MTH1030/35: Assignment 1, 2022 Solutions

Obsessing about cubes

In general, if no serious attempt at solving one of the questions was made assign 0 marks to that question.

Overall, we are marking generously :)

Solution 1 a) (15 marks) Award either 0, 5, 10, or 15 marks.

- **15 marks: Essentially complete solution, well explained, ideally using full English sentences. (Generous interpretation of "complete" applies.)**
- **10 marks: Some shortcomings in the proof and/or exposition.**
- **5 marks: Something but not much.**
- **0 marks: Otherwise.**

$\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ span the parallelogram. Therefore the square of the area of this parallelogram is equal to

$$p^2 = (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{u} \times \mathbf{v}) = (u_1v_2 - u_2v_1)^2 + (u_3v_1 - u_1v_3)^2 + (u_2v_3 - u_3v_2)^2$$

On the other hand, the area m of the orthogonal projection of this parallelogram onto the xy-plane squared is

$$((u_1, u_2, 0)\times(v_1, v_2, 0))\cdot((u_1, u_2, 0)\times(v_1, v_2, 0)) = (0, 0, u_1v_2-u_2v_1)\cdot(0, 0, u_1v_2-u_2v_1) = (u_1v_2-u_2v_1)^2$$

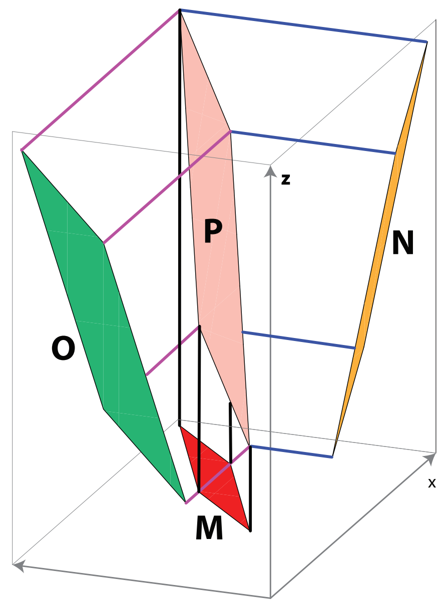
Do the same for n and o and you find that

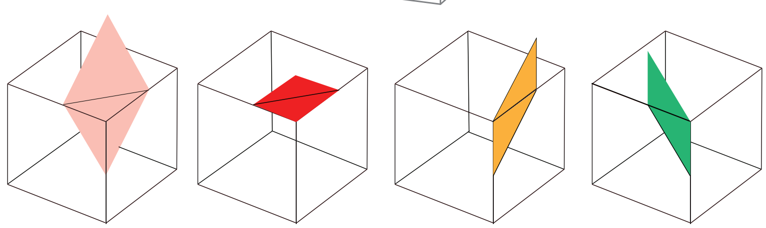
$$m^2 + n^2 + o^2 = p^2.$$

Solution 1 b) (7 marks) Award either 0, 2, 4, or 7 marks.

- **7 marks: Essentially complete solution, well explained, ideally using full English sentences.**
- **5 marks: Some shortcomings in the proof and/or exposition.**
- **2 marks: Something but not much.**
- **0 marks: Otherwise.**

The triangles in question can be interpreted as halves of the parallelograms in question 1a), as shown in the following diagram:





Then, since $m^2 + n^2 + o^2 = p^2$ we also have

$$(m/2)^2 + (n/2)^2 + (o/2)^2 = (p/2)^2 = a^2 + b^2 + c^2 = d^2.$$

If students see this connection, then it's pretty much automatically 7 marks. However, often students don't see the connection and do things from scratch, usually by having the cut vertices on the edges of the cube be $(x, 0, 0)$, $(0, y, 0)$ and $(0, 0, z)$ and then expressing all the areas in terms of x, y and z, leading to pretty much exactly the same proof as in 1 a). In this case, the more detailed marking scheme applies.

Solution 1 c) (3 marks) Just a matter of plugging in the numbers, and so binary marking: 3 marks or nothing.

Letting $m = 3$, $n = 4$ and $o = 2.5$ and using 3d Pythagoras we have

$$(mn/2)^2 + (mo/2)^2 + (no/2)^2 = d^2$$

Therefore $d = 8.66\ m^2$.

Feedback on this set of problems in the form: a+b+c. E.g. full marks:15+7+3=25 marks.

Solution 2

- **Start by giving everybody who did any work on this question 50 marks.**
- **Go straight to the list of results at the end and deduct 2 marks for every incorrect answer. To be counted as correct an answer needs to be correct to at least two decimal places (e.g. 12.56743 m ≈12.57 m). It's okay for the volume to only be correct to one decimal place.**
- **Three different outcomes in terms of quality of explanations: a further 0, 5, or 10 marks deduction. 0 marks deduction if essentially self-contained, complete, easy to follow, well-presented (does not necessarily have to be mathematically correct to get no deductions, this is about how much effort somebody put into explaining what they are doing); 5 marks deduction if any of those four qualifications is definitely not given. 10 marks deduction otherwise.**
- **feedback in the form: 50 - deductions for incorrect answers - deductions for presentation, e.g. 50-4-4=42.**
- **Some free marks in this scenario. I am okay with that.**

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$... be the vectors with the same coordinates as $A, B, C, \ldots$.

First we figure out an equation for the plane that the top square of the cube is contained in. Obviously,

$$\mathbf{n} = \mathbf{a} - \mathbf{a}' = (0.79535, 2.11918, 7.72383)$$

is a normal vector of this plane.

This means that one possible equation of the plane is

$$0.79535x + 2.11918y + 7.72383z = rightside.$$

Plugging in the coordinates of A gives *rightside* (this is the same as calculating the dot product of $\mathbf{a}$ and the normal vector $\mathbf{n}$):

$$0.79535x + 2.11918y + 7.72383z = 157.428.$$

Now we consider the line through B' in the direction of the normal vector. Then B is the point of intersection of this line with the plane. The no-brains-required line equation is

$$\mathbf{b}' + t\mathbf{n} = (25.7441 + 0.79535t, 19.8898 + 2.11918t, 7.72383t)$$

Plugging the three components of the line equation into the plane equation and solving for t gives the t at which the line intersects the plane:

$$t = 1.46343.$$

Plugging this t into the line equation gives the point B.

$$B = (26.908, 22.991, 11.3033).$$

Now that we know A and B, we can calculate the side length of the cube

$$sidelength = |\mathbf{a} - \mathbf{b}| = 18 \text{ metres}.$$

We can navigate to D via A using a vector that is 18 (metres) long and perpendicular to both the normal vector and $\mathbf{a}-\mathbf{b}$. There are two such vectors, one the negative of the other. We want the one with positive z-coordinate because the z-coordinate of D is greater than that of A. We calculate one of these two 18 long vectors using the cross product.

$$18\frac{(\mathbf{a}-\mathbf{b}) \times \mathbf{n}}{|(\mathbf{a}-\mathbf{b}) \times \mathbf{n}|} = (7.62449, -15.9077, 3.57945).$$

The z-coordinate of this vector is positive and so this is actually the vector we are looking for.

$$\textit{point-to-d-vector} = (7.62449, -15.9077, 3.57945).$$

And so

$$D = \mathbf{a} + \textit{point-to-d-vector} = (50.7407, 14.0464, 11.3033).$$

Using the same vector we can navigate from B to the last missing corner C at the top.

$$C = \mathbf{b} + \textit{point-to-d-vector} = (34.5325, 7.08334, 14.8827).$$

Now, we figure out the coordinates of the points C'. Here the point C' is simply the point of intersection of the line in the direction of the normal vector through C with the xy-plane ($z = 0$). A possible line equation is

$$\mathbf{c} + t\mathbf{n} = (34.5325 + 0.79535t, 7.08335 + 2.11918t, 14.8827 + 7.72383t).$$

This means the line intersects the ground $z = 0$ for

$$t = -1.92685.$$

Substituting this value for t back into the line equation then gives

$$C' = (33, 3, 0).$$

Using the same setup we calculate that

$$D' = (18.463, 8.73466, 0).$$

Now that we know $A, B, C, D, A', B', C', D'$, it's easy to calculate $|AA'| = 8.04867\, m$, $|BB'| = 11.7787\, m$, $|CC'| = 15.5086\, m$, $|DD'| = 11.7787\, m$.

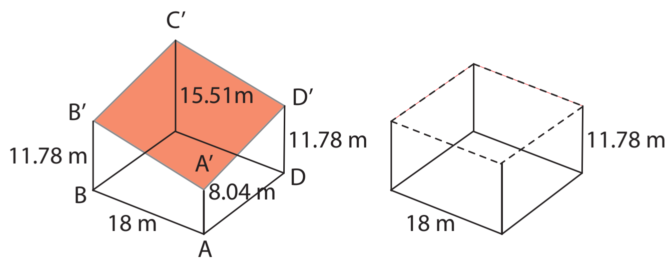


Figure 1: The sunken cube at the Melbourne Museum (drawn upside down).

Note that because B and D are at equal height above the ground, the quadrilateral floor $A'B'C'D'$ is a parallelogram. We calculate its area using the cross-product:

$$area(A'B'C'D') = |(\mathbf{b}' - \mathbf{a}') \times (\mathbf{b}' - \mathbf{c}')| = 337.626\ m^2.$$

To calculate the volume of the part of the cube situated above ground, consider the following diagram.

Since BB' and DD' are of equal lengths the volume in question is equal to the volume of the box on the right, that is $18 \cdot 18 \cdot 11.7787 = 3816.28m^3$.

$$A = (43.1162, 29.9541, 7.72383)$$
$$B = (26.908, 22.991, 11.3033)$$
$$C = (34.5325, 7.08335, 14.8827)$$
$$D = (50.7407, 14.0464, 11.3033)$$
$$A' = (42.3208, 27.8349, 0)$$
$$B' = (25.7441, 19.8898, 0)$$
$$C' = (33, 3, 0)$$
$$D' = (49.5767, 10.9451, 0)$$
$$|AA'| = 8.04867m$$
$$|BB'| = 11.7787m$$
$$|CC'| = 15.5086m$$
$$|DD'| = 11.7787m$$
$$sidelength = 18m$$
$$area(A'B'C'D') = 337.626m^2$$
$$volume\ = 3816.28m^3$$

Solution 3 a) 15 marks. Award either 0, 5, 10, or 15 marks.

- **15 marks: Essentially complete solution, well explained, ideally using full English sentences.**
- **10 marks: Some shortcomings in the proof and/or exposition.**
- **5 marks: Something but not much.**
- **0 marks: Hardly anything worth giving marks for.**

The projections of the three vectors onto the xy-plane are $\mathbf{u}' = (u_1, u_2, 0)$, $\mathbf{v}' = (v_1, v_2, 0)$, and $w' = (w_1, w_2, 0)$

So the area of the projection onto the xy-plane is equal to the sum of the areas of the parallelograms spanned by u' and v', v' and w', and w' and u'. When we calculate these using the cross-product we get

$$|u_1v_2 - u_2v_1| + |v_1w_2 - v_2w_1| + |w_1u_2 - w_2u_1|.$$

On the other hand, the length of the projection onto the z-axis is just the sum of the lengths of the orthogonal projections of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ onto the z-axis, that is,

$$|u_3| + |v_3| + |w_3|.$$

Using $\mathbf{w} = \mathbf{u} \times \mathbf{v}$, $\mathbf{v} = \mathbf{w} \times \mathbf{u}$ and $\mathbf{u} = \mathbf{v} \times \mathbf{w}$ this sum can be seen to be equal to the area in question. E.g. $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ gives $|w_3| = |u_1v_2 - u_2v_1|$.

Solution 3 b) (5 marks: 1 for the square shape, 1 for square area, 2 for hexagon shape, 1 for hexagon area.) Because of a) the shadow of maximum area results when the diagonal of the cube connecting the top and bottom vertices N and S in the diagram is vertical. In this position both N and S project onto the same point and the edges that end in these points will because of the symmetry of this setup cast six shadows of equal length separated by 60 degree.

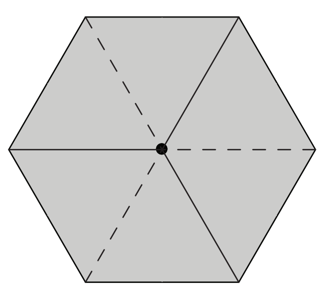


Figure 2: Shadow of the cube when one of the diagonals is vertical.

This means that the hexagon shadow whose spokes these edge shadows are has to be a regular hexagon. Since the length of the diagonal of a unit cube is $\sqrt{3}$, so is the area of the maximum shadow.

Shape of the minimum area projection is a square. This shadow results whenever one of the edges is vertical. The area of this minimal area shadow is 1.

Solution 3 c) (5 marks. Award 0, 3 or 5 marks.) When you scale the unit cube by a factor a to get a cube of side length a, the length in question will be multiplied by a factor of a and the area in question by a factor of a^2. Since for the unit cube we have $length = area$ this means that for a cube of side length a we have

$$area = a \cdot length.$$

Feedback on this set of problems in the form: a+b+c. E.g. full marks:15+5+5=25 marks.