MAT1830 - Discrete Mathematics for Computer Science
Assignment #2
Submit by uploading a pdf to moodle by 11:55pm Wednesday in week 6

Assessment questions/solutions for this unit must not be posted on any website.

For questions 1 and 2, make sure you set out and explain your proofs clearly. To receive marks for question 3, your answers must be exactly right and use correct notation.

(1) Prove by simple induction that, for each integer $n \geq 1$,

$$6 + 6^2 + 6^3 + \cdots + 6^n = \frac{6^{n+1} - 6}{5}.$$

ANS: Let $P(n)$ be the statement "$6 + 6^2 + 6^3 + \cdots + 6^n = \frac{6^{n+1}-6}{5}$".

Base step. The left hand side of $P(1)$ is 6 and the right hand side of $P(1)$ is $\frac{6^2-6}{5} = 6$. So $P(1)$ is true.

Induction step. For some integer $k \geq 1$, suppose that $P(k)$ is true, that is, suppose

$$6 + 6^2 + 6^6 + \cdots + 6^k = \frac{6^{k+1} - 6}{5}.$$

Now we need to prove that $P(k + 1)$ is true. So we must prove

$$6 + 6^2 + 6^6 + \cdots + 6^{k+1} = \frac{6^{k+2} - 6}{5}.$$

Working with the left hand side of this equation we see that

$$\begin{aligned} 6 + 6^2 + 6^6 + \cdots + 6^{k+1} &= (6 + 6^2 + 6^6 + \cdots + 6^k) + 6^{k+1} \\ &= \left(\frac{6^{k+1} - 6}{5}\right) + 6^{k+1} \quad \text{(using } P(k)\text{)} \\ &= \frac{6^{k+1} - 6}{5} + \frac{5(6^{k+1})}{5} \\ &= \frac{6(6^{k+1}) - 6}{5} \\ &= \frac{6^{k+2} - 6}{5}, \end{aligned}$$

which is the right hand side we required. So $P(k + 1)$ is true.

So we have proved by induction that $P(n)$ is true for each integer $n \geq 1$.

(2) Let $S_1, S_2, S_3, S_4, \ldots$ be the sequence of sets defined by $S_1 = \{0,1,2\}$, $S_2 = \{0,2,3\}$, $S_3 = \{0,3,4\}$ and

$$S_i = (S_{i-3} \triangle S_{i-2}) \triangle \big(S_{i-1} \cup \{i-3, i+1\}\big) \text{ for each integer } i \geq 4.$$

Prove by strong induction that $S_n = \{0, n, n+1\}$ for each integer $n \geq 1$.

ANS: Let $P(n)$ be the statement "$S_n = \{0, n, n+1\}$".

Base steps. $P(1)$, $P(2)$ and $P(3)$ say "$S_1 = \{0,1,2\}$", "$S_2 = \{0,2,3\}$" and "$S_3 = \{0,3,4\}$" respectively. Each of these is true by the definition of the sequence.

Induction step. For some integer $k \geq 3$, suppose that $P(1), P(2), \ldots, P(k)$ are all true. We must prove $P(k+1)$ which is the statement "$S_{k+1} = \{0, k+1, k+2\}$".
By the definition of the sequence

$$S_{k+1} = (S_{k-2} \triangle S_{k-1}) \triangle \big(S_k \cup \{k-2, k+2\}\big).$$

Because $P(k-2)$, $P(k-1)$ and $P(k)$ are all true (note $k \geq 3$), we have $S_{k-2} = \{0, k-2, k-1\}$, $S_{k-1} = \{0, k-1, k\}$ and $S_k = \{0, k, k+1\}$. Substituting these into our expression for S_{k+1} above we get

$$\begin{aligned}
S_{k+1} &= (S_{k-2} \triangle S_{k-1}) \triangle \big(S_k \cup \{k-2, k+2\}\big) \\
&= \big(\{0, k-2, k-1\} \triangle \{0, k-1, k\}\big) \triangle \big(\{0, k, k+1\} \cup \{k-2, k+2\}\big) \\
&= \{k-2, k\} \triangle \{0, k-2, k, k+1, k+2\} \\
&= \{0, k+1, k+2\}.
\end{aligned}$$

So $P(k+1)$ is true.

So we have proved by strong induction that $P(n)$ is true for each integer $n \geq 1$.

(3) Let R, S and T be sets defined as follows.

$$
\begin{aligned}
R &= \{2, 4, 6, 7, 8\} \\
S &= \{\{2\}, \{2, 3, 4\}, \{2, 4, 6\}, \{6, 7\}\} \\
T &= \{x \in \mathbb{Z} : x \leq 4 \text{ or } x \geq 8\}
\end{aligned}
$$

Find the following.

(i) $R - T$
(ii) $S - \mathcal{P}(R)$
(iii) $\mathcal{P}(R) \cap \mathcal{P}(T)$
(iv) $(R \cap T) \times (S - \{\{2, 3, 4\}, \{2, 4, 6\}, \{6\}\})$
(v) $|(\mathcal{P}(R) - S) \times S|$

[No explanation required.]

ANS: (i) $\{6, 7\}$
(ii) $\{\{2, 3, 4\}\}$
(iii) $\{\emptyset, \{2\}, \{4\}, \{8\}, \{2, 4\}, \{2, 8\}, \{4, 8\}, \{2, 4, 8\}\}$
(iv) $\{(2, \{2\}), (2, \{6, 7\}), (4, \{2\}), (4, \{6, 7\}), (8, \{2\}), (8, \{6, 7\})\}$
(v) 116
$|R| = 5$, so $|\mathcal{P}(R)| = 2^5 = 32$. Also $|\mathcal{P}(R) \cap S| = |\{\{2\}, \{2, 4, 6\}, \{6, 7\}\}| = 3$.
So $|\mathcal{P}(R) - S| = 32 - 3 = 29$.
So $|(\mathcal{P}(R) - S) \times S| = 29 \times 4 = 116$.

(4) Let A and B be finite sets and let $a = |A|$, $b = |B|$ and $c = |A \cap B|$. Write an expression in terms of a, b and c that is equal to $|(A \times B) \cup (B \times A)|$ for every choice of A and B.

[No explanation required.]

ANS: $2ab - c^2$

If the sets $A \times B$ and $B \times A$ did not share any elements, then we would have $|(A \times B) \cup (B \times A)| = |A \times B| + |B \times A|$. But they may share elements and each element they share appears only once in their union, not twice. So we have

$$
\begin{aligned}
|(A \times B) \cup (B \times A)| &= |A \times B| + |B \times A| - |(A \times B) \cap (B \times A)| \\
&= ab + ba - |(A \times B) \cap (B \times A)| \\
&= 2ab - |(A \times B) \cap (B \times A)|.
\end{aligned}
$$

Now

$$
\begin{aligned}
& (x, y) \in (A \times B) \cap (B \times A) \\
\Leftrightarrow\ & (x, y) \in A \times B \text{ and } (x, y) \in B \times A \\
\Leftrightarrow\ & x \in A \text{ and } y \in B \text{ and } x \in B \text{ and } y \in A \\
\Leftrightarrow\ & x \in A \cap B \text{ and } y \in A \cap B \\
\Leftrightarrow\ & (x, y) \in (A \cap B) \times (A \cap B)
\end{aligned}
$$

This shows that $(A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$. So

$$|(A \times B) \cap (B \times A)| = |(A \cap B) \times (A \cap B)| = |A \cap B|^2 = c^2.$$

Substituting this into our expression for $|(A \times B) \cup (B \times A)|$ above we have

$$|(A \times B) \cup (B \times A)| = 2ab - c^2.$$