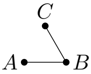
MAT1830

Lecture 31: Degree

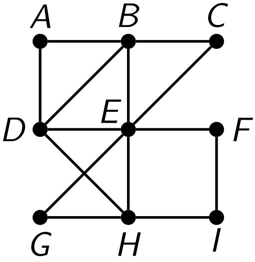
The *degree* of a vertex A in a graph G is the number of edges of G that include A.

For example, if G is



then the degree of B is 2 and the degrees of A and C are both 1.

Question What are the degrees of the vertices in the graph below?



Answer

the degree of A is 2
the degree of B is 4
the degree of C is 2
the degree of D is 4
the degree of E is 6
the degree of F is 2
the degree of G is 2
the degree of H is 4
the degree of I is 2

31.1 The handshaking lemma

In any graph,
sum of degrees $= 2\times$ number of edges.

The reason for the name is that if each edge is viewed as a handshake,

then at each vertex V

$$\text{degree}(V) = \text{number of hands.}$$

Hence

$$\begin{aligned} & \text{sum of degrees} \\ = \; & \text{total number of hands} \\ = \; & 2 \times \text{number of handshakes} \end{aligned}$$

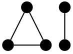
An important consequence
The handshaking lemma implies that *in any graph the sum of degrees is even* (being $2\times$something). Thus it is impossible, e.g. for a graph to have degrees 1,2,3,4,5.

Questions

31.1 For each of the following sequences, construct a graph whose vertices have those degrees, or explain why no such graph exists.

- $1, 2, 3, 4$
- $1, 2, 1, 2, 1$
- $1, 2, 2, 2, 1$

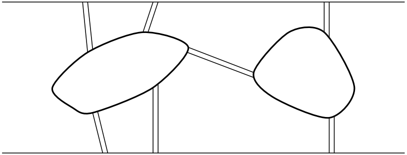
Using the handshaking lemma, we can rule out the second option since the sum of the degrees in any graph must be even, and $1 + 2 + 1 + 2 + 1 = 7$, which is odd.

The third sequence is achieved by  or by 

The first sequence is impossible for a simple graph. Since each vertex can have at most one edge to each other vertex, the largest possible degree would be 3.

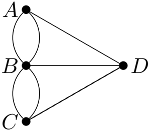
31.2 The seven bridges of Königsberg

In 18th century Königsberg there were seven bridges connecting islands in the river to the banks as follows.



The question came up: is it possible for a walk to cross all seven bridges without crossing the same bridge twice?

An equivalent question is whether there is a trail which includes all edges in the following multigraph.



SOLVTIO PROBLEMATIS
AD
GEOMETRIAM SITVS
PERTINENTIS.

AVCTORE
Leonh. Eulero.

§. 1.

Tabula VIII. PRaeter illam Geometriae partem, quae circa quantitates verſatur, et omni tempore ſummo ſtudio eſt exculta, alterius partis etiamnum admodum ignotae primus mentionem fecit *Leibnitzius*, quam Geometriam ſitus vocauit. Iſta pars ab ipſo in ſolo ſitu determinando, ſitusque proprietatibus eruendis occupata eſſe ſtatuitur; in quo negotio neque ad quantitates reſpiciendum, neque calculo quantitatum vtendum ſit. Cuiusmodi autem problemata ad hanc ſitus Geometriam pertineant, et quali methodo in iis reſoluendis vti oporteat, non ſatis eſt definitum. Quamobrem, cum nuper problematis cuiusdam mentio eſſet facta, quod quidem ad geometriam pertinere videbatur, at ita erat comparatum, vt neque determinationem quantitatum requireret, neque ſolutionem calculi quantitatum ope admitteret, id ad geometriam ſitus referre haud dubitaui: praeſertim quod in eius ſolutione ſolus ſitus in conſiderationem veniat, calculus vero nullius prorſus ſit vſus. Methodum ergo meam quam ad huius generis proble-

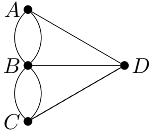
mata

mata ſoluenda inueni, tanquam ſpecimen Geometriae ſitus hic exponere conſtitui.

§. 2. Problema autem hoc, quod mihi ſatis notum eſſe perhibebatur, erat ſequens: Regiomonti in Boruſſia eſſe inſulam A der Kneiphof dictam, fluuiumque eam cingentem in duos diuidi ramos, quemadmodum ex figura videre licet: ramos vero huius fluuii ſeptem inſtructos eſſe pontibus, *a*, *b*, *c*, *d*, *e*, *f*, et *g*. Circa hos pontes iam iſta proponebatur quaeſtio, num quis curſum ita inſtituere queat, vt per ſingulos pontes ſemel et non plus quam ſemel tranſeat. Hocque fieri poſſe, mihi dictum eſt, alios negare alios dubitare; neminem vero affirmare. Ego ex hoc mihi ſequens maxime generale formaui problema; quaecunque ſit fluuii figura et diſtributio in ramos, atque quicunque fuerit numerus pontium, inuenire, vtrum per ſingulos pontes ſemel tantum tranſiri queat, an vero ſecus? Figura 1.

§. 3. Quod quidem ad problema Regiomontanum de ſeptem pontibus attinet, id reſolui poſſet facienda perfecta enumeratione omnium curſuum, qui inſtitui poſſunt; ex his enim innoteſceret, num quis curſus ſatisfaceret, an vero nullus. Hic vero ſoluendi modus propter tantum combinationum numerum et nimis eſſet difficilis atque operoſus, et in aliis quaeſtionibus de multo pluribus pontibus ne quidem adhiberi poſſet. Hoc porro modo ſi operatio ad finem perducatur multa inueniuntur, quae non erant in quaeſtione; in quo procul dubio tantae difficultatis cauſa conſiſtit. Quamobrem miſſa hac methodo, in aliam inquiſiui, quae plus non

 lar-



31.3 Euler's solution

Euler (1737) observed that the answer is no, because

1. Each time a walk enters and leaves a vertex it "uses up" 2 from the degree.

2. Hence if all edges are used by the walk, all vertices except the first and last must have even degree.

3. The seven bridges graph in fact has four vertices of odd degree.

31.4 Euler's theorem

A trail that uses every edge of a graph exactly once is called an *Euler trail*.

The argument from the last section shows in general that

A graph with > 2 odd degree vertices has no Euler trail.

And a similar argument shows

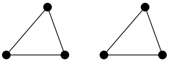
A graph with odd degree vertices has no *closed* Euler trail.

(Because in this case the first and last vertex are the same, and its degree is "used up" by a closed trail as follows: 1 at the start, 2 each time through, 1 at the end.)

31.5 The converse theorem

If, conversely, we have a graph G whose vertices all have even degree, must it have an Euler trail?

Not necessarily. For example, G might be the following disconnected graph.



However, if we also assume the graph is connected then it must have a closed Euler trail.

A connected graph with no odd degree vertices has a closed Euler trail.

Such a closed trail can be constructed as follows:

1. Starting at any vertex V_1 , follow a trail t_1 as long as possible.

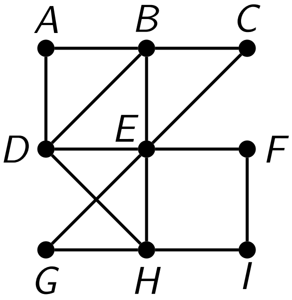
2. The trail t_1 eventually returns to V_1, because it can leave any other vertex it enters. (Immediately after the start, V_1 has one "used" edge, and hence an odd number of "unused" edges. Any other vertex has an even number of "unused" edges.)

3. If t_1 does not use all edges, retrace it to the first vertex V_2 where t_1 meets an edge not in t_1.

4. At V_2 add a "detour" to t_1 by following a trail out of V_2 as long as possible, not using edges in t_1. As before, this trail eventually returns to its starting point V_2, where we resume the trail t_1. Let t_2 be the trail t_1 plus the detour from V_2.

5. If t_2 does not use all the edges, retrace t_2 to the first vertex V_3 where t_2 meets an edge not in t_2. Add a detour at V_3, and so on.

6. Since a graph has only a finite number of edges, this process eventually halts. The result will be a closed trail which uses all the edges (this requires the graph to be connected, since any unused edge would be connected to used ones, and thus would have eventually been used).



connected with degrees 2, 4, 2, 4, 6, 2, 2, 4, 4

Such a closed trail can be constructed as follows:

1. Starting at any vertex V_1 , follow a trail t_1 as long as possible.
2. The trail t_1 eventually returns to V_1, because it can leave any other vertex it enters. (Immediately after the start, V_1 has one "used" edge, and hence an odd number of "unused" edges. Any other vertex has an even number of "unused" edges.)
3. If t_1 does not use all edges, retrace it to the first vertex V_2 where t_1 meets an edge not in t_1.
4. At V_2 add a "detour" to t_1 by following a trail out of V_2 as long as possible, not using edges in t_1. As before, this trail eventually returns to its starting point V_2, where we resume the trail t_1. Let t_2 be the trail t_1 plus the detour from V_2.
5. If t_2 does not use all the edges, retrace t_2 to the first vertex V_3 where t_2 meets an edge not in t_2. Add a detour at V_3, and so on.
6. Since a graph has only a finite number of edges, this process eventually halts. The result will be a closed trail which uses all the edges (this requires the graph to be connected, since any unused edge would be connected to used ones, and thus would have eventually been used).

Question

31.3 A graph H has adjacency matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

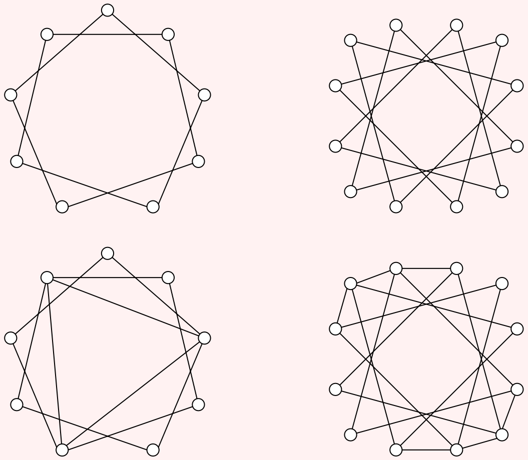
What are the degrees of its vertices? Does H have a closed Euler trail?

Answer

The degree of a vertex is the number of edges that include it. So
the degree of V_1 is 3,
the degree of V_2 is 2,
the degree of V_3 is 2,
the degree of V_4 is 1,
the degree of V_5 is 2.

To have a closed Euler trail, every degree must be even. Here we have two vertices of odd degree, so there is no closed Euler trail.

Flux Exercise (LQMTZZ)



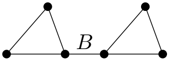
How many of the pictured graphs have an Euler trail?

A. 0 B. 1 C. 2 D. 3 E. 4

Answer C. The two on the left do.

31.6 Bridges

A *bridge* in a connected graph G is an edge whose removal disconnects G. E.g. the edge B is a bridge in the following graph.



The construction of an Euler trail is improved by doing the following (*Fleury's algorithm*).

- Erase each edge as soon as it is used.
- Use a bridge in the remaining graph only if there is no alternative.

It turns out, when this algorithm is used, that it is not necessary to make any detours. The improvement, however, comes at the cost of needing an algorithm to recognise bridges.

Flux Exercise (LQMTZZ)

How many bridges are there in a path of length n? How many bridges are there in a cycle of length n?

A. n, 0

B. 1, 0

C. 1, 2

D. n, 2

E. 1, n

Answer A.

Every edge of a path is a bridge.

No edge of a cycle is a bridge.