MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #3 and Additional Practice Questions

Tutorial Questions

1. Simplify the following statements so that the quantifiers appear before "$\neg$"s.
$\neg\forall x P(x)$
$\neg\exists x\exists y\neg Q(x,y)$
$\neg(\exists x P(x) \vee \exists x\forall y Q(x,y))$

2. Prove using simple induction that $n^2 + 3n$ is even for each integer $n \geq 1$.

3. Let x range over the cupcakes on a tray.
Let $I(x)$ be the predicate "x has pink icing."
Let $S(x)$ be the predicate "x has green sprinkles."

   (a) Say the tray contains the following four cupcakes:
   - pink icing with brown sprinkles,
   - pink icing with green sprinkles,
   - yellow icing with green sprinkles,
   - yellow icing with purple sprinkles.

   Which of the following are true and which are false?
   $\exists x(I(x) \wedge \neg S(x))$
   $\forall x(S(x) \rightarrow I(x))$
   $\exists x I(x) \vee \forall x\neg S(x)$

   (b) Can you invent a tray of cupcakes for which $\forall x(I(x) \vee S(x))$ is true and $\forall x I(x) \vee \forall x S(x)$ is false?

   (c) Can you invent a tray of cupcakes for which $\forall x I(x) \vee \forall x S(x)$ is true and $\forall x(I(x) \vee S(x))$ is false?

4. Prove using simple induction that, for each integer $n \geq 1$,

$$1 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

5. (a) An interior decorator claims that he has one specific type of cushion that will match any sofa. Write his claim in predicate logic and find its negation. What would you have to do to prove the decorator wrong?

   (b) The decorator makes a different claim. He says that for any sofa you might own, he can find a cushion to match. Write his claim in predicate logic and find its negation. What would you have to do to prove the decorator wrong?

(See over for practice questions.)

Practice Questions

1. (a) Consider the following pseudo code.

   for x in $customers$ **do**
       set $flag = 1$
       for y in $products$ **do**
           if not $customerBoughtProduct(x, y)$ **then**
               set $flag = 0$
           end if
       end do
       if $flag = 1$ **then**
           print x
       end if
   end do

   Let x range over all customers, y range over all products and $B(x, y)$ represent "customer x bought product y". Write a logical sentence which is true exactly when the code above will print output.

   (b) Can you write similar code that will produce output exactly when $\exists y \forall x \neg B(x, y)$ is true?

2. Lucia and Rohit are midway through a game of chess - it's Rohit's turn to move. Lucia says that no matter how Rohit plays from now on, she can win the game in exactly three moves (of hers). How would you represent Lucia's claim in predicate logic?

3. The following claims to be a proof that all jelly beans are the same colour. Find what's wrong with it.

   We will prove by induction on n that, for every positive integer n, any group of n jelly beans are all the same colour.

   Base step. When $n = 1$, any one jelly bean is obviously the same colour as itself.

   Induction step. For some integer $k \geq 1$, assume that the result is true for $n = k$ that is, that any group of k jelly beans are all the same colour. Now we need to prove that any group of $k + 1$ jelly beans are all the same colour. Place any group of $k + 1$ jelly beans in a line. Using our assumption, the first k jelly beans are all the same colour and, also, the last k jelly beans are all the same colour. This means that all $k + 1$ jelly beans are the same colour.

   So we have proved by induction on n that, for every positive integer n, any group of n jelly beans are all the same colour.

4. Intuitively, a mathematical function f on the real numbers is continuous at a point c if its graph does not have a "hole" or "jump" at c. The following is often used as the formal definition:

   f is continuous at c if and only if for each real number $\epsilon > 0$ there is a real number $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \epsilon$.

   (a) Can you translate this definition into logic?

   (b) Find the negation of your definition.

   (c) What would you have to do to show that a function f was not continuous at c?

   (d) Can you explain why the formal definition of continuity means there cannot be a "hole" or "jump" at c in the function's graph?