MTH1030/35: Assignment 3, 2022

Step by step to infinity

Due online **Tuesday**, 17 May, 11:55 pm

(Note that this due date is a Tuesday not a Thursday like the previous two assignments!)

The Rules of the Game

Before starting to work on this assignment, please make sure that you understand the instructions on Moodle that go with our assignments (Assignment 3 (=Assignment 1) rules and F.A.Q. plus Academic integrity, plagiarism, and collusion policy).

Your submission for this assignment should be ONE pdf file.

Always justify your answers.

This assignment is worth 100 marks.

Now, let's have some fun!

1 Sequences

1.1 [10 marks]

Show that the following sequence has a limit and find this limit.

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

Hint: Express the terms of this sequence as powers of 2.

1.2 [10 marks]

Let $a_1 = a, a_2 = f(a_1), a_3 = f(a_2), \ldots, a_{n+1} = f(a_n)$, where a is some number and f is a continuous function. If $\lim_{n\to\infty} a_n = L$, show that $f(L) = L$.

Now, let $a = 1$.

Find an example of a function f such that the corresponding sequence converges and such that $f(x) = x$ has exactly 1 solution.

Find an example of a function f such that the corresponding sequence converges and such that $f(x) = x$ has more than one solution.

Find an example of a function f such that the corresponding sequence diverges and such that $f(x) = x$ has no solution.

Find an example of a function f such that the corresponding sequence diverges and such that $f(x) = x$ has a solution.

Hint: This and some of the other questions in this assignment ask you to come up with some examples of functions and series that have certain properties. ALL of these questions have VERY simple functions and series as answers. Please don't overcomplicate things :)

2 Serious series

2.1 [10 marks]

The alternating series test is a theorem which says the following: If a series

$$a_1 - a_2 + a_3 - a_4 + a_5 - ...$$

satisfies (1) $a_n > 0$, (2) $a_n \geq a_{n+1}$ and (3) $\lim_{n\to\infty} a_n = 0$, then the series converges. Very powerful and very useful. For example, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

satisfies (1), (2), and (3).

a) Give an example of a divergent series that satisfies (1) + (3) but not (2).

b) Give an example of a divergent series that satisfies (1) + (2) but not (3).

2.2 [10 marks]

a) If both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ converge and both sums are equal what can you conclude about the two series?

b) If one of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ converges and the other one diverges, which converges and which diverges?

2.3 [5 marks]

Let $\displaystyle\sum_{n=0}^{\infty} a_n$ and $\displaystyle\sum_{n=0}^{\infty} b_n$ be two divergent series. Consider the interlaced series

$$a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + a_4 + b_4 + \cdots .$$

Is it possible that the new series is convergent? If you think that this is not possible give a reason why. If you think that it is possible give an example.

2.4 [10 marks]

For which values of x does the following series converge? And, for those x for which the series converges, what is the sum of the series?

$$\sum_{n=1}^{\infty} \frac{\cos^n(x)}{2^n}.$$

Warning: Careful, we start with $n = 1$ and not $n = 0$.

2.5 [10 marks]

Consider the unit square in the xy-plane whose corners are $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$. Subdivide it into nine equal smaller squares and remove the square in the centre. Next, subdivide each of the remaining eight squares into nine even smaller squares, and remove each of the centre squares. And so on. The following diagram shows what's left after the first three steps of this construction. Give an example of a point in the original square that never gets removed. Show that the area of what is left over when all those squares have been removed is 0, by verifying that the sum of the areas of all the removed squares is 1.

2.6 [10 marks]

Consider the series

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$$

Calculate the first four partial sums of this series. Do some pattern spotting to come up with a simple function $p(n)$ such that $s_n = p(n)$ for $n = 1, 2, 3, 4$. Assuming that $s_n = p(n)$ for all n, show that our series is convergent and evaluate its sum.

3 A cat and ~~mouse~~ dog game [25 marks]

Let's play a game. I am thinking of two differentiable functions $cat(x)$ and $dog(x)$. Both functions can be written as power series

$$cat(x) = \sum_{n=0}^{\infty} c_n x^n, \quad dog(x) = \sum_{n=0}^{\infty} d_n x^n.$$

Both power series converge for all $x \in \mathbf{R}$ which means that both functions are defined everywhere. I am also telling you that

$$cat(0) = 0 \text{ and } dog(0) = 1$$

and that

$$dog'(x) = cat(x) \text{ and } cat'(x) = dog(x).$$

(a) Find $cat(x)$ and $dog(x)$ by calculating the general terms c_n and d_n of their power series.

Hint: To figure out what the coefficients of the two power series are use the fact that two power series are equal if and only if corresponding coefficients are equal and note that $cat''(x) = cat(x)$.

[10 marks]

(b) Write the functions e^x and e^{-x} in terms of of $cat(x)$ and $dog(x)$.

[5 marks]

(c) Conversely, express $cat(x)$ and $dog(x)$ as a combination of e^x and e^{-x}.

[5 marks]

(d) Using *Mathematica* or another piece of software plot $cat(x)$ and the first four *different* partial sums of its power series in the interval $[-2, 2]$.[1]

[5 marks]

[1] In *Mathematica* several functions can be plotted in the same diagram as follows:
`Plot[{Cos[x], Cos[2 x], Cos[3 x]}, {x, 0, 2 Pi}, PlotLegends -> "Expressions"]`