MAT1830

Lecture 7: Predicates and quantifiers

“Every real number is positive or negative.”

False! But we want to be able to write it in logic and do things with it.

For example, we’d like to be able to say that its negation is
“There is a real number which is neither positive nor negative.”

We get a more expressive language than propositional logic by admitting *predicates* like

$$P(n), \quad Q(x,y), \quad R(a,b,c)$$

These stand for properties or relations such as

$$\begin{aligned} P(n) &\ :\ n \text{ is prime} \\ Q(x,y) &\ :\ x \leqslant y \\ R(a,b,c) &\ :\ a+b=c. \end{aligned}$$

Those with one variable, such as "n is prime," are usually called *properties*, while those with two or more variables, such as "$x \leqslant y$," are usually called *relations*.

7.1 Predicates

A predicate such as "n is prime" is not a sentence because it is neither true nor false. Rather, it is a function $P(n)$ of n with the Boolean values T (true) or F (false). In this case, $P(n)$ is a function of natural numbers defined by

$$P(n) = \begin{cases} \mathsf{T} & \text{if } n \text{ is prime} \\ \mathsf{F} & \text{otherwise.} \end{cases}$$

Similarly, the "$x \leqslant y$" predicate is a function of pairs of real numbers, defined by

$$R(x, y) = \begin{cases} \mathsf{T} & \text{if } x \leqslant y \\ \mathsf{F} & \text{otherwise.} \end{cases}$$

Since most of mathematics involves properties and relations such as these, only a language with predicates is adequate for mathematics (and computer science).

$P(n)$: "n is prime"

n	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
$P(n)$	F	T	T	F	T	F	T	F	F	F	T	$\cdots$

$H(x)$: "there is a MAT1830 lecture on x"

x	Mon	Tue	Wed	Thu	Fri	$\cdots$
$H(x)$	T	F	T	T	F	$\cdots$

7.2 Building sentences from predicates

One way to create a sentence from a predicate is to replace its variables by constants. For example, when $P(n)$ is the predicate "n is prime," $P(3)$ is the sentence "3 is prime."

Another way is to use *quantifiers*:

- $\forall$ (meaning "for all") and
- $\exists$ (meaning "there exists" or "there is").

Example. $\exists n P(n)$ is the (true) sentence

there exists an n such that n is prime.

$\forall n P(n)$ is the (false) sentence

for all n, n is prime.

Note that when $\exists n$ is read "there exists an n" we also add a "such that."

The sentence $\forall x P(x)$:

- is true if $P(x)$ is true for every possible x
- is false if $P(x)$ is false for at least one possible x

The sentence $\exists x P(x)$:

- is true if $P(x)$ is true for at least one possible x
- is false if $P(x)$ is false for every possible x

Let $J(x)$ be "x is a jellybean" and $R(x)$ be "x is red"

x	$J(x)$	$R(x)$
yellow jellybean	T	F
red jellybean	T	T
red frog	F	T
red gummi bear	F	T
red snake	F	T

Is $\exists x J(x)$ true? Yes

Is $\forall x R(x)$ true? No

7.3 Quantifiers and connectives

We can also combine quantifiers with connectives from propositional logic.

Example. Let $Sq(n)$ be the predicate "n is a square," and let $Pos(n)$ be the predicate "n is positive" as above. Then we can symbolise the following sentences:

There is a positive square:

$$\exists n(Pos(n) \wedge Sq(n)).$$

There is a positive integer which is not a square:

$$\exists n(Pos(n) \wedge \neg Sq(n))$$

All squares are positive:

$$\forall n(Sq(n) \rightarrow Pos(n))$$

Notice that the "All. . . are" combination in English actually involves an implication. This is needed because we are making a claim only about squares and the implication serves to "narrow down" the set we are examining.

Question 7.1 Write down "roses are red" in predicate logic using
rose(x): "x is a rose."
red(x): "x is red."

$\forall x(\text{rose}(x) \rightarrow \text{red}(x))$

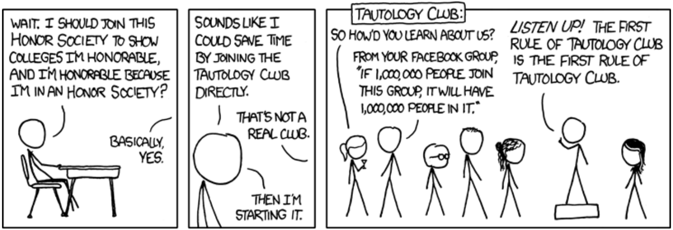
Why does this work?

yellow daffodil	$F \rightarrow F$	T	
red snapdragon	$F \rightarrow T$	T	
white lily	$F \rightarrow F$	T	
red rose	$T \rightarrow T$	T	
red rose	$T \rightarrow T$	T	$\forall$ statement true
white rose	$T \rightarrow F$	F	$\forall$ statement false

Question 7.2

$P(n)$: "n is prime."
$E(n)$: "n is even."
What does $P(n) \wedge \neg E(n)$ mean?

"n is prime and n is not even."   OR
"n is an odd prime."


WAIT. I SHOULD JOIN THIS HONOR SOCIETY TO SHOW COLLEGES I'M HONORABLE, AND I'M HONORABLE BECAUSE I'M IN AN HONOR SOCIETY?
BASICALLY, YES.
SOUNDS LIKE I COULD SAVE TIME BY JOINING THE TAUTOLOGY CLUB DIRECTLY.
THAT'S NOT A REAL CLUB.
THEN I'M STARTING IT.
TAUTOLOGY CLUB:
SO HOW'D YOU LEARN ABOUT US?
FROM YOUR FACEBOOK GROUP, "IF 1,000,000 PEOPLE JOIN THIS GROUP, IT WILL HAVE 1,000,000 PEOPLE IN IT."
LISTEN UP! THE FIRST RULE OF TAUTOLOGY CLUB IS THE FIRST RULE OF TAUTOLOGY CLUB.

Question 7.3

pol(x): "x is a politician."
liar(x): "x is a liar."

"All politicians are liars."
$\forall x(\text{pol}(x) \rightarrow \text{liar}(x))$

"Some politicians are liars."
$\exists x(\text{pol}(x) \wedge \text{liar}(x))$

"No politicians are liars."
$\forall x(\text{pol}(x) \rightarrow \neg\text{liar}(x))$

"Some politicians are not liars."
$\exists x(\text{pol}(x) \wedge \neg\text{liar}(x))$

Flux Exercise (LQMTZZ)

Let n range over the integers.
$P(n)$: "n is prime."
$E(n)$: "n is even."
$G(n)$: "$n \geq 3$."

What does $\forall n\big((P(n) \wedge G(n)) \to \neg E(n)\big)$ mean?

A. "Some primes greater than or equal to 3 are odd."
B. "All odd integers greater than or equal to 3 are prime."
C. "All primes greater than or equal to 3 are odd."
D. "All odd primes are greater than or equal to 3."

Answer: C.

7.4 Alternating quantifiers

Combinations of quantifiers like $\forall x \exists y \ldots$, "for all x there is a y ..." are common in mathematics, and can be confusing. It helps to have some examples in mind to recall the difference between $\forall x \exists y \ldots$ and $\exists y \forall x \ldots$

The relation $x < y$ is convenient to illustrate such combinations; we write $x < y$ as the predicate $L(x, y)$

Then

$$\forall x \exists y L(x, y)$$

is the (true) sentence

for all x there is a y such that $x < y$,

which says that there is no greatest number.

But with the opposite combination of quantifiers we have

$$\exists y \forall x L(x, y)$$

is the false sentence

there is a y such that for all x, $x < y$,

which says there is a number greater than all numbers.

Even though these statements are usually written without brackets they are effectively bracketed "from the centre". So $\forall x \exists y L(x, y)$ means $\forall x(\exists y L(x, y))$ and $\exists y \forall x L(x, y)$ means $\exists y(\forall x L(x, y))$.

Order of quantifiers

Let x and y range over all people.

$\forall x \exists y (x \text{ is friends with } y)$ Think: $\forall x (\exists y (x \text{ is friends with } y))$

$\exists y (x \text{ is friends with } y)$ is saying "x has a friend."
$\forall x \exists y (x \text{ is friends with } y)$ is saying "Everybody has a friend."

$\exists x \forall y (x \text{ is friends with } y)$ Think: $\exists x (\forall y (x \text{ is friends with } y))$

$\forall y (x \text{ is friends with } y)$ is saying "x is friends with everybody."
$\exists x \forall y (x \text{ is friends with } y)$ is saying "There is somebody that is friends with everybody."

7.4 Alternating quantifiers

Combinations of quantifiers like $\forall x \exists y \ldots$, "for all x there is a y ..." are common in mathematics, and can be confusing. It helps to have some examples in mind to recall the difference between $\forall x \exists y \ldots$ and $\exists y \forall x \ldots$

The relation $x < y$ is convenient to illustrate such combinations; we write $x < y$ as the predicate $L(x, y)$

Then

$$\forall x \exists y L(x, y)$$

is the (true) sentence

for all x there is a y such that $x < y$,

which says that there is no greatest number.

But with the opposite combination of quantifiers we have

$$\exists y \forall x L(x, y)$$

is the false sentence

there is a y such that for all x, $x < y$,

which says there is a number greater than all numbers.

Even though these statements are usually written without brackets they are effectively bracketed "from the centre". So $\forall x \exists y L(x, y)$ means $\forall x(\exists y L(x, y))$ and $\exists y \forall x L(x, y)$ means $\exists y(\forall x L(x, y))$.

Question

Let c range over all countries and p range over all people.
$P(p,c)$: "p lives in c."

What does $\forall c \exists p P(p,c)$ mean? Is it true?

"Every country has somebody that lives in it." True.

What does $\exists c \forall p P(p,c)$ mean? Is it true?

"There is one country that everybody lives in." False.

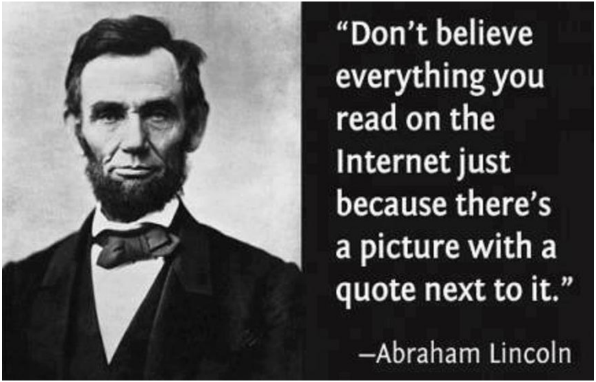
Flux Exercise (LQMTZZ)

Let x and y range over the integers.
$N(x, y)$: "$x + y = 0$."

Is $\exists x \forall y N(x, y)$ true or false? Is $\forall x \exists y N(x, y)$ true or false?

A. True, true
B. True, false
C. False, true
D. False, false

Answer:
The first statement says there is a single integer so that when any other integer is added to it the result is 0. This is false.
The second statement says that for every integer there is another integer so that when the two are added the result is 0. This is true.
So C.


"Don't believe everything you read on the Internet just because there's a picture with a quote next to it."
–Abraham Lincoln

7.5 An example from Abraham Lincoln

You can fool all of the people some of the time
and
you can fool some of the people all of the time
but
you can't fool all of the people all of the time.

Let $F(p, t)$ be the predicate:

person p can be fooled at time t.

Then
$\forall p \exists t F(p, t)$ says
you can fool all of the people some of the time,
$\exists p \forall t F(p, t)$ says
you can fool some of the people all of the time,
$\neg \forall p \forall t F(p, t)$ says
you can't fool all of the people all of the time.

Hence Lincoln's sentence in symbols is:

$$\forall p \exists t F(p, t) \wedge \exists p \forall t F(p, t) \wedge \neg \forall p \forall t F(p, t)$$

Remark. Another way to say "you can't fool all of the people all of the time" is

$$\exists p \exists t \neg F(p, t).$$