MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #3 Solutions

1. $\neg\forall x P(x) \equiv \exists x \neg P(x)$
   $\neg\exists x \exists y \neg Q(x, y) \equiv \forall x \forall y \neg\neg Q(x, y) \equiv \forall x \forall y Q(x, y)$
   $\neg(\exists x P(x) \vee \exists x \forall y Q(x, y)) \equiv \neg\exists x P(x) \wedge \neg\exists x \forall y Q(x, y) \equiv \forall x \neg P(x) \wedge \forall x \exists y \neg Q(x, y)$

2. Let $P(n)$ be the statement "$n^2 + 3n$ is even".

   Base step. When $n = 1$, $n^2 + 3n = 4$. Obviously 4 is even. So $P(1)$ is true.

   Induction step. For some integer $k \geq 1$, assume that $P(k)$ is true. That is, assume that $k^2 + 3k$ is even. Now we need to prove that $P(k+1)$ is true. So we must show that $(k+1)^2 + 3(k+1)$ is even. We have that

$$\begin{aligned}(k+1)^2 + 3(k+1) &= (k^2 + 2k + 1) + 3k + 3 \\ &= k^2 + 5k + 4 \\ &= (k^2 + 3k) + 2k + 4.\end{aligned}$$

   Now $(k^2 + 3k) + 2k + 4$ is even because $k^2 + 3k$ is even by our assumption and $2k$ and 4 are obviously even. Therefore $(k+1)^2 + 3(k+1)$ is even and $P(k+1)$ is true.

   So we have proved by induction that $n^2 + 3n$ is even for each integer $n \geq 1$.

3. (a) True. (There is a cupcake with pink icing but without green sprinkles.)
   False. (It is not true that all the cupcakes with green sprinkles have pink icing.)
   True. ($\exists x I(x)$ is true because there is a cupcake with pink icing. So the statement is true.)

   (b) Yes, for example a tray with the following three cupcakes: pink icing with brown sprinkles, pink icing with green sprinkles, yellow icing with green sprinkles.

   (c) No, if $\forall x I(x) \vee \forall x S(x)$ is true then every cupcake on the tray has pink icing or every cupcake on the tray has green sprinkles, and this means that $\forall x(I(x) \vee S(x))$ would have to be true also.

4. Let $P(n)$ be the statement "$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$".

   Base step. The left hand side of $P(1)$ is 1 and the right hand side of $P(1)$ is $\frac{1(2)(3)}{6} = \frac{6}{6} = 1$. So $P(1)$ is true.

   Induction step. For some integer $k \geq 1$, assume that $P(k)$ is true. That is, assume that

$$1 + 2^2 + 3^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

   Now we need to prove that $P(k+1)$ is true. So we must show that

$$1 + 2^2 + 3^2 + \cdots + (k+1)^2 = \frac{(k+1)(k+2)(2(k+1)+1)}{6}.$$

   Working with the left hand side of this equation we see that

$$\begin{aligned}
1 + 2^2 + 3^2 + \cdots + (k+1)^2 &= (1 + 2^2 + 3^2 + \cdots + k^2) + (k+1)^2 \\
&= \left(\frac{k(k+1)(2k+1)}{6}\right) + (k+1)^2 \quad \text{(using our assumption)} \\
&= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\
&= \frac{k+1}{6}(k(2k+1) + 6(k+1)) \\
&= \frac{k+1}{6}(2k^2 + 7k + 6) \\
&= \frac{k+1}{6}(2k+3)(k+2) \\
&= \frac{(k+1)(k+2)(2k+3)}{6}
\end{aligned}$$

   which is the right hand side we required. Thus $P(k+1)$ is true.

   So we have proved by induction that $1 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$ for each integer $n \geq 1$.

5. Let c range over all cushions, s range over all sofas, and let $M(c, s)$ mean that cushion c matches sofa s.

   (a) Claim: $\exists c \forall s M(c, s)$
   Negation: $\neg \exists c \forall s M(c, s) \equiv \forall c \exists s \neg M(c, s)$
   For every cushion, you'd have to find a sofa which didn't match that cushion.

   (b) Claim: $\forall s \exists c M(c, s)$
   Negation: $\neg \forall s \exists c M(c, s) \equiv \exists s \forall c \neg M(c, s)$
   You'd have to show there was one specific sofa which didn't match any cushion.