MAT1830

Lecture 28: Recursion, lists and sequences

A *list* or *sequence* of objects from a set X is a function f from $\{1, 2, \ldots, n\}$ to X, or (if infinite) from $\{1, 2, 3, \ldots\}$ to X.

We usually write $f(k)$ as x_k and the list as $\langle x_1, x_2, \ldots, x_n \rangle$, or $\langle x_1, x_2, x_3, \ldots \rangle$. Thus

$$\begin{aligned} f(1) &= x_1 = \text{ first item on list} \\ f(2) &= x_2 = \text{ second item on list} \\ &\vdots \end{aligned}$$

The empty list is written $\langle \rangle$.

Example.

$\langle \mathrm{m, a, t, h, s} \rangle$ is a function f from $\{1, 2, 3, 4, 5\}$ into the English alphabet, with $f(1) = m$, $f(2) = a$, etc.

28.1 Sequences

A sequence is also a list, but when we use the term sequence we are usually interested in the rule by which the successive terms $t_1, t_2, \ldots$ are defined.

Often, the rule is a recurrence relation.

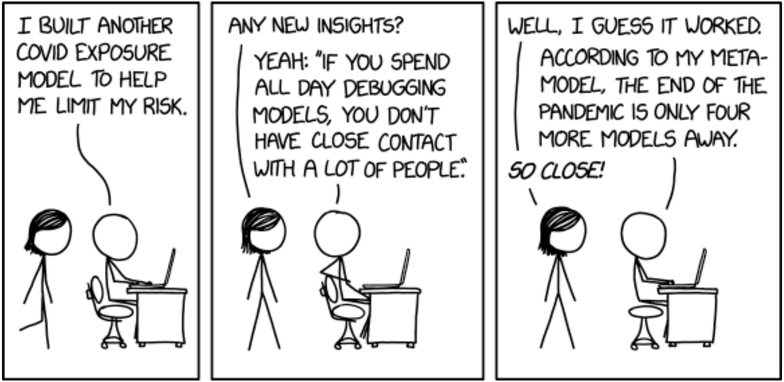
Example. Arithmetic sequence

$$a, a+d, a+2d, a+3d, \ldots$$

This is defined by
Initial value. $t_1 = a$
Recurrence relation. $t_{k+1} = t_k + d$

"Unfolding" this recurrence relation from t_n back to t_1, we see that d gets added $n-1$ times, hence

$$t_n = a + (n-1)d.$$


I BUILT ANOTHER COVID EXPOSURE MODEL TO HELP ME LIMIT MY RISK.
ANY NEW INSIGHTS?
YEAH: "IF YOU SPEND ALL DAY DEBUGGING MODELS, YOU DON'T HAVE CLOSE CONTACT WITH A LOT OF PEOPLE."
WELL, I GUESS IT WORKED.
ACCORDING TO MY META-MODEL, THE END OF THE PANDEMIC IS ONLY FOUR MORE MODELS AWAY.
SO CLOSE!

Example. Geometric sequence

$$a, ar, ar^2, ar^3, \ldots$$

Initial value. $t_1 = a$
Recurrence relation. $t_{k+1} = rt_k$

"Unfolding" t_n, we see that multiplication by r is done $n-1$ times, hence

$$t_n = ar^{n-1}.$$

The above recurrence relations are called *first order* because t_{k+1} depends on only the previous value, t_k. (Or, because the values of all terms follow from one initial value.)

A *second order* recurrence relation requires *two* initial values, and is usually harder to unfold.

Definition The order of a recurrence relation is given by the difference between the highest and lowest subscripts involved in it.

Examples

$t_n = t_{n-1} + t_{n-2}$ is second order

$t_n = 5t_{n-1} + 4t_{n-2} + 7t_{n-3}$ is third order

$t_n = 5t_{n-1} + 2t_{n-6}$ is sixth order

$t_{n+1} = t_n + t_{n-1}$ is second order

The order also gives the number of initial values the need to be given to completely specify the sequence.

Example. A simple sequence in disguise

Initial values. $t_0 = 1, t_1 = 2$
Recurrence relation. $t_{k+1} = 2t_k - t_{k-1}$

Calculating the first values, we find

$$\begin{aligned} t_2 &= 2t_1 - t_0 = 2 \times 2 - 1 = 3, \\ t_3 &= 2t_2 - t_1 = 2 \times 3 - 2 = 4, \\ t_4 &= 2t_3 - t_2 = 2 \times 4 - 3 = 5. \end{aligned}$$

It looks like $t_n = n + 1$, and indeed we can prove this by induction. For the base step we have the initial values $t_0 = 1 = 0 + 1$ and $t_1 = 2 = 1 + 1$. We do the induction step by strong induction: assuming $t_n = n + 1$ for all $n \leqslant k$, we deduce that $t_{k+1} = k + 2$.

In fact we have

$$\begin{aligned} t_{k+1} &= 2t_k - t_{k-1} \\ & \text{by the recurrence relation} \\ &= 2(k+1) - k \\ & \text{by our assumption} \\ &= 2k + 2 - k = k + 2 \\ & \text{as required.} \end{aligned}$$

This completes the induction.

Example Let $t_0, t_1, t_2, \ldots$ be the sequence defined by $t_0 = 1,\ t_1 = 2$ and $t_{k+1} = 2t_k - t_{k-1}$.

Solution Let $P(n)$ be the statement "$t_n = n + 1$".

Base steps.
$t_0 = 1$ and $0 + 1 = 1$, so $P(0)$ is true.
$t_1 = 2$ and $1 + 1 = 2$, so $P(1)$ is true.

Induction step. Suppose that $P(0), \ldots, P(k)$ are true for some integer $k \geq 1$.

We want to prove that $P(k+1)$ is true. We want to show that $t_{k+1} = k + 2$.

$$
\begin{aligned}
t_{k+1} &= 2t_k - t_{k-1} \\
&= 2(k+1) - k && \text{(by } P(k) \text{ and } P(k-1)) \\
&= k + 1
\end{aligned}
$$

So $P(k+1)$ is true.

This proves that $P(n)$ is true for each integer $n \geq 0$.

Flux Exercise (LQMTZZ)

Define sequences $\langle a_1, a_2, a_3, \ldots \rangle$ and $\langle b_1, b_2, b_3, \ldots \rangle$ with initial terms 10, 219, 4796, 105030,
where $a_n = 22a_{n-1} - 3a_{n-2} + 18a_{n-3} - 11a_{n-4}$,
and b_n is $\dfrac{b_{n-1}^2}{b_{n-2}}$ rounded to the nearest integer.
These two sequences agree for their first 10 terms.

A. This is enough to show the two sequences are equal.

B. We should compute more terms. If the sequences agree for 100 terms, they must be the same sequence.

C. We should compute more terms. If the sequences agree for 1000 terms, they must be the same sequence.

D. The sequences might not be the same even if their first 1000 terms agree.

Answer D. In fact the sequences' first disagreement is that $a_{1403} = b_{1403} + 1$ (both terms have 1881 digits)

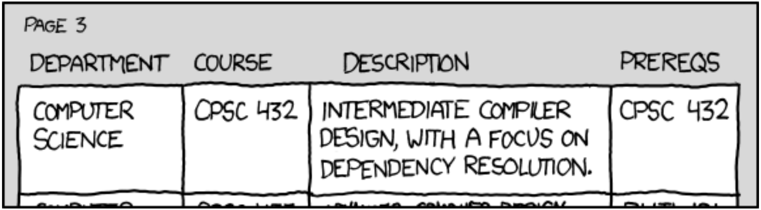
Example. Fibonacci sequence

Initial values. $t_0 = 0, t_1 = 1$
Recurrence relation. $t_{k+1} = t_k + t_{k-1}$

It is possible to write t_n directly as a function of n. The function is not at all obvious, because it involves $\sqrt{5}$:

$$t_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

We do not go into how to find such a formula in this unit. However, if someone gave you such a formula you could prove it is correct by induction.


PAGE 3
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28.2 Relations – homogeneous and inhomogeneous

Recurrence relations such as

$$t_{k+1} = 2t_k$$

and

$$t_{k+1} = t_k + t_{k-1}$$

in which each term is a multiple of some t_j, are called *homogeneous*.

The characteristic property of any homogeneous equation is that if $t_n = f(n)$ is a solution, then so is $t_n = cf(n)$, for any constant c.

E.g. $t_n = 2^n$ is a solution of $t_{k+1} = 2t_k$, and so is $t_n = c2^n$, for any constant c.

Relations like $t_{k+1} = t_k + 3$, in which there is a term other than the t_j terms, are called *inhomogeneous*.

Homogeneous recurrence relations are usually easier to solve, and in fact there is a general method for solving them (which we will not cover in this unit).

There is no general method for solving inhomogeneous recurrence relations, though they can often be solved if the term other than the t_j terms is simple.

Example $t_n = t_{n-1} + 7t_{n-2} + 4t_{n-4}$ is a homogenous recurrence.

Suppose we found a function $f : \mathbb{N} \to \mathbb{R}$ such that

$$f(n) = f(n-1) + 7f(n-2) + 4f(n-4).$$

In other words, f satisfies the recurrence.

Now, for any $c \in \mathbb{R}$, we can define a new function $g : \mathbb{N} \to \mathbb{R}$ by $g(n) = cf(n)$. We will have

$$\begin{aligned} g(n-1) + 7g(n-2) + 4g(n-4) &= cf(n-1) + 7cf(n-2) + 4cf(n-4) \\ &= c(f(n-1) + 7f(n-2) + 4f(n-4)) \\ &= cf(n) \\ &= g(n) \end{aligned}$$

So g also satisfies the recurrence.

The same thing works for any **homogeneous** recurrence.

This can be really useful. For example, we can sometimes try to find a function f that satisfies some recurrence and then choose c to fit some initial conditions.

Question

28.1 Find the next four values of each of the following recurrence relations. What order is each recurrence relation? Which are homogeneous and which are inhomogeneous?

(a) $r_{k+1} = r_k + k^2, \quad r_0 = 0.$

(b) $s_{k+1} = 3s_k - 2s_{k-1}, \quad s_0 = 1,\ s_1 = 2.$

Answer

Order: (a) is first order, (b) is second order

Homogeneous? (a) is not, (b) is

Next terms of (a): $r_1 = 0$, $r_2 = 1$, $r_3 = 5$, $r_4 = 14$.

Next terms of (b): $s_2 = 4$, $s_3 = 8$, $s_4 = 16$, $s_5 = 32$.

Flux Exercise (LQMTZZ)

Is the recurrence relation $t_{k+1} = t_k + t_{k-2} + 1$,

A. homogeneous and second order?

B. inhomogeneous and second order?

C. homogeneous and third order?

D. inhomogeneous and third order?

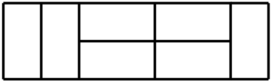
(Once you've answered the poll, find the next four values of the sequence if the initial values are $t_0 = 1,\ t_1 = 1,\ t_2 = 1$).

Answer D.

Next terms are $t_3 = 3,\ t_4 = 5,\ t_5 = 7,\ t_6 = 11.$

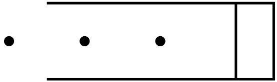
Question

28.2 Let T_n be the number of ways of tiling a $2 \times n$ strip with 2×1 tiles (which may be rotated so they are 1×2). Find T_n for $n = 1, 2, 3, 4$. Find a recurrence relation for T_n.

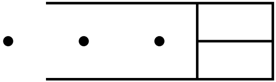
E.g.  is one option counted by T_7.

$T_1 = 1,\ T_2 = 2,\ T_3 = 3,\ T_4 = 5.$

How do we find a recurrence relation for T_n?
Consider what happens at the right hand end:



or



In the first case removing the right most tile leaves us with a tiling of a $2 \times (n-1)$ strip.
In the second case removing the two right most tiles leaves us with a tiling of a $2 \times (n-2)$ strip.
Since these possibilities do not overlap, and between them include every option, we see that $T_n = T_{n-1} + T_{n-2}$.