MAT1830 Sample Exam 1 Solutions

(1) (a) Calculate

$$\begin{array}{rcrcrcr} 504 & = & 1 & \times & 385 & + & 119 \\ 385 & = & 3 & \times & 119 & + & 28 \\ 119 & = & 4 & \times & 28 & + & 7 \\ 28 & = & 4 & \times & 7 & + & 0 \end{array}$$

So $\gcd(504, 385) = 7$. **[6]**

(b) No. If there was an integer y such that $504y \equiv 10 \pmod{385}$, then we would have, for some integer k, $504y = 385k + 10$ or, equivalently, $504y - 385k = 10$. This is impossible because we have just seen that 7 divides both 504 and 385, but obviously 7 does not divide 10. **[4]**

(c) Calculate

$$\begin{array}{rcrcl} 7 & = & 119 - (4 \times 28) & & \\ 7 & = & 119 - (4 \times (385 - 3 \times 119)) & = & -4 \times 385 + 13 \times 119 \\ 7 & = & -4 \times 385 + 13 \times (504 - 1 \times 385) & = & 13 \times 504 - 17 \times 385 \end{array}$$

So $504 \times 13 + 385 \times -17 = 7$ or, equivalently, $504 \times 13 = 385 \times 17 + 7$. So for $z = 13$ we have $504z \equiv 7 \pmod{385}$. **[4]**

(d) *Base step.* When $n = 1$, the left hand side of this equation is 5 and the right hand side is $\frac{5^2-5}{4}$ which equals 5. Thus the result is true for $n = 1$.

Induction step. Now assume that for some integer $k \geq 1$ the result is true for $n = k$. That is,

$$5 + 5^2 + 5^3 + \cdots + 5^k = \frac{5^{k+1} - 5}{4}$$

Now

$$\begin{aligned} 5 + 5^2 + 5^3 + \cdots + 5^{k+1} &= (5 + 5^2 + 5^3 + \cdots + 5^k) + 5^{k+1} \\ &= \frac{5^{k+1} - 5}{4} + 5^{k+1} \quad \text{(by our assumption)} \\ &= \frac{5^{k+1} - 5}{4} + \frac{4(5^{k+1})}{4} \\ &= \frac{5(5^{k+1}) - 5}{4} \\ &= \frac{5^{k+2} - 5}{4}. \end{aligned}$$

So, the result is true for $n = k + 1$.

Thus the statement is true for each integer $n \geq 1$ by mathematical induction. **[6]**

(2) (a) Truth tables:

p	q	r	$\neg q$	$p \to \neg q$	$(p \to \neg q) \wedge r$	$\neg((p \to \neg q) \wedge r)$	$p \wedge q$	$\neg r$	$(p \wedge q) \vee \neg r$
T	T	T	F	F	F	T	T	F	T
T	T	F	F	F	F	T	T	T	T
T	F	T	T	T	T	F	F	F	F
T	F	F	T	T	F	T	F	T	T
F	T	T	F	T	T	F	F	F	F
F	T	F	F	T	F	T	F	T	T
F	F	T	T	T	T	F	F	F	F
F	F	F	T	T	F	T	F	T	T

OR Logic laws:

$$\begin{aligned}\neg((p \to \neg q) \wedge r) &\equiv \neg(p \to \neg q) \vee \neg r \\ &\equiv \neg(\neg p \vee \neg q) \vee \neg r \\ &\equiv (p \wedge q) \vee \neg r\end{aligned}$$

So the propositions are logically equivalent. **[6]**

(b) $p \vee q$ **[2]**

(c) The statement is not valid. Consider the interpretation where x ranges over the integers, $P(x)$ is the predicate "x is negative" and $Q(x)$ is the predicate "x is positive". Under this interpretation $\exists x(P(x) \wedge Q(x))$ is false, (since no integer is both positive and negative). Also, $\exists x P(x)$ is true (since there exits a positive integer) and $\exists x Q(x)$ is true (since there exits a negative integer), and these two facts mean $(\exists x P(x)) \wedge (\exists x Q(x))$ is true. So under this interpretation $(\exists x(P(x) \wedge Q(x))) \leftrightarrow ((\exists x P(x)) \wedge (\exists x Q(x)))$ is false. **[6]**

(d) (i) is true under the given interpretation because there is a positive integer, namely 1, which is less than or equal to every positive integer.
(ii) is true under the given interpretation because for any positive integer, say y, there is a positive integer, for example $x = y$, which is less than or equal to y. **[6]**

(3) (a) (i) $\{3\}$ **[2]**

(ii) $\{3,6\}$ **[2]**

(iii) $\mathbb{N}$ **[2]**

(iv) $A \times \{1,2\}$ has $4 \times 2 = 8$ elements, because A has four elements and $\{1,2\}$ has two. Thus $\mathcal{P}(A \times \{1,2\})$ has $2^8 = 256$ elements. **[2]**

(b) (i) $f(\{2,3,6\}) = 6 - 2 = 4$.
$g(\{2,7,10\}) = \{2,7,10\} \cup \{1,2\} = \{1,2,7,10\}$. **[2]**

(ii) f is not one-to-one. For example, $f(\{1,2\}) = 2 - 1 = 1$ and $f(\{6,7\}) = 7 - 6 = 1$.
g is not one-to-one. For example, $g(\{1\}) = \{1\} \cup \{1,2\} = \{1,2\}$ and $g(\{1,2\}) = \{1,2\} \cup \{1,2\} = \{1,2\}$. **[3]**

(iii) The range of f is $\{0,1,\ldots,9\}$.
For all $X \in A$, $f(X) \geq 0$ because the greatest element in a non-empty subset of $\{1,2,\ldots,10\}$ is obviously greater than or equal to the smallest element in that set.
For all $X \in A$, $f(X) \leq 9$ because the greatest element in a non-empty subset of $\{1,2,\ldots,10\}$ is at most 10 and the smallest element is at least 1.
For any $y \in \{0,1,\ldots,9\}$, $\{1, y+1\} \in X$ and $f(\{1,y+1\}) = (y+1) - 1 = y$. **[3]**

(iv) Yes, because codomain(g) = domain(f).
$f \circ g(\{9\}) = f(g(\{9\})) = f(\{1,2,9\}) = 9 - 1 = 8$. **[2]**

(v) No, because codomain$(f) \neq$ domain(g). **[2]**

(4) (a) (i) R is reflexive because, for all $x \in A$, xRx (there is a loop on every vertex).
R is not symmetric because bRf but $f \not R b$, for example.
R is not antisymmetric because aRe and eRa, for example.
R is not transitive because, aRb and bRf but $a \not R f$, for example. **[4]**

(ii) R is neither a partial order relation nor an equivalence relation. **[1]**

(iii) S is reflexive because, for all $(w,x) \in \mathbb{Z}\times\mathbb{Z}$, $w+x-w-x=0$ and 0 is even, so $(w,x)S(w,x)$.
S is symmetric because, for all $(w,x),(y,z) \in \mathbb{Z}\times\mathbb{Z}$, if $(w,x)S(y,z)$, then $w+x-y-z$ is even and so $y+z-w-x = -(w+x-y-z)$ is even, which means $(y,z)S(w,y)$.
S is not antisymmetric because $(1,1)S(2,2)$ and $(2,2)S(1,1)$, for example.
S is transitive because, for all $(u,v),(w,x),(y,z) \in \mathbb{Z}\times\mathbb{Z}$, if $(u,v)S(w,x)$ and $(w,x)S(y,z)$, then $u+v-w-x$ is even and $w+x-y-z$ is even, so $u+v-y-z = (u+v-w-x)+(w+x-y-z)$ is even, which means $(u,v)S(y,z)$. **[4]**

(iv) S is not a partial order relation, but it is an equivalence relation. **[1]**

(v) S has two equivalence classes, as given below.

$$[(0,0)] = \{(u,v) : (u,v) \in \mathbb{Z}\times\mathbb{Z} \text{ and } u+v \text{ is even}\}$$
$$[(0,1)] = \{(u,v) : (u,v) \in \mathbb{Z}\times\mathbb{Z} \text{ and } u+v \text{ is odd}\}$$

[3]

(b) (i) Yes. Note 9 divides 999 because $9 \times 111 = 999$ and so 9 must also divide $(999)^6$. Similarly, 9 divides $9 \times 9 = 81$ and so 9 must also divide 7×81. Putting these facts together we see that 9 divides $(999)^6 + 7 \times 81$. **[2]**

(ii) Because $x \equiv 3 \pmod{12}$, we have that $x = 12k+3$ for some integer k.
Because $y \equiv 7 \pmod{18}$, we have that $y = 18l+7$ for some integer l.
So $x+y = 12k+3+18l+7 = 6(2k+3l)+10 = 6(2k+3l+1)+4$. Because $2k+3l+1$ is an integer, this means that $x+y \equiv 4 \pmod 6$. **[5]**

(5) (a) $\binom{8}{5} = \frac{8!}{5!3!} = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56.$ **[1]**

(b) The number of 6s rolled obeys a binomial distribution with $p = \frac{1}{6}$ and $n = 100$. Thus the probability that a 6 is rolled exactly 30 times is $\binom{100}{30}(\frac{1}{6})^{30}(\frac{5}{6})^{70}$. **[3]**

(c) (i) Using the formula with $\lambda = 3$, the probability is $\frac{e^{-3}3^0}{0!} = e^{-3}$. **[2]**

(ii) Let Y be the number of calls arriving in the two minute period. We know $E(Y) = 3 + 3 = 6$. So, using the formula with $\lambda = 6$, $\Pr(Y = y) = \frac{e^{-6}6^y}{y!}$. Then

$$\Pr(Y \geq 2) = 1 - \Pr(Y = 1) - \Pr(Y = 0) = 1 - \frac{e^{-6}6^1}{1!} - \frac{e^{-6}6^0}{0!} = 1 - 6e^{-6} - e^{-6} = 1 - 7e^{-6}.$$

[4]

(d) Let X_1 and X_2 be independent random variables that are each selected uniformly at random from the set $\{1, 2, 3\}$. Let $Y = \max(X_1, X_2)$.

(i) If $X_1 = 2 \wedge Y = 2$, then $(X_1, X_2) \in \{(2, 2), (2, 1)\}$. So,

$$\Pr(X_1 = 2 \wedge Y = 2) = \Pr(X_1 = 2 \wedge X_2 = 2) + \Pr(X_1 = 2 \wedge X_2 = 1) = \tfrac{1}{3} \times \tfrac{1}{3} + \tfrac{1}{3} \times \tfrac{1}{3} = \tfrac{2}{9}.$$

[2]

(ii) (X_1, X_2) takes each value in $\{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$ with probability $\frac{1}{9}$.
$Y = 1$ if and only if $(X_1, X_2) = (1, 1)$. So $\Pr(Y = 1) = \frac{1}{9}$.
$Y = 2$ if and only if $(X_1, X_2) \in \{(1, 2), (2, 1), (2, 2)\}$. So $\Pr(Y = 1) = \frac{3}{9}$.
$Y = 3$ if and only if $(X_1, X_2) \in \{(1, 3), (2, 3), (3, 1), (3, 2), (3, 3)\}$. So $\Pr(Y = 1) = \frac{5}{9}$.
So $E(Y) = 1 \times \frac{1}{9} + 2 \times \frac{3}{9} + 3 \times \frac{5}{9} = \frac{22}{9}$. **[4]**

(iii) (X_1, X_2) takes each value in $\{1, 2, \ldots, 100\} \times \{1, 2, \ldots, 100\}$ with probability $\frac{1}{100^2} = \frac{1}{10000}$.
$Y = y$ if and only if $(X_1, X_2) \in \{(1, y), (2, y), \ldots, (y, y)\} \cup \{(y, 1), (y, 2), \ldots, (y, y-1)$, for each $y \in \{1, 2, \ldots, 100\}$. So $\Pr(Y = y) = \frac{2y-1}{10000}$.
So

$$\begin{aligned} E(Y) &= 1 \times \tfrac{2-1}{10000} + 2 \times \tfrac{4-1}{10000} + \cdots + 100 \times \tfrac{100-1}{10000} \\ &= \sum_{i=1}^{100} i \times \frac{2i - 1}{10000} \\ &= \frac{1}{10000} \sum_{i=1}^{100} i(2i - 1). \end{aligned}$$

[4]

(6) (a) (i) The vertices P, Q, R, S, T have degrees $3, 4, 3, 4, 2$, respectively. **[1]**

(ii) The graph has no closed Euler trail because it contains an odd-degree vertex (P, for example). **[2]**

(iii) A spanning tree of the graph containing the edges QS and RS, must contain exactly one of the edges in $\{QT, ST\}$ and exactly one of the edges in $\{PQ, PR, PS\}$. Any choice for the former and any choice for the latter produce a spanning tree. So there are $2 \times 3 = 6$ spanning trees containing the edges QS and RS. **[3]**

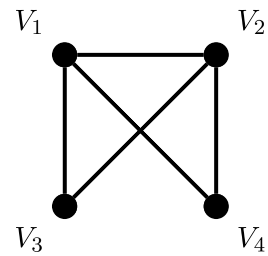
(b) Let X be a set of n elements and x be a fixed element of X. There are $\binom{n-1}{r}$ combinations of r elements of X that do not contain x. There are $\binom{n-1}{r-1}$ combinations of r elements of X that do contain x. Every combination of r elements of x falls into exactly one of these two categories. Thus $\binom{n}{r} = \binom{n}{r-1} + \binom{n-1}{r-1}$.
(A proof via applying the factorial definition of binomial coefficients and manipulating is also acceptable.) **[4]**

(c) (i) $s_1 = 1$ (the only sum is 1), $s_2 = 1$ (the only sum is $1 + 1$), $s_3 = 2$ (the sums are $1 + 1 + 1$ and 3). **[2]**

(ii) Call an ordered sum "legal" if every term in it is a 1, 2 or 3.
If we add a "+1" to the end of a legal sum adding to $n-1$, then we obtain a legal sum adding to n. If we add a "+3" to the end of a legal sum adding to $n-3$, then we obtain a legal sum adding to n. If we add a "+4" to the end of a legal sum adding to $n-4$, then we obtain a legal sum adding to n. Furthermore, every legal sum adding to n can be obtained by exactly one of these three methods (because every legal sum ends in a "+1", a "+3" or a "+4"). So we can see that $s_n = s_{n-1} + s_{n-3} + s_{n-4}$ for all $n \geq 5$. **[6]**

(iii) $s_5 = s_4 + s_2 + s_1 = 4 + 1 + 1 = 6$, $s_6 = s_5 + s_3 + s_2 = 6 + 2 + 1 = 9$. **[2]**

(7) (a) (i)

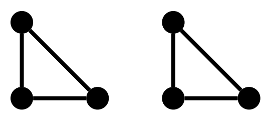


[2]

(ii) Calculate M^6. The entry in the first row and third column of M^6 is the number of walks of length 6 from V_1 to V_3. **[2]**

(b) (i) No such graph exists. If such a graph existed then any spanning tree of it would have 7 vertices (the same number as the graph) and at most 5 edges (some subset of those in the graph). But we know that any tree with n vertices has $n-1$ edges. **[3]**

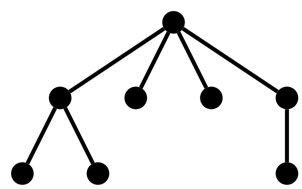
(ii) No. Here is an example of a graph with no odd degree vertices which has no closed Euler trail.



[3]

(iii) No. The sum of these numbers is 19, and the sum of the degrees in any graph is even. **[2]**

(iv) Yes. Here is one example of such a tree.



[3]

(c) Any ternary string of length 5 containing at most two 0s, at most two 1s and at most two 2s, must contain exactly one of the three symbols 0,1,2 and exactly two of each of the other two symbols. There are 3 ways to select the symbol that occurs once, 5 ways to select a position for that symbol, and then $\binom{4}{2}$ ways to select two positions for the smaller of the other two symbols. So the number ternary strings of length 5 containing at most two 0s, at most two 1s and at most two 2s is

$$3 \times 5 \times \tbinom{4}{2} = 90.$$

[5]