Differential equations

I got a bit tired of writing solutions. So, for this final calculus chapter I'll only provide you with the answers and NO worked solutions If you get stuck with any of the following problems, just like with integrals, you can ask *Wolfram Alpha* from within *Mathematica* to provide you with step-by-step solutions.

```
WolframAlpha["y'(x) = 2 x y", IncludePods -> {"Differential equation solution"},
PodStates -> {"Step-by-step solution", "Show all steps"}]
```

Alternatively you can also execute the following

```
WolframAlpha["y'(x) = 2 x y"]
```

and then hit the *Step-by-step solution* button in the *Differential equation solution* box of the answer. Try it!

Separable first order differential equations

Find the general solution for each of the following differential equations.

(a) $y' = 2xy$

(b) $yy' + \sin(x) = 0$

(c) $\sin(x)y' + y\cos(x) = 2\cos(x)$

(d) $\frac{1+y'}{1-y'} = \frac{1-y/x}{1+y/x}$

Integrating factor

Use an integrating factor to find the general solution for each of the following differential equations.

(a) $y' + 2y = 2x$

(b) $y' + \frac{2}{x}y = 1$

(c) $y' + \cos(x)y = 3\cos(x)$

(d) $\sin(x)y' + \cos(x)y = \tan(x)$

Second order homogeneous differential equations

Find the general solution for each of the following differential equations.

(a) $y'' + y' - 2y = 0$

(b) $y'' - 9y = 0$

(c) $y'' + 2y' + 2y = 0$

(d) $y'' + 6y' + 10y = 0$

(e) $y'' - 4y' + 4y = 0$

(f) $y'' + 6y' + 9y = 0$

Find the particular solution, for the corresponding differential equation in the previous question, that satisfies the following boundary conditions.

(a) $y(0) = 1$ and $y(1) = 0$

(b) $y(0) = 0$ and $y(1) = 1$

(c) $y(0) = -1$ and $y(+\pi/2) = +1$

(d) $y(0) = -1$ and $y' = 0$ at $x = 0$

(e) $y(0) = 1$ and $y' = 0$ at $x = 1$

(f) $y' = 0$ at $x = 0$ and $y' = 1$ at $x = 1$

If you want to see the steps in *Mathematica*, e.g., for the first problem, execute the following.

```
WolframAlpha["y''(x) +y'(x) -2 y=0, y(0)=1, y(1)=0",
IncludePods -> {"Differential equation solution"},
PodStates -> {"Step-by-step solution", "Show all steps"}]
```

Second order non-homogeneous differential equations

Find the general solution for each of the following differential equations.

(a) $y'' + y' - 2y = 1 + x$

(b) $y'' - 9y = e^{3x}$

(c) $y'' + 2y' + 2y = \sin(x)$

(d) $y'' + 6y' + 10y = e^{2x}\cos(x)$

(e) $y'' - 4y' + 4y = 2x$

(f) $y'' + 6y' + 9y = \cos(x)$

Non-separable first order differential equations

Each of the following differential equations can be solved using the integrating factor trick. However, since we are dealing with constant coefficients there is an alternative method, just guess a particular solution and add it to the general solution of the homogeneous equation, just like in the case second order DEs with constant coefficients. Give it a try.

(a) $y' + y = 1$

(b) $y' + 2y = 2 + 3x$

(c) $y' - y = e^{2x}$

(d) $y' - y = e^{x}$

(e) $y' + 2y = \cos(2x)$

(f) $y' - 2y = 1 + 2x - \sin(x)$

TEST QUESTIONS

1. The integrating factor trick for solving linear first order differential equations of the form
$$y' + P(x)y = Q(x)$$
is based of the product rule for differentiation. Explain.

2. What is a homogeneous linear differential equation?

3. Show that the sum of two solutions of a homogeneous linear differential equation is also a solution of this differential equation.

4. Show that, given a solution $y_p(x)$ of a non-homogeneous linear differential equation, and a solution $y_h(x)$ of the corresponding homogeneous differential equation, the sum
$$y_h(x) + y_p(x)$$
is also a solution of the non-homogeneous equation.

5. You are given a first order DE with a nicely behaved set of solutions. How many of the solutions do you expect to satisfy the initial condition $y(0) = 666$?

6. What does a typical initial condition for a second order differential equation look like?

7. Derive the general solution of a homogeneous second order linear differential equation from scratch.

8. Check that in the case of a repeated root λ of a homogeneous second order linear differential equation the function $xe^{\lambda x}$ really is a solution.