MTH1030
Techniques for Modelling

Lecture 19, 20

Limits, Sequences, Continuity

Monash University

Semester 1, 2022

Admin stuff

- Lecturer: Kaustav Das, 9 Rainforest Walk, room 347.
- Consultation: Immediately after lectures.
- Unit Coordinators: Burkard Polster (main) & Kaustav Das.
- MTH1035: Andy Hammerlindl (as always!).

Warm welcoming words

Today we'll commence the calculus portion of the course. We'll start with limits and sequences.

Calculus or Analysis?

Roughly speaking, the study of limits in mathematics is called *analysis*. The study of differentiation and integration (which both depend on limits!) is called *calculus*. So, calculus is a subset of analysis. This part of the course is pretty much only on calculus, but technically the portion on sequences and series is a part of analysis.

Limits

Limits are everywhere in calculus. The two most important objects from calculus, the *derivative* and *integral* are defined in terms of a limit (which we will define properly later, but you might remember it!). The concept of continuity can be defined in terms of a limit. So limits are very important!

Sequences

A sequence is an ordered collection of objects. For example,

$$a_1, a_2, a_3, a_4, \ldots$$

is a sequence. Here the *indexing set* is $\mathbb{N} = \{1, 2, 3, \ldots\}$. If we assume $a_1, a_2, \ldots$ are real numbers, then this is called a *real valued sequence*. Often we'll write a sequence like

$$\{a_n\}_{n=1}^{\infty} \text{ or } \{a_n\} \text{ or } (a_n)_{n=1}^{\infty} \text{ or } (a_n)$$

or even just a_n. For example, the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

In mathematics when we talk about sequences, they are usually infinite sequences! Meaning, they go on forever. From now on when we mention sequences, we will always be talking about real valued infinite sequences!

Sequences

What do we do with sequences? Of course we can evaluate a sequence at a given index. For example for the sequence $a_n = \frac{1}{n}$ we have

- $a_5 = 1/5$,
- $a_{10} = 1/10$,
- $a_{123} = 1/123$.

Seems trivial...so why are we spending a week or two on sequences?

Limit of a sequence

In mathematics, it is extremely important to understand the behaviour of your sequence a_n as n becomes large. Sequences often model some sort of real physical phenomenon, and so we often are interested what will occur in the long run. But how can we precisely quantify what the sequence will do in the long run? This is where the notion of limit comes into play!

Definition (Limit of a sequence)

Let a_n be a sequence. We say that a_n *converges* to a limit L if for all $\varepsilon > 0$, there exists a positive integer N such that for all n greater than or equal to N, then $|a_n - L| < \varepsilon$. We say that a_n *diverges* if there is no L that a_n converges to.

This is called the $\varepsilon - N$ definition of a limit of a sequence.

Limit of a sequence

Did you get that? That's okay, no one does initially. Before we explain it, just a couple of notes:

1. If a_n converges to L, we write this as $\lim_{n\to\infty} a_n = L$. In this case we would say that the limit of a_n exists.
2. Divergence of a sequence just means that there is no number L that a_n converges to. That is, Divergence = no convergence. Actually divergence of a sequence can be further classified; more on that later!

Limit of a sequence

Okay, so what does this convergence business mean? Let's break it down. In order to verify that $\lim_{n\to\infty} a_n = L$ we implement the following steps:

1. First we choose any arbitrary number $\varepsilon > 0$. It can be as small as you want it to be! Just not 0.
2. Next, we try to find a positive integer N (which depends on your chosen ε) such that...
3. $|a_n - L| < \varepsilon$ for all $n \geq N$. In other words, if we look at the absolute difference in the sequence and supposed limit L, it is guaranteed to be within ε beyond the index N.
4. If we can repeat the previous steps for any chosen $\varepsilon > 0$, then we say that $\lim_{n\to\infty} a_n = L$.

So basically, convergence occurs to a number L if the sequence and L become arbitrarily close **eventually**. Probably good to illustrate this with an example!

Limit of a sequence

Example

Let

$$a_n = \frac{1}{n}.$$

We feel that $\lim_{n\to\infty} \frac{1}{n} = 0$. Let's show this using the formal ε - N definition of convergence!

Question 1

Question (1)

What does the sequence

$$a_n = sin(n\pi)$$

converge to? Or does it not converge?

Limit of a function

Okay we can talk about limits of sequences, but you may be more familiar with limits of functions. First of all, a sequence $\{a_n\}_{n=1}^{\infty}$ is actually a function:

$$a : \mathbb{N} \to \mathbb{R}.$$

And we simply write $a_n \equiv a(n)$. So, limits of sequences should be special cases of limits of functions. Okay so what's a limit of a function?

Limit of a function

Before considering a limit of a function, let's consider the function $f(x) = x^2$. We know for example that

$$\lim_{x \to 10} f(x) = f(10) = 10^2.$$

Why would this be true? That is, why is the limit as x tends to 10 equal to the function at $x = 10$? This is not always true...but it is for functions which are sufficiently 'smooth'.

Definition (Continuity)

A function $f : I \to \mathbb{R}$ is continuous at the point $x_0 \in I$ if and only if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

So this 'smoothness' property is called continuity. So functions where the limit coincides with the function actually at that value are called *continuous*. Great! But what is a limit of a function???

Limit of a function

Actually, the definition for a limit of a general function is sort of beyond the scope of the course, but for completeness we will give it!

Definition (Limit of a function)

Let $f : I \to \mathbb{R}$ be a function. Then we say that $\lim_{x \to x_0} f(x) = L$ if for any sequence a_n which converges to x_0, we have that the sequence $b_n = f(a_n)$ converges to L.

Pretty abstract...and there are other equivalent definitions which are also pretty abstract ($\varepsilon - \delta$ definition). Maybe you'll explore them in second year.

Continuity of a function

Showing a function is continuous from a direct definition is a little arduous, but not too hard. However, it is beyond the scope of the course. But we 'know' that the following functions are continuous, and for this course we will just trust this is true rather than prove it:

- x^p for any $p \in \mathbb{R}$.
- e^x.
- $\ln(x)$.
- $\sin(x), \cos(x), \tan(x)$.

Before someone complains...continuity is understood over the function's implied domain (aka maximal domain). So yes, $\tan(x)$ is continuous over its implied domain $\mathbb{R}\backslash\{\pi/2 + k\pi : k = \ldots, -2, -1, 0, 1, 2, \ldots\}$.

Continuity of a function

So those functions are continuous...but what about other functions? How do we determine if they are continuous (without brute forcing it via the definition?). Well the following theorem will help!

Theorem

Let f and g be continuous functions. Then the following are continuous:

1. $f(x) + g(x)$.
2. $f(x)g(x)$.
3. $f(x)/g(x)$.
4. $f \circ g(x) = f(g(x))$.

Of course when we say they are continuous we mean continuous over its implied domain. So, $f(x)/g(x)$ is okay! Great, so from continuous functions we can construct more continuous functions!

Continuity of a function

So what's not continuous? Well we can pretty easily define something artificial which isn't continuous

$$f(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

Clearly there's a 'jump' at $x = 0$, where f suddenly goes from 0 to 1. Actually, this function is called the *Heaviside function*, and its pretty important (in other subjects).

This one is slightly more complicated but its the same idea:

$$f(x) = \begin{cases} x^2, & x < 0, \\ x + 1, & x \geq 0. \end{cases}$$

Question 2

Question (2)

Consider the function

$$f(x) = \begin{cases} x - c, & x < c, \\ \sin(x) & x \geq c. \end{cases}$$

where $c \in \mathbb{R}$. What is a value c for which this function is continuous?

Derivative of a function

Whilst we're talking about limits of functions and what not, let's mention derivative and integral. We have that a function $f : I \to \mathbb{R}$ is *differentiable* at a point $x_0 \in I$ if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. Its derivative at x_0 is simply the value of this preceding limit, and we denote it by $f'(x_0)$. If a function is differentiable for all $x_0 \in I$, we simply call it *differentiable*.

See! There's a limit involved here. Actually you may have seen this as

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

It's the same thing. Interpretation: the derivative is equal to a small change in the output divided by a small change in the input. In other words, instantaneous rate of change!

Integral of a function

How about (definite) integral? If $f : I \to \mathbb{R}$ is a nice enough function, then (roughly speaking...) the integral of f over $[a, b]$ is given by

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)(x_k - x_{k-1}).$$

Another limit! But we won't be talking about integrals for a couple of weeks, so let's leave that for now.

Weird functions

In fact there are functions which are continuous but nowhere differentiable. Look up the Weierstrass function and Brownian motion.

If you think about it, differentiability means that if you zoom in on a function, it becomes linear (i.e., it is locally linear). So a continuous but nowhere differentiable function has no locally linear part. It's pretty impossible to draw a plot of such a thing (but we try...).

That such a function exists goes to show you that your intuition is not always correct!

Sequences

Okay we got sidetracked with all this continuity stuff...back to sequences! Remember we had the sequence $a_n = 1/n$ and it obviously converged to 0. In fact we proved this.

It's also pretty obvious that:

- $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for $p > 0$.
- $\lim_{n\to\infty} |\cos(n\pi)| = 1$.
- $\lim_{n\to\infty} e^{-n} = 0$.

Sequences

We said a sequence which does not converge is called divergent. For example, it's pretty clear the following sequences diverge:

- n
- 2^n
- $\ln(n)$

And so do the following!

- $(-1)^n$.
- $(-1)^n n$.

But these ones seem to diverge is a sort of different way...

Sequences

Definition (Divergence to infinity)

Let a_n be a sequence. We say that a_n diverges to infinity or is divergent to infinity if for any $R > 0$, there exists a positive integer N such that $a_n > R$ for all $n \geq N$. We write this as

$$\lim_{n\to\infty} a_n = \infty.$$

A sequence b_n diverges to $-\infty$ if $-b_n$ diverges to ∞.

A silly question...is 'divergence to infinity' really a special case of 'divergence'? That is, is a sequence which 'diverges to infinity' not convergent? Ans: yes! Lesson: always check if your ideas are consistent with one another!

Sequences

So suppose we have a sequence a_n which is divergent...but not divergent to infinity. Then what is it doing? For example:

$$(-1)^n$$

is divergent but not divergent to infinity. This is because the sequence can never settle down! Moral of story: A divergent sequence either goes to positive or negative infinity, or never settles down.

Sequences

Suppose we have a sequence whose convergence we want to study, but it is not obvious if it converges, let alone what the limit would be. How do we determine convergence and its limit? Luckily we have a bunch of theorems!

Theorem (Limit laws for sequences)

Let a_n and b_n be sequences such that $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$. Then

1. $\lim_{n\to\infty} ca_n = cA$.
2. $\lim_{n\to\infty}(a_n + b_n) = A + B$.
3. $\lim_{n\to\infty} a_n b_n = AB$.
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$, *provided* $B \neq 0$.

Limit theorems

Theorem (Substitution law for sequences)

Let a_n be a sequence with $\lim_{n\to\infty} a_n = A$ and let f be a function which is continuous at $x = A$. Then

$$\lim_{n\to\infty} f(a_n) = f(A).$$

Limit theorems

Theorem (Squeeze theorem)

Let a_n, b_n, c_n be sequences such that $a_n \leq b_n \leq c_n$. Suppose in addition that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L.$$

Then

$$\lim_{n\to\infty} b_n = L.$$

The idea is, if the sequence b_n is 'squeezed' or 'sandwiched' between a_n and c_n, and the latter sequences are both converging to L, then b_n has no choice but to converge to L as well!

Limit theorems

Example

Let

$$b_n = \frac{\sin(n)}{n}.$$

We will show via the squeeze theorem that $\lim_{n\to\infty} b_n = 0$.

Question 3

Question (3)

What does the sequence

$$a_n = \frac{\cos(n)}{n}$$

converge to? Or does it not converge?

Monotone sequences

Before we talk about the next theorem we need a definition...

Definition (Monotone sequence)

Let a_n be a sequence such that $a_1 \leq a_2 \leq a_3 \leq \dots$. Then we call a_n a *monotone increasing* sequence. Similarly, let b_n be a sequence such that $b_1 \geq b_2 \geq b_3 \geq \dots$. Then we call b_n a *monotone decreasing* sequence.

In other words, a_n is monotone increasing if $a_n \leq a_{n+1}$. b_n is monotone decreasing if $b_n \geq b_{n+1}$.

A sequence is simply called *monotonic* if it is either monotone increasing or monotone decreasing.

Monotone sequences

Terminology wise, we have the following:

- Monotonic = monotone increasing or monotone decreasing.
- Monotone increasing = non-decreasing = increasing ($a_n \leq a_{n+1}$).
- Monotone decreasing = non-increasing = decreasing ($a_n \geq a_{n+1}$).
- Strictly increasing ($a_n < a_{n+1}$).
- Strictly decreasing ($a_n > a_{n+1}$).

However, unfortunately some people will call strictly increasing sequences increasing...similar for strictly decreasing. We won't do this!

Bounded sequences

One more definition...!

Definition (Bounded sequences)

Let a_n be a sequence. a_n is

1. *bounded from above* if $a_n \leq C$ for some $C \in \mathbb{R}$ independent of n.
2. *bounded from below* if $a_n \geq C$ for some $C \in \mathbb{R}$ independent of n.
3. *bounded* if $|a_n| \leq C$ for some $C > 0$ independent of n.

Note that bounded $\iff$ bounded from below & bounded from above.

Limit theorems

Theorem (Bounded monotonic sequence theorem)

Let a_n be a monotone increasing sequence. Suppose that it is bounded from above. Then $\lim_{n\to\infty} a_n$ exists.

Let b_n be a monotone decreasing sequence. Suppose that it is bounded from below. Then $\lim_{n\to\infty} b_n$ exists.

So if your sequence is monotonic and bounded, it must converge!

Note that this theorem does not tell you what the limit of your sequence is, but just that it exists. That's still pretty good! Sometimes we only care whether convergence occurs.

Sequences

Example

Consider the sequence $a_n = -\frac{1}{n}$. So its elements are

$$-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, -\frac{1}{6}, \ldots$$

And thus it is monotone increasing. It's also bounded above by 0. So by the bounded monotonic sequence theorem, it must converge! Obviously, it converges to 0. But the theorem didn't tell us that.

Sequences

How about this one?

Example

Consider the sequence

$$a_1 = \sqrt{6}, a_2 = \sqrt{6+\sqrt{6}}, a_3 = \sqrt{6+\sqrt{6+\sqrt{6}}}, \ldots,$$

Compactly, this is $a_{n+1} = \sqrt{6 + a_n}$ with $a_0 = 0$. What can we do here?

Limit theorems

Lastly...

Theorem

Let f be a real valued function. Define $a_n = f(n)$. Then

$$\lim_{x\to\infty} f(x) = L$$

implies $\lim_{n\to\infty} a_n = L$.

This makes sense...we constructed a_n by extracting values from f.