# MAT1830

Lecture 20: Pascal's triangle

## 20.1 Pascal's triangle

We can write the binomial coefficients in an (infinite) triangular array as follows:

$$
\begin{array}{ccccccccccccc}
 & & & & & & \binom{0}{0} & & & & & & \\
 & & & & & \binom{1}{0} & & \binom{1}{1} & & & & & \\
 & & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & & \\
 & & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & & \\
 & & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & & \\
 & \binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5} & \\
\binom{6}{0} & & \binom{6}{1} & & \binom{6}{2} & & \binom{6}{3} & & \binom{6}{4} & & \binom{6}{5} & & \binom{6}{6} \\
 & \vdots & & & & & \vdots & & & & & \vdots &
\end{array}
$$

Here are the first ten rows with the entries as integers:

$$
\begin{array}{ccccccccccccccccccccc}
&&&&&&&&&&1&&&&&&&&&& \\
&&&&&&&&&1&&1&&&&&&&&& \\
&&&&&&&&1&&2&&1&&&&&&&& \\
&&&&&&&1&&3&&3&&1&&&&&&& \\
&&&&&&1&&4&&6&&4&&1&&&&&& \\
&&&&&1&&5&&10&&10&&5&&1&&&&& \\
&&&&1&&6&&15&&20&&15&&6&&1&&&& \\
&&&1&&7&&21&&35&&35&&21&&7&&1&&& \\
&&1&&8&&28&&56&&70&&56&&28&&8&&1&& \\
&1&&9&&36&&84&&126&&126&&84&&36&&9&&1& \\
1&&10&&45&&120&&210&&252&&210&&120&&45&&10&&1
\end{array}
$$

This triangular array is often called *Pascal's triangle* (although Pascal was nowhere near the first to discover it).

## 20.2 Patterns

Writing the binomial coeffcients this way reveals a lot of different patterns in them. Perhaps the most obvious is that every row reads the same left-to-right and right-to-left. Choosing $r$ elements from a set of $n$ elements to be in a combination is equivalent to choosing $n - r$ elements from the same set to not be in the combination. So:

$$\binom{n}{r} = \binom{n}{n-r} \text{ for } 0 \leqslant r \leqslant n.$$

This shows that every row reads the same left-to-right and right-to-left.

Suppose you're picking a starting team of 11 players from a squad of 15 players.

There are $\binom{15}{11}$ ways to choose 11 players for the starting team.

There are $\binom{15}{4}$ ways to choose 4 players to leave out of the starting team.

But this is just two different ways of expressing the same choice!
So we must have $\binom{15}{11} = \binom{15}{4}$.

The same logic shows that in general $\binom{n}{r} = \binom{n}{n-r}$ for each $r \in \{0, 1, \ldots, n\}$.

Another pattern is that every "internal" entry in the triangle is the sum of the two entries above it. To see why this is, we'll begin with an example.

**Example.** Why is $\binom{6}{2} = \binom{5}{2} + \binom{5}{1}$?

There are $\binom{6}{2}$ combinations of 2 elements of $\{1, 2, 3, 4, 5, 6\}$. Every such combination either

- does not contain a 6, in which case it is one of the $\binom{5}{2}$ combinations of 2 elements of $\{1, 2, 3, 4, 5\}$; or
- does contain a 6, in which case the rest of the combination is one of the $\binom{5}{1}$ combinations of 1 element from $\{1, 2, 3, 4, 5\}$.

So $\binom{6}{2} = \binom{5}{2} + \binom{5}{1}$.

Remember picking a starting team of 11 players from a squad of 15 players.

Imagine you have a sometimes-brilliant sometimes-terrible star player called Bob.

There are $\binom{14}{11}$ ways to a starting team of 11 players that does not include Bob.
(You need to pick 11 of the 14 other players.)

There are $\binom{14}{10}$ ways to a starting team of 11 players that includes Bob.
(You first pick Bob and then pick 10 of the 14 other players.)

But every possible choice of 11 players either includes Bob or does not!
So by the addition principle $\binom{15}{11} = \binom{14}{11} + \binom{14}{10}$.

The same logic shows that in general $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ for each $r \in \{1, 2, \ldots, n\}$.

We can make a similar argument in general. Let $X$ be a set of $n$ elements and $x$ is a fixed element of $X$. For any $r \in \{1, \dots, n\}$, there are $\binom{n-1}{r}$ combinations of $r$ elements of $X$ that do not contain $x$ and there are $\binom{n-1}{r-1}$ combinations of $r$ elements of $X$ that do contain $x$. So:

$$\boxed{\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1} \text{ for } 1 \leqslant r \leqslant n.}$$

This shows that every internal entry in Pascal's triangle is the sum of the two above it.

## Flux Exercise (LQMTZZ)

The binomial coefficent $\binom{0}{0}$ is the number of ways to choose zero things from the empty set. The value of $\binom{0}{0}$ is

A. 0

B. 1

C. Undefined

D. None of the above

## 20.3 The binomial theorem

$$
\begin{aligned}
(x+y)^0 &= 1\\
(x+y)^1 &= x+y\\
(x+y)^2 &= x^2+2xy+y^2\\
(x+y)^3 &= x^3+3x^2y+3xy^2+y^3\\
(x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4\\
(x+y)^5 &= x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5
\end{aligned}
$$

Notice that the coefficients on the right are exactly the same as the entries in Pascal's triangle. Why does this happen? Think about expanding $(x+y)^3$ and finding the coefficient of $xy^2$, for example.

$$
\begin{aligned}
(x+y)(x+y)(x+y) &= xxx+xxy+xyx+\underline{xyy}\\
&\quad +yxx+\underline{yxy}+\underline{yyx}+yyy\\
&= x^3+3x^2y+3xy^2+y^3
\end{aligned}
$$

The coefficient of $xy^2$ is 3 because we have three terms in the sum above that contain two $y$'s (those underlined). This is because there are $\binom{3}{2}$ ways to choose two of the three factors in a term to be $y$'s.

The same logic holds in general. The coefficient of $x^{n-r}y^r$ in $(x+y)^n$ will be $\binom{n}{r}$ because there will be $\binom{n}{r}$ ways to choose $r$ of the $n$ factors in a term to be $y$'s. This fact is called the *binomial theorem.*

**Binomial theorem** For any $n \in \mathbb{N}$,

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n.$$

# Questions

**20.1** Substitute $x = 1$ and $y = -1$ into the statement of the binomial theorem. What does this tell you about the rows of Pascal's triangle?

$$(1-1)^n = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots \pm \binom{n}{n}.$$

The alternating sum of the terms in each row is zero except the first row!

**20.2** Find a pattern in the sums of the rows in Pascal's triangle. Prove your pattern holds using the binomial theorem. Also prove it holds by considering the powerset of a set.

Substitute $x = 1, y = 1$ in the binomial theorem to get that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = (1+1)^n = 2^n.$$

If $|X| = n$ then $|\mathcal{P}(X)| = 2^n$. We count all subsets by counting those with 0 elements, then those with 1 element, then those with 2 elements and so on, up to those with $n$ elements.

**Flux Exercise** (LQMTZZ)

The number of terms that I would get if I expanded $(x + y)^{2021}$ using the binomial theorem is

A. 2019

B. 2020

C. 2021

D. 2022

## 20.4 Inclusion-exclusion

A school gives out prizes to its best ten students in music and its best eight students in art. If three students receive prizes in both, how many students get a prize? If we try to calculate this as $10 + 8$ then we have counted the three over-achievers twice. To compensate we need to subtract three and calculate $10 + 8 - 3 = 15$.

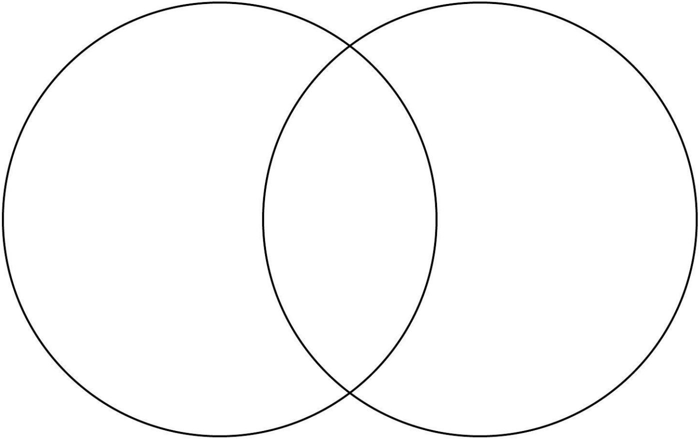
In general, if $A$ and $B$ are finite sets then we have

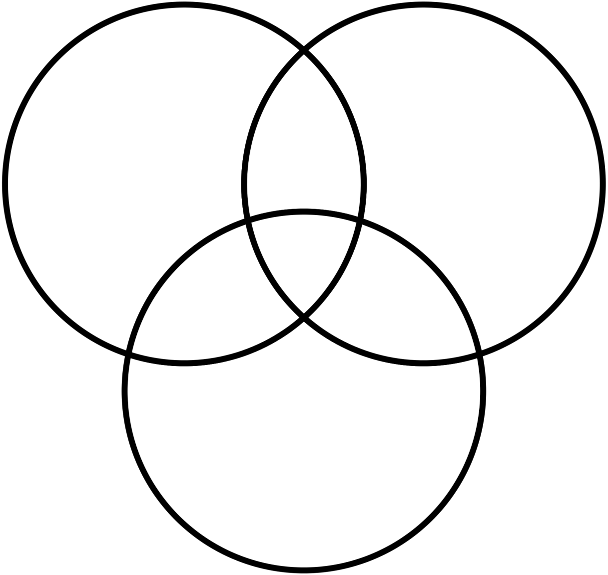
$$|A \cup B| = |A| + |B| - |A \cap B|.$$

With a bit more care we can see that if $A$, $B$ and $C$ are sets then we have

$$\begin{aligned} |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| \\ - |A \cap C| - |B \cap C| + |A \cap B \cap C|. \end{aligned}$$

This is part of a more general law called the *inclusion-exclusion* principle.





Let $X_1, X_2, \ldots, X_t$ be finite sets. To calculate $|X_1 \cup X_2 \cup \cdots \cup X_t|$:

- add the sizes of the sets;
- subtract the sizes of the 2-way intersections;
- add the sizes of the 3-way intersections;
- subtract the sizes of the 4-way intersections;

$$\vdots$$

- add/subtract the size of the $t$-way intersection.

To see why this works, think of an element $x$ that is in $n$ of the sets $X_1, X_2, \ldots, X_t$. It is counted

$$\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \binom{n}{4} + \cdots \pm \binom{n}{n}$$

times. By the Binomial theorem with $x = 1$ and $y = -1$ (see Question 20.1), this is equal to 1. So each element is counted exactly once.

# Question

**20.3** Use inclusion-exclusion to work out how many numbers in the set $\{1, \ldots, 100\}$ are divisible by 2 or 3 or 5.

Let $X_2$ be the set of numbers in $\{1, \ldots, 100\}$ that are divisible by 2.
Let $X_3$ be the set of numbers in $\{1, \ldots, 100\}$ that are divisible by 3.
Let $X_5$ be the set of numbers in $\{1, \ldots, 100\}$ that are divisible by 5.

We want $|X_2 \cup X_3 \cup X_5|$, which P.I.E. tells us is
$|X_2| + |X_3| + |X_5| - |X_2 \cap X_3| - |X_2 \cap X_5| - |X_3 \cap X_5| + |X_2 \cap X_3 \cap X_5|$.

These are much easier sets to count. For example, $|X_2 \cap X_5|$ counts the numbers that are divisible by 10 (of which there are 10).

So $|X_2 \cup X_3 \cup X_5| = 50 + 33 + 20 - 16 - 10 - 6 + 3 = 74$.