Linear systems

Solve each of the following systems of equations using Gaussian elimination with back-substitution by hand.

1. $\begin{array}{rcrcr} J & + & M & = & 75 \\ J & - & 4M & = & 0 \end{array}$

   Answer.
   Turn the system into an augmented matrix and use Gaussian elimination to get the matrix in row echelon form,

$$\begin{array}{rl} & \left[\begin{array}{rr|r} 1 & 1 & 75 \\ 1 & -4 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \end{array} \\ \sim & \left[\begin{array}{rr|r} 1 & 1 & 75 \\ 0 & 5 & 75 \end{array}\right] \begin{array}{l} R_1 \\ R_2 = R_1 - R_2 \end{array} \end{array}$$

   Using back substitution,

$$\begin{aligned} 5M &= 75 \\ M &= 15 \\ J + M &= 75 \\ J + 15 &= 75 \\ J &= 60 \end{aligned}$$

   Therefore $J = 60, M = 15$

2. $\begin{array}{rcrcr} x & + & y & = & 5 \\ 2x & + & 3y & = & 1 \end{array}$

   Answer.
   Turn the system into an augmented matrix and use Gaussian elimination to get the matrix in row echelon form,

$$\begin{array}{rl} & \left[\begin{array}{rr|r} 1 & 1 & 5 \\ 2 & 3 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \end{array} \\ \sim & \left[\begin{array}{rr|r} 1 & 1 & 5 \\ 0 & 1 & -9 \end{array}\right] \begin{array}{l} R_1 \\ R_2 = R_2 - 2 \times R_1 \end{array} \end{array}$$

Using back substitution,

$$\begin{aligned} y &= -9 \\ x + y &= 5 \\ x - 9 &= 5 \\ x &= 14 \end{aligned}$$

Therefore $x = 14, y = -9$

3. $\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ 2x & + & 5y & - & z & = & 13 \\ x & + & 3y & - & 3z & = & 4 \end{array}$

Answer.
Turn the system into an augmented matrix and use Gaussian elimination to get the matrix in row echelon form,

$$\begin{aligned}
&\left[\begin{array}{rrr|r} 1 & 2 & -1 & 6 \\ 2 & 5 & -1 & 13 \\ 1 & 3 & -3 & 4 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \\
\sim &\left[\begin{array}{rrr|r} 1 & 2 & -1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -2 & -2 \end{array}\right] \begin{array}{l} R_1 \\ R_2 = R_2 - 2 \times R_1 \\ R_3 = R_3 - R_1 \end{array} \\
\sim &\left[\begin{array}{rrr|r} 1 & 2 & -1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 = R_2 - R_3 \end{array}
\end{aligned}$$

Using back substitution,

$$\begin{aligned} 3z &= 3 \\ z &= 1 \\ y + z &= 1 \\ y + 1 &= 1 \\ y &= 0 \\ x + 2y - z &= 6 \\ x + 2(0) - 1 &= 6 \\ x &= 7 \end{aligned}$$

Therefore $x = 7, y = 0, z = 1$

4. $\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 6 \\ x & + & 2y & + & 2z & = & 3 \\ 2x & + & 5y & - & z & = & 13 \end{array}$

Answer.
Turn the system into an augmented matrix and use Gaussian elimination to get the matrix in row echelon form,

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 1 & 2 & 2 & 3 \\ 2 & 5 & -1 & 13 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 0 & 3 & -3 \\ 0 & 1 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 = R_2 - R_1 \\ R_3 = R_3 - 2 \times R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 6 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 3 & -3 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \Leftrightarrow R_3 \\ R_3 \Leftrightarrow R_2 \end{array}$$

Using back substitution,

$$\begin{aligned} 3z &= -3 \\ z &= -1 \\ y + z &= 1 \\ y - 1 &= 1 \\ y &= 2 \\ x + 2y - z &= 6 \\ x + 2(2) - (-1) &= 6 \\ x &= 1 \end{aligned}$$

Therefore $x = 1, y = 2, z = -1$

5. $$\begin{array}{rcrcrcr} 2x & + & 3y & - & z & = & 4 \\ x & + & y & + & 3z & = & 1 \\ x & + & 2y & - & z & = & 3 \end{array}$$

Answer.
Turn the system into an augmented matrix and use Gaussian elimination to get the matrix in row echelon form,

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 1 & 1 & 3 & 1 \\ 1 & 2 & -1 & 3 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & 1 & -7 & 2 \\ 0 & 1 & -1 & 2 \end{array}\right] \begin{array}{l} R_1 \\ R_2 = R_1 - 2 \times R_2 \\ R_3 = 2 \times R_3 - \times R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & 3 & -1 & 4 \\ 0 & 1 & -7 & 2 \\ 0 & 0 & 6 & 0 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 = R_3 - R_2 \end{array}$$

Using back substitution,

$$\begin{aligned}
6z &= 0 \\
z &= 0 \\
y - 7z &= 2 \\
y - 7(0) &= 2 \\
y &= 2 \\
2x + 3y - z &= 4 \\
2x + 3(2) - (0) &= 4 \\
x &= -1
\end{aligned}$$

Therefore $x = -1, y = 2, z = 0$

6. You all know how to translate these linear systems into augmented matrices? Just checking!

   Answer. Trivial

THE FOLLOWING PROBLEMS ARE FROM KUTTLER'S BOOK

7. You have a system of k equations in two variables, $k \geq 2$. Explain what it means in terms of the corresponding lines in the plane for this system to have

   (a) no solution;

   (b) a unique solution;

   (c) an infinite number of solutions;

   (d) exactly two solutions.

   Answer. a) The lines do not have any point in common. b) The only point common to all lines is a single point. c) All equations describe the same line. d) If you ever come up with anything but a), b) or c) in terms of numbers of solutions to a linear system you've made a mistake. Exactly two solutions is not possible.

8. Here is an augmented matrix in which $*$ stands for an arbitrary number and $\blacksquare$ stands for a non-zero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccccc|c}
\blacksquare & * & * & * & * & * \\
0 & \blacksquare & * & * & 0 & * \\
0 & 0 & \blacksquare & * & * & * \\
0 & 0 & 0 & 0 & \blacksquare & *
\end{array}\right)$$

   Answer. The system is consistent and has infinitely many solutions (one free variable).

9. Here is an augmented matrix in which $*$ stands for an arbitrary number and $\blacksquare$ stands a non-zero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array}\right)$$

Answer. The system is consistent and has exactly one solution (no free variables).

10. Here is an augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a non-zero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & 0 & * & \blacksquare \end{array}\right)$$

Answer. If the star in the last row is equal to zero, then the system is clearly inconsistent. If the star in the last row is non-zero you can subtract (*/square) times row (3) from row (4) and get $0 =$ non-zero, which is inconsistent. This means that the system is always inconsistent.

11. Suppose a system of equations has fewer equations than variables. Must such a system be consistent? If so, explain why and if not, give an example which is not consistent.

    Answer. No. Consider the linear system consisting of the two equations $x + y + z = 0$ and $x + y + z = 1$.

12. If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, give a convincing argument why this is not possible.

    Answer. Such systems can have solutions. For example, the equations in the system $x+y = 1, 2x+2y = 2, 3x+3y = 3$ all describe the same line. Hence there are infinitely many solutions to this system.

13. Find h such that

$$\left(\begin{array}{cc|c} 2 & h & 4 \\ 3 & 6 & 7 \end{array}\right)$$

is the augmented matrix of an inconsistent matrix.

Answer. $h = 4$

14. Find h such that

$$\left(\begin{array}{cc|c} 1 & h & 3 \\ 2 & 4 & 6 \end{array}\right)$$

is the augmented matrix of a consistent matrix.

Answer. Any h will work.

15. Find h such that
$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & h & 12 \end{array}\right)$$
is the augmented matrix of a consistent matrix.

Answer. Any h will work.

16. Choose h and k such that the augmented matrix shown has one solution. Then choose h and k such that the system has no solutions. Finally, choose h and k such that the system has infinitely many solutions.
$$\left(\begin{array}{cc|c} 1 & h & 2 \\ 2 & 4 & k \end{array}\right).$$

Answer. If $h \neq 2$ there will be a unique solution for any k. If $h = 2$ and $k \neq 4$, there are no solutions. If $h = 2$ and $k = 4$, then there are infinitely many solutions.

17. Choose h and k such that the augmented matrix shown has one solution. Then choose h and k such that the system has no solutions. Finally, choose h and k such that the system has infinitely many solutions.
$$\left(\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & h & k \end{array}\right).$$

Answer. If $h \neq 4$, then there is exactly one solution. If $h = 4$ and $k \neq 4$, then there are no solutions. If $h = 4$ and $k = 4$, then there are infinitely many solutions.

18. Here is the reduced row echelon form of some linear system. What can you tell me about its solutions?
$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Answer. The system has no solutions, since in the last row we have only zeros to the left of the vertical stroke and a non-zero number on the right.

19. Consider the linear system whose augmented matrix is
$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 3 & 2 & 1 & 3 \end{array}\right)$$
and whose reduced row echelon form is
$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Mark the pivots and pivot columns and find the general solution of this linear system.

Answer. There a two pivots, the leading 1s in the first and second rows. This means that the pivot columns in both matrices are the first two columns. There is one free variable corresponding to the third column. Let this free variable be t. Then the solutions are

$$x = \frac{1}{2} - \frac{1}{2}t, y = \frac{3}{4} + \frac{1}{4}t, z = t,$$

where $t \in \mathbf{R}$. Other ways of recording these solutions would be

$$\left(\frac{1}{2} - \frac{1}{2}t, \frac{3}{4} + \frac{1}{4}t, t\right), t \in \mathbf{R}$$

or

$$\left(\frac{1}{2}, \frac{3}{4}, 0\right) + t\left(-\frac{1}{2}, \frac{1}{4}, 1\right), t \in \mathbf{R}.$$

20. Consider the linear system whose augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 4 & 2 \end{array}\right)$$

and whose reduced row echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & -4 & -1 \end{array}\right)$$

Mark the pivots and pivot columns and find the general solution of this linear system.

Answer. There a two pivots, the leading 1s in the first and second rows. This means that the pivot columns in both matrices are the first two columns. There is one free variable corresponding to the third column. Let this free variable be t. Therefore the solutions are $x = 2 - 4t, y = -1 + 4t, z = t, t \in \mathbf{R}$.

21. Consider the linear system whose augmented matrix is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 & 2 \end{array}\right)$$

and whose reduced row echelon form is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 9 & 3 \\ 0 & 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & -7 & -1 \\ 0 & 0 & 0 & 1 & 6 & 1 \end{array}\right).$$

Mark the pivots and pivot columns and find the general solution of this linear system.

Answer. There a four pivots. The pivot columns in both matrices are the first four columns. There is one free variable corresponding to the fifth column. Let this free variable be t. Solutions

$$x = 3 - 9t, y = 4t, z = -1 + 7t, u = 1 - 6t, v = t.$$

22. Consider the linear system whose augmented matrix is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 & 1 & 3 \\ 1 & -1 & 2 & 2 & 2 & 0 \end{array}\right).$$

and whose reduced row echelon form is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Mark the pivots and pivot columns and find the general solution of this linear system.

Answer. There are three pivots, the leading 1s in the first, second and third rows. This means that the pivot columns in both matrices are column one, two and four. There two free variables corresponding column three and five. Let's call these free variables s and t. Solutions

$$x = \frac{5}{2} + \frac{1}{2}t - 2s, y = \frac{3}{2} - t\frac{1}{2}, z = s, u = -\frac{1}{2} - \frac{3}{2}t, v = t, s, t \in \mathbf{R}.$$

SOME TEST QUESTIONS

23. You are facing a linear system of n linear equations in m unknowns. Somebody tells you that that the coefficients of this system have been chosen randomly. How many solutions do you expect this system to have if: $n = m, n < m, n > m$.

24. You have a system of three linear equations in three unknowns that has infinitely many solutions. How are the corresponding three planes in space situated with respect to each other (two essentially different scenarios!)? Explain using some diagrams.

25. Consider an arbitrary system of two linear equations in two unknowns and the corresponding two lines in the plane. Describe the essentially different configurations of the two lines (there are three, one of which is a bit tricky) and the corresponding numbers of solutions of the system of equations.

26. You have two linear equations (1) and (2), say in two unknowns. Why does the system of equations (1), (1)+(2) have exactly the same solutions as the first one?

27. Explain what it means for a system of linear equations to be consistent.
28. A system of 666 linear equations in 666 unknowns has a unique solution. How many pivots does it have?