MAT1830

Lecture 3: Congruences

We're used to classifying the integers as either even or odd. The even integers are those that can be written as $2k$ for some integer k. The odd integers are those that can be written as $2k + 1$ for some integer k.

even	$\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots$
odd	$\ldots, -5, -3, -1, 1, 3, 5, \ldots$

This classification is useful because even and odd integers have particular properties. For example, the sum of any two odd integers is even.

Similarly we can split the integers into three classes: those that are $3k$ for some integer k, those that are $3k+1$ for some integer k, and those that are $3k+2$ for some integer k.

$3k$	$\ldots, -9, -6, -3, 0, 3, 6, 9, \ldots$
$3k+1$	$\ldots, -8, -5, -2, 1, 4, 7, 10, \ldots$
$3k+2$	$\ldots, -7, -4, -1, 2, 5, 8, 11, \ldots$

These classes also have particular properties. For example, the sum of an integer in the second class and an integer in the third class will always be in the first class.

We don't have to stop with 3. We could divide integers into 4 different classes according to their remainders when divided by 4, and so on.

3.1 Congruences

Let $n \geq 2$ be an integer. We say integers a and b are *congruent modulo* n and write

$$a \equiv b \pmod{n}$$

when n divides $a - b$.

Example.

$19 \equiv 13 \pmod{6}$ because 6 divides $19 - 13$

$12 \equiv 20 \pmod{4}$ because 4 divides $12 - 20$

$22 \equiv 13 \pmod{3}$ because 3 divides $22 - 13$

Let n be a positive integer and let a and b be integers.

Basically $a \equiv b \pmod{n}$ means that a and b have the same remainder when you divide them by n.

Definition We say $a \equiv b \pmod{n}$ if n divides $a - b$.

Equivalent definition We say $a \equiv b \pmod{n}$ if $a = kn + b$ for some integer k.

Note we're talking about "congruence modulo n" as a relation here, which is not quite the same as using a mod operation.

Really "$a \underset{n}{\equiv} b$" would be better notation than "$a \equiv b \pmod{n}$".

If $\mathrm{mod}(x, n)$ is the remainder of x when divided by n, then "$a \equiv b \pmod{n}$" means the same thing as "$\mathrm{mod}(a, n) = \mathrm{mod}(b, n)$".

Questions

Is $25 \equiv 9 \pmod 4$? Yes (because 4 divides $25 - 9$)

Is $9 \equiv 16 \pmod 3$? No (because 3 doesn't divide $16 - 9$)

What integers are congruent to 3 modulo 4?
$\ldots, -9, -5, -1, 3, 7, 11, \ldots$

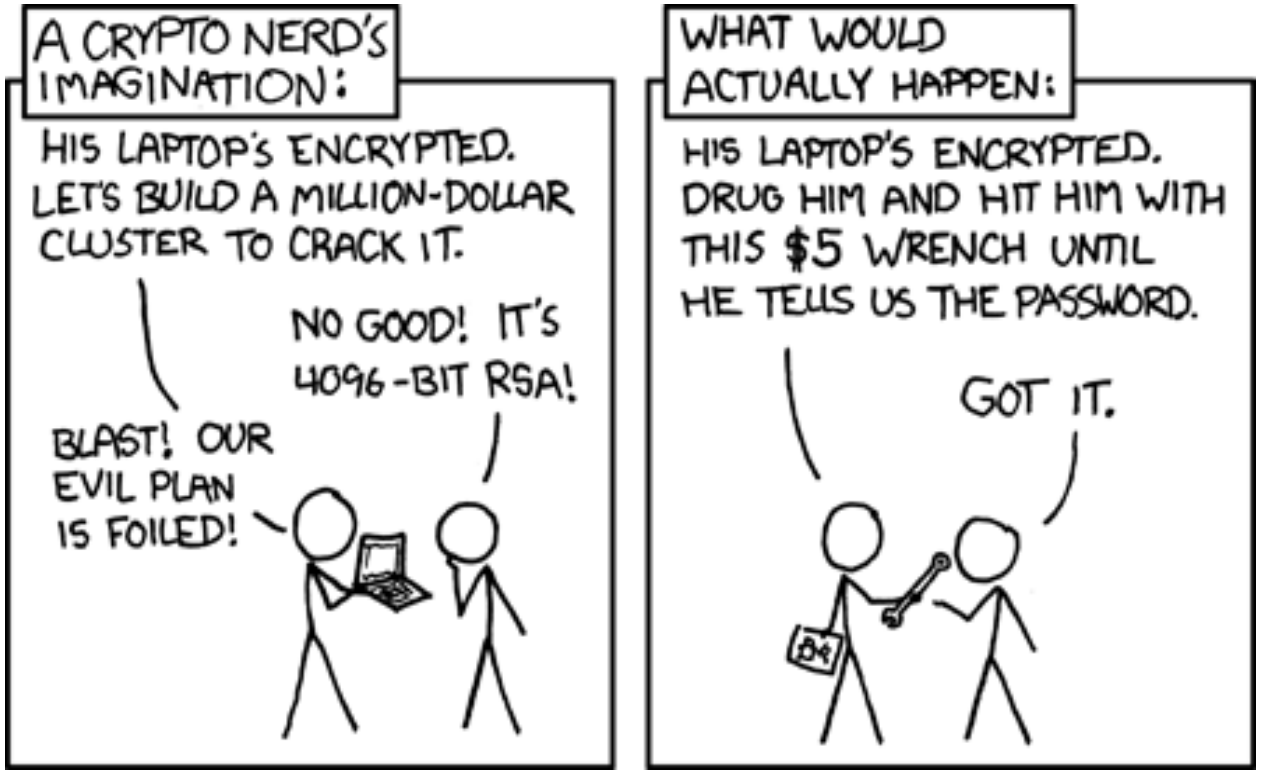
Question 3.1

Is $6 \equiv 3 \pmod 3$? Yes.

Flux Exercise (LQMTZZ)

Is $9 \equiv 18 \pmod 8$?
Is $5x + 6 \equiv 2x \pmod 3$? (where x is an integer)

Answer: No, Yes
(8 doesn't divide $18 - 9 = 9$ but 3 does divide $5x + 6 - 2x = 3x + 6$)


A CRYPTO NERD'S IMAGINATION:
HIS LAPTOP'S ENCRYPTED. LET'S BUILD A MILLION-DOLLAR CLUSTER TO CRACK IT.
NO GOOD! IT'S 4096-BIT RSA!
BLAST! OUR EVIL PLAN IS FOILED!
WHAT WOULD ACTUALLY HAPPEN:
HIS LAPTOP'S ENCRYPTED. DRUG HIM AND HIT HIM WITH THIS $5 WRENCH UNTIL HE TELLS US THE PASSWORD.
GOT IT.

3.2 Working with congruences

When working with congruences modulo some fixed integer n, we can "substitute in" just like we can with equalities.

> If $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$, then
>
> $$a \equiv c \pmod{n}.$$

Example. Suppose $x \equiv 13 \pmod{7}$. Then $x \equiv 6 \pmod{7}$ because $13 \equiv 6 \pmod{7}$.

We can add, subtract and multiply congruences just like we can with equations.

If $a_1 \equiv b_1 \pmod{n}$ and $a_2 \equiv b_2 \pmod{n}$, then

- $a_1 + a_2 \equiv b_1 + b_2 \pmod{n}$
- $a_1 - a_2 \equiv b_1 - b_2 \pmod{n}$
- $a_1 a_2 \equiv b_1 b_2 \pmod{n}$.

Example. If $x \equiv 3 \pmod{8}$ and $y \equiv 2 \pmod{8}$, then

- $x + y \equiv 5 \pmod{8}$
- $x - y \equiv 1 \pmod{8}$
- $xy \equiv 6 \pmod{8}$.

We can also deduce that $x + 4 \equiv 7 \pmod{8}$, that $4x \equiv 12 \pmod{8}$ and so on, because obviously $4 \equiv 4 \pmod{8}$. Note as well that $4x \equiv 12 \pmod{8}$ can be simplified to $4x \equiv 4 \pmod{8}$.

Examples

Suppose we know that $x \equiv 3 \pmod 4$ and $y \equiv 2 \pmod 4$.

Adding these, we see $x + y \equiv 5 \pmod 4$.
So $x + y \equiv 1 \pmod 4$ (because $5 \equiv 1 \pmod 4$).

Question

You probably knew that you could add congruences modulo 2 for a long time before you learned what congruences were. How?

You knew even+even=even, odd+odd=even, even+odd=odd, and odd+even=odd.

Examples

Suppose we know that $3x \equiv 1 \pmod 5$ and $2x \equiv 4 \pmod 5$.

Subtracting the second from the first, we see $x \equiv -3 \pmod 5$.

So $x \equiv 2 \pmod 5$ (because $-3 \equiv 2 \pmod 5$).

Questions

What does the fact we can multiply congruences modulo 2 tell us about multiplying evens and odds?

That even$\times$anything=even, anything$\times$even=even, and odd$\times$odd=odd.

If a is an integer such that $a \equiv 0 \pmod 6$, then $ab \equiv 0 \pmod 6$ for any integer b. What's another way of saying this?

If you take a multiple of 6 and multiply it by any number, then the result is also a multiple of 6.

Question 3.2 (one part)

Fact. If $a_1 \equiv b_1 \pmod{n}$ and $a_2 \equiv b_2 \pmod{n}$, then $a_1 + a_2 \equiv b_1 + b_2 \pmod{n}$.

Proof.
Because $a_1 \equiv b_1 \pmod{n}$, n divides $a_1 - b_1$
This means that $a_1 - b_1 = k_1 n$ for some integer k_1.
Because $a_2 \equiv b_2 \pmod{n}$, n divides $a_2 - b_2$
This means that $a_2 - b_2 = k_2 n$ for some integer k_2.

$$\text{So,} \quad (a_1 - b_1) + (a_2 - b_2) = k_1 n + k_2 n.$$
$$\text{So,} \quad (a_1 + a_2) - (b_1 + b_2) = (k_1 + k_2) n.$$

Because $k_1 + k_2$ is an integer, this means n divides $(a_1 + a_2) - (b_1 + b_2)$.
So $a_1 + a_2 \equiv b_1 + b_2 \pmod{n}$. $\square$

Substituting in

In the two most common situations, "substituting in" using congruences is legal:

Fact If $a \equiv b + c \pmod{n}$ and $c \equiv d \pmod{n}$, then $a \equiv b + d \pmod{n}$.

Proof Because $c \equiv d \pmod{n}$ and $b \equiv b \pmod{n}$, we have $b + c \equiv b + d \pmod{n}$.
So because $a \equiv b + c \pmod{n}$, we have $a \equiv b + d \pmod{n}$.

Fact If $a \equiv bc \pmod{n}$ and $c \equiv d \pmod{n}$, then $a \equiv bd \pmod{n}$.

Proof Because $c \equiv d \pmod{n}$ and $b \equiv b \pmod{n}$, we have $bc \equiv bd \pmod{n}$.
So because $a \equiv bc \pmod{n}$, we have $a \equiv bd \pmod{n}$.

But you can't substitute in to exponents, logarithm bases, etc:

Example We know $6 \equiv 1 \pmod{5}$, but $2^6 \not\equiv 2^1 \pmod{5}$.

In some situations we can also "divide through" a congruence by an integer.

If $a \equiv b \pmod{n}$ and d divides a, b and n, then

$$\frac{a}{d} \equiv \frac{b}{d} \pmod{\frac{n}{d}}.$$

Examples

Suppose that $7x \equiv 21 \pmod{28}$ for an integer x.

Dividing through by 7, we see $x \equiv 3 \pmod{4}$.

Be careful!

Remember to divide the modulus as well.

If we have $2x \equiv 0 \pmod{10}$ for an integer x, we cannot conclude that $x \equiv 0 \pmod{10}$.

Flux Exercise (LQMTZZ)

If we know that $2x + 4y \equiv 4 \pmod 8$ and $y \equiv 3 \pmod 4$ what can we say about x?

A. $x \equiv 3 \pmod 4$
B. $x \equiv 3 \pmod 8$
C. $x \equiv 0 \pmod 4$
D. $x \equiv 0 \pmod 8$

Answer:

$$\begin{aligned} x + 2y &\equiv 2 \pmod 4 && \text{(dividing the first congruence by 2)} \\ x + 6 &\equiv 2 \pmod 4 && \text{(substituting } y \equiv 3 \pmod 4\text{)} \\ x &\equiv -4 \pmod 4 && \text{(subtracting 6 from both sides)} \\ x &\equiv 0 \pmod 4 && \text{(because } -4 \equiv 0 \pmod 4\text{)} \end{aligned}$$

So C. (To see that D is wrong think about $x = 4$, $y = 3$.)

```
int getRandomNumber()
{
    return 4;  // chosen by fair dice roll.
               // guaranteed to be random.
}
```

3.3 Solving linear congruences

Think of a congruence like $7x \equiv 5 \pmod{9}$. This will hold if 9 divides $7x - 5$ or in other words if there is an integer y such that $7x - 5 = 9y$. So to solve our original congruence we can find an integer solution to $7x - 9y = 5$.

Some congruences don't have solutions. For example, there is no solution to $10x \equiv 6 \pmod{20}$ because there are no integers x and y such that $10x - 20y = 6$.

To find an expression for all the integers x that satisfy a congruence like $ax \equiv b \pmod{n}$, first find $d = \gcd(a, n)$ and then act as follows.

If $d = 1$: Find integers x' and y' such that $ax' - ny' = b$. The integers x that satisfy the original congruence are exactly those for which $x \equiv x' \pmod{d}$.

If $d > 1$ and d divides b: The method above will still work but it will only give *some* of the solutions. To find *all* of the solutions, first divide through the congruence by d to get the equivalent congruence $\frac{a}{d}x \equiv \frac{b}{d} \pmod{\frac{n}{d}}$ and then use the method above on the new congruence.

If d doesn't divide b: The congruence has no solutions.

Question 3.3 Find an expression for all the integers x that satisfy $9x \equiv 12 \pmod{60}$.

First calculate $\gcd(9, 60) = 3$.

3 does divide 12 so there are solutions.

Divide through by 3 to get $3x \equiv 4 \pmod{20}$.

We now want to find x' and y' such that $3x' - 20y' = 4$.

Using the extended Euclidean algorithm, $x' = 28$ and $y' = 4$ work.

So the integers x that satisfy $9x \equiv 12 \pmod{60}$ are exactly those for which $x \equiv 28 \pmod{20}$ or, equivalently, $x \equiv 8 \pmod{20}$.

Example. Find all integers x such that $36x \equiv 10 \pmod{114}$.

Using the Euclidean algorithm we find $\gcd(36, 114) = 6$. So 6 divides $36x - 114y$ for any integers x and y, and consequently $36x - 114y \neq 10$. This means that there are no integers x such that $36x \equiv 10 \pmod{114}$.

Example. Find all integers x such that $24x \equiv 8 \pmod{44}$.

Using the Euclidean algorithm we find $\gcd(24, 44) = 4$. So we divide through by 4 to get the equivalent congruence $6x \equiv 2 \pmod{11}$. Using the extended Euclidean algorithm we see that $2 \times 6 - 1 \times 11 = 1$, and hence $4 \times 6 - 2 \times 11 = 2$. Thus the integers x such that $24x \equiv 8 \pmod{44}$ are exactly the integers $x \equiv 4 \pmod{11}$.

3.4 Modular inverses

A modular multiplicative inverse of an integer a modulo n is an integer x such that

$$ax \equiv 1 \pmod{n}.$$

From the last section we know that such an inverse will exist if and only if $\gcd(a, n) = 1$. If inverses do exist then we can find them using the extended Euclidean algorithm (there will be lots of inverses, but they will all be in one congruence class modulo n). These inverses have important applications to cryptography and random number generation.

Example: 7 is a multiplicative inverse of 4 modulo 9, because $4 \times 7 \equiv 1 \pmod{9}$. (Note $28 \equiv 1 \pmod{9}$.)

Example: 9 is its own multiplicative inverse modulo 10, because $9 \times 9 \equiv 1 \pmod{10}$. (Note $81 \equiv 1 \pmod{10}$.)

Example. 8 should have a multiplicative inverse modulo 45 because $\gcd(8, 45) = 1$. Using the extended Euclidean algorithm we see that $-3 \times 45 + 17 \times 8 = 1$. So $8 \times 17 \equiv 1 \pmod{45}$. This means that 17 is a multiplicative inverse of 8 modulo 45.