MAT1830

Lecture 34: Revision and exam preparation 1

Relations

A relation R on a set X is:

reflexive if xRx for all $x \in X$.

symmetric if $xRy \Rightarrow yRx$ for all $x, y \in X$.

antisymmetric if $xRy \wedge yRx \Rightarrow x = y$ for all $x, y \in X$.
this NOT the same as "not symmetric"

transitive if $xRy \wedge yRz \Rightarrow xRz$ for all $x, y, z \in X$.

an equivalence relation if it is reflexive, symmetric and transitive.

a partial order relation if it is reflexive, antisymmetric and transitive.

a total order relation if it is a partial order, and xRy or yRx for all $x, y \in X$.

a well order relation if it is a total order, and each subset of X has a least element.
(A least element is an element ℓ such that ℓRx for all $x \in X$.)

examples of equivalence relations: congruence modulo n on $\mathbb{Z}$, parallel-ness on the set of lines in the plane

examples of partial order relations: $\subseteq$ on $\mathcal{P}(\mathbb{Z})$, divides on $\mathbb{N}$
(these two are not total orders)

examples of total order relations: $\leq$ on $\mathbb{Z}$ (not a well order)

examples of well order relations: $\leq$ on $\mathbb{N}$, alphabetical order on words

Question Let R, S and T be relations on $\{1,2,3\} \times \{1,2,3\}$ defined by

$$\begin{aligned} (a,b)R(c,d) &\quad \text{if } a = c \\ (a,b)S(c,d) &\quad \text{if } b \leq d \\ (a,b)T(c,d) &\quad \text{if } a = c \text{ and } b \leq d \end{aligned}$$

Which of the following describes when each of these relations does NOT have a property out of reflexive, symmetric, transitive, antisymmetric?

A. R: not antisymmetric, S: not symmetric, T: not symmetric, not transitive
B. R: not antisymmetric, not transitive, S: not antisymmetric, T: not symmetric
C. R: not antisymmetric, S: not symmetric, T: not symmetric
D. R: not antisymmetric, S: not symmetric, not antisymmetric, T: not symmetric

Solution

R is not antisymmetric because $(1,1)R(1,2)$ and $(1,2)R(1,1)$ but $(1,1) \neq (1,2)$.
S is not symmetric because $(1,1)S(1,2)$ and $(1,2) \not S(1,1)$.
S is not antisymmetric because $(1,1)S(2,1)$ and $(2,1)S(1,1)$ but $(1,1) \neq (2,1)$.
T is not symmetric because $(1,1)T(1,2)$ and $(1,2) \not T(1,1)$.

And that's all, so D

Question Which of R, S and T are equivalence relations? partial order relations?

Question Let R and T be relations on $\{1,2,3\} \times \{1,2,3\}$ defined by

$$\begin{aligned} (a,b)R(c,d) &\quad \text{if } a = c \\ (a,b)T(c,d) &\quad \text{if } a = c \text{ and } b \leq d \end{aligned}$$

How many equivalence classes does R have? What sizes are the equivalence classes? Is T a total order relation?

A. 3 classes each of size 3, yes
B. 9 classes each of size 1, no
C. 3 classes each of size 3, no
D. 3 classes of sizes 2,3,4, yes

Solution

The equivalence classes of R are $\{(1,1),(1,2),(1,3)\}$ and $\{(2,1),(2,2),(2,3)\}$ and $\{(3,1),(3,2),(3,3)\}$.

T is not a total order because $(1,2) \not T (2,2)$ and $(2,2) \not T (1,2)$.

So C.

Question Let R, S, T, U be relations on $\mathbb{Z}$ defined by

aRb if and only if $a \leq b$
aSb if and only if $|a| \leq |b|$
aTb if and only if either $|a| < |b|$ or $|a| = |b|$ and $a \leq b$
aUb if and only if $b - 5 \leq a \leq b$

Which of R, S, T, U are partial orders? Which are total orders? Which are well orders?

Solution

R is a partial order and a total order but not a well order.
For example the set of even integers doesn't have least element.

S isn't a partial order because it's not antisymmetric.
For example $6S(-6)$ and $(-6)S6$ but $6 \neq -6$.

T is a well order.
If there's one element with smallest absolute value in a set, it's the least element of that set. If there's two, then the negative one is the least element of the set.

U isn't a partial order because it's not transitive.
For example $1U5$ and $5U9$ but $1 \not{U} 9$.