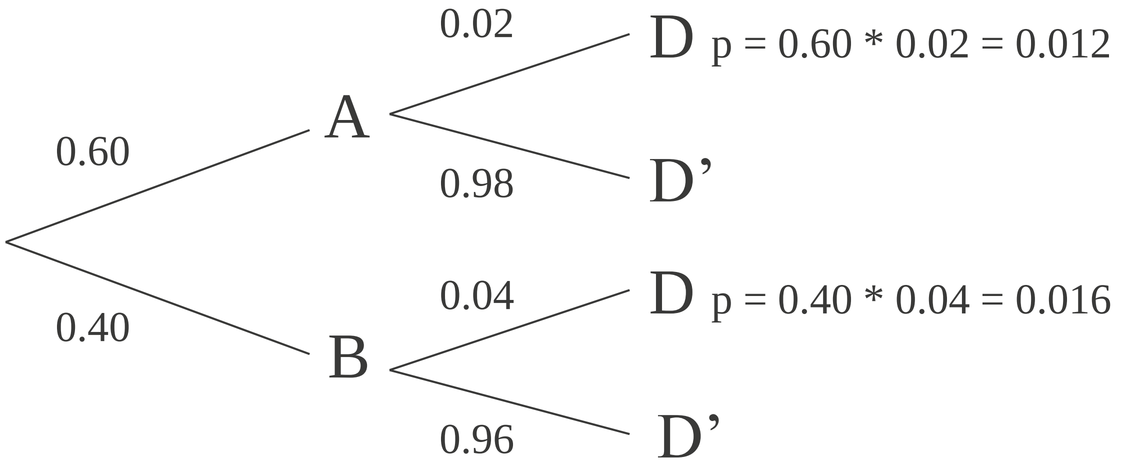


$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0.2}{0.4} = 0.5$$

Example 1

- A plant has two machines.
- Machine A produces 60% of the output with the fraction defective being 0.02.
- Machine B produces 40% of the output with the fraction defective being 0.04.
- The quality control inspector finds a defective part awaiting pack and ship.
- What is the probability that it was produced by Machine A?

Answer



$$P(A|D) = \frac{0.012}{0.012 + 0.016}$$

$$= 0.429$$

Example 2

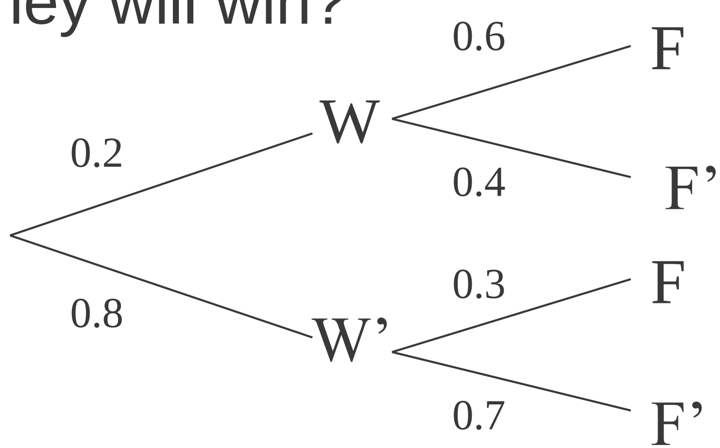
- The local football team has a 20% chance of winning any given match (W).

$P(F|W) = 0.6$

- The local commentator is ‘quite’ good at predicting winners: he correctly forecasts the team will win (F) in 60% of cases and correctly predicts the team will lose in 70% of cases.

- What is the probability the team will win when the commentator predicts they will win?

$P(W|F)$??



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Question 4

For the football problem shown below, without knowing the forecast, the team's probability of winning is:

P(W)??

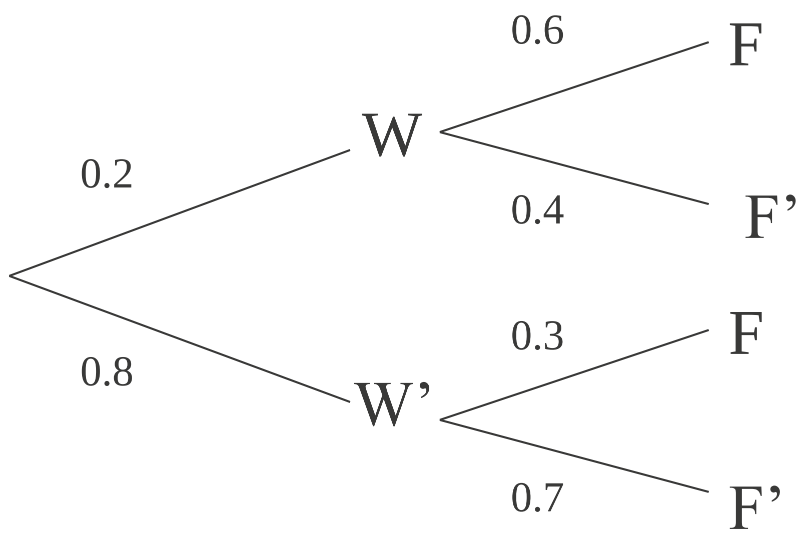
✓ A. 0.2

B. 0.3

C. 0.6

D. 0.8

E. None of these.



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Question 5

For the football problem shown below, the probability the team wins and the commentator forecasts they win is:

$P(W \cap F)$??

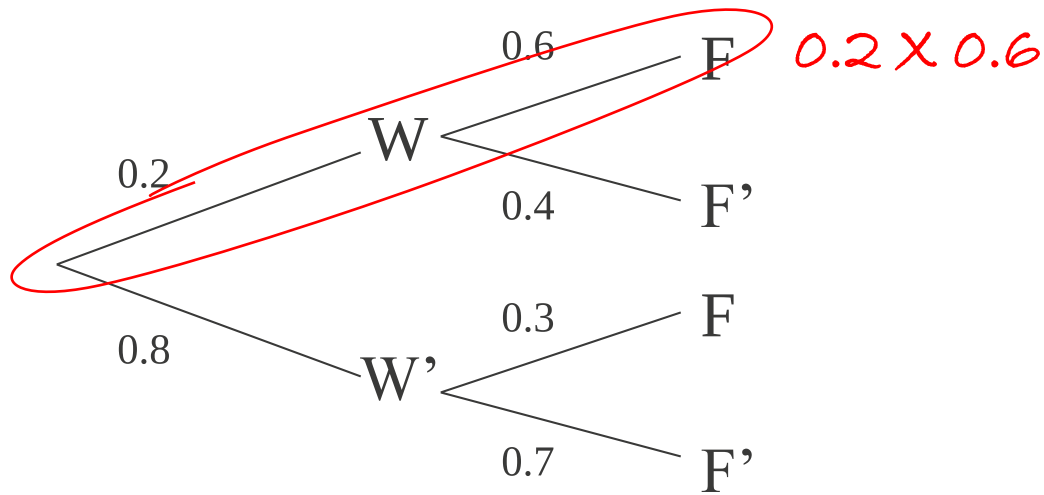
✓ A. 0.12

B. 0.08

C. 0.24

D. 0.56

E. None of these.



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Question 6

For the football problem shown below, the probability the commentator forecasts the team will win is:

P(F)??

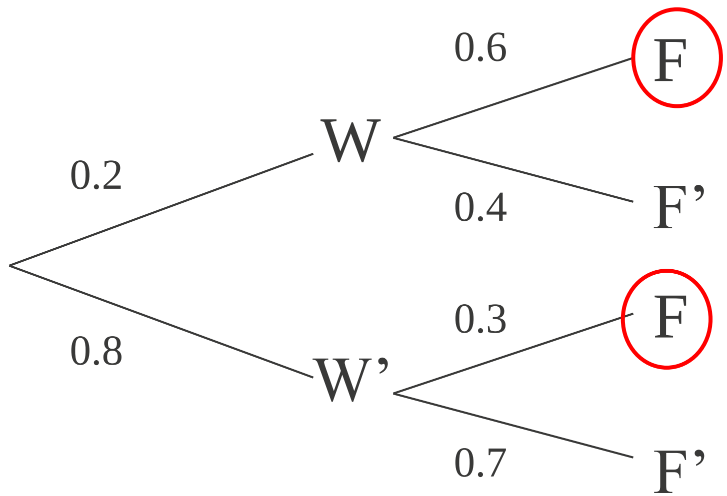
A. 0.12

B. 0.20

C. 0.24

✓ D. 0.36   0.12 + 0.24

E. None of these.



0.2 X 0.6 =0.12

0.8 X 0.3 =0.24

https://flux.qa (Feed code: SJ6KGV)

Question 7

For the football problem shown below, the probability the team wins <u>when</u> the commentator forecasts they win is:

$P(W|F)$??

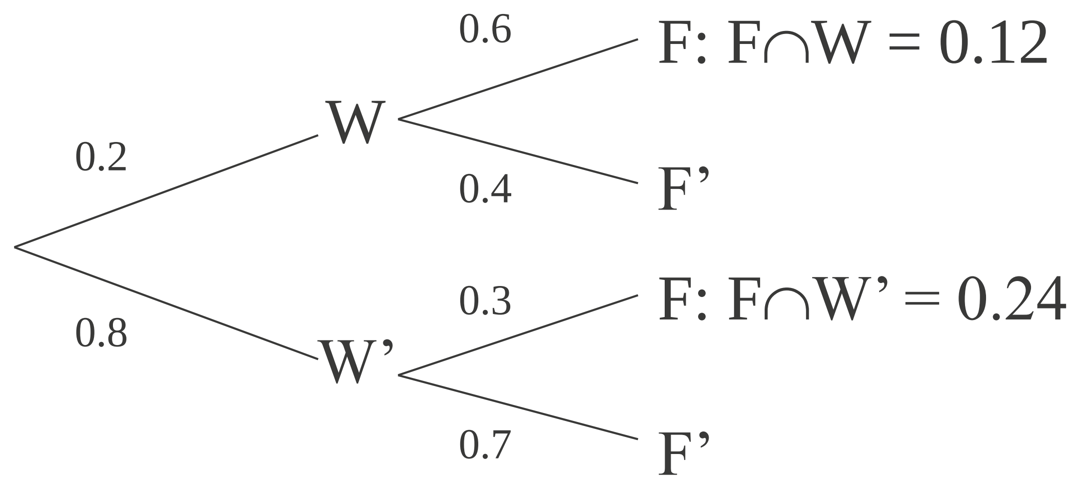
A. 0.12

B. 0.20

C. 0.24

✓ D. 0.33

E. 0.67.



$$P(W\,|\,F)=\frac{P(F\cap W)}{P(F)} = \frac{0.12}{0.12+0.24} = 0.33$$

Answer

Match	Prior	Forecast	Conditional	Joint	Posterior after predicting win.
Win	0.2	Win	0.6	0.1200	0.3333
		Lose	~~0.4~~	~~0.0800~~	
Lose	0.8	Win	0.3	0.2400	0.6667
		Lose	~~0.7~~	~~0.5600~~	
			Pr Win	0.3600	

Reading:

- For you to read:
- Formal statement of Bayes' Theorem.
- The following slides take you through the components of the formal statement.

Independent Events

- If two events are independent then the probability of one event occurring has no effect on the other.
- Thus, for independent events A and B,
  - $P(B|A) = P(B)$ and $P(A|B) = P(A)$
  - $P(A \cap B) = P(A)*P(B)$
- For tosses of a coin, let A be the outcome of a head with the first toss and B the outcome of a head with the second toss.
- Then $P(A \cap B) = P(A)*P(B) = 0.5*0.5 = 0.25$

Mutually Exclusive Events

- Mutually exclusive events are events which cannot all occur in the same trial.
    - For example, a person tosses a coin, the outcome of head and tails is mutually exclusive.
    - A person may choose product A or B or C.
    - A person may be infected or not infected.
- For mutually exclusive events A and B $P(A \cap B) = 0$.

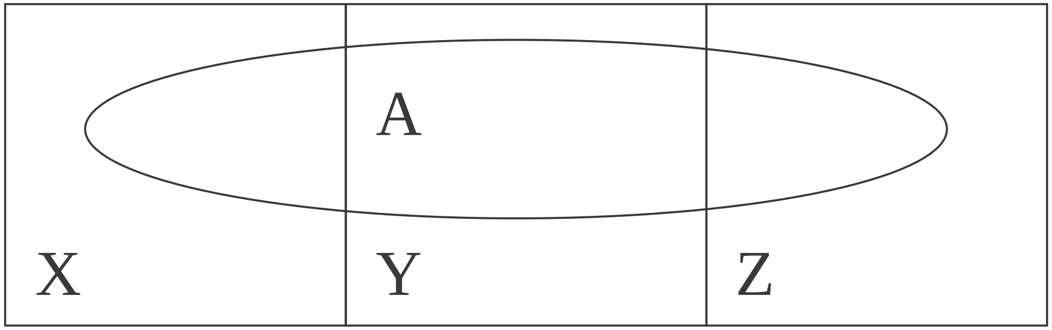
Collectively Exhaustive Events

- Collectively exhaustive events cover the whole sample space.
    - For example Head or Tails
    - Infected or Not Infected.

The Law of Total Probability

- X, Y and Z are *mutually exclusive* and *collectively exhaustive* because they do not intersect and together they cover the total sample space (or universe).
- Let A = $(A \cap X) \cup (A \cap Y) \cup (A \cap Z)$. Then by the law of total probability:

$$P(A) = P(A \cap X) + P(A \cap Y) + P(A \cap Z).$$



Conditional Probabilities

- From earlier work with Venn and Tree Diagrams the following is true:

$$P(\mathrm{B}|\mathrm{A}) = \frac{P(\mathrm{A} \cap \mathrm{B})}{P(\mathrm{A})}$$

$$\textit{thus}$$

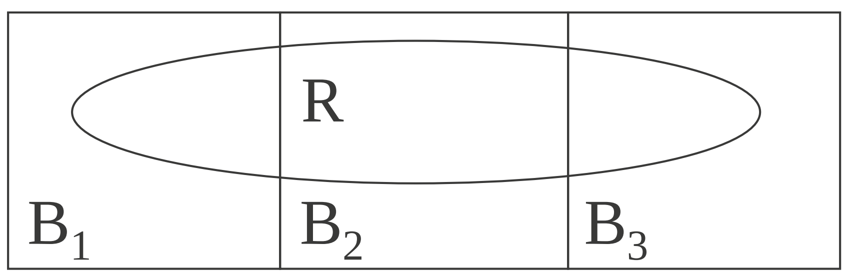
$$P(\mathrm{A} \cap \mathrm{B}) = P(\mathrm{B}|\mathrm{A})\, P(\mathrm{A})$$

Bayes' Theorem

- Formal statement:

For an event with outcomes $B_1, B_2 \ldots B_n$, and event R. what is the probability that outcome B_x occurred given that event R has occurred?

$$P(B_x \mid R) = \frac{P(B_x)P(R \mid B_x)}{\sum_{j=1}^{n} P(B_j)P(R \mid B_j)} = \frac{P(B_x \cap R)}{P(R)}$$



Probability Distributions

- For many commonly occurring situations we don't have to create a probability distribution from scratch but instead use well understood mathematical models. Next week we cover three of the most important:
- <u>Binomial</u> and <u>Poisson</u> Distributions. Both are discrete, where the random variable can take on natural (counting) number values, and the
- <u>Normal</u> Distribution, the most important continuous distribution.

Background for next week…

- *If you intend doing calculations by hand you will need to know some basic mathematical functions:*

- Exponential

- Factorial

- Combinatorial

Exponents

We describe the notation a^b as a raised to the power of b.

This is defined formally as $a^b = \underbrace{(a \cdot a \cdot a \cdot a \cdot a \cdot a \cdot a \ \cdots \ a)}_{b \text{ times}}$

a and b can take on non-integer values and we often use the number 'e' as a base. $e \approx 2.7182\ldots$

Using a calculator you should be able to calculate expressions such as: $6^2,\ e^2,\ e^{-0.5},\ 0.3^{10},\ 0.994^{10},\ e^{-0.44},\ 8^8,\ -0.001^2,\ e^{-20}$.

You should be able to use the $\left[y^x\right]$ and $\left[e^x\right]$ or $[\exp]$ keys on your calculator.

Factorial

Factorial notation, '!' is easiest to understand with an example.

$5! \; = \; 5 \times 4 \times 3 \times 2 \times 1 = 120, \; 1! \; = \; 1, \; \textit{and} \; 0! \; = \; 1$ by convention.

Formally, $\mathrm{n}! \; = \; \mathrm{n} \cdot \; (\mathrm{n} - 1) \cdot (n-2) \cdot (n-3) \; \cdots \; 3 \cdot 2 \cdot 1$

Combinations

We use the notation ${}^{n}C_x$ *or* $\binom{n}{x}$ to describe the number of different ways we can select *x* objects at a time from a group of *n* objects.

$${}^{n}C_x = \frac{n!}{x!(n-x)!}$$

The number of ways that we can select 4 students from a class of 10 students is given by ${}^{10}C_4 = \frac{10!}{4!(10-4)!} = \frac{10!}{4!\ 6!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$

The number of ways that we can select 3 cards from a deck of 52 cards is given by ${}^{52}C_3 = \frac{52!}{3!(52-3)!} = \frac{52!}{3!\ 49!} = \frac{52 \cdot 51 \cdot 50}{3 \cdot 2 \cdot 1} = 22100$

Reading/Questions

- Reading:
  - 7^{th} Ed. Sections 6.2 – 6.6.
- Questions:
  - 7^{th} Ed. Questions 6.18, 6.22, 6.32, 6.45, 6.52, 6.77, 6.79, 6.81, 6.84.