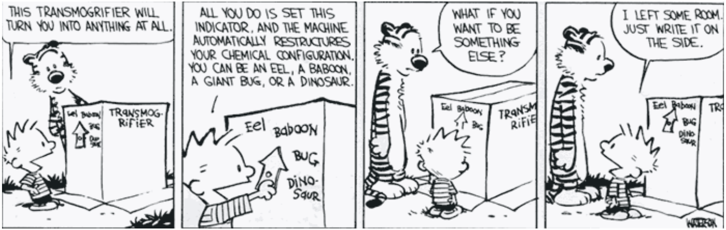
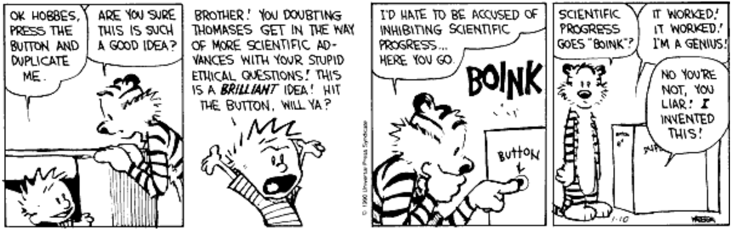
# MAT1830

Lecture 15: Composition and Inversion


THIS TRANSMOGRIFIER WILL TURN YOU INTO ANYTHING AT ALL.
TRANSMOG-RIFIER
ALL YOU DO IS SET THIS INDICATOR, AND THE MACHINE AUTOMATICALLY RESTRUCTURES YOUR CHEMICAL CONFIGURATION. YOU CAN BE AN EEL, A BABOON, A GIANT BUG, OR A DINOSAUR.
EEL
BABOON
BUG
DINO-SAUR
WHAT IF YOU WANT TO BE SOMETHING ELSE?
I LEFT SOME ROOM. JUST WRITE IT ON THE SIDE.
WATTERSON


OK HOBBES, PRESS THE BUTTON AND DUPLICATE ME.
ARE YOU SURE THIS IS SUCH A GOOD IDEA?
BROTHER! YOU DOUBTING THOMASES GET IN THE WAY OF MORE SCIENTIFIC ADVANCES WITH YOUR STUPID ETHICAL QUESTIONS! THIS IS A BRILLIANT IDEA! HIT THE BUTTON, WILL YA?
I'D HATE TO BE ACCUSED OF INHIBITING SCIENTIFIC PROGRESS... HERE YOU GO.
BOINK
BUTTON
SCIENTIFIC PROGRESS GOES "BOINK"?
IT WORKED! IT WORKED! I'M A GENIUS!
NO YOU'RE NOT, YOU LIAR! I INVENTED THIS!
1-10
WATTERSON

Complicated functions are often built from simple parts. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x^2 + 1)^3$ is computed by doing the following steps in succession:

- square,
- add 1,
- cube.

We say that $f(x) = (x^2 + 1)^3$ is the composite of the functions (from $\mathbb{R}$ to $\mathbb{R}$)

- square($x$)=$x^2$,
- successor($x$)=$x + 1$,
- cube($x$)=$x^3$.

## 15.1 Notation for composite functions

In the present example we write

$$f(x) = \text{cube}(\text{successor}(\text{square}(x))),$$

or

$$f = \text{cube} \circ \text{successor} \circ \text{square}.$$

In general, if $f(x) = g(h(x))$ we write $f = g \circ h$ and say $f$ is the *composite* of $g$ and $h$.

> **Warning:** Remember that $g \circ h$ means "do $h$ first, then $g$." $g \circ h$ is usually different from $h \circ g$.

**Example.**
$\text{square}(\text{successor}(x)) = (x+1)^2 = x^2 + 2x + 1$
$\text{successor}(\text{square}(x)) = x^2 + 1$

**Question 15.1** Let $f$, $m$ and $s$ be functions on the set of people defined by
$m(x) =$ mother of $x$
$f(x) =$ father of $x$
$s(x) =$ spouse of $x$.
What are the following?

(Note $s$ is not actually a valid function on the set of people.)

| | |
|---|---|
| $m \circ s(x)$ | mother in law of $x$ |
| $f \circ s(x)$ | father in law of $x$ |
| $m \circ m(x)$ | grandmother (maternal) of $x$ |
| $f \circ m(x)$ | grandfather (maternal) of $x$ |
| $s \circ s(x)$ | $x$ |

**Question 15.2** Write the following as composites of square($x$), sqrt($x$), successor($x$) and cube($x$).

(Assume that all of these have domain and codomain $\{x : x \in \mathbb{R}$ and x $\geq 0\}$.)

$$
\begin{aligned}
\sqrt{1+x^3} &= \text{sqrt}(\text{successor}(\text{cube}(x))) = \text{sqrt} \circ \text{successor} \circ \text{cube}(x) \\
x^{\frac{3}{2}} &= \text{sqrt}((\text{cube}(x)) = \text{sqrt} \circ \text{cube}(x) \\
(1+x)^3 &= \text{cube}(\text{successor}(x)) = \text{cube} \circ \text{successor}(x) \\
(1+x^3)^2 &= \text{square}(\text{successor}(\text{cube}(x))) = \text{square} \circ \text{successor} \circ \text{cube}(x)
\end{aligned}
$$

**Note** Composition of functions is associative: $(f \circ g) \circ h = f \circ (g \circ h)$. So we don't bother with the brackets.

## 15.2 Conditions for composition

Composite functions do not always exist.

**Example.** If reciprocal : $\mathbb{R} - \{0\} \to \mathbb{R}$ is defined by reciprocal$(x) = \frac{1}{x}$ and predecessor : $\mathbb{R} \to \mathbb{R}$ is defined by predecessor$(x) = x - 1$, then reciprocal $\circ$ predecessor does not exist, because predecessor$(1) = 0$ is not a legal input for reciprocal.

To avoid this problem, we demand that the codomain of $h$ be equal to the domain of $g$ for $g \circ h$ to exist. This ensures that each output of $h$ will be a legal input for $g$.

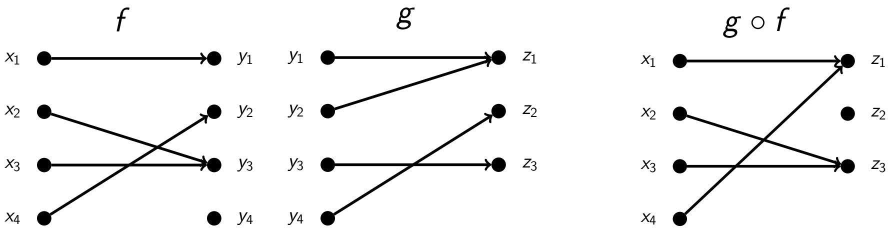
Let $h : A \to B$ and $g : C \to D$ be functions. Then $g \circ h : A \to D$ exists if and only if $B = C$.

Let $g : C \to D$ and $h : A \to B$ be functions.

The function $g \circ h$ exists if and only if $C = B$.

If it exists, $g \circ h : A \to D$ and is defined by $g \circ h(x) = g(h(x))$.

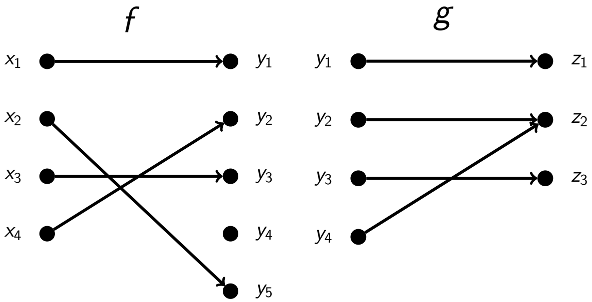
**Question** Let $f : A \to B$ and $g : C \to D$ be the functions pictured below.




Does $g \circ f$ exist?

codomain$(f) = \{y_1, y_2, y_3, y_4\}$ and domain$(g) = \{y_1, y_2, y_3, y_4\}$.
So $g \circ f$ does exist because codomain$(f) =$ domain$(g)$.
$g \circ f : A \to D$

**Question** Let $f : A \to B$ and $g : C \to D$ be the functions pictured below.




Does $g \circ f$ exist?

codomain$(f) = \{y_1, y_2, y_3, y_4, y_5\}$ and domain$(g) = \{y_1, y_2, y_3, y_4\}$.
So $g \circ f$ does not exist because codomain$(f) \neq$ domain$(g)$.

## Flux Exercise (LQMTZZ)

Let $f$, $g$ and $h$ be the functions

$f : \mathbb{R} \to \mathbb{Z}$ defined by $f(x) = \lfloor x \rfloor$. ("$x$ rounded down")

$g : \mathbb{Z} \to \mathbb{R}$ defined by $g(x) = \frac{x}{2}$.

$h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^2 + 7$.

Which of the following statements is *false*?

A. $g \circ f$ exists, $g \circ f : \mathbb{R} \to \mathbb{R}$

B. $f \circ g$ exists, $f \circ g : \mathbb{Z} \to \mathbb{Z}$

C. $g \circ h$ does not exist

D. $g \circ f \circ g$ does not exist

### Answer

A is true because codomain($f$) = domain($g$), domain($f$) = $\mathbb{R}$, codomain($g$) = $\mathbb{R}$

B is true because codomain($g$) = domain($f$), domain($g$) = $\mathbb{Z}$, codomain($f$) = $\mathbb{Z}$

C is true because codomain($h$) $\neq$ domain($g$)

D is false because codomain($f \circ g$) = domain($g$). In fact, $g \circ f \circ g : \mathbb{Z} \to \mathbb{R}$.

So D.

## 15.3 The identity function

On each set $A$ the function $i_A : A \to A$ defined by

$$i_A(x) = x,$$

is called the *identity function* (on $A$).

## 15.4 Inverse functions

Functions $f : A \to A$ and $g : A \to A$ are said to be inverses (of each other) if

$$f \circ g = g \circ f = i_A.$$

**Example.** square and sqrt are inverses of each other on the set $\mathbb{R}^{\geqslant 0}$ of reals $\geqslant 0$.

$$\mathrm{sqrt}(\mathrm{square}(x)) = x \text{ and } \mathrm{square}(\mathrm{sqrt}(x)) = x.$$

In fact, this is exactly what sqrt is supposed to do – reverse the process of squaring. However, this works only if we restrict the domain to $\mathbb{R}^{\geqslant 0}$. On $\mathbb{R}$ we do not have $\mathrm{sqrt}(\mathrm{square}(x)) = x$ because, for example,

$$\mathrm{sqrt}(\mathrm{square}(-1)) = \mathrm{sqrt}(1) = 1.$$

This problem arises whenever we seek an inverse for a function which is not one-to-one. The squaring function on $\mathbb{R}$ sends both 1 and $-1$ to 1, but we want a single value 1 for sqrt(1). Thus we have to restrict the squaring function to $\mathbb{R}^{\geqslant 0}$.

## 15.5 Conditions for inversion

A function $f$ can have an inverse without its domain and codomain being equal.

The inverse of a function $f : A \to B$ is a function $f^{-1} : B \to A$ such that

$$f^{-1} \circ f = i_A \quad \text{and} \quad f \circ f^{-1} = i_B.$$

Note that $f^{-1} \circ f$ and $f \circ f^{-1}$ are both identity functions but they have different domains.

Not every function has an inverse, but we can neatly classify the ones that do.

Let $f : A \to B$ be a function. Then $f^{-1} : B \to A$ exists if and only if $f$ is one-to-one and onto.

Let $f : A \to B$.

The function $f^{-1} : B \to A$ exists if and only if $f$ is one-to-one and onto.

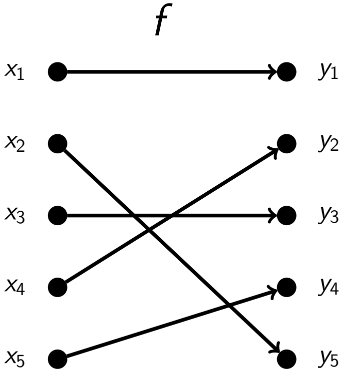
(Remember onto means image$(f) = B$.)

If it exists, $f^{-1} : B \to A$ is defined by $f^{-1}(y)$ equals the unique $x \in A$ such that $f(x) = y$.

We have $f^{-1} \circ f = i_A$ and $f \circ f^{-1} = i_B$.

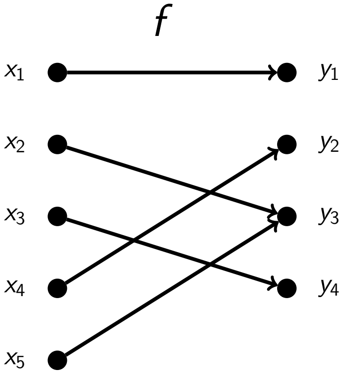
**Note** $f^{-1}$ is just a notation for "the inverse function of $f$".
It is *not* an exponential.

**Question** Let $f : A \to B$ be the function pictured below.




Does $f^{-1}$ exist? Yes.

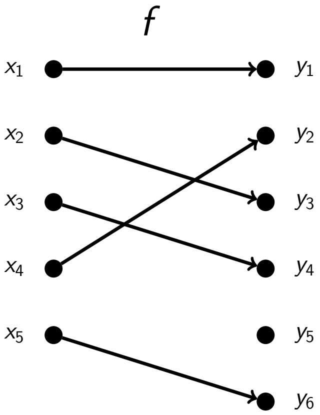
**Question** Let $f : A \to B$ be the function pictured below.




Does $f^{-1}$ exist? No.

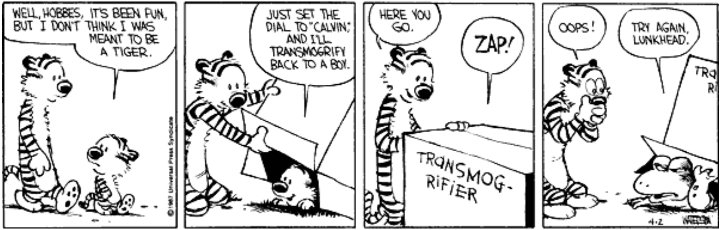
$f$ is not one-to-one.

**Question** Let $f : A \to B$ be the function pictured below.




Does $f^{-1}$ exist? No.

$f$ is not onto.


WELL, HOBBES, IT'S BEEN FUN, BUT I DON'T THINK I WAS MEANT TO BE A TIGER.
© 1987 Universal Press Syndicate
JUST SET THE DIAL TO "CALVIN," AND I'LL TRANSMOGRIFY BACK TO A BOY.
HERE YOU GO.
ZAP!
TRANSMOG-RIFIER
OOPS!
TRY AGAIN, LUNKHEAD.
4-2
WATTERSON

**Question 15.4** What feature do

$\neg : \mathbb{B} \to \mathbb{B}$ defined by $\neg(x) = \neg x$;

$f(x) : \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ defined by $f(x) = \frac{1}{x}$; and

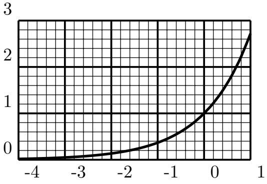
$g(x) : \mathbb{R} - \{1\} \to \mathbb{R} - \{1\}$ defined by $g(x) = \frac{x}{x-1}$;

have in common?

They are their own inverses.

**Example:** $e^x$ **and** $\log$

Consider $f : \mathbb{R} \to \mathbb{R}^{\geqslant 0} - \{0\}$ defined by $f(x) = e^x$. We know that $e^x$ is one-to-one (e.g. because it is strictly increasing), and onto. So it has an inverse $f^{-1}$ on $\mathbb{R}^{\geqslant 0} - \{0\}$.




Plot of $y = e^x$.

In fact, $f^{-1} = \log(y)$ where

$$\log : \mathbb{R}^{\geqslant 0} - \{0\} \to \mathbb{R}.$$

Now

$$e^{\log x} = x \quad \text{and} \quad \log(e^x) = x,$$

so $e^{\log x}$ and $\log(e^x)$ are both identity functions, but they have different domains.

The domain of $e^{\log x}$ is $\mathbb{R}^{\geqslant 0} - \{0\}$ (note $\log$ is defined only for reals $> 0$). The domain of $\log(e^x)$ is $\mathbb{R}$.

**Question** Let $f : \{x : x \text{ is a Monash student}\} \to \mathbb{N}$ be the function defined by $f(x)$ equals the ID number of $x$. Does $f^{-1}$ exist?

**Answer**
No. $f$ is not onto. (E.g. there is no student with ID number $10^{200}$.)

## Flux Exercise (LQMTZZ)

Let $g$ and $h$ be the functions

$g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined by $g(a, b) = ab$

$h : \{C : C \text{ is a circle in the plane with centre } (0, 0)\} \to \mathbb{R}$ defined by $h(C)$ is the area of $C$.

Does $g^{-1}$ exist? Does $h^{-1}$ exist?

A. Yes, yes

B. Yes, no

C. No, yes

D. No, no

**Answer**

$g$ is not one-to-one. E.g. $g((2, 3)) = g((1, 6))$.
So $g^{-1}$ doesn't exist.

$h$ is not onto. E.g. there is no circle of area $-1$ centred at $(0, 0)$.
So $h^{-1}$ doesn't exist.

So D.