MAT1830

Lecture 5: Tautologies and logical equivalence

A major problem in logic is to recognise statements that are “always true” or “always false”.

5.1 Tautologies and contradictions

A *sentence* ϕ in propositional logic is a formula with variables $p, q, r, \ldots$ which can take the values T and F. The possible *interpretations* of ϕ are all possible assignments of values to its variables.

A sentence in propositional logic is

- a *tautology* if it has value T under all interpretations;
- a *contradiction* if it has value F under all interpretations.

We can check whether ϕ is a tautology, a contradiction, or neither by computing its value for all possible values of its variables.

Example. $(\neg p) \vee p$
For $p = \mathsf{T}$ we have

$$(\neg \mathsf{T}) \vee \mathsf{T} = \mathsf{F} \vee \mathsf{T} = \mathsf{T},$$

using the known values of the $\neg$ and $\vee$ functions.

For $p = \mathsf{F}$ we have

$$(\neg \mathsf{F}) \vee \mathsf{F} = \mathsf{T} \vee \mathsf{F} = \mathsf{T},$$

hence $(\neg p) \vee p$ is a tautology. (It is known as the *law of the excluded middle*).

We can similarly compute the values of any truth function ϕ, so this is an algorithm for recognising tautologies. However, if ϕ has n variables, they have 2^n sets of values, so the amount of computation grows rapidly with n. One of the biggest unsolved problems of logic and computer science is to find an efficient algorithm for recognising tautologies.

"If Mitchell Starc is limping and Pat Cummins is looking grumpy, then Australia are losing the test match."

p: "Mitchell Starc is limping."
q: "Pat Cummins is looking grumpy."
r: "Australia are losing the test match."

$(p \wedge q) \rightarrow r$

p could be T or F, q could be T or F, and r could be T or F.

So there are $2 \times 2 \times 2 = 8$ ways to assign truth values to this statement.

That's why there would be eight rows in the truth table.

Question How many ways are there to assign truth values to the following statement?

$$\neg((\neg p_1 \wedge p_2) \vee ((p_3 \vee \neg p_4) \wedge p_1)) \rightarrow (p_2 \wedge p_5)$$

$2 \times 2 \times 2 \times 2 \times 2 = 32$

Question 5.1 There are two possibilities for each of the variables and this gives a total of $2 \times 2 \times \cdots \times 2 = 2^n$ possibilities.

5.2 Logical equivalence

Sentences ϕ and ψ are *logically equivalent* if they are the same truth function, which also means $\phi \leftrightarrow \psi$ is a tautology. This relation between sentences is written $\phi \Leftrightarrow \psi$ or $\phi \equiv \psi$.

Example. $p \to q \equiv (\neg p) \vee q$
We know $p \to q$ has the truth table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Now $(\neg p) \vee q$ has the truth table

p	q	$\neg p$	$(\neg p) \vee q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

So $p \to q$ and $(\neg p) \vee q$ have the same truth table (looking just at their columns). It follows from this that $p \to q$ can always be rewritten as $(\neg p) \vee q$. In fact, all truth functions can be expressed in terms of $\wedge, \vee$, and $\neg$.

Why should $p \to q$ mean the same as $\neg p \vee q$?

"If you train hard, then you'll finish the marathon."

"You won't train hard or you'll finish the marathon."

"If that's a lion, then we're dead."

"That's not a lion or we're dead."

The expression

$$\frac{x^3 - 3x^2 + 3x + 1}{(x-1)^4} \times \frac{x^3 - x}{x+1}$$

is actually the same as the expression

$$x$$

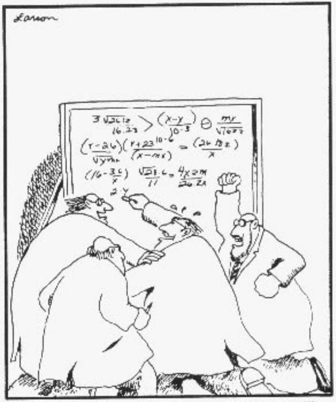
(for $x \neq -1, 1$), but the second expression is nicer.

Similarly

$$((p \wedge (p \vee q)) \rightarrow (\neg p \wedge q)) \wedge (p \vee \neg q)$$

is logically equivalent to

$$\neg p \wedge \neg q.$$



"Go for it, Sidney! You've got it! You've got it!
Good hands! Don't choke!"

5.3 Useful equivalences

The following equivalences are the most frequently used in this "algebra of logic".

Equivalence law

$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Implication law

$p \to q \equiv (\neg p) \vee q$

Double Negation law

$\neg\neg p \equiv p$

Idempotent laws

$p \wedge p \equiv p$

$p \vee p \equiv p$

Commutative laws

$p \wedge q \equiv q \wedge p$

$p \vee q \equiv q \vee p$

Associative laws

$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

$p \vee (q \vee r) \equiv (p \vee q) \vee r$

Distributive laws

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

De Morgan's laws

$\neg(p \wedge q) \equiv (\neg p) \vee (\neg q)$

$\neg(p \vee q) \equiv (\neg p) \wedge (\neg q)$

Identity laws

$p \wedge \mathsf{T} \equiv p$

$p \vee \mathsf{F} \equiv p$

Annihilation laws

$p \wedge \mathsf{F} \equiv \mathsf{F}$

$p \vee \mathsf{T} \equiv \mathsf{T}$

Inverse laws

$p \wedge (\neg p) \equiv \mathsf{F}$

$p \vee (\neg p) \equiv \mathsf{T}$

Absorption laws

$p \wedge (p \vee q) \equiv p$

$p \vee (p \wedge q) \equiv p$

"Mr. Osborne, may I be excused? My brain is full."

Flux Exercise (LQMTZZ)

What simpler sentence is $((p \wedge \neg\neg p) \vee \neg p) \vee (q \wedge r)$ logically equivalent to?

A. p

B. $q \wedge r$

C. T

D. F

Answer:

$$
\begin{aligned}
& ((p \wedge \neg\neg p) \vee \neg p) \vee (q \wedge r) \\
\equiv\ & ((p \wedge p) \vee \neg p) \vee (q \wedge r) \qquad \text{(double negation law)} \\
\equiv\ & (p \vee \neg p) \vee (q \wedge r) \qquad \text{(idempotent law)} \\
\equiv\ & \mathrm{T} \vee (q \wedge r) \qquad \text{(inverse law)} \\
\equiv\ & \mathrm{T} \qquad \text{(annihilation law)}
\end{aligned}
$$

So C.

Question 5.2 First show that $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

The columns for $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are the same so $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

Question 5.2 (cont) Next show that $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

The columns for $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are the same so $\neg(p \vee q) \equiv \neg p \wedge \neg q$.

Absorption laws left as an exercise.

Remarks

1. The commutative laws are used to rearrange terms, as in ordinary algebra. The law $p \vee q \equiv q \vee p$ is like $p + q = q + p$ in ordinary algebra, and $p \wedge q \equiv q \wedge p$ is like $pq = qp$.

2. The associative laws are used to remove brackets. Since $p \vee (q \vee r) \equiv (p \vee q) \vee r$, we can write either side as $p \vee q \vee r$. This is like $p+(q+r) = (p+q)+r = p+q+r$ in ordinary algebra.

3. The distributive laws are used to "expand" combinations of $\wedge$ and $\vee$.

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

is like

$$p(q + r) = pq + pr.$$

The other distributive law is *not* like anything in ordinary algebra.

4. Some of these laws are redundant, in the sense that other laws imply them. For example, the absorption law

$$p \wedge (p \vee q) \equiv p$$

follows from the distributive, idempotent, identity and annihilation laws:

$$\begin{aligned}
p \wedge (p \vee q) &\equiv (p \wedge p) \vee (p \wedge q) \\
&\qquad \text{by distributive law} \\
&\equiv p \vee (p \wedge q) \\
&\qquad \text{by idempotent law} \\
&\equiv (p \wedge \mathsf{T}) \vee (p \wedge q) \\
&\qquad \text{by identity law} \\
&\equiv p \wedge (\mathsf{T} \vee q) \\
&\qquad \text{by distributive law} \\
&\equiv p \wedge \mathsf{T} \\
&\qquad \text{by annihilation law} \\
&\equiv p \\
&\qquad \text{by identity law}
\end{aligned}$$

Question 5.3 First show that $p \wedge (q \veebar r) \equiv (p \wedge q) \veebar (p \wedge r)$.

p	q	r	$q \veebar r$	$p \wedge (q \veebar r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \veebar (p \wedge r)$
T	T	T	F	F	T	T	F
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	F	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

The columns for $p \wedge (q \veebar r)$ and $(p \wedge q) \veebar (p \wedge r)$ are the same so $p \wedge (q \veebar r) \equiv (p \wedge q) \veebar (p \wedge r)$.

Second statement left as an exercise (but they're not logically equivalent).

SAT*

* this last part of the lecture is not assessable

Problem (SAT): Given a sentence in propositional logic using $\wedge$, $\vee$, $\neg$ and n variables, is there an assignment of truth values to the variables that makes the sentence evaluate to T?

SAT is an example of a *decision problem*.

P is the class of all decision problems which there is an efficient (polynomial-time) algorithm for solving.

NP is the class of all decision problems for which there is an efficient (polynomial-time) algorithm for checking that a given solution is correct.

Obviously every problem in P is also in NP.

SAT is in NP, but we don't know whether it's in P.

Huge question: Is P=NP?

Everybody thinks P $\neq$ NP, but no-one has proven it.

SAT*

* this last part of the lecture is not assessable

In fact, SAT is what we call *NP-complete*.

This means that if we had an efficient algorithm for solving it we could "translate" that algorithm to make it solve any problem in NP efficiently, and we would have P = NP.

Examples of other NP-complete problems are the "travelling salesman problem" and the problem of finding a hamilton path in a graph.