FIT3158
Business Decision Modelling

SEMESTER 2, 2022

Lecture 3

- Sensitivity Analysis

Topics covered:

- Exploring the Solver's Reports
- Interpreting the Sensitivity Report
- Exploring the Sensitivity Assistant – Spider Plot & Solver Tables
- A brief look at the Simplex Method

Introduction

- When solving an LP problem, we assume that values of all model coefficients are known with certainty.
- Such certainty rarely exists.
- Sensitivity analysis helps answer questions about how sensitive the optimal solution is to changes in various coefficients in a model.

General Form of a Linear Programming (LP) Problem

c_i's are costs or prices

MAX (or MIN): $c_1X_1 + c_2X_2 + \ldots + c_nX_n$

Subject to: $a_{11}X_1 + a_{12}X_2 + \ldots + a_{1n}X_n <= b_1$

a_{ij}'s are resource requirements

$$\vdots$$

$$a_{k1}X_1 + a_{k2}X_2 + \ldots + a_{kn}X_n <= b_k$$

$$\vdots$$

$$a_{m1}X_1 + a_{m2}X_2 + \ldots + a_{mn}X_n = b_m$$

- **How sensitive is a solution to changes in the c_i, a_{ij}, and b_i?**

b_i's indicate resource availability

Approaches to Sensitivity Analysis

- Change the data and re-solve the model!
  - Sometimes this is the only practical approach.
- Solver also produces sensitivity reports that can answer various questions…

Solver's Sensitivity Report

- Answers questions about:
  - Amounts by which objective function coefficients can change without changing the optimal solution.
  - The impact on the optimal objective function value of changes in constrained resources.
  - The impact on the optimal objective function value of forced changes in decision variables.
  - The impact changes in constraint coefficients will have on the optimal solution.

Software Note

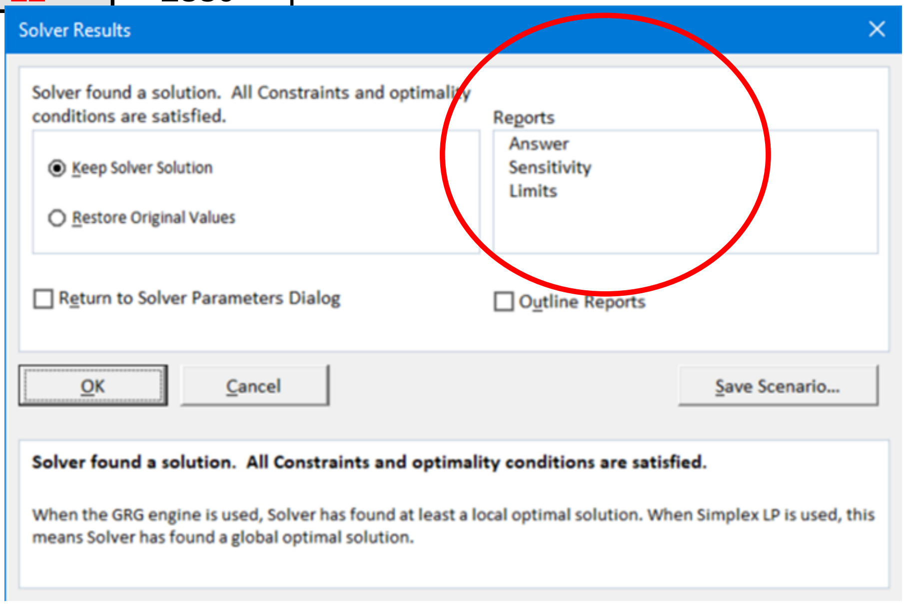
{When solving LP problems, be sure to use an LP algorithm/engine as this allows Solver to provide more sensitivity information than it could otherwise do}

Once Again, We'll Use The Blue Ridge Hot Tubs Example...

$$
\begin{array}{lll}
\text{MAX:} & 350X_1 + 300X_2 & \} \text{ profit} \\
\text{S.T.:} & 1X_1 + 1X_2 <= 200 & \} \text{ pumps} \\
 & 9X_1 + 6X_2 <= 1566 & \} \text{ labor} \\
 & 12X_1 + 16X_2 <= 2880 & \} \text{ tubing} \\
 & X_1, X_2 >= 0 & \} \text{ nonnegativity}
\end{array}
$$

Solver Results

	Blue Ridge Hot Tubs			
	Aqua-Spas	Hydro-Luxes		
Number to Make	122	78	Total Profit	
Unit Profits	$350	$300	$66,100	
Constraints			Used	Available
- Pumps Required	1	1	200	200
- Labor Required	9	6	1,566	1566
- Tubing Required	12	16	2,712	2880



Solver Reports

- Answer Report
- Sensitivity Report
- Limits Report

See file Lecture3.xlsm

Answer Report

- 3 sections:
  - Section 1: Final value of optimal solution (obj.function).
  - Section 2: Final values of decision variables.
  - Section 3: How much of each resource is used.
  - Names/labels are determined by the worksheet names we have used.
  - Note that the Answer Report gives us the same information as the worksheet output but in a more useful format for reporting purposes.

Answer Report

Value of Objective Function

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$D$6	Unit Profits Total Profit	$66,100	$66,100

Optimal Solution

Decision Variables

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$5	Number to Make Aqua-Spas	122	122	Contin
$C$5	Number to Make Hydro-Luxes	78	78	Contin

Resource constraints

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$9	Pumps Req'd Used	200	$D$9<=$E$9	Binding	0
$D$10	Labor Req'd Used	1,566	$D$10<=$E$10	Binding	0
$D$11	Tubing Req'd Used	2,712	$D$11<=$E$11	Not Binding	168
$B$5	Number to Make Aqua-Spas	122	$B$5>=0	Not Binding	122
$C$5	Number to Make Hydro-Luxes	78	$C$5>=0	Not Binding	78

Non-negativity conditions

Answer report notes

- Section 1 and 2 together tell us the optimal values for the decision variables and the optimal value of the objective function (e.g. profit or return).
- Section 3 gives info. on the constraints:
  - Resource constraints:
  - Which are binding – i.e. which resources are all used up
  - Which are non-binding – i.e. some of the resource is not used
  - Non-negativity constraints
  - Which decision variables have values greater than their lower bound of zero

Answer Report for Hot Tubs

- Sections 1 and 2 tell us the optimal values for the decision variables
  - X_1= Aqua-spas = 122
  - X_2 = Hydro-Luxe = 78
  - The optimal value of the objective function = $66,100
- Section 3 gives info on the constraints:
  - Resource constraints- Which are binding/non-binding (i.e. which resources are all used up and which are not.)
  - Pumps and labour are binding.
  - Tubing is non-binding.
  - Non-negativity constraints: Both decision variables (Aqua-spas and Hydro-Luxes) are greater than their lower bound of zero.

The Sensitivity Report

The sensitivity report consists of 2 sections:

- The variable cells section
  - What happens when we change the values of the coefficients in the objective function
  - What happens if we include variables which are not part of the optimal solution
- The constraints section
  - What happens to the value of the objective function if we increase (or decrease) the amount of available resources

The Sensitivity Report

Microsoft Excel 16.0 Sensitivity Report
Worksheet: [Lecture 3.xlsm]Production Report (Solution)
Report Created: 4/08/2018 12:24:15 AM

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number to Make Aqua-Spas	122	0	350	100	50
$C$5	Number to Make Hydro-Luxes	78	0	300	50	66.6667

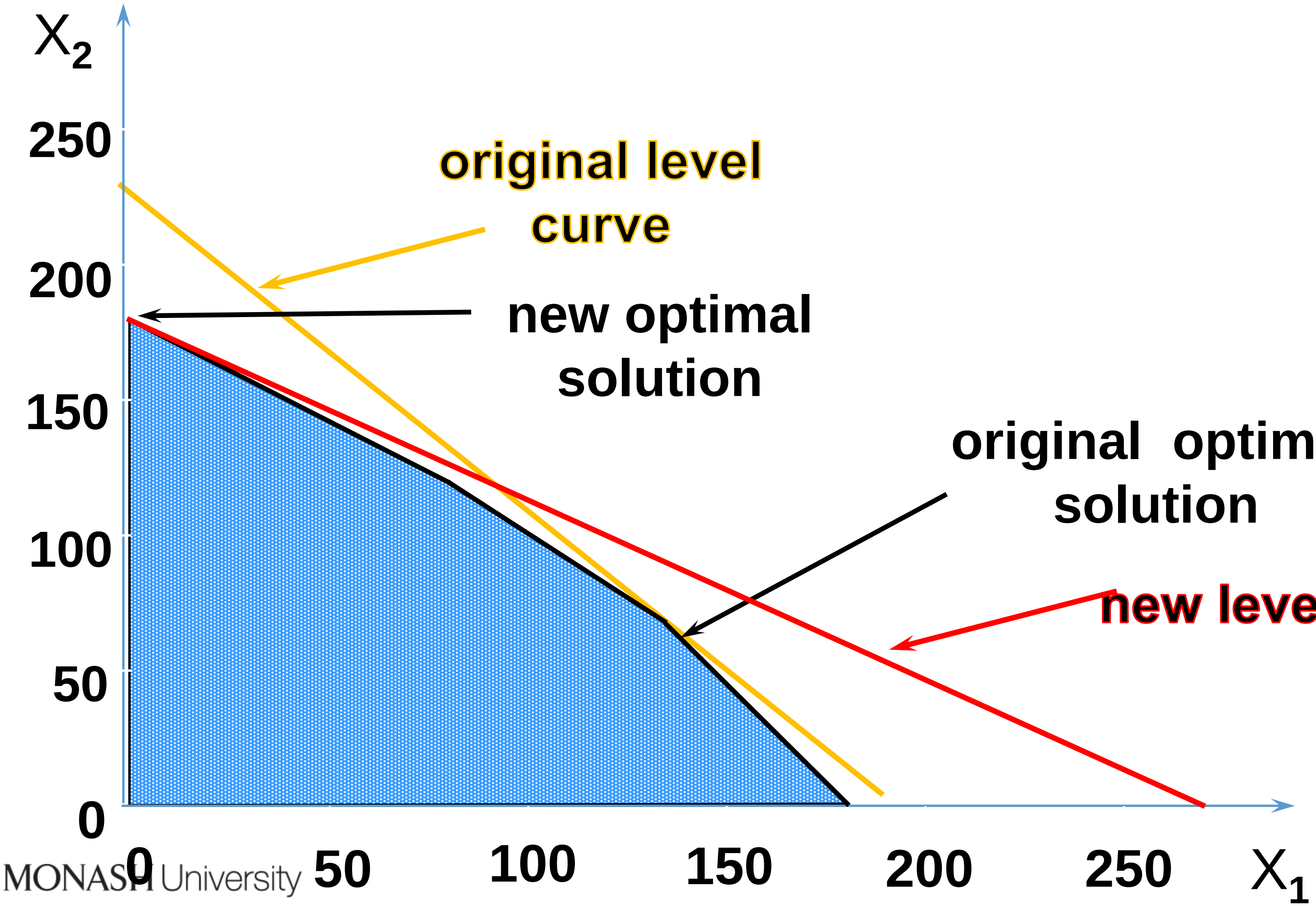
Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$9	Pumps Req'd Used	200	200	200	7	26
$D$10	Labor Req'd Used	1566	16.6667	1566	234	126
$D$11	Tubing Req'd Used	2712	0	2880	1E+30	168

Changing the objective function coefficients

- The objective function is: MAX: 350 X_1 + 300 X_2.
- Suppose we are not sure of the profit for X_1 and X_2.
- I.e. what happens if we change one of the coefficients?
- We can see this diagrammatically and by looking at the sensitivity report.

How Changes in Objective Coefficients Change the Slope of the Level Curve



How Changes in Objective Coefficients Change the Slope of the Level Curve

See file Hot Tub Analysis.xlsm

Sensitivity Report: Variable Cells

- Tells the amount by which each objective function coefficient can change without affecting the optimal solution – assuming all other coefficients remain constant.
- Max. increase in profit for Aqua-Spas is $100 Max. decrease in profit for Aqua-Spas is $50
- In the absence of degeneracy, any zero value in Allowable Increase or Decrease for any objective function coefficient indicates alternate optimal solutions exist.

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number to Make Aqua-Spas	122	0	350	100	50
$C$5	Number to Make Hydro-Luxes	78	0	300	50	66.6667

Changes in Objective Function Coefficients

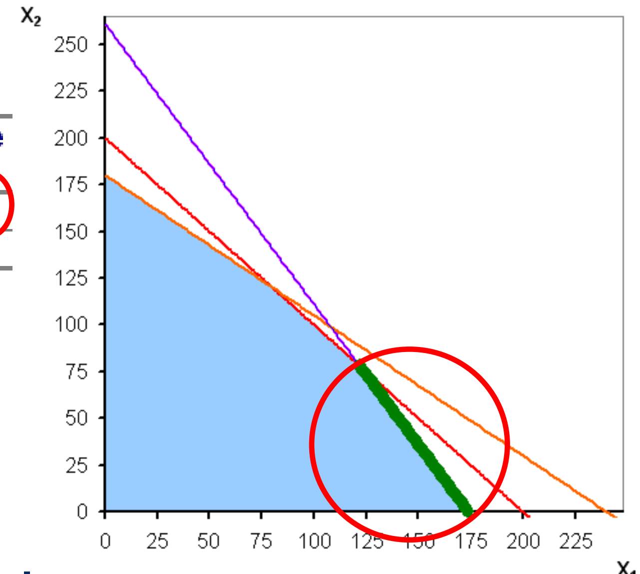
- Values in the “Allowable Increase” and Allowable Decrease” columns for the Decision Cells indicate the amounts by which an objective function coefficient can change without changing the optimal solution (i.e. the optimal values of the decision variables), *assuming all other coefficients remain constant*.
- The value of the optimal solution (obj. function) however, does change.

Alternate Optimal Solutions

Note: Values of zero (0) in "Allowable Increase" or "Allowable Decrease" columns for the Changing Cells indicate that an alternate optimal solution exists

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number to make Aqua-Spas	174	0	450	1E+30	0
$C$5	Number to make Hydro-Luxes	0	0	300	0	1E+30



e.g. (122,78) and (174,0) are both optimal

Changes in Constraint RHS Values

- The *shadow price* of a constraint indicates the amount by which the objective function value changes given a unit *increase* in the RHS value of the constraint, *assuming all other coefficients remain constant*.
- Shadow prices holds if RHS value of constraint falls within Allowable Increase or Decrease values.
- Shadow prices for non-binding constraints are always zero.

So, what is shadow price…

- The shadow prices of resources equate the marginal value of the resources consumed with the marginal benefit of the goods being produced.
- The shadow prices of a resource is the amount over and above what you are currently paying for and which you would be willing to pay to acquire an additional unit of that resource.
- Resources in excess supply have a shadow price (or marginal value) of zero.

Sensitivity Report for Hot Tubs

- 1 unit increase in labour increases OF by $16.67.
- 10 units increase in labour increases by $16.67*10 = $166.67.
- 1 unit decrease in labour decreases OF by $16.67.
- Tubing is a non-binding constraint as there is a slack of 168 units (2880 – 2712) and therefore, has zero shadow price.

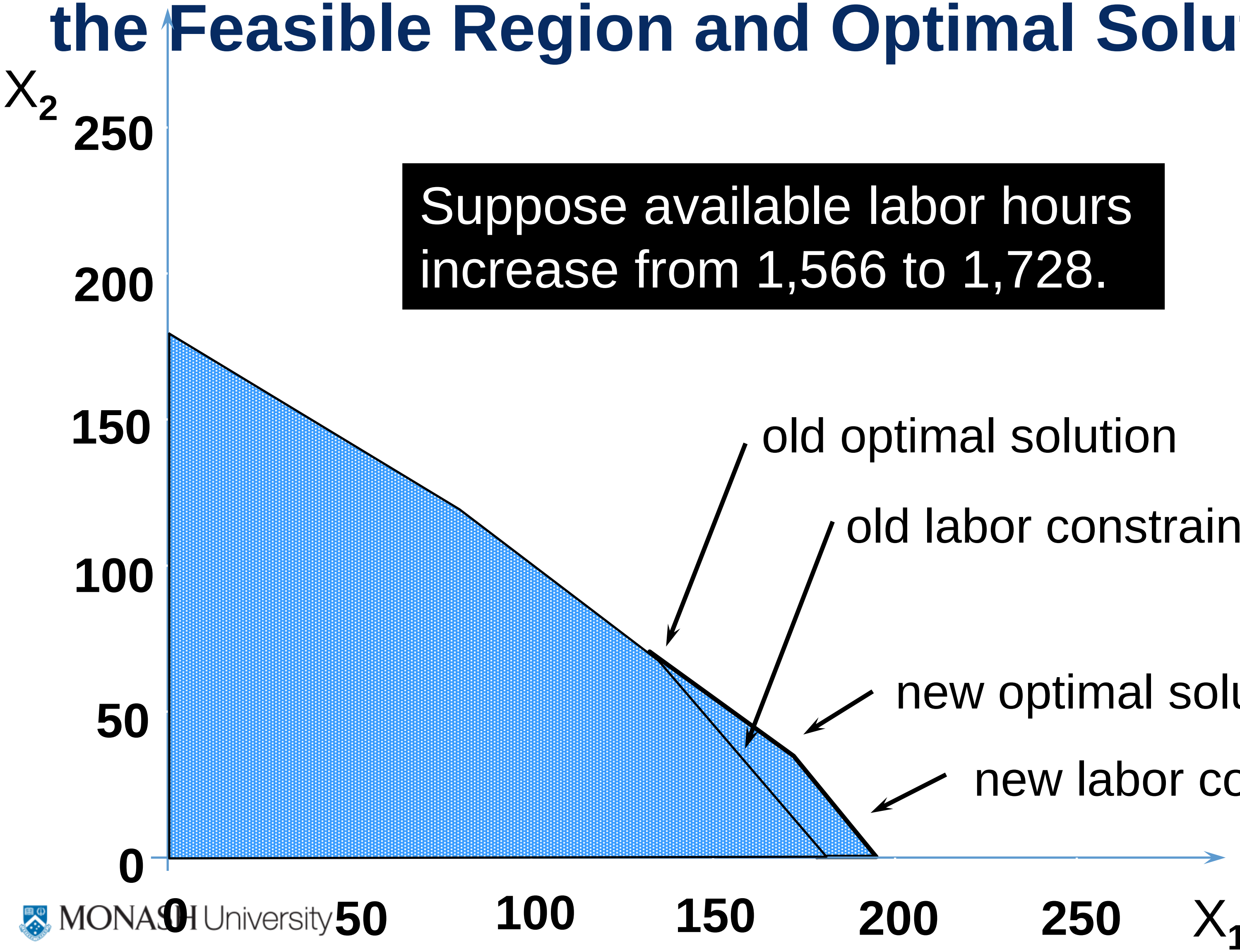
Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$9	- Pumps Required Used	200	200.00	200	7	26
$D$10	- Labor Required Used	1,566	16.67	1566	234	126
$D$11	- Tubing Required Used	2,712	0.00	2880	1E+30	168

Comments About Changes in Constraint RHS Values

- Shadow prices only indicate the changes that occur in the objective function value as RHS values change.
- Changing a RHS value for a binding constraint also changes the feasible region and the optimal solution (see graph on following slide).
- To find the optimal solution after changing a binding RHS value, you must re-solve the problem.

How Changing an RHS Value Can Change the Feasible Region and Optimal Solution



How Changing an RHS Value Can Change the Feasible Region and Optimal Solution

See file Hot Tub Analysis.xlsm

Sensitivity Report - Constraints

- Binding constraint:
  - A change in RHS value changes the feasible solutions & optimal solution. You have to re-solve model to find new optimal solution (the report does not tell you this!)
- Non-binding constraint:
  - The shadow price is always zero (because adding more of a resource that has not been used up is no help!)

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$9	- Pumps Required Used	200	200.00	200	7	26
$D$10	- Labor Required Used	1,566	16.67	1566	234	126
$D$11	- Tubing Required Used	2,712	0.00	2880	1E+30	168

Other Uses of Shadow Prices

- Suppose a new Hot Tub (the Typhoon-Lagoon) is being considered. It generates a marginal profit of $320 and requires:
  - 1 pump (shadow price = $200)
  - 8 hours of labor (shadow price = $16.67)
  - 13 feet of tubing (shadow price = $0)
- Q: Would it be profitable to produce any?

Negative reduced cost

A: $320 - $200*1 - $16.67*8 - $0*13 = -$13.33 = *No!*

Reduced cost = Profit - ∑(shadow price x units of resources required)

What solver says…

Microsoft Excel 12.0 Sensitivity Report
Worksheet: [Fig4-8.xls]Production
Report Created: 6/03/2009 4:24:12

Aqua spas = 350 - 200*1 - 16.67*9 - 0*12 = 0
Hydro-luxes = 300 - 200*1 - 16.67*6 - 0*16 = 0
Typhoon-lagoons = 320 - 200*1 - 16.67*8 - 0*13 = -13.33

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number to make Aqua-Spas	122	0	350	100	20
$C$5	Number to make Hydro-Luxes	78	0	300	50	40
$D$5	Number to make Typhoon-Lagoons	0	-13	320	13.33333	1E+30

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$9	Pumps Req'd Used	200	200	200	7	26
$E$10	Labor Req'd Used	1566	17	1566	234	126
$E$11	Tubing Req'd Used	2712	0	2880	1E+30	168

The Meaning of Reduced Costs

- The *reduced cost* for each product equals its per-unit marginal profit minus the per-unit value of the resources it consumes (priced at their shadow prices).
- Products whose marginal profits are less than the marginal value of the goods required for their production will not be produced in an optimal solution (unless a constraint forces the product to be produced)
- Reduced costs tell us by how much we would have to increase the profitability of an item before it would be included in the optimal solution (for a maximisation problem). And vice-versa for a minimisation problem.

Interpretation of Reduced Costs

- Current solution will remain optimal provided marginal profit on Typhoon-Lagoons is less than $320 + $13.33 = $333.33.
- If marginal profit for Typhoon-Lagoons is greater than $333.33, producing this product would be profitable and optimal solution would change.

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Number to make Aqua-Spa	122	0.00	350	100	20
$C$5	Number to make Hydro-Luxe	78	0.00	300	50	40
$D$5	Number to make Typhoon-Lagoon	0	-13.33	320	13.333333	1E+30

Analyzing Changes in Constraint Coefficients

- Q: Suppose a Typhoon-Lagoon required only 7 labor hours rather than 8. Is it now profitable to produce any?

  A: $320 - $200*1 - $16.67*7 - $0*13 = $3.31 = *Yes!*

- Q: What is the maximum amount of labor Typhoon-Lagoons could require and still be profitable?

  A: We need $320 - $200*1 - $16.67*$L_3$ - $0*13 >=0

  The above is true if L_3 <= $120/$16.67 = 7.20 hrs

Simultaneous Changes in Objective Function Coefficients

- The *100% Rule* can be used to determine if the optimal solutions changes when more than one objective function coefficient changes.
- Two cases can occur:
  - Case 1: All variables with changed obj. coefficients have non-zero reduced costs.
  - Case 2: At least one variable with changed obj. coefficient has a reduced cost of zero.

Simultaneous Changes in Objective Function Coefficients: Case 1

- The current solution remains optimal provided the obj. coefficient changes are all within their Allowable Increase or Decrease.

(All variables with changed obj. coefficients have non-zero reduced costs.)

Simultaneous Changes in Objective Function Coefficients: Case 2

(At least one variable with changed obj. coefficient has a reduced cost of zero.)

- **For each variable compute:**

$$r_j = \begin{cases} \dfrac{\Delta c_j}{I_j}, \text{if } \Delta c_j \geq 0 \\ \dfrac{-\Delta c_j}{D_j}, \text{if } \Delta c_j < 0 \end{cases}$$

- If more than one objective function coefficient changes, the current solution remains optimal provided the r_j sum to <= 1.
- If the r_j sum to > 1, the current solution, might remain optimal, but this is not guaranteed.

A Warning About Degeneracy

- The solution to an LP problem is degenerate if the Allowable Increase or Decrease on any constraint is zero (0).
- When the solution is degenerate:

  1. The methods mentioned earlier for detecting alternate optimal solutions cannot be relied upon.

  2. The reduced costs for the changing cells may not be unique. Also, the objective function coefficients for changing cells must change by at least as much as (and possibly more than) their respective reduced costs before the optimal solution would change.

- When the solution is degenerate (cont'd):

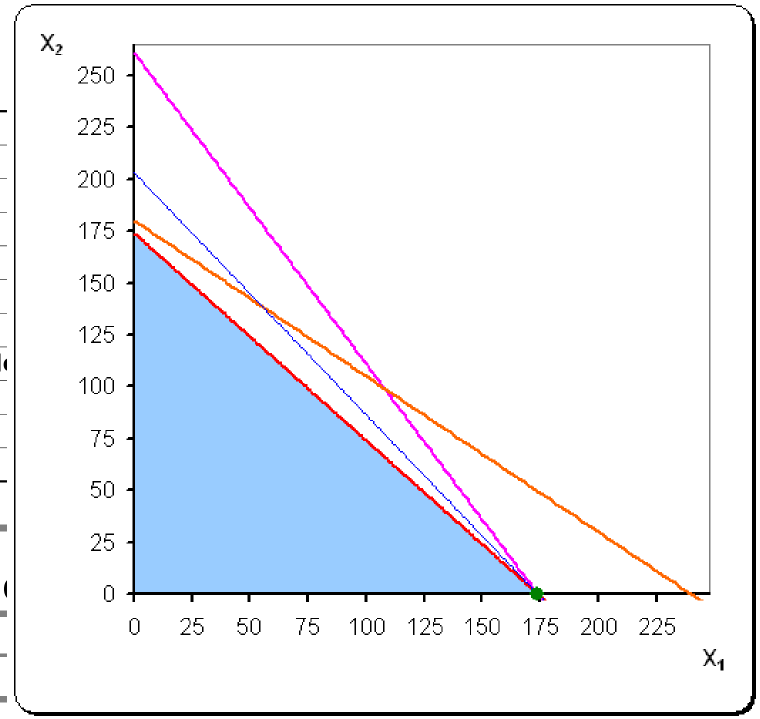
  3. The allowable increases and decreases for the objective function coefficients still hold and, in fact, the coefficients may have to be changed beyond the allowable increase and decrease limits before the optimal solution changes.

  4. The given shadow prices and their ranges may still be interpreted in the usual way but they may not be unique. That is, a different set of shadow prices and ranges may also apply to the problem (even if the optimal solution is unique!).

Degenerate case

- Setting # pumps to 174.

	Blue Ridge Hot Tubs			
	Aqua-Spas	**Hydro-Luxes**		
Number to Make	**174**	**0**	**Total Profit**	
Unit Profits	**$350**	**$300**	**$60,900**	
Constraints			**Used**	**Availabl**
- Pumps Required	**1**	**1**	**$174**	**174**
- Labor Required	**9**	**6**	**$1,566**	**1566**
- Tubing Required	**12**	**16**	**$2,088**	**2880**



Adjustable Cells

Cell	Name	Final Value	Reduced Cost
$B$5	Number to Make Aqua-Spas	174	0
$C$5	Number to Make Hydro-Luxes	0	0

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$9	- Pumps Required Used	$174	$200	174	33	0
$D$10	- Labor Required Used	$1,566	$17	1566	0	522
$D$11	- Tubing Required Used	$2,088	$0	2880	1E+30	792

Limits report

- Summarises the value of the objective function as each variable cell ranges through its limits (and the others remain constant).

See file [Hot Tub Analysis.xlsm]

Cell	Target Name	Value
$D$6	Unit Profits Total Profit	$66,100

Cell	Adjustable Name	Value	Lower Limit	Target Result	Upper Limit	Target Result
$B$5	Number to Make Aqua-Spas	122	0	23400	122	66100
$C$5	Number to Make Hydro-Luxes	78	0	42700	78	66100

The Sensitivity Assistant

- Sensitivity.xla is an add-in that allows you to create:
  - Spider Tables & Plots
    - Summarize the optimal value for one output cell as individual changes are made to various input cells.
  - Solver Tables
    - Summarize the optimal value of multiple output cells as changes are made to a single input cell.

The Sensitivity Assistant

See files:

[Spider Plot and Solver Table.xlsx]

(Also refer to file: “How to install sensitivity assistant.pdf” for instructions)

Note: Sensitivity Assistant add-in works for Windows systems only.

The Simplex Method

- To use the simplex method, we first convert all inequalities to equalities by adding slack variables to <= constraints and subtracting slack variables from >= constraints.

For example: $a_{k1}X_1 + a_{k2}X_2 + \ldots + a_{kn}X_n <= b_k$

converts to: $a_{k1}X_1 + a_{k2}X_2 + \ldots + a_{kn}X_n + S_k = b_k$

And: $a_{k1}X_1 + a_{k2}X_2 + \ldots + a_{kn}X_n >= b_k$

converts to: $a_{k1}X_1 + a_{k2}X_2 + \ldots + a_{kn}X_n - S_k = b_k$

For Our Example Problem...

$$
\begin{array}{lll}
\text{MAX: } 350X_1 + 300X_2 & & \} \text{ profit} \\
\text{S.T.: } & 1X_1 + 1X_2 + S_1 = 200 & \} \text{ pumps} \\
& 9X_1 + 6X_2 + S_2 = 1566 & \} \text{ labor} \\
& 12X_1 + 16X_2 + S_3 = 2880 & \} \text{ tubing} \\
& X_1, X_2, S_1, S_2, S_3 >= 0 & \} \text{ non-negativity}
\end{array}
$$

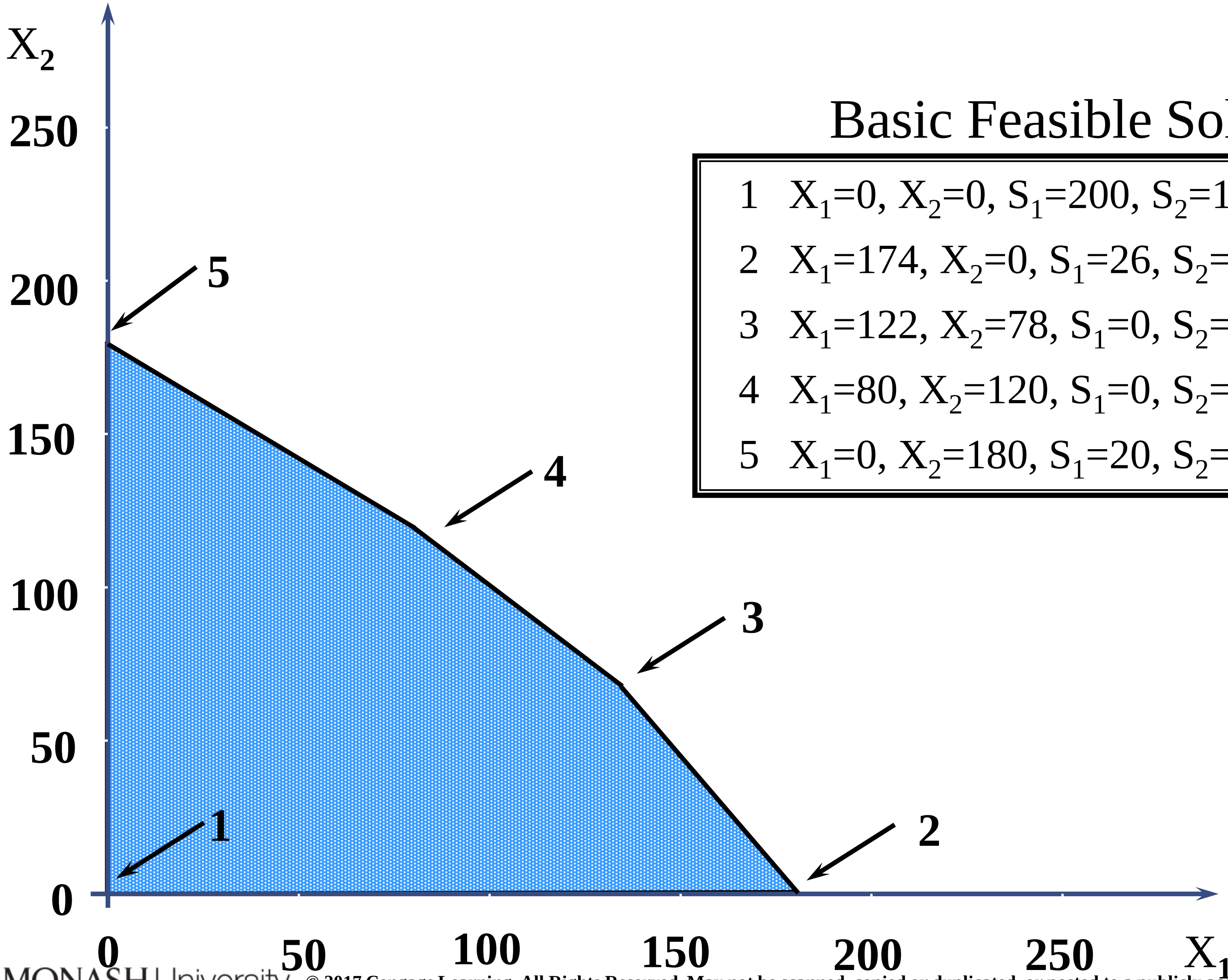
- If there are n variables in a system of m equations (where $n \geq m$) we can select any m variables and solve the equations (setting the remaining n-m variables to zero.)

Possible Basic Feasible Solutions

	Basic Variables	Nonbasic Variables	Solution	Objective Value
1	S_1, S_2, S_3	X_1, X_2	$X_1=0, X_2=0, S_1=200, S_2=1566, S_3=2880$	0
2	X_1, S_1, S_3	X_2, S_2	$X_1=174, X_2=0, S_1=26, S_2=0, S_3=792$	60,900
3	X_1, X_2, S_3	S_1, S_2	$X_1=122, X_2=78, S_1=0, S_2=0, S_3=168$	66,100
4	X_1, X_2, S_2	S_1, S_3	$X_1=80, X_2=120, S_1=0, S_2=126, S_3=0$	64,000
5	X_2, S_1, S_2	X_1, S_3	$X_1=0, X_2=180, S_1=20, S_2=486, S_3=0$	54,000
6*	X_1, X_2, S_1	S_2, S_3	$X_1=108, X_2=99, S_1=-7, S_2=0, S_3=0$	67,500
7*	X_1, S_1, S_2	X_2, S_3	$X_1=240, X_2=0, S_1=-40, S_2=-594, S_3=0$	84,000
8*	X_1, S_2, S_3	X_2, S_1	$X_1=200$, X2=0, $S_1=0, S_2=-234, S_3=480$	70,000
9*	X_2, S_2, S_3	X_1, S_1	$X_1=0, X_2=200, S_1=0, S_2=366, S_3=-320$	60,000
10*	X_2, S_1, S_3	X_1, S2	$X_1=0, X_2=261, S_1=-61, S_2=0, S_3=-1296$	78,300

* denotes infeasible solutions

Basic Feasible Solutions & Extreme Points



Basic Feasible Solutions

1	$X_1=0, X_2=0, S_1=200, S_2=1566, S_3=2880$
2	$X_1=174, X_2=0, S_1=26, S_2=0, S_3=792$
3	$X_1=122, X_2=78, S_1=0, S_2=0, S_3=168$
4	$X_1=80, X_2=120, S_1=0, S_2=126, S_3=0$
5	$X_1=0, X_2=180, S_1=20, S_2=486, S_3=0$

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Simplex Method Summary

- Identify any basic feasible solution (or extreme point) for an LP problem, then moving to an adjacent extreme point if such a move improves the value of the objective function.
- Moving from one extreme point to an adjacent one occurs by switching one of the basic variables with one of the nonbasic variables to create a new basic feasible solution (for an adjacent extreme point).
- When no adjacent extreme point has a better objective function value, stop -- the current extreme point is optimal.

End of Lecture 3

Content References:

Ragsdale, C. Spreadsheet Modeling & Decision Analysis: A Practical Introduction to Business Analytics (8e/9e) Cengage Learning: Chapter 4

Tutorial 2 this week:

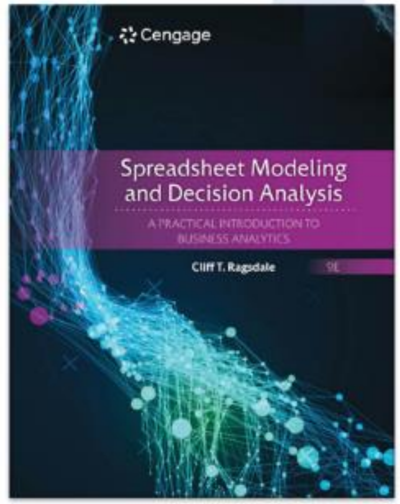
- LP with Spreadsheet Modelling
  - Goals for Good Spreadsheet Design
  - Implementing LP Problems in Spreadsheet
  - Solution Analysis

Homework

- Go through the Sensitivity, Answer and Limits reports in today's examples.
- Make sure you know how to interpret the results generated in those reports.
- Read Ragsdale Chapter 4.

☞ ***Readings for next week Lecture:***

Ragsdale, C. Spreadsheet Modeling & Decision Analysis: A Practical Introduction to Business Analytics (8e/9e) Cengage Learning: Chapter 6



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