MAT1830 - Discrete Mathematics for Computer Science
Tutorial Sheet #8 and Additional Practice Questions

1. The sample space is $\{\mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}, \mathrm{THH}, \mathrm{THT}, \mathrm{TTH}, \mathrm{TTT}\}$ (where HTH means heads on the first flip, tails on the second, heads on the third, and so on). Each of these outcomes occurs with probability $(\frac{1}{2})^3$ because the three flips are independent.

   $X = 0$ if the outcome is TTT.

   $X = 1$ if the outcome is in $\{\mathrm{HHT}, \mathrm{HTH}, \mathrm{THH}\}$.

   $X = 2$ if the outcome is in $\{\mathrm{HTT}, \mathrm{THT}, \mathrm{TTH}\}$.

   $X = 3$ if the outcome is HHH.

   Thus the probability distribution of X is given by

x	0	1	2	3
$\Pr(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

.

2. (a) Without any further information, the best the doctor can answer is to say that about one in every three pairs of twins worldwide is a pair of identical twins and hence the probability is about $\frac{3}{10}$.

   (b) Let I be the event the twins are identical and M be the event they're both male.
   $\Pr(I) = \frac{3}{10}$ from the question.
   $\Pr(M|I) = \frac{1}{2}$ from the question.
   $\Pr(M|\overline{I}) = \frac{1}{4}$ from the question.
   By Bayes' theorem,

$$\begin{aligned}\Pr(I|M) &= \frac{\Pr(M|I)\Pr(I)}{\Pr(M|I)\Pr(I) + \Pr(M|\overline{I})\Pr(\overline{I})} \\ &= \frac{\frac{1}{2}\times\frac{3}{10}}{(\frac{1}{2}\times\frac{3}{10}) + (\frac{1}{4}\times(1-\frac{3}{10}))} \\ &= \frac{6}{13}.\end{aligned}$$

   So the doctor can say that the probability is about $\frac{6}{13}$.

3. (a) Using the definition of expected value,

$$\mathrm{E}[X] = \tfrac{1}{2} \times 0 + \tfrac{1}{3} \times 1 + \tfrac{1}{6} \times 2 = \tfrac{2}{3}.$$

Now, using $\mathrm{E}[X] = \frac{2}{3}$,

$$\mathrm{Var}[X] = \tfrac{1}{2} \times (0 - \tfrac{2}{3})^2 + \tfrac{1}{3} \times (1 - \tfrac{2}{3})^2 + \tfrac{1}{6} \times (2 - \tfrac{2}{3})^2 = \tfrac{2}{9} + \tfrac{1}{27} + \tfrac{8}{27} = \tfrac{5}{9}.$$

(b) Because $E[Y] = 2$ we have

$$2 = E[Y] = p \times 0 + \tfrac{1}{12} \times 1 + \tfrac{1}{3} \times 2 + q \times 3 = \tfrac{3}{4} + 3q.$$

Solving $2 = \frac{3}{4} + 3q$, we see $q = \frac{5}{12}$.
Then, because $p + \frac{1}{12} + \frac{1}{3} + q = 1$, we have that $p = \frac{1}{6}$.

4. (a) $\Pr(X = 8) = \frac{1}{256}$ because exactly when the string is 11111111.
$\Pr(Y = 8) = \frac{1}{256}$ because exactly when the string is 00000000.
$\Pr(X = 8 \wedge Y = 8) = 0$ because there is no binary string of length 8 with 8 0s and 8 1s.
Thus $\Pr(X = 8 \wedge Y = 8) \neq \Pr(X = 8)\Pr(Y = 8)$ and so X and Y are not independent.
(There are many other examples that will show this, as well).

(b) Because the string has length 8, $Z = X + Y$ is always 8. So the probability distribution of Z is given by

z	8
$\Pr(Z = z)$	1

.

5. (a) One example is a random variable X with probability distribution given by

x	0	1
$\Pr(X = x)$	$\frac{1}{2}$	$\frac{1}{2}$

Then $E[X] = \frac{1}{2}$, but $\Pr(X = \frac{1}{2}) = 0$.

(b) One example is a random variable Y with probability distribution given by

y	-1000000	1
$\Pr(Y = y)$	$\frac{1}{1000}$	$\frac{999}{1000}$

Then $E[Y] = \frac{1}{1000} \times -1000000 + \frac{999}{1000} \times 1 = -999.001$, but $\Pr(Y > 0) = \frac{999}{1000}$.

(c) The best you'll manage is $\frac{1}{3}$ (google "Markov's inequality" for why). One example would be

z	0	3
$\Pr(Z = z)$	$\frac{2}{3}$	$\frac{1}{3}$

Then $E[Z] = \frac{1}{3} \times 3 = 1$, and $\Pr(Z \geq 3\mathrm{E}[Z]) = \Pr(Z \geq 3) = \frac{1}{3}$.