MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #1 Solutions

1. (a) False (because 8 does not divide $20 - 14$).
   (b) True (because $-15 = -5 \times 3$).
   (c) True (because 3 divides $-11 - 1$).
   (d) False (because $9 = 3 \times 3$).
   (e) False (because $1000 \equiv 0 \pmod 5$ and $12544 \equiv 4 \pmod 5$).
   (f) False (because there is no integer k such that $66k = 22$).

2. (a) We use the Euclidean algorithm.

$$\begin{array}{rcrcrcr} 1022 & = & 2 & \times & 400 & + & 222 \\ 400 & = & 1 & \times & 222 & + & 178 \\ 222 & = & 1 & \times & 178 & + & 44 \\ 178 & = & 4 & \times & 44 & + & 2 \\ 44 & = & 22 & \times & 2 & + & 0 \end{array}$$

   So $\gcd(1022, 400) = 2$.

   (b) Now we use the extended Euclidean algorithm.

$$\begin{array}{rcrcr} 2 & = & 178 - 4 \times 44 & & \\ 2 & = & 178 - 4 \times (222 - 178) & = & -4 \times 222 + 5 \times 178 \\ 2 & = & 5 \times (400 - 222) - 4 \times 222 & = & 5 \times 400 - 9 \times 222 \\ 2 & = & 5 \times 400 - 9 \times (1022 - 2 \times 400) & = & -9 \times 1022 + 23 \times 400 \end{array}$$

   So $2 = -9 \times 1022 + 23 \times 400$. So $a = -9$, $b = 23$ is one solution.

   (c) We just saw that

$$2 = -9 \times 1022 + 23 \times 400.$$

   Multiplying through by 4 gives

$$\begin{aligned} 4 \times 2 &= (4 \times -9) \times 1022 + (4 \times 23) \times 400 \\ 8 &= -36 \times 1022 + 92 \times 400. \end{aligned}$$

   So $a = -36$, $b = 92$ is one solution.

3. (a) x must be prime.
   If x were not prime then there would be an integer d such that $2 \leqslant d < x$ and d divides x. But then d would also divide n, and this would mean that x wasn't the smallest integer such that $x \geqslant 2$ and x divides n (because d would be smaller).
   (b) Because $y \equiv 2 \pmod 3$, $y = 3k + 2$ for some integer k. For some integer l, the integer k is $4l$, $4l + 1$, $4l + 2$ or $4l + 3$.
   If $k = 4l$, then $y = 3(4l) + 2 = 12l + 2$ and $y \equiv 2 \pmod{12}$.
   If $k = 4l + 1$, then $y = 3(4l + 1) + 2 = 12l + 5$ and $y \equiv 5 \pmod{12}$.
   If $k = 4l + 2$, then $y = 3(4l + 2) + 2 = 12l + 8$ and $y \equiv 8 \pmod{12}$.
   If $k = 4l + 3$, then $y = 3(4l + 3) + 2 = 12l + 11$ and $y \equiv 11 \pmod{12}$.
   So y might be congruent to 2, 5, 8 or 11 modulo 12.

4. (a) 7 and 21 and 28 are all divisible by 7, and by using these jugs (adding and subtracting multiples of these numbers) we can only get numbers of litres of water that are divisible by 7. Of course, 4 isn't divisible by 7. Send Batman.

   (b) If we add 400 lots of 1022 to a number and subtract 1022 lots of 400, then the number doesn't change. This means that the pair $(-9 + 400, 23 - 1022) = (391, -999)$ works as well as $(-9, 23)$ and we can keep doing this getting $(791, -2021), (1191, -3043), \ldots$. We can also go the other way and get $(-409, 1045), (-809, 2067), \ldots$.