MTH1030
Techniques for Modelling

Lecture 31 & 32

Differential equations (part 1)

Monash University

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Warm welcoming words

Differential equations are equations which involve derivatives of a function $y(x)$. Solving them (that is, finding $y(x)$) is in general, non-trivial! In fact, these things can become exceedingly difficult very quickly.

Differential equations

What's a differential equation (DE)? Here's an example

$$y''(x) + 10(y(x))^2 = x^3.$$

The goal here is to figure out a way to express $y(x)$ in terms of elementary functions (usually), this is called solving the DE.

Naively, we can try and solve this by using integration, as integration (via the fundamental theorem of calculus) undoes integration.

Differential equations

Here's the DE:

$$y''(x) + 10(y(x))^2 = x^3.$$

Okay so

$$y'(x) + 10 \int (y(x))^2 \mathrm{d}x = \frac{1}{4}x^4 + C,$$

and thus

$$y(x) + 10 \int \left(\int (y(x))^2 \mathrm{d}x \right) \mathrm{d}x = \frac{1}{20}x^5 + Cx + D.$$

Hence

$$y(x) = -10 \int \left(\int (y(x))^2 \mathrm{d}x \right) \mathrm{d}x + \frac{1}{20}x^5 + Cx + D.$$

Is this what we want? (Ans: no).

Differential equations

A couple of points:

- We will often suppress the argument of y, so e.g., we would write

$$y'' + 10y^2 = x^3.$$

- Notice that these constants of integration C and D turn up. Apparently that means the solution to the DE will depend on C, D...does that mean there are infinitely many solutions?

Differential equations

That one seemed complicated. Here's a really easy example of a differential equation.

$$y' = 5.$$

The solution to this is $y(x) = 5x + C$...for some C. So how do we know which C?

What if I told you not only the DE, but also that $y(0) = 3$? Then we get $C = 3$.

To formalise this, the solution to the following so-called initial value problem

$$\begin{aligned} y' &= 5, \\ y(0) &= 3, \end{aligned}$$

is $y(x) = 5x + 3$.

Differential equations

In general a DE is the following:

Definition (Differential equation)

An equation of the form

$$H(x, y, y', y'', \dots, y^{(n)}) = 0$$

is called a *differential equation* for the function $y(x)$.

Remark

If in addition we have a condition $y(0) = c$ for some real number c, then the DE is called an *initial value problem* (IVP).

Remark

The *order* of a DE is the number corresponding to the highest order derivative in the DE.

Differential equations

Here are some examples of DEs. Let's determine their order.

$$y' + xy + x^2 = 0.$$

$$y'' + \cos(x)y' + x^2 = 5.$$

$$y^{(5)} + \cos(x)(y'')^3 = x^2.$$

$$y' + xy^8 + x^2 = y.$$

Differential equations

Here are some examples of IVPs:

$$\begin{aligned} y' + xy + x^2 &= 0, \\ y(0) &= 8. \end{aligned}$$

$$\begin{aligned} y'' + \cos(x)y' + x^2 &= 5, \\ y(0) &= 3. \end{aligned}$$

$$\begin{aligned} y^{(5)} + \cos(x)(y'')^3 &= x^2, \\ y(0) &= 1. \end{aligned}$$

$$\begin{aligned} y' + xy^8 + x^2 &= y, \\ y(0) &= 2. \end{aligned}$$

Question 1

Question (1)

The order of the following DE is

$$8y'' + y^{10} = x^3$$

is equal to:

1. 10.
2. 2.
3. 3.
4. 8.

Differential equations

A couple of points:

- A DE of order n will require n constraints in order to obtain a unique solution (if everything is nice). If you have less, you'll only have a family of solutions.
- Actually, what we are studying are often referred to as *ordinary differential equations* (ODEs). This is to contrast them with *partial differential equations* (PDEs). We won't study PDEs at all, but it is good to know the terminology in case you come across this somewhere else!

Differential equations

Example

Consider the second-order IVP

$$\begin{aligned} y'' &= xe^x, \\ y(0) &= 3, \\ y'(0) &= 1. \end{aligned}$$

This is not too hard to solve!

Question 2

Question

Consider the second order IVP

$$\begin{aligned} y'' &= \cos(x), \\ y(0) &= 4, \\ y'(0) &= 1. \end{aligned}$$

The solution to this is

1. $y(x) = -\sin(x) + x + 5.$
2. $y(x) = -\cos(x) + 5x + 1.$
3. $y(x) = -\cos(x) + x + 5.$
4. $y(x) = \cos(x) + x + 5.$

Differential equations

Example

Consider the first order IVP

$$y' + 2xy = 6x,$$
$$y(0) = 4.$$

The solution to this is $y(x) = 3 + e^{-x^2}$...Why? How?

Differential equations

Answering ‘why’ is easy, just verify that the proposed solution works!
Answering ‘how’ is much much harder.

First-order DEs are rather manageable. Second-order are more complicated but also manageable. Beyond this it gets a bit complex. We’ll have a better look at this next time!