FIT3158
Business Decision Modelling

Lecture 8

Inventory Modelling under Uncertainty

Topics Covered:

Inventory Management: Stochastic Demand

- Review of the Normal Distribution
- Single-Period Order Quantity Model
- Reorder-Point Quantity Model
- Periodic-review Order Quantity Model

Probabilistic Models

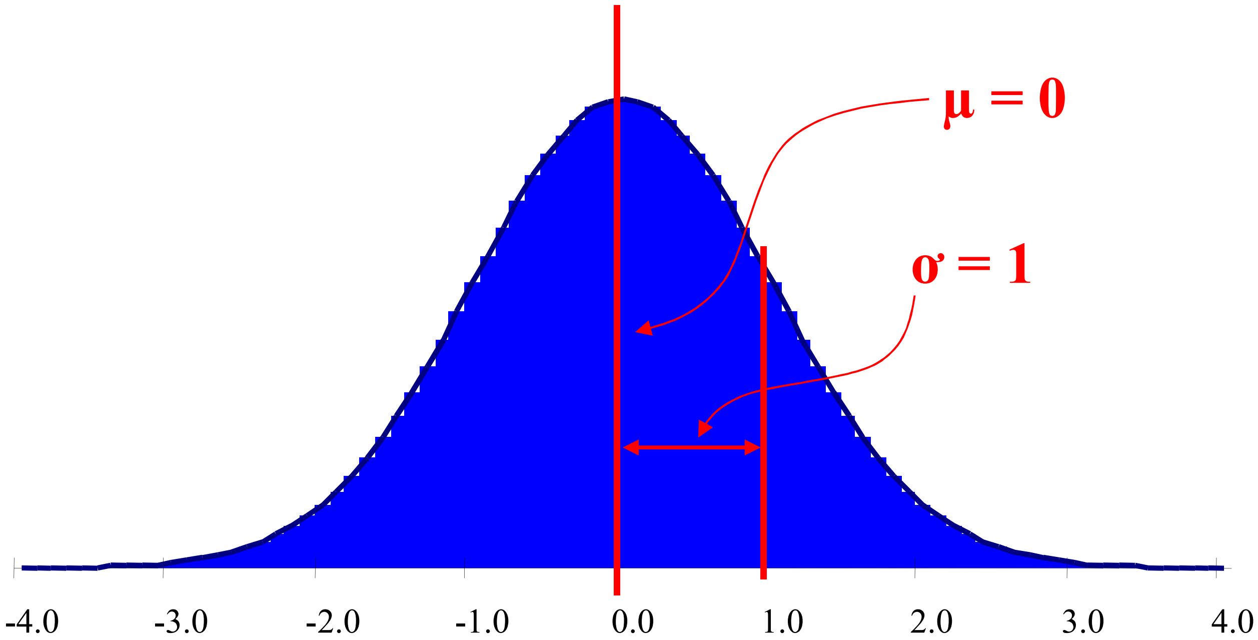
- In many cases demand (or some other factor) is not known with a high degree of certainty and a probabilistic inventory model should be used.
- These models tend to be more complex than deterministic models.
- Although in theory, any (non-negative integer-valued) probability distribution may define customer demand, the Normal distribution is a good general-purpose distribution, and forms the basis of the following examples.
- We will first review the Normal distribution … .

The Normal Distribution

- Arguably the most important distribution in statistics
- Arises when we measure a large number of nearly identical objects subject to random fluctuations – e.g., height, weight, response time, etc. (Used a lot in biometric population studies.)
- The Normal Distribution arises when we take the sums or means of a large number of observations from any distribution and thus provides the basis for sampling theory (through the Central Limit Theorem).

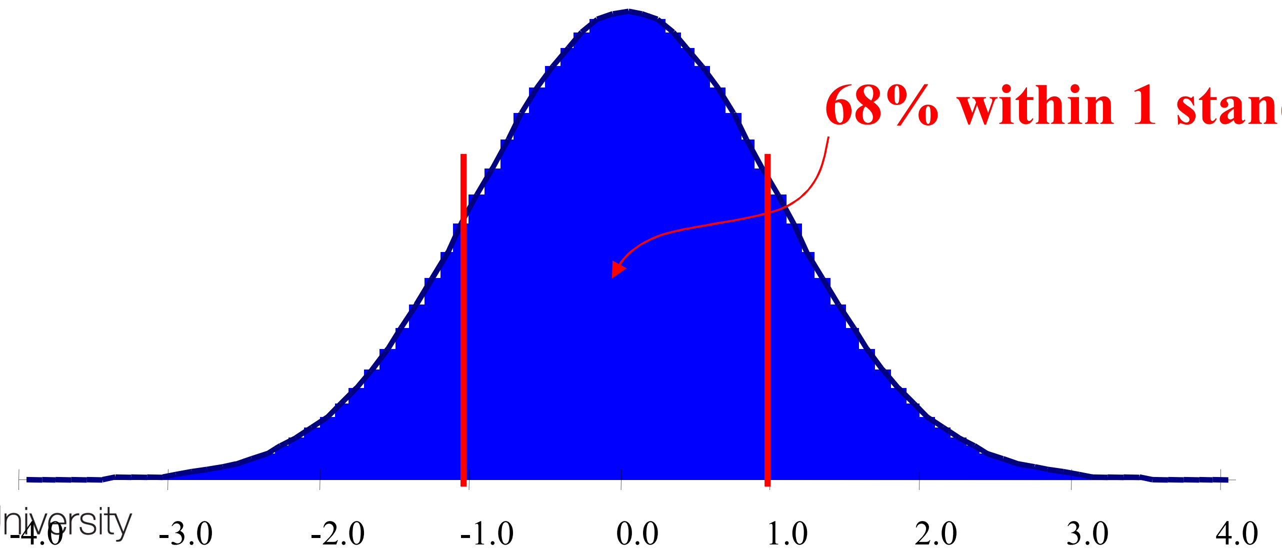
The Bell Curve

- The shape of the normal distribution can be seen below. The shape is often referred to as the bell curve (or bell-shaped curve). The distribution below has a mean of 0 and a standard deviation of 1 and is called the ***Standard Normal***.



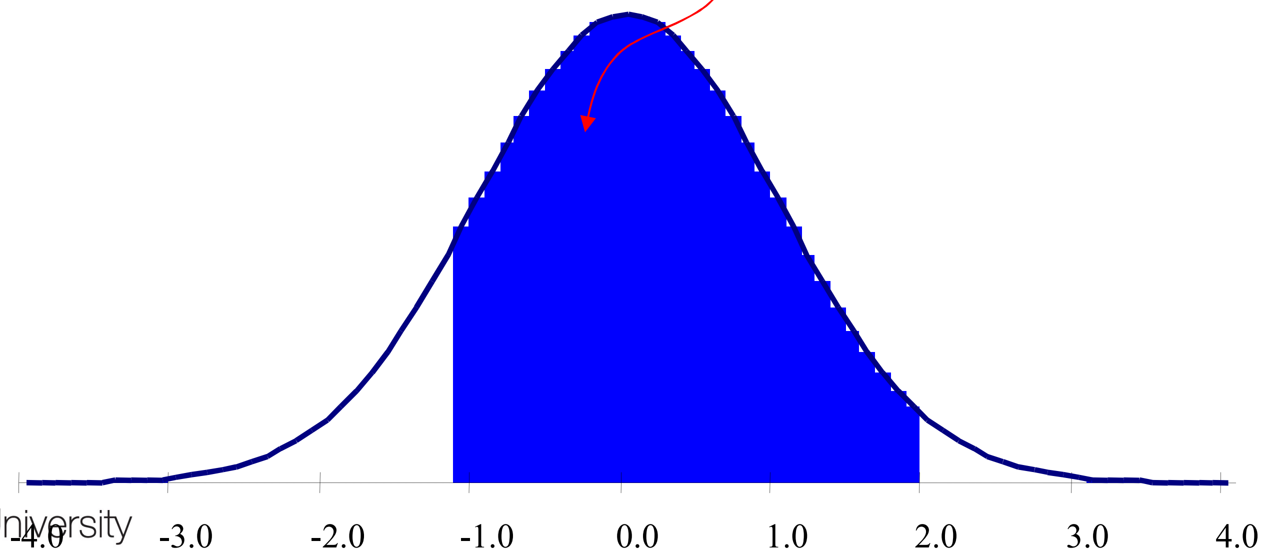
General Properties 1

- The total area under the curve = 1,
  - ❖ 68% of the area is within 1 standard deviation of the mean.
  - ❖ 95% of the area is within 2 standard deviations of the mean.
  - ❖ 99.7% of the area is within 3 standard deviations of the mean.



General Properties 2

- As the Normal Distribution is a continuous distribution, the probability that *Z* equals a particular value is zero. Instead we find the probability that *Z* lies within a particular range by calculating the area under the curve.
- E.g., $P(Z < 2)$, $P(Z > 1.2)$, $P(-1.1 < Z < 2.0)$, etc.



Calculating Normal Probabilities

- The formula of the Normal Distribution will not be given and does not need to be learned for this unit.
- Normal probabilities are calculated as the area under the curve.
- As the curve has an integral which does not yield analytical solutions, a table of probabilities based on the Cumulative Standard Normal Distribution is usually used for determining Normal probabilities.
- The Excel Function `=NORMDIST(Z,Mean,Stdev,TRUE)’ can also be used for determining Cumulative Normal Probabilities.

Standard Normal CDF Table

Cumulative Probabilities for the Standard Normal Distribution

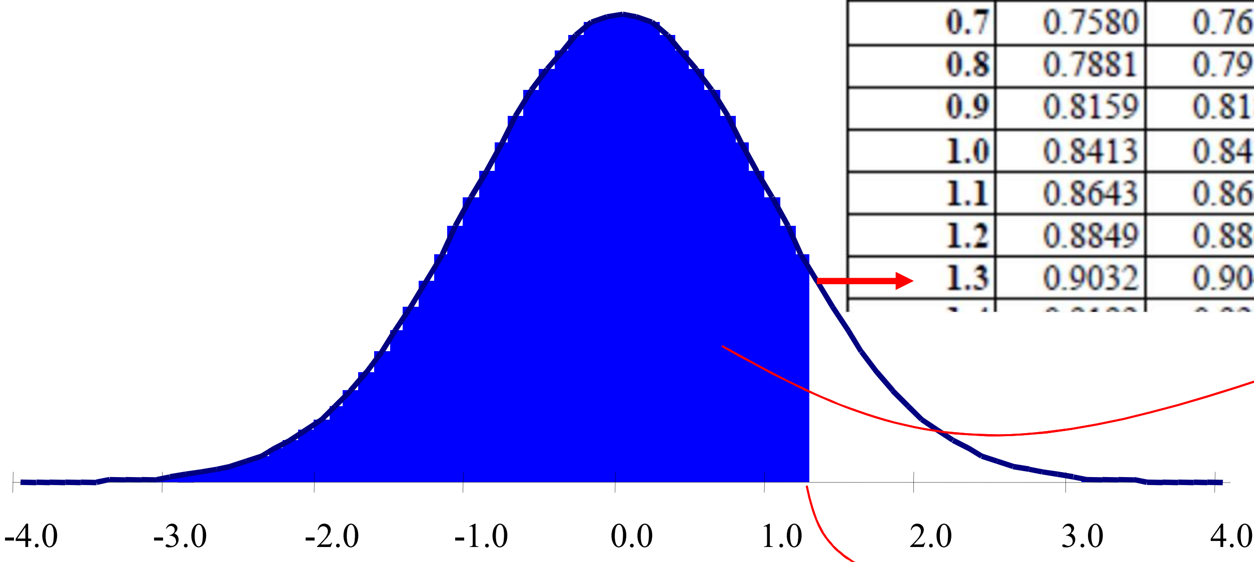
Table gives P(Z<z) for Z = N(0,1)

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319

P(Z < a) when a is positive

P(Z < 1.33) = 0.9082

z	0.00	0.01	0.02	0.03
0.0	0.5000	0.5040	0.5080	0.5120
0.1	0.5398	0.5438	0.5478	0.5517
0.2	0.5793	0.5832	0.5871	0.5910
0.3	0.6179	0.6217	0.6255	0.6293
0.4	0.6554	0.6591	0.6628	0.6664
0.5	0.6915	0.6950	0.6985	0.7019
0.6	0.7257	0.7291	0.7324	0.7357
0.7	0.7580	0.7611	0.7642	0.7673
0.8	0.7881	0.7910	0.7939	0.7967
0.9	0.8159	0.8186	0.8212	0.8238
1.0	0.8413	0.8438	0.8461	0.8485
1.1	0.8643	0.8665	0.8686	0.8708
1.2	0.8849	0.8869	0.8888	0.8907
1.3	0.9032	0.9049	0.9066	0.9082



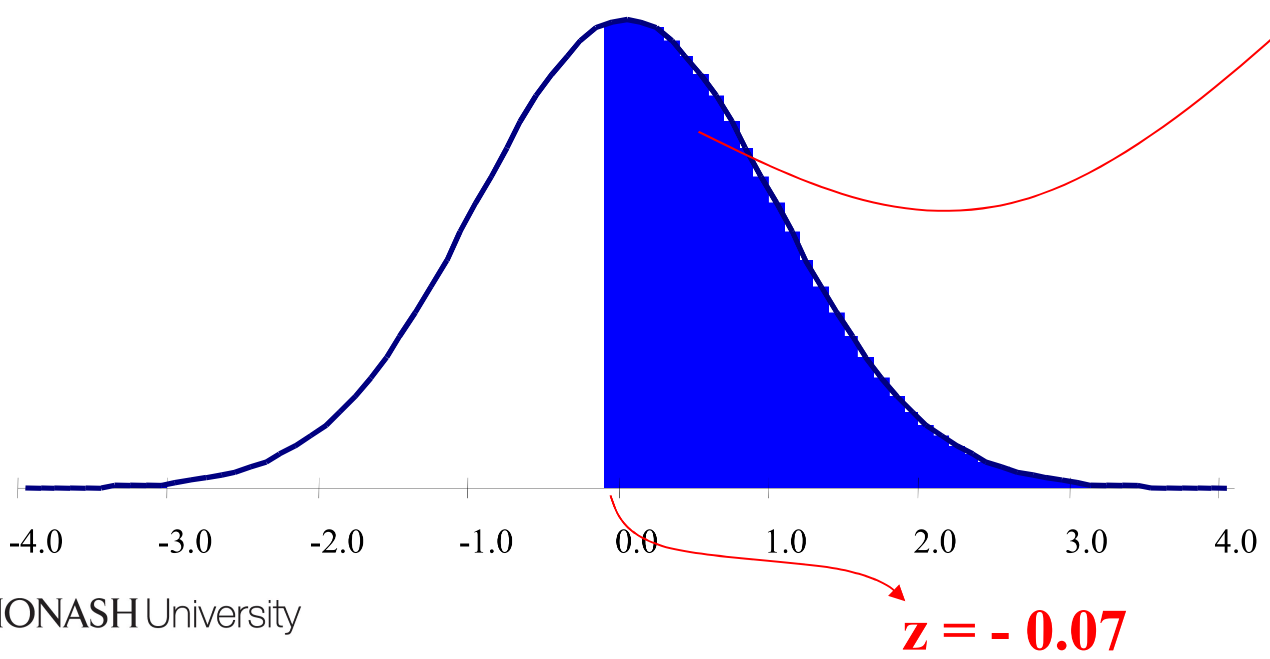
z = 1.33

P(Z > a) when a is negative

P(Z > -0.07) = P(Z < 0.07) = 0.5279

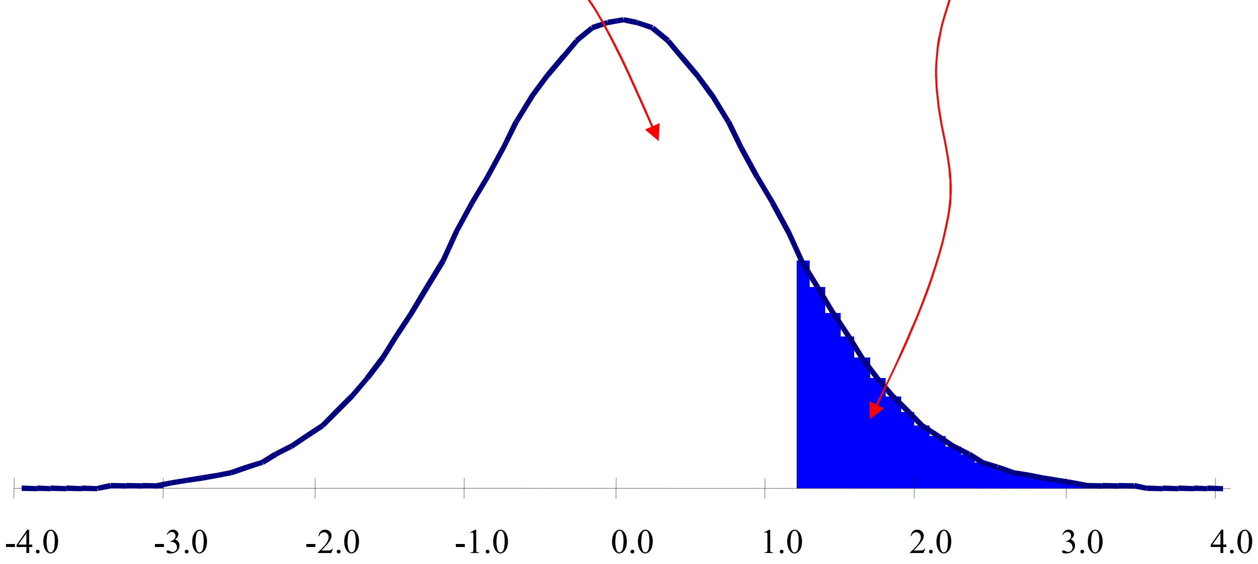
when a is negative.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675



P(Z > a) when a is positive

$P(Z > 1.21) = 1 - P(\color{red}{\mathbf{Z < 1.21}}) \sim 1 - 0.8869 \sim 0.1131$

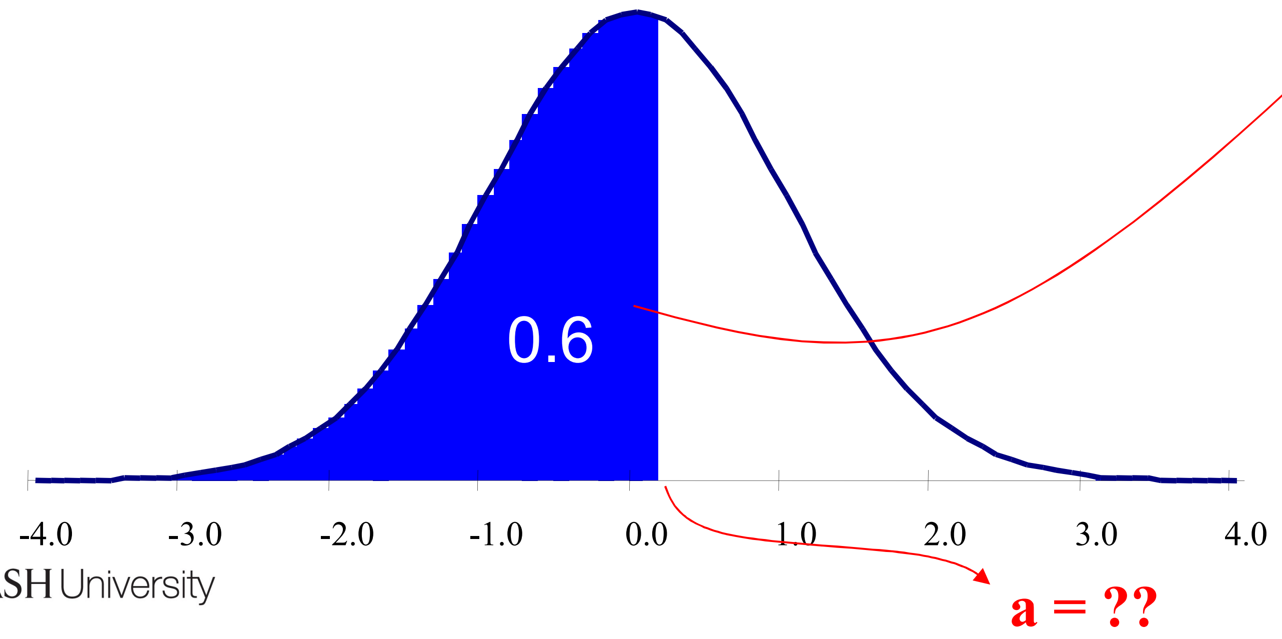


Inverse CDF

We frequently want to find a value of Z for a given probability.
For example find ***a*** such that *P(Z < a) = 0.6*
We read the tables in reverse and find that *a* ~ 0.255

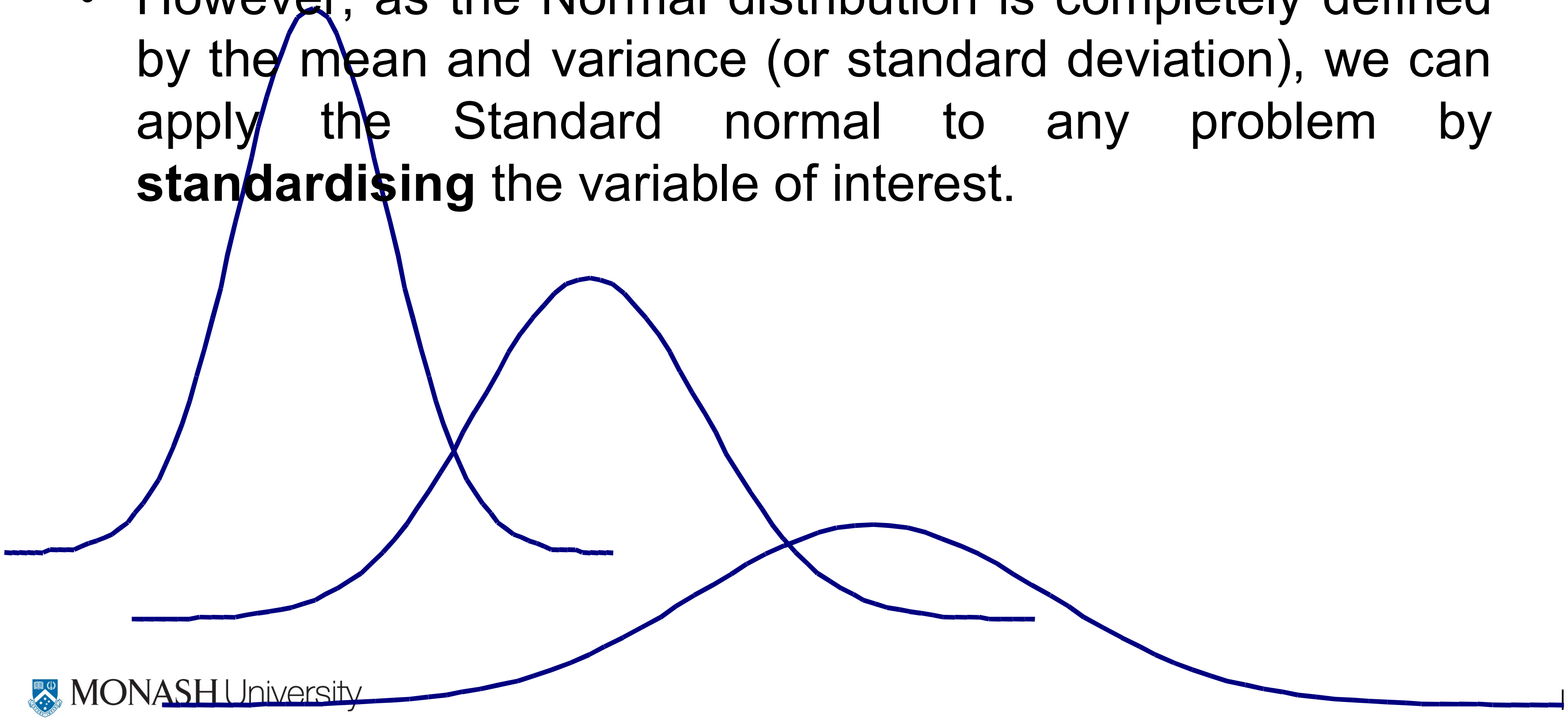
z	0.00	0.01	0.02	0.03	0.04	0.05	0.06
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026

a ~ 0.255



The Normal Distribution in Practice

- The Standard Normal distribution has mean = 0 and variance = 1. This might seem to limit the applications of the Standard Normal to everyday problems.
- However, as the Normal distribution is completely defined by the mean and variance (or standard deviation), we can apply the Standard normal to any problem by **standardising** the variable of interest.



Standardising Variables

Let X be a normal variable with mean μ and variance σ^2.

We can write $X \approx \mathrm{N}(\mu, \sigma^2)$. We can standarise X by use of the formula: $z = \dfrac{x - \mu}{\sigma}$ and $\mathrm{Z} \approx \mathrm{N}(0,1)$. In this way we can apply the Standard Normal probabilities to any problem.

Example

- The number of customers entering a shop is known from historical information to be normally distributed with a mean of 365 and a standard deviation of 33. What is the probability that on any given day the number of customers will be less than 320?

From the problem, $X \approx \mathrm{N}(\mu = 365, \sigma^2 = 33^2)$. thus $P(X < 320)$

becomes $\mathrm{P}\left(\mathrm{Z} < \dfrac{320 - 365}{33}\right)$ and $\mathrm{Z} \approx \mathrm{N}(0,1)$.

Thus we calculate $\mathrm{P}(\mathrm{Z} < -1.36) = 1 - 0.9131 = 0.0869$

recall approx. slide 10

Example

- The proprietor of the shop described in the previous question wants to set her staffing levels. She wants to be 80% sure of being able to meet customer demand. What number of customers per day should she plan for?

From the problem, $X \approx \mathrm{N}(\mu = 365, \sigma^2 = 33^2)$. We want to find α such that $\mathrm{P}\left(Z < \dfrac{\alpha - 365}{33}\right) = 0.8$. From the tables, $\mathrm{P}(Z < 0.845) = 0.8$ and so $\alpha = 0.845 \times 33 + 365 = 392.85$, approximately 393 customers per day.

recall approx. slide 10

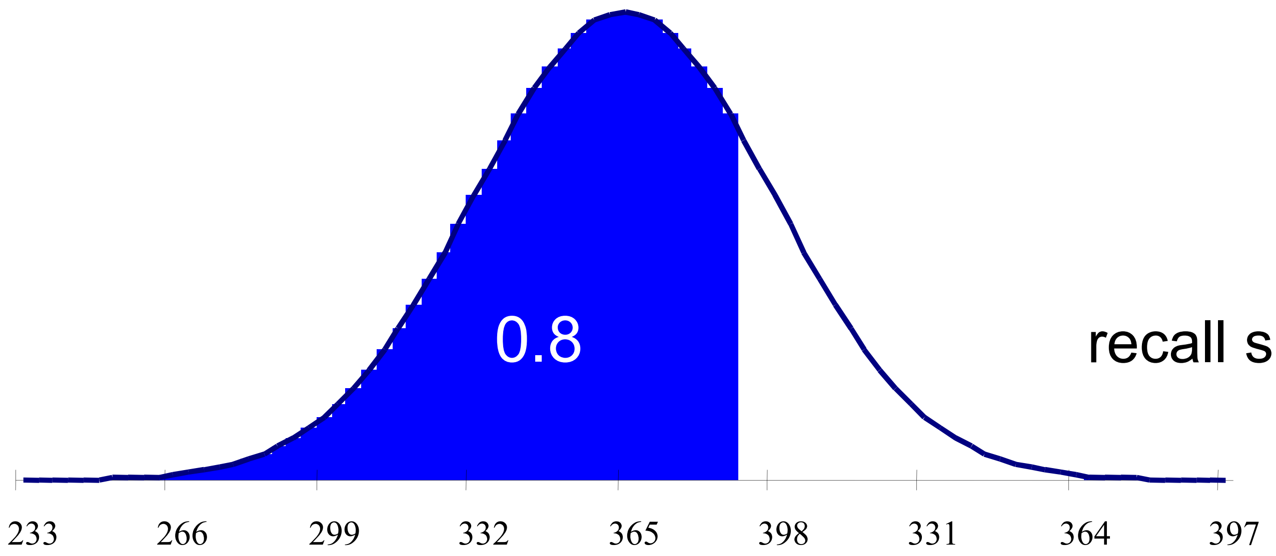
Therefore:

What is the probability that she gets less than 393 customers?

– $P(X < 393)$ with $X \sim N(365,33)$

$P(X < 393) = P(Z < (393-365)/33)$

$\sim P(Z < 0.845) = 0.8,$ where $Z \sim N(0,1)$



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Single-Period Order Quantity

- A single-period order quantity model (sometimes called the newsboy/newsvendor problem) deals with a situation in which only one order is placed for the item and the demand is probabilistic.
- If the period’s demand exceeds the order quantity, the demand is not back-ordered and revenue (profit) will be lost.
- If demand is less than the order quantity, the surplus stock is sold at the end of the period (usually for less than the original purchase price).

Single-Period Order Quantity

Assumptions:

- Period demand follows a known probability distribution. For example:
  - normal: mean is μ, standard deviation is σ
  - uniform: minimum is a, maximum is b
- Cost of over-estimating demand: $\$c_v$
- Cost of under-estimating demand: $\$c_u$
- Shortages are not back-ordered.
- Period-end stock is sold for salvage or discarded and is not sold in the following period.

Single-Period Order Quantity: Formulas

– Optimal probability of no shortage:

$$\mathrm{P}\left(\text{demand} < \mathrm{Q}^{*}\right) = \frac{C_u}{C_u + C_v}$$

– Optimal probability of shortage:

$$\mathrm{P}\left(\text{demand} > \mathrm{Q}^{*}\right) = 1 - \frac{C_u}{C_u + C_v}$$

– Optimal order quantity, based on demand distribution:

$$\text{Normal}\left(\mu, \sigma^{2}\right) : \mathrm{Q}^{*} = \mu + z\sigma$$

$$\text{Uniform}\left[a, b\right] : \mathrm{Q}^{*} = a + P\left(demand \le Q^{*}\right)\left(b - a\right)$$

Single-Period Order Quantity

- Calculating C_u and C_v

$$\text{Cost of overestimating demand} = C_v$$

$$\text{Cost of underestimating demand} = C_u$$

$$let$$

$$\text{Unit Cost} = c$$

$$\text{Penalty for item held at end of inventory cycle} = h_E$$

$$\text{Penalty for each item short (goodwill etc.)} = P_S$$

$$\text{Selling Price} = P_R$$

$$then$$

$$C_v = h_E + c$$

$$C_u = P_S + P_R - c$$

Single-Period Order Quantity

- Consequently, we can rewrite
  - Optimal probability of no shortage:

$$\mathrm{P}\left(\text{demand} < \mathrm{Q}^*\right) = \frac{C_u}{C_u + C_v} = \frac{\left(P_S + P_R - c\right)}{\left(P_S + P_R - c\right) + \left(h_E + c\right)}$$

  - Example:

    A Gotham City company sells Christmas trees. Trees have a demand with mean = μ = 2,000, and s.d. = σ = 500 trees. Trees sell for an average of \$9 and cost \$3 each. The loss of goodwill from a lost sale is estimated to be \$1. Unsold trees are converted to pulp at a cost of \$.50 = 50c per tree.

Christmas tree example (continued) …

- So, as per calculations below, the company should order Q* = 2,216 Christmas trees to maximise their profit. You are not required to calculate the value of lost sales or expected profit for Normal Distribution problems. mean = μ = 2,000, s.d. = σ = 500.

$$h_E = 0.50 \quad c = 3 \quad P_S = 1 \quad P_R = 9$$

$$C_v = h_E + c = 0.50 + 3 = 3.5$$

$$C_u = P_S + P_R - c = 1 + 9 - 3 = 7$$

$$p(\text{demand} < Q^*) = \frac{7}{7 + 3.5} = 0.6667$$

$$Q^* = 0.4307 \sim N(0,1)$$

$$Q^* = 2{,}000 + 0.4307(500) \sim N(2{,}000, 500^2)$$

$$= 2{,}215.37$$

Adaptation for the Uniform Distribution

- Let us now assume that the demand for Christmas trees is uniformly distributed between 1,500 and 2,500 trees.
- As before:

$$C_v = h_E + c = 0.50 + 3 = 3.5$$

$$C_u = P_S + P_R - c = 1 + 9 - 3 = 7$$

$$p(\text{demand} < Q^*) = \frac{7}{7 + 3.5} = 0.6667$$

$$Q^* = 0.6667 \sim U(0,1)$$

$$Q^* = 1{,}500 + 0.6667(2{,}500 - 1{,}500)$$

$$= 2{,}166.7 \sim U(1{,}500,\ 2{,}500)$$

- **So the company should order 2,167 Christmas trees to maximise their profit.**

Adaptation for an Empirical Discrete Distribution

Let us now assume that the demand for Christmas trees has the distribution given on the right.

- As before:

$$p(\text{demand} < Q^*)$$

$$= \frac{7}{7 + 3.5} = 0.6667$$

Demand	Probability	Cumulative Probability
1500	0.05	0.05
1600	0.06	0.11
1700	0.09	0.20
1800	0.12	0.32
1900	0.17	0.49
2000	0.20	0.69
2100	0.12	0.81
2200	0.08	0.89
2300	0.06	0.95
2400	0.04	0.99
2500	0.01	1.00

- **So the company should order 2,000 Christmas trees to maximise their profit as this is the smallest demand exceeding the cumulative probability corresponding to the maximum profit.**

Under Stock/Overstock Costs

Refer to Extra notes in Lecture 8.xlsx

- For a given order quantity, (say, Q* = 2000) we can work out the expected costs of being under-stocked and over-stocked during the order period. This would, in turn, allow us to calculate the expected profit over each order period.

Demand	Probability	Surplus (Q*=2000)	Surplus*P	Shortage (Q*=2000)	Shortage*P
1500	0.05	500.0	25.0		
1600	0.06	400.0	24.0		
1700	0.09	300.0	27.0		
1800	0.12	200.0	24.0		
1900	0.17	100.0	17.0		
2000	0.20	0.0	0.0	0.0	0.0
2100	0.12			100.0	12.0
2200	0.08			200.0	16.0
2300	0.06			300.0	18.0
2400	0.04			400.0	16.0
2500	0.01			500.0	5.0
Expected			117.0		67.0
Cost		Overstock	409.5	Understock	469.0

Example: McHardee Press

- McHardee Press publishes the Fast Food Restaurant Menu Book and wishes to determine how many copies to print. There is a fixed cost of $5,000 to produce the book and the incremental profit per copy is $0.45 = 45c. Any unsold copies of the book can be sold at salvage at a $0.55 = 55c loss.
- Sales for this edition are estimated to be Normally distributed. The most likely sales volume is 12,000 copies and they believe there is a 5% chance that sales will exceed 20,000.
- How many copies should be printed?

Example: McHardee Press

- **Single-Period Order Quantity**

$$m = 12{,}000$$

- To find σ, note that $z = 1.65$ corresponds to a 5% tail probability. Therefore,

  $8000 = (20{,}000 - 12{,}000) = 1.65\sigma$, or $\sigma \sim 4848$ books.

- Using incremental analysis (or recall slide 22)

  with $C_v = .55$ and $C_u = .45$,

$$(C_u/(C_u+C_v)) = .45/(.45 + .55) = .45$$

- Find Q * such that $P(D < Q^*) = 0.45$. The probability of 0.45 corresponds to $z = -0.12$. Thus,

$$Q^* = 12{,}000 - 0.12\ (4848) = 11{,}418 \text{ books}$$

Example: McHardee Press

Single-Period Order Quantity (revised)

- If any unsold copies of the book can be sold at salvage at a \$0.65 = 65c loss, how many copies should be printed?

$$C_v = .65,\ (C_u/(C_u + C_v)) = .45/(.45 + .65) = .4091$$

- Find Q * such that P($D < Q$ *) = .4091.

z ~ -0.23 gives this probability. Thus,

$$Q^* \sim 12{,}000 - 0.23\ (4848) = 10{,}885 \text{ books}$$

- However, since this is less than the break-even volume of 11,111 (= 5000/0.45) books, <u>no copies should be printed</u> because if the company produced only 10,885 copies it will not recoup its \$5,000 fixed cost of producing the book.

Re-order Point Quantity

- A firm's inventory position consists of the on-hand inventory plus on-order inventory (all amounts previously ordered but not yet received).
- An inventory item is re-ordered when the item's inventory position reaches a predetermined value, referred to as the re-order point.
- The re-order point represents the quantity available to meet demand during lead time.
- Lead time is the time span starting when the replenishment order is placed and ending when the order arrives.

Re-order Point Quantity

- Under deterministic conditions, when both demand and lead time are constant, the re-order point associated with EOQ-based models is set equal to lead time demand.
- Under probabilistic conditions, when demand and/or lead time varies, the re-order point often includes safety stock.
- <u>Safety stock</u> is the amount by which the re-order point exceeds the expected (average) lead time demand.

Safety Stock and Service Level

- The amount of safety stock in a re-order point determines the chance of a stockout during lead time.
- The complement of this chance is called the service level.
- Service level, in this context, is defined as the probability of not incurring a stockout during any one lead time.
- Service level, in this context, also is the long-run proportion of lead times in which no stockouts occur.
- We look at two models, a re-order point, and a periodic review model.

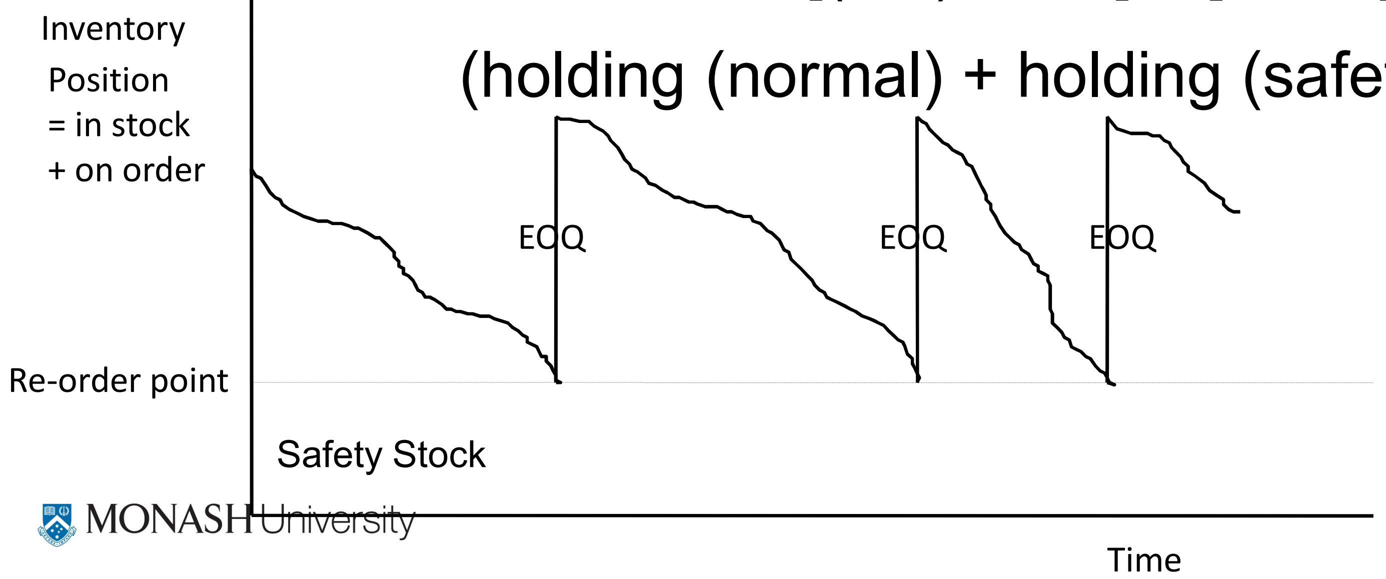
Re-order Point

- **Assumptions**
  - Lead-time demand is normally distributed with mean μ and standard deviation σ.
  - Approximate optimal order quantity: EOQ
  - Service level is defined in terms of the probability of no stockouts during lead time and is reflected in z.
  - Shortages are not back-ordered.
  - Inventory position is reviewed continuously.

Re-order Point

- Formulas
  - Re-order point: $r = \mu + z\sigma$
  - Safety stock: $z\sigma$
  - Average inventory: $1/2(Q) + z\sigma$
  - Total annual cost: $[(1/2)Q^{*}ch] + [z\sigma\, ch] + [Ak/Q^{*}]$

    (holding (normal) + holding (safety) + ordering)



Example: Robert's Pharmacy

Re-order Point Model

- Robert's Pharmacy is a Pharmacy wholesaler supplying 55 independent Pharmacy stores. Robert's wishes to determine an optimal inventory policy for *Comfort* brand headache remedy. Sales of *Comfort* are relatively constant as the past 10 weeks of data indicate:

Week	Sales (cases)	Week	Sales (cases)
1	110	6	120
2	115	7	130
3	125	8	115
4	120	9	110
5	125	10	130

Example: Robert's Pharmacy

- Each case of *Comfort* costs Robert's \$10 and Robert's uses a 14% annual holding cost rate for its inventory. The cost to prepare a purchase order for *Comfort* is \$12. What is Roberts' optimal order quantity?

- The average weekly sales over the 10 week period is 120 cases.

  Hence A = 120 x 52 = 6,240 cases per year;

  ch = (.14)(10) = 1.40; k = 12. The EOQ is:

$$Q^* = \sqrt{\frac{2Ak}{ch}} = \sqrt{\frac{2 \times 6240 \times 12}{1.40}} = 327$$

Example: Robert’s Pharmacy

- The lead time for a delivery of *Comfort* has averaged four working days. Lead time demand has therefore been *estimated* as having a Normal distribution with a mean of 80 cases and a standard deviation of 10 cases.

- Roberts wants at most a 2% probability of selling out of *Comfort* during this lead time. What re-order point should Robert’s use?

Example: Robert's Pharmacy

<u>*Optimal Re-order Point*</u>

- Lead time demand is Normally distributed with $\mu = 80$, $\sigma = 10$.
- Since Roberts wants at most a 2% probability of selling out of *Comfort*, the corresponding *z* value is 2.06.
- That is, $P(z > 2.06) \sim 0.0197$ (about 0.02). Hence, Robert's should reorder *Comfort* when supply reaches

$$\mu + z\sigma = 80 + 2.06(10) = 101 \text{ cases.}$$

- The safety stock is 21 cases.

101 – 80

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Example: Robert's Pharmacy

Total Annual Inventory Cost

- Ordering:

  $$(Ak/Q^*) = ((6240)(12)/327) \quad = \$229$$

- Holding-Normal:

  $$(1/2)Q^*ch = (1/2)(327)(1.40) \quad = 229$$

- Holding-Safety Stock:

  $$ch(21) = (1.40)(21) \quad = 29$$

- TOTAL = $487

Periodic Review System

- A <u>periodic review system</u> is one in which the inventory level is checked and re-ordering is done only at specified points in time (at fixed intervals usually).
- Assuming the demand rate varies, the order quantity will vary from one review period to another. (This is in contrast to the continuous review system in which inventory is monitored continuously and a fixed-quantity order can be placed whenever the re-order point is reached.)
- At the time a periodic-review order quantity is being decided, the concern is that the on-hand inventory and the quantity being ordered is enough to satisfy demand from the time this order is placed until the next order is *received* (not placed).

Periodic Review Order Quantity

Assumptions

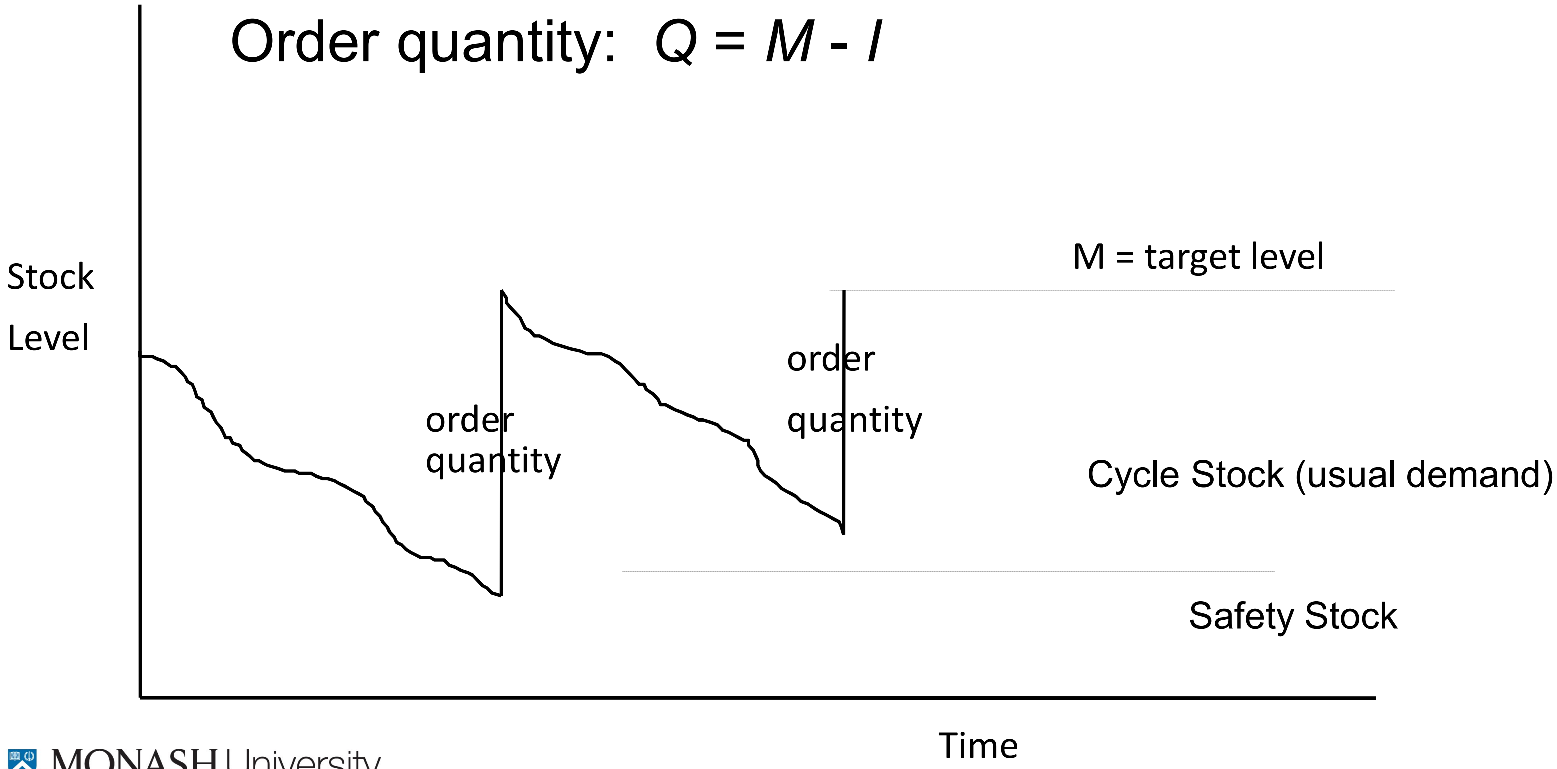
- Inventory position is reviewed at constant intervals (periods).
- Demand during review period plus lead time period is Normally distributed with mean μ and standard deviation σ.
- Service level is defined in terms of the probability of no stockouts during a review period plus lead time period and is reflected in z.
- On-hand inventory at ordering time: I
- Shortages are not back-ordered.
- Lead time is less than the length of the review period.

Periodic Review Order Quantity

- **Formulas**

Replenishment level: $M = \mu + z\sigma$

Order quantity: $Q = M - I$



Example: Ace Brush

Periodic Review Order Quantity Model

- Joe Walsh is a salesman for the Ace Brush Company. Every four weeks he contacts Dollar Department Store so that they may place an order to replenish their stock. Weekly demand for Ace brushes at Dollar approximately follows a Normal distribution with a mean of 60 brushes and a standard deviation of 9 brushes.
- Once Joe submits an order, the lead time until Dollar receives the brushes is one week. Dollar would like at most a 2% chance of running out of stock during any replenishment period. If Dollar has 75 brushes in stock when Joe contacts them, how many should they order?

Example: Ace Brush

Demand During Uncertainty Period (Lead time)

- The review period plus the following lead time totals 4 weeks. This is the amount of time that will elapse before the next shipment of brushes will arrive.
- Weekly demand is normally distributed with:

  Mean weekly demand, μ = 60

  Weekly standard deviation, σ = 9

  Weekly variance, σ^2 = 81

- Demand for 4 weeks is normally distributed with:

  Mean demand over 4 weeks, $\mu = 4 \times 60$ = 240

  Variance of demand over 4 weeks, $\sigma^2 = 4 \times 81$ = 324

  Standard deviation over 4 weeks, $\sigma = \sqrt{(324)}$ = 18

Example: Ace Brush

Replenishment Level

- $M = \mu + z\sigma$ where z is determined by the desired stockout probability. For a 2% stockout probability (2% tail area), $z = 2.05$. Thus,

  $$M = 240 + 2.05(18) = 277 \text{ brushes}$$

- As the store currently has 75 brushes in stock, Dollar should order:

  $$277 + 60 - 75 = 262 \text{ brushes}$$

- The safety stock is:

  $$z\sigma = (2.05)(18) = 37 \text{ brushes}$$

Refer to Extra notes in Lecture 8.xlsx

End of Lecture 8

References:

Lapin, L. and Whisler, W., Quantitative Decision Making with Spreadsheet Applications 7th Ed., Wadsworth (Thomson Learning) Belmont, 2002: Chapter 16

Tutorial 7 (week 8) this week:

- Economic Order Quantity (EOQ) Model
- Quantity Discounts for the EOQ Model
- An Inventory Model with Planned Shortages
- Economic Production Lot Size Model

Homework

- ➢ Go through today's lecture examples for inventory modelling (under stochastic demand)
- ➢ Familiarise yourself with the following:
  - ✓ The Normal distribution
  - ✓ Single-period order quantity model
  - ✓ Reorder-point model
  - ✓ Periodic-review order model

☞ ***Readings for next week's Lecture:***

☞ Ragsdale, C. (2021). Spreadsheet Modeling & Decision Analysis: A Practical Introduction to Business Analytics (9e) Cengage Learning: Chapter 14

☞ [or Ragsdale, C. (2017). Spreadsheet Modeling & Decision Analysis: A Practical Introduction to Business Analytics (8e) Cengage Learning: Chapter 15]