Linear systems

Solve each of the following systems of equations using Gaussian elimination with back-substitution by hand.

1.
$$\begin{aligned} J + M &= 75 \\ J - 4M &= 0 \end{aligned}$$

2.
$$\begin{aligned} x + y &= 5 \\ 2x + 3y &= 1 \end{aligned}$$

3.
$$\begin{aligned} x + 2y - z &= 6 \\ 2x + 5y - z &= 13 \\ x + 3y - 3z &= 4 \end{aligned}$$

4.
$$\begin{aligned} x + 2y - z &= 6 \\ x + 2y + 2z &= 3 \\ 2x + 5y - z &= 13 \end{aligned}$$

5.
$$\begin{aligned} 2x + 3y - z &= 4 \\ x + y + 3z &= 1 \\ x + 2y - z &= 3 \end{aligned}$$

6. You all know how to translate these linear systems into augmented matrices? Just checking!

THE FOLLOWING PROBLEMS ARE FROM KUTTLER'S BOOK

7. You have a system of k equations in two variables, $k \geq 2$. Explain what it means in terms of the corresponding lines in the plane for this system to have

   (a) no solution;

   (b) a unique solution;

   (c) an infinite number of solutions;

   (d) exactly two solutions.

8. Here is an augmented matrix in which $*$ stands for an arbitrary number and $\blacksquare$ stands for a non-zero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * \end{array}\right)$$

9. Here is an augmented matrix in which $*$ stands for an arbitrary number and $\blacksquare$ stands a non-zero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccc|c} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & \blacksquare & * \end{array}\right)$$

10. Here is an augmented matrix in which $*$ denotes an arbitrary number and $\blacksquare$ denotes a non-zero number. Determine whether the given augmented matrix is consistent. If consistent, is the solution unique?

$$\left(\begin{array}{ccccc|c} \blacksquare & * & * & * & * & * \\ 0 & \blacksquare & * & * & 0 & * \\ 0 & 0 & 0 & 0 & \blacksquare & 0 \\ 0 & 0 & 0 & 0 & * & \blacksquare \end{array}\right)$$

11. Suppose a system of equations has fewer equations than variables. Must such a system be consistent? If so, explain why and if not, give an example which is not consistent.

12. If a system of equations has more equations than variables, can it have a solution? If so, give an example and if not, give a convincing argument why this is not possible.

13. Find h such that

$$\left(\begin{array}{cc|c} 2 & h & 4 \\ 3 & 6 & 7 \end{array}\right)$$

is the augmented matrix of an inconsistent matrix.

14. Find h such that

$$\left(\begin{array}{cc|c} 1 & h & 3 \\ 2 & 4 & 6 \end{array}\right)$$

is the augmented matrix of a consistent matrix.

15. Find h such that

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 3 & h & 12 \end{array}\right)$$

is the augmented matrix of a consistent matrix.

16. Choose h and k such that the augmented matrix shown has one solution. Then choose h and k such that the system has no solutions. Finally, choose h and k such that the system has infinitely many solutions.

$$\left(\begin{array}{cc|c} 1 & h & 2 \\ 2 & 4 & k \end{array}\right).$$

17. Choose h and k such that the augmented matrix shown has one solution. Then choose h and k such that the system has no solutions. Finally, choose h and k such that the system has infinitely many solutions.

$$\left(\begin{array}{cc|c} 1 & 2 & 2 \\ 2 & h & k \end{array}\right).$$

18. Here is a reduced row echelon form of some linear system. What can you tell me about its solutions?

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

19. Consider the linear system whose augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 2 & 0 & 1 & 1 \\ 3 & 2 & 1 & 3 \end{array}\right)$$

and whose reduced row echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Mark the pivots and pivot columns and find the general solution of this linear system.

20. Consider the linear system whose augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 4 & 2 \end{array}\right)$$

and whose reduced row echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & 4 & 2 \\ 0 & 1 & -4 & -1 \end{array}\right)$$

Mark the pivots and pivot columns and find the general solution of this linear system.

21. Consider the linear system whose augmented matrix is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & 0 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 & 2 \end{array}\right)$$

and whose reduced row echelon form is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 9 & 3 \\ 0 & 1 & 0 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & -7 & -1 \\ 0 & 0 & 0 & 1 & 6 & 1 \end{array}\right).$$

Mark the pivots and pivot columns and find the general solution of this linear system.

22. Consider the linear system whose augmented matrix is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 & 1 & 3 \\ 1 & -1 & 2 & 2 & 2 & 0 \end{array}\right).$$

and whose reduced row echelon form is

$$\left(\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 1 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Mark the pivots and pivot columns and find the general solution of this linear system.

SOME TEST QUESTIONS

23. You are facing a linear system of n linear equations in m unknowns. Somebody tells you that that the coefficients of this system have been chosen randomly. How many solutions do you expect this system to have if: $n = m, n < m, n > m$.

24. You have a system of three linear equations in three unknowns that has infinitely many solutions. How are the corresponding three planes in space situated with respect to each other (two essentially different scenarios!)? Explain using some diagrams.

25. Consider an arbitrary system of two linear equations in two unknowns and the corresponding two lines in the plane. Describe the essentially different configurations of the two lines (there are three, one of which is a bit tricky) and the corresponding numbers of solutions of the system of equations.

26. You have two linear equations (1) and (2), say in two unknowns. Why does the system of equations (1), (1)+(2) have exactly the same solutions as the first one?

27. Explain what it means for a system of linear equations to be consistent.

28. A system of 666 linear equations in 666 unknowns has a unique solution. How many pivots does it have?