MAT1830

Lecture 25: Discrete distributions

In this lecture we'll introduce some of the most common and useful (discrete) probability distributions. These arise in various different real-world situations.

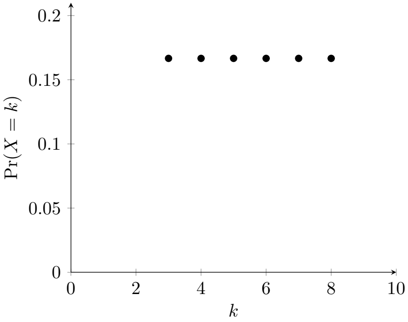
25.1 Discrete uniform distribution

This type of distribution arises when we choose one of a set of consecutive integers so that all choices are equally likely.

The *discrete uniform distribution* with parameters $a, b \in \mathbb{Z}$ ($a \leqslant b$) is given by
$\Pr(X = k) = \frac{1}{b-a+1}$ for $k \in \{a, a+1, \ldots, b\}$.

We have $\mathrm{E}[X] = \frac{a+b}{2}$ and $\mathrm{Var}[X] = \frac{(b-a+1)^2-1}{12}$.

Uniform distribution with $a = 3$, $b = 8$



Question Let X be a number selected uniformly at random from $\{100, 101, \ldots, 200\}$. What are $\mathrm{E}[X]$ and $\mathrm{Var}[X]$?

Answer

X has a discrete uniform distribution with $a = 100$ and $b = 200$. According to our formulas:

$\mathrm{E}[X] = \frac{a+b}{2} = \frac{100+200}{2} = 150$

$\mathrm{Var}[X] = \frac{(b-a+1)^2-1}{12} = \frac{101^2-1}{12} = 850.$

Checking the formulas

$\mathrm{E}[X] = \frac{1}{101} \times (100 + 101 + \cdots + 200) = \frac{1}{101} \times (\frac{101\times300}{2}) = 150.$

$$\begin{aligned}\mathrm{Var}[X] &= \tfrac{1}{101} \times \big((100-150)^2 + (101-150)^2 + \cdots + (200-150)^2\big) \\ &= \tfrac{1}{101} \times 2\big(1^2 + 2^2 + \cdots + 50^2\big) = \tfrac{1}{101} \times \tfrac{2(50\times51\times101)}{6} = 850.\end{aligned}$$

distribution	very rough intuition
discrete uniform	all outcomes equally likely

25.2 Bernoulli distribution

This type of distribution arises when we have a single process that succeeds with probability p and fails otherwise. Such a process is called a *Bernoulli trial*.

The *Bernoulli distribution* with parameter $p \in [0,1]$ is given by

$$\Pr(X = k) = \begin{cases} p & \text{for } k = 1 \\ 1 - p & \text{for } k = 0. \end{cases}$$

We have $\mathrm{E}[X] = p$ and $\mathrm{Var}[X] = p(1-p)$.

Think of a Bernoulli distribution as just a (possibly biased) coin flip.
The coin gives 1 with probability p and gives 0 with probability $1 - p$.

These are pretty boring by themselves but we can build more interesting distributions from them...

distribution	very rough intuition
discrete uniform	all outcomes equally likely
Bernoulli	biased 0/1 coin flip

25.3 Geometric distribution

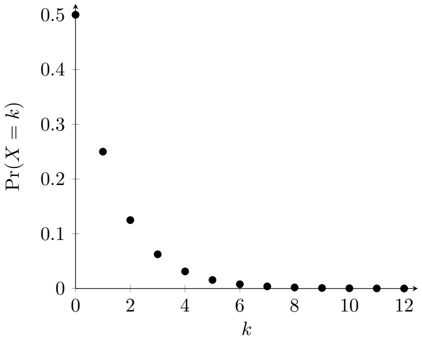
This distribution gives the probability that, in a sequence of independent Bernoulli trials, we see exactly k failures before the first success.

The *geometric distribution* with parameter $p \in [0, 1]$ is given by

$$\Pr(X = k) = p(1 - p)^k \text{ for } k \in \mathbb{N}.$$

We have $\mathrm{E}[X] = \frac{1-p}{p}$ and $\mathrm{Var}[X] = \frac{1-p}{p^2}$.

Geometric distribution with $p = 0.5$



Example. If every minute there is a 1% chance that your internet connection cuts out then the probability of staying online for exactly x consecutive minutes is approximated by a geometric distribution with $p = 0.01$. It follows that the expected value is $\frac{1-0.01}{0.01} = 99$ minutes and the variance is $\frac{1-0.01}{(0.01)^2} = 9900$.

Question

25.1 There is a 95% chance of a packet being received after being sent down a noisy line, and the packet is resent until it is received. What is the probability that the packet is received within the first three attempts?

Answer

Each time the packet is sent is an independent Bernoulli trial, with probability of successful transmission $p = 0.95$.

Let X be a random variable whose value is the number of times transmission fails. The probability distribution of X is geometric with parameter $p = 19/20$.

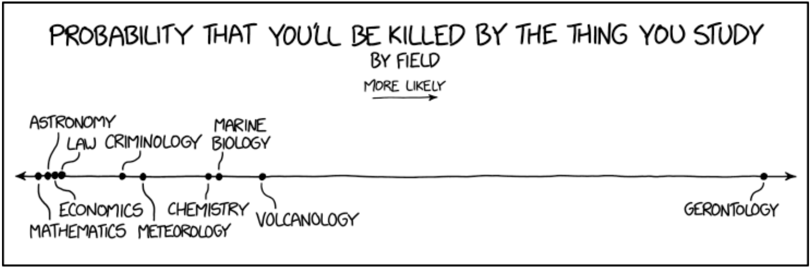
(Note this means $\mathrm{E}[X] = (1 - p)/p = 1/19$ so we don't expect many failures.)

We seek

$$\begin{aligned} \Pr(X \leq 2) = & \Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2) \\ & = p(1-p)^0 + p(1-p)^1 + p(1-p)^2 = \tfrac{19}{20}\left(1 + \tfrac{1}{20} + \left(\tfrac{1}{20}\right)^2\right) = \tfrac{7999}{8000}. \end{aligned}$$

We could also have answered this by $\Pr(X \leq 2) = 1 - \Pr(X \geq 3)$. The probability of 3 failures is $(1 - p)^3 = (1/20)^3 = 1/8000$.

distribution	very rough intuition
discrete uniform	all outcomes equally likely
Bernoulli	biased 0/1 coin flip
geometric	number of ‘losses’ before first ‘win’


PROBABILITY THAT YOU'LL BE KILLED BY THE THING YOU STUDY
BY FIELD
MORE LIKELY
ASTRONOMY
LAW
CRIMINOLOGY
MARINE BIOLOGY
MATHEMATICS
ECONOMICS
METEOROLOGY
CHEMISTRY
VOLCANOLOGY
GERONTOLOGY

25.4 Binomial distribution

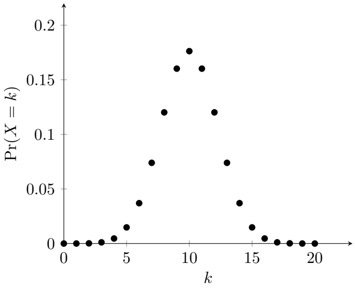
This distribution gives the probability that, in a sequence of n independent Bernoulli trials, we see exactly k successes.

The *binomial distribution* with parameters $n \in \mathbb{Z}^+$ and $p \in [0, 1]$ is given by

$$\Pr(X = k) = \tbinom{n}{k} p^k (1 - p)^{n-k} \quad \text{for } k \in \{0, \ldots, n\}.$$

We have $\mathrm{E}[X] = np$ and $\mathrm{Var}[X] = np(1 - p)$.

Binomial distribution with $n = 20$, $p = 0.5$



If X in binomially distributed with parameters n and p, then $\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k \in \{0, \ldots, n\}$. **Why?**

Let X be the number of 6s rolled on five standard dice. Then $\Pr(X = 3) = \binom{5}{3} (\frac{1}{6})^3 (\frac{5}{6})^2$.

$(\frac{1}{6})^3 (\frac{5}{6})^2$ is the probability of rolling 666LL in that order (where L stands for "less than 6").

But we could also roll three 6s as 66L6L, 66LL6, 6L66L, 6L6L6, 6LL66, L666L, L66L6, L6L66, LL666.

There are $\binom{5}{3}$ possibilities in total and so $\Pr(X = 3) = \binom{5}{3} (\frac{1}{6})^3 (\frac{5}{6})^2$.

Running the same argument in general shows that $\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$ for $k \in \{0, \ldots, n\}$.

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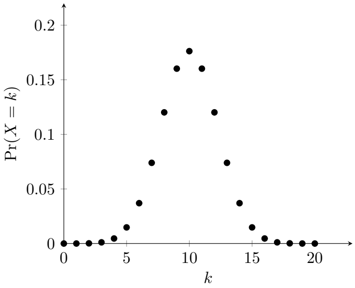


Example. If 1000 people search a term on a certain day and each of them has a 10% chance of clicking a sponsored link, then the number of clicks on that link is approximated by a binomial distribution with $n = 1000$ and $p = 0.1$. It follows that the expected value is $1000 \times 0.1 = 100$ clicks and the variance is $1000 \times 0.1 \times 0.9 = 90$.

Questions

25.2 A factory aims to have at most 2% of the components it makes be faulty. What is the probability of a quality control test of 20 random components finding that 2 or more are faulty, if the factory is exactly meeting its 2% target?

Each component fails or succeeds independently of the result of testing other components, so we have 20 independent Bernoulli trials. Together these combine to give a binomial distribution. Let X be the random variable that counts how many of the $n = 20$ components fail their test.

Each component fails its test with a probability $p = 0.02$. We want to know $\Pr(X \geq 2)$. We can find this using $\Pr(X \geq 2) = 1 - \Pr(X \leq 1)$.

$$\begin{aligned}\Pr(X \leq 1) &= \Pr(X = 0) + \Pr(X = 1) = \binom{20}{0} p^0 (1-p)^{20} + \binom{20}{1} p^1 (1-p)^{19} \\ &= (0.98)^{20} + 20(0.02)(0.98)^{19} \approx 0.94.\end{aligned}$$

Hence $\Pr(X \geq 2) = 1 - \Pr(X \leq 1) \approx 0.06$.

distribution	very rough intuition
discrete uniform	all outcomes equally likely
Bernoulli	biased 0/1 coin flip
geometric	number of ‘losses’ before first ‘win’
binomial	number of ‘wins’ from a fixed number of tries

Flux Exercise (LQMTZZ)

Which of the following distributions is the best model for the number spun on a roulette wheel?

A. Discrete uniform distribution

B. Bernoulli distribution

C. Geometric distribution

D. Binomial distribution

Answer A.

Unless it's a really dodgy casino the numbers should be equally likely.

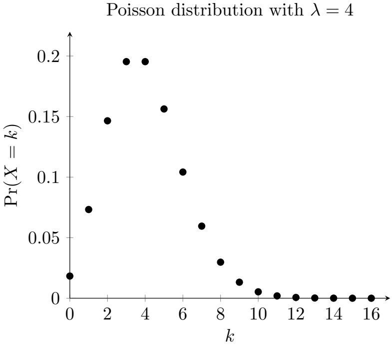
25.5 Poisson distribution

In many situations where we know that an average of λ events occur per time period, this distribution gives a good model of the probability that k events occur in a time period.

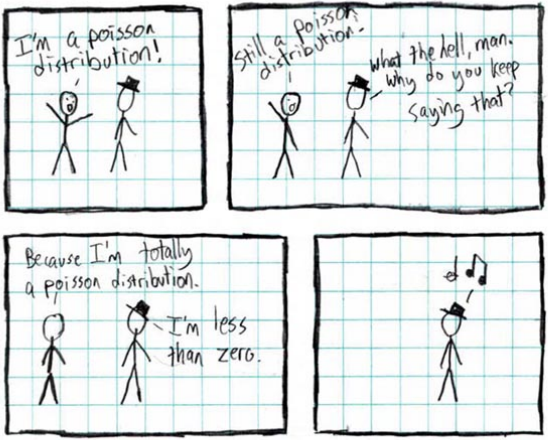
The *Poisson distribution* with parameter $\lambda \in \mathbb{R}$ (where $\lambda > 0$) is given by

$$\Pr(X = k) = \frac{\lambda^k e^{-\lambda}}{k!} \text{ for } k \in \mathbb{N}.$$

We have $\mathrm{E}[X] = \lambda$ and $\mathrm{Var}[X] = \lambda$.



Example. If a call centre usually receives 6 calls per minute, then a Poisson distribution with $\lambda = 6$ approximates the probability it receives k calls in a certain minute. It follows that the expected value is 6 calls and the variance is 6.


I'm a poisson distribution!
Still a poisson distribution.
What the hell, man. Why do you keep saying that?
Because I'm totally a poisson distribution.
I'm less than zero.

Question

25.3 The number of times a machine needs adjusting during a day approximates a Poisson distribution, and on average the machine needs to be adjusted three times per day. What is the probability it does not need adjusting on a particular day?

Let X be a random variable that counts the number of adjustments required during a day. So we want to know $\Pr(X = 0)$.

We are told that X has a Poisson distribution and that $\mathrm{E}[X] = 3$. Hence the parameter $\lambda = 3$.

Then we know that $\Pr(X = 0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-\lambda} = e^{-3} \approx 0.05$.

Important fact: If the number of times something happens in a time interval follows a Poisson distribution (and this continues to hold) then the number of times it happens in a different time interval will also follow a Poisson distribution.

But, of course, the values of λ will be different.

Example The number of requests to a server in any hour of a day follows a Poisson distribution with $\lambda = 100$.

The number of requests to the server over the 24 hours will follow a Poisson distribution with $\lambda = 24 \times 100 = 2400$.

The number of requests to the server in any minute of the day will follow a Poisson distribution with $\lambda = \frac{100}{60} = \frac{5}{3}$.

etc. etc.

distribution	very rough intuition
discrete uniform	all outcomes equally likely
Bernoulli	biased 0/1 coin flip
geometric	number of ‘losses’ before first ‘win’
binomial	number of ‘wins’ from a fixed number of tries
Poisson	(often) number of events in a fixed time period

Flux Exercise (LQMTZZ)

Which of the following distributions is the best model for the number of babies born in Melbourne during this lecture?

A. Discrete uniform distribution

B. Bernoulli distribution

C. Geometric distribution

D. Binomial distribution

E. Poisson distribution

Answer E.

The Poisson distribution models random arrivals.