MAT1830 Sample Exam 2

When you are instructed to "write down" something, no explanation is required. Everywhere else, you must justify your answers. Marks will be allocated for clarity of explanation. It is not enough to get the right answer.

(1) (a) Use the Euclidean algorithm to find the greatest common divisor of 633 and 255. **[5]**

(b) Find integers x and y such that $633x + 255y = 6$, or explain why none exist. **[4]**

(c) Is there an integer z such that $255z \equiv 7 \pmod{633}$? If there is such a z, find one. If there is no such z, explain why not. **[4]**

(d) Prove using induction that, for each integer $n \geq 1$,

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.$$ **[7]**

(2) (a) Determine whether the propositions $p \to (q \vee \neg r)$ and $(p \wedge \neg q) \to \neg r$ are logically equivalent using either a truth table or laws of logic. **[6]**

(b) Let A, B and C be sets. If a is the proposition "$x \in A$", b is the proposition "$x \in B$" and c is the proposition "$x \in C$", write down a proposition involving a, b and c that is logically equivalent to "$x \in A \cup (B - C)$". **[2]**

(c) Consider the statement $\forall x \exists y \neg P(x, y)$. Write down a negation of the statement that does not use the symbol $\neg$.

[2]

(d) Under the interpretation where x and y are in $\mathbb{R} - \{0\}$ and $P(x, y)$ is "$xy \geq 0$", is the original statement in (c) true or is its negation true?

[5]

(e) Is the statement $(\,\exists x(P(x) \vee Q(x))\,) \to (\,(\exists x P(x)) \vee (\exists x Q(x))\,)$ valid? If it is, explain why. If it isn't, give an interpretation under which it is false.

[5]

(3) (a) (i) Write down whether $-2 \in \mathbb{R} \cap \mathbb{N}$. **[1]**

(ii) Write down whether $\{(1,2),(2,2)\} \subseteq \{1,2,3\} \times \{2,3\}$. **[1]**

(iii) Given that A and B are sets such that $A \subseteq B$, write down $A \cup B$ in a simplified form. **[2]**

(iv) Given that A and B are sets such that $A \subseteq B$, write down $\mathcal{P}(A) - \mathcal{P}(B)$ in a simplified form. **[2]**

(v) How many elements does $\{5,6\} \times \mathcal{P}(\{1,2,3\})$ contain? **[2]**

(b) Let F, G and H be the following sets of ordered pairs.

$$\begin{aligned} F &= \{(1,1),(2,2),(3,7),(4,1)\} \\ G &= \{(1,1),(2,1),(3,2),(3,3),(4,2)\} \\ H &= \{(1,1),(2,3),(3,4),(4,2)\} \end{aligned}$$

(i) Does F define a function $f : \{1,2,3,4\} \to \mathbb{Z}$? **[1]**

(ii) Does G define a function $g : \{1,2,3,4\} \to \mathbb{Z}$? **[1]**

(iii) Does H define a function $h : \{1,2,3,4\} \to \{1,2,3,4\}$? **[1]**

(iv) For those of f, g and h that are functions, write down their ranges. **[2]**

(v) For those of f, g and h that are functions, do they have inverse functions? **[3]**

(vi) Does $f \circ h$ exist? If it does exist, write down the set of ordered pairs that defines $f \circ h$. **[2]**

(vii) Does $h \circ f$ exist? If it does exist, write down the set of ordered pairs that defines $h \circ f$. **[2]**

(4) (a) Let R be a relation defined on $\mathbb{R} \times \mathbb{R}$ by $(a,b)R(c,d)$ if and only if $ab \leq cd$.
Let S be a relation defined on $\mathbb{R}$ by aSb if and only if $a - b$ is an integer.

(i) Is R reflexive? Is R symmetric? Is R antisymmetric? Is R transitive? **[4]**

(ii) Is R a partial order relation? Is R an equivalence relation? **[1]**

(iii) Is S reflexive? Is S symmetric? Is S antisymmetric? Is S transitive? **[4]**

(iv) Is S a partial order relation? Is S an equivalence relation? **[1]**

(v) If R is an equivalence relation, then give its equivalence classes. If S is an equivalence relation, then give its equivalence classes. **[3]**

(b) (i) Rewrite the expression $\displaystyle\sum_{i=1}^{4} 2ix^i$ without using $\sum$. **[3]**

(ii) Prove that if x and y are integers such that $x \equiv 6 \pmod 8$ and $y \equiv 3 \pmod 8$, then 8 divides $2(x+y)^{21} + 3xy$. **[4]**

(5) (a) A soccer squad contains 3 goalkeepers, 7 defenders, 9 midfielders and 4 forwards.

(i) In how many ways can a team of 1 goalkeeper, 4 defenders, 4 midfielders, and 2 attackers be chosen from this squad? **[2]**

(ii) Two of the defenders refuse to play together. In how many ways can a team be chosen that contains at most one of these two defenders? **[3]**

(b) Let p and q be real numbers. A random variable X has expected value 1 and its probability distribution is given by the table below.

x	-1	0	1	3
$\Pr(X = x)$	$\frac{3}{16}$	p	$\frac{1}{4}$	q

(i) Find p and q. **[4]**

(ii) What is the variance of X? **[2]**

(iii) What is $E[8X]$? **[1]**

(c) A binary string of length 4 is selected uniformly at random. Let Y and Z be random variables so that Y is the numbers of 1s in the entire string and Z is the number of 1s in the first two bits of the string.

(i) Find $\Pr(Y = 1)$ and $\Pr(Z = 2)$. **[2]**

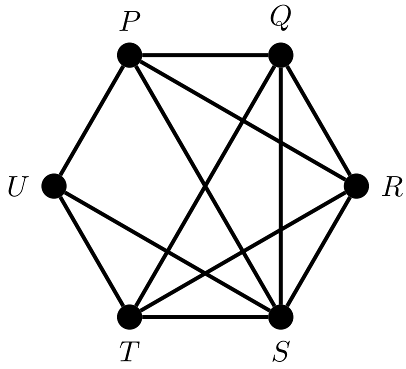
(ii) Find $\Pr(Y = 1 \wedge Z = 2)$. **[2]**

(iii) Are the events "$Y = 2$" and "$Z = 2$" independent? **[1]**

(iv) Are Y and Z independent random variables? **[2]**

(v) Find $\Pr(Y = 1 \vee Z = 2)$. **[1]**

(6) (a) Consider the following graph.



(i) What are the degrees of the vertices in the graph? **[1]**

(ii) Does the graph have a closed Euler trail? If so, give an example of a closed Euler trail in the graph. If not, explain why no closed Euler trail exists. **[2]**

(iii) Give an example of a spanning tree in the graph. **[1]**

(iv) Two identical looking bags are on a table. One contains 30 green marbles and 30 black marbles, and the other contains 10 green marbles, 10 blue marbles and 10 red marbles. One of the bags is randomly selected (each has a 50% chance of being chosen) and a marble is drawn uniformly at random from that bag. Given that the marble drawn is green, what is the probability it comes from the bag with green and black marbles? **[6]**

(b) For each integer $n \geq 1$, let r_n be the number of binary strings of length n that do not contain three consecutive 1s.

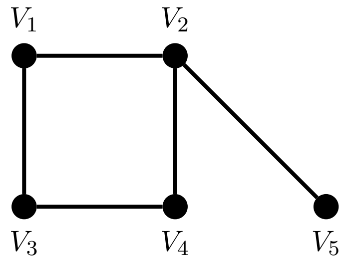
(i) Find r_1, r_2, r_3 and r_4. **[2]**

(ii) Find a recurrence for r_n that holds for all integers $n \geq 4$. Explain why your recurrence gives r_n. **[4]**

For each integer $n \geq 5$, let s_n be the number of binary strings of length n that do not contain three consecutive 1s, do not begin with 1 and end with two consecutive 1s, and do not begin with two consecutive 1s and end with 1.

(iii) Find an expression for s_n in terms of $r_1, r_2, \ldots, r_{n-1}$ that holds for all integers $n \geq 5$. Explain why your expression gives s_n. **[4]**

(7) (a) Consider the following graph.



(i) Give the adjacency matrix of the graph, where columns 1, 2, 3, 4 and 5 correspond to vertices V_1, V_2, V_3, V_4 and V_5 respectively. **[2]**

(ii) Explain how you could use the adjacency matrix to find the number of walks of length 10 that begin at V_5. (You don't have to actually find the number.) **[3]**

(b) (i) Draw a simple graph with 8 vertices and 7 edges that is not a tree, or explain why this is impossible. **[2]**

(ii) Is there a simple graph with 7 vertices in which every vertex has degree 3? If so, draw one. If not, explain why not. **[2]**

(iii) Prove that $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$ for all integers n and r with $n > r > 0$. **[5]**

(iv) A simple graph is formed randomly on the vertices $V_1, V_2, \ldots, V_{20}$ in the following way. For each unordered pair of these vertices $\{V_i, V_j\}$, the edge V_iV_j is in the graph with probability $\frac{1}{3}$ and is not in the graph with probability $\frac{2}{3}$. What is the expected number of triangles contained in this graph?

(A triangle is a simple graph with three vertices such that each pair of vertices is joined by an edge.) **[6]**