MTH1030
Techniques for Modelling

Lecture 28 & 29

Integration (part 1)

Monash University

Semester 1, 2022

Warm welcoming words

Integration is an operation which returns the (signed) area underneath the plot of a (nice) function. Somehow this operation is closely related to the derivative...

Derivative

First let's recall the derivative. A function $f : I \to \mathbb{R}$ is *differentiable* at a point $x_0 \in I$ if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. Equivalently,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

If it exists, we say that the derivative of f at x_0 is precisely this limit, and we denote it by $f'(x_0)$.

A function is called differentiable if it is differentiable for all $x_0 \in I$. The operator $\frac{\mathrm{d}}{\mathrm{d}x}$ is called the derivative operator. Specifically

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) := f'(x).$$

Derivative

Common derivatives include:

- $$\frac{\mathrm{d}}{\mathrm{d}x} x^n = nx^{n-1}.$$

- $$\frac{\mathrm{d}}{\mathrm{d}x} C = 0,$$

  where $C \in \mathbb{R}$.

- $$\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x.$$

- $$\frac{\mathrm{d}}{\mathrm{d}x} \ln(x) = 1/x.$$

Derivative

- $$\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = \cos(x).$$
- $$\frac{\mathrm{d}}{\mathrm{d}x}\cos(x) = -\sin(x).$$
- $$\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = 1/\cos^2(x).$$

Differentiation rules

If we have two functions f, g which are differentiable, then:

- d/dx is linear, meaning

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(af(x)+bg(x)\right) &= a\frac{\mathrm{d}}{\mathrm{d}x}f(x)+b\frac{\mathrm{d}}{\mathrm{d}x}g(x)\\ &= af'(x)+bg'(x).\end{aligned}$$

- The *chain rule* states that

$$\frac{\mathrm{d}}{\mathrm{d}x}f(g(x)) = f'(g(x))g'(x).$$

- The *product rule* states that

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x)g(x) = f'(x)g(x)+f(x)g'(x).$$

- The *quotient rule* states that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}.$$

Differentiation rules

That was all assumed knowledge. The proofs follow directly from the definition of derivative.

Integration

What actually is an integral?

Definition (Riemann integral)

Let $f : I \to \mathbb{R}$ be a piecewise continuous and bounded function. The Riemann integral of f over the interval $[a, b]$ is given by

$$\int_a^b f(x)\mathrm{d}x = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(x_i)\left(x_{i+1} - x_i\right),$$

where the limit runs over the partitions $\{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$.

Actually, the definition is a bit more complicated than this, but the idea is not too hard to comprehend.

Integration

It's not hard to realise that if we have numbers $a < b < c$ then the integral over $[a, c]$ can be decomposed as the integral over $[a, b]$ and then $[b, c]$. Specifically,

$$\int_a^c f(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_b^c f(x)\mathrm{d}x.$$

Integration

A couple of notes:

1. A Riemann integral is often called a *definite integral*. This in contrast to an indefinite integral (more on that later).
2. If we say integral we mean Riemann integral/definite integral. But in school you might have said integral to mean indefinite integral...Depends on context!

Integration

Example

Okay let's look at finding the integral of $f(x) = x^2$ over the interval $[1, 3]$. This is

$$\int_1^3 x^2 \mathrm{d}x = \lim_{n \to \infty} \sum_{i=0}^{n-1} x_i^2 (x_{i+1} - x_i).$$

Integration

Looks horrible! Apparently we need to evaluate some weird limit in order to compute integrals. Is there a better way?

Integration

Theorem (Fundamental theorem of calculus)

Let $f : I \to \mathbb{R}$ be a continuous and bounded function. Let F be a function such that $F'(x) = f(x)$. Then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Remark

The function F above is called an *antiderivative* of f. Antiderivatives are non-unique. If F is a antiderivative, then so is $F(x) + C$ for any $C \in \mathbb{R}$.

Integration

The fundamental theorem of calculus is huge! It states that the act of differentiation (instantaneous rate of change) is fundamentally linked to integration (signed area under plot). We'll look at an idea of a proof later.

Indefinite integral

Because antidifferentiation is so connected to integration, we have the following object:

Definition (Indefinite integral)

Let $f : I \to \mathbb{R}$. Then the *indefinite integral* of f is

$$\int f(x)\mathrm{d}x = F(x) + C,$$

where F is an antiderivative of f, i.e., $F'(x) = f(x)$.

So indefinite integral is like antiderivative, but you really need to put this $+C$ business for an indefinite integral.

Indefinite integral

Common indefinite integrals include:

- $$\int x^n \mathrm{d}x = \frac{1}{n+1} x^{n+1} + C$$

  for $n \neq -1$.
- $$\int K \mathrm{d}x = xK + C$$

  where $C \in \mathbb{R}$.
- $$\int e^x \mathrm{d}x = e^x + C.$$
- $$\int \frac{1}{x} \mathrm{d}x = \log(|x|).$$

Indefinite integral

- $$\int \sin(x)\mathrm{d}x = -\cos(x) + C.$$

- $$\int \cos(x)\mathrm{d}x = \sin(x) + C.$$

Indefinite integral

Lastly, integration (both definite and indefinite) is linear, meaning

$$\int_a^b [c_1 f(x) + c_2 g(x)]\,\mathrm{d}x = c_1 \int_a^b f(x)\mathrm{d}x + c_2 \int_a^b g(x)\mathrm{d}x,$$

and

$$\int [c_1 f(x) + c_2 g(x)]\,\mathrm{d}x = c_1 \int f(x)\mathrm{d}x + c_2 \int g(x)\mathrm{d}x.$$

Integration

What if we want to integrate a more complicated function, like for example

$$\int x \ln(x) \mathrm{d}x.$$

What can we do?

Integration by parts

We have a couple of indispensable tools, the first being:

Proposition (Integration by parts)

Let f, g be functions which are both differentiable. Then

1. $$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x.$$

2. $$\int_a^b f(x)g'(x)\mathrm{d}x = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\mathrm{d}x.$$

Integration by parts

Let's use this in a couple of examples before proving it.

Example

We wanted to evaluate

$$\int x \ln(x) \mathrm{d}x.$$

Integration by parts

Example

How about

$$\int xe^x \mathrm{d}x.$$

Integration by parts

Example

This one is a little more tricky:

$$\int e^x \cos(x) \mathrm{d}x.$$

Question 1

Question (1)

The following integral

$$\int_1^2 \ln(x)\mathrm{d}x$$

is equal to

1. $x\ln(x) - x$.
2. $2\ln(1) - 1$.
3. $\ln(4) - 1$.
4. $\ln(2) - 2$.

Integration by substitution

Our next tool is:

Proposition (Integration by substitution)

Let f, g be functions such that g is differentiable. Then:

1. $$\int f(g(x))g'(x)\mathrm{d}x = \int f(u)\mathrm{d}u.$$

2. $$\int_a^b f(g(x))g'(x)\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\mathrm{d}u.$$

Honestly looks more complicated than it actually is...let's see examples!

Integration by substitution

Example

We want to calculate

$$\int \sin(3x)\mathrm{d}x.$$

Integration by substitution

Example

How about this?

$$\int 2x\cos(x^2+1)\mathrm{d}x.$$

Integration by substitution

Example

And this...

$$\int \cos(x)\ln(\sin(x))\,\mathrm{d}x.$$

Question 2

Question (2)

The following integral

$$\int_0^2 xe^{-x^2}\,\mathrm{d}x$$

is equal to

1. $\frac{1}{2}\left(1 - e^{-2}\right)$.
2. $\frac{1}{2}\left(1 - e^{-4}\right)$.
3. $-\frac{1}{2}e^{-x^2/2}$.
4. $\frac{1}{2}\left(e^{-2} - 1\right)$.

Integration by substitution

Maybe you could tell, but integration by parts and integration by substitution look very similar to some of the differentiation rules we've stumbled upon. That's no accident!

1. Integration by parts = Reverse product rule.
2. Integration by substitution = Reverse chain rule.

How?

Integration

Sounds great! Seems like integration is as easy as differentiation, as to do it, you just need to 'reverse' differentiation.

But unfortunately that is not the case. The following integral

$$\int_a^b e^{-x^2} \mathrm{d}x$$

can not be computed in a closed-form sense. This is because there *does not* exist an elementary function $F(x)$ such that $F'(x) = e^{-x^2}$.

Remark

In general, **there is no guarantee** that an arbitrary function f will possess an elementary antiderivative F.

Improper integration

We've seen the following integral before

$$\int_1^\infty \frac{1}{x}\mathrm{d}x.$$

But what does this actually mean? Actually, this is technically not a Riemann integral!

This is an *improper integral*, and is actually defined by

$$\lim_{a\to\infty}\int_1^a \frac{1}{x}\mathrm{d}x.$$

More on this next time!