MAT1830

Lecture 14: Examples of Functions

The functions discussed in the last lecture were familiar functions of real numbers. Many other examples occur elsewhere, however.

14.1 Functions of several variables

We might define a function

$$\mathrm{sum} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \quad \text{by} \quad \mathrm{sum}(x, y) = x + y.$$

Because the domain of this function is $\mathbb{R} \times \mathbb{R}$, the inputs to this function are ordered pairs (x, y) of real numbers. Because its codomain in $\mathbb{R}$, we are guaranteed that each output will be a real number. This function can be thought of as a function of two variables x and y.

Similarly we might define a function

$$\mathrm{binomial} : \mathbb{R} \times \mathbb{R} \times \mathbb{N} \to \mathbb{R}$$

by

$$\mathrm{binomial}(a, b, n) = (a + b)^n.$$

Here the inputs are ordered triples (x, y, n) such that x and y are real numbers and n is a natural number. We can think of this as a function of three variables.

Question What are the ordered pairs which define the function $\text{sum} : \{1, 2\} \times \{1, 2\} \to \mathbb{N}$ defined by $\text{sum}(x, y) = x + y$?

Answer

We have $\text{sum}((1, 1)) = 2$, $\text{sum}((1, 2)) = 3$, $\text{sum}((2, 1)) = 3$, and $\text{sum}((2, 2)) = 4$.

So $\{\ ((1, 1), 2),\ ((1, 2), 3),\ ((2, 1), 3),\ ((2, 2), 4)\ \}$.

Note We often abbreviate $f((x, y))$ to $f(x, y)$ and so on when dealing with multivariable functions.

Question 14.1 Suggest domains and codomains for the following functions.

gcd domain: $\mathbb{Z} \times \mathbb{Z} - \{(0,0)\}$ codomain: $\mathbb{N}$

reciprocal domain: $\mathbb{R} - \{0\}$ codomain: $\mathbb{R} - \{0\}$

Flux Exercise (LQMTZZ)

Suggest a domain and a codomain for a $\cap$ (intersection) function for sets of real numbers.

A. domain: $\mathbb{R} \times \mathbb{R}$, codomain: $\mathbb{R}$

B. domain: $\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R})$, codomain: $\mathcal{P}(\mathbb{R})$

C. domain: $\mathcal{P}(\mathbb{R})$, codomain: $\mathcal{P}(\mathbb{R})$

D. domain: $\mathbb{R} \times \mathbb{R}$, codomain: $\mathcal{P}(\mathbb{R})$

Example Input: $(\{1,2,3,4\},\{2,3,\pi\})$ Ouput: $\{2,3\}$

Answer

The function must output a *set* of real numbers.
So the codomain must be the set containing all sets of real numbers.
$\mathbb{R}$ is the set of real numbers, so $\mathcal{P}(\mathbb{R})$ is the set of all sets of real numbers.

The function must accept a *pair* of *sets* of real numbers.
So the domain must be the set containing all pairs of sets of real numbers.
$\mathcal{P}(\mathbb{R})$ is the set of all sets of real numbers, so $\mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R})$ is the set of pairs of sets of real numbers.

So B.

14.2 Sequences

An infinite sequence of numbers, such as

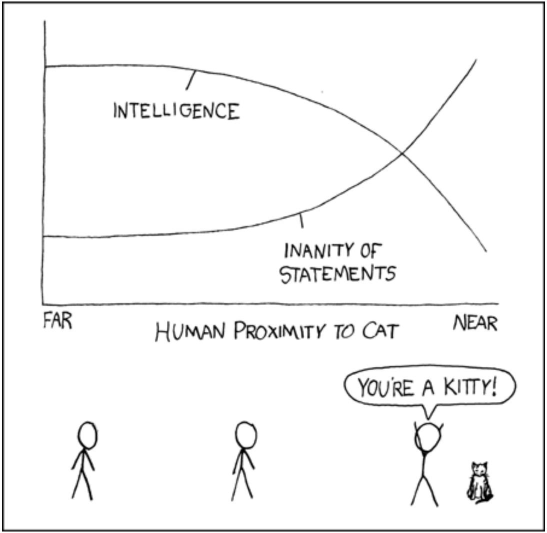
$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \ldots,$$

can be viewed as the function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = 2^{-n}$. In this case, the inputs to f are natural numbers, and its outputs are real numbers.

Any infinite sequence $a_0, a_1, a_2, a_3, \ldots$ can be viewed as a function $g(n) = a_n$ from $\mathbb{N}$ to some set containing the values a_n.

Question For each of the following sequences, find a function f such that the sequence is $f(0), f(1), f(2), \ldots$.

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots$	$f : \mathbb{N} \to \mathbb{Q}$, $f(n) = \frac{1}{n+1}$
$5, 1, -3, -7, -11, -15, \ldots$	$f : \mathbb{N} \to \mathbb{Z}$, $f(n) = 5 - 4n$
$4, 12, 36, 108, 324, 972, \ldots$	$f : \mathbb{N} \to \mathbb{Z}$, $f(n) = 4(3^n)$


INTELLIGENCE
INANITY OF STATEMENTS
FAR
HUMAN PROXIMITY TO CAT
NEAR
YOU'RE A KITTY!

14.3 Characteristic functions

A subset of $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ can be represented by its characteristic function. For example, the set of squares is represented by the function $\chi : \mathbb{N} \to \{0, 1\}$ defined by

$$\chi(n) = \begin{cases} 1 & \text{if } n \text{ is a square} \\ 0 & \text{if } n \text{ is not a square} \end{cases}$$

which has the following sequence of values

$$110010000100000010000000010000000000100\ldots$$

(with 1s at the positions of the squares $0, 1, 4, 9, 16, 25, 36, \ldots$).

Any property of natural numbers can likewise be represented by a characteristic function. For example, the function χ above represents the property of being a square.

Thus any set or property of natural numbers is represented by a function

$$\chi : \mathbb{N} \to \{0, 1\}.$$

Characteristic functions of two or more variables represent relations between two or more objects. For example, the relation $x \leqslant y$ between real numbers x and y has the characteristic function $\chi : \mathbb{R} \times \mathbb{R} \to \{0, 1\}$ defined by

$$\chi(x, y) = \begin{cases} 1 & \text{if } x \leqslant y \\ 0 & \text{otherwise.} \end{cases}$$

Question 14.2 If A and B are subsets of $\mathbb{N}$ with characteristic functions $\chi_A(n)$ and $\chi_B(n)$, then what set does the function $\chi_A(n)\chi_B(n)$ represent?

Answer

If $n \in A$ and $n \in B$ then $\chi_A(n)\chi_B(n) = 1 \times 1 = 1$.

If $n \in A$ and $n \notin B$ then $\chi_A(n)\chi_B(n) = 1 \times 0 = 0$.

If $n \notin A$ and $n \in B$ then $\chi_A(n)\chi_B(n) = 0 \times 1 = 0$.

If $n \notin A$ and $n \notin B$ then $\chi_A(n)\chi_B(n) = 0 \times 0 = 0$.

So $\chi_A(n)\chi_B(n)$ is the characteristic function of $A \cap B$.

Question Let d be a positive integer. If $\chi_d : \mathbb{N} \to \{0, 1\}$ is a function defined by

$$\chi_d(x) = \begin{cases} 1, & \text{if } x \text{ divides } d; \\ 0, & \text{if } x \text{ does not divide } d. \end{cases}$$

then what is $1\chi_d(1) + 2\chi_d(2) + 3\chi_d(3) + \cdots + d\chi_d(d)$?

Answer

The sum of the positive divisors of d.

Question How many functions are there with domain $\{1, 2, 3, 4\}$ and codomain $\{-1, 0, 1\}$?

Answer
The domain has 4 elements. (There are 4 possible inputs.)
For each input, we can decide if it's mapped to -1 or 0 or 1.
We can do this in $\underbrace{3 \times 3 \times \cdots \times 3}_{4} = 3^4 = 81$ ways.

Question How many functions are there with domain X and codomain Y?

Answer
The domain has $|X|$ elements. (There are $|X|$ possible inputs.)
For each input, we have $|Y|$ options for where it's mapped to.
We can do this in $\underbrace{|Y| \times |Y| \times \cdots \times |Y|}_{|X|} = |Y|^{|X|}$ ways.

14.4 Boolean functions

The connectives $\wedge$, $\vee$ and $\neg$ are functions of variables whose values come from the set $\mathbb{B} = \{\mathsf{T}, \mathsf{F}\}$ of Boolean values (named after George Boole).

$\neg$ is a function of one variable, so

$$\neg : \mathbb{B} \to \mathbb{B}$$

and it is completely defined by giving its values on T and F, namely

$$\neg\mathsf{T} = \mathsf{F} \quad \text{and} \quad \neg\mathsf{F} = \mathsf{T}.$$

This is what we previously did by giving the truth table of $\neg$.

$\wedge$ and $\vee$ are functions of two variables, so

$$\wedge : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$$

and

$$\vee : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$$

They are completely defined by giving their values on the pairs $\{\mathsf{T}, \mathsf{T}\}, \{\mathsf{T}, \mathsf{F}\}, \{\mathsf{F}, \mathsf{T}\}, \{\mathsf{F}, \mathsf{F}\}$ in $\mathbb{B} \times \mathbb{B}$, which is what their truth tables do.

Flux Exercise (LQMTZZ)

Let $\mathbb{B} = \{0, 1\}$. How many functions are there with domain $\underbrace{\mathbb{B} \times \mathbb{B} \times \cdots \times \mathbb{B}}_{n}$ and codomain $\mathbb{B}$?

A. n^2

B. $2^{(n^2)}$

C. 2^n

D. $2^{(2^n)}$

Answer

The domain has 2^n elements. (There are 2^n possible inputs.)
For each input, we can decide if it's mapped to 0 or 1.
We can do this in $\underbrace{2 \times 2 \times \cdots \times 2}_{2^n} = 2^{(2^n)}$ ways.

So D.

For $n = 2$ there are $2^{(2^2)} = 2^4 = 16$.
For $n = 3$ there are $2^{(2^3)} = 2^8 = 256$.
For $n = 4$ there are $2^{(2^4)} = 2^{16} = 65\,536$.

Example (Hamming distance)

Let B_n be the set of all binary strings of length n.

Hamming distance is a function $h : B_n \times B_n \to \mathbb{N}$ defined by $h(s, t)$ equals the number of places in which s and t disagree.

For example,
$h(000, 101) = 2$,
$h(011, 010) = 1$,
$h(10111, 01000) = 5$.

A set of binary strings of length n such that any two different strings in the set have Hamming distance at least d is called a *binary error correcting code of length n and distance d*.

These are useful in sending information across noisy channels.

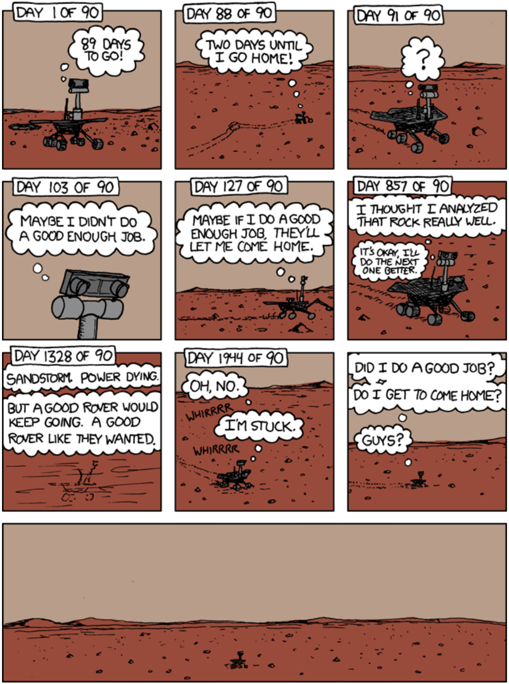
$\{0000, 0011, 0101, 0110, 1001, 1010, 1100, 1111\}$
is a binary code of length 4 and distance 2.

If we only send strings in this set across a channel and at most one error occurs in each string then we will be able to detect the errors.

$$\{0000000, 1110000, 1001100, 0111100, 0101010, 1011010, 1100110, 0010110,$$
$$1101001, 0011001, 0100101, 1010101, 1000011, 0110011, 0001111, 1111111\}$$
is a binary code of length 7 and distance 3.

If we only send strings in this set across a channel and at most one error occurs in each string then we will be able to *correct* the errors on the fly.

(* this slide is not assessable)


DAY 1 OF 90
89 DAYS TO GO!
DAY 88 OF 90
TWO DAYS UNTIL I GO HOME!
DAY 91 OF 90
?
DAY 103 OF 90
MAYBE I DIDN'T DO A GOOD ENOUGH JOB.
DAY 127 OF 90
MAYBE IF I DO A GOOD ENOUGH JOB, THEY'LL LET ME COME HOME.
DAY 857 OF 90
I THOUGHT I ANALYZED THAT ROCK REALLY WELL.
IT'S OKAY, I'LL DO THE NEXT ONE BETTER.
DAY 1328 OF 90
SANDSTORM. POWER DYING.
BUT A GOOD ROVER WOULD KEEP GOING. A GOOD ROVER LIKE THEY WANTED.
DAY 1944 OF 90
OH, NO.
WHIRRRR
I'M STUCK.
WHIRRRR
DID I DO A GOOD JOB?
DO I GET TO COME HOME?
GUYS?

14.5* Characteristic functions and subsets of $\mathbb{N}$

Mathematicians say that two (possibly infinite) sets A and B have the same *cardinality* (size) if there is a one-to-one and onto function from A to B. This function associates each element of A with a unique element of B and vice-versa. With this definition, it is not too hard to show that, for example, $\mathbb{N}$ and $\mathbb{Z}$ have the same cardinality (they are both "countably infinite").

It turns out, though, that $\mathcal{P}(\mathbb{N})$ has a strictly greater cardinality than $\mathbb{N}$. We can prove this by showing: *no sequence* $f_0, f_1, f_2, f_3, \ldots$ *includes all characteristic functions for subsets of* $\mathbb{N}$. (This shows that there are more characteristic functions than natural numbers.)

In fact, for any infinite list $f_0, f_1, f_2, f_3, \ldots$ of characteristic functions, we can define a characteristic function f which is *not* on the list. Imagine each function given as the infinite sequence of its values, so the list might look like this:

$$
\begin{array}{lll}
f_0 & \text{values} & \underline{0}101010101\ldots \\
f_1 & \text{values} & 0\underline{0}00011101\ldots \\
f_2 & \text{values} & 11\underline{1}1111111\ldots \\
f_3 & \text{values} & 000\underline{0}000000\ldots \\
f_4 & \text{values} & 1001\underline{0}01001\ldots \\
 & \vdots &
\end{array}
$$

Now if we switch each of the underlined values to its opposite, we get a characteristic function

$$
f(n) = \begin{cases} 1 & \text{if } f_n(n) = 0 \\ 0 & \text{if } f_n(n) = 1 \end{cases}
$$

which is *different* from each function on the list. In fact, it has a different value from f_n on the number n.

For the given example, f has values

$$11011\ldots$$

The construction of f is sometimes called a "diagonalisation argument", because we get its values by switching values along the diagonal in the table of values of $f_0, f_1, f_2, f_3, \ldots$.