Matrices

Matrix operations

1. Evaluate each of the following

   $$2\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix},$$

   $$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -4 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

2. Given

   $$A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix},$$

   compute A^2, A^3 and hence write down A^n for $n > 1$.

Inverse

1. Compute the inverse A^{-1} of the following matrices by hand.

   $$A = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$$

   Verify that $A^{-1}A = I$ and $AA^{-1} = I$.

2. Consider the following pair of matrices

   $$A = \begin{pmatrix} 11 & 18 & 7 \\ a & 6 & 3 \\ -3 & -5 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 1 & 12 \\ b & -1 & -5 \\ -2 & 1 & -6 \end{pmatrix}$$

   Compute the values of a and b so that A is the inverse of B while B is the inverse of A.

3. Let
$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$
Show that
$$A^2 - 6A + I = 0,$$
where I is the 2×2 identity matrix and 0 stands for the 2×2 zero matrix. Use this result to compute A^{-1}.

MORE PROBLEMS ON MATRIX OPERATIONS AND INVERSES FROM KUTTLER'S BOOK

1. Suppose A and B are square matrices of the same size. Which of the following are correct?
   (a) $(A - B)^2 = A^2 - 2AB + B^2$
   (b) $(AB)^2 = A^2B^2$
   (c) $(A + B)^2 = A^2 + 2AB + B^2$
   (d) $(A + B)^2 = A^2 + AB + BA + B^2$
   (e) $A^2B^2 = A\ (AB)\ B$
   (f) $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$
   (g) $(A + B)(A - B) = A^2 - B^2$
2. $A = \begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix}$. Find ALL 2×2 matrices B such that AB is the 2×2 zero matrix.
3. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 1 & k \end{pmatrix}$. Is it possible to choose k such that $AB = BA$?
4. Let A be a square matrix. Show that A equals the sum of a symmetric and a skew symmetric matrix. (A matrix M is skew symmetric if $M = -M^T$. M is symmetric if $M^T = M$.) Hint: Show that $\frac{1}{2}(A^T + A)$ is symmetric and then consider using this as one of the matrices.
5. Prove that every skew symmetric matrix has all zeros down the main diagonal. The main diagonal consists of every entry of the matrix which is of the form a_{ii}.
6. Prove that if $AB = AC$ and A^{-1} exists, then $B = C$.
7. Give an example of matrices A and B such that neither A nor B is equal to a zero matrix and yet AB is equal to a zero matrix.
8. Prove that if A^{-1} exists and $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$.
9. Prove that if A is an invertible $n \times n$ matrix, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

10. Prove that $(AB)^{-1} = B^{-1}A^{-1}$ by verifying that $AB(B^{-1}A^{-1}) = I$ and $B^{-1}A^{-1}(AB) = I$.

11. Prove that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

12. If A is invertible, prove that $(A^2)^{-1} = (A^{-1})^2$.

13. Why is the product of two upper triangular $n \times n$ matrices A and B upper triangular?

14. Given two upper triangular matrices A and B of the same dimension, express the coefficients on the diagonal of AB in terms of the coefficients of A and B.

Determinants

1. For the matrix
$$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$$
compute the determinant twice, first by expanding along the top row and second by expanding along the second column.

2. Given
$$A = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix},$$
compute $\det(A)$, $\det(B)$ and $\det(AB)$. Verify that $\det(AB) = \det(A)\det(B)$.

3. Compute the following determinants using expansions along a suitable row or column.
$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 0 & 9 & 8 \end{vmatrix}, \quad \begin{vmatrix} 4 & 3 & 2 \\ 1 & 7 & 8 \\ 3 & 9 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 3 \\ 4 & 0 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 5 & 1 & 3 \\ 2 & 1 & 7 & 5 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{vmatrix}.$$

4. Compute the following determinants as fast as you can (under one second per determinant, if possible).
$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{vmatrix}, \quad \begin{vmatrix} 0 & 0 & 3 \\ 0 & 2 & 2 \\ 1 & 9 & 8 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}.$$

5. Which of the following statements are true?

   (a) If A is a 3×3 matrix with a zero determinant, then one row of A must be a multiple of some other row.

   (b) Even if any two rows of a square matrix are equal, the determinant of that matrix may be non-zero.

(c) If any two columns of a square matrix are equal then the determinant of that matrix is zero.

(d) For any pair of $n \times n$ matrices, A and B, we always have $\det(A + B) = \det(A) + \det(B)$

(e) Let A be a 3×3 matrix. Then $\det(7A) = 7^3 \det(A)$.

(f) If A^{-1} exists, then $\det(A^{-1}) = \det(A)$.

6. Assume that A is square matrix with inverse A^{-1}. Prove that $\det(A^{-1}) = 1/\det(A)$

MORE PROBLEMS ON DETERMINANTS FROM KUTTLER'S BOOK

7. An elementary operation is applied to transform the first matrix into the second. Which elementary operation is being used and how does it affect the value of the determinant.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & b \\ a+c & b+d \end{pmatrix}$$

8. An elementary operation is applied to transform the first matrix into the second. Which elementary operation is being used and how does it affect the value of the determinant.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & b \\ 2c & 2d \end{pmatrix}$$

9. An elementary operation is applied to transform the first matrix into the second. Which elementary operation is being used and how does it affect the value of the determinant.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

10. Prove from scratch that $\det(AB) = \det(A)\det(B)$ if A and B are 2×2 matrices.

11. An $n \times n$ matrix is called **nilpotent** if for some positive integer k we have $A^k = 0$. If A is a nilpotent matrix and k is the smallest possible integer such that $A^k = 0$, what are the possible values of $\det(A)$?

12. A matrix is said to be **orthogonal** if $A^T A = I$. This means that the inverse of an orthogonal matrix is just its transpose. What are the possible values of $\det(A)$ if A is an orthogonal matrix?

13. Fill in the missing entries to make the following matrix orthogonal.

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \_ & \_ \\ \_ & \frac{\sqrt{6}}{3} & \_ \end{pmatrix}.$$

14. Let A and B be two $n \times n$ matrices. $A \sim B$ (A is **similar** to B) means there exists an invertible matrix, S such that $A = S^{-1}BS$. Prove that if $A \sim B$, then $B \sim A$. Show also that $A \sim A$ and that if $A \sim B$ and $B \sim C$, then $A \sim C$.

15. Continuing on with the last problem prove that if $A \sim B$, then $\det(A) = \det(B)$.

16. Tell whether the statement is true or false.

    (a) If A is a 3×3 matrix with a 0 determinant, then one column must be a multiple of some other column.

    (b) If any two columns of a square matrix are equal, then the determinant of the matrix equals zero.

    (c) For A and B two $n \times n$ matrices, $\det(A + B) = \det(A) + \det(B)$.

    (d) For A an $n \times n$ matrix, $\det(3A) = 3\det(A)$.

    (e) If A^{-1} exists then $\det(A^{-1}) = \det(A)^{-1}$.

    (f) If B is obtained by multiplying a single row of A by 4, then $\det(B) = 4\det(A)$.

    (g) For A a square matrix, $\det(-A) = (-1)^n \det(A)$.

    (h) If A is a square matrix, then $\det\left(A^T A\right) \geq 0$.

    (i) Cramer's rule is useful for finding solutions to systems of linear equations in which there is an infinite set of solutions.

    (j) If $A^k = 0$ for some positive integer k, then $\det(A) = 0$.

    (k) If $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$, then $\det(A) = 0$.

17. Use Cramer's rule to find the solution to

$$\begin{aligned} x + 2y &= 1 \\ 2x - y &= 2 \end{aligned}$$

18. Here is a matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix}$$

    Does there exist a value of t for which this matrix fails to have an inverse? Explain.

19. Here is a matrix,

$$\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{pmatrix}$$

    Does there exist a value of t for which this matrix fails to have an inverse? Explain.

20. Use the formula for the inverse in terms of the cofactor matrix to find the inverse of the matrix,

$$A = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t \cos(t) & e^t \sin(t) \\ 0 & e^t \cos(t) - e^t \sin(t) & e^t \cos(t) + e^t \sin(t) \end{pmatrix}.$$

21. Suppose A is an upper triangular matrix. Show that A^{-1} exists if and only if all elements of the main diagonal are non-zero. Is it true that A^{-1} will also be upper triangular? Explain. Is everything the same for lower triangular matrices?

22. If A, B, and C are each $n \times n$ matrices and ABC is invertible, why are each of A, B, and C invertible.

SOME TEST QUESTIONS

23. Let A be an $n \times n$ matrix with zero determinant. Is it possible by looking at A alone to determine whether a system of linear equations $A\mathbf{x} = \mathbf{b}$ has no or infinitely many solutions.

24. State the two universal laws of swaps.

25. Describe an algorithm/a computer program that can turn any permutation into the identity permutation using only swaps.

26. Which of the following is a term of the general formula for the 5×5 determinant: (a) $a_{1,1}a_{2,3}a_{3,4}a_{4,2}a_{5,5}$; (b) $a_{1,1}a_{2,3}a_{1,4}a_{4,2}a_{5,5}$, (c) $a_{1,1}a_{2,2}a_{3,6}a_{4,4}a_{5,5}a_{6,3}$.

27. In the 4×4 determinant formula which sign precedes $a_{1,4}a_{2,3}a_{3,2}a_{4,1}$.

28. Why is the product of two upper triangular matrices of the same dimension upper triangular?

29. Show that that among the permutations of $123...n$ exactly half are odd and half are even.

30. Based only on our definition of the determinant in the lecture notes show that a square matrix with a row of zeros has determinant equal to zero.

31. Based only on our definition of the determinant in the lecture notes show that the determinant of an upper triangular matrix is equal to the product of the coefficients on the diagonal.

32. What is the minimum number of non-zero coefficients of an $n \times n$ matrix with non-zero determinant.

33. What are the elementary 5×5 matrices that: a) swap rows 1 and 5; b) multiply the 3rd row by 27; c) add -5 times the first row to the 3rd row.

34. Show that elementary matrices are invertible and that their inverses are also elementary matrices.

35. What are the inverses of the three elementary matrices above?

36. Find a 4×4 matrix P that inverts the rows of any other 4×4 matrix A in the sense that the 1st, 2nd, 3rd and 4th row of the matrix PA are the 4th, 3rd, 2nd, and 1st row, respectively, of the matrix A.

37. Someone tells you that the product of 666 6×6 matrices is invertible. How many of the 666 matrices do you expect to be invertible, too?

38. Describe three ways to calculate the inverse of a square matrix.