Sequences and series

Sequences

For the following sequences determine whether they converge or diverge. If possible also determine the limits for those among the sequences that converge.

1.
$$\left\{ \frac{n-1}{n^2-1} \right\}$$

2.
$$\left\{ 2 - \left(-\frac{1}{2} \right)^n \right\}$$

3.
$$\left\{ \sin\left(\frac{1}{n}\right) \right\}$$

4.
$$\left\{ \frac{1+(-1)^n}{\sqrt{n}} \right\}$$

5.
$$\left\{ \frac{2+\cos(n)}{n} \right\}$$

6.
$$\left\{ (0.001)^{-\frac{1}{n}} \right\}$$

7.
$$\left\{ \frac{f_{n+1}}{f_n} \right\}, \text{ where } f_n \text{ denotes the } n\text{th Fibonacci number.}$$

8. Turn the following infinite expression into a sequence.

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+...}}}}$$

If this sequence converges, what is its limit?

9. Turn the following infinite expression into two different sequences

$$...+\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

If these sequences converge, what are their limits?

10. What is the limit of the sequence

$$3, 3.1, 3.14, 3.141, 3.1415, ...$$

11. Prove from first principles, that is, based solely on Definition 3.1.1. in the lecture notes, that the sequence

$$\left\{\frac{1}{n^2}\right\}$$

has the limit $L = 0$.

12. What is the limit of the sequence

$$\left\{\frac{\sin(5+\frac{1}{n})-\sin(5)}{\frac{1}{n}}\right\}$$

13. If you are bored by all this and/or if you are up for a real challenge, try your hand at finding reasonable interpretations/limits of the crazy infinite expression that I listed in the lecture notes.

    Or, what about this one: For which x do the infinite expressions

$$\cdots(((x^x)^x)^x)^x\cdots \text{ and } \cdots x^{\left(x^{(x^x)}\right)}$$

    make sense/converge? If they converge what are their values?

    No solutions given for these problems! If you are interested in discussing your attempts at solving these problems, come and have a chat with me sometime.

Series

For the following infinite series determine whether they converge or diverge. If possible also determine the sum for those among the series that converge.

1.
$$1 + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi^3} + \frac{1}{\pi^4} + \frac{1}{\pi^5} + \cdots$$

2.
$$1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \frac{1}{3^8} + \frac{1}{3^{10}} + \cdots$$

3.
$$1 - 2 + 4 - 8 + 16 - 32 + \cdots$$

4.
$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n+1}$$

5.
$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[4]{4}} + \frac{1}{\sqrt[5]{5}} \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$$

6.
$$1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{5}} \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

7.
$$1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} \cdots = \sum_{n=1}^{\infty} \frac{1}{n^5}$$

8.
$$\sum_{n=0}^{\infty} \frac{3^n - 2^n}{4^n}$$

9.
$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{2^n} \right)$$

10. Write the rational number that is equal to the following infinite repeating decimal as a fraction.
$$0.\overline{123}\ldots$$

11. Write the rational number that is equal to the following infinite repeating decimal as a fraction.
$$4.34\overline{123}\ldots$$

12. For which values of x is the following series a convergent infinite geometric series. For those values for which this is the case find the sum of the series.
$$\sum_{n=0}^{\infty} \left(\frac{x-1}{2} \right)^n$$

13. Find a closed formula for the kth partial sum of the following infinite series and, based on this formula, establish whether or not the series converges. (Hint: This is an example of a telescoping sum in action.)

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$$

14. Find a closed formula for kth partial sum of the following infinite series and, based on this formula, establish whether or not the series converges. (Hint: This is another example of a telescoping sum in action.)

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

15. Use the integral test to figure out whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

16. Use the integral test to figure out whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$$

17. Use the integral test to figure out whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

18. Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^2}{n^2-1}$$

by finding a closed formula for the partial product

$$P_k = \prod_{n=2}^{k} \frac{n^2}{n^2-1}$$

and then letting k go to infinity.

19. Explain why the integral test does not apply to the following two series.

$$(i) \sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad (ii) \sum_{n=1}^{\infty} \frac{2+\sin(n)}{n^2}$$

20. Determine the convergence or divergence of the following series using the suggested series for comparison.

$$(i) \sum_{n=0}^{\infty} \frac{n+2}{n+1} \text{ compare with } \sum_{n=0}^{\infty} 1$$

$$(ii) \sum_{n=0}^{\infty} \frac{1}{\left(2+\frac{1}{n+1}\right)^{n+1}} \text{ compare with } \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}$$

$$(iii) \sum_{n=0}^{\infty} \frac{2+\sin(n)}{n+1} \text{ compare with } \sum_{n=0}^{\infty} \frac{1}{n+1}$$

TEST QUESTIONS

21. Define what it means for an infinite sequence to converge or diverge. Be able to reproduce the formal definitions in the lecture notes.

22. Define what it means for an infinite series to converge or diverge. Be able to reproduce the formal definitions in the lecture notes.

23. What does the squeeze law for sequences say?

24. What is $\lim_{x \to a} f(x)$ of a function that is continuous at $x = a$?

25. What is a monotone sequence, an increasing sequence, a decreasing sequence, a positive sequence, a non-negative sequence, a bounded sequence, an unbounded sequence, a constant sequence?

26. Can one determine whether a given infinite series converges or diverges merely by computing a sufficiently large number of partial sums?

27. Let $\{a_n\}$ be an increasing sequence with lower bound B. Does this sequence converge?

28. Let $\{a_n\}$ be an increasing sequence with upper bound B. Does this sequence converge?

29. Give a characterization of irrational numbers/rational numbers in terms of their decimal expansions.

30. How do we make sense of infinite expressions like

$$\cdots + \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6}}}} \quad ?$$

31. What is a geometric series, the harmonic series, a p-series, a telescoping sum, a positive-term series, the Riemann zeta function?

32. Derive the formula for the partial sums of the infinite geometric series

$$\sum_{n=0}^{\infty} ar^n$$

from scratch and use it to decide for which combinations of a and r the infinite series converges and diverges and what the sum of a convergent infinite geometric series is.

33. Given a positive-term series, there are only two possibilities: (1) The series converges or (2) it diverges to infinity. Explain in your own words why other forms of divergence are not possible.

34. Prove from scratch that the harmonic series diverges, i.e. make sure you understand the proof given in the lecture notes and be able to reproduce it.

35. Say the integral test applied to $\sum_{n=0}^{\infty} f(n)$ tells you that the series converges. Describe in your own words how you can use an integral to estimate how well the kth partial sum of this series approximates the sum of the series.

36. What does it means for one positive-term series to dominate another positive-term series?

37. How does the comparison test for positive-term series work?

38. You replace the first 666 terms of the harmonic series by 6s. Does the resulting series converge or diverge?