# Linear transformations

### A LOT OF THE FOLLOWING ARE ADAPTED FROM KUTTLER'S BOOK

1. Find the matrix of the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\pi/3$ (=60 degrees) in the counterclockwise direction.
2. Find the matrix of the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\pi/4$ (=45 degrees) in the *clockwise* direction.
3. Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\pi/3$ in the counterclockwise direction and then reflects across the $x$-axis.
4. Find the matrix of the linear transformation which reflects every vector in $\mathbf{R}^2$ through the $x$-axis and then rotates every vector through an angle of $\pi/3$ in the counterclockwise direction.
5. Find the matrix of the linear transformation which reflects every vector in $\mathbf{R}^2$ through the line containing the origin and making an angle of 45 degrees with the $x$-axis.
6. Find the matrix of the linear transformation $\mathbf{R}^2 \to \mathbf{R}^2$ that first rotates through an angle $\alpha$ in the clockwise direction and then rotates through an angle $\beta$ in the counterclockwise direction.
7. Find the matrix of the linear transformation which rotates every vector in $\mathbf{R}^3$ counter-clockwise around the $z$-axis (when viewed from the positive $z$-axis) through an angle of $\pi/3$ and then reflects through the $xy$-plane.
8. Find the matrix for $\mathbf{proj_u}(\mathbf{v})$ where $\mathbf{u} = (1, 5, 3)^T$.
9. Find the matrix for $\mathbf{proj_u}(\mathbf{v})$ where $\mathbf{u} = (1, 0, 3)^T$
10. What is the matrix that describes the orthogonal projection onto the $xy$-plane in $\mathbf{R}^3$.
11. How would you construct the matrix that describes the orthogonal projection onto a plane given by a normal vector $\mathbf{u}$ in $\mathbf{R}^3$?
12. What is the determinant of a rotation of $\mathbf{R}^2$ and what is the determinant of a rotation of $\mathbf{R}^3$.

13. The columns of the matrix $S$ that we used to construct the 3d rotation matrices are three mutually orthogonal unit vectors.

   (a) Prove that
   $$S^T S = I,$$
   that is, the inverse of $S$ is just its transpose.

   (b) A matrix $B$ with the property that $B^T B = I$ is called an **orthogonal** matrix. Prove that the columns of a matrix are mutually orthogonal unit vectors if and only if the matrix is orthogonal.

   (c) For an orthogonal matrix $B$ what are the possible values for $\det(B)$.

   (d) Prove that the rows of an orthogonal matrix also form a set of mutually orthogonal unit vectors.

   (e) Prove that the product of two orthogonal matrices $A$ and $B$ is an orthogonal matrix.

   (f) Prove that all rotations are othogonal matrices. (In fact, it turns out that in $\mathbf{R}^2$ and $\mathbf{R}^3$ the orthogonal matrices with determinant 1 are exactly the rotation matrices. All other orthogonal matrices describe rotations followed by some reflection through a line in $\mathbf{R}^2$ or a plane in $\mathbf{R}^3$.)

   (g) Let $\mathbf{u}$ be a unit column vector. Prove that the matrix $I - 2\mathbf{u}\mathbf{u}^T$ is an orthogonal matrix. (This matrix is called a **Householder matrix**. In $\mathbf{R}^2$ and $\mathbf{R}^3$ these matrices are the matrices of reflections through lines and planes, respectively, containing the origin and orthogonal to $\mathbf{u}$. If you are keen try to prove this, too!)

   (h) Prove that an orthogonal matrix preserves distances, that is,
   $$|A\mathbf{x}| = |\mathbf{x}|$$
   for all possible vectors $\mathbf{x}$.

14. You have a linear transformation $T$ and
$$T\begin{pmatrix} 1 \\ 1 \\ -8 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} -1 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}$$
   Find the matrix of $T$.

15. Prove that the function $T_{\mathbf{u}}$ defined by $T_{\mathbf{u}}(\mathbf{x}) = \mathbf{v} - \mathbf{proj}_{\mathbf{u}}(\mathbf{x})$ is a linear transformation.

16. Here are some descriptions of functions $\mathbf{R}^n \to \mathbf{R}^n$.

   (a) $T$ multiplies the $j^{th}$ component of the vector $\mathbf{x}$ by a non-zero number $b$.

   (b) $T$ replaces the $i^{th}$ component of the vector $\mathbf{x}$ with $b$ times the $j^{th}$ component added to the $i^{th}$ component.

   (c) $T$ switches two components of the vector.

   Show that these functions are linear transformations and describe their matrices.

**SOME TEST QUESTIONS**

17. Construct from scratch a $2 \times 2$ matrix that describes a counterclockwise rotation around the origin by an angle $\alpha$.

18. How would you go about constructing a $3 \times 3$ rotation matrix from scratch that rotates in the counterclockwise direction around a given unit vector by a certain angle $\alpha$?