Differential equations

I got a bit tired of writing solutions. So, for this final calculus chapter I'll only provide you with the answers and NO worked solutions If you get stuck with any of the following problems, just like with integrals, you can ask *Wolfram Alpha* from within *Mathematica* to provide you with step-by-step solutions.

```
WolframAlpha["y'(x) = 2 x y", IncludePods -> {"Differential equation solution"},
PodStates -> {"Step-by-step solution", "Show all steps"}]
```

Alternatively you can also execute the following

```
WolframAlpha["y'(x) = 2 x y"]
```

and then hit the *Step-by-step solution* button in the *Differential equation solution* box of the answer. Try it!

Separable first order differential equations

Find the general solution for each of the following differential equations.

(a) $y' = 2xy$

(b) $yy' + \sin(x) = 0$

(c) $\sin(x)y' + y\cos(x) = 2\cos(x)$

(d) $\frac{1+y'}{1-y'} = \frac{1-y/x}{1+y/x}$

Answer.

(a)

$$\begin{aligned}
\frac{dy}{dx} &= 2xy \\
\int \frac{dy}{y} &= \int 2x \ dx \\
\log(y) &= x^2 + c \\
y &= Ce^{x^2}
\end{aligned}$$

(b)

$$\begin{aligned}
y\frac{dy}{dx} &= -\sin(x) \\
\int y \ dy &= -\int \sin(x) \ dx \\
\frac{y^2}{2} &= \cos(x) + c \\
y &= \pm\sqrt{2\cos(x) + C}
\end{aligned}$$

(c)

$$\begin{aligned}
\sin(x)\frac{dy}{dx} &= (2-y)\cos(x) \\
\int \frac{1}{2-y} \ dy &= \int \frac{\cos(x)}{\sin(x)} \ dx
\end{aligned}$$

Let $u = 2 - y$ and let $t = \sin(x)$. We get that $du/dy = -1$ and $dt/dx = \cos(x)$, and that $dy = -du$ and $dt = \cos(x)dx$. Hence

$$\begin{aligned}
-\int \frac{1}{u} \ du &= \int \frac{1}{t} \ dt \\
-\log|u| &= \log|t| + c \\
-\log|2-y| &= \log|\sin(x)| + c \\
2 - y &= -\frac{C}{\sin(x)} \\
y &= 2 + \frac{C}{\sin(x)}
\end{aligned}$$

(d)

$$
\begin{aligned}
\frac{1+dy/dx}{1-dy/dx} &= \frac{1-y/x}{1+y/x} \\
\left(1+\frac{dy}{dx}\right)\left(1+\frac{y}{x}\right) &= \left(1-\frac{dy}{dx}\right)\left(1-\frac{y}{x}\right) \\
1+\frac{y}{x}+\frac{dy}{dx}+\frac{y}{x}\frac{dy}{dx} &= 1-\frac{y}{x}-\frac{dy}{dx}+\frac{y}{x}\frac{dy}{dx} \\
2\frac{y}{x}+2\frac{dy}{dx} &= 0 \\
\frac{dy}{dx} &= -\frac{y}{x} \\
\int \frac{dy}{y} &= -\int \frac{dx}{x} \\
\log|y| &= -\log(x)+c \\
y &= \frac{C}{x}
\end{aligned}
$$

Integrating factor

Use an integrating factor to find the general solution for each of the following differential equations.

(a) $y' + 2y = 2x$

(b) $y' + \frac{2}{x}y = 1$

(c) $y' + \cos(x)y = 3\cos(x)$

(d) $\sin(x)y' + \cos(x)y = \tan(x)$

Answer.

(a) In the form $y' + P(x)y = Q(x)$, we have that $P(x) = 2$. Using the integrating factor method, we solve

$$
\begin{aligned}
\frac{dI}{dx} &= 2I \\
\int \frac{dI}{I} &= 2\int\ dx \\
\ln|I| &= 2x \\
I(x) &= e^{2x}
\end{aligned}
$$

Therefore our solution is

$$
\begin{aligned}
y &= \frac{1}{I(x)}\int Q(x)I(x)\ dx \\
&= \frac{1}{e^{2x}}\int 2xe^{2x}\ dx
\end{aligned}
$$

Using integration by parts, let $f(x) = 2x$ and $g'(x) = e^{2x}$. Then $f'(x) = 2$ and $g(x) = \frac{1}{2}e^{2x}$. Hence

$$\begin{aligned} y &= \frac{1}{e^{2x}}\left[xe^{2x} - \int e^{2x}\ dx\right] \\ &= \frac{1}{e^{2x}}\left[xe^{2x} - \frac{1}{2}e^{2x} + C\right] \\ &= x - \frac{1}{2} + Ce^{-2x} \end{aligned}$$

(b) In the form $y' + P(x)y = Q(x)$, we have that $P(x) = \frac{2}{x}$. Hence

$$\begin{aligned} \frac{dI}{dx} &= \frac{2I}{x} \\ \int \frac{dI}{I} &= 2\frac{dx}{x} \\ \log|I| &= \log|x^2| \\ I &= x^2 \end{aligned}$$

Therefore our solution is

$$\begin{aligned} y &= \frac{1}{x^2}\int x^2\ dx \\ &= \frac{1}{x^2}\left(\frac{x^3}{3} + C\right) \\ &= \frac{x}{3} + \frac{C}{x^2} \end{aligned}$$

(c) In the form $y' + P(x)y = Q(x)$, we have that $P(x) = \cos(x)$. Hence

$$\begin{aligned} \frac{dI}{dx} &= \cos(x)I \\ \int \frac{dI}{I} &= \int \cos(x)\ dx \\ \log(I) &= \sin(x) \\ I &= e^{\sin(x)} \end{aligned}$$

Therefore our solution is

$$y = \frac{1}{e^{\sin(x)}}\int 3e^{\sin(x)}\cos(x)\ dx$$

Using substitution, let $t = \sin(x)$, therefore $dt = \cos(x)\ dx$. This gives

$$\begin{aligned} y &= 3e^{-\sin(x)}\int e^t\ dt \\ &= 3e^{-\sin(x)}(e^{\sin(x)} + C) \\ &= 3 + Ce^{-\sin(x)} \end{aligned}$$

(d) In the form $y' + P(x)y = Q(x)$, we have that $P(x) = \frac{\cos(x)}{\sin(x)}$. Hence

$$\begin{aligned} \frac{dI}{dx} &= \frac{\cos(x)}{\sin(x)} I \\ \int \frac{dI}{I} &= \int \frac{\cos(x)}{\sin(x)}\ dx \end{aligned}$$

To solve integral on the right, let $t = \sin(x)$ and $dt = \cos(x)\ dx$. This implies that

$$\int \frac{\cos(x)}{\sin(x)}\ dx = \int \frac{dt}{t} = \log(t)$$

By substituting $t = \sin(x)$ back into the integral we get

$$\begin{aligned} \log(I) &= \log(\sin(x)) \\ I &= \sin(x) \end{aligned}$$

Therefore our solution is

$$y = \frac{1}{\sin(x)} \int \frac{\sin(x)}{\cos(x)}\ dx$$

Again, using substitution, let $u = \cos(x)$ and $du = -\sin(x)\ dx$. This implies that

$$y = -\frac{1}{\sin(x)} \int \frac{1}{u}\ du$$

By substituting $u = \cos(x)$ back in we get

$$\begin{aligned} y &= -\frac{1}{\sin(x)}(\log(\cos(x)) + c) \\ &= \frac{C - \log(\cos(x))}{\sin(x)} \end{aligned}$$

Second order homogeneous differential equations

Find the general solution for each of the following differential equations.

(a) $y'' + y' - 2y = 0$

(b) $y'' - 9y = 0$

(c) $y'' + 2y' + 2y = 0$

(d) $y'' + 6y' + 10y = 0$

(e) $y'' - 4y' + 4y = 0$

(f) $y'' + 6y' + 9y = 0$

Answer.

(a) Assume a solution of the form $y = e^{\lambda x}$. Therefore $y' = \lambda e^{\lambda x}$ and $y'' = \lambda^2 e^{\lambda x}$. Substituting these into our differential equation gives

$$e^{\lambda x}(\lambda^2 + \lambda - 2) = 0$$

And we know that $e^{\lambda x} \neq 0 \; \forall \; x \in \mathbb{R}$. Our characteristic equation is

$$\begin{aligned}
&\lambda^2 + \lambda - 2 = 0 \\
&(\lambda - 1)(\lambda + 2) = 0 \\
&\lambda_1 = 1, \lambda_2 = -2
\end{aligned}$$

Therefore our solution is

$$y = Ae^{x} + Be^{-2x}$$

(b) Assuming a solution of the form $y = e^{\lambda x}$, our characteristic equation is

$$\begin{aligned}
&\lambda^2 - 9 = 0 \\
&(\lambda - 3)(\lambda + 3) = 0 \\
&\lambda_1 = 3, \lambda_2 = -3
\end{aligned}$$

Therefore our solution is

$$y = Ae^{3x} + Be^{-3x}$$

(c) Assuming a solution of the form $y = e^{\lambda x}$, our characteristic equation is

$$\begin{aligned}
&\lambda^2 + 2\lambda + 2 = 0 \\
&\lambda_1 = \frac{-2 + \sqrt{4 - 8}}{2} = -1 + \imath \\
&\lambda_2 = \frac{-2 - \sqrt{4 - 8}}{2} = -1 - \imath
\end{aligned}$$

Therefore our solution is

$$\begin{aligned}
y &= \bar{A}e^{(-1+\imath)x} + \bar{B}e^{(-1-\imath)x} \\
&= \bar{A}e^{-x}(\cos(x) + \imath\sin(x)) + \bar{B}e^{-x}(\cos(x) - \imath\sin(x)) \\
&= e^{-x}[(\bar{A} + \bar{B})\cos(x) + \imath(A - B)\sin(x)] \\
&= e^{-x}(A\cos(x) + B\sin(x))
\end{aligned}$$

where $A = \bar{A} + \bar{B}$ and $B = \imath(\bar{A} - \bar{B})$.

(d) Assuming a solution of the form $y = e^{\lambda x}$, our characteristic equation is

$$\begin{aligned}
&\lambda^2 + 6\lambda + 10 = 0 \\
&\lambda_1 = \frac{-6 + \sqrt{36 - 40}}{2} = -3 + \imath \\
&\lambda_2 = \frac{-6 - \sqrt{36 - 40}}{2} = -3 - \imath
\end{aligned}$$

Therefore our solution is (using the same simplification as part (c))

$$y = e^{-3x}(A\cos(x) + B\sin(x))$$

(e) Assuming a solution of the form $y = e^{\lambda x}$, our characteristic equation is

$$\begin{aligned}
&\lambda^2 - 4\lambda + 4 = 0 \\
&(\lambda - 2)^2 = 0 \\
&\lambda_1 = \lambda_2 = 2
\end{aligned}$$

We require two linearly independent solutions in this case, so our solution is

$$y = (A + Bx)e^{2x}$$

(f) Assuming a solution of the form $y = e^{\lambda x}$, our characteristic equation is

$$\begin{aligned}
&\lambda^2 + 6\lambda + 9 = 0 \\
&(\lambda + 3)^2 = 0 \\
&\lambda_1 = \lambda_2 = -3
\end{aligned}$$

We require two linearly independent solutions in this case, so our solution is

$$y = (A + Bx)e^{-3x}$$

Find the particular solution, for the corresponding differential equation in the previous question, that satisfies the following boundary conditions.

(a) $y(0) = 1$ and $y(1) = 0$

(b) $y(0) = 0$ and $y(1) = 1$

(c) $y(0) = -1$ and $y(+\pi/2) = +1$

(d) $y(0) = -1$ and $y' = 0$ at $x = 0$

(e) $y(0) = 1$ and $y' = 0$ at $x = 1$

(f) $y' = 0$ at $x = 0$ and $y' = 1$ at $x = 1$

If you want to see the steps in *Mathematica*, e.g., for the first problem, execute the following.

```
WolframAlpha["y''(x) +y'(x) -2 y=0, y(0)=1, y(1)=0",
IncludePods -> {"Differential equation solution"},
PodStates -> {"Step-by-step solution", "Show all steps"}]
```

Answer.

(a) We have the solution to this differential equation

$$y = Ae^{x} + Be^{-2x}$$

Using the boundary conditions $y(0) = 1$ and $y(1) = 0$ we get the following equations

$$\begin{aligned} A + B &= 1 \\ Ae + Be^{-2} &= 0 \end{aligned}$$

We get $B = 1 - A$ from the first equation, which we can substitute into the second equation to get

$$\begin{aligned} &Ae + (1-A)e^{-2} = 0 \\ &A(e - e^{-2}) = -e^{-2} \\ &A = \frac{-1}{e^3 - 1} \end{aligned}$$

Substitute our result for A back into the first equation to get

$$\begin{aligned} B &= 1 - A \\ &= 1 + \frac{1}{e^3 - 1} \\ &= \frac{e^3}{e^3 - 1} \end{aligned}$$

Our solution becomes

$$\begin{aligned} y &= \frac{-1}{e^3 - 1}e^{x} + \frac{e^3}{e^3 - 1}e^{-2x} \\ &= \frac{1}{e^3 - 1}(-e^{-x} + e^{3-2x}) \end{aligned}$$

(b) We have the solution to this differential equation

$$y = Ae^{3x} + Be^{-3x}$$

Using the boundary conditions $y(0) = 0$ and $y(1) = 1$ we get the following equations

$$\begin{aligned} A + B &= 0 \\ Ae^3 + Be^{-3} &= 1 \end{aligned}$$

We get $B = -A$ from the first equation, which we can substitute into the second equation to get

$$
\begin{aligned}
A(e^{-3} - e^{3}) &= 1 \\
A &= \frac{1}{e^{-3} - e^{3}}
\end{aligned}
$$

Substitute our result for A back into the first equation to get

$$B = -A = \frac{1}{e^{3} - e^{-3}}$$

Our solution becomes

$$y = \frac{-1}{e^{3} - e^{-3}}(e^{3x} - e^{-3x})$$

(c) We have the solution to this differential equation

$$y = e^{x}(A\cos(x) + B\sin(x))$$

Using the boundary conditions $y(0) = -1$ and $y(\pi/2) = 1$ we get the following equations

$$
\begin{aligned}
A &= -1 \\
Be^{-\pi/2} &= 1
\end{aligned}
$$

Rearranging the second equation we get $B = e^{\pi/2}$.
Our solution becomes

$$y = e^{-x}(-\cos(x) + e^{\pi/2}\sin(x))$$

(d) We have the solution to the differential equation

$$y = e^{-3x}(A\cos(x) + B\sin(x))$$

Using the boundary conditions $y(0) = -1$ and $y' = 0$ at $x = 0$, we get the following equations

$$
\begin{aligned}
A &= -1 \\
y'(x) &= e^{-3x}(-A(\sin(x){+}3\cos(x)) + B(\cos(x) - 3\sin(x))) \\
y'(0) &= B - 3A = 0 \\
B &= 3A = -3
\end{aligned}
$$

Our solution becomes

$$y = -e^{-3x}(\cos(x) + 3\sin(x))$$

(e) We have the solution to the differential equation

$$y = (A + Bx)e^{2x}$$

Using the boundary conditions $y(0) = 1$ and $y' = 0$ at $x = 1$, we get the following equations

$$\begin{aligned} y(0) &= A = 1 \\ y'(x) &= 2Ae^{2x} + Be^{2x}(1 + 2x) \\ y'(1) &= 2Ae^2 + 3Be^2 = 0 \end{aligned}$$

Substituting our result for A into the bottom equation gives

$$\begin{aligned} 2e^2 + 3Be^2 &= 0 \\ 3Be^2 &= -2e^2 \\ B &= -\frac{2}{3} \end{aligned}$$

Our solution becomes

$$y = (1 - \frac{2}{3}x)e^{2x}$$

(f) We have the solution to the differential equation

$$y = (A + Bx)e^{-3x}$$

Using the boundary conditions $y' = 0$ at $x = 0$, and $y' = 1$ at $x = 1$, we get the following equations

$$\begin{aligned} y'(x) &= -3Ae^{-3x} + Be^{-3x}(1 - 3x) \\ y'(0) &= -3A + B = 0 \\ B &= 3A \\ y'(1) &= -3Ae^{-3} - 2Be^{-3} = 1 \end{aligned}$$

Substituting our result for B into the bottom equation gives

$$\begin{aligned} -3Ae^{-3} - 6Ae^{-3} &= 1 \\ A &= -\frac{e^3}{9} \\ B = 3A &= -\frac{e^3}{3} \end{aligned}$$

Our solution becomes

$$y = -\frac{1}{9}(1 + 3x)e^{3-3x}$$

Second order non-homogeneous differential equations

Find the general solution for each of the following differential equations.

(a) $y'' + y' - 2y = 1 + x$

(b) $y'' - 9y = e^{3x}$

(c) $y'' + 2y' + 2y = \sin(x)$

(d) $y'' + 6y' + 10y = e^{2x}\cos(x)$

(e) $y'' - 4y' + 4y = 2x$

(f) $y'' + 6y' + 9y = \cos(x)$

Answer.

(a) The solution for the homogenous differential equation $y_h'' + y_h' - 2y_h = 0$ is

$$y_h = Ae^x + Be^{-2x}$$

Using the method of undetermined coefficients, guess a particular solution of the form $y_p = C + Dx$. We get that $y_p' = D$ and $y_p'' = 0$, which gives

$$D - 2(C + Dx) = 1 + x$$

Equating the coefficients gives the equations

$$\begin{aligned} D - 2C &= 1 \\ -2D &= 1 \end{aligned}$$

We can solve this to get $C = -\dfrac{3}{4}$ and $D = -\dfrac{1}{2}$, which gives a particular solution $y_p = -\dfrac{3}{4} - \dfrac{1}{2}x$. Our final solution is

$$y = y_p + y_h = -\frac{3}{4} - \frac{1}{2}x + Ae^x + Be^{-2x}$$

(b) The solution for the homogenous differential equation $y_h'' - 9y_h = 0$ is

$$y_h = Ae^{3x} + Be^{-3x}$$

Using the method of undetermined coefficients, guess a particular solution of the form $y_p = (C + Dx)e^{3x}$. Note in this case we choose a solution that is linearly independent to the homogenous solution. We get that $y_p' = De^{3x} + 3(C + Dx)e^{3x}$ and $y_p'' = 6De^{3x} + 9(C + Dx)e^{3x}$, which gives

$$6De^{3x} + 9(C + Dx)e^{3x} - 9(C + Dx)e^{3x} = e^{3x}$$

Equating the coefficients of e^{3x} gives the equations

$$6De^{3x} = e^{3x}$$

We can solve this to get $D = \dfrac{1}{6}$, which gives a particular solution $y_p = \dfrac{1}{6}xe^{3x}$. Our final solution is

$$y = y_p + y_h = \left(A + \frac{1}{6}x\right)e^{3x} + Be^{-3x}$$

(c) The solution for the homogenous differential equation $y_h'' + 2y_h' + 2y_h = 0$ is

$$y_h = (A\cos(x) + B\sin(x))e^{-x}$$

Using the method of undetermined coefficients, guess a particular solution of the form $y_p = C\sin(x) + D\cos(x)$. We get that $y_p' = C\cos(x) - D\sin(x)$ and $y_p'' = -C\sin(x) - D\cos(x)$, which gives

$$(C - 2D)\sin(x) + (2C + D)\cos(x) = \sin(x)$$

Equating the coefficients of $\sin(x), \cos(x)$ gives the equations

$$\begin{aligned} C - 2D &= 1 \\ 2C + D &= 0 \end{aligned}$$

We can solve this to get $C = \dfrac{1}{5}$ and $D = -\dfrac{2}{5}$, which gives a particular solution $y_p = \dfrac{1}{5}(\sin(x) - 2\cos(x))$. Our final solution is

$$y = y_p + y_h = (A\cos(x) + B\sin(x))e^{-x} + \frac{1}{5}(\sin(x) - 2\cos(x))$$

(d) The solution for the homogenous differential equation $y_h'' + 6y_h' + 10y_h = 0$ is

$$y_h = (A\cos(x) + B\sin(x))e^{-3x}$$

Using the method of undetermined coefficients, guess a particular solution of the form $y_p = e^{2x}(C\cos(x) + D\sin(x))$. We get that

$$\begin{aligned} y_p' &= e^{2x}(-C\sin(x) + D\cos(x)) + 2e^{2x}(C\cos(x) + D\sin(x)) \\ y_p'' &= e^{2x}(-C\cos(x) - D\sin(x)) + 2e^{2x}(-C\sin(x) + D\cos(x)) \\ &\quad + 2e^{2x}(-C\sin(x) + D\cos(x)) + 4e^{2x}(C\cos(x) + D\sin(x)) \end{aligned}$$

Which gives

$$e^{2x}\cos(x)(25C + 10D) + e^{2x}\sin(x)(-10C + 25D) = e^{2x}\cos(x)$$

Equating the coefficients of $e^{2x}\sin(x), e^{2x}\cos(x)$ gives the equations

$$\begin{aligned} 25C + 10D &= 1 \\ -10C + 25D &= 0 \end{aligned}$$

We can solve this to get $C = \dfrac{1}{29}$ and $D = -\dfrac{2}{145}$, which gives a particular solution $y_p = \dfrac{1}{145}(5\cos(x) + 2\sin(x))$. Our final solution is

$$y = y_p + y_h = (A\cos(x) + B\sin(x))e^{-3x} + \frac{1}{145}(5\cos(x) + 2\sin(x))$$

(e) The solution for the homogenous differential equation $y_h'' - 4y_h' + 4y_h = 0$ is

$$y_h = (A + Bx)e^{2x}$$

Using the method of undetermined coefficients, guess a particular solution of the form $y_p = C + Dx$. We get that $y_p' = D$ and $y_p'' = 0$, which gives

$$(4D)x + (4C - 4D) = 2x$$

Equating the coefficients gives the equations

$$\begin{aligned} 4D &= 2 \\ 4C - 4D &= 0 \end{aligned}$$

We can solve this to get $C = \frac{1}{2}$ and $D = -\frac{1}{2}$, which gives a particular solution $y_p = \frac{1}{2}(1 + x)$. Our final solution is

$$y = y_p + y_h = (A + Bx)e^{2x} + \frac{1}{2}(1 + x)$$

(f) The solution for the homogenous differential equation $y_h'' + 6y_h' + 9y_h = 0$ is

$$y_h = (A + Bx)e^{-3x}$$

Using the method of undetermined coefficients, guess a particular solution of the form $y_p = C\cos(x) + D\sin(x)$. We get that $y_p' = -C\sin(x) + D\cos(x)$ and $y_p'' = -C\cos(x) - D\sin(x)$, which gives

$$(8C + 6D)\cos(x) + (-6C + 8D)\sin(x) = \cos(x)$$

Equating the coefficients gives the equations

$$\begin{aligned} 8C + 6D &= 1 \\ -6C + 8D &= 0 \end{aligned}$$

We can solve this to get $C = \frac{4}{50}$ and $D = \frac{3}{50}$, which gives a particular solution $y_p = \frac{1}{50}(4\cos(x) + 3\sin(x))$. Our final solution is

$$y = y_p + y_h = (A + Bx)e^{-3x} + \frac{1}{50}(4\cos(x) + 3\sin(x))$$

Non-separable first order differential equations

Each of the following differential equations can be solved using the integrating factor trick. However, since we are dealing with constant coefficients there is an alternative method, just guess a particular solution and add it to the general solution of the homogeneous equation, just like in the case second order DEs with constant coefficients. Give it a try.

(a) $y' + y = 1$

(b) $y' + 2y = 2 + 3x$

(c) $y' - y = e^{2x}$

(d) $y' - y = e^{x}$

(e) $y' + 2y = \cos(2x)$

(f) $y' - 2y = 1 + 2x - \sin(x)$

Answer.

(a) The solution for the homogenous differential equation $y_h' + y_h = 0$ can be solved by separation of variables. We have that

$$\begin{aligned}
\frac{dy_h}{dx} &= -y_h \\
\int \frac{dy_h}{y_h} &= -\int dx \\
\log(y_h) &= -x + c \\
y_h &= Ce^{-x}
\end{aligned}$$

Using the method of undetermined coefficients, guess a particular solution of the form $y_p = A$. We get that $y_p' = 0$, which gives $A = 1$. Therefore $y_p = 1$ and our solution is

$$y = y_h + y_p = Ce^{-x} + 1$$

(b) The solution for the homogenous differential equation $y_h' + 2y_h = 0$ can be solved by separation of variables. We have that

$$\begin{aligned}
\frac{dy_h}{dx} &= -2y_h \\
\int \frac{dy_h}{y_h} &= -2\int dx \\
\log(y_h) &= -2x + c \\
y_h &= Ce^{-2x}
\end{aligned}$$

Using the method of undetermined coefficients, guess a particular solution of the form $y_p = A + Bx$. We get that $y_p' = B$, which gives

$$B + 2(A + Bx) = 2 + 3x$$

Matching the coefficients gives

$$\begin{aligned} B + 2A &= 2 \\ 2B &= 3 \end{aligned}$$

We can solve this to get $A = \dfrac{1}{4}$ and $B = \dfrac{3}{2}$, therefore $y_p = \dfrac{1}{4} + \dfrac{3}{2}x$ and our solution is

$$y = y_h + y_p = Ce^{-2x} + \frac{1}{4} + \frac{3}{2}x$$

(c) The solution for the homogenous differential equation $y_h' - y_h = 0$ can be solved by separation of variables. We have that

$$\begin{aligned} \frac{dy_h}{dx} &= y_h \\ \int \frac{dy_h}{y_h} &= \int \, dx \\ \log(y_h) &= x + c \\ y_h &= Ce^x \end{aligned}$$

Using the method of undetermined coefficients, guess a particular solution of the form $y_p = Ae^{2x}$. We get that $y_p' = 2Ae^{2x}$, which gives

$$Ae^{2x} = e^{2x}$$

Matching the coefficients gives $A = 1$, therefore $y_p = e^{2x}$ and our solution is

$$y = y_h + y_p = Ce^x + e^{2x}$$

(d) The solution for the homogenous differential equation $y_h' - y_h = 0$ can be solved by separation of variables. We have that

$$\begin{aligned} \frac{dy_h}{dx} &= y_h \\ \int \frac{dy_h}{y_h} &= \int \, dx \\ \log(y_h) &= x + c \\ y_h &= Ce^x \end{aligned}$$

Using the method of undetermined coefficients, guess a particular solution of the form $y_p = (A + Bx)e^x$, as we need to choose y_p such that it is linearly independent to y_h. We get that $y_p' = Be^x + (A + Bx)e^x$, which gives

$$Be^x = e^x$$

Matching the coefficients gives $B = 1$, therefore $y_p = xe^x$ and our solution is

$$y = y_h + y_p = e^x(C + x)$$

(e) The solution for the homogenous differential equation $y_h' + 2y_h = 0$ can be solved by separation of variables. We have that

$$\begin{aligned}
\frac{dy_h}{dx} &= -2y_h \\
\int \frac{dy_h}{y_h} &= -2\int \, dx \\
\log(y_h) &= -2x + c \\
y_h &= Ce^{-2x}
\end{aligned}$$

Using the method of undetermined coefficients, guess a particular solution of the form $y_p = A\cos(2x) + B\sin(2x)$. We get that $y_p' = -2A\sin(2x) + 2B\cos(2x)$, which gives

$$(2B + 2A)\cos(2x) + (-2A + 2B)\sin(2x) = \cos(2x)$$

Matching the coefficients gives

$$\begin{aligned}
2A + 2B &= 1 \\
-2A + 2B &= 0
\end{aligned}$$

We can solve this to get $A = \frac{1}{4}$ and $B = \frac{1}{4}$, therefore $y_p = \frac{1}{4}(\cos(2x) + \sin(2x))$, and our solution is

$$y = y_h + y_p = Ce^{-2x} + \frac{1}{4}(\cos(2x) + \sin(2x))$$

(f) The solution for the homogenous differential equation $y_h' - 2y_h = 0$ can be solved by separation of variables. We have that

$$\begin{aligned}
\frac{dy_h}{dx} &= 2y_h \\
\int \frac{dy_h}{y_h} &= 2\int \, dx \\
\log(y_h) &= 2x + c \\
y_h &= Ce^{2x}
\end{aligned}$$

Using the method of undetermined coefficients, guess a particular solution of the form $y_p = A + Bx + C\cos(x) + D\sin(x)$. We get that $y_p' = B - C\sin(x) + D\cos(x)$, which gives

$$(B - 2A) - (2B)x + (D - 2C)\cos(x) + (-C - 2D)\sin(x) = 1 + 2x - \sin(x)$$

Matching the coefficients gives

$$\begin{aligned}
B - 2A &= 1 \\
-2B &= 2 \\
D - 2C &= 0 \\
-C - 2D &= -1
\end{aligned}$$

We can solve this to get $A = -1$, $B = -1$, $C = \frac{1}{5}$ and $D = \frac{2}{5}$, therefore $y_p = -1 - x + \frac{1}{5}(\cos(x) + 2\sin(x))$, and our solution is

$$y = y_h + y_p = Ce^{2x} - 1 - x + \frac{1}{5}(\cos(x) + 2\sin(x))$$

TEST QUESTIONS

1. The integrating factor trick for solving linear first order differential equations of the form
$$y' + P(x)y = Q(x)$$
is based of the product rule for differentiation. Explain.

2. What is a homogeneous linear differential equation?

3. Show that the sum of two solutions of a homogeneous linear differential equation is also a solution of this differential equation.

4. Show that, given a solution $y_p(x)$ of a non-homogeneous linear differential equation, and a solution $y_h(x)$ of the corresponding homogeneous differential equation, the sum
$$y_h(x) + y_p(x)$$
is also a solution of the non-homogeneous equation.

5. You are given a first order DE with a nicely behaved set of solutions. How many of the solutions do you expect to satisfy the initial condition $y(0) = 666$?

6. What does a typical initial condition for a second order differential equation look like?

7. Derive the general solution of a homogeneous second order linear differential equation from scratch.

8. Check that in the case of a repeated root λ of a homogeneous second order linear differential equation the function $xe^{\lambda x}$ really is a solution.