MTH1030/MTH1035 Problem sets

These notes have only been revised recently. Please e-mail me about any typos or other nonsense that you stumble across while using them:
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The current reward for finding anything in this respect is one Freddo frog.

Burkard Polster

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Introduction

Hello again,

This is a collection of problems that you can use to test your understanding of the material covered in the lectures. Try to solve as many of these problems as you can in the support classes and as part of your exam preparation. Short solutions of the basic problems are provided in a separate document. Full worked solutions of the more challenging problems are also provided once you've talked about the problem sets they are part of in the support classes.

You should always bring a copy of the problems you expect to work on to your support class.

This collection incorporates many of the problems that were used in previous years. These were compiled by my colleague Dr Leo Brewin. Also included in this document are many exercises adapted from Professor Kuttler's linear algebra book *Elementary Linear Algebra*.[1] and from the calculus textbook by Edwards and Penney, Calculus - Early Transcendentals, 7th edition.

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[1]Download it for free from here `http://tinyurl.com/cu7fsph`

Contents

Part I

Linear algebra

Vectors, dot product, cross product

IMPORTANT: In these problem sets, the more advanced problems, especially those involving proofs, have been included mainly to challenge those of you enrolled in MTH1035 and MTH1040. If you are enrolled in MTH1030, in the first instance please make absolutely sure that you understand the worked problems in the lecture notes and that you can solve the basic problems in every one of the following problem sets and you'll be fine.

1. Find all the vectors whose tips and tails are among the three points with coordinates $(2, -2, 3)$, $(3, 2, 1)$ and $(0, -1, -4)$.

2. Let $\mathbf{v} = (3, 2, -2)$. How long is $-2\mathbf{v}$. Find a unit vector (a vector of length 1) in the direction of $\mathbf{v}$.

3. For each pair of vectors given below, calculate the dot product and the angle θ between the vectors.

   (a) $\mathbf{v} = (3, 2, -2)$ and $\mathbf{w} = (1, -2, -1)$

   (b) $\mathbf{v} = (0, -1, 4)$ and $\mathbf{w} = (4, 2, -2)$

   (c) $\mathbf{v} = (2, 0, 2)$ and $\mathbf{w} = (-3, -2, 0)$

4. Given the two vectors $\mathbf{v} = (\cos(\theta), \sin(\theta), 0)$ and $\mathbf{w} = (\cos(\phi), \sin(\phi), 0)$, use the dot product to derive the trigonometric identity $\cos(\theta - \phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)$.

5. Use the dot product to determine which of the following two vectors are perpendicular to one another: $\mathbf{u} = (3, 2, -2), \mathbf{v} = (1, 2, -2), \mathbf{w} = (2, -1, 2)$.

6. For each pair of vectors given below, calculate the cross product by hand. Calculate the area of the parallelogram spanned by the two vectors.

   (a) $\mathbf{v} = (3, 2, -2), \mathbf{w} = (1, -2, -1)$

   (b) $\mathbf{v} = (0, -1, 4), \mathbf{w} = (4, 2, -2)$

   (c) $\mathbf{v} = (2, 0, 2), \mathbf{w} = (-3, -2, 0)$

7. Find the area of the triangle determined by the three points, $(1, 2, 3), (4, 2, 0)$ and $(-3, 2, 1)$.

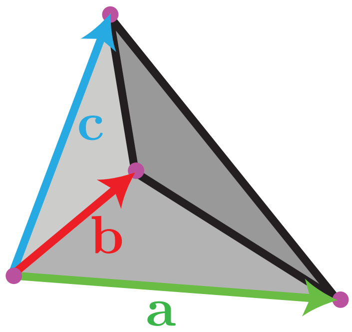
8. Calculate the volume of the parallelepiped defined by the three vectors $\mathbf{u} = (3, 2, -2)$, $\mathbf{v} = (1, 2, -2)$, $\mathbf{w} = (2, -1, 2)$.

EXTRA QUESTIONS

9. Prove that, given any two vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$, the following holds
$$||\mathbf{u}| - |\mathbf{v}|| \leq |\mathbf{u} - \mathbf{v}|.$$
(Hint: Start by writing $\mathbf{u} = \mathbf{u} - \mathbf{v} + \mathbf{v}$ and then apply the triangle inequality.)

10. Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$. Define as in the lecture notes
$$\mathbf{proj_u}(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2}\right) \mathbf{u} \text{ (the projection of } \mathbf{v} \text{ onto } \mathbf{u})$$
and
$$\mathbf{perp_u}(\mathbf{v}) = \mathbf{v} - \mathbf{proj_u}(\mathbf{v})$$
Prove that $\mathbf{perp_u}(\mathbf{v}) \cdot \mathbf{proj_u}(\mathbf{v}) = 0$, that is, $\mathbf{perp_u}(\mathbf{v})$ is perpendicular to $\mathbf{proj_u}(\mathbf{v})$.

11. Prove that the tetrahedron determined by the three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ (as shown in the following diagram) is equal to one sixth of the volume of the parallelepiped determined by the same vectors.



12. If $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 5$ what is the value of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$?

THE FOLLOWING PROBLEMS ARE FROM KUTTLER'S BOOK

13. Does it make sense to write $(1, 2) + (3, 4, 5)$?

14. Three forces are applied to a point which does not move. Two of the forces are $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$ Newtons and $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ Newtons. Find the third force.

15. Does it make sense to speak of $\mathbf{proj_0}(\mathbf{v})$?

16. Show that
$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{4}[|\mathbf{a} + \mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2].$$

17. Prove the parallelogram identity,

$$|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2.$$

18. Show that if $\mathbf{a} \times \mathbf{u} = \mathbf{0}$ for all unit vectors $\mathbf{u}$, then $\mathbf{a} = \mathbf{0}$.

19. Suppose $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are three vectors whose components are all integers. Can you conclude that the volume of the parallelepiped determined by these three vectors will always be an integer?

20. Is $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$? (Hint: Try $(\mathbf{i} \times \mathbf{j}) \times \mathbf{j}$.)

SOME MORE SHORT QUESTIONS TO TEST YOUR UNDERSTANDING OF THIS MATERIAL.

21. Two vectors $(1, 2)$ and $(2, 3)$ are added by adding corresponding components. Interpreted as arrows in the plane, the two vectors get added by arranging them tip to tail. Explain why this gives the same result.

22. Give an algebraic and a geometric argument that shows why $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

23. What has the distance between two points in $\mathbf{R}^n$ to do with Pythagoras' theorem?

24. Can you justify the individual steps in the proof of the Cauchy-Schwarz inequality $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ given in the lecture notes?

25. Prove the triangle inequality $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$ using the Cauchy-Schwarz inequality.

26. Why is it called the triangle inequality?

27. Use the Cosine Rule

$$a^2 + b^2 - 2ab\cos(\theta) = c^2$$

to show that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$ for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and the angle θ between them.