MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #7 Solutions

Unless you're told otherwise, it's always OK to leave the answers to "counting" questions as mathematical expressions rather than evaluating them as (sometimes huge) numbers. I give the numbers below as well as the expressions just to give you an idea of the sizes involved.

1. (a) $4! = 4 \times 3 \times 2 \times 1 = 24$
   (b) $\frac{10!}{8!} = 10 \times 9 = 90$
   (c) $\binom{10}{8} = \frac{10!}{8!2!} = \frac{10 \times 9}{2} = 45$
   (d) $\binom{7}{3} = \frac{7!}{3!4!} = \frac{7 \times 6 \times 5}{3!} = 35$

2. (a) This is the number of ordered selections without repetition of 3 elements chosen from a set of 10 elements. So it is $\frac{10!}{(10-3)!} = \frac{10!}{7!} = 10 \times 9 \times 8 = 720$.
   (b) This is the number of unordered selections without repetition of 3 elements chosen from a set of 10 elements. So it is $\binom{10}{3} = \frac{10!}{7! \times 3!} = \frac{10 \times 9 \times 8}{3 \times 2} = 120$.
   (c) The selection in (a) is ordered (because president, treasurer and secretary are different roles), while the selection in (b) is unordered.

   Noticing this means you know that the answer to (a) will be bigger than the answer to (b) without calculation: there are always more ways to take an ordered selection (of at least two things) than an unordered selection. In this case for every 3 person team there are $3! = 6$ ways to appoint them as president, treasurer and secretary and so the answer for (a) is 6 times the answer for (b).
   (d) This is the number of unordered selections with repetition of 3 items from a set of 10 items. So it is $\binom{10+3-1}{3} = \binom{12}{3} = \frac{12!}{3! \times 9!} = \frac{12 \times 11 \times 10}{3 \times 2} = 220$.
   (e) Each possible way to divide the prizes corresponds to a sequence of length 5 with each term in the set $\{\mathrm{A}, \mathrm{B}, \mathrm{C}\}$ with 3 elements (for example giving the first three prizes to Anastasia and the last two to Cadel corresponds to AAACC). So there are 3^5 possible ways.
   (f) This is the number of ordered selections without repetition of 6 elements chosen from a set of 6 elements (or permutations of length 6). So it is $6! = 6 \times 5 \times 4 \times 3 \times 2 = 720$.

3. If it fires 80 (or fewer) missiles, then each A-wing could have 5 (or fewer) missiles locked on to it. But if it fires 81, then the pigeonhole principle guarantees that at least one A-wing will have at least $\lceil \frac{81}{16} \rceil = 6$ missiles locked on.

4. (a) By the binomial theorem, the terms of this expansion will be $\binom{20}{i}x^i 2^{20-i}$ for $i = 0, 1, \ldots, 20$. So the relevant term will be for $i = 9$. This term is $\binom{20}{9}x^9 2^{11} = \binom{20}{9}2^{11}x^9$. So the coefficient is $\binom{20}{9}2^{11}$.

   (b) By the binomial theorem, the terms of this expansion will be $\binom{20}{i}(3x)^i 2^{20-i}$ for $i = 0, 1, \ldots, 20$. Because $(3x)^9 = 3^9 x^9$, the relevant term will be for $i = 9$. This term is $\binom{20}{9}(3x)^9 2^{11} = \binom{20}{9}2^{11}3^9 x^9$. So the coefficient is $\binom{20}{9}2^{11}3^9$.

   (c) By the binomial theorem, the terms of this expansion will be $\binom{20}{i}x^i(2)^{20-i}$ for $i = 0, 1, \ldots, 20$. Because $(3x^3)^i = 3^i x^{3i}$, the relevant term will be for $3i = 9$, so for $i = 3$. This term is $\binom{20}{3}(3x^3)^3 2^{17} = \binom{20}{3}2^{17}3^3 x^9$. So the coefficient is $\binom{20}{3}2^{17}3^3$.

5. (a) $\Pr(A) = \frac{1}{6}$ because one of the six sides is marked 0.
   $\Pr(B) = \frac{3}{6} = \frac{1}{2}$ because three of the six sides are marked 3.
   $\Pr(C) = \frac{1}{2} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{2} = \frac{1}{3}$ because the only ways the sum of the rolls can be 5 is if the first is 2 and the second is 3 or if the first is 3 and the second is 2. The probability that the first is 2 and the second is 3 is $\frac{1}{2} \times \frac{1}{3}$ because the two rolls are independent. Similarly, the probability that the first is 3 and the second is 2 is $\frac{1}{3} \times \frac{1}{2}$.

   (b) $\Pr(A \cap B) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$ because the first and second rolls are independent.
   $\Pr(A \cap C) = 0$ because if the first roll is 0 then the sum of the rolls cannot be 5.
   $\Pr(B \cap C) = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6}$ because $B \cap C$ can only occur if the first roll is 2 and the second is 3.

   (c) A and C are not independent because $\Pr(A \cap C) \neq \Pr(A)\Pr(C)$.
   From above, $\Pr(A \cap C) = 0$ and $\Pr(A)\Pr(C) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$.
   B and C are independent because $\Pr(B \cap C) = \Pr(B)\Pr(C)$.
   From above, $\Pr(B \cap C) = \frac{1}{6}$ and $\Pr(B)\Pr(C) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

   (d) $\Pr(A \cup C) = \Pr(A) + \Pr(C) - \Pr(A \cap C) = \frac{1}{6} + \frac{1}{3} - 0 = \frac{1}{2}$ using our answers above.
   $\Pr(B \cup C) = \Pr(B) + \Pr(C) - \Pr(B \cap C) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$ using our answers above.