MAT1830 - Discrete Mathematics for Computer Science
Assignment #4
Submit by uploading a pdf to moodle by 11:55pm Wednesday in week 10

Assessment questions/solutions for this unit must not be posted on any website.

Full justifications are required for Questions 1(i) and 3.

(1) (i) A bag contains one trick coin that always shows heads when flipped, and nine fair coins (that show heads with probability $\frac{1}{2}$ and tails with probability $\frac{1}{2}$ when flipped). A coin is selected uniformly at random from the bag and flipped four times. What is the probability that the coin selected was the trick coin, given that it shows heads all four times? **[5]**

(ii) A bag contains n coins, one of which is a trick coin and the rest of which are fair (the coins are as described in (i)). A coin is selected uniformly at random from the bag and flipped k times. The coin shows heads all k times and, given this, you calculate that there is an exactly 50% probability that the coin selected was the trick coin. What is n? (Write your answer as an expression in terms of k.) **[2]**

[Fully explain your answer for (i). Answer only required for (ii).]

ANS: (i) Let C be the event that the trick coin was selected.
Let H be the event that the coins showed heads all four times.
(Because every other coin in the bag was fair, $\overline{C}$ is the event a fair coin was selected.)

$\Pr(C) = \frac{1}{10}$ because the coin was selected uniformly at random from the bag.
$\Pr(H|C) = 1$ because the trick coin always flips heads.
$\Pr(H|\overline{C}) = (\frac{1}{2})^4 = \frac{1}{16}$ because the probability of a fair coin flipping heads is $\frac{1}{2}$ and the 4 flips are independent.

By Bayes' theorem,

$$\begin{aligned}\Pr(C|H) &= \frac{\Pr(H|C)\Pr(C)}{\Pr(H|C)\Pr(C) + \Pr(H|\overline{C})\Pr(\overline{C})}\\ &= \frac{1 \times \frac{1}{10}}{(1 \times \frac{1}{10}) + (\frac{1}{16} \times (1 - \frac{1}{10}))}\\ &= \frac{16}{16+9}\\ &= \frac{16}{25}.\end{aligned}$$

So, given that the coin showed heads all four times, the probability that the coin selected was the trick coin is $\frac{16}{25}$.

(ii) $n = 2^k + 1$

By a very similar argument to the one in (i), we can calculate that the probability that the coin selected was the trick coin, given that it shows heads all k times it is flipped, is

$$\frac{1 \times \frac{1}{n}}{(1 \times \frac{1}{n}) + (\frac{1}{2^k} \times (1 - \frac{1}{n}))} = \frac{2^k}{2^k + n - 1}.$$

Setting this to be exactly 50%, we can calculate

$$\begin{aligned} \frac{2^k}{2^k+n-1} &= \frac{1}{2}\\ \Leftrightarrow \quad 2^{k+1} &= 2^k + n - 1\\ \Leftrightarrow \quad 2^k + 1 &= n.\end{aligned}$$

(2) Let X and Y be independent random variables with distributions

x	0	1	2	4
$\Pr(X=x)$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

y	0	1	2
$\Pr(Y=y)$	$\frac{1}{6}$	$\frac{2}{3}$	$\frac{1}{6}$

.

Let $W = 8X + 3Y$ and $Z = XY$.

(i) What is $\mathrm{E}[X]$? **[1]**

(ii) What is $\mathrm{Var}[X]$? **[1]**

(iii) What is $\mathrm{E}[W]$? **[2]**

(iv) Write down a table giving the probability distribution for Z. **[3]**

[Answers only required.]

ANS: (i) $\mathrm{E}[X] = \frac{1}{8}\times 0 + \frac{1}{2}\times 1 + \frac{1}{4}\times 2 + \frac{1}{8}\times 4 = 0 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$.

(ii) $\mathrm{Var}[X] = \frac{1}{8}(0-\frac{3}{2})^2 + \frac{1}{2}(1-\frac{3}{2})^2 + \frac{1}{4}(2-\frac{3}{2})^2 + \frac{1}{8}(4-\frac{3}{2})^2 = \frac{9}{32} + \frac{1}{8} + \frac{1}{16} + \frac{25}{32} = \frac{5}{4}$.

(iii) $\mathrm{E}[W] = \mathrm{E}[8X+3Y] = 8\mathrm{E}[X] + 3E[Y] = 8\times\frac{3}{2} + 3\times 1 = 15$.
In the above, we used linearity of expectation, our answer from (i) and the fact that $\mathrm{E}[Y] = \frac{1}{6}\times 0 + \frac{2}{3}\times 1 + \frac{1}{6}\times 2 = 0 + \frac{2}{3} + \frac{1}{3} = 1$.

(iv)

z	0	1	2	4	8
$\Pr(Z=z)$	$\frac{13}{48}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{48}$

This table can be found as follows. Note that the possible values of Z are $0,1,2,3,4,8$. We work out the probability for each value in turn. In every case, we use the fact that $\Pr(X=x \wedge Y=y) = \Pr(X=x)\Pr(Y=y)$ because X and Y are independent.

$$\Pr(Z=0) = \Pr(X=0 \vee Y=0) = \Pr(X=0) + \Pr(Y=0) - \Pr(X=0 \wedge Y=0)$$
$$= \frac{1}{8} + \frac{1}{6} - \frac{1}{8}\times\frac{1}{6} = \frac{13}{48}.$$
$$\Pr(Z=1) = \Pr(X=1 \wedge Y=1) = \frac{1}{2}\times\frac{2}{3} = \frac{1}{3}.$$
$$\Pr(Z=2) = \Pr(X=1 \wedge Y=2) + \Pr(X=2 \wedge Y=1) = \frac{1}{2}\times\frac{1}{6} + \frac{1}{4}\times\frac{2}{3} = \frac{1}{4}.$$
$$\Pr(Z=4) = \Pr(X=2 \wedge Y=2) + \Pr(X=4 \wedge Y=1) = \frac{1}{4}\times\frac{1}{6} + \frac{1}{8}\times\frac{2}{3} = \frac{1}{8}.$$
$$\Pr(Z=8) = \Pr(X=4 \wedge Y=2) = \frac{1}{8}\times\frac{1}{6} = \frac{1}{48}.$$

(3) One of the 256 subsets of $\{1,2,3,4,5,6,7,8\}$ is chosen uniformly at random. Let X be the number of elements of this subset. Let Y be 0 if the subset is empty and be the least element of the subset otherwise.

(i) Are the events "$X = 1$" and "$Y = 3$" independent? **[2]**

(ii) Are the events "$X = 2$" and "$Y = 6$" independent? **[2]**

(iii) What is $\Pr(X = 2 \mid Y \leq 4)$? **[2]**

[Full justification required.]

ANS: There are $2^8 = 256$ subsets of $\{1,2,3,4,5,6,7,8\}$, so each is chosen with probability $\frac{1}{256}$.

(i) $\Pr(X = 1) = \frac{8}{256} = \frac{1}{32}$ because there are 8 subsets with one element.
$\Pr(Y = 3) = \frac{32}{256} = \frac{1}{8}$ because there are 32 subsets with least element 3 (they are the subsets that can be written $\{3\} \cup A$ for a subset A of $\{4,5,6,7,8\}$.)
So $\Pr(X = 1)\Pr(Y = 3) = \frac{1}{32} \times \frac{1}{8} = \frac{1}{256}$.

$\Pr(X = 1 \wedge Y = 3) = \frac{1}{256}$ because the event "$X = 1 \wedge Y = 3$ means that the subset chosen had one element and that its least element was 3, which means the subset was $\{3\}$.

So the events "$X = 1$" and "$Y = 3$" are independent because $\Pr(X = 1)\Pr(Y = 3) = \Pr(X = 1 \wedge Y = 3)$

(ii) $\Pr(X = 2) = \frac{28}{256} = \frac{7}{64}$ because there are $\binom{8}{2} = 28$ subsets with two elements.
$\Pr(Y = 6) = \frac{1}{64}$ because there are 4 subsets with least element 6 (they are the subsets that can be written $\{6\} \cup A$ for a subset A of $\{7,8\}$.)
So $\Pr(X = 2)\Pr(Y = 6) = \frac{7}{64} \times \frac{1}{64} = \frac{7}{4096}$.

$\Pr(X = 2 \wedge Y = 6) = \frac{2}{256} = \frac{1}{128}$ because the event "$X = 2 \wedge Y = 6$ means that the subset chosen had two elements and that its least element was 6, which means the subset was $\{6,7\}$ or $\{6,8\}$.

So the events "$X = 2$" and "$Y = 6$" are not independent because $\Pr(X = 2)\Pr(Y = 6) \neq \Pr(X = 2 \wedge Y = 6)$

(iii) $\Pr(Y \leq 4) = 1 - \Pr(Y \geq 5) = 1 - \frac{15}{256} = \frac{241}{256}$ because there are $2^4 - 1$ subsets for which $Y \geq 5$ (the nonempty subsets of $\{5,6,7,8\}$).

$\Pr(X = 2 \wedge Y \leq 4) = \frac{28-6}{256} = \frac{11}{128}$ because there are $\binom{8}{2} = 28$ subsets for which $X = 2$ and $\binom{4}{2} = 6$ subsets for which $X = 2$ and $Y \geq 5$ (the subsets of $\{5,6,7,8\}$ with two elements).

So

$$\Pr(X = 2 \mid Y \leq 4) = \frac{\Pr(X = 2 \wedge Y \leq 4)}{\Pr(Y \leq 4)} = \frac{11/128}{241/256} = \frac{22}{241}.$$