MTH1030 A1

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Part 1 The weirdest 3d Pythagoras yet

A:

Set u and v which are span the parallelogram

```
In[ ]:= u = {u1, u2, u3}
        v = {v1, v2, v3}
```

Plane P

We can calculate the area of P and we can get P^2 which is the right hand side of equation

```
In[ ]:= AreaP = Norm[Cross[u, v]]
```

```
In[ ]:= AreaP^2
```

Out[ ]= $\text{Abs}[-u2\ v1 + u1\ v2]^2 + \text{Abs}[u3\ v1 - u1\ v3]^2 + \text{Abs}[-u3\ v2 + u2\ v3]^2$

Plane O

Due to the plane O is the orthogonal projection of P on yz plane, we can know the u and v projection on yz is:

```
In[ ]:= uo = {0, u2, u3}
        vo = {0, v2, v3}
```

Then we use the cross product of uo and vo calculate the area of O, then we can know the area O square

```
In[ ]:= AreaO = Norm[Cross[uo, vo]]
        AreaO^2
```

Out[ ]= $\text{Abs}[-u3\ v2 + u2\ v3]$

Out[ ]= $\text{Abs}[-u3\ v2 + u2\ v3]^2$

Plane N

Due to the plane N is the orthogonal projection of P on zx plane, we can know the u and v projection on zx is:

```
In[ ]:= un = {u1, 0, u3}
        vn = {v1, 0, v3}
```

Then we use the cross product of un and vn calculate the area of N, then we can know the area N square

In[∘]:=
```
AreaN = Norm[Cross[un, vn]]
AreaN^2
```

Out[∘]= $\text{Abs}[u3\ v1 - u1\ v3]$

Out[∘]= $\text{Abs}[u3\ v1 - u1\ v3]^2$

Plane M

Due to the plane M is the orthogonal projection of P on xy plane, we can know the u and v projection on xy is:

In[∘]:=
```
um = {u1, u2, 0}
vm = {v1, v2, 0}
```

Then we use the cross product of um and vm calculate the area of M, then we can know the area M square

In[∘]:=
```
AreaM = Norm[Cross[um, vm]]
AreaM^2
```

Out[∘]= $\text{Abs}[-u2\ v1 + u1\ v2]$

Out[∘]= $\text{Abs}[-u2\ v1 + u1\ v2]^2$

Final result

We can make a equation and and let Mathematica to verify it is True

In[∘]:=
```
AreaM^2 + AreaN^2 + AreaO^2 == AreaP^2
```

Out[∘]= True

B:

Set the highest point of tetrahedron is point D, and other point is point A, point B and point C
Then we can get vectors: DA, DB, DC
The bottom triangle side vectors: CA, CB ,AB

Area of ADC, ADB and CDB

We can using the cross product to calculate the area, which is the left hand side

In[∘]:=
```
TopArea = 1/2 * (Norm[Cross[DA, DC]] + Norm[Cross[DB, DC]] + Norm[Cross[DA, DB]])
```

Out[∘]= $\frac{1}{2}(\text{Norm}[DA\times DB] + \text{Norm}[DA\times DC] + \text{Norm}[DB\times DC])$

Area of ABC

We also use cross product of two vectors to calculate the area of bottom

$\frac{1}{2}\text{Norm}[CA\times CB] = \frac{1}{2}$Norm[(DA-DC)x(DB-DC)]

$=\frac{1}{2}$Norm[(DA-DC)x(DB-DC)]

$=\frac{1}{2}$Norm[(DAxDB)+ -(DAxDC)+ -(DCxDB)+(DCxDC)]

$=\frac{1}{2}$Norm[(DAxDB)+ -(DAxDC)+ -(DCxDB)]

$=\frac{1}{2}$Norm[(DAxDB)]+Norm[(DAxDC)]+Norm[(DCxDB)] = Left hand side

C:

We can firstly get the area of plane, then get the area of bottom plane, and add up

```
In[•]:= upper = 1 / 2 * (2.5 * 3 + 3 * 4 + 2.5 * 4)  == 3.75 + 6 + 5
Out[•]= True

In[•]:= upper = 1 / 2 * (2.5 * 3 + 3 * 4 + 2.5 * 4)
Out[•]= 14.75

In[•]:= bottom = (3.75^2 + 6^2 + 5^2) ^ (1 / 2)
Out[•]= 8.66386

In[•]:= theSum = upper + bottom
Out[•]= 23.4139
```

Part 2 The Big Cube

Find Aa length and it vector

Let’s tell *Mathematica* about **A** and **a, b**. (a is A’)

```
In[263]:= A = {43.1162, 29.9541, 7.72383}
          a = {42.3208, 27.8349, 0}
          b = {25.7441, 19.8898, 0}
```

and we make vector Aa

```
In[266]:= Aa = a - A
Out[266]= {-0.7954, -2.1192, -7.72383}
```

and we figure out the length of Aa

```
In[166]:= Aa_length = Norm[Aa]
Out[166]= 8.04868
```

Find Plane ABCD equation

aA is vertical with plane ABCD, so we can **set aA as normal vector of ABCD**, and we put the coordinate of A in it to get the equation(A is on the plane)

```
In[169]:= ABCD = 0.7954 x + 2.1192 y + 7.72383 z == 157.431
```

Find the distance of Bb

We start with finding the distance of Bb
Find point p on the plane setting y and z =0

```
In[170]:= p = {157.431 / 0.7954, 0, 0}
Out[170]= {197.927, 0, 0}

In[171]:= v = b - p
```

Out[171]= {-172.183, 19.8898, 0}

In[172]:=
```
distanceBb = Norm[Projection[v, n]]
```

Out[172]= 11.7788

Find point B

We set B as {Bx,By,Bz}

In[176]:=
```
B = {Bx, By, Bz}
```

In[173]:=
```
Bb = {25.7441 - Bx, 19.8898 - By, -Bz}
```

In[174]:=
```
Norm[Bb] == distanceBb
```

$$\sqrt{[25.7441` - Bx]^2 + [19.8898` - By]^2 + Abs[Bz]^2} == 11.778788659188484`$$

We find AB and AB vertical Bb

In[193]:=
```
AB = B - A
Dot[AB, Bb] == 0
```

Out[194]= -16.2081 (25.7441 - Bx) - 6.96297 (19.8898 - By) - 3.57956 Bz == 0

Then we set up 3 equations for solving the coordinate of B

1. The B is on the plane ABCD
2. Norm[Bb]==11.77
3. Dot[AB,Bb]==0

In[195]:=
$$\text{Solve}\Big[0.7954\,Bx + 2.1192\,By + 7.72383\,Bz == 157.431\ \&\&\ \sqrt{(25.7441 - Bx)^2 + (19.8898 - By)^2 + (Bz)^2} == 11.7788\ \&\&\ (25.7441 - Bx)\ (-43.1162 + Bx) + (19.8898 - By)\ (-29.9541 + By) - (-7.72383 + Bz)\ Bz == 0,\ \{Bx, By, Bz\}\Big]$$

Out[195]= {{Bx → 26.9081, By → 22.9911, Bz → 11.3034}}

In[190]:=
```
B = {26.90813027581234`, 22.991133262610784`, 11.303390101606857`}
```

In[196]:=
```
Bb = b - B
```

Out[196]= {-1.16403, -3.10133, -11.3034}

Find point D (Due to character D is protected, we set variable as pointD)

In[210]:=
```
pointD = {Dx, Dy, 11.303390101606857`}
```

Out[210]= {Dx, Dy, 11.3034}

In[211]:=
```
AD = pointD - A
```

Out[211]= {-43.1162 + Dx, -29.9541 + Dy, 3.57956}

We set up 2 equations for solving the coordinate of D

1. AB vertical AD
2. D on plane ABCD

In[212]:=
```
Dot[AB, AD] == 0
```

Out[212]= 12.8133 - 16.2081 (-43.1162 + Dx) - 6.96297 (-29.9541 + Dy) == 0

In[213]:=
```
0.7954 Dx + 2.1192 Dy + 7.72383 * 11.3034 == 157.431
```
Out[213]=
```
87.3055 + 0.7954 Dx + 2.1192 Dy == 157.431
```

In[214]:=
```
Solve[0.7954 Dx + 2.1192 Dy + 7.72383 * 11.3034 == 157.431 && Dot[AB, AD] == 0, {Dx, Dy}]
```
Out[214]=
```
{{Dx → 50.7409, Dy → 14.0459}}
```

In[215]:=
```
pointD = {50.740870303284865`, 14.045947404099287`, 11.303390101606857`}
```
Out[215]=
```
{50.7409, 14.0459, 11.3034}
```

Find point d

The point d in plane abcd, that mean dz = 0

In[219]:=
```
pointd = {dx, dy, 0}
```

In[220]:=
```
Dd = pointd - pointD
```
Out[220]=
```
{-50.7409 + dx, -14.0459 + dy, -11.3034}
```

We set up 2 equations for solving the coordinate of d

1. Aa//Dd
2. Dd vertical AD

In[232]:=
```
AD = pointD - A
Solve[Norm[Cross[Dd, Aa]] == 0 && Dot[Dd, AD] == 0, {dx, dy}]
```
Out[232]=
```
{7.62467, -15.9082, 3.57956}
```
Out[233]=
```
{{dx → 49.5768, dy → 10.9446}}
```

In[236]:=
```
pointd = {49.57684784855689`, 10.944616098272855`, 0}
```
Out[236]=
```
{49.5768, 10.9446, 0}
```

Find point C

In[240]:=
```
pointC = {Cx, Cy, Cz}
```

In[260]:=
```
CD = pointD - pointC
CB = B - pointC
```
Out[260]=
```
{50.7409 - Cx, 14.0459 - Cy, 11.3034 - Cz}
```
Out[261]=
```
{26.9081 - Cx, 22.9911 - Cy, 11.3034 - Cz}
```

We set up 3 equations for solving the coordinate of B

1. CD vertical CB
2. The C is on the plane ABCD
3. CD//AB

In[262]:=
```
Solve[Dot[CD, CB] == 0 && 0.7954 Cx + 2.1192 Cy + 7.72383 Cz == 157.431 &&
  Norm[Cross[CD, AB]] == 0, {Cx, Cy, Cz}]
```
Out[262]=
```
{{Cx → 50.7409, Cy → 14.0459, Cz → 11.3034}, {Cx → 34.5328, Cy → 7.08298, Cz → 14.8829}}
```

Due to the height which is **Cz must larger than Az (7.72383)**

In[268]:=
```
pointC = {34.532800339947386`, 7.082980029533567`, 14.882947880911194`}
```
Out[268]=
```
{34.5328, 7.08298, 14.8829}
```

Find point c

In[237]:=
```
c = {cx, cy, 0}
```

In[287]:=
```
cd = d - c
cb = b - c
Cc = c - pointC
CD = pointD - pointC
Dd = d - pointD
```

Out[287]= {49.5768 - cx, 10.9446 - cy, 0}

Out[288]= {25.7441 - cx, 19.8898 - cy, 0}

Out[289]= {- 34.5328 + cx, - 7.08298 + cy, - 14.8829}

Out[290]= {16.2081, 6.96297, - 3.57956}

Out[291]= {- 1.16402, - 3.10133, - 11.3034}

We set up 2 equations for solving the coordinate of c

1. Cc vertical CD
2. Dd //Cc

In[292]:=
```
Solve[Norm[Cross[Dd, Cc]] == 0 && Dot[Cc, CD] == 0, {cx, cy}]
```

Out[292]= {{cx → 33.0001, cy → 2.99952}}

In[295]:=
```
c = {33.00014769626712`, 2.9995154578492857`, 0}
```

Out[295]= {33.0001, 2.99952, 0}

Find side length

In[304]:=
```
Cc = c - pointC
cd = d - c
```

Out[304]= {- 1.53265, - 4.08346, - 14.8829}

Out[305]= {16.5767, 7.9451, 0}

In[307]:=
```
sidelength = Norm[AB] * 4 + Norm[Aa] + Norm[Cc] + Norm[Bb] * 2 + Norm[ab] * 2 + Norm[cd] * 2
```

Out[307]= 192.644

Find size of plane abcd

abcd is not a square, **but ab = ad, bc = dc**

so we can draw auxiliary lines **ac and bd**

Two lines separate abcd into two triangle

In[309]:=
```
ac = c - a
bd = d - b
```

Out[309]= {- 9.32065, - 24.8354, 0}

Out[310]= {23.8327, - 8.94518, 0}

In[311]:=
```
abcd = 1 / 2 * ( Norm[ac] * (1 / 2 * Norm[bd])) * 2
```

Out[311]= 337.635

Find volume

If we make a rotate of this cube and set it as cube 2, the combination of cube and cube 2 will look like a cuboid, and we just need to divide the volume of combination by 2.

```
In[315]:= areaOfABCD = Norm[AB] * Norm[BC] / 2
Out[315]= 168.532

In[318]:= Height = Norm[Bb] * 2
Out[318]= 23.5576

In[320]:= volume = areaOfABCD * Height / 2
Out[320]= 1985.1
```

Final Result

In[336]:=
```
A
B
C = pointC
D = pointD
a
b
c
d
Norm[Aa]
Norm[Bb]
Norm[Cc]
Norm[Dd]
sidelength
abcd
volume
```

Out[336]= {43.1162, 29.9541, 7.72383}

Out[337]= {26.9081, 22.9911, 11.3034}

Out[338]= {34.5328, 7.08298, 14.8829}

Out[339]= {50.7409, 14.0459, 11.3034}

Out[340]= {42.3208, 27.8349, 0}

Out[341]= {25.7441, 19.8898, 0}

Out[342]= {33.0001, 2.99952, 0}

Out[343]= {49.5768, 10.9446, 0}

Out[344]= 8.04868

Out[345]= 11.7788

Out[346]= 15.5089

Out[347]= 11.7788

Out[348]= 192.644

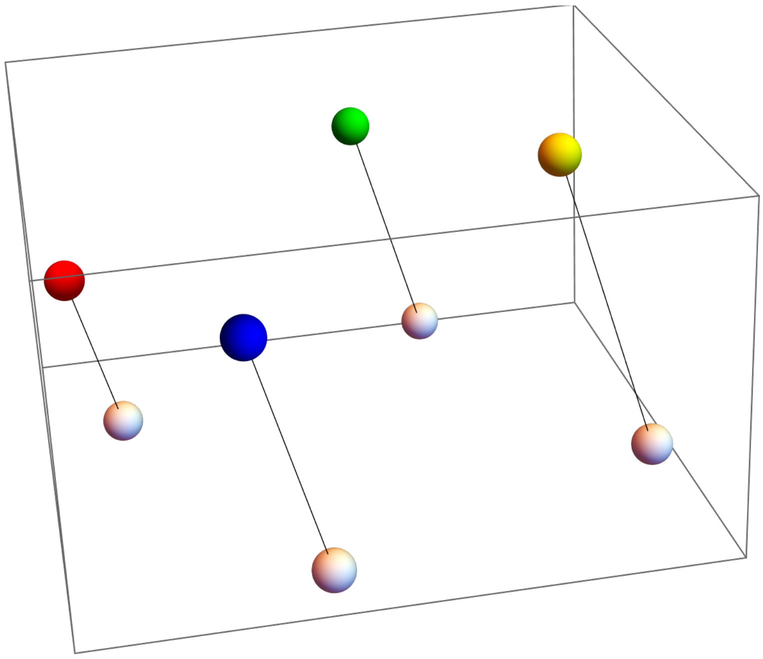
Out[349]= 337.635

Out[350]= 1985.1

In[296]:=
```
r = 1
Graphics3D[{
  {Red, Sphere[A, r]},
  {Blue, Sphere[B, r]},
  {Green, Sphere[pointD, r]},
  {Yellow, Sphere[pointC, r]},
  Sphere[b, r],
  Sphere[a, r],
  Sphere[d, r],
  Sphere[c, r],
  Line[{A, a}],
  Line[{B, b}],
  Line[{pointD, d}],
  Line[{pointC, c}]
 }]
```

Out[296]= 1

Out[297]=



Part 3 An amazing property of unit cubes

A:

First we set up the vectors (xyw, xyv and xyu means the orthogonal projection of vector on xy-plane)

In[∘]:=
```
xyw = {w1, w2, 0}
xyv = {v1, v2, 0}
xyu = {u1, u2, 0}
w = {w1, w2, w3}
v = {v1, v2, v3}
u = {u1, u2, u3}
```

and we can find the area of orthogonal projection of the cube onto the xy-plane

In[∘]:=
```
area = Norm[Cross[xyw, xyv]] + Norm[Cross[xyw, xyu]] + Norm[Cross[xyu, xyv]]
```

Out[∘]= Abs[-u2 v1 + u1 v2] + Abs[u2 w1 - u1 w2] + Abs[v2 w1 - v1 w2]

Then we start calculate with the vector SN
Due to the vector can be replace by the cross product of other two vectors
so we replace w, v and u vector with the cross product

```
In[∘]:= SN = Cross[v, u] + Cross[w, v] + Cross[w, u]
Out[∘]= {u3 v2 - u2 v3 + u3 w2 + v3 w2 - u2 w3 - v2 w3,
  -u3 v1 + u1 v3 - u3 w1 - v3 w1 + u1 w3 + v1 w3, u2 v1 - u1 v2 + u2 w1 + v2 w1 - u1 w2 - v1 w2}
```

Find the length of its orthogonal projection onto the z-axis

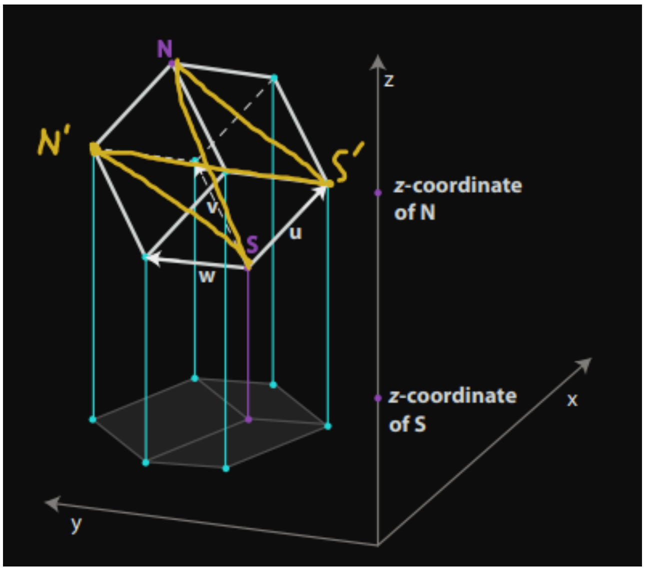
```
In[∘]:= z = {0, 0, 1}
Out[∘]= {0, 0, 1}

In[∘]:= Norm[Projection[SN, z]]
Out[∘]= Abs[u2 v1 - u1 v2 + u2 w1 + v2 w1 - u1 w2 - v1 w2]
```

Then we find Norm[Projection[SN,z]] can be in form:
=Abs[u2 v1-u1 v2]+Abs[u2 w1-u1 w2]+Abs[v2 w1-v1 w2]=area
So Norm[SN] = area

B:

Set the start of vector u is S and the end of u is S'
Set the other side of N is N'

The minimum area of projection in xy-plane can deduced by the projection of SS'NN'plane on z-coordinates



The minimum length on z-coordinates is while SS' // z, this is because after the cube roll over SS'// z, the Norm[SN] is not equal to the area on xy-plane anymore, it turn to Norm[S'N'] = area.

While the SS' // z, the shape on xy-plane is a square, it is the minimum area

```
In[∘]:= minimum = 1 * 1
Out[∘]= 1
```

Maximum happen while SN // z, that means SN show it longest length on z-coordinates
we can use Pythagorean theorem to find the Norm[SN]

In[ ]:=
```
maximum = Square[Square[1^2 + 1^2] + 1^2]
```

Out[ ]= □ (1 + □2)

C:

If we change the side length, the relationship will not be change