MAT1830

Lecture 18: Order Relations

18.1 Partial order relations

A *partial order relation* R on a set A is a binary relation with the following three properties.

1. Reflexivity.

$$aRa$$
$$\text{for all } a \in A.$$

2. Antisymmetry.

$$aRb \text{ and } bRa \Rightarrow a = b$$
$$\text{for all } a, b \in A.$$

3. Transitivity.

$$aRb \text{ and } bRc \Rightarrow aRc$$
$$\text{for all } a, b, c \in A.$$

For a binary relation R on a set A.

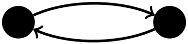
Antisymmetry: For all $x, y \in A$, if xRy and yRx then $x = y$.

This definition is useful for proofs but I think the contrapositive is more intuitive.

Antisymmetry (equivalent defn): For all $x, y \in A$, if $x \neq y$ then it is not the case that xRy and yRx.

Antisymmetry (For a binary relation R on a set A.)

I never see:



To prove R is antisymmetric, show that...

For all $x, y \in A$, if xRy and yRx then $x = y$.

To prove R is not antisymmetric, show that...

There are some $x, y \in A$ such that $x \neq y$, xRy and yRx.

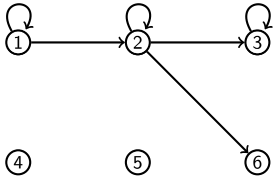
Warning Antisymmetric does not mean "not symmetric"!

An example which is neither symmetric nor antisymmetric:

$$\begin{tikzcd} x \arrow[r, bend left] & y \arrow[l, bend left] \arrow[r] & z \end{tikzcd}$$

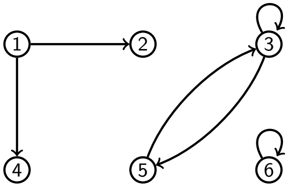
Technically, "$=$" is both symmetric and antisymmetric.

Question Let R be the relation on A pictured below. Is R antisymmetric?



Yes. For all $x, y \in A$, if xRy and yRx then $x = y$.

Question Let S be the relation on A pictured below. Is S antisymmetric?



No. $3S5$ and $5S3$ (and $3 \neq 5$).

Examples.

1. $\leqslant$ on $\mathbb{R}$.
   Reflexive: $a \leqslant a$ for all $a \in \mathbb{R}$.
   Antisymmetric: $a \leqslant b$ and $b \leqslant a \Rightarrow a = b$ for all $a, b \in \mathbb{R}$.
   Transitive: $a \leqslant b$ and $b \leqslant c \Rightarrow a \leqslant c$ for all $a, b, c \in \mathbb{R}$.

2. $\subseteq$ on $\mathcal{P}(\mathbb{N})$.
   Reflexive: $A \subseteq A$ for all $A \in \mathcal{P}(\mathbb{N})$.
   Antisymmetric: $A \subseteq B$ and $B \subseteq A \Rightarrow A = B$ for all $A, B \in \mathcal{P}(\mathbb{N})$.
   Transitive: $A \subseteq B$ and $B \subseteq C \Rightarrow A \subseteq C$ for all $A, B, C \in \mathcal{P}(\mathbb{N})$.

3. Divisibility on $\mathbb{N}$.
   The relation "a divides b" on natural numbers is reflexive, antisymmetric and transitive. We leave checking this as an exercise.

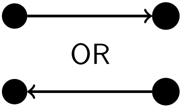
4. Alphabetical order of words.
   Words on the English alphabet are alphabetically ordered by comparing the leftmost letter at which they differ. We leave checking that this relation is reflexive, antisymmetric and transitive as an exercise.

Definition A binary relation R on a set A is a *total order relation* if

- it is a partial order relation; and
- for any $x, y \in A$ we have xRy or yRx.

Everywhere I see:

I actually see:



Example $\leq$ on $\mathbb{R}$ is a total order relation (because for any $x, y \in \mathbb{R}$ we have that $x \leq y$ or $y \leq x$).

Example $\subseteq$ on $\mathcal{P}(\{1, 2, 3\})$ is not a total order relation (for example, $\{1\} \not\subseteq \{2, 3\}$ and $\{2, 3\} \not\subseteq \{1\}$).

18.2 Total order relations

A total order relation is a special kind of partial order relation that “puts everything in order”.

A *total order relation* R on a set A is a partial order relation that also has the property aRb or bRa for all $a, b \in A$.

Examples.

1. $\leqslant$ on $\mathbb{R}$
   This is a total order relation because for all real numbers a and b we have $a \leqslant b$ or $b \leqslant a$.

2. $\subseteq$ on $\mathcal{P}(\mathbb{N})$.
   This is not a total order because, for example, $\{1, 2\} \nsubseteq \{1, 3\}$ and $\{1, 3\} \nsubseteq \{1, 2\}$.

3. Divisibility on $\mathbb{N}$.
   This is not a total order because, for example, 2 does not divide 3 and 3 does not divide 2.

4. Alphabetical order of words.
   This is a total order because given any two different words, one will appear before the other in alphabetical order.

Flux Exercise (LQMTZZ)

Let R be the partial order relation on $\mathbb{N} \times \mathbb{N}$ defined by $(m_1, n_1)R(m_2, n_2)$ if and only if $m_1 \leq m_2$ and $n_1 \leq n_2$.
Is R a total order?

A. No because $(1, 4) \not R (2, 2)$ and $(2, 2) \not R (1, 4)$.
B. No because $(2, 3) \not R (2, 2)$ and $(2, 2) R (2, 3)$.
C. No because $(1, 4) \not R (2, 8)$ and $(2, 8) \not R (1, 4)$.
D. Yes because $(a_1, b_1)R(a_2, b_2)$ or $(a_2, b_2)R(a_1, b_1)$ for any $(a_1, b_1), (a_2, b_2) \in \mathbb{N} \times \mathbb{N}$.

Examples

$(2, 3)R(5, 4)$ because $2 \leq 5$ and $3 \leq 4$
$(2, 3) \not R (5, 2)$ because $2 \leq 5$ and $3 > 2$

Answer

B is silly: to show something is not a total order we must find two things that are not related in either direction.

C is wrong because $(1, 4)R(2, 8)$.

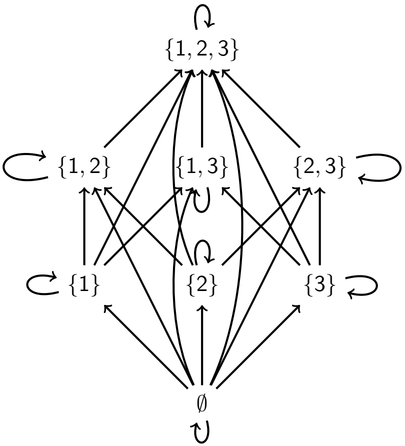
A gives a correct example of why R is not a total order. So A.

(The example in A, of course, shows that D is wrong.)

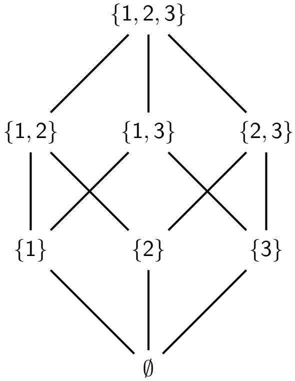
Hasse Diagrams

Example The relation $\subseteq$ on $\mathcal{P}(\{1,2,3\})$ is a partial order relation.

Arrow diagram



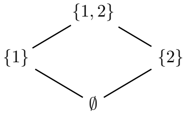
Hasse diagram



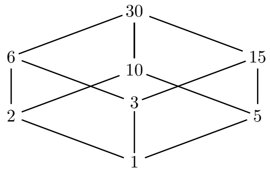
18.3 Hasse diagrams

A partial order relation R on a finite set A can be represented as a Hasse diagram. The elements of A are written on the page and connected by lines so that, for any $a, b \in A$, aRb exactly when b can be reached from a by travelling upward along the lines.

Example. A Hasse diagram for the relation $\subseteq$ on the set $\mathcal{P}(\{1, 2\})$ can be drawn as follows.



Example. A Hasse diagram for the relation "divides" on the set $\{1, 2, 3, 5, 6, 10, 15, 30\}$ can be drawn as follows.

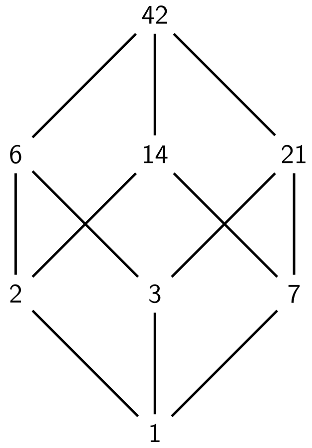


Example. A Hasse diagram for the relation $\leqslant$ on the set $\{1, 2, 3, 4, 5\}$ can be drawn as follows.

Notice how this last Hasse diagram can be simply drawn as a vertical chain, when the previous two are "wider" and more complicated. This corresponds to the fact that the last example was of a total order relation but the previous two were not of total order relations.

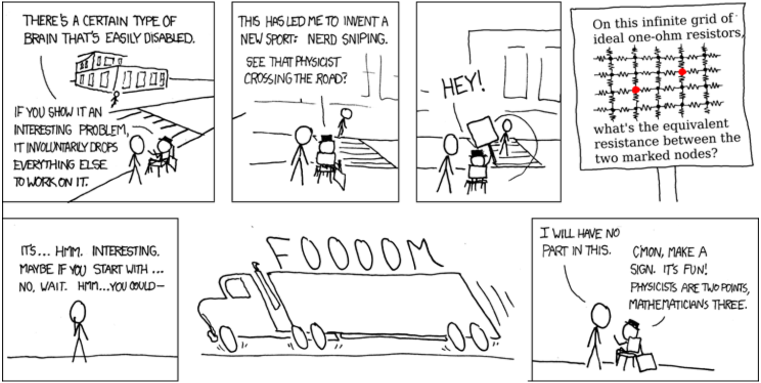
Question 18.2 Draw a Hasse diagram for the set of divisors of 42.

The set of divisors is $\{1, 2, 3, 6, 7, 14, 21, 42\}$.



Question 18.2 (cont) Why does the Hasse diagram for the set of divisors of 42 look like the diagram for the set of the divisors of 30?

Because $42 = 2 \times 3 \times 7$ and $30 = 2 \times 3 \times 5$. They both look like the Hasse diagram for the subsets of $\{a, b, c\}$ ($a = 2$, $b = 3$, $c = 7$ for 42, and $a = 2$, $b = 3$, $c = 5$ for 30).



THERE'S A CERTAIN TYPE OF BRAIN THAT'S EASILY DISABLED.
IF YOU SHOW IT AN INTERESTING PROBLEM, IT INVOLUNTARILY DROPS EVERYTHING ELSE TO WORK ON IT.
THIS HAS LED ME TO INVENT A NEW SPORT: NERD SNIPING.
SEE THAT PHYSICIST CROSSING THE ROAD?
HEY!
On this infinite grid of ideal one-ohm resistors, what's the equivalent resistance between the two marked nodes?
IT'S... HMM. INTERESTING. MAYBE IF YOU START WITH ... NO, WAIT. HMM...YOU COULD—
FOOOOM
I WILL HAVE NO PART IN THIS.
C'MON, MAKE A SIGN. IT'S FUN! PHYSICISTS ARE TWO POINTS, MATHEMATICIANS THREE.

Definition A binary relation R on a set A is a *well-order relation* if

- it is a total order relation; and
- every non-empty $S \subseteq A$ has a least element.

We could write this second condition formally as

- for every non-empty $S \subseteq A$ there is an $\ell \in S$ such that $\ell R y$ for all $y \in S$.

Example $\leq$ on $\mathbb{N}$ is a well-order relation (because every non-empty set of natural numbers has a least element).

Example $\leq$ on $\mathbb{R}$ is not a well-order relation (for example, the set $\{x \in \mathbb{R} : x > 2\}$ has no least element).

18.4 Well-ordering

A well-order relation on a set is a total order relation that also has the property that each nonempty set of its elements contains a least element.

> A *well-order relation* R on a set A is a total order relation such that, for all nonempty $S \subseteq A$, there exists an $\ell \in S$ such that $\ell R a$ for all $a \in S$.

Example. The relation $\leqslant$ on $\mathbb{N}$ is a well-order relation because every nonempty subset of $\mathbb{N}$ has a least element.

The well-ordering of $\mathbb{N}$ is the basis of proofs by induction.

Example. The relation $\leqslant$ on $\mathbb{Z}$ is not a well-order relation. For example, $\mathbb{Z}$ itself has no least element.

Example. The relation $\leqslant$ on $\{x : x \in \mathbb{R}, x \geqslant 0\}$ is not a well-order relation. For example, the subset $\{x : x \in \mathbb{R}, x > 3\}$ has no least element.

Flux Exercise (LQMTZZ)

Let R be the partial order relation on $\mathbb{N} \times \mathbb{N}$ defined by $(m_1, n_1)R(m_2, n_2)$ if and only if either

- $m_1 < m_2$; or
- $m_1 = m_2$ and $n_1 \leq n_2$.

Is R a total order? Is R a well order?

Hint Roughly the definition of R says "order by the first coordinate and break ties using the second coordinate."
E.g. $(3, 4)R(4, 1)$ because $3 < 4$, and $(3, 4)R(3, 7)$ because $3 = 3$ and $4 \leq 7$.

Answer
It's a total order. To prove this, let $(a_1, b_1), (a_2, b_2) \in \mathbb{N} \times \mathbb{N}$.
We must show $(a_1, b_1)R(a_2, b_2)$ or $(a_2, b_2)R(a_1, b_1)$.

- If $a_1 < a_2$, then $(a_1, b_1)R(a_2, b_2)$.
- If $a_2 < a_1$, then $(a_2, b_2)R(a_1, b_1)$.
- If $a_1 = a_2$ and $b_1 \leq b_2$, then $(a_1, b_1)R(a_2, b_2)$.
- If $a_1 = a_2$ and $b_2 < b_1$, then $(a_2, b_2)R(a_1, b_1)$.

It's also a well order. To prove this, let S be a nonempty subset of $\mathbb{N} \times \mathbb{N}$.
We must show S has a least element.
Let a_0 be the least first coordinate amongst all the pairs in S.
Let b_0 be the least second coordinate amongst all the pairs in S with first coordinate a_0. Then (a_0, b_0) is the least element in S.