MAT1830 Sample Exam 1

When you are instructed to "write down" something, no explanation is required. Everywhere else, you must justify your answers. Marks will be allocated for clarity of explanation. It is not enough to get the right answer.

(1) (a) Use the Euclidean algorithm to find the greatest common divisor of 504 and 385. **[6]**

(b) Is it possible to find an integer y such that $504y \equiv 10 \pmod{385}$? If it is, find one. If it isn't, explain why not. **[4]**

(c) Is it possible to find an integer z such that $504z \equiv 7 \pmod{385}$? If it is, find one. If it isn't, explain why not. **[4]**

(d) Prove using induction that, for each integer $n \geq 1$,

$$5 + 5^2 + 5^3 + \cdots + 5^n = \frac{5^{n+1} - 5}{4}.$$

[6]

(2) (a) Determine whether the propositions $\neg((p \to \neg q) \wedge r)$ and $(p \wedge q) \vee \neg r$ are logically equivalent using either a truth table or the laws of logic. **[6]**

(b) Let A and B be sets. If p is the proposition "$x \in A$" and q is the proposition "$x \in B$", write down a proposition involving p and q which is logically equivalent to "$x \in A \cup B$". **[2]**

(c) Is the statement $(\exists x(P(x) \wedge Q(x))) \leftrightarrow ((\exists x P(x)) \wedge (\exists x Q(x)))$ valid? If it is, explain why. If it isn't, give an interpretation under which it is false. **[6]**

(d) Consider the following sentences.

(i) $\exists x \forall y P(x, y)$

(ii) $\forall y \exists x P(x, y)$

For each sentence, state whether it is true or false under the interpretation where x and y range over the positive integers and $P(x, y)$ is "$x \leq y$", and explain why this is. **[6]**

(3) (a) Let P be the set of all prime numbers, let T be the set of all natural numbers which are divisible by 3, and let $A = \{3, 4, 5, 6\}$.

(i) Write down $P \cap T$. **[2]**

(ii) Write down $T \cap A$. **[2]**

(iii) Write down $T \cup \mathbb{N}$. **[2]**

(iv) How many elements does $\mathcal{P}(A \times \{1, 2\})$ have? **[2]**

(b) Let A be the set of all non-empty subsets of $\{1, 2, \ldots, 10\}$ and let f and g be the following functions.

$f : A \to \mathbb{Z}$ defined by $f(X) = a - b$, where a is the largest element of X and b is the smallest element of X.

$g : A \to A$ defined by $g(X) = X \cup \{1, 2\}$.

(i) Write down $f(\{2, 3, 6\})$. Write down $g(\{2, 7, 10\})$. **[2]**

(ii) Is f one-to-one? Is g one-to-one? **[3]**

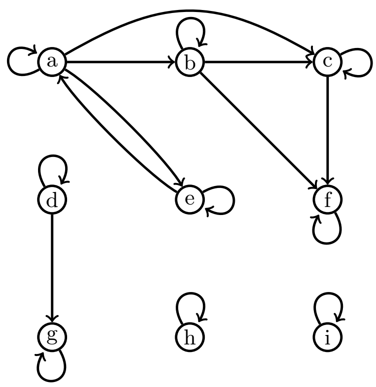
(iii) What is the range of f? **[3]**

(iv) Does $f \circ g$ exist? If it does exist, write down $f \circ g(\{9\})$. **[2]**

(v) Does $g \circ f$ exist? Explain. If it does exist, write down $g \circ f(\{9\})$. **[2]**

(4) (a) Let R and S be binary relations defined as follows.

R is defined on $A = \{a, b, c, d, e, f, g, h, i\}$ by the following arrow diagram.



S is defined on $\mathbb{Z} \times \mathbb{Z}$ by $(w, x)S(y, z)$ if and only if $w + x - y - z$ is even.

(i) Is R reflexive? Is R symmetric? Is R antisymmetric? Is R transitive? **[4]**

(ii) Is R a partial order relation? Is R an equivalence relation? **[1]**

(iii) Is S reflexive? Is S symmetric? Is S antisymmetric? Is S transitive? **[4]**

(iv) Is S a partial order relation? Is S an equivalence relation? **[1]**

(v) If R is an equivalence relation, then give its equivalence classes. If S is an equivalence relation, then give its equivalence classes. **[3]**

(b) (i) Is $(999)^6 + 7 \times 81$ divisible by 9? **[2]**

(ii) Prove that if x and y are integers such that $x \equiv 3 \pmod{12}$ and $y \equiv 7 \pmod{18}$, then $x + y \equiv 4 \pmod{6}$. **[5]**

(5) (a) Evaluate $\binom{8}{5}$. **[1]**

(b) A standard die is rolled 100 times. What is the probability that a 6 is rolled exactly 30 times? **[3]**

(c) The number of calls received by a call center forms a Poisson distribution. An average of 3 call per minute are received.
(Remember that if X is a Poisson random variable with $E(X) = \lambda$, then $\Pr(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}$.)

(i) Find the probability that no calls come in a certain 1 minute period. (Leave your answer as a mathematical expression.) **[2]**

(ii) Find the probability that at least two calls will arrive in a certain two minute period. (Leave your answer as a mathematical expression.) **[4]**

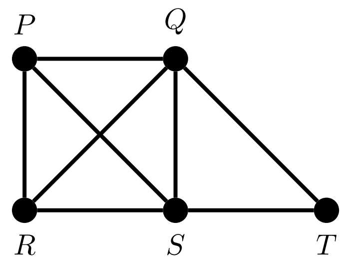
(d) Let X_1 and X_2 be independent random variables that are each selected uniformly at random from the set $\{1, 2, 3\}$. Let $Y = \max(X_1, X_2)$.

(i) Find $\Pr(X_1 = 2 \wedge Y = 2)$. **[2]**

(ii) Find $E(Y)$. **[4]**

(iii) What would $E(Y)$ be if X_1 and X_2 were instead selected uniformly at random from the set $\{1, 2, \dots, 100\}$? (Leave your answer as a mathematical expression in $\sum$ notation.) **[4]**

(6) (a) Consider the following graph.



(i) What are the degrees of the vertices in the graph? **[1]**

(ii) Does the graph have a closed Euler trail? If so, give an example of a closed Euler trail in the graph. If not, explain why no closed Euler trail exists. **[2]**

(iii) How many spanning trees of the graph contain the edges QS and RS? **[3]**

(b) Let n and r be integers such that $n \geq r \geq 1$. Prove that $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$. **[4]**

(c) For $n \geq 1$, let s_n be the number of ways of writing n as a sum of 1s, 3s and 4s (order being important). For example, $s_4 = 4$ because 4 can be written in four ways:

$$1+1+1+1, \quad 1+3, \quad 3+1, \quad 4.$$

(i) Find s_1, s_2 and s_3. **[2]**

(ii) Find a recurrence for s_n which holds for all $n \geq 5$. Explain why your recurrence gives s_n. **[6]**

(iii) Use your answer to (ii) to find s_5 and s_6. **[2]**

(7) (a) (i) Draw a graph whose adjacency matrix is

$$M = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

using V_1, V_2 V_3 and V_4 as the names for the vertices corresponding to columns 1,2,3 and 4 respectively. **[2]**

(ii) Explain how you could use M to find the number of walks of length 6 from V_1 to V_3 in the graph you drew in part (i). (You don't have to actually find the number.) **[2]**

(b) (i) Draw a simple graph with 7 vertices and 5 edges that has a spanning tree, or explain why no such graph exists. **[3]**

(ii) Does every simple graph with no odd degree vertices have a closed Euler trail? If so, prove it. If not, give a counterexample. **[3]**

(iii) Is there a simple graph with 9 vertices with degrees 4, 4, 3, 2, 2, 1, 1, 1, 1? If so, draw one. If not, explain why not. **[2]**

(iv) Is there a tree with 8 vertices with degrees 4, 3, 2, 1, 1, 1, 1, 1? If so, draw one. If not, explain why not. **[3]**

(c) How many ternary strings (that is, strings made up of 0s, 1s and 2s) of length 5 contain at most two 0s, at most two 1s and at most two 2s? **[5]**