MAT1830

Lecture 4: Logic

Logic - why should you care?

Logic is at the heart of maths and computer science.

In computer science, it finds applications in:

- digital circuit design (logic gates)
- programming (Boolean variables, Prolog, ASP)
- artificial intelligence
- software engineering (specification and verification)
- theory of computation

More generally, logic is useful for reasoning through things in formal settings and in everyday life.

The simplest and most commonly used part of logic is the logic of “and”, “or” and “not”, which is known as *propositional logic*.

A proposition is any sentence which has a definite truth value (true= T or false= F), such as

$$1 + 1 = 2, \text{ or}$$

$$\text{11 is a prime number.}$$

but not

What is your name? or

This sentence is false.

Propositions are denoted by letters such as $p, q, r, \ldots$, and they are combined into compound propositions by *connectives* such as $\wedge$ (and), $\vee$ (or) and $\neg$ (not).

Question 4.1 Which of the following are propositions?

$1 + 1 = 3$	Yes	(false)
$1 + 1$	No	
3 divides 9	Yes	(true)
$3 \div 7$	No	

4.1 Connectives $\wedge, \vee$ and $\neg$

$\wedge, \vee$ and $\neg$ are called "connectives" because they can be used to connect two sentences p and q into one. These particular connectives are defined so that they agree with the most common interpretations of the words "and", "or" and "not."

To define $p \wedge q$, for example, we only have to say that $p \wedge q$ is true only when p is true and q is true.

We define $\wedge$ by the following *truth table:*

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Similarly, $p \vee q$ is true when p is true or q is true, but now we have to be more precise, because "or" has at least two meanings in ordinary speech.

We define $\vee$ by the truth table

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

This is the inclusive sense of "p or q" (often written "p and/or q" and meaning at least one of p, q is true).

Finally, "not" $\neg$ (also called negation) is defined as follows.

We define $\neg$ by the truth table

p	$\neg p$
T	F
F	T

The connectives $\wedge$, $\vee$ and $\neg$, are functions of the propositional variables p and q, which can take the two values T and F. For this reason, $\wedge$, $\vee$ and $\neg$ are also called *truth functions*.

Notation

$\wedge$	"and"
$\vee$	"or" (inclusive)
$\neg$	"not"

Order of precedence

"$\neg$" has precedence over the other connectives.

For example, $\neg p \vee q$ means $(\neg p) \vee q$.

For other connectives we'll always use brackets to make the meaning clear.

Example Find the truth tables for $p \wedge (q \vee r)$ and $(p \wedge q) \vee (p \wedge r)$.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Flux Exercise (LQMTZZ)

What must the truth values of p and q be to make $p \wedge (\neg p \vee \neg q)$ true?

A. p must be T, q must be T
B. p must be T, q must be F
C. p must be F, q must be T
D. p must be F, q must be F

Answer:

p must be T because otherwise the expression would evaluate to F.
So the expression is then $\mathrm{T} \wedge (\mathrm{F} \vee \neg q)$.
So to make the bracketed part evaluate to T we need $\neg q$ to be T.
Or in other words we need q to be F.

So B. (Or you can draw a truth table.)

4.2 Implication

Another important truth function is $p \to q$, which corresponds to "if p then q" or "p implies q" in ordinary speech.

In ordinary speech the value of $p \to q$ depends only on what happens when p is true. For example to decide whether

$$\text{MCG flooded} \to \text{the cricket is off}$$

it is enough to see what happens when the MCG is flooded. *Thus we agree that $p \to q$ is true when p is false.*

We define $\to$ by the truth table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Notation

$\wedge$	"and"
$\vee$	"or" (inclusive)
$\neg$	"not"
$\rightarrow$	"implies"

Why do F $\rightarrow$ F and F $\rightarrow$ T evaluate to T?

"For all integers $x \geq 3$, if x is prime then x is odd."

For $x = 3$ we have T $\rightarrow$ T
For $x = 4$ we have F $\rightarrow$ F
For $x = 9$ we have F $\rightarrow$ T

So if we want the above statement to be true we have to live with F $\rightarrow$ F and F $\rightarrow$ T evaluating to T.

But it's not a perfect representation of regular English:

"If I am a mouse then I chase cats."

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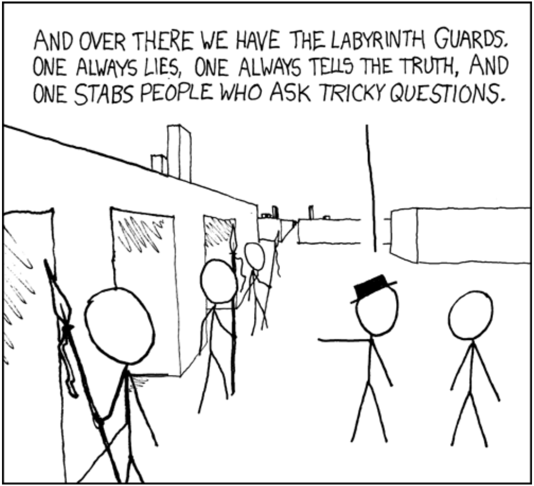
We define $\to$ by the truth table

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Question 4.2

f: "foo"
b: "bar"

"if foo, then bar"	$f \rightarrow b$
"bar if foo"	$f \rightarrow b$
"bar only if foo"	$b \rightarrow f$
"foo implies not bar"	$f \rightarrow \neg b$
"foo is sufficient for bar"	$f \rightarrow b$
"foo is necessary for bar"	$b \rightarrow f$


AND OVER THERE WE HAVE THE LABYRINTH GUARDS. ONE ALWAYS LIES, ONE ALWAYS TELLS THE TRUTH, AND ONE STABS PEOPLE WHO ASK TRICKY QUESTIONS.

4.3 Other connectives

Two other important connectives are $\leftrightarrow$ ("if and only if") and $\veebar$ ("exclusive or").

The sentence $p \leftrightarrow q$ is true exactly when the truth values of p and q agree.

We define $\leftrightarrow$ by the truth table

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

We could also write $p \leftrightarrow q$ as $(p \to q) \wedge (q \to p)$. We'll see how to prove this in the next lecture.

The sentence $p \veebar q$ is true exactly when the truth values of p and q disagree.

We define $\veebar$ by the truth table

p	q	$p \veebar q$
T	T	F
T	F	T
F	T	T
F	F	F

$\wedge$	"and"
$\vee$	"or" (inclusive)
$\neg$	"not"
$\rightarrow$	"implies"
$\leftrightarrow$	"if and only if"
$\underline{\vee}$	"exclusive or"

Question 4.3

"Would you like coffee or tea?"	exclusive
"Oranges or lemons are a good source of vitamin C."	inclusive
"He will arrive in a minute or two."	exclusive

Flux Exercise (LQMTZZ)

What must the truth values of p and q be to make $q \to (p \veebar q)$ false?

A. p must be T, q must be T

B. p must be T, q must be F

C. p must be F, q must be T

D. p must be F, q must be F

Answer:

For $q \to (p \veebar q)$ to be F we need q to be T and $p \veebar q$ to be F.
So q must be T and $p \veebar \mathrm{T}$ must be F.
This means that p must be T (checking both cases for p).

So A. (Or you can draw a truth table.)

4.4 Remarks

1. The symbols $\wedge$ and $\vee$ are intentionally similar to the symbols $\cap$ and $\cup$ for set intersection and union because

$$\begin{aligned} x \in A \cap B \quad &\Leftrightarrow \quad (x \in A) \wedge (x \in B) \\ x \in A \cup B \quad &\Leftrightarrow \quad (x \in A) \vee (x \in B) \end{aligned}$$

(We study sets later.)

2. The "exclusive or" function $\underline{\vee}$ is written XOR in some programming languages.

3. If we write 0 for F and 1 for T then $\underline{\vee}$ becomes the function

p	q	$p \underline{\vee} q$
1	1	0
1	0	1
0	1	1
0	0	0

This is also known as the "mod 2 sum", because $1 + 1 = 2 \equiv 0 \pmod 2$. (It could also be called the "mod 2 difference" because $a + b$ is the same $a - b$, mod 2).

4. The mod 2 sum occurs in many homes where two switches p, q control the same light. The truth value of $p \underline{\vee} q$ tells whether the light is on or not, and the light can be switched to the opposite state by switching the value of either p or q.