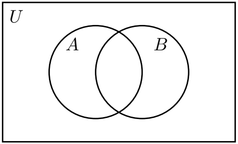
MAT1830

Lecture 12: Operations on Sets

There is an "arithmetic" of sets similar to ordinary arithmetic. There are operations similar to addition, subtraction and multiplication.

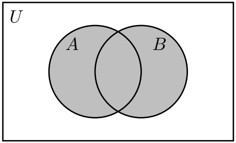
12.1 Venn diagrams

The simple operations on sets can be visualised with the help of *Venn diagrams*, which show sets $A, B, C, \ldots$ as disks within a rectangle representing the universal set U.



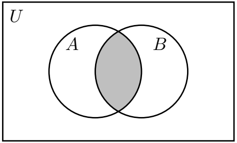
12.2 Union $A \cup B$

The union $A \cup B$ of sets A and B consists of the elements in A *or* B, and is indicated by the shaded region in the following Venn diagram.



12.3 Intersection $A \cap B$

The intersection $A \cap B$ of sets A and B consists of the elements in A *and* B, indicated by the shaded region in the following Venn diagram.

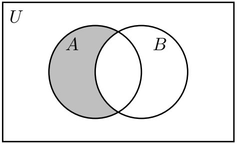


Questions

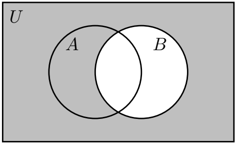
What is $\{1, 2, 3\} \cup \{2, 5, 7\}$?	$\{1, 2, 3, 5, 7\}$
What is $\{1, 2, 3\} \cap \{2, 3, 6, 7\}$?	$\{2, 3\}$
What is $\{1, 2, 3\} \cup \{2, 3\}$?	$\{1, 2, 3\}$
What is $\{1, 2, 3\} \cap \{3\}$?	$\{3\}$
What is $\{1, 2, 3\} \cap \{7, 8\}$?	$\{\}$ (or $\emptyset$ if you prefer)

12.4 Difference $A - B$

The difference $A - B$ of sets A and B consists of the elements in A and *not* in B, indicated by the shaded region in the following Venn diagram.

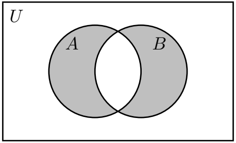


The difference $U - B$ relative to the universal set U is called the *complement* $\overline{B}$ of B. Here is the Venn diagram of $\overline{B}$.



12.5 Symmetric difference $A \triangle B$

The union of $A - B$ and $B - A$ is called the *symmetric difference* $A \triangle B$ of A and B.



$A \triangle B$ consists of the elements of *one of* A, B but not the other.

It is clear from the diagram that we have not only

$$A \triangle B = (A - B) \cup (B - A),$$

but also

$$A \triangle B = (A \cup B) - (A \cap B).$$

Questions

What is $\{1,2,3\} - \{2,5,7\}$?	$\{1,3\}$
What is $\{1,2,3\} \triangle \{2,3,6,7\}$?	$\{1,6,7\}$
What is $\{2,3\} - \{1,2,3\}$?	$\{\}$
What is $\{1,2,3\} \triangle \{3\}$?	$\{1,2\}$
What is $\{1,2,3\} - \{7,8\}$?	$\{1,2,3\}$

Flux Exercise (LQMTZZ)

Let $S = \{-2, -1, 0, 1, 2\} \cap \mathbb{N}$. If we know that $S \subseteq (\{-1, 0, 1\} \cup T)$, what can we say about T?

A. T must equal $\{2\}$

B. T must equal $\mathbb{N}$

C. T can be any set such that $-2 \in T$

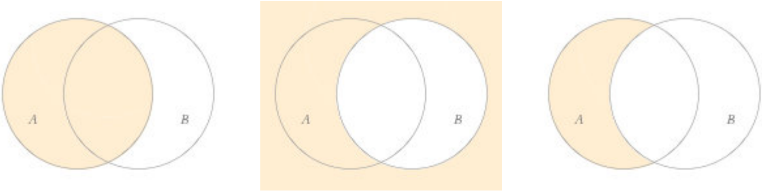
D. T can be any set such that $2 \in T$

Answer

Note $S = \{0, 1, 2\}$.
So for S to be a subset of $\{-1, 0, 1, \} \cup T$ we need that each of $0, 1, 2$ is an element of $\{-1, 0, 1\} \cup T$.
This is definitely true for 0 and 1 because they're in $\{-1, 0, 1\}$.
So we just need that 2 be an element of T.

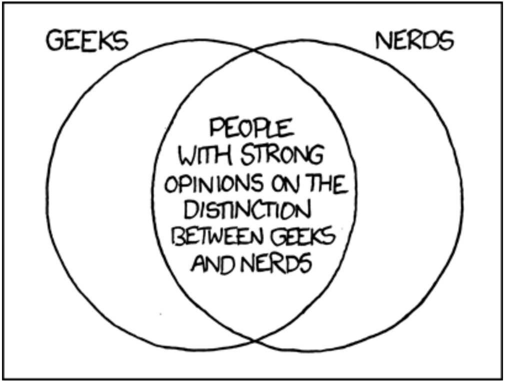
So D.

Question 12.1 Draw a Venn diagram for $A \cap \overline{B}$. What is another name for this set?



A $\qquad$ $\overline{B}$ $\qquad$ $A \cap \overline{B}$

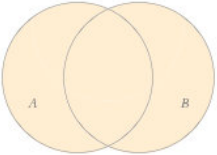
So $A \cap \overline{B} = A - B$.


GEEKS
NERDS
PEOPLE WITH STRONG OPINIONS ON THE DISTINCTION BETWEEN GEEKS AND NERDS

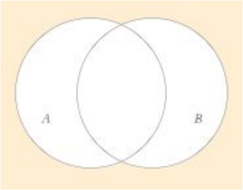
Set operations and logic operations

$$
\begin{array}{lcl}
x \in A \cup B & \text{if and only if} & (x \in A) \vee (x \in B) \\
x \in A \cap B & \text{if and only if} & (x \in A) \wedge (x \in B) \\
x \in A - B & \text{if and only if} & (x \in A) \wedge (x \notin B) \\
x \in A \triangle B & \text{if and only if} & (x \in A) \veebar (x \in B)
\end{array}
$$

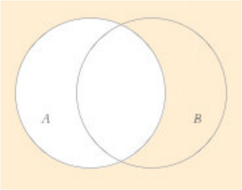
Question 12.2 Show that $\overline{A \cup B} = \overline{A} \cap \overline{B}$ is true using Venn diagrams.



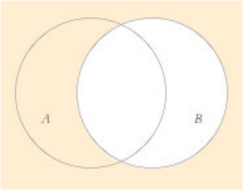
$A \cup B$



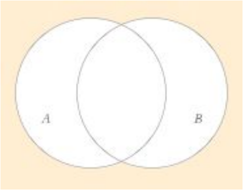
$\overline{A \cup B}$



$\overline{A}$



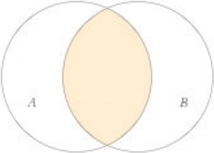
$\overline{B}$



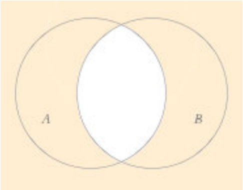
$\overline{A} \cap \overline{B}$

So $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

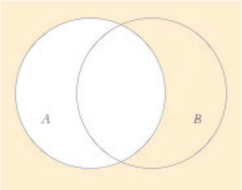
Question 12.2 (cont) Show that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ is true using Venn diagrams.



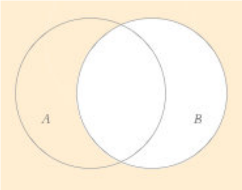
$A \cap B$



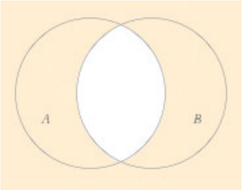
$\overline{A \cap B}$



$\overline{A}$



$\overline{B}$



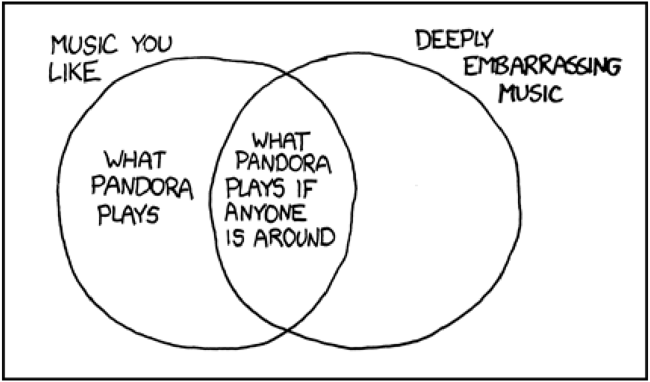
$\overline{A} \cup \overline{B}$

So $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Question Show that $\overline{A \cup B} = \overline{A} \cap \overline{B}$ is true using logic.

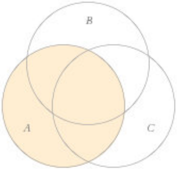
$$
\begin{aligned}
x \in \overline{A \cup B} &\equiv \neg(x \in A \cup B) \\
&\equiv \neg((x \in A) \vee (x \in B)) \\
&\equiv \neg(x \in A) \wedge \neg(x \in B) \\
&\equiv (x \in \overline{A}) \wedge (x \in \overline{B}) \\
&\equiv x \in \overline{A} \cap \overline{B}
\end{aligned}
$$

So $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

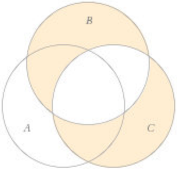


MUSIC YOU LIKE
DEEPLY EMBARRASSING MUSIC
WHAT PANDORA PLAYS
WHAT PANDORA PLAYS IF ANYONE IS AROUND

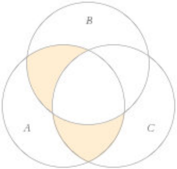
Question 12.3 Find whether $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$ is true using Venn diagrams.



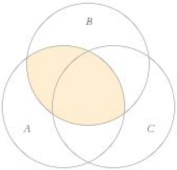
A



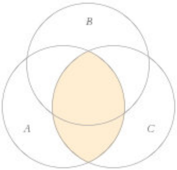
$B \triangle C$



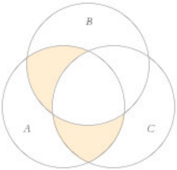
$A \cap (B \triangle C)$



$A \cap B$



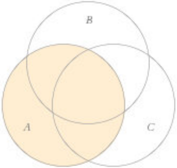
$A \cap C$



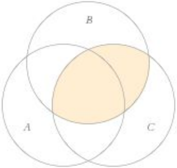
$(A \cap B) \triangle (A \cap C)$

So $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

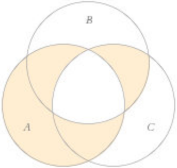
Question 12.3 (cont) Find whether $A \triangle (B \cap C) = (A \triangle B) \cap (A \triangle C)$ is true using Venn diagrams.



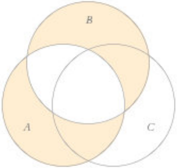
A



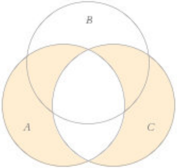
$B \cap C$



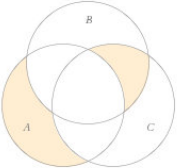
$A \triangle (B \cap C)$



$A \triangle B$

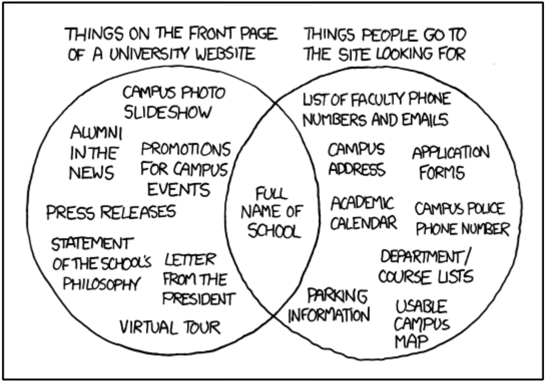


$A \triangle C$



$(A \triangle B) \cap (A \triangle C)$

So $A \triangle (B \cap C) = (A \triangle B) \cap (A \triangle C)$ is not true *in general*.
(But it will be true for some choices of A, B and C.)


THINGS ON THE FRONT PAGE OF A UNIVERSITY WEBSITE
THINGS PEOPLE GO TO THE SITE LOOKING FOR
CAMPUS PHOTO SLIDESHOW
ALUMNI IN THE NEWS
PROMOTIONS FOR CAMPUS EVENTS
PRESS RELEASES
STATEMENT OF THE SCHOOL'S PHILOSOPHY
LETTER FROM THE PRESIDENT
VIRTUAL TOUR
FULL NAME OF SCHOOL
LIST OF FACULTY PHONE NUMBERS AND EMAILS
CAMPUS ADDRESS
APPLICATION FORMS
ACADEMIC CALENDAR
CAMPUS POLICE PHONE NUMBER
DEPARTMENT/COURSE LISTS
PARKING INFORMATION
USABLE CAMPUS MAP

Ordered pairs

For sets we have $\{a, b\} = \{b, a\}$.

But sometimes order is important.

So we define ordered pairs (a, b), where the order is significant: $(a, b) \neq (b, a)$.

We can of course also define ordered triples (a, b, c) etc.

12.6 Ordered Pairs

Sometimes we do want order to be important. In computer science arrays are ubiquitous examples of ordered data structures. In maths, *ordered pairs* are frequently used. An ordered pair (a, b) consists simply of a first object a and a second object b. The objects a and b are sometimes called the *entries* or *coordinates* of the ordered pair.

Two ordered pairs (a, b) and (c, d) are equal if and only if $a = c$ and $b = d$.

Example. $\{0, 1\} = \{1, 0\}$ but $(0, 1) \neq (1, 0)$.

There's no reason we need to stop with pairs. We can similarly define ordered triples, quadruples, and so on. When there are k coordinates, we call the object an *ordered k-tuple*. Two ordered k-tuples are equal if and only if their ith coordinates are equal for $i = 1, 2, \ldots, k$.

How can we represent ordered pairs using sets?*

If we want to define ordered pairs just using sets, how can we introduce an order?

One way is to define $(a, b) = \{\{a, b\}, \{a\}\}$.

The 2-element set tells us what the two things in the ordered pair are (but not their order).

The 1-element set tells us which of them comes first.

* this method of defining of ordered pairs is not assessable, but your ability to work with ordered pairs is.

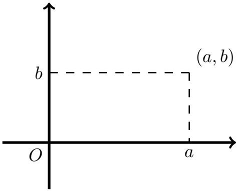
12.7 Cartesian product $A \times B$

The set of ordered pairs

$$A \times B = \{(a, b) \ : \ a \in A \text{ and } b \in B\}$$

is the *cartesian product* of sets A and B.

The commonest example is where $A = B = \mathbb{R}$ (the set of real numbers, or the number line). Then the pairs (a, b) are points in the plane, so $\mathbb{R} \times \mathbb{R}$ is the plane.



Because Descartes used this idea in geometry, the cartesian product is named after him.

Remember the elements of $A \times B$ are always *ordered pairs*.

Question If $A = \{0, 1\}$ and $B = \{1, 2, 3\}$ what is $A \times B$?

$\{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3)\}$

Question If $A = \{0, 1\}$ what is $A \times \mathbb{N}$?

$\{(0, 0), (1, 0), (0, 1), (1, 1), (0, 2), (1, 2), \ldots\}$

Flux Exercise (LQMTZZ)

Let $S = \{-1, 1\} \times \{0, 1, 2\}$. Is $(-1, 1) \in S$? Is $(0, 1) \in S$?

A. Yes, yes

B. Yes, no

C. No, yes

D. No, no

Answer

$(-1, 1) \in S$ because $-1 \in \{-1, 1\}$ and $1 \in \{0, 1, 2\}$.
$(0, 1) \notin S$ because $0 \notin \{-1, 1\}$.

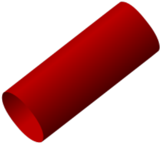
So B.

12.8 $A \times B$ and multiplication

If A has $|A|$ elements and B has $|B|$ elements, then $A \times B$ has $|A| \times |B|$ elements.

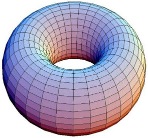
Similarly, if L is a line of length l, and W is a line of length w, then $L \times W$ is a rectangle of area $l \times w$. In fact, we call it an "$l \times w$ rectangle." This is probably the reason for using the $\times$ sign, and for calling $A \times B$ a "product."

Question 12.4 If line $\times$ line $=$ plane, what is line $\times$ circle?



A cylinder.

Question 12.4 (cont) What is circle $\times$ circle?



A torus.