MTH1030/35: Assignment 3, 2022

Step by step to infinity

Due online Tuesday, 17 May, at 5 pm. 11:55 pm.

The Rules of the Game

In general, if no serious attempt at solving one of the questions was made assign 0 marks to that question.

As much as possible binary marking following the guidelines given below.

At the end of each question please note the number of marks awarded.

Overall, we are marking generously :)

1 Sequences

1.1 [10 marks]

Show that the following sequence has a limit and find this limit.

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$$

Hint: Express the terms of this sequence as powers of 2.

Solution:

. For $\left\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\right\}$, $a_1 = 2^{1/2}$, $a_2 = 2^{3/4}$, $a_3 = 2^{7/8}$, …, so $a_n = 2^{(2^n-1)/2^n} = 2^{1-(1/2^n)}$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{1-(1/2^n)} = 2^1 = 2.$

Alternate solution: Let $L = \lim_{n\to\infty} a_n$. (We could show the limit exists by showing that $\{a_n\}$ is bounded and increasing.)

Then L must satisfy $L = \sqrt{2 \cdot L} \quad \Rightarrow \quad L^2 = 2L \quad \Rightarrow \quad L(L-2) = 0$. $L \neq 0$ since the sequence increases, so $L = 2$.

3 marks for writing the terms of the sequence as powers of 2. Then 2 marks for spotting the right pattern in the exponents and 5 marks for figuring out the correct limit.

1.2 [10 marks]

Let $a_1 = a, a_2 = f(a_1), a_3 = f(a_2), \ldots, a_{n+1} = f(a_n)$, where a is some number and f is a continuous function. If $\lim_{n\to\infty} a_n = L$, show that $f(L) = L$.

Now, let $a = 1$.

Find an example of a function f such that the corresponding sequence converges and such that $f(x) = x$ has 1 solution.

Find an example of a function f such that the corresponding sequence converges and such that $f(x) = x$ has more than one solution.

Find an example of a function f such that the corresponding sequence diverges and such that $f(x) = x$ has no solution.

Find an example of a function f such that the corresponding sequence diverges and such that $f(x) = x$ has a solution.

Hint: This and some of the other questions in this assignment ask you to come up with some examples of functions and series that have certain properties. ALL of these questions have VERY simple functions and series as answers.

Solution:

If $\lim_{n\to\infty} a_n = L$, then, since f is continuous, $f(L) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_{n+1}$. But since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$ we conclude that $L = f(L)$.

2 marks for anything resembling this.

$f(x) = 0$: constant sequence, $L = 0$ (obviously), one solution

Any example that obviously works: 2 marks. Any example that clearly does not work 0 marks. Same with any crazy example that comes without any explanation. Crazy example with some merit 1 mark.

$f(x) = x^2$, constant sequence, $L = 1$ (obviously), two solutions ($x = 0, 1$)

Any example that obviously works: 2 marks. Any example that clearly does not work 0 marks. Same with any crazy example that comes without any explanation. Crazy example with some merit 1 mark.

$f(x) = x + 1$, obviously diverges, no solution.

Any example that obviously works: 2 marks. Any example that clearly does not work 0 marks. Same with any crazy example that comes without any explanation. Crazy example with some merit 1 mark.

$f(x) = 2x$, obviously diverges, one solution ($x = 0$)

Any example that obviously works: 2 marks. Any example that clearly does not work 0 marks. Same with any crazy example that comes without any explanation. Crazy example with some merit 1 mark.

2 Serious series

2.1 [10 marks]

The alternating series test is a theorem which says the following: If a series

$$a_1 - a_2 + a_3 - a_4 + a_5 - ...$$

satisfies (1) $a_n > 0$, (2) $a_n \geq a_{n+1}$ and (3) $\lim_{n\to\infty} a_n = 0$, then the series converges. Very powerful and very useful. For example, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

satisfies (1), (2), and (3).

a) Give an example of a divergent series that satisfies (1) + (3) but not (2).

b) Give an example of a divergent series that satisfies (1) + (2) but not (3).

Solution:

a) Make the odd terms coincide with those of the harmonic series and the even terms with a converging geometric series.

Any example that obviously works: 5 marks. Any example that clearly does not work 0 marks. Same with any crazy example that comes without any explanation. Crazy example with some merit 3 marks.

b) Set $a_n = 1$ for all n will do, or anything decreasing that does not converge to 0.

Any example that obviously works: 5 marks. Any example that clearly does not work 0 marks. Same with any crazy example that comes without any explanation. Crazy example with some merit 3 marks.

2.2 [10 marks]

a) If both $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ converge and both sums are equal what can you conclude about the two series?

b) If one of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} |a_n|$ converges and the other one diverges, which converges and which diverges?

Solution:

a) If any of the a_n is negative, then $a_n < |a_n|$. This cannot be the case because since it would imply that $\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} |a_n|$. Hence all a_n are non-negative and the two series are identical.

2 marks for spelled out reasoning, 3 marks for correct conclusion.

b) By Theorem 2.9.1 in the lecture notes, if $\sum_{n=1}^{\infty} |a_n|$ converges, then both converge. Hence $\sum_{n=1}^{\infty} |a_n|$ diverges and $\sum_{n=1}^{\infty} a_n$ converges.

2 marks for spelled out reasoning, 3 marks for correct conclusion.

2.3 [5 marks]

Let $\displaystyle\sum_{n=0}^{\infty} a_n$ and $\displaystyle\sum_{n=0}^{\infty} b_n$ be two divergent series. Consider the interlaced series

$$a_1 + b_1 + a_2 + b_2 + a_3 + b_3 + a_4 + b_4 + \cdots .$$

Is it possible that the new series is convergent? If you think that this is not possible give a reason why. If you think that it is possible give an example.

Solution:

Yes, e.g., zip together the harmonic series and its negative:

$$1 - 1 + 1/2 - 1/2 + 1/3 - 1/3 + ...$$

Any example that obviously works: 5 marks. Any example that clearly does not work 0 marks. Same with any crazy example that comes without any explanation. Crazy example with some merit 3 marks.

2.4 [10 marks]

For which values of x does the following series converge? And, for those x for which the series converges, what is the sum of the series?

$$\sum_{n=1}^{\infty} \frac{\cos^n(x)}{2^n}.$$

Warning: Careful we start with $n = 1$ and not $n = 0$.

Solution:

$\sum_{n=1}^{\infty} \frac{\cos^n(x)}{2^n}$ is a geometric series with ratio $r = \frac{\cos(x)}{2}$ and so it converges if and only if $|r| < 1$. But $\frac{|\cos(x)|}{2} < \frac{1}{2}$ for all x. This means that the series converges for all real values of x. Since the series starts with $n = 1$ and not with $n = 0$, its sum is

$$\frac{1}{1 - \frac{\cos(x)}{2}} - 1 = \frac{\cos(x)}{2 - \cos(x)}.$$

3 marks each for correct answers: $\frac{\cos(x)}{2-\cos(x)}$ AND for all real values of x. 4 marks for correct reasoning: geometric, $\frac{|\cos(x)|}{2} < \frac{1}{2}$ for all x.

2.5 [10 marks]

Consider the unit square in the xy-plane whose corners are $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$. Subdivide it into nine equal smaller squares and remove the square in the centre. Next, subdivide each of the remaining eight squares into nine even smaller squares, and remove each of the centre squares. And so on. The following diagram shows what's left after the first three steps of this construction. Give an example of a point in the original square that never gets removed. Show that the area of what is left over when all those squares have been removed is 0, by verifying that the sum of the areas of all the removed squares is 1.

Solution:

Corners of the original square never get removed.

The area removed at the first step is $\frac{1}{9}$; at the second step, $8 \cdot \left(\frac{1}{9}\right)^2$; at the third step, $(8)^2 \cdot \left(\frac{1}{9}\right)^3$. In general, the area removed at the nth step is $(8)^{n-1}\left(\frac{1}{9}\right)^n = \frac{1}{9}\left(\frac{8}{9}\right)^{n-1}$, so the total area of all removed squares is

$$\sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{8}{9}\right)^{n-1} = \frac{1/9}{1 - 8/9} = 1.$$

2 marks for some points of the square that obviously do not get removed. 8

marks for setting up and summing the correct geometric series (3 or 6 partial marks for attempts of different merit.

2.6 [10 marks]

Consider the series

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$$

Calculate the first four partial sums of this series. Do some pattern spotting to come up with a simple function $p(n)$ such that $s_n = p(n)$ for $n = 1, 2, 3, 4$. Assuming that $s_n = p(n)$ for all n, show that our series is convergent and evaluate its sum.

Solution:

For $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$, $s_1 = \frac{1}{1\cdot 2} = \frac{1}{2}$, $s_2 = \frac{1}{2} + \frac{2}{1\cdot 2\cdot 3} = \frac{5}{6}$, $s_3 = \frac{5}{6} + \frac{3}{1\cdot 2\cdot 3\cdot 4} = \frac{23}{24}$,

$s_4 = \frac{23}{24} + \frac{4}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \frac{119}{120}$. The denominators are $(n+1)!$, so a guess would be $p(n) = \frac{(n+1)!-1}{(n+1)!}$.

$\lim_{n\to\infty} p(n) = \lim_{n\to\infty} \frac{(n+1)!-1}{(n+1)!} = \lim_{n\to\infty} \left[1 - \frac{1}{(n+1)!}\right] = 1$ and so $\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1$.

4 marks for correct partial sums, 3 marks for guessing the correct general term, 3 marks for finding the correct limit.

3 A cat and ~~mouse~~ dog game [25 marks]

Let's play a game. I am thinking of two differentiable functions $cat(x)$ and $dog(x)$. Both functions can be written as power series

$$cat(x) = \sum_{n=0}^{\infty} c_n x^n, \quad dog(x) = \sum_{n=0}^{\infty} d_n x^n.$$

Both power series converge for all $x \in \mathbf{R}$ which means that both functions are defined everywhere. I am also telling you that

$$cat(0) = 0 \text{ and } dog(0) = 1$$

and that

$$dog'(x) = cat(x) \text{ and } cat'(x) = dog(x).$$

(a) Find $cat(x)$ and $dog(x)$ by calculating the general terms c_n and d_n of their power series. Hint: To figure out what the coefficients of the two power series are use the fact that two power series are equal if and only if corresponding coefficients are equal and note that $cat''(x) = cat(x)$.

[10 marks]

(b) Write the functions e^x and e^{-x} in terms of of $cat(x)$ and $dog(x)$.

[5 marks]

(c) Conversely, express $cat(x)$ and $dog(x)$ as a combination of e^x and e^{-x}.

[5 marks]

(d) Using *Mathematica* or another piece of software plot $cat(x)$ and the first four *different* partial sums of its power series in the interval $[-2, 2]$.[1]

[5 marks]

Solution: a) $cat''(x) = cat(x)$ and $dog''(x) = dog(x)$. Hence, by deriving the corresponding power series termwise and comparing coefficients we get:

$$a_{2n} = 0, b_{2n} = \frac{1}{(2n)!}.$$

But because $cat'(x) = dog(x)$ and $dog'(x) = cat(x)$ we also get $b_{2n-1} = 0$ and $a_{2n-1} = \frac{1}{(2n-1)!}$

$$cat(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + ...$$

$$dog(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ...$$

5 marks for cat and 5 for cat. Some partial marks okay in this case.

Of course, *cat* and *dog* are just the hyperbolic functions *sinh* and *cosh*.

b)

$$e^x = cat(x) + dog(x) \text{ and } e^{-x} = dog(x) - cat(x).$$

3 marks for getting one right 5 marks for both.

[1]In *Mathematica* several functions can be plotted in the same diagram as follows:

```
Plot[{Cos[x], Cos[2 x], Cos[3 x]}, {x, 0, 2 Pi}, PlotLegends -> "Expressions"]
```

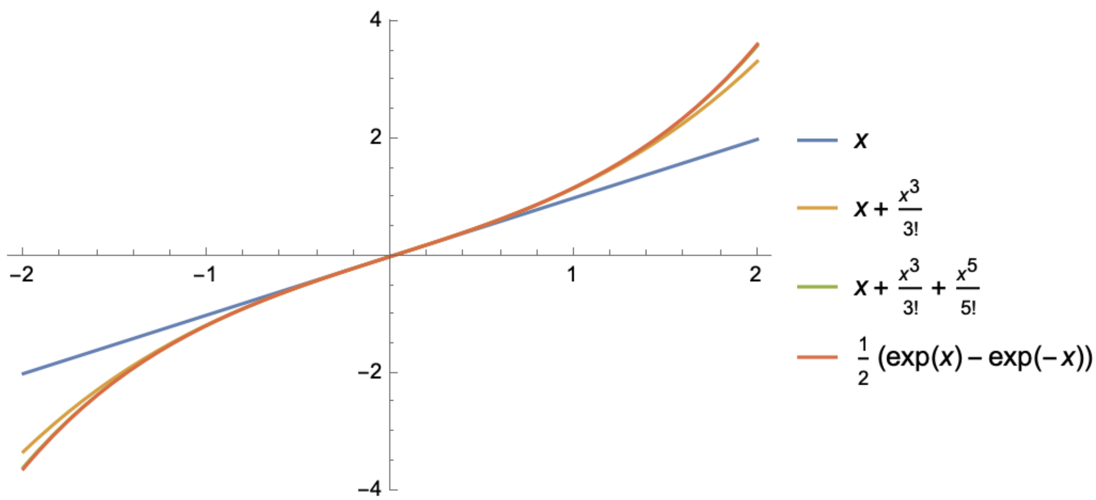
c)

$$cat(x) = \frac{e^x - e^{-x}}{2} \text{ and } dog(x) = \frac{e^x + e^{-x}}{2}.$$

3 marks for getting one right 5 marks for both.

d)

```
Plot[{x, x + x^3 / Factorial[3], x + x^3 / Factorial[3] + x^5 / Factorial[5],
  (Exp[x] - Exp[-x]) / 2}, {x, -2, 2 }, PlotLegends → "Expressions"]
```



1 mark per partial sum plus 1 for the correct domain.