MAT1830

Lecture 6: Rules of inference

Last time we saw how to recognise tautologies and logically equivalent sentences by computing their truth tables. Another way is to *infer* new sentences from old by *rules of inference*.

6.1 Replacement

Any sentence may be replaced by a logically equivalent sentence. Any series of such replacements therefore leads to a sentence equivalent to the one we started with.

Using replacement is like the usual method of proving identities in algebra – make a series of replacements until the left hand side is found equal to the right hand side.

Why can we say that $(\frac{2x}{2})^2 = x^2$?

Because $\frac{2x}{2} = x$.

And there's a rule of replacement.

It's the same in logic except with $\equiv$ instead of $=$.

So we can say that $p \wedge \neg(q \vee r) \equiv p \wedge (\neg q \wedge \neg r)$ because $\neg(q \vee r) \equiv \neg q \wedge \neg r$.

Example. Prove that $x \to y \equiv (\neg y) \to (\neg x)$.

$$\begin{aligned}
x \to y &\equiv (\neg x) \vee y \\
&\quad \text{by implication law} \\
&\equiv y \vee (\neg x) \\
&\quad \text{by commutative law} \\
&\equiv (\neg\neg y) \vee (\neg x) \\
&\quad \text{by law of double negation} \\
&\equiv (\neg y) \to (\neg x) \\
&\quad \text{by implication law}
\end{aligned}$$

6.2 Contrapositives

$$x \to y \equiv (\neg y) \to (\neg x)$$

$(\neg y) \to (\neg x)$ is the *contrapositive* of $x \to y$.

Example. The contrapositive of

$$\text{MCG flooded} \to \text{cricket is off}$$

is

$$\text{Cricket is on} \to \text{MCG not flooded.}$$

An implication and its contrapositive are equivalent: they mean the same thing!

Question 6.1 What does "no pain, no gain" mean as an implication?

"no pain" $\rightarrow$ "no gain"

Question 6.2 What is its contrapositive?

$\neg$"no gain" $\rightarrow$ $\neg$"no pain"
OR "gain" $\rightarrow$ "pain"

Flux Exercise (LQMTZZ)

What is the contrapositive of "If $x \equiv 0 \pmod 6$, then $2x \equiv 0 \pmod 6$"?

A. "If $2x \not\equiv 0 \pmod 6$, then $x \not\equiv 0 \pmod 6$."
B. "If $2x \equiv 0 \pmod 6$, then $x \equiv 0 \pmod 6$."
C. "If $2x \not\equiv 0 \pmod 3$, then $x \equiv 0 \pmod 6$."
D. "If $x \equiv 0 \pmod 6$ and $2x \not\equiv 0 \pmod 6$."

Answer: A.

Contrapositives are not negations!

Don't confuse contrapositives with negations.

We've seen that the contrapositive of $p \to q$ is $\neg q \to \neg p$ and that it is logically equivalent to the original statement.

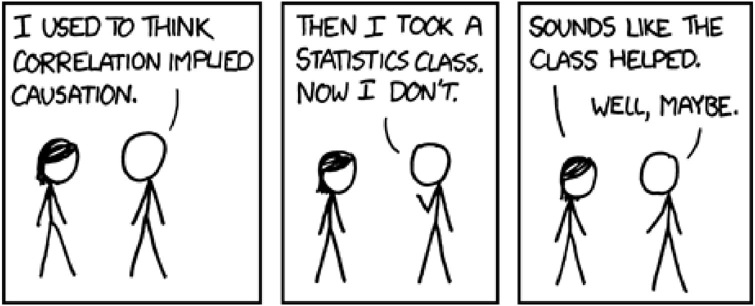
The negation of $p \to q$ is $\neg(p \to q)$. It is not logically equivalent to the original statement.

$$\begin{aligned} \neg(p \to q) &\equiv \neg(\neg p \vee q) \\ &\equiv \neg\neg p \wedge \neg q \\ &\equiv p \wedge \neg q \end{aligned}$$

"If he was famous then he'd fly first class."

Contrapositive: "If he doesn't fly first class then he isn't famous."
Logically equivalent to original statement!

Negation: "He's famous and he doesn't fly first class."
True exactly when the original statement is false!


I USED TO THINK CORRELATION IMPLIED CAUSATION.
THEN I TOOK A STATISTICS CLASS. NOW I DON'T.
SOUNDS LIKE THE CLASS HELPED.
WELL, MAYBE.

Question 6.3 Write down the following sentences as implications and then write their contrapositives.

Sentence: "You can't make an omelette without breaking eggs."
Implication: "you made an omelette" $\rightarrow$ "you broke eggs"
Contrapositive: "you didn't break eggs" $\rightarrow$ "you didn't make an omelette"

Sentence: "If n is even, so is n^2."
Implication: "n is even" $\rightarrow$ "n^2 is even"
Contrapositive: "n^2 is odd" $\rightarrow$ "n is odd"

Sentence: "Haste makes waste."
Implication: "haste" $\rightarrow$ "waste"
Contrapositive: "no waste" $\rightarrow$ "no haste"

Flux Exercise (LQMTZZ)

Suppose "If x is mimsy, then x is not frumious" is true. What can we say about whether the following statements are true?

(1) "If x is not frumious, then x is mimsy."
(2) "x is mimsy and x is frumious."
(3) "If x is frumious, then x is not mimsy."

A. (1) true, (2) false, (3) true
B. (1) false, (2) false, (3) true
C. (1) maybe, (2) false, (3) false
D. (1) maybe, (2) false, (3) true

Answer:

(1) may or may not be true (we don't know if non-mimsy, non-frumious things exist).
(2) is the negation of the original statement. It must be false because the original is true.
(3) is the contrapositive of the original statement. It means exactly the same as the original so must be true.

So D.

6.3 Using logic laws

Example. Prove that $p \to (q \to p)$ is a tautology.

$$\begin{aligned}
& p \to (q \to p) \\
\equiv\ & (\neg p) \vee (q \to p) \\
& \text{by implication law} \\
\equiv\ & (\neg p) \vee ((\neg q) \vee p) \\
& \text{by implication law} \\
\equiv\ & (\neg p) \vee (p \vee (\neg q)) \\
& \text{by commutative law} \\
\equiv\ & ((\neg p) \vee p) \vee (\neg q) \\
& \text{by associative law} \\
\equiv\ & (p \vee (\neg p)) \vee (\neg q) \\
& \text{by commutatve law} \\
\equiv\ & \mathsf{T} \vee (\neg q) \quad \text{by inverse law} \\
\equiv\ & \mathsf{T} \quad \text{by annihilation law}
\end{aligned}$$

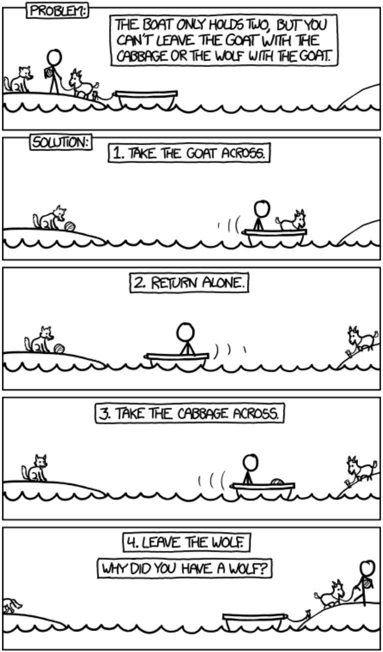
Question 6.4 Show that $p \to (q \to (r \to p))$ is a tautology.

$$
\begin{aligned}
p \to (q \to (r \to p)) &\equiv \neg p \vee (q \to (r \to p)) && \text{(implication law)} \\
&\equiv \neg p \vee (\neg q \vee (r \to p)) && \text{(implication law)} \\
&\equiv \neg p \vee (\neg q \vee (\neg r \vee p)) && \text{(implication law)} \\
&\equiv \neg p \vee \neg q \vee \neg r \vee p && \text{(associative law)} \\
&\equiv (\neg p \vee p) \vee \neg q \vee \neg r && \text{(commutative law)} \\
&\equiv T \vee \neg q \vee \neg r && \text{(inverse law)} \\
&\equiv T && \text{(annihilation law)}
\end{aligned}
$$

So the statement is logically equivalent to T and so is a tautology.

Question 6.5 Find a tautology form with n variables which is $p \to (q \to p)$ for $n = 2$ and $p \to (q \to (r \to p))$ for $n = 3$.

$$p_1 \to (p_2 \to (p_3 \to (p_4 \to \cdots\cdots (p_{n-1} \to (p_n \to p_1)) \cdots)))$$


PROBLEM:
THE BOAT ONLY HOLDS TWO, BUT YOU CAN'T LEAVE THE GOAT WITH THE CABBAGE OR THE WOLF WITH THE GOAT.
SOLUTION:
1. TAKE THE GOAT ACROSS.
2. RETURN ALONE.
3. TAKE THE CABBAGE ACROSS.
4. LEAVE THE WOLF.
WHY DID YOU HAVE A WOLF?

Example. Prove that $((p \to q) \land p) \to q$ is a tautology.

$$
\begin{aligned}
& ((p \to q) \land p) \to q \\
\equiv\ & \neg((p \to q) \land p) \lor q \\
& \text{by implication law} \\
\equiv\ & (\neg(p \to q) \lor (\neg p)) \lor q \\
& \text{by de Morgan's law} \\
\equiv\ & \neg(p \to q) \lor ((\neg p) \lor q) \\
& \text{by associative law} \\
\equiv\ & \neg(p \to q) \lor (p \to q) \\
& \text{by implication law} \\
\equiv\ & (p \to q) \lor \neg(p \to q) \\
& \text{by commutative law} \\
\equiv\ & \mathsf{T} \quad \text{by inverse law}
\end{aligned}
$$

This tautology says that "if p implies q and p is true then q is true".

6.4 Logical consequence

A sentence ψ is a *logical consequence* of a sentence ϕ, if $\psi = \mathsf{T}$ whenever $\phi = \mathsf{T}$. We write this as $\phi \Rightarrow \psi$.

It is the same to say that $\phi \to \psi$ is a tautology, but $\phi \Rightarrow \psi$ makes it clearer that we are discussing a relation between the sentences ϕ and ψ.

Any sentence ψ logically *equivalent* to ϕ is a logical consequence of ϕ, but not all consequences of ψ are equivalent to it.

It might help to think that:

$$\Rightarrow \text{ corresponds to } \rightarrow \quad \text{in the same way} \quad \equiv \text{ corresponds to } \leftrightarrow.$$

We saw last lecture that, for sentences ϕ and ψ,
$\phi \equiv \psi$ exactly when $\phi \leftrightarrow \psi$ is a tautology.

In the same way,
$\phi \Rightarrow \psi$ exactly when $\phi \rightarrow \psi$ is a tautology.

Example. $p \wedge q \Rightarrow p$

p is a logical consequence of $p \wedge q$, because $p = \mathsf{T}$ whenever $p \wedge q = \mathsf{T}$. However, we can have $p \wedge q = \mathsf{F}$ when $p = \mathsf{T}$ (namely, when $q = \mathsf{F}$). Hence $p \wedge q$ and p are not equivalent.

This example shows that $\Rightarrow$ is not symmetric:

$$(p \wedge q) \Rightarrow p \quad \text{but} \quad p \not\Rightarrow (p \wedge q)$$

This is where $\Rightarrow$ differs from $\equiv$, because if $\phi \equiv \psi$ then $\psi \equiv \phi$.

In fact, we build the relation $\equiv$ from $\Rightarrow$ the same way $\leftrightarrow$ is built from $\rightarrow$:

$$\phi \equiv \psi \;\text{ means }\; (\phi \Rightarrow \psi) \text{ and } (\psi \Rightarrow \phi).$$

Example Show that $p \wedge (q \vee r) \Rightarrow (p \wedge q) \vee r$ using a truth table.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$(p \wedge q) \vee r$	
T	T	T	T	T	T	T	(!)
T	T	F	T	T	T	T	(!)
T	F	T	T	T	F	T	(!)
T	F	F	F	F	F	F	
F	T	T	T	F	F	T	
F	T	F	T	F	F	F	
F	F	T	T	F	F	T	
F	F	F	F	F	F	F	