Eigenvalues and eigenvectors

1. Here is a matrix
$$A = \begin{pmatrix} -3 & -7 & 19 \\ -2 & -1 & 8 \\ -2 & -3 & 10 \end{pmatrix}.$$

Determine which of the following vectors are eigenvectors of A. If a vector is an eigenvector also figure out what the corresponding eigenvalue is.

$$(a) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (b) \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \quad (c) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad (d) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (e) \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Answer. The vectors are eigenvectors if we can find an eigenvalue λ that satisfies the equation, $A\mathbf{x} = \lambda\mathbf{x}$. This is done by calculating $A\mathbf{x}$, then factorising out a constant such that we get $\lambda\mathbf{x}$.

(a) $A\mathbf{x} = \begin{pmatrix} 9 \\ 5 \\ 5 \end{pmatrix} \neq \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. This is not an eigenvector of A.

(b) $A\mathbf{x} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 1 \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$. This is an eigenvector of A with eigenvalue 1.

(c) $A\mathbf{x} = \begin{pmatrix} 2 \\ 4 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$. This is an eigenvector of A with eigenvalue 2.

(d) $A\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. This is the trivial eigenvector of A which has infinite eigenvalues.

(e) $A\mathbf{x} = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$. This is an eigenvector of A with eigenvalue 3.

2. Here is a matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

Find the eigenvalues and corresponding eigenvectors of A. Determine whether or not the matrix can be diagonalized. If it can be diagonalized do so.

Answer. Since the matrix is upper triangular the eigenvalues are the entries on the main diagonal: 1, 4, 6 and the matrix can be diagonalized.

The eigenvectors $\mathbf{x}$ can be found by solving $(A - \lambda I)\mathbf{x} = 0$ three times for each unique eigenvalue.

Examples of eigenvectors corresponding to the eigenvalues 1, 4 and 6 are

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 16 \\ 25 \\ 10 \end{pmatrix}.$$

(All other eigenvectors are multiples of these three vectors.) To diagonalise a matrix, we seek to turn it into the form $A = PDP^{-1}$, where P is a matrix of the three eigenvectors, and D is a diagonal matrix with the eigenvalues as the diagonal entries. Hence this matrix can be diagonalized as follows

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 16 \\ 0 & 3 & 25 \\ 0 & 0 & 10 \end{pmatrix}^{-1}$$

3. Compute the eigenvalues and eigenvectors of the following matrices. If possible use the eigenvalues and eigenvectors to diagonalize the matrices.

$$(a) \begin{pmatrix} 4 & -2 \\ 5 & -3 \end{pmatrix}, \quad (b) \begin{pmatrix} 6 & 1 \\ -3 & 2 \end{pmatrix}, \quad (c) \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}.$$

Answer. For each matrix, the eigenvalues are found by solving the polynomial given by $\det(A - \lambda I) = 0$.

Each eigenvector is found by solving $(A - \lambda I)\mathbf{x} = 0$ for each eigenvalue.
(a) Two eigenvalues 2 and -1. Corresponding eigenvectors are

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 2 \\ 5 \end{pmatrix}.$$

Using these eigenvectors the matrix can be diagonalized as follows.

$$\begin{pmatrix} 4 & -2 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 5 \end{pmatrix}^{-1}$$

(b) Two eigenvalues 3 and 5. Corresponding eigenvectors are

$$\begin{pmatrix} -1 \\ 3 \end{pmatrix} \text{ and } \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Using these eigenvectors the matrix can be diagonalized as follows.

$$\begin{pmatrix} 6 & 1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 3 & 1 \end{pmatrix}^{-1}$$

(c) This matrix only has one eigenvalue, the number 2. All corresponding eigenvectors are multiples of

$$\begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

This matrix cannot be diagonalized because any 2×2 matrix with column vectors that are multiples of each other does not have an inverse.

4. Prove that a matrix A has 0 as an eigenvalue if and only if $A\mathbf{x} = \mathbf{0}$ has non-zero solutions.

   Proof. This is an immediate consequence of the definition of eigenvalues. ■

5. The following are characteristic equations of some matrices.

   (a)
   $$t^5(t-1)^3(t+1)^4 = 0$$

   (b)
   $$t^3 + 3t^2 + 3t + 1 = 0$$

   (c)
   $$(t-2)(t-3)(t-4)(t-5) = 0$$

   (d)
   $$t^2 + 1 = 0$$

   For every one of these equations determine the dimensions of the matrices they come from and the eigenvalues of the matrices.

   Answers. The dimension of each matrix is $n \times n$ where n is the highest power of the polynomial. The eigenvalues are found by solving each characteristic equation.

   a) dimensions 12×12, eigenvalues $0, 1, -1$.

   b) dimensions 3×3, eigenvalue -1.

   c) dimensions 4×4, eigenvalues $2, 3, 4, 5$.

   d) dimensions 2×2, no (real) eigenvalues.

6. What are the eigenvalues and eigenvectors of the following diagonal matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Answer. The eigenvalues are 1, 2, and 3. The corresponding eigenvectors are the non-zero multiples of $\mathbf{e}_1$, $a\mathbf{e}_1 + b\mathbf{e}_2$ not both a and b equal to 0, and the non-zero multiples of $\mathbf{e}_4$.

7. If A is the matrix of a linear transformation which rotates all vectors in $\mathbf{R}^2$ through $30^{\mathbf{o}}$, explain why A cannot have any eigenvalues.

   Answer. If A did have an eigenvalue $\lambda \in \mathbf{R}$ with corresponding eigenvector $\mathbf{u}$, then $A\mathbf{u} = \lambda\mathbf{u}$. Obviously, $\lambda \neq 0$ since rotations turn non-zero vectors into non-zero vectors. But then $A\mathbf{u} = \lambda\mathbf{u}$ would imply that $\mathbf{u}$ and its image under $30^{\mathbf{o}}$-rotation would be parallel, which is clearly impossible. We conclude that our rotation has not eigenvalues.

8. Let A be the matrix of a 3d rotation through 30° around the vector $(1, 2, 3)$. What are its eigenvalues and eigenvectors?

   Answer. Clearly, the only eigenvectors of this matrix are the non-zero multiples of $(1, 2, 3)$ and the eigenvalue corresponding to these eigenvectors is 1.

9. Let A be the matrix of the reflection through the x-axis of $\mathbf{R}^2$. Find A and its eigenvalues and corresponding eigenvectors.

   Answer. The matrix of T is
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
   Eigenvalues are 1 and -1. Eigenvectors corresponding to these eigenvalues are the unit coordinate vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

10. If $char(\lambda) = \det(\lambda I - A)$ is the characteristic polynomial of an $n \times n$ matrix A what is $char(0)$?

    Answer. $char(0) = \det(-A) = (-1)^n \det(A)$.

11. If A is an $n \times n$ matrix and c is a non-zero constant, compare the eigenvalues of A and cA.

    Answer. If $A\mathbf{u} = \lambda\mathbf{u}$, then $cA\mathbf{u} = c\lambda\mathbf{u}$ and so the eigenvalues of cA are just the eigenvalues of A multiplied by c.

12. If A is an invertible $n \times n$ matrix, compare the eigenvalues of A and A^{-1}. More generally, for m an arbitrary integer, compare the eigenvalues of A and A^m.

    Answer. If $A\mathbf{u} = \lambda\mathbf{u}$, then
$$A^{-1}A\mathbf{u} = \lambda A^{-1}\mathbf{u}.$$

Hence

$$\mathbf{u} = \lambda A^{-1}\mathbf{u},$$

or

$$A^{-1}\mathbf{u} = \lambda^{-1}\mathbf{u}.$$

Similarly, $A\mathbf{u} = \lambda\mathbf{u}$ implies

$$A^m\mathbf{u} = \lambda^m\mathbf{u}$$

for any integer m.

13. Suppose A is a square matrix and it satisfies $A^m = A$ for some m a positive integer larger than 1. Show that if λ is an eigenvalue of A then $|\lambda|$ equals either 0 or 1.

    Answer. Let $\mathbf{u}$ be the eigenvector. Then

    $$A^m\mathbf{u} = \lambda^m\mathbf{u}.$$

    In addition, $A^m = A$ implies that

    $$A^m\mathbf{u} = A\mathbf{u} = \lambda\mathbf{u},$$

    and so

    $$\lambda^m = \lambda.$$

    Hence if $\lambda \neq 0$, then

    $$\lambda^{m-1} = 1$$

    and so $|\lambda| = 1$.

SOME TEST QUESTIONS

14. Derive the determinant formula for the characteristic equation of a square matrix.
15. What does it mean for a matrix to be defective?
16. Sketch a proof of the Cayley-Hamilton theorem in the case of non-defective matrices.
17. Sketch a proof that non-defective matrices can be diagonalized.
18. Why is finding the eigenvalues of a matrix a "hard" problem?
19. What are some practical applications of eigenvalues?
20. Describe how the power method for finding eigenvectors and eigenvalues works.
21. How many complex eigenvalues does an $n \times n$ matrix with complex coefficients have?