MAT1830 - Discrete Mathematics for Computer Science
Assignment #4 Solutions

(1) Let $P(n)$ be the statement "2^n divides a_n".

Base steps. $a_1 = 4$ and $2^1 = 2$ divides 4, so $P(1)$ is true.
$a_2 = 12$ and $2^2 = 4$ divides 12, so $P(2)$ is true. **[2]**

Induction step. For some integer $k \geq 2$, assume that $P(1), P(2), \ldots, P(k)$ are true. We need to prove that $P(k+1)$ is true, that is, that 2^{k+1} divides a_{k+1}.

Because $P(k)$ and $P(k-1)$ are true 2^k divides a_k and 2^{k-1} divides a_{k-1}. So we know that $a_k = r2^k$ and $a_{k-1} = s2^{k-1}$ for some integers r and s. **[2]**

Now,

$$\begin{aligned} a_{k+1} &= 10a_k - 12a_{k-1} && \text{by the definition of } a_{k+1} \\ &= 10r2^k - 12s2^{k-1} && \text{by } P(k) \text{ and } P(k-1) \\ &= 5r2^{k+1} - 3s2^{k+1} \\ &= 2^{k+1}(5r - 3s). \end{aligned}$$ **[2]**

Note that $5r - 3s$ is an integer because r and s are integers.
So 2^{k+1} divides a_{k+1} and thus $P(k+1)$ is true.

So we have proved by induction that $P(n)$ is true for each integer $n \geq 1$. **[2]**

(2) (i) $\{-6, -5, 3, 4, 5\}$ **[1]**

(ii) $\{x : x \in \mathbb{Z} \text{ and } x \leq -4\}$ **[1]**

(iii) $\{x : x \in \mathbb{Z} \text{ and either } x \leq -7 \text{ or } x = -4 \text{ or } x = -3 \text{ or } x \geq 6\}$ **[2]**

(iv) $\{\{-6, -5, 3\}, \{5\}, \{\}\}$ **[1]**

(v) $|S \cap T| = |\{3, 4, 5\}| = 3$. So $|\mathcal{P}(S \cap T)| = 2^3 = 8$. So $|\mathcal{P}(\mathcal{P}(S \cap T))| = 2^8 = 256$. **[2]**

(3) (i) Yes. Let A, B and C be any sets. Now

$$\begin{aligned} (x, y) \in (A \cup B) \times C &\equiv x \in A \cup B \wedge y \in C \\ &\equiv (x \in A \vee x \in B) \wedge y \in C \\ &\equiv (x \in A \wedge y \in C) \vee (x \in B \wedge y \in C) \\ &\equiv (x, y) \in A \times C \vee (x, y) \in B \times C \\ &\equiv (x, y) \in (A \times C) \cup (B \times C). \end{aligned}$$

So $(x, y) \in (A \cup B) \times C$ is logically equivalent to $(x, y) \in (A \times C) \cup (B \times C)$ and it follows that $(A \cup B) \times C = (A \times C) \cup (B \times C)$. **[3]**

(ii) No. For example, let $A = \{1\}$ and $B = \{2\}$. Then

$$\begin{aligned} \mathcal{P}(A) \triangle \mathcal{P}(B) &= \{\emptyset, \{1\}\} \triangle \{\emptyset, \{2\}\} = \{\{1\}, \{2\}\} \\ \mathcal{P}(A \triangle B) &= \mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}. \end{aligned}$$ **[2]**