MAT1830

Lecture 17: Equivalence Relations

An *equivalence relation* R on a set A is a binary relation with the following three properties.

1. Reflexivity.

$$aRa$$

for all $a \in A$.

2. Symmetry.

$$aRb \Rightarrow bRa$$

for all $a, b \in A$.

3. Transitivity.

$$aRb \text{ and } bRc \Rightarrow aRc$$

for all $a, b, c \in A$.

Equality and congruence mod n (for fixed n) are examples of equivalence relations.

Reflexivity (For a binary relation R on a set A.)

Everywhere I see:

I actually see:

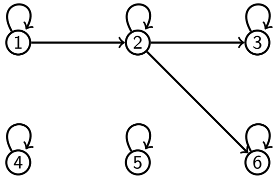
To prove R is reflexive, show that...

For all $x \in A$, xRx.

To prove R is not reflexive, show that...

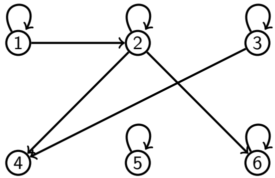
There is an $x \in A$ such that $x \not R x$.

Question Let R be the relation on A pictured below. Is R reflexive?



Yes. xRx for all $x \in A$.

Question Let S be the relation on A pictured below. Is S reflexive?

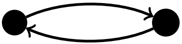


No. $4 \not S 4$.

Symmetry (For a binary relation R on a set A.)

Everywhere I see:

I actually see:



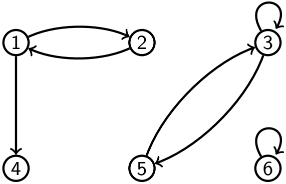
To prove R is symmetric, show that...

For all $x, y \in A$, if xRy then yRx.

To prove R is not symmetric, show that...

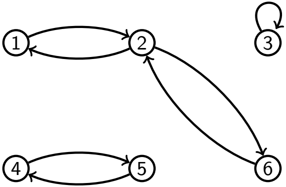
There are some $x, y \in A$ such that xRy but $y \not R x$.

Question Let R be the relation on A pictured below. Is R symmetric?

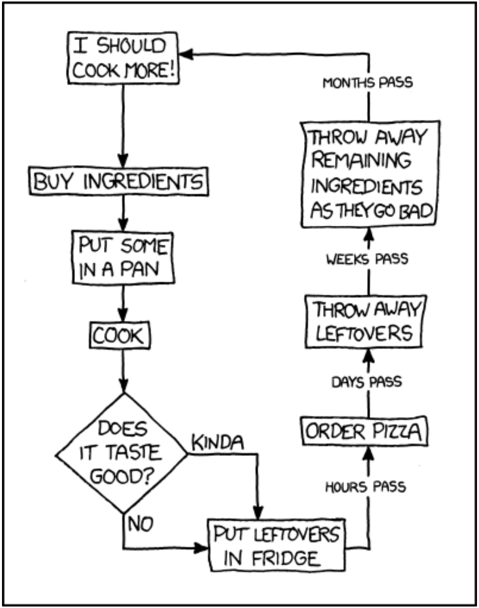


No. $1R4$ but $4 \not R 1$.

Question Let S be the relation on A pictured below. Is S symmetric?

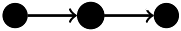


Yes. For all $x, y \in A$ if xSy then ySx.

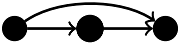

I SHOULD COOK MORE!
BUY INGREDIENTS
PUT SOME IN A PAN
COOK
DOES IT TASTE GOOD?
KINDA
NO
PUT LEFTOVERS IN FRIDGE
HOURS PASS
ORDER PIZZA
DAYS PASS
THROW AWAY LEFTOVERS
WEEKS PASS
THROW AWAY REMAINING INGREDIENTS AS THEY GO BAD
MONTHS PASS

Transitivity (For a binary relation R on a set A.)

Everywhere I see:



I actually see:



Everywhere I see:

I actually see:

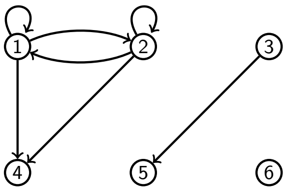
To prove R is transitive, show that...

For all $x, y, z \in A$, if xRy and yRz then xRz.

To prove R is not transitive, show that...

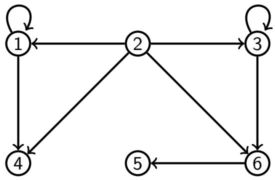
There are some $x, y, z \in A$ such that xRy and yRz but $x \not R z$.

Question Let R be the relation on A pictured below. Is R transitive?



Yes. For all $x, y, z \in A$, if xRy and yRz then xRz.

Question Let S be the relation on A pictured below. Is S transitive?



No, because $3S6$ and $6S5$ but $3 \not S 5$.

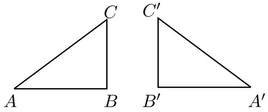
17.1 Other equivalence relations

1. Equivalence of fractions.

Two fractions are equivalent if they reduce to the same fraction when the numerator and denominator of each are divided by their gcd. E.g. $\frac{2}{4}$ and $\frac{3}{6}$ are equivalent because both reduce to $\frac{1}{2}$.

2. Congruence of triangles.

Triangles ABC and $A'B'C'$ are congruent if $AB = A'B'$, $BC = B'C'$ and $CA = C'A'$. E.g. the following triangles are congruent.

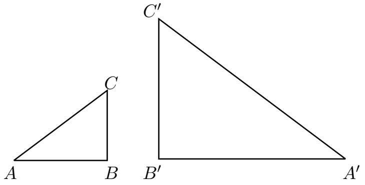


3. Similarity of triangles.

Triangles ABC and $A'B'C'$ are similar if

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}.$$

E.g the following triangles are similar



4. Parallelism of lines.

The relation $L \| M$ (L is parallel to M) is an equivalence relation.

Remark

In all these cases the relation is an equivalence because it says that objects are the *same* in some respect.

1. Equivalent fractions have the same reduced form.

2. Congruent triangles have the same side lengths.

3. Similar triangles have the same shape.

4. Parallel lines have the same direction.

Sameness is always reflexive (a is the same as a), symmetric (if a is the same as b, then b is the same as a) and transitive (if a is the same as b and b is the same as c, then a is the same as c).

Flux Exercise (LQMTZZ)

Which of the following relations are equivalence relations on $\mathbb{Z}$?

(1) R defined by xRy if and only if $|x| = |y|$

(2) S defined by xSy if and only if $x^3 - y^3 = 1$

(3) T defined by xTy if and only if x divides y

(4) U defined by xUy if and only if 5 divides $x - y$

A. Just (1) and (3)
B. Just (1)
C. Just (1) and (4)
D. Just (1), (2) and (4)

Answer

(2) is not reflexive. E.g. $1 \not S 1$ because $1^3 - 1^3 \neq 1$.
So (2) is not an equivalence relation.

(3) is not symmetric. E.g. $3T6$ but $6 \not T 3$ (3 divides 6 but 6 doesn't divide 3).
So (3) is not an equivalence relation.

(1) and (4) are equivalence relations (details on next two slides).

So C.

(1) R defined on $\mathbb{Z}$ by xRy if and only if $|x| = |y|$

Reflexive: Yes. $|a| = |a|$ for all $a \in \mathbb{Z}$.
Symmetric: Yes. If $|a| = |b|$, then $|b| = |a|$ for all $a, b \in \mathbb{Z}$.
Transitive: Yes. If $|a| = |b|$ and $|b| = |c|$, then $|a| = |c|$ for all $a, b, c \in \mathbb{Z}$.
So it is an equivalence relation.

(2) S defined on $\mathbb{Z}$ by xSy if and only if $x^3 - y^3 = 1$

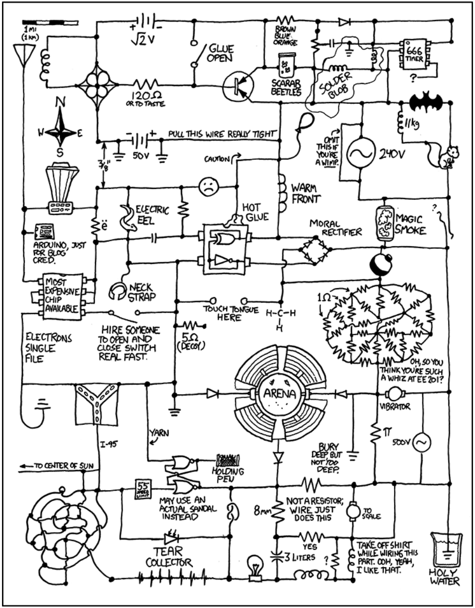
Reflexive: No. $1^3 - 1^3 \neq 1$ so $1 \not S 1$.
Symmetric: No. $1^3 - 0^3 = 1$ but $0^3 - 1^3 \neq 1$, so $1S0$ but $0 \not S 1$.
Transitive: No. $1^3 - 0^3 = 1$ and $0^3 - (-1)^3 = 1$ but $1^3 - (-1)^3 \neq 1$, so $1S0$ and $0S(-1)$ but $1 \not S (-1)$.
So it is not an equivalence relation.

(3) $\quad T$ defined on $\mathbb{Z}$ by xTy if and only if x divides y

Reflexive: Yes. a divides a for all $a \in \mathbb{Z}$.
Symmetric: No. 3 divides 6 but 6 does not divide 3, so $3T6$ but $6 \not T 3$.
Transitive: Yes. If a divides b and b divides c, then a divides c for all $a, b, c \in \mathbb{Z}$.
So it is not an equivalence relation.

(4) $\quad U$ defined on $\mathbb{Z}$ by xUy if and only if 5 divides $x - y$

Yes. This relation is the same as $x \equiv y \pmod 5$ and we know that's an equivalence relation.



1 MI
(1 KM)
+ −
√2 V
GLUE OPEN
BROWN
BLUE
ORANGE
666
TIMER
?
120Ω
OR TO TASTE
SCARAB
BEETLES
SOLDER
BLOB
N
W
E
S
− +
50V
PULL THIS WIRE REALLY TIGHT
3/8"
CAUTION
OMIT
THIS IF
YOU'RE
A WIMP.
240V
11kg
WARM
FRONT
ë
ELECTRIC
EEL
HOT
GLUE
MORAL
RECTIFIER
MAGIC
SMOKE
?
ARDUINO, JUST
FOR BLOG
CRED.
NECK
STRAP
MOST
EXPENSIVE
CHIP
AVAILABLE
TOUCH TONGUE
HERE
H–C–H
H
1Ω
HIRE SOMEONE
TO OPEN AND
CLOSE SWITCH
REAL FAST.
ELECTRONS
SINGLE
FILE
5Ω
(DECOY)
ARENA
OH, SO YOU
THINK YOU'RE SUCH
A WHIZ AT EE 201?
VIBRATOR
π
500V
BURY
DEEP, BUT
NOT TOO
DEEP.
YARN
I-95
TO CENTER OF SUN
HOLDING
PEN
55
SPEED
LIMIT
MAY USE AN
ACTUAL SANDAL
INSTEAD
8mm
NOT A RESISTOR;
WIRE JUST
DOES THIS
TO
SCALE
TEAR
COLLECTOR
YES
3 LITERS
?
TAKE OFF SHIRT
WHILE WIRING THIS
PART. OOH, YEAH,
I LIKE THAT.
HOLY
WATER

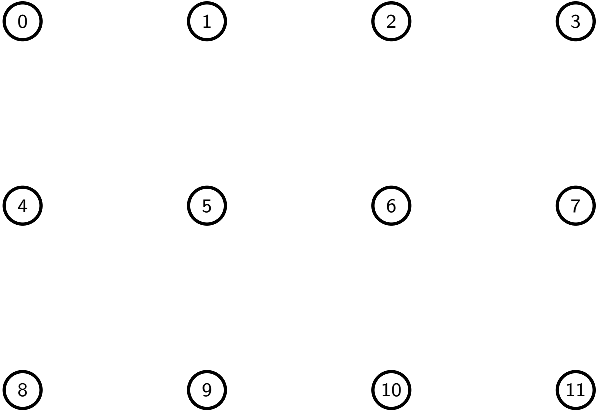
Question What is the same about the equivalent objects for the equivalence relations (1) and (4)?

(1) R defined on $\mathbb{Z}$ by xRy if and only if $|x| = |y|$

x and y have the same "magnitude".

(4) U defined on $\mathbb{Z}$ by xUy if and only if 5 divides $x - y$

x and y have the same remainder when divided by 5.


0
1
2
3
4
5
6
7
8
9
10
11

17.2 Equivalence classes

Conversely, we can show that if R is a reflexive, symmetric and transitive relation then aRb says that a and b are the same in some respect: *they have the same R-equivalence class.*

If R is an equivalence relation we define the *R-equivalence class* of a to be

$$[a] = \{s : sRa\}.$$

Thus $[a]$ consists of all the elements related to a. It can also be defined as $\{s : aRs\}$, because sRa if and only if aRs, by symmetry of R.

Examples

- The parallel equivalence class of a line L consists of all lines parallel to L.
- The equivalence class of 1 for congruence mod 2 is the set of all odd numbers.

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17.3 Equivalence class properties

Claim. *If two elements are related by an equivalence relation R on a set A, their equivalence classes are equal.*

Proof. Suppose $a, b \in A$ and aRb. Now

$$\begin{aligned} s \in [a] &\Rightarrow sRa \text{ by definition of } [a] \\ &\Rightarrow sRb \text{ by transitivity of } R \\ &\quad\ \text{since } sRa \text{ and } aRb \\ &\Rightarrow s \in [b] \text{ by definition of } [b]. \end{aligned}$$

Thus all elements of $[a]$ belong to $[b]$. Similarly, all elements of $[b]$ belong to $[a]$, hence $[a] = [b]$. $\square$

Claim. *If R is an equivalence relation on a set A, each element of A belongs to exactly one equivalence class.*

Proof. Suppose $a, b, c \in A$, and $c \in [a] \cap [b]$.

$$
\begin{aligned}
& c \in [a] \text{ and } c \in [b] \\
\Rightarrow \quad & cRa \text{ and } cRb \\
& \text{by definition of } [a] \text{ and } [b] \\
\Rightarrow \quad & aRc \text{ and } cRb \text{ by symmetry} \\
\Rightarrow \quad & aRb \text{ by transitivity} \\
\Rightarrow \quad & [a] = [b] \\
& \text{by the previous claim.}
\end{aligned}
$$

17.4 Partitions and equivalence classes

A *partition* of a set S is a set of subsets of S such that each element of S is in exactly one of the subsets.

Using what we showed in the last section, we have the following.

If R is an equivalence relation on a set A, then the equivalence classes of R form a partition of A. Two elements of A are related if and only if they are in the same equivalence class.

Example. Let R be the relation on $\mathbb{Z}$ defined by aRb if and only if $a \equiv b \pmod 3$. The three equivalence classes of R are

$$\{x : x \equiv 0 \pmod 3\} = \{3k : k \in \mathbb{Z}\}$$
$$\{x : x \equiv 1 \pmod 3\} = \{3k + 1 : k \in \mathbb{Z}\}$$
$$\{x : x \equiv 2 \pmod 3\} = \{3k + 2 : k \in \mathbb{Z}\}.$$

These partition the set $\mathbb{Z}$.

Examples of partitions

$\{\{1, 6\}, \{2\}, \{3, 4, 5\}\}$ is a partition of $\{1, 2, 3, 4, 5, 6\}$

$\{\{x : x \in \mathbb{Z}$ and x is even$\}, \{x : x \in \mathbb{Z}$ and x is odd$\}\}$ is a partition of $\mathbb{Z}$.

Question What are the equivalence classes of the equivalence relations (1) and (4)?

(1) R defined on $\mathbb{Z}$ by xRy if and only if $|x| = |y|$

$\{0\}, \{1, -1\}, \{2, -2\}, \{3, -3\} \ldots.$

(4) U defined on $\mathbb{Z}$ by xUy if and only if 5 divides $x - y$

$\{\ldots, -10, -5, 0, 5, 10, \ldots\}$,
$\{\ldots, -9, -4, 1, 6, 11, \ldots\}$,
$\{\ldots, -8, -3, 2, 7, 12, \ldots\}$,
$\{\ldots, -7, -2, 3, 8, 13, \ldots\}$,
$\{\ldots, -6, -1, 4, 9, 14, \ldots\}$

Flux Exercise (LQMTZZ)

Let X be the set $\{\text{there}, \text{was}, \text{a}, \text{wall}, \text{it}, \text{did}, \text{not}, \text{look}, \text{important}\}$.

Let R be the equivalence relation on X defined by xRy if and only if x and y contain the same number of 'o's.

What are the equivalence classes of R?

A. $\{\text{there}, \text{was}, \text{a}, \text{wall}, \text{it}, \text{did}\}$, $\{\text{not}, \text{look}, \text{important}\}$

B. $\{\text{there}, \text{was}, \text{a}, \text{wall}\}$, $\{\text{not}, \text{important}\}$, $\{\text{look}\}$

C. $\{\text{there}, \text{was}, \text{a}, \text{wall}\}$, $\{\text{it}, \text{did}\}$, $\{\text{not}, \text{important}\}$, $\{\text{look}\}$

D. $\{\text{there}, \text{was}, \text{a}, \text{wall}, \text{it}, \text{did}\}$, $\{\text{not}, \text{important}\}$, $\{\text{look}\}$

Answer

Not A because it has 'not' and 'look' in the same class when they are not related.

Not B because there is no class containing 'it'.

Not C because it has 'wall' and 'it' in different classes when they are related.

In D, two words are in the same class exactly when they're related.

So D.