MTH1030/1035 Lecture Notes 2022
Part 1: Linear Algebra

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Contents

Hello everybody,

I used to teach MTH1030 in second semester from 2005 to 2009. Since then quite a few things have changed in terms of the topics covered in this unit. When I was put in charge of the unit in 2015 I decided to rewrite the lecture notes rather than relying on the notes of my colleagues who taught the unit in recent years. Writing new notes is always a huge job but one that really pays off, especially if one teaches a unit for a number of years.

Anyway, this first part of the notes you are looking at are based on some old notes of mine, the free **Elementary Linear Algebra** book by Kenneth Kuttler which I had been using for a couple of years[1] and a lot of new material. In particular, I've incorporated many of the worked examples from Kuttler's book in these notes and in some places I am referring to this book as a source for more details and some proofs that we'll skip. We'll also be using quite a few of the exercises in Kuttler's book in the applied classes.

Please let me know about any mistakes that you stumble across, clumsy and hard-to-understand parts of the text, etc.

Finally, just a quick note on how to use these lecture notes. What I've tried to do by writing these notes is to provide you with a very comprehensive account of everything we will be discussing in MTH1030, including proofs or at least sketches of proofs of most of the results we'll come across.

Not all of the material in these notes will be covered in full detail in the lectures. Some of it I'll expect you to read on your own, and some material is optional and mainly meant for those among you who really want to know everything.

In the first instance, when you study these notes I'd like you to focus on understanding what the various results mean and how they are applied in practice. First and foremost, this involves studying the examples contained in these notes and discussed in the lectures. Once you understand HOW something works you could and should move on to understanding exactly WHY things work the way they do. This means understanding the proofs.

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[1] A copy of Kuttler's book is available for free here `http://tinyurl.com/cu7fsph`

Chapter 1

Vectors in $\mathbf{R}^n$

You are all familiar with the set of real numbers $\mathbf{R}$ (which is often represented by the 1-dimensional number line), the 2-dimensional plane $\mathbf{R}^2$ and the 3-dimensional space $\mathbf{R}^3$. In fact, you've spent most of your mathematical life so far playing in these worlds. In the first part of this unit we'll also venture into higher dimensions and get to know the 4-dimensional space $\mathbf{R}^4$, the 5-dimensional space $\mathbf{R}^5$, the 6-dimensional space $\mathbf{R}^6$, and so on.

Formally, for any natural number n we define $\mathbf{R}^n$ as follows (buckle your seat belts and at the same time relax!):

$$\mathbf{R}^n = \{(x_1, \cdots, x_n) \mid x_j \in \mathbf{R} \text{ for } j = 1, \cdots, n\}.$$

To translate the expression enclosed in the curly brackets into English say "the set of" when you come across the first curly bracket. Then spell out what precedes the vertical stroke. The stroke itself translates into "such that". Then spell out what comes after the stroke. So, here we are looking at the "set of all **n-tuples** of the form $(x_1, \cdots, x_n)$ such that all entries x_j are real numbers." For $n = 2$ and $n = 3$ we have

$$\mathbf{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbf{R}\} \text{ and } \mathbf{R}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbf{R}\}.$$

Here a 2-tuple or 3-tuple of numbers is what you would usually refer to as a pair or a triple of numbers, respectively.

The elements of $\mathbf{R}^n$ are called **points** and the individual entries of a point are called its **coordinates**. The set

$$\{(0, \cdots, 0, t, 0, \cdots, 0) \mid t \in \mathbf{R}\},$$

for t in the i^{th} slot is called the i^{th} **coordinate axis** of $\mathbf{R}^n$. The point

$$(0, \cdots, 0)$$

is called the **origin** of $\mathbf{R}^n$.

No real surprises here except for the fancy language and, maybe, how unexciting and straightforward the introduction of higher dimensions pans out to be.

1.1 Vectors

In the following we will not only think of the elements of $\mathbf{R}^n$ as points in this space but also as **vectors** that can be added and multiplied by real numbers. Interpreted as vectors the elements of $\mathbf{R}^n$ are usually denoted by bold face letters,

$$\mathbf{x} = (x_1, \ \cdots, \ x_n).$$

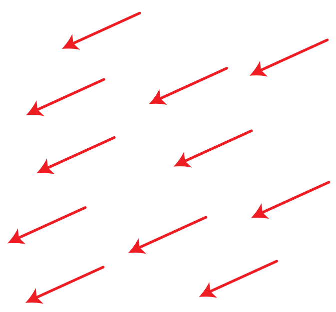
The coordinates of the vector $\mathbf{x}$ are also called the **components** of $\mathbf{x}$ and the origin interpreted as a vector is called the **zero vector,**

$$\mathbf{0} = (0, \ \cdots, \ 0).$$

1.2 Vectors as arrows in space

In $\mathbf{R}^2$ and $\mathbf{R}^3$ you probably already have a good idea of what is meant by interpreting a triple like $(1, 2, 4)$ as a vector—you think of it as an arrow that starts out at the origin and ends at the point $(1, 2, 4)$.

And this is basically the right idea, except that when mathematicians speak of the vector $(1, 2, 4)$ they think of all the infinitely many arrows that share the same direction and length with the arrow that starts out at the origin and points at $(1, 2, 4)$. In other words, a vector does not live at any one place in space—it stays the same when it is translated (but not rotated!) anywhere. This means that all the arrows in the following diagram represent the same vector.[1]

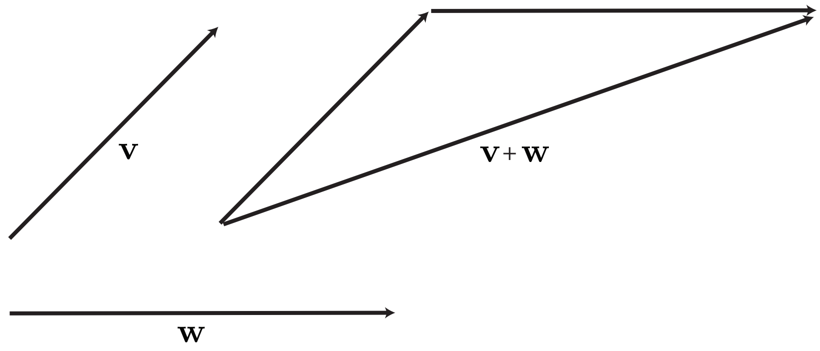


Then you all know how to stretch and how to add vectors:

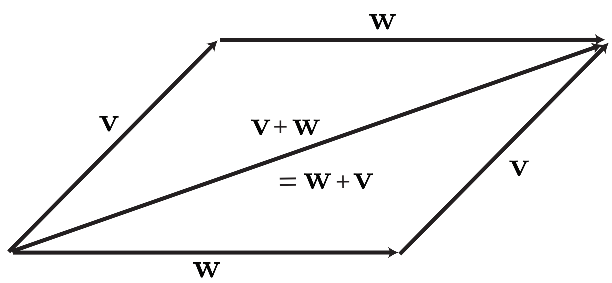
[1]At first glance it may appear strange that a vector has infinitely many representatives. However, you are already familiar with this phenomenon in a different context, namely infinitely many different looking fractions representing the same rational number: $1/2 = 2/4 = 3/6 = \cdots$.

- **Stretching**
  If λ is a real number, the vector $\lambda\mathbf{v}$ is the vector $\mathbf{v}$ stretched by the factor λ. This means that if $\lambda = 0$, then $\lambda\mathbf{v}$ is the zero vector. Also, if λ is a negative number, then $\lambda\mathbf{v}$ is pointing in the opposite direction as $\mathbf{v}$. If two vectors can be transformed into each other by stretching we call them **parallel**. Note that any vector is parallel to itself and to the zero vector.

- **Addition**
  To add two vectors $\mathbf{v}$ and $\mathbf{w}$, arrange them so that they are tip to tail. Then $\mathbf{v} + \mathbf{w}$ is the vector that starts at the first vector's tail and ends at the second vector's tip as shown in the following diagram.



  As you can see in the next diagram, the order in which you add two vectors does not matter: $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.



Here is how addition and stretching of vectors can be expressed algebraically. Let $\mathbf{v} = (1, 2, 7)$ and $\mathbf{w} = (a, b, c)$.

To stretch a vector by a factor λ simply means multiplying all its components by λ. So, $\lambda(1, 2, 7) = (\lambda, 2\lambda, 7\lambda)$.

To add two vectors, simply add respective components:

$(1, 2, 7) + (a, b, c) = (1 + a, 2 + b, 7 + c)$.

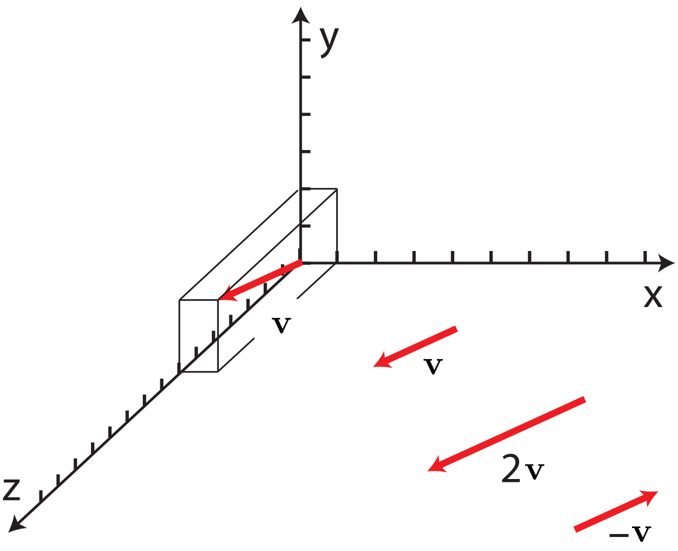
Example 1.2.1 *If* $\mathbf{v} = (3, 4, 2)$ *and* $\mathbf{w} = (1, 2, 3)$, *then*

$$\mathbf{v} + \mathbf{w} = (3 + 1, 4 + 2, 2 + 3) = (4, 6, 5)$$

and

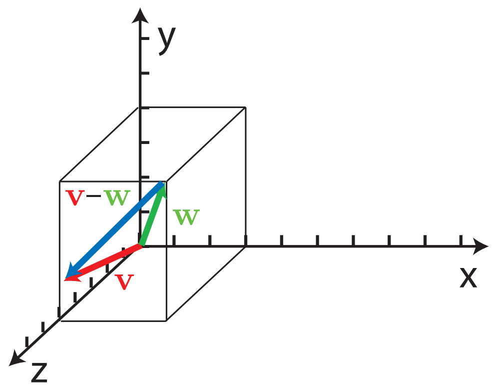
$$2\mathbf{v} + 7\mathbf{w} = 2(3, 4, 2) + 7(1, 2, 3) = (6 + 7, 8 + 14, 4 + 21) = (13, 22, 25).$$

Example 1.2.2 *Given* $\mathbf{v} = (1, 2, 7)$ *let's draw* $\mathbf{v}$, $2\mathbf{v}$ *and* $-\mathbf{v}$.



Example 1.2.3 *Given* $\mathbf{v} = (1, 2, 7)$ *and* $\mathbf{w} = (3, 4, 5)$ *let's draw and compute* $\mathbf{v} - \mathbf{w}$.

$$\mathbf{v} - \mathbf{w} = (1, 2, 7) - (3, 4, 5) = (-2, -2, 2)$$



1.3 Unit coordinate vectors

In $\mathbf{R}^3$, instead of $\mathbf{v} = (1, 7, 3)$ people also often write $\mathbf{v} = 1\mathbf{i} + 7\mathbf{j} + 3\mathbf{k}$. Here $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$ and $\mathbf{k} = (0, 0, 1)$. This notation helps remind us that the three numbers in $(1, 7, 3)$

refer to directions parallel to the three coordinate axes (with $\mathbf{i}$ parallel to the x-axis, $\mathbf{j}$ parallel to the y-axis and $\mathbf{k}$ parallel to the z-axis). The three vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are usually referred to as the **unit coordinate vectors**.

In $\mathbf{R}^n$ the unit coordinate vectors are denoted as $\mathbf{e}_1 = (1, 0, 0, \ldots, 0), \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \mathbf{e}_3 = (0, 0, 1, 0 \ldots, 0)$, and so on.

1.4 Vectors in $\mathbf{R}^n$

Although it still makes sense to think of vectors as arrows in a higher-dimensional space such as the "devilish" $\mathbf{R}^{666}$ we usually don't bother drawing pictures. On the other hand, our algebraic componentwise ways of adding, subtracting and multiplying vectors by real numbers generalizes in a straightforward way to higher dimensions. In this context a real number used for multiplying a vector is also often referred to as a **scalar.**[2]

It is then easy to convince yourself that the addition of vectors and the multiplication of vectors by scalars have the following properties.

Proposition 1.4.1 (Vector addition and multiplication by scalars) *For* $\mathbf{v}, \mathbf{w}$, *and* $\mathbf{z}$ *vectors in* $\mathbf{R}^n$ *and* α, β *scalars (= real numbers) the following hold:*

Addition of vectors is **commutative***,*

$$\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}.$$

Addition of vectors is **associative***,*

$$(\mathbf{v} + \mathbf{w}) + \mathbf{z} = \mathbf{v} + (\mathbf{w} + \mathbf{z}).$$

The zero vector $\mathbf{0}$ *is an* **additive identity***,*

$$\mathbf{v} + \mathbf{0} = \mathbf{v},$$

Every vector $\mathbf{v}$ *has an* **additive inverse** $-\mathbf{v}$*,*

$$\mathbf{v} + (-\mathbf{v}) = \mathbf{0},$$

Also

$$\alpha(\mathbf{v} + \mathbf{w}) = \alpha\mathbf{v} + \alpha\mathbf{w},$$
$$(\alpha + \beta)\mathbf{v} = \alpha\mathbf{v} + \beta\mathbf{v},$$
$$\alpha(\beta\mathbf{v}) = (\alpha\beta)\mathbf{v},$$
$$1\mathbf{v} = \mathbf{v}.$$

[2]This name comes from the fact that multiplying a vector by a number corresponds to *scaling* the corresponding arrow using the same number.

The main message to take away from this long list of properties is that addition of vectors and multiplication by scalars essentially follows the same rules as addition and multiplication of numbers.

The above properties of vector addition and scalar multiplication identify $\mathbf{R}^n$ as a so-called **vector space over the real numbers** or a **real vector space**.[3] In fact, $\mathbf{R}^n$ is the prototype of an **n-dimensional** real vector space. Here are a few examples of other naturally occurring sets endowed with an addition and multiplication by scalars that share the above properties with $\mathbf{R}^n$ and are therefore also real vector spaces:

- The set of polynomials in the variable x with real coefficients and degree at most n together with the natural ways of adding polynomials and multiplying them by real numbers.
- The set of infinite sequences of real numbers together with the natural ways of adding them elementwise and multiplying them by real numbers.
- The set of continuous functions $[0, 1] \to \mathbf{R}$.

The fact that all these mathematical structures are real vector spaces means that everything about $\mathbf{R}^n$ that we'll derive from our basic list of properties also applies to these structures. Abstract vector spaces are one of the main themes of a separate unit dedicated to linear algebra here at Monash.

1.5 Pythagoras' theorem, distance in $\mathbf{R}^n$ and the length of a vector

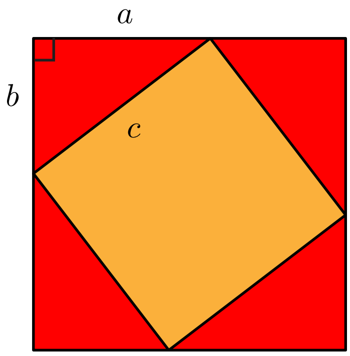
Pythagoras' theorem

Pythagoras' theorem states that in a right-angled triangle the square of the hypotenuse c (the side opposite the right angle) is equal to the sum of the squares of the other two sides a and b:

$$a^2 + b^2 = c^2.$$

As a serious maths student you should know how to prove this fundamental theorem. Here is just one of the many pretty proofs of Pythagoras that is easy to remember.

[3] There are also vector spaces over the complex numbers and other number fields.



Proof of Pythagoras' Theorem:

From the diagram it is clear that

$$\boxed{\begin{array}{c}\text{area}\\\text{big square}\end{array}} = \boxed{\begin{array}{c}\text{area}\\\text{small square}\end{array}} + 4\cdot\boxed{\begin{array}{c}\text{area}\\\text{triangle}\end{array}}$$

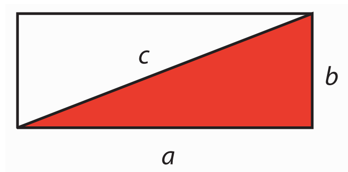
This just means that $$(a+b)^2 = c^2 + 4\cdot\frac{ab}{2}$$

Expanding gives $$a^2+2ab+b^2 = c^2 + 2ab$$

We conclude that $$a^2+b^2 = c^2$$

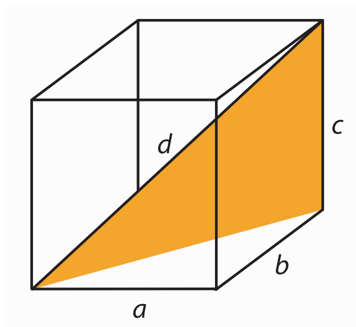
QED

Another way to express Pythagoras' theorem is to say the following: In a rectangle of sides a and b and diagonal c we have $a^2+b^2=c^2$.



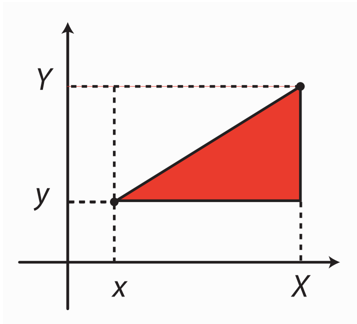
This version of everybody's favourite theorem has the following 3d counterpart: In a rectangular box of sides a, b, c and diagonal d we have

$$a^2+b^2+c^2=d^2.$$



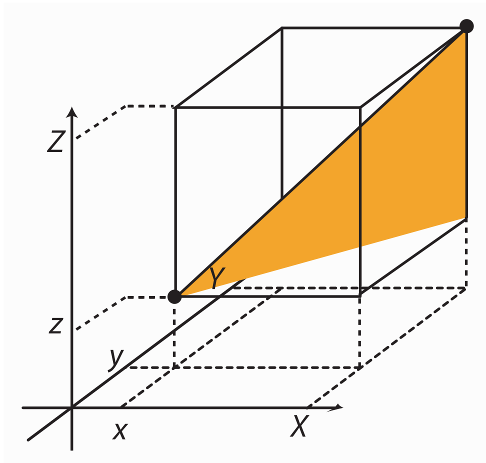
We can use Pythagoras' theorem to measure the distance between two points in $\mathbf{R}^2$. Note that two points span a rectangle the length of whose diagonal is the distance between the two points. If the coordinates of the points are (x, y) and (X, Y), the lengths of the sides of the rectangle are $|x - X|$ and $|y - Y|$. Now, Pythagoras tells us that the distance between the two points is

$$\sqrt{|x - X|^2 + |y - Y|^2}.$$



Similarly, two points in space span a rectangular box. Then, if (x, y, z) and (X, Y, Z) are the coordinates of these points, the distance between them is

$$\sqrt{|x - X|^2 + |y - Y|^2 + |z - Z|^2}.$$



The 3d counterpart of Pythagoras' theorem and therefore also the 3d distance formula can be proved using Pythagoras' theorem. Together, the 2d and the 3d versions of the distance formula then suggest the following *definition* for the distance between two points in higher dimensions.

Definition 1.5.1 (The distance formula) *Let* $\mathbf{x} = (x_1, \dots, x_n)$ *and* $\mathbf{y} = (y_1, \dots, y_n)$ *be two points/vectors in* $\mathbf{R}^n$*. Then* $|\mathbf{x} - \mathbf{y}|$ *denotes the* **distance** *between these points,*

$$|\mathbf{x} - \mathbf{y}| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2} = \sqrt{\sum_{k=1}^{n} (x_k - y_k)^2}.$$

The **length** $|\mathbf{x}|$ *(or* **norm***) of the vector* $\mathbf{x}$ *is the distance between the point* $\mathbf{x}$ *and the origin. So,*

$$|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\sum_{k=1}^{n} x_k^2}.$$

A vector $\mathbf{x}$ *is called a* **unit vector** *if it is one unit long,*

$$|\mathbf{x}| = 1.$$

Example 1.5.1 *Let's find the distance in* $\mathbf{R}^4$ *between the points,*

$$\mathbf{a} = (1, 2, -4, 6)$$

and

$$\mathbf{b} = (2, 3, -1, 0).$$

Use the distance formula and write

$$|\mathbf{a} - \mathbf{b}|^2 = (1-2)^2 + (2-3)^2 + (-4-(-1))^2 + (6-0)^2 = 47.$$

Therefore, $|\mathbf{a} - \mathbf{b}| = \sqrt{47}$.

Example 1.5.2 *The length of the vector* $(1, 2, -1, 1)$ *is* $\sqrt{1^2 + 2^2 + (-1)^2 + 1^2} = \sqrt{7}$.

Example 1.5.3 *Let's describe the points which are at the same distance from* $(1, 2, 3)$ *and* $(0, 1, 2)$.

Let $(x,\ y,\ z)$ *be such a point. Then*

$$\sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x-0)^2 + (y-1)^2 + (z-2)^2}.$$

Squaring both sides gives

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = x^2 + (y-1)^2 + (z-2)^2.$$

Therefore,

$$x^2 - 2x + 14 + y^2 - 4y + z^2 - 6z = x^2 + y^2 - 2y + 5 + z^2 - 4z.$$

This implies

$$-2x + 14 - 4y - 6z = -2y + 5 - 4z$$

and, finally,

$$2x + 2y + 2z = 9.$$

What this means is that a point (x, y, z) *is at equal distance from the two points we started with* **if and only if**[4] *it satisfies this equation. In a short while we'll see that the set of these points is a plane.*

[4] By saying "Statement A is true **if and only if** statement B is true" we mean that the two statements are equivalent, that B follows from A and A follows from B.

There are certain properties of the distance which are obvious. Three which follow directly from the definition are

$$|\mathbf{x}-\mathbf{y}| = |\mathbf{y}-\mathbf{x}|$$

and

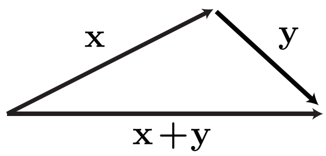
$$|\mathbf{x}-\mathbf{y}| \geq 0$$

and

$$|\mathbf{x}-\mathbf{y}| = 0 \text{ if and only if } \mathbf{x}=\mathbf{y}.$$

A fourth fundamental property of distance is known as the triangle inequality. Recall that in any triangle the sum of the lengths of two sides is always at least as large as the length of the third side,

$$|\mathbf{x}+\mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|.$$



In the following chapter we'll give a proof that this inequality holds true for vectors in any $\mathbf{R}^n$ and not just in $\mathbf{R}^2$ and $\mathbf{R}^3$.

Chapter 2

Vector Products

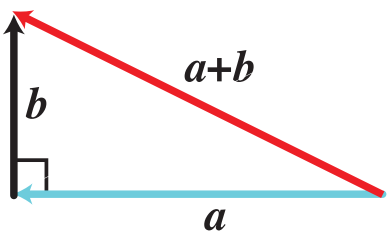
In this chapter we'll introduce the two most useful ways of multiplying vectors. The first of these is called the dot product. The second product is called the cross product.

2.1 The dot product

Now that we know how to calculate distances between two points in $\mathbf{R}^n$ in terms of their coordinates it is natural to look for good ways to recognize and construct right (and other) angles between vectors, measure areas and volumes of shapes described by vectors, and all this in terms of coordinates.

Definition and basic properties

To motivate the first vector product, let's figure out when two vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ are at right angles to each other.



This is the case if and only if the triangle in the diagram is a right triangle. Then Pythagoras tells us that

$$|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$$

Expressing this equation in terms of coordinates gives

$$(a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2 = (a_1^2 + a_2^2 + a_3^2) + (b_1^2 + b_2^2 + b_3^2).$$

Expanding on the left gives

$$(a_1^2 + 2a_1b_1 + b_1^2) + (a_2^2 + 2a_2b_2 + b_2^2) + (a_3^2 + 2a_3b_3 + b_3^2) = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2.$$

Finally this simplifies to

$$a_1b_1 + a_2b_2 + a_3b_3 = 0.$$

This last equation provides a simple test to decide when our two vectors are at right angles and motivates the following definition.

Definition 2.1.1 (Dot product) *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two vectors in* $\mathbf{R}^n$. *Then their* **dot product** *is*

$$\mathbf{a} \cdot \mathbf{b} = \sum_{k=1}^{n} a_k b_k.$$

The dot product $\mathbf{a} \cdot \mathbf{b}$ *is sometimes called* **inner product** *and is denoted by* $\langle \mathbf{a}, \mathbf{b} \rangle$.

Example 2.1.1 *The dot product of* $(1,\ 2,\ 0,\ -1)$ *and* $(0, 1, 2, 3)$ *is*

$$(1,\ 2,\ 0,\ -1) \cdot (0, 1, 2, 3) = 1 \cdot 0 + 2 \cdot 1 + 0 \cdot 2 + (-1) \cdot 3 = 0 + 2 + 0 - 3 = -1.$$

The dot product has a number of important properties.

Proposition 2.1.1 (Properties of the dot product) *Let* α *and* β *be scalars and let* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *be vectors in* $\mathbf{R}^n$. *Then*

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a},$$

$$\mathbf{a} \cdot \mathbf{a} \geq 0 \text{ and } \mathbf{a} \cdot \mathbf{a} = 0 \text{ if and only if } \mathbf{a} = \mathbf{0},$$

$$(\alpha\mathbf{a} + \beta\mathbf{b}) \cdot \mathbf{c} = \alpha\ (\mathbf{a} \cdot \mathbf{c}) + \beta(\mathbf{b} \cdot \mathbf{c}),$$

$$\mathbf{c} \cdot (\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha(\mathbf{c} \cdot \mathbf{a}) + \beta(\mathbf{c} \cdot \mathbf{b}),$$

$$|\mathbf{a}|^2 = \mathbf{a} \cdot \mathbf{a}.$$

Together with our long list of properties of vector addition and multiplication by scalars, the main message to take away from this long list is that when it comes to manipulating simple algebraic expressions involving the dot product you can simply pretend that you are dealing with numbers instead of vectors.

Example 2.1.2 *Let's expand* $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})$*. Here we go (on autopilot)*

$$\begin{aligned}&(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})\\&=\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}\\&=\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{b}\\&=\mathbf{a}\cdot\mathbf{a}+2\ (\mathbf{a}\cdot\mathbf{b})+\mathbf{b}\cdot\mathbf{b}.\end{aligned}$$

You should NOT abbreviate $\mathbf{a}\cdot\mathbf{a}$ as $\mathbf{a}^2$. However, note that you can write $\mathbf{a}\cdot\mathbf{a}$ as $|\mathbf{a}|^2$. Why? Because the dot product of a vector with itself is simply the sum of the squares of all its components, which is the square of the length of the vector. This is the last of the properties of the dot product listed above. This means that

$$(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=|\mathbf{a}|^2+2\ (\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2.$$

Pretty neat that formally this works out the same way as if $\mathbf{a}$ and $\mathbf{b}$ had been numbers.

Example 2.1.3 *Let's find the length of* $\mathbf{a}=(2,1,4,2)$*. That is, find* $|\mathbf{a}|$*.*

$$|\mathbf{a}|=\sqrt{(2,1,4,2)\cdot(2,1,4,2)}=5.$$

The dot product satisfies a fundamental inequality known as the Cauchy-Schwarz inequality.

Theorem 2.1.1 (The Cauchy-Schwarz inequality) *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be vectors in* $\mathbf{R}^n$*. Then*

$$|\mathbf{a}\cdot\mathbf{b}|\leq|\mathbf{a}||\mathbf{b}|.$$

Furthermore $|\mathbf{a}\cdot\mathbf{b}|=|\mathbf{a}||\mathbf{b}|$ *if and only if one of* $\mathbf{a}$ *or* $\mathbf{b}$ *is a scalar multiple of the other.*

Proof. Note that if $\mathbf{b}=\mathbf{0}$ then $|\mathbf{a}\cdot\mathbf{b}|=|\mathbf{a}||\mathbf{b}|=0$ and therefore the inequality holds in this case. We may therefore assume in everything that follows that $\mathbf{b}\neq\mathbf{0}$.

Define the function

$$f(t)=(\mathbf{a}+t\mathbf{b})\cdot(\mathbf{a}+t\mathbf{b}).$$

Then, as a consequence of the basic properties of the dot product listed above, $f(t)\geq 0$ for all $t\in\mathbf{R}$ and we can expand the dot product on the right just like we would expand it if $\mathbf{a}$ and $\mathbf{b}$ were numbers. So

$$\begin{aligned}f(t)&=\mathbf{a}\cdot(\mathbf{a}+t\mathbf{b})+t\mathbf{b}\cdot(\mathbf{a}+t\mathbf{b})\\&=\mathbf{a}\cdot\mathbf{a}+t\ (\mathbf{a}\cdot\mathbf{b})+t\ (\mathbf{b}\cdot\mathbf{a})+t^2\ \mathbf{b}\cdot\mathbf{b}\\&=|\mathbf{a}|^2+2t\ (\mathbf{a}\cdot\mathbf{b})+|\mathbf{b}|^2t^2.\end{aligned}$$

This means that $f(t)$ is a quadratic polynomial in the variable t. Then the quadratic formula tells us that the equation $f(t) = 0$ has the following solutions

$$\frac{-2(\mathbf{a} \cdot \mathbf{b}) \pm \sqrt{4(\mathbf{a} \cdot \mathbf{b})^2 - 4|\mathbf{a}|^2|\mathbf{b}|^2}}{2|\mathbf{b}|^2}.$$

However, we already noted that $f(t) \geq 0$ everywhere. This means that the equation $f(t) = 0$ can have at most one solution. We conclude that the expression under the square root sign (called the **discriminant**) has to be less than or equal to 0. So,

$$(\mathbf{a} \cdot \mathbf{b})^2 - |\mathbf{a}|^2|\mathbf{b}|^2 \leq 0$$

and, therefore,

$$|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|.$$

Finally, tracing this chain of conclusions backwards, it is clear that we have equality if and only if the discriminant is equal to zero. This is the case if and only if

$$f(t) = (\mathbf{a} + t\mathbf{b}) \cdot (\mathbf{a} + t\mathbf{b}) = 0$$

has a solution. In turn this is the case if and only if $\mathbf{a} + t\mathbf{b} = 0$, which is the same as saying that $\mathbf{a}$ is a scalar multiple of $\mathbf{b}$. ■

We now use the Cauchy-Schwarz inequality to prove the triangle inequality for distances in $\mathbf{R}^n$. You already know this inequality and you may even have seen a proof for vectors in $\mathbf{R}^3$ in school. The proof we give below works, just like the one above, for vectors in $\mathbf{R}^n$ no matter what n.

Theorem 2.1.2 (The triangle inequality) *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be vectors in* $\mathbf{R}^n$. *Then*

$$|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$$

and equality holds if and only if one of the vectors is a nonnegative scalar multiple of the other.

Proof. Using the basic properties of the dot product and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|\mathbf{a} + \mathbf{b}|^2 &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) \\
&= (\mathbf{a} \cdot \mathbf{a}) + (\mathbf{a} \cdot \mathbf{b}) + (\mathbf{b} \cdot \mathbf{a}) + (\mathbf{b} \cdot \mathbf{b}) \\
&= |\mathbf{a}|^2 + 2\ (\mathbf{a} \cdot \mathbf{b}) + |\mathbf{b}|^2 \\
&\leq |\mathbf{a}|^2 + 2|\mathbf{a} \cdot \mathbf{b}| + |\mathbf{b}|^2 \\
&\leq |\mathbf{a}|^2 + 2|\mathbf{a}||\mathbf{b}| + |\mathbf{b}|^2 \\
&= (|\mathbf{a}| + |\mathbf{b}|)^2
\end{aligned}$$

Taking square roots on both sides we arrive at the inequality that we are after. Note that to get the first inequality we simply use the fact that the absolute value of a number is always greater than or equal to the number.[1] The second inequality comes from applying the Cauchy-Schwarz inequality.

We still need to convince ourselves that equality holds if and only if one of the vectors is a nonnegative scalar multiple of the other. Since this is certainly the case if at least one of the two vectors is the zero vector, let's assume that neither is the zero vector.

Then it is clear that equality occurs if and only if the two inequality signs above are equal signs. We know that the second (Cauchy-Schwarz) inequality becomes an equality if and only if one of $\mathbf{a}$ or $\mathbf{b}$ is a scalar multiple of the other. So, $\mathbf{a} = \alpha\mathbf{b}$ for some non-zero number α. The first inequality is an equality if and only if $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a} \cdot \mathbf{b}|$. However, this is only possible if the factor α is positive. This completes the proof of the triangle inequality. ■

This was an example of a formal proof. Note that it is a custom in mathematics to start a proof by saying "Proof" and to mark its end by saying **Q.E.D.**,[2] drawing a black square like I did, or drawing one of a number of other symbols. When you are asked to do a proof in an assignment please mark the beginning and the end of your proof in the same way.

The Cauchy-Schwarz inequality is one of the most important inequalities in mathematics and makes its appearance in very diverse areas of mathematics. Just to give you a taste remember that there are many sets that satisfy the same basic properties as our vector space $\mathbf{R}^n$. One example was the set of all continuous real-valued functions defined on the closed interval $[a, b]$. Let's define a product of two such functions f, g as follows:

$$f \cdot g = \int_a^b f(x)g(x)\ dx.$$

Now it is easy to check that this strange product satisfies all the fundamental properties of the dot product listed in proposition 2.1.1. Since our proof of the Cauchy-Schwarz inequality was solely based on the basic properties of vectors and the vector product and our abstract vector space and its product also have these properties, we know that the Cauchy-Schwarz inequality also applies to this abstract setting and can be expressed like this:

$$|f \cdot g| = \left|\int_a^b f(x)g(x)\ dx\right| \leq \left|\int_a^b f^2(x)\ dx\right|^{\frac{1}{2}} \left|\int_a^b g^2(x)\ dx\right|^{\frac{1}{2}} = |f||g|.$$

So for example, choose $f(x) = \sin(x)$ and $g(x) = \cos(x)$. Then Cauchy-Scharz tells us that

$$\left|\int_a^b \sin(x)\cos(x)\ dx\right| \leq \left|\int_a^b \sin^2(x)\ dx\right|^{\frac{1}{2}} \left|\int_a^b \cos^2(x)\ dx\right|^{\frac{1}{2}}.$$

[1] For example, take any negative number, like -5. Then $-5 \leq |-5| = 5$. Or take any positive number, like 3. Then $3 \leq |3| = 3$. And, of course, $0 \leq |0| \leq 0$.

[2] Q.E.D. abbreviates "quod erat demonstrandum" which is the Latin expression for "what had to be demonstrated".

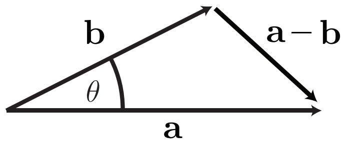
The angle between two vectors in $\mathbf{R}^3$

The included angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{R}^3$ is the angle between these two vectors which is less than or equal to 180 degrees. We can use the dot product to determine this angle.

Theorem 2.1.3 (The angle made by two vectors) *Let θ be the included angle between two vectors $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{R}^3$. Then*

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta).$$

Proof. Let's consider the following diagram.



The **Cosine Rule** tells us that

$$|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos(\theta).$$

On the other hand,

$$\begin{aligned} |\mathbf{a} - \mathbf{b}|^2 &= (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\ (\mathbf{a} \cdot \mathbf{b}). \end{aligned}$$

Therefore

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\ (\mathbf{a} \cdot \mathbf{b}) = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\ |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

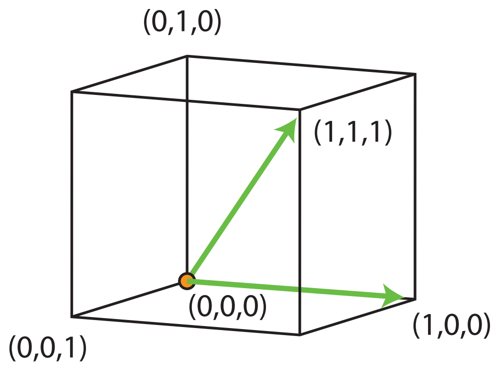
and we conclude that

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta). \qquad \blacksquare$$

Of course, if both vectors are non-zero we can also solve for $\cos(\theta)$,

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}.$$

Example 2.1.4 *Let's calculate the angle between an edge and a diagonal of a cube meeting in one of the vertices of the cube.*



For this we consider the unit cube which has $(0,0,0),(1,0,0),(0,1,0)$ *and* $(0,0,1)$ *as four of its vertices. Then the vertex opposite* $(0,0,0)$ *is the point* $(1,1,1)$. *This means that we can let the vectors* $(1,0,0)$ *and* $(1,1,1)$ *stand for the edge and the diagonal meeting in the vertex* $(0,0,0)$ *of this cube. So, let's find the angle between the vectors* $(1,0,0)$ *and* $(1,1,1)$. *The dot product of these two vectors equals 1 and the lengths of these vectors are 1 and* $\sqrt{3}$. *Therefore, the cosine of the included angle equals*

$$\cos(\theta) = \frac{1}{\sqrt{3}} \approx 0.57735.$$

Now that we know the cosine, we can calculate the angle that we are after by solving the equation, $\cos(\theta) = 0.57735$ *using a calculator. The answer is* $\theta = 54.74$ *degrees.*

The angle between two vectors in $\mathbf{R}^n$

Remember that we started the last section by restricting ourselves to vectors in $\mathbf{R}^3$. However, we can also use the formula

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

to **define** angles between non-zero vectors in $\mathbf{R}^n$. For this to make sense we just have to make sure that the expression on the right is always a number between -1 and 1 (since the cosine only takes on values in this range). However, this follows immediately from the Cauchy-Schwarz inequality $|\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$.

Is there any need for angles in higher dimensions? Well, we'll see that any two vectors in $\mathbf{R}^n$ also live inside a **2-dimensional subspace of** $\mathbf{R}^n$ which is really just a copy of $\mathbf{R}^2$ and angles definitely make sense in $\mathbf{R}^2$, right? Just wait and see.

A simple test for right angles

Just looking at our formula for $\cos(\theta)$, it is clear that two non-zero vectors in $\mathbf{R}^3$ make a right angle if and only if their dot product is equal to zero. This is our simple way to check whether two vectors are at right angles that we derived at the beginning of this chapter. Also motivated by this fact, we define two vectors in $\mathbf{R}^n$ to be **perpendicular** if their dot product is equal to zero. Instead of saying vectors are perpendicular people also often say that the vectors are **at right angles** or **orthogonal**.

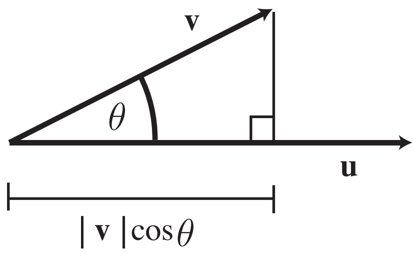
Example 2.1.5 *We already know that any two of the unit coordinate vectors* $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0)$ *and* $\mathbf{k} = (0, 0, 1)$ *are perpendicular. This is also easy to see at a glance using the dot product. For example,*

$$\mathbf{i} \cdot \mathbf{j} = (1, 0, 0) \cdot (0, 1, 0) = 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0 = 0.$$

Projecting vectors

There are two closely related ways of projecting one vector onto another which are both important tools that we'll need in the following. Let's again start by restricting ourselves to vectors in $\mathbf{R}^3$ and then generalize to $\mathbf{R}^n$ later.

Definition 2.1.2 (Scalar projection) *Let θ be the included angle between the two non-zero vectors* $\mathbf{u}$ *and* $\mathbf{v}$*. The* **scalar projection** *of* $\mathbf{v}$ *onto* $\mathbf{u}$ *is equal to* $|\mathbf{v}|\cos(\theta)$*. Note that, as you can see in the following diagram, the absolute value of the scalar projection is simply the length of the "shadow" cast by* $\mathbf{v}$ *onto* $\mathbf{u}$*.*



Combining the expression of the scalar projection with our formula for $\cos(\theta)$,

$$\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

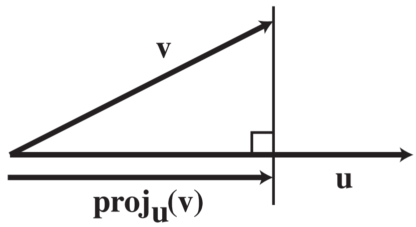
yields the following simple formula for calculating scalar projections:

Proposition 2.1.2 (Scalar projection) *Let* $\mathbf{u}$ *be a non-zero vector and let* $\mathbf{v}$ *be an arbitrary vector. Then the scalar projection of* $\mathbf{v}$ *onto* $\mathbf{u}$ *is given by*

$$\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|}.$$

Example 2.1.6 *Let's calculate the scalar projection of* $\mathbf{v} = (1, 2, 7)$ *onto* $\mathbf{u} = (2, 3, 4)$*. First we calculate* $|\mathbf{u}| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$ *and* $\mathbf{v}\cdot\mathbf{u} = 2 + 6 + 28 = 36$*. Then the scalar projection is equal to* $\frac{36}{\sqrt{29}} \approx 6.685$.

Definition 2.1.3 (Vector projection) *The* **vector projection** $\mathbf{proj}_{\mathbf{u}}(\mathbf{v})$ *of* $\mathbf{v}$ *onto* $\mathbf{u}$ *is the vector shown in the following diagram.*



It is easy to construct this vector: First multiply $\mathbf{u}$ by $\frac{1}{|\mathbf{u}|}$. This gives a unit vector pointing in the same direction as $\mathbf{u}$. Second, multiply this unit vector by the scalar projection to create the vector projection that we are after. This translates into the following formula:

Proposition 2.1.3 (Vector projection) *Let* $\mathbf{u}$ *be a non-zero vector and* $\mathbf{v}$ *an arbitrary vector. The vector projection of* $\mathbf{v}$ *onto* $\mathbf{u}$ *is*

$$\mathbf{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\right)\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}.$$

[3]

[3] Note that originally we've restricted the scalar projection to projecting a non-zero vector $\mathbf{v}$ onto a non-zero vector $\mathbf{u}$. In the subsequent discussion we then snuck in a natural extension of our original definition to also allow for the vector $\mathbf{v}$ to be the zero vector—the scalar projection of the zero vector onto any non-zero vector is equal to 0.

This extension is of no importance when it comes to practical applications of the scalar projection, but it does become important when we consider projections of vectors onto other vectors in terms of linear transformations in Chapter 8.

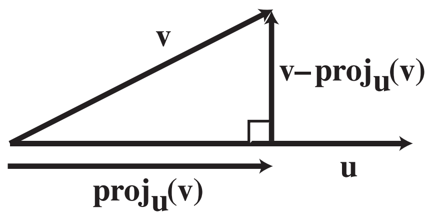
We only remark that there is no natural extension of our original definition in which we also allow for $\mathbf{u}$, the vector that is being projected onto, to be the zero vector.

Example 2.1.7 *Let's find the vector projection of* $\mathbf{v} = (1, 2, 7)$ *onto* $\mathbf{u} = (2, 3, 4)$*. In the previous exercise we calculated that* $|\mathbf{u}| = \sqrt{29}$ *and* $\mathbf{v} \cdot \mathbf{u} = 36$*. Therefore,* $\mathbf{proj_u(v)} = \frac{36}{29}(2, 3, 4)$*.*

Example 2.1.8 *Given* $\mathbf{v} = (1, 2, 7)$ *and* $\mathbf{u} = (2, 3, 4)$*, let's express* $\mathbf{v}$ *as the sum of a vector pointing in the same direction as* $\mathbf{u}$ *and a vector perpendicular to* $\mathbf{u}$*.*

The following diagram shows that the answer to this question is

$$\mathbf{v} = \mathbf{proj_u(v)} + (\mathbf{v} - \mathbf{proj_u(v)}).$$



This example shows how a vector may be **resolved** into two parts, one parallel and one perpendicular to another given vector.

This does not only work for vectors in $\mathbf{R}^3$ but also for vectors in $\mathbf{R}^n$: Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{R}^n$ define

$$\mathbf{proj_u(v)} = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2}\right) \mathbf{u} \text{ (the projection of } \mathbf{v} \text{ onto } \mathbf{u})$$

and

$$\mathbf{perp_u(v)} = \mathbf{v} - \mathbf{proj_u(v)}$$

Then clearly

- $\mathbf{proj_u(v)}$ is parallel to $\mathbf{u}$;
- $\mathbf{perp_u(v)} + \mathbf{proj_u(v)} = \mathbf{v}$;
- $\mathbf{perp_u(v)} \cdot \mathbf{proj_u(v)} = 0$, that is, $\mathbf{perp_u(v)}$ is perpendicular to $\mathbf{proj_u(v)}$.

Interpreting Cauchy-Schwarz geometrically

Most people are aware of the nice geometrical interpretation of the triangle inequality for vectors in terms of the sides of a triangle: the sum of the lengths of two sides of a triangle is always greater than the length of the remaining side. Not many people I know seem to be aware of the following equally nice geometrical interpretation of the Cauchy-Schwarz inequality

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}| :$$

For non-zero $\mathbf{u}$ we can rewrite Cauchy-Schwarz as

$$\frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}|} \leq |\mathbf{v}|.$$

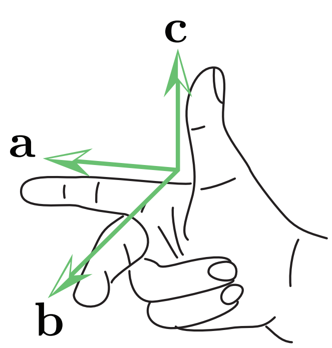
Cast in this form the inequality just says that the length of the projection of $\mathbf{v}$ onto $\mathbf{u}$ is never longer than $\mathbf{v}$ (with equality occurring if and only if the two vectors are parallel). In other words, projection never lengthens vectors.

2.2 The cross product

The cross product is another very useful way of multiplying two vectors in $\mathbf{R}^3$. It is very different from the dot product in many ways. Most importantly, while the dot product is defined between any two vectors in $\mathbf{R}^n$ and produces a real number, the cross product is only defined for vectors in $\mathbf{R}^3$ and produces another vector in $\mathbf{R}^3$. While the dot product is a great tool for calculating angles, the cross product can be used to construct right angles as well as measure areas and volumes of basic shapes given in terms of vectors.

Definition 2.2.1 (Right-handed and left-handed systems of vectors) *Three vectors* $\mathbf{a}, \mathbf{b}, \mathbf{c}$*, in that order and not all contained in a plane, form a right-handed system (a left-handed system) if, when you extend the index finger of your right (left) hand along the vector* $\mathbf{a}$ *and the middle finger along the vector* $\mathbf{b}$*, the thumb points roughly in the direction of* $\mathbf{c}$*.*

In the following diagram our three vectors form a right-handed system.



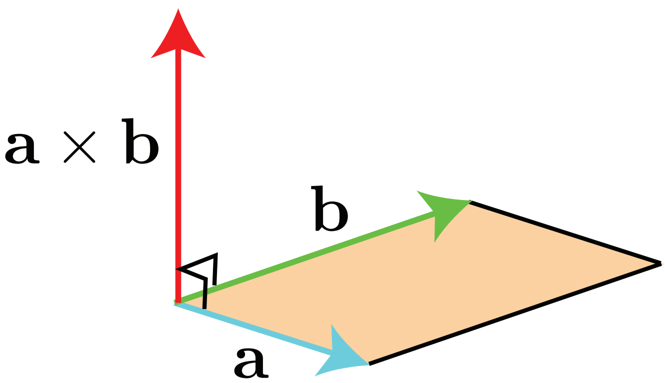
Definition 2.2.2 (Right-handed and left-handed coordinate systems) *A coordinate system for* $\mathbf{R}^3$ *is called right-handed or left-handed if the three coordinate vectors* $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *form a right-handed or left-handed system of vectors.*

It is a convention followed in most texts on mathematics to only work with right-handed coordinate systems. We will do the same. Please check that whenever the coordinate axes in these notes are visible in diagrams the vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ really form a right-handed system.

The following is a geometric definition of the cross product.

Definition 2.2.3 (The cross product) *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be two vectors in* $\mathbf{R}^3$. *Then the* **cross product** *of the two vectors is the vector* $\mathbf{a} \times \mathbf{b}$ *which is determined by the following three properties:*

- *The length of* $\mathbf{a} \times \mathbf{b}$ *is equal to the area of the parallelogram spanned by* $\mathbf{a}$ *and* $\mathbf{b}$. *In particular, this implies that* $\mathbf{a} \times \mathbf{b}$ *is equal to the zero vector if and only if* $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel.*[4]
- $\mathbf{a} \times \mathbf{b}$ *is perpendicular to both* $\mathbf{a}$ *and* $\mathbf{b}$.[5]
- **If** $\mathbf{a} \times \mathbf{b}$ **is not the zero vector,** *then the three vectors* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{a} \times \mathbf{b}$ *form a right-handed system.*



From this definition it is clear that once we have a good way of calculating the cross product it will be a great tool for constructing vectors that are at right angles to two given vectors and for calculating the areas of parallelograms in space and related shapes such as triangles (a triangle is half a parallelogram).

[4] Remember that the zero vector is parallel to any other vector. Hence, $\mathbf{a} \times \mathbf{0} = \mathbf{0} \times \mathbf{a} = \mathbf{0}$.

[5] This is the same as saying $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$.

Example 2.2.1 *Since the coordinate vectors* $\mathbf{i}$ *and* $\mathbf{j}$ *are perpendicular unit vectors, the parallelogram spanned by these two vectors is a square of side length* 1*. Since the unit vector* $\mathbf{k}$ *is perpendicular to both vectors and since we have agreed that* $\mathbf{i}$*,* $\mathbf{j}$ *and* $\mathbf{k}$ *form a right-handed system it is clear that* $\mathbf{i} \times \mathbf{j} = \mathbf{k}$*.*

The cross product satisfies the following properties:

Proposition 2.2.1 (Basic properties of the cross product) *Let* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *be vectors in* $\mathbf{R}^3$ *and let* α *be a scalar. Then*

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= -(\mathbf{b} \times \mathbf{a}) \quad \text{(!!!!)} \\ \mathbf{a} \times \mathbf{a} &= \mathbf{0} \\ (\alpha \mathbf{a}) \times \mathbf{b} &= \alpha(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (\alpha \mathbf{b}) \\ \mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} \\ (\mathbf{b} + \mathbf{c}) \times \mathbf{a} &= \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}. \end{aligned}$$

That the cross product has the first three properties follows immediately from its definition. On the other hand, the last two properties are not obvious at all. These last two properties are called the **distributive law** of the cross product. We won't prove the distributive law. If you are keen please look up a proof in Kuttler's book (referred to in the introduction to these notes). Also note that, unlike our other two long lists of basic properties that we encountered in previous chapters, this one does contain a surprise (indicated by lots of exclamation marks).

The following theorem gives the coordinate description of the cross product.

Theorem 2.2.1 (Coordinate description of the cross product) *Let* $\mathbf{a} = (a_1, a_2, a_3)$ *and* $\mathbf{b} = (b_1, b_2, b_3)$ *be two vectors. Then*

$$\mathbf{a} \times \mathbf{b} = (a_2 b_3 - a_3 b_2, -(a_1 b_3 - a_3 b_1), a_1 b_2 - a_2 b_1).$$

Proof. We showed earlier on that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. Here is the complete list of cross products between the unit coordinate vectors.

$$\mathbf{i} \times \mathbf{j} = \mathbf{k} \ , \ \mathbf{j} \times \mathbf{i} = -\mathbf{k}$$

$$\mathbf{k} \times \mathbf{i} = \mathbf{j} \ , \ \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

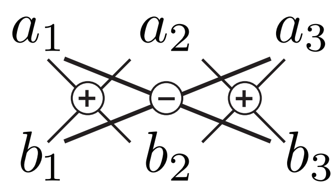
$$\mathbf{j} \times \mathbf{k} = \mathbf{i} \ , \ \mathbf{k} \times \mathbf{j} = -\mathbf{i}$$

$$\mathbf{i} \times \mathbf{i} = \mathbf{0} \ , \ \mathbf{j} \times \mathbf{j} = \mathbf{0} \ , \ \mathbf{k} \times \mathbf{k} = \mathbf{0}$$

With this information and using the basic properties of the cross product, we calculate

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \times (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
&= a_1b_2\mathbf{i} \times \mathbf{j} + a_1b_3\mathbf{i} \times \mathbf{k} + a_2b_1\mathbf{j} \times \mathbf{i} + a_2b_3\mathbf{j} \times \mathbf{k} + a_3b_1\mathbf{k} \times \mathbf{i} + a_3b_2\mathbf{k} \times \mathbf{j} \\
&= a_1b_2\mathbf{k} - a_1b_3\mathbf{j} - a_2b_1\mathbf{k} + a_2b_3\mathbf{i} + a_3b_1\mathbf{j} - a_3b_2\mathbf{i} \\
&= (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} \\
&= (a_2b_3 - a_3b_2, -(a_1b_3 - a_3b_1), a_1b_2 - a_2b_1). \qquad \blacksquare
\end{aligned}$$

A simple graphical way to remember the coordinate description of the cross product is to write the second vector below the first one and then multiply and combine as indicated by the three crosses:



Important:

1. The cross on the **right** corresponds to the first component of the cross product (on the **left**), and the cross on the left corresponds to the third component on the right.

2. Don't forget the minus in the middle!!!! [6]

Example 2.2.2 *Using the coordinate description of the cross product we calculate*

$$(1, 1, 1) \times (5, 6, 7) = (1, -2, 1).$$

Now, let's check that the cross product is really perpendicular to both $(1, 1, 1)$ *and* $(5, 6, 7)$*:*

$$(1, 1, 1) \cdot (1, -2, 1) = 0 \text{ and } (5, 6, 7) \cdot (1, -2, 1) = 0.$$

This really works!

[6] Those among you who are familiar with determinants of 3×3 matrices may be interested to know that the coordinate description of the cross product can also be expressed as the following "formal determinant":

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}.$$

What this means is that if you pretend that the determinant on the right, with its strange mix of vector and number entries, makes sense and evaluate it you really get $(a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$. All this also makes for a very nice way of remembering the coordinate description of the cross product once you know how to evaluate determinants by expanding along rows and columns. We'll come back to this in the chapter on determinants.

Example 2.2.3 *Let's find the area of the parallelogram spanned by the vectors* $(1, 1, 1)$ *and* $(5, 6, 7)$ *(the same vectors as in the previous example). By definition this area is equal to the length of the cross product* $(1, -2, 1)$ *of the two vectors. This length is*

$$\sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}.$$

Example 2.2.4 *Let's find the area of the triangle whose vertices are the points* $(1, 2, 3)$, $(0, 2, 5)$, *and* $(5, 1, 2)$. *The vector that starts at* $(1, 2, 3)$ *and ends at* $(0, 2, 5)$ *is* $\mathbf{a} = (0, 2, 5) - (1, 2, 3) = (-1, 0, 2)$ *and the vector that starts at* $(1, 2, 3)$ *and ends at* $(5, 1, 2)$ *is* $(4, -1, -1)$. *Then the area of the triangle is half the area of the parallelogram determined by* $\mathbf{a}$ *and* $\mathbf{b}$. *Now,*

$$\mathbf{a} \times \mathbf{b} = (-1, 0, 2) \times (4,\ -1,\ -1) = (2, 7, 1)$$

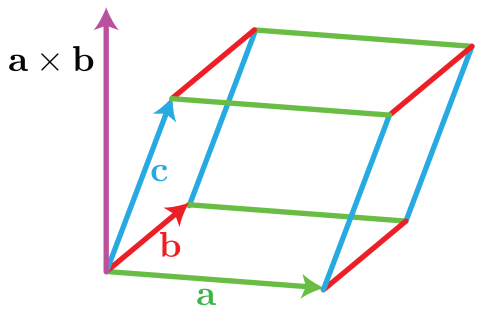
and so the area of the triangle is

$$\frac{1}{2}\sqrt{4 + 49 + 1} = \frac{3}{2}\sqrt{6}.$$

2.3 The box product

The vector product is a great tool for constructing right angles and measuring area. Together with the dot product you can also use it to measure volumes.

Definition 2.3.1 (Parallelepiped) *A* **parallelepiped** *is the 3d counterpart of a parallelogram. Just as the parallelogram in our definition of the cross product is spanned by two vectors, a parallelepiped is* **spanned** *by three vectors* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$, *as shown in the following diagram.*



The volume of the parallelepiped is simply the

$$(\text{area parallelogram spanned by } \mathbf{a} \text{ and } \mathbf{b}) \cdot (\text{height in the direction of } \mathbf{a} \times \mathbf{b}).$$

However, this height is just the absolute value of the scalar projection of the vector $\mathbf{c}$ onto $\mathbf{a} \times \mathbf{b}$. This means that the volume of the shape is

$$|\mathbf{a} \times \mathbf{b}| \frac{|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|}{|\mathbf{a} \times \mathbf{b}|} = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|.$$

Note that the meaning of two vertical strokes enclosing an expression depends on the nature of the expression. For example, in the above equation $|\mathbf{a} \times \mathbf{b}|$ is the length of the vector $\mathbf{a} \times \mathbf{b}$. On the other hand, $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$ is the absolute value of the number $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$.

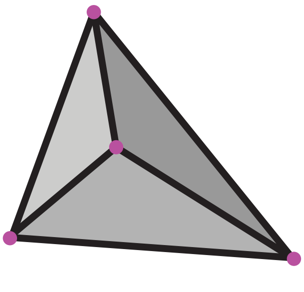
Definition 2.3.2 (Box product) *The expression* $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ *is called the* **box product** *of* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$. *Another name for the box product is* **scalar triple product.**

On close inspection it turns out that the absolute value of the box product is independent of the order in which we multiply the three vectors. The sign of the box product is positive if $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ form a right-handed system and negative if they form a left-handed system. The box product is equal to zero if the three vectors are contained in a plane.

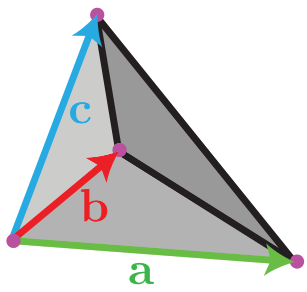
Proposition 2.3.1 (Box product) *The volume of the parallelepiped spanned by the vectors* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *is the absolute value of the box product* $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ *of the three vectors.*

Example 2.3.1 $(\mathbf{i} \times \mathbf{j}) \cdot \mathbf{k} = 1$, *the volume of the unit cube spanned by the three unit coordinate vectors.*

The 3d shape determined by four points in space is called a **tetrahedron**. A tetrahedron is the 3d counterpart of a triangle (which is the shape determined by three points).



As we've seen it is easy to calculate the area of a triangle in space using the cross product. Similarly, the box product provides a convenient way to calculate the volume of a tetrahedron. We use the three vectors connecting one vertex of the tetrahedron with the other three vertices of the tetrahedron shown in the following diagram.



Then the volume of the tetrahedron is simply one sixth of the volume of the parallelepiped spanned by the three vectors.

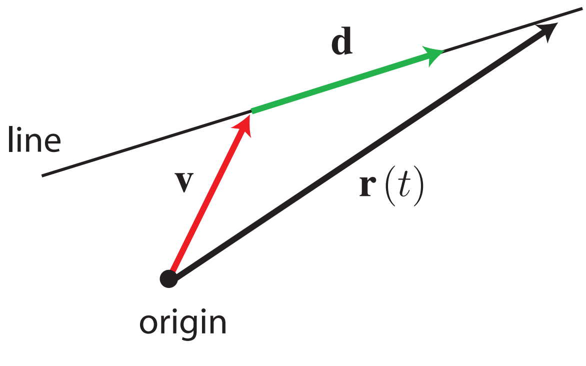
Chapter 3

Lines and planes in $\mathbf{R}^3$

Points, lines and planes are the basic building blocks for doing geometry in three dimensions. And vectors combined with the dot product and the cross product are some of the best tools to tame these basic building blocks, as we'll find out in this chapter.

3.1 Vector equations of lines

Lines in $\mathbf{R}^3$ are easily described using two vectors $\mathbf{v}$ and $\mathbf{d}$, where $\mathbf{d}$ is pointing in the direction of the line and $\mathbf{v}$ connects the origin with some arbitrary point of the line.



Then

$$\mathbf{r}(t) = \mathbf{v} + t\mathbf{d}$$

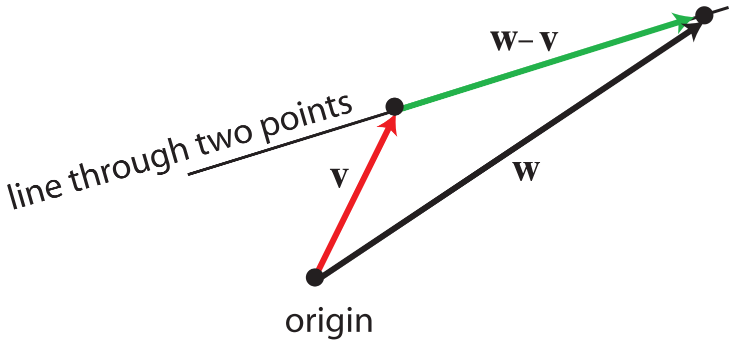
is a vector that depends on the variable t. If you think of the tail of this vector as being anchored at the origin, it is easy to see that its tip will always be a point on the line and that every point of the line corresponds to exactly one choice of t. In this sense the above equation is an **equation for the line**. It is called a **vector equation** for the line.

If $\mathbf{r}(t) = (x(t), y(t), z(t)), \mathbf{d} = (p, q, r)$ and $\mathbf{v} = (a, b, c)$, then we can also write this vector equation in the following **parametric form**:

$$\begin{aligned} x(t) &= a + pt \\ y(t) &= b + qt \\ z(t) &= c + rt \end{aligned}$$

Example 3.1.1 *Let's find the vector equation of the line joining* $(2, 4, 0)$ *and* $(2, 4, 7)$*. We start with the two vectors* $\mathbf{v} = (2, 4, 0)$ *and* $\mathbf{w} = (2, 4, 7)$*. For our equation we need one vector that corresponds to a point on the line. Either* $\mathbf{v}$ *or* $\mathbf{w}$ *will do. Let's choose* $\mathbf{v}$*. We also need a vector that points in the direction of the line. As you can see from the following diagram, one such vector is*

$$\mathbf{d} = \mathbf{w} - \mathbf{v} = (2, 4, 7) - (2, 4, 0) = (0, 0, 7).$$



Therefore a vector equation of the line is

$$\mathbf{r}(t) = (2, 4, 0) + t(0, 0, 7) = (2, 4, 7t),$$

or in parametric form

$$\begin{aligned} x(t) &= 2 \\ y(t) &= 4 \\ z(t) &= 7t \end{aligned}$$

Important: It does not make sense to talk about **the** vector equation of a line as there are infinitely many different vector equations that describe the same line (since there are infinitely many different possible choices for both $\mathbf{v}$ and $\mathbf{d}$).

3.2 "Fun" things to do with points and lines

Point on a line

Given a vector equation of a line and the coordinates of a point it is easy to check whether the point is contained in the line, or not. For example, let's check whether the point $(1, 2, 3)$ is contained in the line $(3, 4, 5) + t(2, 2, 2)$. For this we simply have to check whether there is a common solution for the system of equations

$$\begin{aligned} 1 &= 3 + 2t \\ 2 &= 4 + 2t \\ 3 &= 5 + 2t \end{aligned}$$

The first equation yields $t = -1$, the second equation yields $t = -1$ and the third equation also yields $t = -1$. Since all solutions are equal, the point is contained in the line. If not all solutions had been the same, we would have been able to conclude that the point is not contained in the line. For example, the point $(1, 2, 4)$ is not contained in the line. Check this!

Distance between a point and a line

Let's figure out the distance[1] between the point

$$(1, 2, 3)$$

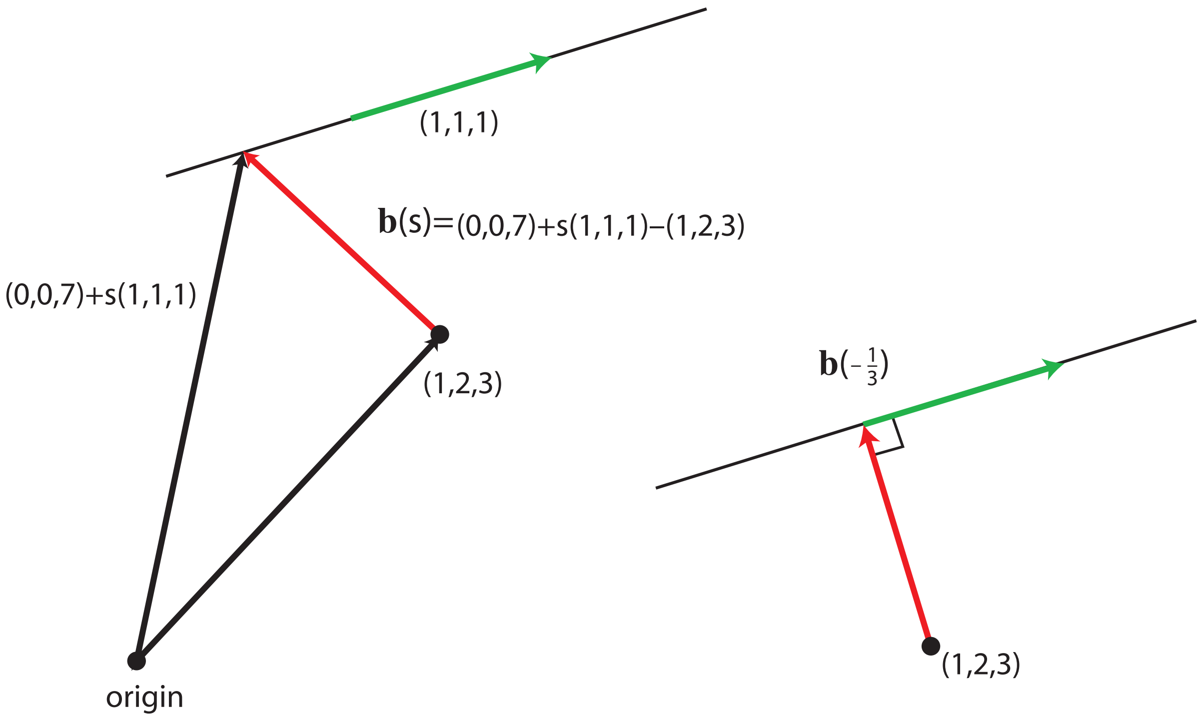
and the line given by the equation

$$(0, 0, 7) + s(1, 1, 1).$$

1. Subtract the vector $(1, 2, 3)$ from the equation of the line. This gives a vector $\mathbf{b}(s)$ that depends on the variable s (see the following diagram):

$$\mathbf{b}(s) = (0, 0, 7) + s(1, 1, 1) - (1, 2, 3) = (-1, -2, 4) + s(1, 1, 1) = (-1 + s, -2 + s, 4 + s).$$

[1]The distance between two sets of points A and B is the shortest distance between the two sets, that is, the minimum of all distances between points in A and points in B. There is a bit of a problem with this definition because for certain sets this minimum does not exist. This won't be a problem with the sets we'll be considering, though. But just in case you are curious, take A to be the closed interval $[0, 1]$ and let B be the open interval $(2, 3)$. Then the shortest distance between these two sets does not exist. If you'd like to know more ask me!



Think of the tail of this vector being fixed at the point $(1,2,3)$ and its tip running along the line as s changes. We want to find the value of s for which the length of $\mathbf{b}(s)$ is as short as possible. This shortest length is equal to the distance that we are after.

2. Note that the shortest $\mathbf{b}(s)$ will be perpendicular to the direction vector $(1,1,1)$ of the line; see the diagram on the right. This means that the dot product between $(1,1,1)$ and $\mathbf{b}(s)$ is zero. Hence

$$(1,1,1)\cdot(-1+s,-2+s,4+s) = -1+s-2+s+4+s = 1+3s = 0.$$

This equation has the solution $s=-\frac{1}{3}$, which means that the distance we are looking for is the length of the vector $\mathbf{b}(-\frac{1}{3})$, and you know how to calculate that length, don't you? Also, if you plug $s=-\frac{1}{3}$ into the vector equation of the line you get the coordinates of the point on the line that is closest to the point $(1,2,3)$.

Two lines and four scenarios

Given vector equations of two lines in space, there are the following four possible scenarios:

a) The two equations describe the same line.

b) The two equations describe two different lines that intersect in a point.

c) The two equations describe two different lines that are parallel but not the same.

d) The two equations describe two different lines that are neither parallel nor have a common point.

You can use vector equations of the two lines to decide which scenario you are dealing with and, in the last two scenarios, how far apart the two lines are.

Parallel lines

When you are dealing with two lines, the first and simplest thing to check is whether the two lines are parallel. This is the case if and only if the direction vectors in the two equations are parallel. For example, the lines described by the equations

$$(1,2,3) + t(1,1,1)$$

and

$$(0,0,7) + s(3,3,3)$$

are parallel because you get one of the direction vectors by stretching the other one:

$$(3,3,3) = 3(1,1,1).$$

In general, if the direction vectors of two vector equations are parallel, then either the two equations describe the same line (scenario a), or the two lines are parallel and don't meet (scenario c). To decide which of these two cases you are dealing with, simply check whether or not some point of the first line is contained in the second line. If it is, then the two equations describe the same line, otherwise they do not.

Lines that are not parallel

The direction vectors of the two lines

$$(1,2,3) + t(1,1,2)$$

and

$$(0,0,7) + s(1,1,1)$$

are not pointing in the same direction. Therefore, the lines are not parallel. To figure out whether the two lines have a point in common, equate their corresponding parametric equations

$$\begin{aligned} 1 + 1t &= 0 + 1s \\ 2 + 1t &= 0 + 1s \\ 3 + 2t &= 7 + 1s \end{aligned}$$

There is a point of intersection if and only if this system of equations in the two variables s and t has a solution. We'll learn very soon how to solve this sort of system in full generality. However, even at this stage it is easy to see that this system of equations has no solution, since the first and second equations contradict each other.

On the other hand, the two lines

$$(1, 0, 0) + t(1, 0, 0)$$

and

$$(0, 0, 0) + s(0, 1, 0)$$

do intersect because the corresponding system of equations

$$\begin{aligned} 1 + 1t &= 0 + 0s \\ 0 + 0t &= 0 + 1s \\ 0 + 0t &= 0 + 0s \end{aligned}$$

has the solution $t = -1, s = 0$. To find the point of intersection, simply plug the t value in the first equation (or the s value in the second equation):

$$(1, 0, 0) + (-1)(1, 0, 0) = (0, 0, 0).$$

So, the point of intersection of these two lines is the origin.

Note:

A second way of deciding whether two lines intersect is to calculate their distance. If this distance is 0, then the two lines intersect. Otherwise they don't intersect. In the following section we will describe how the distance between two lines can be calculated.

The distance between two lines

If two lines are parallel, then it is easy to calculate their distance from each other. Simply pick a point on one of the lines and calculate its distance from the other line as we've described it earlier on in this chapter!

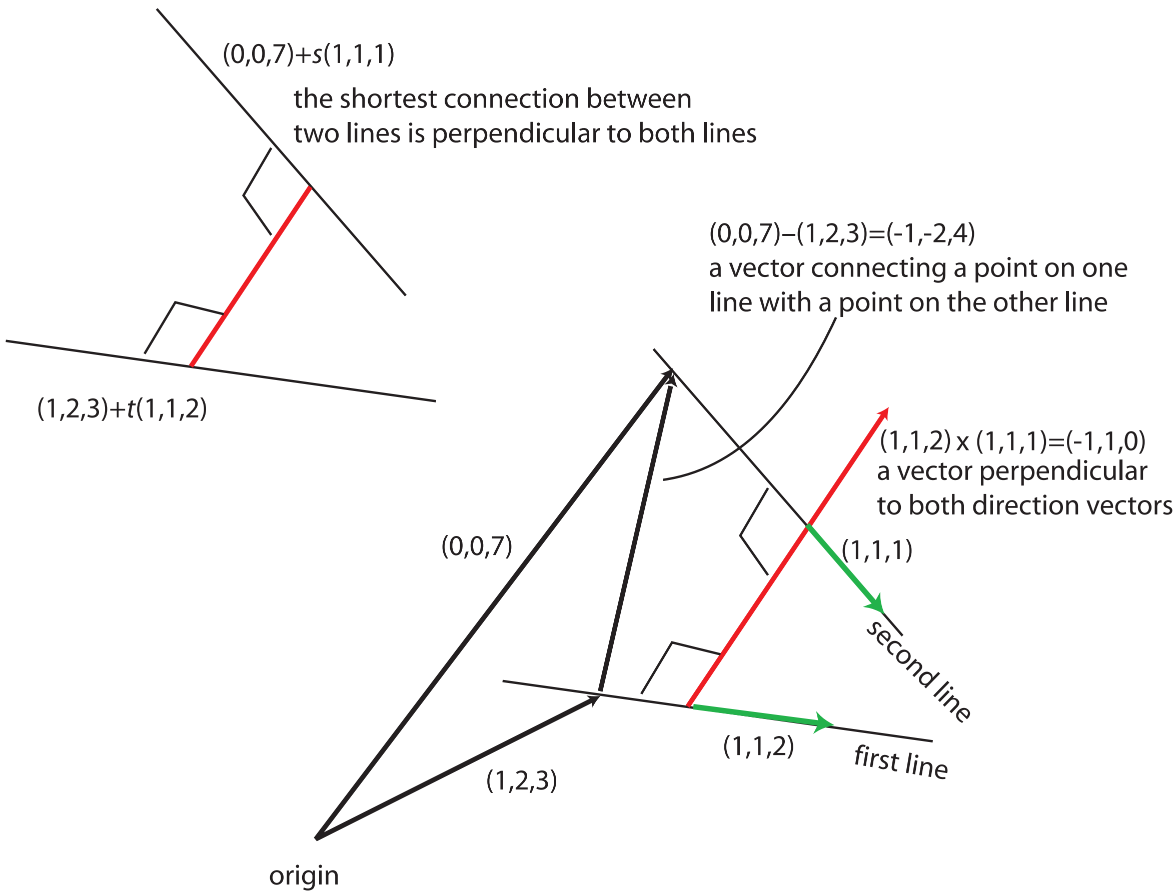
Let's find the distance between the two lines

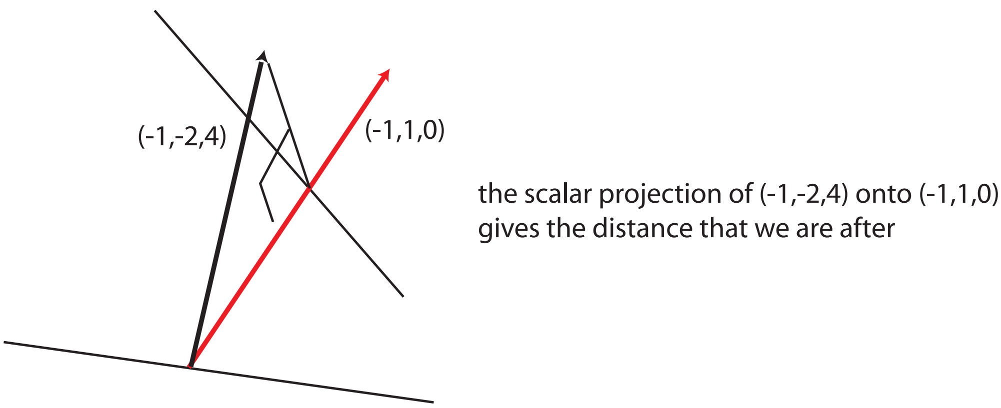
$$(1, 2, 3) + t(1, 1, 2)$$

and

$$(0, 0, 7) + s(1, 1, 1).$$

Remember that these two lines are not parallel. The following sequence of diagrams illustrates the individual steps necessary to find this distance.





The first diagram illustrates that the shortest connection between the two lines is perpendicular to both lines. Therefore the vector product of the direction vectors of the two lines

$$\mathbf{w} = (1,1,2) \times (1,1,1) = (-1,1,0)$$

is pointing in the direction of this connection.

Also construct a vector that connects a point of the first line with a point of the second line, for example,

$$\mathbf{v} = (0,0,7) - (1,2,3) = (-1,-2,4).$$

Now, form the scalar projection of this last vector onto the cross product that we calculated earlier (third diagram):

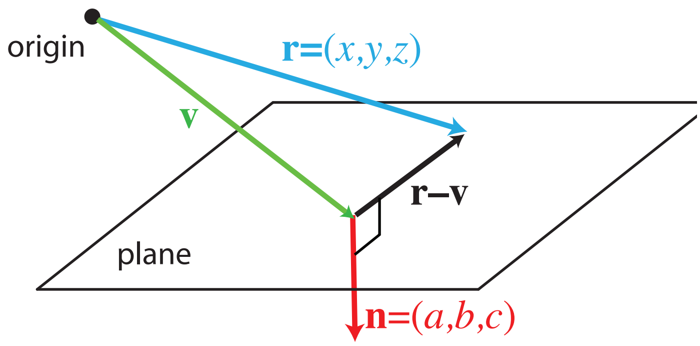
$$\frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|} = \frac{-1}{\sqrt{2}}.$$

Then the absolute value of this scalar projection is the distance that we are after. This absolute value is

$$\frac{1}{\sqrt{2}} \approx 0.7071.$$

3.3 Planes in $\mathbf{R}^3$

Let $\mathbf{n} = (a,b,c)$ be a vector that is perpendicular to a plane in space. This means that the vector is perpendicular to all vectors contained in this plane. We call such a vector a **normal vector** of a plane. Let $\mathbf{v}$ be an arbitrary point on the plane, and let $\mathbf{r} = (x,y,z)$ be the "general" point on the plane. Then we can read off the following diagram that $\mathbf{n}$ is perpendicular to the vector $\mathbf{r}-\mathbf{v}$ that is "contained" in the plane.



Hence

$$\mathbf{n}\cdot(\mathbf{r}-\mathbf{v}) = 0,$$

or

$$\mathbf{n}\cdot\mathbf{r} = \mathbf{n}\cdot\mathbf{v}.$$

The right side of this equation is just some number, let's call it d. The left side is $(a,b,c)\cdot(x,y,z)$. This means that the above equation can also be written as

$$ax + by + cz = d.$$

Again, from the diagram it is clear that an arbitrary point in space is contained in the plane if and only if its x-, y- and z-coordinates form a solution of this equation. Conversely, given an arbitrary equation of the form $ax + by + cz = d$ it is easy to see that its solution set is a plane with normal vector (a, b, c). Therefore it makes sense to call $ax + by + cz = d$ an equation of a plane in space. Furthermore the plane corresponding to this equation is the set of points

$$\{(x, y, z) \in \mathbf{R}^3 \mid ax + by + cz = d\}.$$

Note that multiplying our equation by a non-zero number gives a new equation that does have exactly the same solutions as the equation we started with. This means that both equations describe the same plane. For example, multiplying our equation by 2 gives another equation describing the same plane:

$$2ax + 2by + 2cz = 2d.$$

Conversely, two equations describe the same plane if one is a multiple of the other.

This also means that it does not make sense to speak of **the** equation of a plane, since there are really infinitely many different equations that describe the same plane.

We summarize our discussion as follows

Proposition 3.3.1 (Planes and their equations) *If $ax + by + cz = d$ is the equation of a plane, then the vector (a, b, c) is a normal vector of the plane. Conversely, if (a, b, c) is a normal vector of a plane, then there is a number d such that $ax + by + cz = d$ is an equation for the plane.*

[2]

The left side of an equation of a plane corresponds to a normal vector to the plane. What does the right side tell you? Not that much, except that a plane passes through the origin if and only if this right side is equal to 0.

Example 3.3.1 *Let's have a look at some plane equations. The plane $z = 1$ is the plane parallel to the xy-plane that intersects the z-axis at 1. Similarly, the plane $y = 3$ is the plane parallel to the xz-plane that intersects the y-axis at 3. Finally, the plane $x = 1$ is the plane parallel to the yz-plane that intersects the x-axis at 1.*

---

[2] In analogy to what we just did in three dimensions, in two dimensions the equation $\mathbf{n} \cdot (\mathbf{r} - \mathbf{v}) = 0$ describes a line. Rewriting this equation using coordinates then gives

$$ax + by = c$$

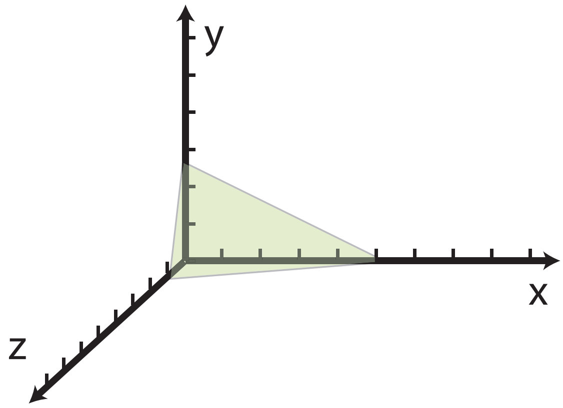
as an equation of a line in the plane.

Doing the same in $\mathbf{R}^n$ yields

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = d,$$

the equation of a **hyperplane,** the n-dimensional counterpart of lines in $\mathbf{R}^2$ and planes in $\mathbf{R}^3$.

To get an idea of the position of a generic plane like $x + 2y + 5z = 5$, we can figure out where it intersects the x-, y- and z-axes. The y- and z-coordinates of a point on the x-axis are both 0. Therefore, setting $y = z = 0$ in the equation and solving for x, tells us where the plane intersects the x-axis. We find that the point at which the plane intersects the x-axis is $(5, 0, 0)$. Similarly, the other two points of intersection with the coordinate axes are $(0, \frac{5}{2}, 0)$ and $(0, 0, 1)$ and we can sketch part of the plane as follows:



Constructing an equation of a plane

It is very easy to construct a normal vector of a plane using the cross product of two vectors that are contained in the plane.

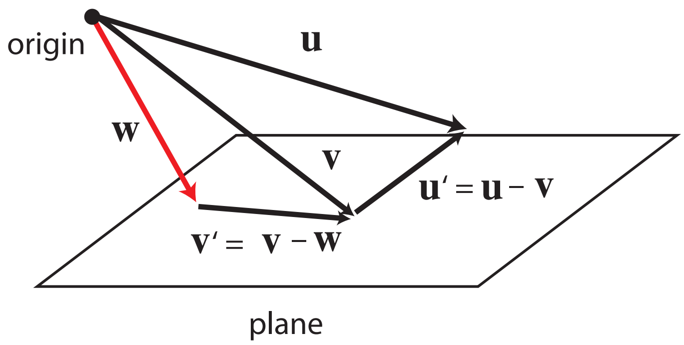
Using this idea here is a simple recipe for determining an equation of a plane starting with three vectors $\mathbf{u} = (1, 2, 7)$, $\mathbf{v} = (2, 3, 4)$ and $\mathbf{w} = (-1, 4, 1)$ that correspond to three points that determine a plane (this means that the three points are not contained in a line):

1. Form the vectors

$$\mathbf{u}' = \mathbf{u} - \mathbf{v} = (1, 2, 7) - (2, 3, 4) = (-1, -1, 3)$$

and

$$\mathbf{v}' = \mathbf{v} - \mathbf{w} = (2, 3, 4) - (-1, 4, 1) = (3, -1, 3).$$



2. As you can see in the diagram, these two vectors are contained in the plane and, as we know, their cross product is perpendicular to both vectors. This means that it is a normal vector of the plane. Therefore, we can set

$$(a, b, c) = \mathbf{u}' \times \mathbf{v}' = (0, 12, 4),$$

and therefore the equation that we are after has the form

$$0x + 12y + 4z = d.$$

3. So, the only thing left to determine is the value of d. But that is easy. We just plug in the coordinates for one of the points of the plane, say $(1, 2, 7)$, and we are done:

$$0 \cdot 1 + 12 \cdot 2 + 4 \cdot 7 = 24 + 28 = 52 = d.$$

4. Therefore an equation of our plane is

$$12y + 4z = 52.$$

Easy!

3.4 Fun things to do with points, lines and planes

A point and a plane

Let's figure out the distance between the plane given by the equation

$$x + 2y + 3z = 4$$

and the point $(2, 3, 4)$.

1. We start by finding a point on the plane. Setting $y = z = 0$ gives

$$x + 2 \cdot 0 + 3 \cdot 0 = x = 4,$$

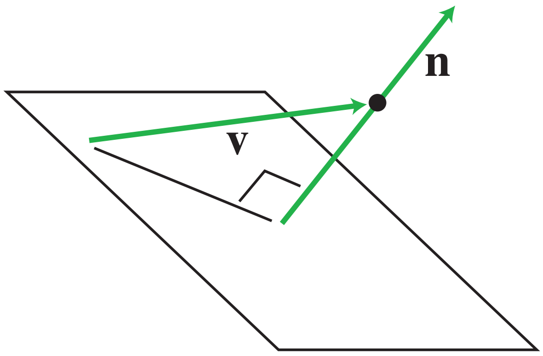
so $(4, 0, 0)$ is a point of the plane.

2. Now we extract a normal vector

$$\mathbf{n} = (1, 2, 3)$$

to the plane from its equation and define the vector that captures the change in position between the point on the plane, $(4, 0, 0)$ and the point that we started with.

$$\mathbf{v} = (2, 3, 4) - (4, 0, 0).$$



3. The absolute value of the scalar projection of $\mathbf{v}$ onto the normal vector $\mathbf{n}$ (see page 26) is equal to the distance that we are after.

A line and a plane

There are three different possible relative positions of a line and a plane in space:

a) The line is contained in the plane;

b) The line intersects the plane in a point;

c) The line is parallel to but not contained in the plane.

It is easy to figure out which of the three scenarios you are dealing with. Consider the example of the plane given by

$$x + 2y + 3z = 4$$

and the line given in parametric form

$$\begin{aligned} x(s) &= 2 - s \\ y(s) &= 3 \\ z(s) &= 4 + 3s \end{aligned}$$

Simply plug the expressions for $x(s), y(s)$ and $z(s)$ into the equation of the plane and try to solve for s. Depending on whether there is no, exactly one or infinitely many solutions, we are dealing with scenario c), b) or a), respectively.

In this case we get

$$2 - s + 2 \cdot 3 + 3(4 + 3s) = 20 + 8s = 4.$$

Hence, $s = -2$. This means that the line intersects the plane in exactly one point. To figure out which point this is, just plug this value of s into the vector equation of the line. The point of intersection is

$$(2+2, 3, 4-3\cdot 2) = (4, 3, -2).$$

It is also possible to calculate the angle at which the line intersects the plane at this point. We will have a look at this in the lab classes.

Two planes

Given equations for two planes, the planes are parallel if and only if the two corresponding normal vectors are parallel. For example, the planes given by the two equations

$$x + 2y + 3z = 4$$

and

$$3x + 6y + 9z = 1$$

are parallel because the normal vectors $(1, 2, 3)$ and $(3, 6, 9)$ are parallel $(3(1, 2, 3) = (3, 6, 9))$.

If two planes are not parallel, they intersect in a line. Let's determine a vector equation for the line of intersection of the planes given by the two equations

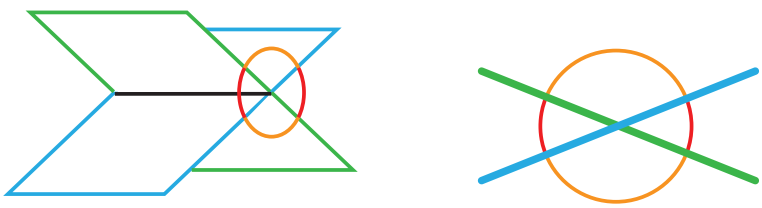
$$x + y + z = 1$$

and

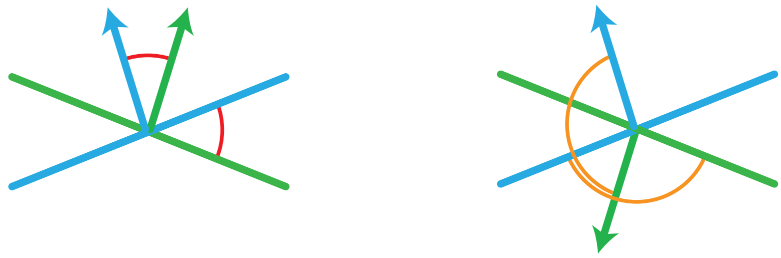
$$x + y + 2z = 1.$$

Let $\mathbf{n} = (1, 1, 1)$ and $\mathbf{n}' = (1, 1, 2)$ be the normal vectors to the two planes extracted from these equations. Then it is clear that the line of intersection is perpendicular to both vectors. Therefore, the cross product of these two vectors $(1, -1, 0)$ is a direction vector of the line. So, the only thing left to find is one point on the line, which amounts to finding one common solution to the two equations. We'll figure out how to do this in the next lecture.

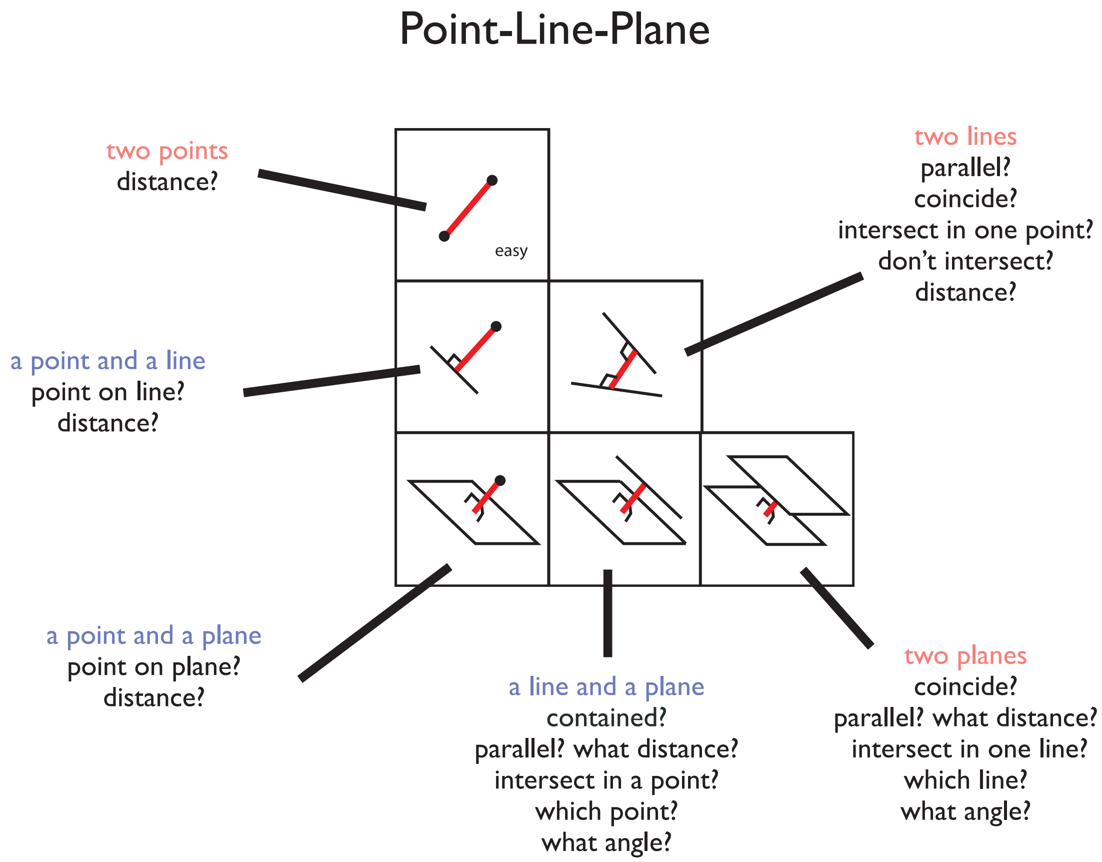
The following diagram shows two planes intersecting in a line. It also shows four angles. It is clear that the two orange angles are equal and that the two red angles are equal. Also, an orange and a red angle clearly add up to 180 degrees. When asked to calculate the angle between the two planes either of these two angles counts as the correct answer.



The next diagram illustrates that to find one of these angles simply calculate the angle between any two normal vectors of the planes.



Here is a concise summary of the different types of questions regarding the relative positions of points, lines and planes in space that we can answer using the techniques introduced so far.



Chapter 4

Systems of Linear Equations

A **system of linear equations** or **linear system** is a collection of linear equations sharing the same variables. Here are two examples of cooked up "real-world" problems that give rise to linear systems.

Bags of coins

We have three bags with a mixture of gold, silver and copper coins. We are given the following information

Bag 1	contains	10 gold, 3 silver, 1 copper	and weighs 60g
Bag 2	contains	5 gold, 1 silver and 2 copper	and weighs 30g
Bag 3	contains	3 gold, 2 silver, 4 copper	and weighs 25g

The question is: What are the respective weights of the gold, silver and copper coins?

Let G, S and C denote the weight of each of the gold, silver and copper coins. Then we can pose our problem in the form of the following system of linear equations:

$$\begin{array}{rcrcrcr} 10G & + & 3S & + & C & = & 60 \\ 5G & + & S & + & 2C & = & 30 \\ 3G & + & 2S & + & 4C & = & 25 \end{array}$$

Puzzles

John's and Mary's ages add to 75 years. When John was half his present age John was twice as old as Mary. How old are they?

This puzzle translates into the following linear system:

$$\begin{array}{rcrcr} J & + & M & = & 75 \\ \frac{1}{2}J & - & 2M & = & 0 \end{array}$$

Who cares?

These are the sort of examples that you are typically presented with in school. If these were really the only systems of linear equations that need to be solved in practice you may justifiably wonder why we spend so much time on studying this sort of problem. Well, the linear systems that pop up in science and engineering tend to be much more complex, involving thousands of equations and variables and it does not make sense to study anything like this as part of an introduction to the topic. Also, if you know how to deal with simple systems like the above you do know, at a least in theory, how to tackle systems of real importance.

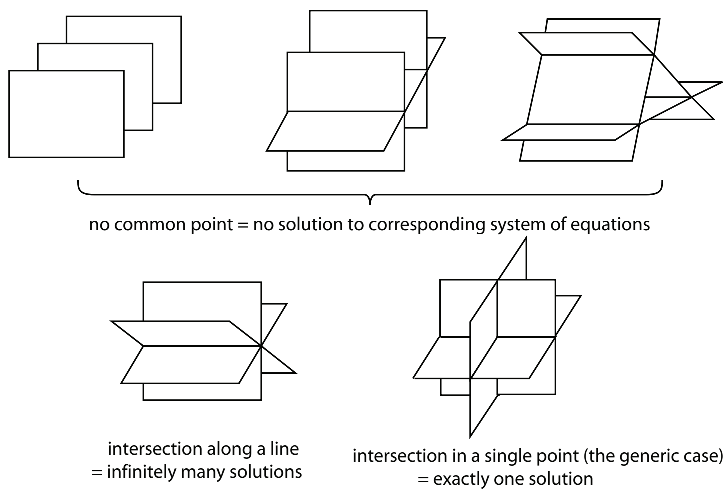
4.1 Intersections of planes

One of the best ways to get a feel for linear systems is to interpret them geometrically. For example, the individual equations in the system of linear equations

$$\begin{array}{rcrcrcr} 3x & + & 7y & - & 2z & = & 0 \\ 6x & + & 16y & - & 3z & = & -1 \\ 3x & + & 9y & + & 3z & = & 3 \end{array}$$

correspond to three planes in space. Depending on the relative position of these three planes, this system of equations will have no, infinitely many, or exactly one solution.

The following diagram lists the essentially different possible scenarios:



We have to add a little bit of a disclaimer here: The pictures show everything that can happen if the three equations describe **different** planes. But, as we've pointed out previously, two or even all three equations may describe the same plane. In the first case, we are really only dealing with the intersection of two planes and therefore have either no or infinitely many solutions. In the second case **all** points of the plane correspond to solutions, so, there are infinitely many solutions.

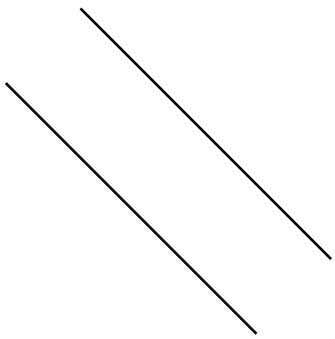
Important: It is very easy to figure out whether a system of linear equations corresponds to three different, two different or just one plane because two equations describe the same plane if and only if they differ by a factor. For example, $x+y+2z=3$ and $3x+3y+6z=9$ describe the same plane because the second equation is the first equation "times three".

What about systems of linear equations that involve only two variables such as the system of linear equation corresponding to our puzzle,

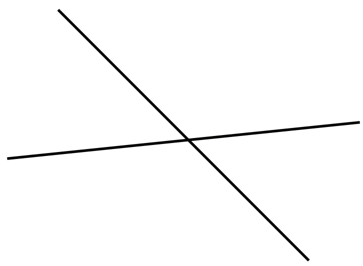
$$\begin{array}{rcrcr} x & + & y & = & 75 \\ \frac{1}{2}x & - & 2y & = & 0 \end{array}$$

In this case the two equations involved correspond to two lines in the plane. These two lines can be

1. identical if the two equations describe the same line (=infinitely many solutions);
2. intersect in exactly one point (= exactly one solution);
3. parallel but different (= no solutions).



parallel lines = no solution



intersection in a single point
= exactly one solution

Anyway, these are the pictures to keep in mind when dealing with systems of linear equations in three or two unknowns/variable.

Just to get you used to this sort of thing, here is a formal definition of linear systems.

Definition 4.1.1 (Linear systems) *A system of linear equations is a list of equations,*

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 \ &+ \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 \ &+ \cdots + a_{2n}x_n = b_2 \\
&\cdots \\
a_{m1}x_1 + a_{m2}x_2 &+ \cdots + a_{mn}x_n = b_m
\end{aligned}$$

where a_{ij} and b_i are real numbers. The above is a system of m equations in the n variables $x_1,\ x_2, \ldots,\ x_n$. In terms of summation notation, the above can also be written very concisely in the form

$$\sum_{j=1}^{n} a_{ij}x_j = b_i,\ \ i = 1, 2, 3,\ \ \cdots,\ \ m$$

Linear systems that have at least one solution are called **consistent** *and those that don't have a solution are called* **inconsistent***.*

Note that the number of equations and variables does not have to be the same. For example, just one equation also qualifies as a linear system and it is also possible to have more equations than variables.

The special case of linear systems with three variables that we considered above suggests (correctly) what you can expect in terms of the number of solutions of a linear system.

In general, you either get exactly one, none or infinitely many different solutions. Exactly one solution is only possible if there are at least as many equations as there are variables. For a generic linear system with the same number of equations and variables you can expect a unique solution. Again, for a generic linear system, you can expect infinitely many solutions if there are fewer equations than variables and no solutions if there are more equations than variables. Having said this, none and infinitely many solutions do occur for certain (mostly cooked up) linear systems, no matter whether the number of equations exceeds, is equal, or is less than the number of variables.

We'll give a proof for everything we just said once we've discussed how linear systems are solved.

4.2 Solving by Gaussian elimination

We'll demonstrate the general strategy for solving linear systems using the following fairly typical example,

$$\begin{aligned}
x + 3y + \ 6z &= 25 \\
2x + 7y + 14z &= 58 \\
2y + \ 5z &= 19
\end{aligned}$$

We start by replacing the second equation by (-2) times the first equation added to the second. This yields the system

$$\begin{aligned} x+3y+6z &= 25 \\ y+2z &= 8 \\ 2y+5z &= 19 \end{aligned}$$

We then replace the third equation with (-2) times the second added to the third. This yields the system

$$\begin{aligned} x+3y+6z &= 25 \\ y+2z &= 8 \\ z &= 3 \end{aligned}$$

At this point it is important to point out that this last system has **exactly the same solutions** as the system we started with. This is not obvious and we'll have a closer look on the next page to convince ourselves of this fact. In any case, the last system which is now in **upper triangular form**[1] is much easier to solve than the one we started with. In fact, at this point we already know that $z = 3$. Then substituting this in the second equation, it follows $y + 6 = 8$ and therefore $y = 2$. Finally, substituting the values for y and z in the top equation yields $x + 6 + 18 = 25$ and so $x = 1$. The process to go from the system in upper triangular form to the final solution is called **back substitution**.

Instead of switching to back substitution we could also have continued as follows. In the system in upper triangular form add (-2) times the bottom equation to the middle and then add (-6) times the bottom to the top. This gives

$$\begin{aligned} x+3y &= 7 \\ y &= 2 \\ z &= 3 \end{aligned}$$

Now add (-3) times the second equation to the top one. This yields

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

a system which, just like all the other systems we've come across, has the same solution as the original system.

[1] Can you see the triangle?

4.3 Elementary operations

Our original solution of this linear system involved just one type of operation:

1. Replace a first equation with a multiple of a second equation added to the first equation.

Sometimes it is also convenient/necessary to

2. Replace an equation by a non-zero multiple of this equation.
3. Swap two equations.

In fact, these three operations are all that is needed to solve any linear system. They are usually referred to as **elementary operations**.

We already mentioned that applying an elementary operation to a linear system yields a linear system that has exactly the same solutions as the original system. Why is that so?

Well, you might ask: Why shouldn't this be the case? Let's consider the following super-simple equation:

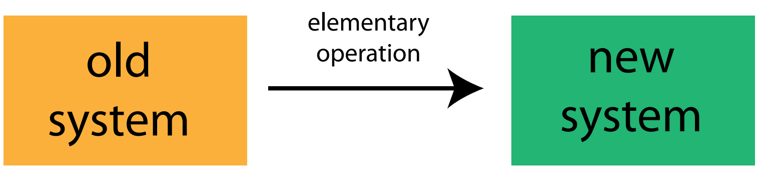
$$2x + 1 = 3.$$

Obviously it has the solution $x = 1$. Now, let's square both sides. This gives the equation

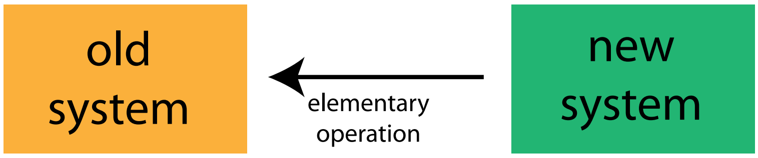
$$(2x + 1)^2 = 3^2$$

which ends up having the solutions $x = 1$ and $x = -2$. That is, the original solution is still a solution of the new equation. However, the new equation also has a solution that is not a solution of the original equation. This means that the operation of squaring may change the number of solutions when applied to an equation. On the other hand, we are claiming that we don't have to worry about this sort of problem manifesting itself when it comes to applying elementary operations to linear systems.

Okay, here we go. It is easy to see that, just as with our squaring example, we do not lose any of our solutions when we apply an elementary operation. That is, all the solutions of our original system are also solutions of a system that results after applying one of our elementary operations. Here is the picture to keep in mind.



What we have to worry about is that we don't get any new solutions popping up in the process. The key to showing that the new system won't have any extra solutions is to observe that all elementary operations are reversible.



In particular, if the elementary operation under consideration is swapping two rows, then the same swap applied a second time will turn the new system back into the old system.

If the elementary operation is multiplying a row by a non-zero constant c, then multiplying the same row in the new system by $1/c$ will take us back to the old system.

Finally, let's say the elementary operation is of the complicated type 1. In order not to lose ourselves in generalities, let's consider a specific example. Let's say the elementary operation consists in adding 6 times the first row to the second row. Then this operation can be reversed by adding (-6) times the first row of the new system to its second row.

Okay, so what does all this tell us? Well, let's assume for the moment that the new system has an extra solution. Then since we can get back to the old system by applying an elementary operation to the new system and we don't lose solutions in this way, we conclude that this extra solution was already a solution of the original system. This is obviously impossible (otherwise it wouldn't be "extra"). We conclude that both the old and the new system have exactly the same solutions.

4.4 Augmented matrix

All the important information about our linear system

$$\begin{aligned} x+3y+\ 6z &=25 \\ 2x+7y+14z &=58 \\ 2y+\ 5z &=19 \end{aligned}$$

can be summarized concisely in the following **augmented matrix**.

$$\left(\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right)$$

As you can see, the rows correspond to the equations in the system, the first three columns to the coefficients in front of the three variables and the last column to the right sides of the equations.

Now it also makes sense to solve our system by simply manipulating its augmented matrix, thus cutting down on carrying around a lot of superfluous information (all those xs, the '$=$' signs, etc.). Let's give this a go.

At the level of the augmented matrix our elementary operations are

1. Replace a first row with a multiple of a second row added to the first row.
2. Replace a row of the matrix by a non-zero multiple of this row.
3. Swap two rows.

Thus the first step in solving the system would be to take (-2) times the first row of the augmented matrix above and add it to the second row,

$$\left(\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 2 & 5 & 19 \end{array}\right)$$

In the second step, we take (-2) times the second row and add to the third,

$$\left(\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

This augmented matrix corresponds to the system

$$\begin{aligned} x+3y+6z &= 25 \\ y+2z &= 8 \\ z &= 3 \end{aligned}$$

which is the same as the simple system that we solved by back substitution.

4.5 Gaussian elimination - for real

Now we are ready to describe **Gaussian elimination** in full generality. Gaussian elimination is an algorithm that takes augmented matrices as input and transforms them using elementary operations into simpler augmented matrices from which the solutions of the underlying linear system can be easily read off.

We now describe what the simple matrices that Gaussian elimination produces look like.

Definition 4.5.1 (Echelon form) *The* **leading coefficient** *of a non-zero row of an augmented matrix is the first non-zero entry in this row from the left. An augmented matrix is in* **echelon form** *if*

1. All nonzero rows are above any rows of zeros.

2. The leading entry of a row is in a column to the right of the leading entries of any rows above it.

The following two examples show typical examples of augmented matrices in echelon form. As you can see, a matrix in echelon form is divided by a staircase descending from left to right as the ones highlighted in our examples. All the entries below such a staircase are zeros, steps are always just one row high and the steps themselves occur just before the first non-zero entries in a row.

$$\left(\begin{array}{ccc|c} 3 & 3 & 6 & 25 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 2 & 3 & 7 & 5 & 8 & 0 \\ 0 & 0 & 6 & 2 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The systems of equations corresponding to augmented matrices in row echelon form can be solved easily using back substitution. However, it is possible to take the process of Gaussian elimination even further. The end result is a matrix in reduced row echelon form.

Definition 4.5.2 (Reduced row echelon form) *An augmented matrix is in* **reduced row echelon form** *if it is in echelon form and, in addition,*

1. the leading entry in any non-zero row is a 1; and

2. every leading entry in a non-zero row is the only non-zero entry in its column.

The following two examples show typical instances of augmented matrices in reduced row echelon form.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 45 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right) \quad \left(\begin{array}{ccccc|c} 1 & 3 & 0 & 5 & 8 & 0 \\ 0 & 0 & 1 & 2 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Example 4.5.1 *Here are some augmented matrices which are not in echelon form.*

$$\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{cc|r} 1 & 2 & 3 \\ 2 & 4 & -6 \\ 4 & 0 & 7 \end{array}\right), \left(\begin{array}{ccc|c} 0 & 2 & 3 & 3 \\ 1 & 5 & 0 & 2 \\ 7 & 5 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

The following is a description of the so-called **Gauss-Jordan algorithm** for turning the augmented matrix into reduced row echelon form. To describe the algorithm we'll again use a fairly general example in order to not get lost in notation. Okay, so here is a matrix. The stars are supposed to stand for numbers which to start with are of no interest to us.

$$\left(\begin{array}{ccccc|c} 5 & * & * & * & * & * \\ 2 & * & * & * & * & * \\ 3 & * & * & * & * & * \\ 6 & * & * & * & * & * \end{array}\right)$$

We start the algorithm by focussing on the first column. The top entry in this column is non-zero. (If it is equal to zero, swap the first row with a row that features a non-zero entry in this column).

1. Divide the first row by 5 to make its leading coefficient equal to 1.

$$\left(\begin{array}{ccccc|c} 1 & * & * & * & * & * \\ 2 & * & * & * & * & * \\ 3 & * & * & * & * & * \\ 6 & * & * & * & * & * \end{array}\right)$$

2. Now we use this 1 to wipe out all the non-zero entries below it. To get rid of the 2 in the second row add -2 times the first row to the second row. Then add -3 times the first row to the third row to get rid of the 3 in the third row and, finally, add -6 times the first row to the last row to wipe out the 6 there. At this stage our augmented matrix looks like this:

$$\left(\begin{array}{ccccc|c} 1 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{array}\right)$$

3. Now focus on the submatrix contained in the green box.

$$\left(\begin{array}{ccccc|c} 1 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \\ 0 & * & * & * & * & * \end{array}\right)$$

Let's first assume that one entry in the left column of this submatrix is non-zero. Then you can arrange for the top left corner to be non-zero by swapping rows if necessary. So, maybe the matrix looks like this.

$$\left(\begin{array}{ccccc|c} 1 & * & * & * & * & * \\ 0 & 4 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 2 & * & * & * & * \end{array}\right)$$

Then we make the 4 into a 1 by dividing the second row by 4 and then we use this 1 to wipe out all the non-zero coefficients BELOW it as before and ABOVE it.

$$\left(\begin{array}{ccccc|c} 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{array}\right)$$

Now shrink the green box and consider the submatrix contained in it.

$$\left(\begin{array}{ccccc|c} 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \end{array}\right)$$

If the first column of this submatrix contains non-zero entries deal with this column as we just did. Otherwise, if the first column only has zero entries skip this column and consider the following smaller box.

$$\left(\begin{array}{ccccc|c} 1 & 0 & * & * & * & * \\ 0 & 1 & * & * & * & * \\ 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * & * \end{array}\right)$$

From now on everything repeats until you are left with an augmented matrix in reduced row echelon form. Just to finish this example, maybe the new left column in the green box has non-zero elements, then we get

$$\left(\begin{array}{ccccc|c} 1 & 0 & * & 0 & * & * \\ 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & * & * \end{array}\right)$$

Finally, let's say the left entry of the small green box is also non-zero. Then we finish up with the following matrix.

$$\left(\begin{array}{ccccc|c} 1 & 0 & * & 0 & 0 & * \\ 0 & 1 & * & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \end{array}\right)$$

This algorithm applies to both augmented matrices and non-augmented matrices. There is nothing special about the augmented column on the right with respect to the algorithm.

Example 4.5.2 *Let's apply our algorithm to the following matrix.*

$$\begin{pmatrix} 0 & 0 & 2 & 3 & 2 \\ 0 & 1 & 1 & 4 & 3 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

This gives.

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

There are infinitely many ways to use elementary operations to turn a matrix into a matrix in reduced row echelon form. However, all these different ways yield one and the same reduced row echelon form. In other words.

Theorem 4.5.1 (THE reduced row echelon form) *The reduced row echelon form of a matrix is uniquely determined.*

When you think about this for a moment, you may come to the (correct) conclusion that this is quite amazing. We only mention this pretty theorem without giving a proof. Well, if there is time you are going to get the proof anyway.

Before we move on let's have another look at our last detailed example. We ended up with the following reduced row echelon form.

$$\left(\begin{array}{ccccc|c} 1 & 0 & * & 0 & 0 & * \\ 0 & 1 & * & 0 & 0 & * \\ 0 & 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 & 1 & * \end{array}\right)$$

The circled 1s, which are just the leading non-zero entries in the rows that they are contained in, turn out to be important. Let's give them a name.

Definition 4.5.3 (Pivots and pivot columns) *The distinguished leading 1s in a reduced row echelon form (circled in red in our example) are called its* **pivots***. The columns these 1s are contained in are called the* **pivot columns** *of the reduced row echelon form and the corresponding columns in the original matrix are called the pivot columns of the original matrix.*

For example, here is an augmented matrix and its reduced row echelon form. We've highlighted the pivots in the reduced row echelon form and the pivot columns in both matrices.

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

We'll refer to the pivots in the next section and then use them in a fairly major way in the chapter on subspaces. But let's move on for now.

These days nobody in their right mind would turn a matrix into reduced row echelon form by hand. Computers can do this much faster and more reliably.[2]

[2] It is important to note that if your aim is to solve a linear system by hand you will usually use Gaussian elimination to turn a linear system or its augmented matrix into upper triangular form/echelon form and then complete the solution using back substitution. Also, if the original system has integer coefficients you will try to avoid division for as long as possible. And even a dedicated computer program will solve a monstrous linear system using a slightly different sequence of elementary operations than the one produced by our algorithm (which was mainly chosen to get us to the reduced echelon form in conceptually as slick a way as possible). Considerations like numerical stability of the different paths to a solution will play a major role in the design of practical algorithms.

For example, in **Mathematica** or **Wolfram Alpha** the command

```
RowReduce[{{ 1, 2, -1, 1, 3}, {1, 1, -1, 1, 1}, {1, 3, -1, 1, 5}}]
```

will generate the reduced row echelon form of the matrix

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{array}\right).$$

Still, it is important to know how this algorithm works, to recognize whether or not a matrix is in reduced row echelon form, and most importantly, and we'll learn this in the next section, how to read the solutions to a linear system off its reduced row echelon form. In a little while we'll also use the reduced row echelon form to find the inverse of a matrix.

4.6 Reading the solutions of linear systems off the reduced row echelon form

Inconsistent linear system: no solutions.

Let's find all solutions to the system of equations,

$$\begin{aligned} 2x + 4y \ -3z &= -1 \\ 5x + 10y - 7z &= -2 \\ 3x + 6y \ +5z &= \ \ 9 \end{aligned}$$

The augmented matrix for this system is

$$\left(\begin{array}{ccc|c} 2 & 4 & -3 & -1 \\ 5 & 10 & -7 & -2 \\ 3 & 6 & 5 & 9 \end{array}\right).$$

This matrix has the following reduced row echelon form

$$\left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Now, remember that the original linear system has exactly the same solutions as the linear system that corresponds to the reduced row echelon form

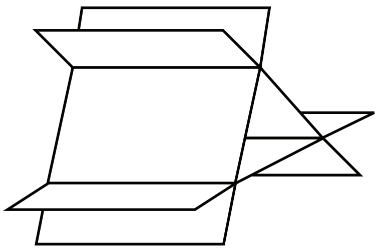
$$\begin{aligned} 1x + 2y + 0z &= 0 \\ 0x + 0y \ +z &= 0 \\ 0x + 0y + 0z &= 1 \end{aligned}$$

However, it is clear that this last system does not have any solution since the last of its equations

$$0x + 0y + 0z = 1$$

does not have a solution. This means that the original equation is **inconsistent**, that is, does not have any solution. In general, by looking at the last non-zero row in the reduced row echelon form of a matrix you can tell at a glance whether or not the system is consistent or not: If all entries of the row to the left of the vertical bar are equal to zero and the entry on the right is non-zero, then the system is inconsistent, otherwise the system is **consistent**.

By the way, a closer look at this system of equations reveals that the three equations correspond to three planes any two of which intersect in a line and all three lines are parallel and don't have a point in common.



Consistent systems: exactly one solution

Let's have another look at the linear system

$$\begin{aligned} x + 3y + 6z &= 25 \\ 2x + 7y + 14z &= 58 \\ 2y + 5z &= 19 \end{aligned}$$

Its augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \\ 0 & 2 & 5 & 19 \end{array}\right)$$

row reduces to

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

This augmented matrix corresponds to the system

$$\begin{aligned} x &= 1 \\ y &= 2 \\ z &= 3 \end{aligned}$$

which means we have a unique solution. In general, a unique solution corresponds to a reduced row echelon form that looks like the following matrix,

$$\left(\begin{array}{ccc|c} \boxed{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}} & & & \begin{array}{c} * \\ * \\ * \end{array} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

That is, after removing the 0 rows at the bottom, the left side of the resulting matrix (the part to the left of the vertical stroke) is a square matrix full of zeros, except for the 1s on the diagonal from the upper left to the lower right.

Consistent system: infinitely many solutions

Let's remove one equation from the linear system in the last section. This leaves us with the following system of two equations in three unknowns.

$$\begin{aligned} x+3y+\ 6z &= 25 \\ 2x+7y+14z &= 58 \end{aligned}$$

Its augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 3 & 6 & 25 \\ 2 & 7 & 14 & 58 \end{array}\right)$$

row reduces to

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 8 \end{array}\right).$$

This corresponds to the equations $x = 1$ and $y + 2z = 8$. This means that if we let $z = t$ be an arbitrary number, then $(1, 8 - 2t, t)$ is a solution of our linear system. We can also write this solution as

$$(1, 8, 0) + t(0, -2, 1).$$

This should look familiar to you as it is just the vector equation of a line, the line in which the two planes corresponding to the equations in our original system intersect. The set of solutions is

$$\{(1, 8, 0) + t(0, -2, 1) \big| t \in \mathbf{R}\}.$$

Here is a more complex example, which illustrates what needs to be done in general.

$$\begin{aligned} x + 2y - z + w &= 3 \\ x + y\ \ - z + w &= 1 \\ x + 3y - z + w &= 5 \end{aligned}$$

The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 3 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & 3 & -1 & 1 & 5 \end{array}\right).$$

This row reduces to

$$\begin{array}{c} \begin{array}{cccc} \color{red}{x} & \color{red}{y} & \color{green}{z} & \color{green}{w} \end{array} \\ \left(\begin{array}{cccc|c} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

Note that we've circled the pivots and have written the variables that correspond to the pivot columns of the matrix in red. The remaining variables have been written in green. They are called the **free variables of the system** as substituting any numbers for these variables corresponds to a solution of the system. To illustrate, let $z = s$ and $w = t$. Then the first row of the reduced row echelon form tells us that

$$x = -1 + s - t.$$

The second row tells us that

$$y = 2.$$

Then, since $z = s$ and $w = t$, the set of solutions to our system is

$$\{(-1 + s - t, 2, s, t) | s, t \in \mathbf{R}\}.$$

To summarize, here is what you have to do when it comes to finding the solutions of a linear system. To start with, just calculate the reduced row echelon form and determine whether there is a solution. If there is, see if there are free variables. If this is the case, there will be infinitely many solutions. Find them by assigning different parameters to the free variables and obtain the solutions. If there are no free variables, then there will be a unique solution. Anyway, once you've done this a couple of times, finding solutions is not a problem (at least not for smallish systems).

4.7 A useful trick

Say you need to solve a couple of linear systems that have the same "left sides" like the following three linear systems

$$\begin{aligned} x + 3y + 6z &= 25 \\ 2x + 7y + 14z &= 58 \\ 2y + 5z &= 19 \end{aligned}$$

$$\begin{aligned} x+3y+\ 6z &= 3 \\ 2x+7y+14z &= 8 \\ 2y+\ 5z &= 9 \end{aligned}$$

$$\begin{aligned} x+3y+\ 6z &= 7 \\ 2x+7y+14z &= 7 \\ 2y+\ 5z &= 7 \end{aligned}$$

Then it should be clear that when you row reduce the corresponding augmented matrices you'll be using the exact same elementary operations in the exact same order. This means that you can achieve the same by simply row reducing the following modified augmented matrix. Its left side corresponds to the common left side of the three systems and its right side has the right sides of the systems side by side.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 6 & 25 & 3 & 7 \\ 2 & 7 & 14 & 58 & 8 & 7 \\ 0 & 2 & 5 & 19 & 9 & 7 \end{array}\right)$$

The reduced row echelon form of this matrix is

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -3 & 28 \\ 0 & 1 & 0 & 2 & -8 & -49 \\ 0 & 0 & 1 & 3 & 5 & 21 \end{array}\right).$$

And you can now read off the solutions to the different systems on the right side. So, $x = 1$, $y = 2$ and $z = 3$ is a solution to the first system, $x = -3$, $y = -8$ and $z = 5$ is a solution to the second system and $x = 28$, $y = -49$ and $z = 21$ is a solution to the third system. Neat! We'll use this trick in the next chapter to calculate the inverse of a matrix.

Chapter 5

Matrices

An $\mathbf{m \times n}$ **matrix** is a rectangular array of numbers with m rows and n columns. For example,

$$\begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

is a 3×3 matrix. A matrix like this, with the same number of rows and columns, is called a **square** matrix. Square matrices are special in many ways and we'll spend a lot of time "torturing" them.

Vectors like $(1, 2, 3)$ are 1×3 matrices. On the other hand,

$$\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

is a 3×1 matrix. Matrices having just one row are also sometimes called **row vectors** and matrices with just one column **column vectors**. We'll often talk about **row and column vectors of a matrix**. By this we mean the rows and columns of a matrix considered as row and column vectors. For example, the above 3×3 matrix has three row vectors, one of which is $(3, 2, -1)$.

The **dimensions** of an $m \times n$ matrix are the numbers m and n.

5.1 Addition and scalar multiplication

It makes sense to consider matrices as straightforward generalizations of vectors and just like in the case of a vector, you multiply a matrix by a number/scalar by simply multiplying all its entries by this number and you add two matrices of the same dimensions by adding corresponding entries to form a matrix of the same dimension:

$$2\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 2 & -1 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

The counterpart of zero vectors in the case of matrices are the **zero matrices**. These are the matrices all of whose entries are zeros.

Just like vectors, matrices behave very much like numbers when you add them or when you multiply them by numbers.

Proposition 5.1.1 (Matrix addition and multiplication by scalars) *Let A, B, C be matrices of the same dimensions, let O be the zero matrix of the same dimension and let $\alpha,\ \beta \in \mathbf{R}$. Then the following hold:*

Addition of matrices is commutative,

$$A + B = B + A.$$

Addition of matrices is associative,

$$(A + B) + C = A + (B + C).$$

The zero matrix O is an additive identity,

$$A + O = A.$$

Every matrix A has an additive inverse $-A$.

$$A + (-A) = O.$$

Also

$$\alpha(A + B) = \alpha A + \alpha B,$$
$$(\alpha + \beta)A = \alpha A + \beta A,$$
$$\alpha(\beta A) = (\alpha\beta)A,$$
$$1A = A.$$

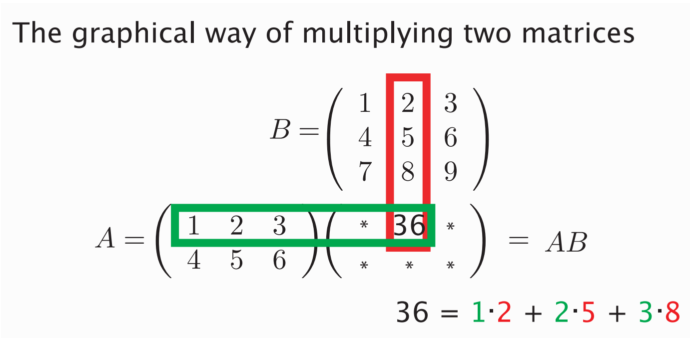
Just like in the case of vectors, the main message to take away from this long list of properties is that addition of matrices and multiplication by numbers follows the same rules as addition and multiplication of numbers.

5.2 Matrix multiplication

Even the dot product of two vectors generalizes to matrices. However, this generalization is a bit more tricky than the generalization of addition and scalar multiplication. If A and B are matrices, then the product AB of A and B is defined **only if** A has the same number of columns as B has rows. Furthermore, AB is a matrix that has the same number of rows as A and the same number of columns as B. For example, the two matrices

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

can be multiplied because A has three columns and B has three rows. The following diagram illustrates how AB is calculated in practice.



Arrange A and B as shown. Then AB automatically fits into the rectangular array on the bottom right that is boxed in by the two matrices. Now the individual entries of AB are calculated as shown for one of the entries. In this way we calculate

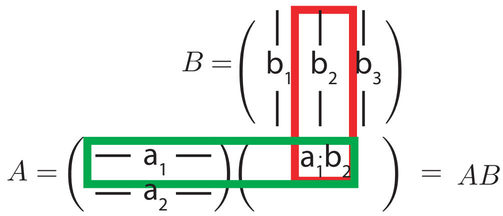
$$AB = \begin{pmatrix} 30 & 36 & 42 \\ 66 & 81 & 96 \end{pmatrix}.$$

Have a closer look at the expression

$$36 = 1 \cdot 2 + 2 \cdot 5 + 3 \cdot 8.$$

This is just the dot product of the vector $(1, 2, 3)$ and the vector $(2, 5, 8)$. The first vector is one of the row vectors of A and the second is one of the column vectors of B. This means that every entry in AB is the dot product of one of the rows of A with one of the columns of B. In this way, the product for matrices is a generalization of the dot product for vectors.

We can modify our diagram above to summarize matrix multiplication in terms of dot products of the row vectors $\mathbf{a}_1, \mathbf{a}_2$, ... of A and the column vectors $\mathbf{b}_1, \mathbf{b}_2$, ... of B.



Before we move on, practice multiplying matrices a bit, by checking that you can do the following products.

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 20 \\ 4 & 30 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 7 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 30 & 10 \\ 8 & 2 \end{pmatrix}$$

When we multiply numbers (or vectors) A and B, we always have $AB = BA$. For example, $5 \cdot 4 = 4 \cdot 5 = 20$. As the above example shows, this is not necessarily true when we are dealing with matrices A and B.

Here is an even more extreme example of $AB \neq BA$.

$$(1, 2, 3) \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = (32).$$

$$\begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} (1, 2, 3) = \begin{pmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{pmatrix}.$$

In fact, a moment's thought will convince you that if $AB = BA$, then both A and B are square matrices.

And here is an example of AB being not defined and BA being defined. The product

$$\begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 7 \\ 0 & 2 \\ 4 & 1 \end{pmatrix}$$

is not defined since the width of the first matrix is not equal to the height of the second matrix. On the other hand,

$$\begin{pmatrix} 1 & 7 \\ 0 & 2 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 30 & 10 \\ 8 & 2 \\ 12 & 13 \end{pmatrix}.$$

Two useful ways to interpret a matrix product

Here are two more useful ways of thinking about the matrix product which follow immediately from the definition.

$$B = \begin{pmatrix} | & | & | \\ \mathsf{b}_1 & \mathsf{b}_2 & \mathsf{b}_3 \\ | & | & | \end{pmatrix}$$

$$A = \begin{pmatrix} & & \end{pmatrix} \begin{pmatrix} | & | & | \\ A\mathsf{b}_1 & A\mathsf{b}_2 & A\mathsf{b}_3 \\ | & | & | \end{pmatrix} = AB$$

What this says is that the first, second, third, etc., column vector of the product AB is equal to A times the first, second, third, etc., respectively, column vector of B.

Similarly,

$$B = \begin{pmatrix} & & \end{pmatrix}$$

$$A = \begin{pmatrix} — \mathsf{a}_1 — \\ — \mathsf{a}_2 — \end{pmatrix} \begin{pmatrix} —\mathsf{a}_1 B— \\ —\mathsf{a}_2 B— \end{pmatrix} = AB$$

says that the first, second, third, etc., row vector of the product AB is equal to the first, second, third, etc., respectively, row vector of A times B.

5.3 Special matrices

Transpose and symmetric matrices

The **transpose** A^T of A is the matrix that we get when we flip the rows and columns of A as in the following examples.

$$\begin{pmatrix} 1 & 2 & 7 \\ 0 & 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 0 \\ 2 & 3 \\ 7 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

A matrix A is called a **symmetric matrix** if $A = A^T$.

Here is an example of a symmetric matrix.

$$\begin{pmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{pmatrix}$$

Theorem 5.3.1 (Transpose of a product) *If A and B are matrices such that AB exists, then*

$$((A)^T)^T = A$$

and

$$(AB)^T = B^T A^T.$$

Proof. The identity

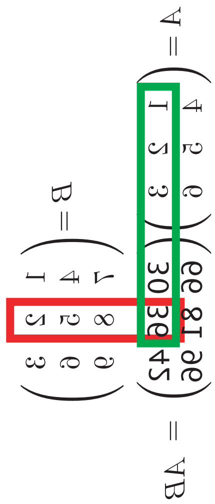
$$((A)^T)^T = A$$

is an immediate consequence of the definition of the transpose of a matrix.

To prove the second identity consider again our graphical way of multiplying two matrices.

$$\begin{array}{ccc} & B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} & \\ A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} & \begin{pmatrix} 30 & 36 & 42 \\ 66 & 81 & 96 \end{pmatrix} & = AB \end{array}$$

Now flip the whole diagram like you would flip a matrix to turn it into its transpose.



Then if you ignore the weird orientation of the numbers and brackets, what we see here is a perfectly fine matrix product. At the same time it is clear that the three matrices involved in this product are the transposes of the original matrices, that is, B^T, A^T and $(AB)^T$.

$$\begin{array}{cc} & A^T \\ & \| \\ & \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \\ B^T & \\ \| & \\ \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} & \begin{pmatrix} 30 & 66 \\ 36 & 81 \\ 42 & 96 \end{pmatrix} \\ & \| \\ & (AB)^T \end{array}$$

But now what does the diagram tell you as to how these three matrices are connected? Well,

$$B^T A^T = (AB)^T.$$

Neat! ■

Identity matrix

The following are the 1×1, 2×2, 3×3 and 4×4 **identity matrices**.

$$(1), \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In general, the $n \times n$ identity matrix is the $n \times n$ matrix that continues this pattern; so 1s down the **main diagonal** and 0s everywhere else.

Proposition 5.3.1 (Identity matrix) *Let A be an $m \times n$ matrix and let I be the $n \times n$ identity matrix. Then*

$$AI = A.$$

Let B be an $m \times n$ matrix and let I be the $m \times m$ identity matrix. Then

$$IB = B.$$

Of course, if C is a square matrix and I is the identity matrix with the same dimensions, then

$$CI = IC = C.$$

Proof. Let's prove $AI = A$. Remember the following diagram that tells you what the columns of a product AB are in terms of the column vectors of B (see page 73):

$$\begin{array}{rcl} & & B = \begin{pmatrix} | & | & | \\ \mathsf{b}_1 & \mathsf{b}_2 & \mathsf{b}_3 \\ | & | & | \end{pmatrix} \\ A = \begin{pmatrix} & & \end{pmatrix} & & \begin{pmatrix} | & | & | \\ A\mathsf{b}_1 & A\mathsf{b}_2 & A\mathsf{b}_3 \\ | & | & | \end{pmatrix} = AB \end{array}$$

So, the first column vector of the product AI that we are interested in is just

$$A \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}.$$

However, it is easy to see that this is equal to the first column vector of A. For example, if

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

then

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}$$

In general, we conclude that the first column of AI is equal to the first column of A. Arguing the same way, we find that all the columns of AI are the same as those of A and that therefore the two matrices have to be the same.

To show that

$$IB = B,$$

note that

$$IB = ((IB)^T)^T = (B^T I^T)^T = (B^T I)^T = (B^T)^T = B.$$

In this string of equalities we use the two basic properties of the transpose that we proved earlier in this chapter, the fact that the transpose of a symmetric matrix like I is the matrix itself, and finally, the first part of this theorem. ■

Some more important properties of the matrix product

We've already seen that even if both AB and BA exist and have the same dimension it is generally not true that $AB = BA$. This means that in general matrix multiplication is not commutative. However, apart from this lack of commutativity the matrix product does behave very similar to the product of numbers.

Proposition 5.3.2 (Properties of the matrix product) *If all multiplications and additions make sense, the following hold for matrices A, B, C.*

Matrix multiplication is distributive,

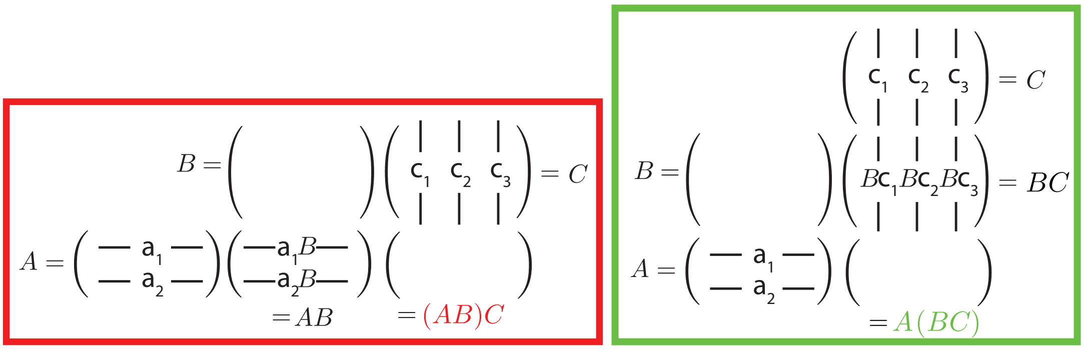
$$A(B + C) = AB + AC,$$
$$(A + B)C = AC + BC.$$

Matrix multiplication is associative,

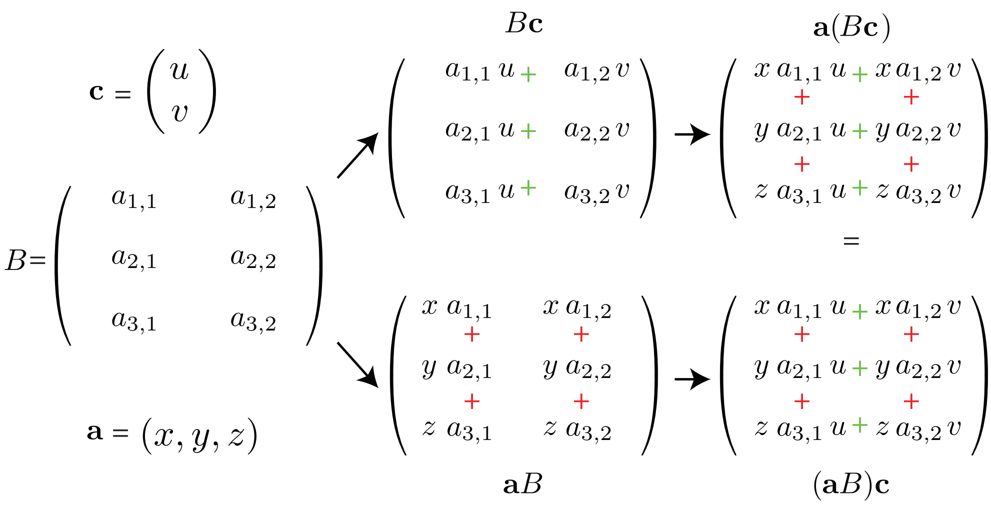
$$A(BC) = (AB)C.$$

We'll only give a sketch of a proof for associativity. See whether you can come up with a similar proof for distributivity. For some formal proofs, check our Kuttler's book.

Sketch of a proof. The following two diagrams amount to reducing the proof of $A(BC) = (AB)C$ for matrices A, B, C to the proof that $\mathbf{a}(B\mathbf{c}) = (\mathbf{a}B)\mathbf{c}$, where $\mathbf{a}$ is a row vector and $\mathbf{c}$ is a column vector. Just note that if $\mathbf{a}_i$ is the ith row of A and $\mathbf{c}_j$ is the jth column of C then the $(i, j)th$ entries of $(AB)C$ and $A(BC)$ can be seen to be equal to $(\mathbf{a}_i B)\mathbf{c}_j$ and $\mathbf{a}_i(B\mathbf{c}_j)$, respectively.



In order to see at a glance that $\mathbf{a}(B\mathbf{c}) = (\mathbf{a}B)\mathbf{c}$ ponder the following diagrams for a minute.



Chapter 6

The inverse of a square matrix

As a first application of matrices, and a nice introduction to this chapter, we use them to express linear systems in a very concise manner:

$$\begin{array}{rcrcrcl} 3x & + & 2y & - & z & = & 3 \\ x & - & y & + & z & = & 1 \\ 2x & + & y & - & z & = & 0 \end{array}$$

Letting

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix},$$

we can express this linear system in **matrix form**,

$$A\mathbf{x} = \mathbf{b}.$$

Let's check this:

$$A\mathbf{x} = \begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3x + 2y - 1z \\ 1x - 1y + 1z \\ 2x + 1y - 1z \end{pmatrix} = \begin{pmatrix} 3x + 2y - z \\ x - y + z \\ 2x + y - z \end{pmatrix}$$

Works!

If A and $\mathbf{b}$ were numbers and $\mathbf{x}$ an unknown we could solve the equation $A\mathbf{x} = \mathbf{b}$ by simply dividing by A (if $A \neq 0$). Now wouldn't it be great if something similar was possible for matrices? It turns out that this is indeed possible if A is an **invertible matrix**.

Definition 6.0.1 (Inverse) *Given a square matrix A, suppose that B is a matrix such that $AB = BA = I$. Then we say that B is an* **inverse** *of A. A matrix that has an inverse is called* **invertible**.

Is it possible that a matrix A has two different inverses B and C, that is, $AB = BA = I$ and $AC = CA = I$? Well let's see.

$$B = BI = B(AC) = (BA)C = IC = C.$$

We conclude that if A has an inverse, then it is uniquely determined. Therefore it makes sense to speak about THE inverse of A. The inverse of A is called A^{-1}.

It is also true that if we find a **one-sided** inverse of a square matrix A, that is, a matrix B such that $AB = I$ OR $BA = I$, then A is invertible and B is the inverse of A. This is not obvious, but let's assume for the moment that this is true. Time permitting we'll supply a proof at a later stage.

Using the inverse we can solve our linear system

$$A\mathbf{x} = \mathbf{b}$$

like this

$$\mathbf{x} = I\mathbf{x} = (A^{-1}A)\mathbf{x} = A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}.$$

We'll return to this way of solving linear systems at the end of this chapter.

The only real number that does not have an inverse is 0. So it is tempting to suspect that all square matrices except the zero matrices are invertible. However, while it is true that no zero square matrix is invertible, the zero square matrices are not the only square matrices that are not invertible.

Example 6.0.1 *Let's show that the matrix*

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is not invertible. To see this, we multiply A by a general 2×2 matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}.$$

Clearly, the product on the right is never the 2×2 identity matrix. Hence A is not invertible.

Here is a neat way to figure out whether the inverse of a given matrix exists and, if it does, to calculate this inverse.

Theorem 6.0.1 (Existence of the inverse and finding the inverse) *Let A be an $n \times n$ matrix and let I be the identity matrix having the same dimensions. Find the reduced row echelon form of the augmented $n \times 2n$ matrix*

$$(A \mid I).$$

Let

$$(B \mid C)$$

be this reduced row echelon form. Then A is invertible if and only if B equal to the identity matrix I. If A is invertible then C is its inverse.

Okay, that sounds a bit complicated so before we give a proof here are some examples.

Let's consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix}.$$

The augmented matrix that the theorem talks about is

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right).$$

As you can see, we've got the matrix we wish to invert on the left and the identity matrix on the right of the vertical stroke. Now, the reduced row echelon form of this matrix pans out to be

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array}\right).$$

We see that now the identity matrix is on the left which means that our matrix is invertible. Furthermore, the inverse is the square matrix on the right

$$\begin{pmatrix} -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{pmatrix}.$$

Magic!

Here is a second example. Let's see whether the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

has an inverse.

The corresponding augmented matrix is

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right).$$

Its reduced row echelon form is

$$\left(\begin{array}{cc|cc} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Since the identity matrix does not appear on the left side we conclude that our original matrix does not have an inverse.

As a final example, let's revisit our first example of a non-invertible matrix,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

The corresponding augmented matrix is

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right).$$

Note that this augmented matrix coincides with its reduced row echelon form. So, in this case, using our theorem, we can see at a glance that our matrix is not invertible.

That still leaves us with the task of actually proving that all this works. So here is a

Sketch of a proof. Let's again start with the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix}.$$

Does A have an inverse? What we are looking for is a matrix

$$B = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$$

such that

$$\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let's interpret the matrix product in terms of columns as we have done before

$$B = \begin{pmatrix} | & | & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \\ | & | & | \end{pmatrix}$$

$$A = \begin{pmatrix} & & \end{pmatrix} \begin{pmatrix} | & | & | \\ A\mathbf{b}_1 & A\mathbf{b}_2 & A\mathbf{b}_3 \\ | & | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then we can see that our problem of finding the inverse splits up into three subproblems, one per column, namely to find the first, second and third column of our mystery matrix B such that

$$Ab_1 = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$Ab_2 = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$Ab_3 = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 2 \\ 3 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

However, as we've seen at the very beginning of this chapter solving these three problems amounts to solving three linear systems. In addition, at the level of their augmented matrices these three linear systems have the same left side and this is just the matrix A. On page 67 we learned that these three linear systems can be solved simultaneously using the augmented matrix that the theorem talks about.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 1 & -1 & 0 & 0 & 1 \end{array}\right).$$

So, we attempt to solve the linear system by finding the reduced row echelon form of the matrix.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{7} & \frac{2}{7} & \frac{2}{7} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{14} & \frac{5}{14} & -\frac{1}{7} \end{array}\right).$$

Then it is clear that the inverse exists and is equal to the right side of this matrix if the left side is the identity matrix. On the other hand, it is also clear that A does not have an inverse if the left side is not the identity matrix. ■

For the sake of completeness, here is a list of the most important properties of the inverse.

Theorem 6.0.2 (Properties of the inverse) *Let A and B be two invertible square matrices of the same dimensions. Then the following hold:*

$$\begin{aligned} (A^{-1})^{-1} &= A, \\ (AB)^{-1} &= B^{-1}A^{-1}, \\ (A^T)^{-1} &= (A^{-1})^T. \end{aligned}$$

I'll leave the easy proof as an exercise for you.

6.1 Solving linear systems using the inverse

Let's solve the linear system from the start of this chapter using using an inverse. So, again

$$\begin{array}{rcrcrcr} 3x & + & 2y & - & z & = & 3 \\ x & - & y & + & z & = & 1 \\ 2x & + & y & - & z & = & 0 \end{array}$$

Letting

$$A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix},$$

this system can be written as

$$A\mathbf{x} = \mathbf{b}.$$

The matrix A turns out to have an inverse

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 0 & 1 & 1 \\ 4 & -1 & -5 \\ 4 & 1 & -7 \end{pmatrix}$$

And so the system's unique solution is

$$\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{4}\begin{pmatrix} 1 \\ 11 \\ 13 \end{pmatrix}.$$

The following theorem is a consequence of our discussion.

Theorem 6.1.1 (Solving linear systems via the inverse) *Let A be an $n \times n$ matrix. Then A has an inverse if and only if a linear system of the form*

$$A\mathbf{x} = \mathbf{b}.$$

has a unique solution.

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Note that once you have calculated an inverse of a matrix, solutions of the equation $A\mathbf{x} = \mathbf{b}$ for different $\mathbf{b}$ can be found quickly.

Having said this, the inverse is not used in practice to solve linear systems as it takes a lot more calculations to solve a linear system using an inverse than to simply use Gaussian elimination (and it's got a few other disadvantages). However, the inverse does play a very important role in the theory of matrices.

Chapter 7

Determinants

Before we start let's agree that all matrices in this chapter will be square matrices.

7.1 Motivating determinants

The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

depend in a fairly complicated manner on the three coefficients a, b and c. Usually, it is very hard to determine by just looking at an equation like this how many solutions it has: 0, 1, or 2. However, the three coefficients combine into the very useful **discriminant**, $\Delta = b^2 - 4ac$, which is just the term under the square root sign in the quadratic formula. Looking at this single number it is possible to tell straightaway how many solutions our equation has: no solution if $\Delta < 0$, 1 solution if $\Delta = 0$ or 2 solutions if $\Delta > 0$.

Similarly, given an arbitrary square matrix A, it is usually impossible to tell just by looking at A how many solutions a system of linear equations $A\mathbf{x} = \mathbf{b}$ has. Luckily, there exists something like a discriminant for matrices. It is called the **determinant** of the matrix A and is denoted either by $\det(A)$ or $|A|$.

7.1.1 Determinants of 2×2 matrices

You are all already familiar with the determinant of a general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

It is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$= a \cdot d - b \cdot c$$

Also, you already know from school that a system of linear equations $A\mathbf{x} = \mathbf{b}$ in two unknowns has a unique solution if and only if this determinant is not equal to zero.[1] This will be true in general. To motivate the definition of determinants of general square matrices, let me first tell you what the 2×2 determinant means geometrically:

The two row vectors (a, b) and (c, d) of A span a parallelogram whose area is equal to the absolute value of the determinant.[2]

Let's check this. The area of the parallelogram in question is simply the length of the cross product of the vectors $(a, b, 0)$ and $(c, d, 0)$.[3] Then,

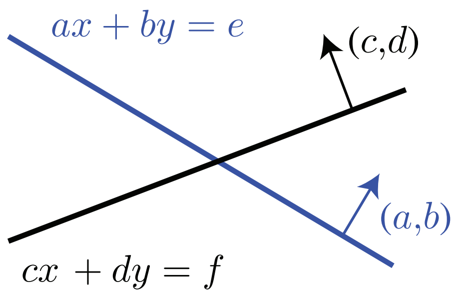
$$(a, b, 0) \times (c, d, 0) = (0, 0, a \cdot d - b \cdot c)$$

and the length of this vector is

$$|a \cdot d - b \cdot c|.$$

This is the absolute value of our determinant. So, this works.

Now, let's have a look at the system of linear equations $ax + by = e, cx + dy = f$. This system corresponds to two lines in the plane with normal vectors (a, b) and (c, d).



For the determinant to be zero just means that these two normal vectors are multiples of each other, or equivalently that the two lines are parallel. This corresponds to the system having no or infinitely many solutions. On the other hand, if the determinant is not equal to zero, then the two lines are clearly not parallel and we get a unique point of intersection, that is, a unique solution to our system of equations.

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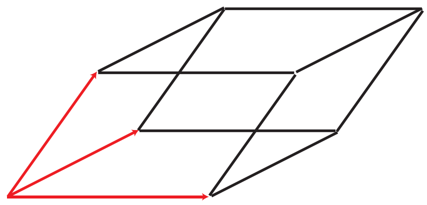
[1] Or maybe you know, which amounts to the same thing, that the inverse $\frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ of A exists if and only if A's determinant is not equal to zero.

[2] This also means that you can use this determinant to calculate areas of parallelograms, triangles, etc., in the plane.

[3] Remember that the cross product only works for 3d vectors! There is no cross product of 2d vectors.

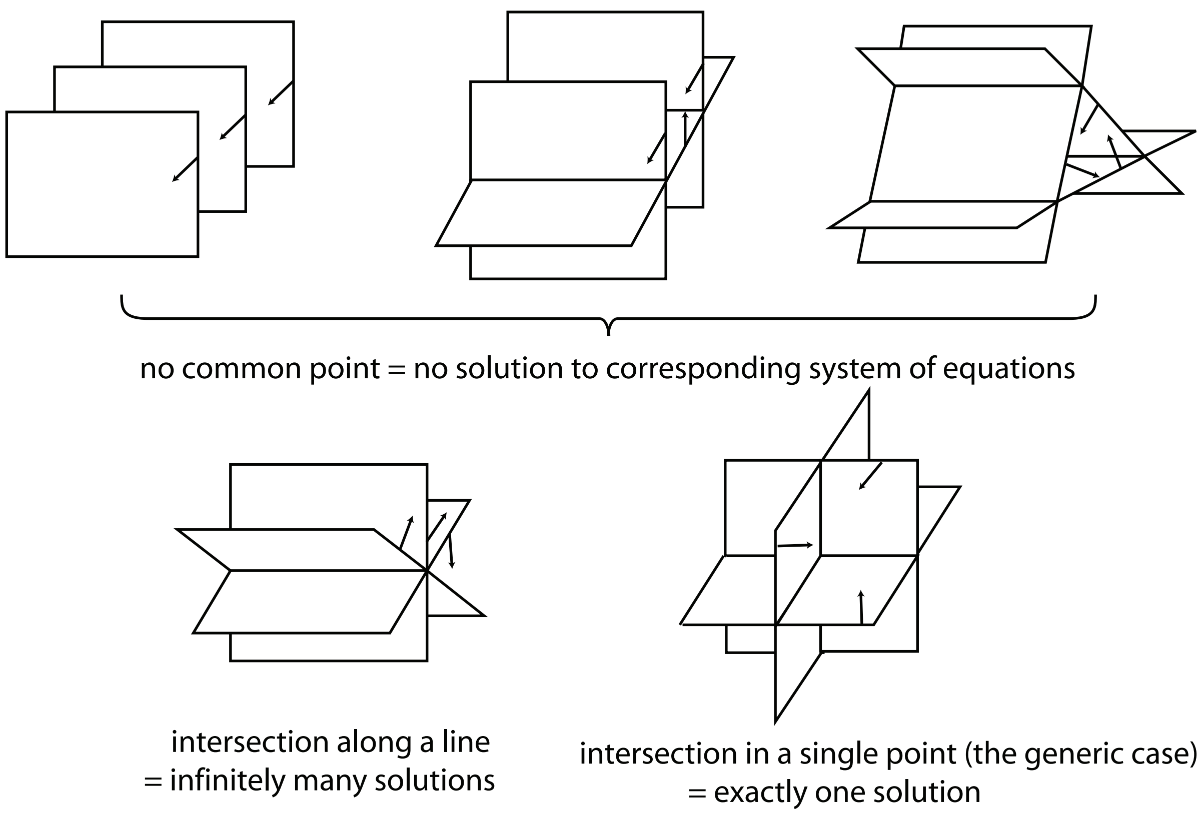
7.1.2 Determinants of 3×3 matrices

How would this geometric setup generalize for 3×3 matrices? Well, now A is a 3×3 matrix with three row vectors, and the 3d counterpart of the parallelogram is the parallelepiped spanned by these three vectors. And, the area of the parallelogram becomes the volume of this parallelepiped.



And, remember, we already know how to calculate this volume: Use the box product! So, it should not come as too much of a surprise that the determinant of A is simply the box product of its three row vectors.

To check whether this 3×3 determinant is as useful as its 2×2 counterpart, let's have a look at what this determinant can tell us about the solutions of $A\mathbf{x} = \mathbf{b}$. Here we are dealing with a system of three linear equations in three unknowns corresponding to three planes in space, with the three row vectors being normal vectors of these planes. The following pictures show the essentially different scenarios.



For example, the diagram in the upper left corresponds to the case of three parallel planes. In this case the three normal vectors point along the same line and therefore the determinant will be zero. The same is the case for all but the scenario depicted in the lower right. This means that, again, just like in the 2×2 case, the determinant of a 3×3 matrix is non-zero if and only if the corresponding systems of linear equations have unique solutions.

7.1.3 Determinants of $n \times n$ matrices

At this stage you are probably convinced that you know how this story continues. And you are (probably) right. Basically, the determinant of an $n \times n$ matrix A is the n-dimensional volume of the n-dimensional counterpart of a parallelogram spanned by the row vectors of A.[4] And, $A\mathbf{x} = \mathbf{b}$ has a unique solution if and only if this determinant is not equal to zero.

If we were 4-dimensional beings we could use our 4d intuition to deal with the 4×4 determinants geometrically, just like we did in the cases of 2×2 and 3×3 matrices. However, very few of us ever learn to think in four or higher dimension and geometric intuition is not really an option for us at this stage. Instead, we'll proceed with algebra.

To get us started, let's have a look at the determinant formula for the 3×3 matrices. Since this determinant is equal to the box product of the three row vectors it is easy to derive this formula.

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

$$= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}$$

Note that the different coloured strokes superimposed onto the determinant correspond to the six products that this formula consists of. Also the red strokes of "negative slope" correspond to the $+$ terms and those of "positive slope" to the $-$ terms. Just memorize the flower pattern formed by these strokes and you'll always be able to easily calculate 3×3 determinants.[5]

[4] We have not talked about what the sign of the determinant means geometrically. That will become clear later when we interpret matrices as linear transformations of space. But, just to give you something to hold on to until then, let's just say that any ordered set of vectors, like our set of row vectors that corresponds to a non-zero determinant is either right- or left-handed, very much like in the case of three vectors that we considered earlier. If it is right-handed the determinant is positive, otherwise it is negative.

[5] Another popular mnemonic is the so-called **Rule of Sarrus**. Google it!

So, the formula consists of six products. For every one of these six products

$$a_{1,\mathbf{1}}a_{2,\mathbf{2}}a_{3,\mathbf{3}}, \quad a_{1,\mathbf{2}}a_{2,\mathbf{3}}a_{3,\mathbf{1}}, \quad a_{1,\mathbf{3}}a_{2,\mathbf{1}}a_{3,\mathbf{2}}, \quad a_{1,3}a_{2,2}a_{3,1}, \quad a_{1,1}a_{2,3}a_{3,2}, \quad a_{1,2}a_{2,1}a_{3,3}$$

let's pick out the right indices of their three factors.[6] This gives:

$$123, 231, 312, 321, 132, 213.$$

Of course, you immediately recognize the six permutations of the three digits 1, 2 and 3. Let's do the same for the formula for the 2×2 determinant

$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix} = a_{1,\mathbf{1}}a_{2,\mathbf{2}} - a_{1,\mathbf{2}}a_{2,\mathbf{1}}$$

and we get the two permutations, 12 and 21 of the two digits 1 and 2.

Okay, there are $4! = 24$ permutations of the numbers 1, 2, 3, and 4. So, what this suggests is that the formula for the determinant of 4×4 matrices should have 24 terms, each of the form $a_{1,e}a_{2,f}a_{3,g}a_{4,h}$ where $efgh$ is one of these permutations. And, in general, the formula for $n \times n$ matrices should have $n!$ terms, each a product of n factors.

Looks like we are almost there. However, there is still the question of when one of the terms is preceded by a $+$ and when by a $-$. The simple rule for this is based on what I like to call the two **universal laws of swaps**:

- A **swap** inside a permutation consists of an exchange of two of its entries. For example, by exchanging the first and third number of 2341 we can turn this permutation into 4321. Now the first law of swaps says that it is always possible to turn any permutation of the first n numbers into the **identity permutation** $1234\dots n$ using a few swaps. This is actually quite easy to see. For example, we can transform 4321 as follows: 4321 (swap first and fourth) 1324 (swap second and third) 1234. Done!
- The second law of swaps is really quite amazing (and not that easy to prove). There are lots of different ways to turn a permutation into the identity permutation using swaps. However, either **all of these ways involve an even number of swaps or all consist of an odd number of swaps**. For example, in our previous example it took two swaps to transform 4321 into 1234. Let's do it another way: 4321 (swap first and second) 3421 (swap second and third) 3241 (swap third and fourth) 3214 (swap first and third) 1234. Four swaps. And 4, just like 2, is an even number.

Because of these laws it makes sense to call a permutation **odd** or **even** depending on how many swaps it takes to transform it into the identity permutation.

For example, the permutations $123, 231, 312$ are even, whereas the permutations $321, 132, 213$ are odd. Note that all the terms in the 3×3 determinant that correspond to an even permutation are preceded by a $+$ and the odd ones are preceded by a $-$.

[6] The left indices in each product are 1,2,3, in this order.

And this is also the general rule for assigning signs to the different terms of the determinant formulas. For example, one term in the determinant formula for 4×4 matrices is $a_{1,4}a_{2,3}a_{3,2}a_{4,1}$. The corresponding permutation is 4321, which is even. Therefore this term will be preceded by a $+$.

So, in principle, you should now be able to write down the determinant formula for 100×100 matrices.

Just for fun, here is the general formula for 4×4 matrices.

$$a_{1,4}a_{2,3}a_{3,2}a_{4,1} - a_{1,3}a_{2,4}a_{3,2}a_{4,1} - a_{1,4}a_{2,2}a_{3,3}a_{4,1} + a_{1,2}a_{2,4}a_{3,3}a_{4,1}$$
$$+a_{1,3}a_{2,2}a_{3,4}a_{4,1} - a_{1,2}a_{2,3}a_{3,4}a_{4,1} - a_{1,4}a_{2,3}a_{3,1}a_{4,2} + a_{1,3}a_{2,4}a_{3,1}a_{4,2}$$
$$+a_{1,4}a_{2,1}a_{3,3}a_{4,2} - a_{1,1}a_{2,4}a_{3,3}a_{4,2} - a_{1,3}a_{2,1}a_{3,4}a_{4,2} + a_{1,1}a_{2,3}a_{3,4}a_{4,2}$$
$$+a_{1,4}a_{2,2}a_{3,1}a_{4,3} - a_{1,2}a_{2,4}a_{3,1}a_{4,3} - a_{1,4}a_{2,1}a_{3,2}a_{4,3} + a_{1,1}a_{2,4}a_{3,2}a_{4,3}$$
$$+a_{1,2}a_{2,1}a_{3,4}a_{4,3} - a_{1,1}a_{2,2}a_{3,4}a_{4,3} - a_{1,3}a_{2,2}a_{3,1}a_{4,4} + a_{1,2}a_{2,3}a_{3,1}a_{4,4}$$
$$+a_{1,3}a_{2,1}a_{3,2}a_{4,4} - a_{1,1}a_{2,3}a_{3,2}a_{4,4} - a_{1,2}a_{2,1}a_{3,3}a_{4,4} + a_{1,1}a_{2,2}a_{3,3}a_{4,4}$$

And here is the general formula for the determinant of an $n \times n$ matrix

$$\det(A) = \sum_{\sigma \in \{\text{permuations of } 123...n\}} sgn(\sigma) a_{1,\sigma(1)}a_{2,\sigma(2)}a_{3,\sigma(3)}...a_{n,\sigma(n)},$$

where $sgn(\sigma) = \pm 1$, depending on whether σ is even or odd. Also, if, for example, $\sigma = 4123$, then $\sigma(1) = 4, \sigma(2) = 1, \sigma(3) = 2, \sigma(4) = 3$.

7.1.4 Determinants of special matrices

What is $\det(aA)$ where a in an arbitrary number and A is an $n \times n$ matrix? Well, every coefficient of A gets multiplied by the number a and every term of the determinant has n factors. Hence we get the determinant of the matrix aA by multiplying the determinant of A by a^n, or

$$\det(aA) = a^n \det(A).$$

For example, in the case of the general 2×2 determinant we have

$$\det(aA) = (a \cdot a_{1,1})(a \cdot a_{2,2}) - (a \cdot a_{1,2})(a \cdot a_{2,1}) = a^2(a_{1,1}a_{2,2} - a_{1,2}a_{2,1}) = a^2 \det(A).$$

Let's have another look at the flower pattern for the 3×3 determinant.

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

$$= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}$$

Here is a very important observation: **Given any of the terms of this formula, like $a_{1,3}a_{2,1}a_{3,2}$, every row and every column of the matrix contains exactly one of the factors of this term.**

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

This is true in general and is a consequence of the fact that the left indices of the factors in a term correspond to the identity permutation and the right indices of the factors translate into permutations. For another example, let's highlight the factors of the term $a_{1,3}a_{2,4}a_{3,2}a_{4,1}$ in the general 4×4 matrix/determinant.

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{vmatrix}$$

Again, as you can see, every row and column contains exactly one of the factors.

Zero rows and columns, diagonal and triangular matrices

Here are some simple consequences of this last observation:

1. If a matrix contains a column or a row consisting entirely of 0s, then its determinant is 0. (*Proof.* For a matrix like this every term in the determinant formula is equal to zero.) For example,

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 3 & 2 & 1 \end{vmatrix} = 1 \cdot 0 \cdot 1 + 2 \cdot 0 \cdot 3 + 3 \cdot 0 \cdot 2 - 3 \cdot 0 \cdot 3 - 1 \cdot 0 \cdot 2 - 2 \cdot 0 \cdot 1 = 0$$

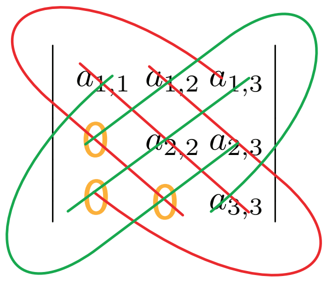
because the second row of the matrix consists of 0s.

2. The **main diagonal** of a matrix consists of the entries $a_{1,1}, a_{2,2}, ..., a_{n,n}$. A **diagonal** matrix is a matrix in which all coefficients off the diagonal are equal to zero. An **upper triangular matrix** is a matrix all of whose coefficients below the main diagonal are zero. A **lower triangular matrix** is a matrix all of whose coefficients above the main diagonal are zero. Here are three examples.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$$

The matrix on the left is a diagonal matrix, that in the middle an upper triangular matrix and the one on the right a lower triangular matrix.

The determinant of a diagonal, upper triangular or lower triangular matrix is equal to the product of the coefficients on the diagonal. (*Proof.* For these types of matrices, all terms of the determinant formula except for the one that corresponds to the main diagonal are equal to zero. In the case of 2×2 determinants and 3×3 determinants this is very easy to see. For example, here is our flower pattern superimposed on an upper triangular 3×3 matrix. As you can see, except for the curve that runs along the main diagonal, all of our curves pass through one of the orange 0s.



This means that all terms in the determinant formula except for the one corresponding to the main diagonal are equal to 0.)

We conclude that the determinant of any identity matrix is equal to 1. The determinants of the above matrices are

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 1 \cdot 2 \cdot 3 = 6, \qquad \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \cdot 4 \cdot 6 = 24, \qquad \begin{vmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{vmatrix} = 1 \cdot 3 \cdot 6 = 18.$$

Elementary matrices

The three types of elementary operations that we used to turn A into its reduced row echelon form can be performed by multiplying A by certain **elementary matrices from the left**. There are three types of elementary matrices corresponding to the three types of elementary operations:

1. **Swapping rows**: For example to exchange the third and fifth row of a 5×5 matrix A you multiply it by

$$E_{\text{swap rows 3 and 5}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

from the left, that is, you calculate the product $E_{\text{swap rows 3 and 5}}A$.
The only term of the determinant of this matrix different from zero is

$$a_{1,1}a_{2,2}, a_{3,5}, a_{4,4}, a_{5,3},$$

the term whose factors correspond to the positions of the 1s in this matrix. Its permutation is 12543, that is, the permutation that arises from the identity permutation by swapping 3 and 5. This means that we are dealing with an odd permutation and therefore the determinant of this elementary matrix is -1. Similarly, it is easy to see that the determinant of any elementary matrix like this is equal to -1.

2. **Multiplying a row by a number** a: For example, to multiply the second row of a 5×5 matrix by a, simply multiply it by the following elementary matrix:

$$E_{\text{multiply second row by } a} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since this is a diagonal matrix, its determinant is a.

3. **Adding a multiple of one row to another row**: For example, to add a times the third row of a 5×5 matrix to the fifth row, multiply by the following elementary matrix of the third kind:

$$E_{\text{add } a \text{ times the third row to fifth row}} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & a & 0 & 1 \end{pmatrix}$$

This sort of matrix is a triangular matrix with 1s on the diagonal. Hence its determinant is equal to 1.

Finding determinants by elementary row operations

Now, here is an extremely important observation: **If E is an elementary matrix and EA makes sense as a matrix product, then** $\det(EA) = \det(E)\det(A)$.[7]

This means that if you swap two rows of a matrix, then the determinant of the new matrix is simply -1 times the determinant of the original matrix. If you multiply one of the rows of a matrix by a number a, then the determinant of the new matrix is equal to a times the determinant of the original matrix. Finally, if you add a multiple of a row to another row, both the new and the original matrix have the same determinant.

This result then translates immediately into the quick method for computing the determinants of an arbitrary matrix: Simply row reduce the matrix using elementary operations/matrices to echelon form (or reduced row echelon form), keeping track of the determinants of the elementary matrices that you come across on the way. So, let's say we've row reduced A to R using the elementary matrices $E_1, E_2, E_3, ...E_k$, that is,

$$E_k \cdots E_3 E_2 E_1 A = R.$$

Then

$$\det(E_k \cdots E_3 E_2 E_1 A) = \det(E_k) \cdots \det(E_3)\det(E_2)\det(E_1)\det(A) = \det(R).$$

The determinant of the reduced matrix R on the right side is easy to find since it is upper triangular. So, the determinant of A is simply the determinant of R divided by the product of the determinants of the elementary matrices that you used.

Here is an example. Let's row reduce

$$\begin{pmatrix} 1 & 3 & 6 \\ 2 & 7 & 14 \\ 0 & 2 & 5 \end{pmatrix}$$

[7] I won't prove this here, but here are some hints if you feel like giving this a go yourself: To prove this is pretty straightforward if E swaps rows or multiplies a row by a non-zero number. If E adds a multiple of one row to another row, say two times the third to the fifth, then it is easy to check that $\det(EA) = \det(A) + 2\det(A')$, where A' results from A by replacing the fifth row by a copy of the third row of A. This means that the third and fifth row of A' are the same. In turn this means that exchanging the third and fifth row of A' does not change the matrix A' therefore also does not change the determinant. However, we already know that exchanging two rows reverses the sign of the determinant of a matrix. This means we can conclude that $\det(A') = -\det(A')$. Hence $\det(A') = 0$ and $\det(EA) = \det(A) + 2\det(A') = \det(A)$.

to echelon form. We start by replacing the second row by (-2) times the first equation added to the second row.

$$\begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

We then replace the third row of the new matrix with (-2) times the second row added to the third row. This gives

$$\begin{pmatrix} 1 & 3 & 6 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Now the two elementary operations that we used to get from the first matrix to the last matrix both correspond to elementary matrices with determinant 1. This means that all three matrices listed above have the same determinant. And, of course, since the last matrix is an upper diagonal matrix, this determinant is simply the product of the entries on the main diagonal, that is, the determinant is equal to 1.

The meaning of a zero determinant

Let's have another look at the last equation:

$$\det(E_k \cdots E_3 E_2 E_1 A) = \det(E_k) \cdots \det(E_3)\det(E_2)\det(E_1)\det(A) = \det(R).$$

Let's also assume that we've gone all the way, that is, that R, the matrix on the right is the reduced row echelon form of A. Since the determinants of all elementary matrices are non-zero their product is non-zero as well. Hence,

$$\text{some non-zero number} \cdot \det(A) = \det(R).$$

This means that $\det(A)$ is non-zero if and only if $\det(R)$ is. However, we already know that R is the identity matrix if A is invertible. Otherwise, it is an upper triangular matrix with at least one 0 on the diagonal. This means that $\det(A)$ is non-zero if and only if A is invertible, or equivalently, if and only if any of the systems of equations $A\mathbf{x} = \mathbf{b}$ has a unique solution.

Great, so our general determinants can really tell us when systems of linear equations have unique solutions. To summarize

Theorem 7.1.1 (Non-zero determinant) *The following properties of a square matrix A are equivalent:*

- $\det(A) \neq 0$
- *A is invertible.*

- *The reduced row echelon form of A is the identity matrix.*
- *Any system of linear equations of the form $A\mathbf{x} = \mathbf{b}$ has a unique solution.*
- *All systems of linear equations of the form $A\mathbf{x} = \mathbf{b}$ have unique solutions.*

The determinant of a product is the product of the determinants

Given an invertible $n \times n$ matrix A and the identity matrix I of the same dimensions, we've learned to transform the $n \times 2n$ matrix $(A|I)$ into the matrix $(I|A^{-1})$ using elementary operations. This also means that there are elementary matrices such that

$$E_k \cdots E_2 E_1 A = I$$

or, equivalently, $A^{-1} = E_k \cdots E_2 E_1$. We conclude that the inverse of an invertible matrix is the product of elementary matrices. However, since A is the inverse of A^{-1} we know that the same is true for A itself. We conclude that **every invertible matrix is the product of elementary matrices**.

Given two invertible matrices A and B of the same dimensions this implies that

$$\det(AB) = \det(A)\det(B).$$

In fact, this is true for any two matrices A and B of the same dimensions no matter whether they are invertible of not.[8]

Let's summarize all this discussion and just mention another interesting property of determinants in the following theorem.

Theorem 7.1.2 (The determinant of a product) *Let A and B be two $n \times n$ matrices. Then*

$$\det(AB) = \det(A) \det(B).$$

Furthermore,

$$\det(A) = \det(A^T).$$

To prove that $\det(A) = \det(A^T)$ is again straightforward if A is invertible, or equivalently, if A is the product of elementary matrices

$$A = E_1 E_2 E_3 \cdots E_{k-1} E_k.$$

[8] The only thing missing here is to prove that $\det(A) = 0$ or $\det(B) = 0$ implies that $\det(AB) = 0$.

Because then we have

$$\begin{aligned}
\det(A) &= \det(E_1 E_2 E_3 \cdots E_{k-1} E_k) \\
&= \det(E_1)\det(E_2)\det(E_3)\cdots\det(E_{k-1})\det(E_k) \\
&= \det(E_k)\det(E_{k-1})\cdots\det(E_3)\det(E_2)\det(E_1) \\
&= \det(E_k^T)\det(E_{k-1}^T)\cdots\det(E_3^T)\det(E_2^T)\det(E_1^T) \\
&= \det(E_k^T E_{k-1}^T \cdots E_3^T E_2^T E_1^T) \\
&= \det((E_1 E_2 E_3, \cdots, E_{k-1} E_k)^T) \\
&= \det(A^T)
\end{aligned}$$

Here the equality highlighted in red can be justified by observing that for every elementary operation E we have $\det(E) = \det(E^T)$. Can you justify all the other equal signs in this chain of equalities?

7.2 Calculating determinants

Since we know what the general formula for an $n \times n$ determinant is, in principle we now know how to calculate the determinant of every square matrix. However, apart from the formulas for 2×2 and 3×3 matrices, it is a bad idea to calculate determinants using the general formulas. There are much better ways.

For some special matrices we've already figured out explicitly what their determinants are. Here is a summary of our findings so far.

Theorem 7.2.1 (Determinants of special matrices) *Let A be a square matrix.*

If A is an upper triangular, lower triangular or diagonal matrix, then the determinant of A is equal to the product of its entries on the main diagonal. In particular, the determinant of an identity matrix is 1 and the determinant of a zero matrix is 0.

Elementary matrices that swap rows have determinant -1, elementary matrices that correspond to multiplying a row by a constant c have determinant c and elementary matrices which add a multiple of a row to another row have determinant 1.

A matrix with a column or row of 0s has determinant 0.

A matrix in which one row is a multiple of another row, or one column is a multiple of another column has determinant 0.

We now describe a method which reduces the problem of finding the determinant of an $n \times n$ matrix to that of finding the determinants of a number of $(n-1) \times (n-1)$ matrices.

Definition 7.2.1 (Cofactors) *Let A be an $n \times n$ matrix. Then the $(i, j)^{th}$* **cofactor** *is the determinant of the $(n-1) \times (n-1)$ matrix which results from deleting the i^{th} row and the j^{th} column from A and multiplying the resulting number by $(-1)^{i+j}$.*

That's quite a mouthful, so let's have a look at an example. Here is a matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$

To find the $(1, 2)$th cofactor we first create a 2×2 matrix by deleting the first row and the second column of this 3×3 matrix.

$$\begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 4 & \cancel{3} & 2 \\ 3 & \cancel{2} & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$$

The determinant of this 2×2 matrix is

$$\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} = -2.$$

To arrive at our answer we still have to multiply this number by $(-1)^{1+2} = -1$. This means that the $(1, 2)$th cofactor of our matrix is equal to $(-2) \cdot (-1) = 2$. Note that the factor $(-1)^{i+j}$ is either $+1$ or -1. In fact, the (i, j)th entry of the following checkerboard pattern tells you at a glance what factor we are dealing with.

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Theorem 7.2.2 (Calculating determinants by expanding along rows and columns) *Given a square matrix, pick a row (or column). You arrive at the determinant of A by taking the product of each entry in that row (or column) with the corresponding cofactor and adding up these products.*

Okay, let's find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 3 & 2 & 1 \end{pmatrix}.$$

by expanding along the first row.

$$1\,(+1)\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 2\,(-1)\begin{vmatrix} 4 & 2 \\ 3 & 1 \end{vmatrix} + 3\,(+1)\begin{vmatrix} 4 & 3 \\ 3 & 2 \end{vmatrix} = -1+4-3=0.$$

Let's check that we really get the same result when we expand along the first column.

$$1\,(+1)\begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} + 4\,(-1)\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} + 3\,(+1)\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = -1+16-15=0.$$

Expanding along any of the other rows or columns will always give the same result.

Let's have a look at the 4×4 matrix

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 3 & 4 & 5 \\ 3 & 4 & 3 & 2 \end{pmatrix}$$

The checkerboard pattern to keep in mind here is

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

Let's expand this matrix along the second row. Then we get

$$5\,(-1)\begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 3 & 2 \end{vmatrix} + 4\,(+1)\begin{vmatrix} 1 & 3 & 4 \\ 1 & 4 & 5 \\ 3 & 3 & 2 \end{vmatrix} + 2\,(-1)\begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} + 3\,(+1)\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 3 \end{vmatrix}.$$

Now you can calculate the 3×3 determinants either by using the explicit formula that I showed you earlier or by expanding them along rows or columns. Putting it all together you then arrive at the answer -12.

When it comes to calculating a 5×5 determinant (on a desert island with no access to a computer) you should now know what to do. Expand along a row or column to get a sum involving five 4×4 determinants, etc.

The method of evaluating a determinant by expanding along a row or a column is called the **method of Laplace expansion.**

This method is particularly useful when a matrix has a few zeros in a row or column. For example, the following 4×4 matrix has only one non-zero entry in the second column.

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 0 & 4 & 5 \\ 3 & 0 & 3 & 2 \end{pmatrix}$$

This suggests to expand along this column which then reduces the problem of finding the determinant of this 4×4 matrix to that of finding the determinant of just one 3×3 matrix.

$$\begin{vmatrix} 1 & 0 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 0 & 4 & 5 \\ 3 & 0 & 3 & 2 \end{vmatrix} = 4(+1) \begin{vmatrix} 1 & 3 & 4 \\ 1 & 4 & 5 \\ 3 & 3 & 2 \end{vmatrix} = -16.$$

Proof. The proof that the method of Laplace expansion really works follows in a fairly straightforward (if messy) way from the general formula for the determinant of $n \times n$ matrices. Just to give you an idea for the work that goes into a formal proof, here is how you show that expanding along the first row of a 3×3 matrix really works.

$$\begin{aligned} &\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} \\ &= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} \\ &= a_{1,1}(a_{2,2}a_{3,3} - a_{2,3}a_{3,2}) - a_{1,2}(a_{2,1}a_{3,3} - a_{2,3}a_{3,1}) + a_{1,3}(a_{2,1}a_{3,2} - a_{2,2}a_{3,1}) \\ &= a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} \end{aligned}$$

■

7.3 A formula for the inverse

The cofactors and the determinant of a matrix can also be used to find the inverse of a matrix.

Definition 7.3.1 (Cofactor matrix) *The* **cofactor matrix** *of an $n \times n$ matrix is the $n \times n$ matrix whose (i, j)th entry is the (i, j)th cofactor of A. We'll call the cofactor matrix* $\mathbf{cof}(A)$.

The following theorem says that to find the inverse of a matrix (if it exists), take the transpose of its cofactor matrix and divide it by the determinant of the matrix.

Theorem 7.3.1 (Formula for the inverse) *Let A be a square matrix. If A is invertible, then*

$$A^{-1} = \frac{1}{|A|}\mathbf{cof}(A)^T.$$

Remember that the T stands for flipping the matrix across its main diagonal to arrive at the **transpose** of the matrix. Also, previously, we already convinced ourselves that a matrix is invertible if and only if its determinant is non-zero. This result is also reflected by the fact that in this formula for the inverse we are dividing by the determinant. So the formula only makes sense for matrices with non-zero determinant.

For an invertible 2×2 matrix this gives the formula for the inverse that you are all familiar with.

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{|A|} \begin{pmatrix} +d & -c \\ -b & +a \end{pmatrix}^T = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

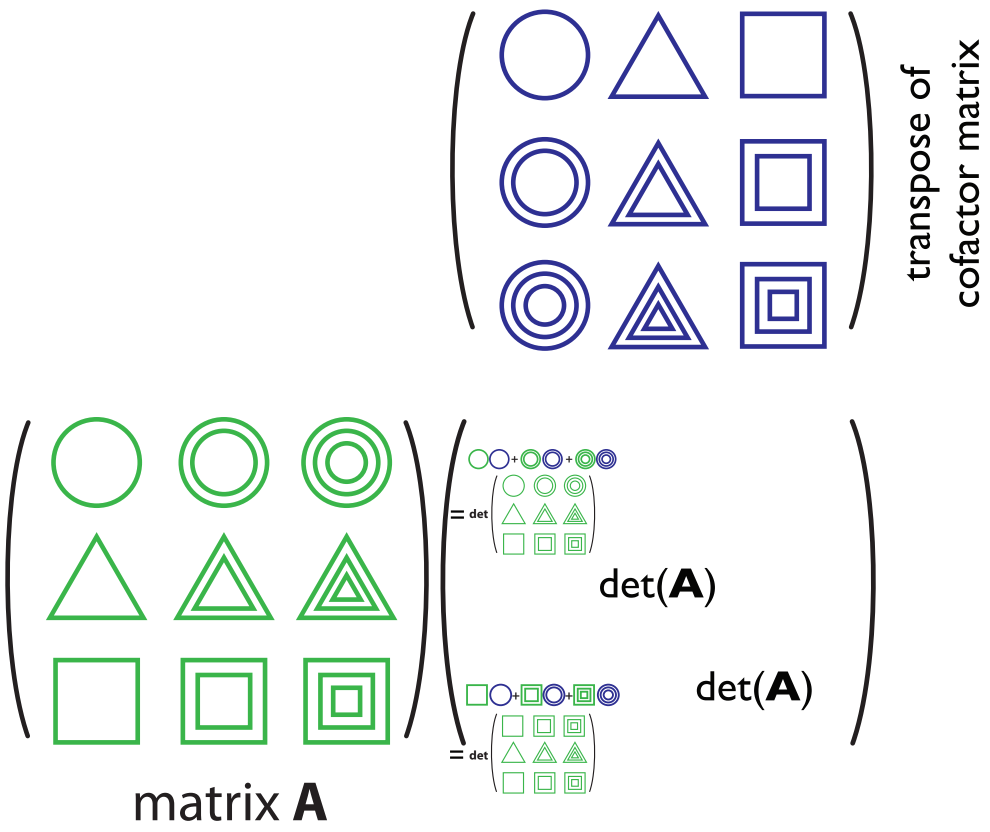
For 3×3 matrices we find

$$A^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1} = \frac{1}{|A|} \begin{pmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}^T.$$

The following diagrams show how I would go about filling in the big matrix on the right (if my life depended on it and I did not have access to a computer): 1) I fill in the checkerboard pattern of +'s and −'s highlighted in blue. 2) I pick an entry in the original matrix and cross out the row and column it is contained in. Just using two pencils to cover them up works well. 3) I figure out the determinant of the resulting 2×2 matrix in my head and write the result in the position that corresponds to the position of the entry that I picked in step 2. 4) Repeat for all entries of A.

$$\begin{pmatrix} \cancel{a} & \cancel{b} & \cancel{c} \\ \cancel{d} & e & f \\ \cancel{g} & h & i \end{pmatrix}^{-1} = \frac{1}{|A|} \begin{pmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}^T$$

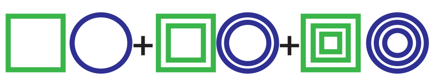
Proof. For those of you who are interested in a proof of the last theorem, the following diagram illustrates quite nicely what is going on. It shows the matrix A being multiplied by the transpose of its cofactor matrix. Clearly, the theorem is true if and only if we can show that the product of these two matrices is a diagonal matrix all of whose entries on the main diagonal are equal to $\det(A)$.



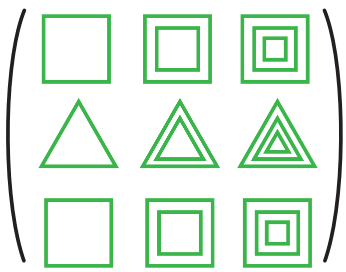
Entries that occupy the same position in A and the cofactor matrix are represented by the same symbol.

Let's first figure out what happens on the diagonal of the product. The entry in the upper left corner of the product is

However, this is just A expanded along its first row and is therefore equal to $\det(A)$. Similarly, we can see that all entries on the diagonal of the product are equal to $\det(A)$. So far so good. What about the entries off the diagonal? Let's consider the entry in the lower left corner as a representative example.



On close inspection this corresponds to the determinant of the following matrix expanded along the top row.



However, this determinant is equal to 0 since it contains two identical rows. We can conclude in a similar fashion that the other entries off the diagonal of the product are equal to 0 as well. This then completes the proof. ■

7.4 Cramer's rule

Remember that if the linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution, then $\mathbf{x} = A^{-1}\mathbf{b}$ is this solution. If we substitute our formula for the inverse A^{-1} from the previous section into the product $A^{-1}\mathbf{b}$ and then have a really close look we arrive at **Cramer's rule** for solving the linear system $A\mathbf{x} = \mathbf{b}$.

Theorem 7.4.1 (Cramer's rule) *Let $A\mathbf{x} = \mathbf{b}$ be a linear system in the unknowns $x_1, x_2, \ldots, x_n$ with a unique solution (this means that A is square and has non-zero determinant). Let A_i be the matrix that results from A by replacing the ith column of A by $\mathbf{b}$. Then*

$$x_i = \frac{\det(A_i)}{\det(A)}.$$

Let's have look at an example. We'd like to solve the linear system

$$\begin{pmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

The determinant of the matrix A is 3 and therefore this linear system has a unique solution. According to Cramer's rule this solution is

$$x_1 = \frac{\begin{vmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}} = \frac{1}{3}, \; x_2 = \frac{\begin{vmatrix} 3 & 3 & -1 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}} = \frac{8}{3}, \; x_3 = \frac{\begin{vmatrix} 3 & 2 & 3 \\ 1 & -1 & 1 \\ 2 & 1 & 0 \end{vmatrix}}{\begin{vmatrix} 3 & 2 & -1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{vmatrix}} = \frac{10}{3}.$$

Neither our formula for the inverse of a matrix nor Cramer's rule are used in practice when it comes to calculating inverses of large matrices or solving equations with zillions of equations and unknowns. Finding inverses and solutions by Gaussian elimination is then always the way to go.

However, both our formula for the inverse and Cramer's rule are very good tricks to know when you are dealing with small(ish) matrices having integer coefficients. Note that in this case all the determinants that need to be calculated produce integer values and the eventual output of both methods are nice fractions. Also, sometimes you may be confronted with matrices whose coefficients are functions instead of numbers. In such a situation our formula for the inverse or Cramer's rule may very well be the way to go.

Chapter 8

Linear transformations

To every $m \times n$ matrix A corresponds a function

$$T_A : \mathbf{R}^n \to \mathbf{R}^m, \text{ where } \mathbf{x} \mapsto A\mathbf{x}.$$

You are all familiar with the most trivial case of such a function, the case where $m = n = 1$. This corresponds to a simple linear function of the form

$$\mathbf{R} \to \mathbf{R}, \text{ where } x \mapsto ax.$$

Definition 8.0.1 (Linear transformations) *A function $\mathbf{R}^n \to \mathbf{R}^m$ of the form T_A for some matrix A is called a* **linear transformation**.

Our focus will be on linear transformations for which $m = n$. These are described by square matrices. In particular, we'd like to get a firm handle on the following fundamental transformations of $\mathbf{R}^2$ and $\mathbf{R}^3$ that turn out to be linear transformations (just take my word for it for the moment). These are

1. Rotations of $\mathbf{R}^2$ around the origin.
2. Rotations of $\mathbf{R}^3$ around an axis through the origin.
3. Reflections through the origin, through lines and through planes.
4. Projections onto vectors.

To be able to construct the matrices that describe these transformations, let's have another really close look at what happens when we multiply a matrix

$$A = \begin{pmatrix} 3 & 7 & -2 \\ 6 & 16 & -3 \\ 3 & 9 & 3 \end{pmatrix}$$

by a vector

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_2 \end{pmatrix}.$$

Writing the vector $\mathbf{x}$ as a linear combination of the three standard coordinate vectors

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

we get

$$\mathbf{x} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3}.$$

Now the product can be seen to have the form

$$A\mathbf{x} = x_1A\mathbf{e_1} + x_2A\mathbf{e_2} + x_3A\mathbf{e_3}.$$

Conclusion 1. For every vector x the product Ax is simply the sum of the three vectors Ae$_1$, Ae$_2$, and Ae$_3$ scaled by the components of x.

On the other hand, the three vectors $A\mathbf{e_1}$, $A\mathbf{e_2}$, and $A\mathbf{e_3}$ are simply the column vectors of A. This means that we can rewrite the above conclusion as follows.

Conclusion 2. For every vector x the product Ax is simply the sum of the three column vectors of A scaled by the components of x.

For example,

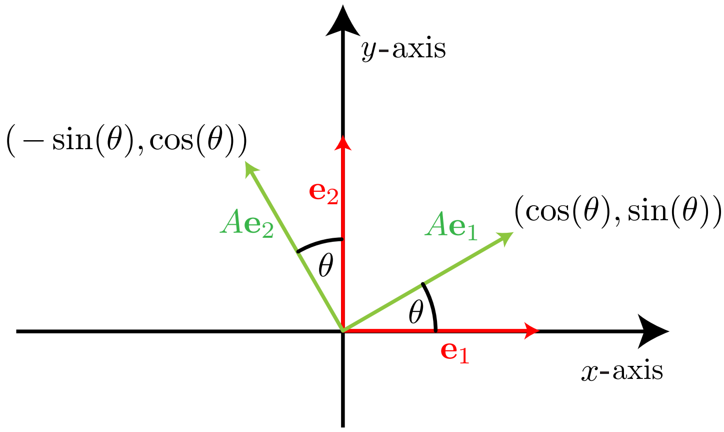
$$\begin{pmatrix} 3 & 7 & -2 \\ 6 & 16 & -3 \\ 3 & 9 & 3 \end{pmatrix} \begin{pmatrix} 37 \\ 66 \\ 89 \end{pmatrix} = 37 \begin{pmatrix} 3 \\ 6 \\ 3 \end{pmatrix} + 66 \begin{pmatrix} 7 \\ 16 \\ 9 \end{pmatrix} + 89 \begin{pmatrix} -2 \\ -3 \\ 3 \end{pmatrix}$$

Obviously, our conclusions will be the same no matter which matrix and vector we start with (as long as they can be multiplied). Then our two conclusions combine into the following very useful theorem:

Theorem 8.0.1 (Reconstructing the matrix of a linear transformation) *Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation (that is possibly given in a form that does not involve matrices). Then the columns of the matrix A of this linear transformation are simply the images of the unit coordinate vectors of $\mathbf{R}^n$ under this transformation.*

8.1 Rotations in $\mathbf{R}^2$

Let's put this pretty theorem to work. We start by figuring out the matrix of the counter-clockwise rotation through the angle θ around the origin of $\mathbf{R}^2$.[1] By our theorem we just have to figure out what happens to the unit coordinate vectors under such a rotation. The following diagram illustrates what is going on.



So, to get the answer we simply interpret the coordinates in the picture as column vectors

$$\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \text{ and } \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \end{pmatrix}.$$

We conclude that the matrix we are looking for is

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

8.2 Reflections in $\mathbf{R}^2$

The simplest of all reflections of $\mathbf{R}^n$ is the **reflection through the origin**. This reflection turns every vector into its negative. For example, in $\mathbf{R}^2$, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$ get mapped to $-\mathbf{e}_1$, $-\mathbf{e}_2$. Therefore, the matrix of this reflection is

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The reflection through the x-axis leaves $\mathbf{e}_1$ unchanged and takes $\mathbf{e}_2$ to $-\mathbf{e}_2$. This means that the matrix of this reflection is

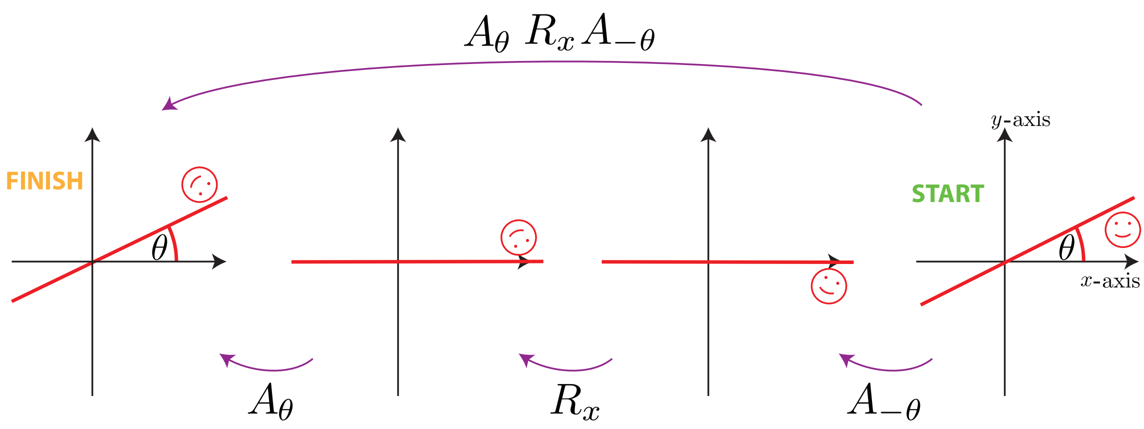
$$R_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

[1]Again, for the moment you are just taking my word that rotations are linear transformations, that is, can be described by matrices.

Let's construct the matrix of the reflection through a line containing the origin that makes an angle of θ with the x-axis. Here the idea is to reduce this complicated reflection to the simple reflection through the x-axis using a *setup move* involving our rotation matrix

$$A_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

We first rotate around the origin to align the line with the x-axis using the matrix $A_{-\theta}$. Then we reflect through the x-axis using the matrix R_x and finally we rotate the line back to its original position using A_θ. Here is the picture to keep in mind.



So, the matrix that we are looking for is simply

$$A_\theta R_x A_{-\theta} = \begin{pmatrix} \cos^2(\theta) - \sin^2(\theta) & 2\cos(\theta)\sin(\theta) \\ 2\cos(\theta)\sin(\theta) & \sin^2(\theta) - \cos^2(\theta) \end{pmatrix}.$$

Of course, if θ is a multiple of $\pi/4$ this matrix simplifies considerably.[2] For example, for $\pi/4$ (=45 degrees) this matrix becomes

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So, this reflection swaps $\mathbf{e}_1$ and $\mathbf{e}_2$. What does the matrix look like if we choose our angle to be equal to π?

[2] In fact, if you know your trigonometry you'll have noticed that our general reflection matrix can also be written as follows

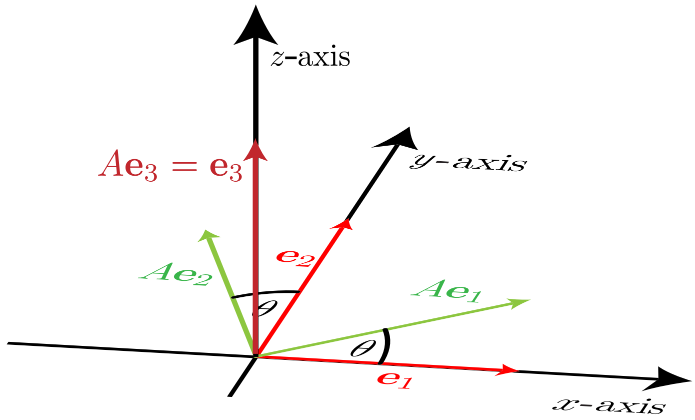
$$\begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix}.$$

8.3 Rotations in $\mathbf{R}^3$

Let's move on to 3d rotations. To start with let's figure out the matrix of the 3d rotation that extends the 2d rotation with matrix

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

that we considered before. This is a rotation around the z-axis in the counterclockwise direction (when viewed from the positive z-axis). Here is the picture to keep in mind.



The images of $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ under the 3d rotation are

$$\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix}, \begin{pmatrix} -\sin(\theta) \\ \cos(\theta) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

This means that the rotation matrix that we are after is

$$A_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrices that capture the rotations around $\mathbf{e}_1$ and $\mathbf{e}_2$ look very similar. What about the matrix that rotates around another unit vector $\mathbf{w}$

$$\mathbf{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \sqrt{a^2 + b^2 + c^2} = 1 \quad ?$$

The strategy for finding the matrix of the rotation around this vector is the same that we used in the previous section to find the matrices of reflections through lines. Really take your time to digest the following highlighted text (with your mathematical seat belts on!)

We start by constructing two orthogonal unit vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} \times \mathbf{v} = \mathbf{w}$. This means that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ form a right-handed system of mutually orthogonal unit vectors, just like the unit coordinate vectors $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$. Then we define a matrix S whose columns are $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

Viewed as a linear transformation this matrix takes the unit coordinate vectors $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$ to the vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, respectively. Since there is a 3d rotation that does the same and rotations are linear it is clear that our matrix is the matrix of this 3d rotation.

Now, it is also clear that the inverse of S is a rotation which takes $\mathbf{w}$ to $\mathbf{e_3}$.

This means that we arrive at the rotation around $\mathbf{w}$ that we are after by first aligning $\mathbf{w}$ with $\mathbf{e_3}$ using S^{-1}. Then we rotate around $\mathbf{e_3}$ using the matrix A_θ above. Finally, we reverse our setup move S^{-1} by multiplying with S. So, the rotation we are after is

$$SA_\theta S^{-1}.$$

To work! We first need to construct *unit* vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u} \times \mathbf{v} = \mathbf{w}$. For $\mathbf{u}$ any unit vector perpendicular to $\mathbf{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ will do. Clearly, the vector

$$\begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}$$

is perpendicular to $\mathbf{w}$. Dividing by its length we find a unit vector perpendicular to $\mathbf{w}$,

$$\mathbf{u} = \begin{pmatrix} \frac{-b}{\sqrt{a^2+b^2}} \\ \frac{a}{\sqrt{a^2+b^2}} \\ 0 \end{pmatrix}.$$

Now, we are looking for the unit vector $\mathbf{v}$ such that $\mathbf{u} \times \mathbf{v} = \mathbf{w}$. If three mutually perpendicular unit vectors satisfy $\mathbf{u} \times \mathbf{v} = \mathbf{w}$, they automatically also satisfy $\mathbf{w} \times \mathbf{u} = \mathbf{v}$ and $\mathbf{v} \times \mathbf{w} = \mathbf{u}$.

Since we know $\mathbf{w}$ and $\mathbf{u}$ we use the second identity $\mathbf{w} \times \mathbf{u} = \mathbf{v}$ to calculate $\mathbf{v}$. We find that

$$\mathbf{v} = \begin{pmatrix} \frac{-ca}{\sqrt{a^2+b^2}} \\ \frac{-cb}{\sqrt{a^2+b^2}} \\ \frac{a^2+b^2}{\sqrt{a^2+b^2}} \end{pmatrix}.$$

Now let S be the matrix with $\mathbf{u}, \mathbf{v}, \mathbf{w}$ as column vectors

$$S = \begin{pmatrix} \frac{-b}{\sqrt{a^2+b^2}} & \frac{-ca}{\sqrt{a^2+b^2}} & a \\ \frac{a}{\sqrt{a^2+b^2}} & \frac{-cb}{\sqrt{a^2+b^2}} & b \\ 0 & \frac{a^2+b^2}{\sqrt{a^2+b^2}} & c \end{pmatrix}.$$

Then here is the matrix we have been chasing,

$$SA_\theta S^{-1} = \begin{pmatrix} a^2(1-\cos(\theta))+\cos(\theta) & ab(1-\cos(\theta))-c\sin(\theta) & ac(1-\cos(\theta))+b\sin(\theta) \\ ab(1-\cos(\theta))+c\sin(\theta) & b^2(1-\cos(\theta))+\cos(\theta) & bc(1-\cos(\theta))-a\sin(\theta) \\ ac(1-\cos(\theta))-b\sin(\theta) & bc(1-\cos(\theta))+a\sin(\theta) & c^2(1-\cos(\theta))+\cos(\theta) \end{pmatrix}.$$

What a monster! But what a useful monster. If you want to describe anything turning in space you'll need this matrix.[3]

8.4 Reflections in $\mathbf{R}^3$

The reflection through the origin is given by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

What about the reflection through the xy-plane? This reflection fixes the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ and takes $\mathbf{e}_3$ to $-\mathbf{e}_3$. This means that the matrix of this reflection is

$$R_{xy} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

To construct the matrix through a plane containing the origin given by one of its unit normal vectors $\mathbf{u}$, we can proceed in the same way in which we handled rotations around this unit vector. So, we first apply the same matrix S^{-1} as in the section on rotations that takes $\mathbf{u}$ to $\mathbf{e}_3$. Then we reflect through the xy-plane by multiplying with the reflection matrix R_{xy} above and, finally, we take $\mathbf{e}_3$ back to $\mathbf{u}$ by multiplying with S. The resulting matrix $SR_{xy}S^{-1}$ is the reflection matrix that we are after.

There are also reflections through lines. For example, the reflection through the z-axis is described by the matrix

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

And, again, we can reduce the task of finding a reflection through the line given by the vector $\mathbf{u}$ using the, by now, usual trick.

[3] For this argument to work we actually have to assume that $|c| \neq 1$ to make sure that our choices don't result in $\mathbf{u}$ being the zero vector. However, on close inspection, it turns out that the resulting matrix still works even if $|c| = 1$. So our solution really is completely general.

8.5 What is a linear transformation?

We already gave an answer to this question. A linear transformation is a function $\mathbf{R}^n \to \mathbf{R}^m$ that can be described by an $m \times n$ matrix. But how can you tell whether a transformation given in a weird way is really of this form? Well, you could just pretend that it is a linear transformation, calculate the images of the unit coordinate vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots$ and combine these image into a matrix, as we've now done a number of times. And once you've got that matrix it may be possible to check whether it actually does the same for all possible input vectors as the transformation you are interested in.

An alternative way is to use the following useful criterion which is also often used to define linear transformations.

Theorem 8.5.1 (Recognizing linear transformations) *Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a function. Then f is a linear transformation if and only if it satisfies the following two properties:*

1.

$$T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \text{ for all vectors } \mathbf{u}, \mathbf{v} \in \mathbf{R}^n$$

2.

$$T(a\mathbf{u}) = aT(\mathbf{u}) \text{ for all vectors } \mathbf{u} \in \mathbf{R}^n \text{ and scalars } a \in \mathbf{R}.$$

It is straightforward to use this criterion to check that a rotation is really a linear transformations. For example, to check (1) just note that if you've got two vectors, you get the same result no matter whether you add them first and then rotate the sum, or whether you rotate both vectors and then add them.

Of course, to make everything we've said self-contained we'd still have to prove this last theorem. This is actually completely straightforward.

Proof. First, given a linear transformation with matrix A it is clear that properties (1) and (2) are satisfied since $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$ and $A(a\mathbf{u}) = aA\mathbf{u}$.

On the other hand, if you are dealing with a function T that satisfies (1) and (2) write a vector $\mathbf{u}$ as a linear combination of $\mathbf{e_1}, \mathbf{e_2}, \ldots$ like this: $\mathbf{u} = x_1\mathbf{e_1} + x_2\mathbf{e_2} + x_3\mathbf{e_3} + \ldots$ Then (1) and (2) imply that

$$T(\mathbf{u}) = x_1T(\mathbf{e_1}) + x_2T(\mathbf{e_2}) + x_3T(\mathbf{e_3}) + \ldots$$

Of course, this is the same as saying that

$$T(\mathbf{u}) = A\mathbf{u},$$

where A is the matrix with column vectors $T(\mathbf{e_1}), T(\mathbf{e_2}), \ldots$. Consequently, T is a linear transformation. ■

8.6 Projections

Here is an example of a transformation that in the first place is not given by a matrix, but which can be recognized as being linear using Theorem 8.5.1.

On page 27 we defined what it means to project the vector $\mathbf{v}$ onto another vector $\mathbf{u}$:

$$\mathbf{proj_u}(\mathbf{v}) = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

Then the projection map $P_{\mathbf{u}}$ corresponding to the vector $\mathbf{u}$ is

$$P_{\mathbf{u}} : \mathbf{R}^n \to \mathbf{R}^n, \text{ where } \mathbf{x} \mapsto \mathbf{proj_u}(\mathbf{x}).$$

Theorem 8.6.1 (Projections are linear) *The projection map $P_{\mathbf{u}}$ is a linear transformation.*

Proof. By Theorem 8.5.1, given two arbitrary vectors $\mathbf{v}, \mathbf{w} \in \mathbf{R}^n$ and a scalar $a \in \mathbf{R}$ we have to show that

$$P_{\mathbf{u}}(\mathbf{v}+\mathbf{w}) = P_{\mathbf{u}}(\mathbf{v}) + P_{\mathbf{u}}(\mathbf{w})$$

and

$$P_{\mathbf{u}}(a\mathbf{v}) = aP_{\mathbf{u}}(\mathbf{v}).$$

Okay, here we go.

$$P_{\mathbf{u}}(\mathbf{v}+\mathbf{w}) = \left(\frac{(\mathbf{v}+\mathbf{w})\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} + \left(\frac{\mathbf{w}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = P_{\mathbf{u}}(\mathbf{v}) + P_{\mathbf{u}}(\mathbf{w}).$$

Check! And

$$P_{\mathbf{u}}(a\mathbf{v}) = \left(\frac{a\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = a\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = aP_{\mathbf{u}}(\mathbf{v}).$$

■

Now that we know that the projection map $P_{\mathbf{u}}$ is a linear transformations, let's calculate the corresponding matrix for one particular vector,

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

As usual, we have to figure out what happens to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. So,

$$\left(\frac{\mathbf{e_1}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \frac{1}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \left(\frac{\mathbf{e_2}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \frac{2}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \left(\frac{\mathbf{e_3}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \frac{3}{14}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Therefore, the matrix of our projection is

$$\frac{1}{14}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}.$$

Okay and now that we have this matrix, when it comes to actually projecting another vector $\mathbf{v}$ onto $\mathbf{u}$, we can do this in two different ways, either by using the usual formula or by multiplying this matrix by $\mathbf{v}$.

Chapter 9

Eigenvectors and eigenvalues

Definition 9.0.1 *Let A be a square matrix and let $\mathbf{x}$ be a* **NONZERO (!!!!)** *vector such that*

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some $\lambda \in \mathbf{R}$. Then $\mathbf{x}$ is called an **eigenvector** *with* **eigenvalue** *λ of the matrix A.*

In other words, an eigenvector of A is a vector $\mathbf{x}$ for which multiplication by A results in a vector in the same or opposite direction to $\mathbf{x}$.[1] Since the zero vector $\mathbf{0}$ has no direction this would make no sense for this special vector.

Let's have a look at an example you will be able to relate to. Let

$$A = \begin{pmatrix} 3/5 & 1/5 \\ 2/5 & 4/5 \end{pmatrix}.$$

This is a transition matrix for a Markov chain, something you will already have encountered in school. The steady state vector of this matrix is

$$\mathbf{x} = \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

which, of course, tells you that

$$A\mathbf{x} = \mathbf{x}.$$

In the language of eigenvectors and eigenvalues this means that the steady state vector of a transition matrix of a Markov chain is an eigenvector with eigenvalue 1.

[1] Multiplication of vectors will be from the right. So what we are really talking about are **right eigenvectors.** It also makes sense to talk about left eigenvectors. Their theory is basically the same as that of right eigenvalues.

Here are two more examples:

$$\begin{pmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{pmatrix} \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix} = \begin{pmatrix} -50 \\ -40 \\ 30 \end{pmatrix} = 10 \begin{pmatrix} -5 \\ -4 \\ 3 \end{pmatrix}.$$

This means that the vector we are multiplying here is an eigenvector with eigenvalue 10. Then

$$\begin{pmatrix} 0 & 5 & -10 \\ 0 & 22 & 16 \\ 0 & -9 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This means that the vector we are multiplying here is an eigenvector with eigenvalue 0. Note that whereas the zero vector $\mathbf{0}$ cannot be an eigenvector, the number 0 can be an eigenvalue.

9.1 Finding eigenvectors and eigenvalues

Okay, let's assume that

$$A\mathbf{x} = \lambda\mathbf{x},$$

for some vector $\mathbf{x} \neq \mathbf{0}$. This can also be written as

$$A\mathbf{x} = \lambda I\mathbf{x},$$

or

$$(\lambda I - A)\mathbf{x} = \mathbf{0}.$$

Since $\mathbf{x} \neq \mathbf{0}$ this implies that

$$\det(\lambda I - A) = 0.$$

Definition 9.1.1 (Characteristic equation) *If A is an $n \times n$ matrix,* $\det(\lambda I - A)$ *pans out to be an nth degree polynomial in the variable λ called the* **characteristic polynomial** *of A. The above equation is called the* **characteristic equation** *of A.*

This means that we can find all candidates for eigenvalues of a matrix by solving the characteristic equation of the matrix. On the other hand, if a number λ_0 is a solution of the characteristic equation, then

$$\det(\lambda_0 I - A) = 0.$$

This implies that the equation we started with has a non-zero solution which then automatically is an eigenvector with eigenvalue λ_0.

We summarize what we just said in the following theorem.

Theorem 9.1.1 (Eigenvectors and eigenvalues) *Let A be a square matrix. Then $\lambda_0 \in$ $\mathbf{R}$ is an eigenvalue of A if and only if it is a solution of the characteristic equation of A,*

$$\det(\lambda I - A) = 0.$$

The eigenvectors of A corresponding to an eigenvalue λ_0 are the non-zero solutions of the linear system

$$(\lambda_0 I - A)\mathbf{x} = \mathbf{0}.$$

9.2 A first example

Here is an example. Let

$$A = \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}.$$

We first go after the eigenvalues by finding the characteristic equation of A,

$$\det(\lambda I - A) = 0.$$

Now,

$$\det(\lambda I - A) = \det\left(\lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}\right) = 0.$$

After expanding this determinant and simplifying, you find the equation you need to solve is

$$\lambda^3 - 25\lambda^2 + 200\lambda - 500 = 0.$$

The solutions of this characteristic equation and therefore also the eigenvalues of A are 5, 10 and 10. We have listed 10 twice because it is a solution of the characteristic equation of multiplicity 2.

To find the eigenvectors corresponding to the eigenvalue 5 we have to solve the linear system

$$(5I - A)\mathbf{x} = \mathbf{0}$$

or

$$\left(5 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}\right) \mathbf{x} = \mathbf{0}.$$

Let's rewrite this linear system in the standard form

$$\begin{pmatrix} 0 & 10 & 5 \\ -2 & -9 & -2 \\ 4 & 8 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

At this point we go on autopilot by first writing down the corresponding augmented matrix

$$\left(\begin{array}{rrr|r} 0 & 10 & 5 & 0 \\ -2 & -9 & -2 & 0 \\ 4 & 8 & -1 & 0 \end{array}\right)$$

and then finding its reduced row echelon form

$$\left(\begin{array}{rrr|r} 1 & 0 & -\frac{5}{4} & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This means that the eigenvectors are the vectors

$$t \cdot \begin{pmatrix} 5/4 \\ -1/2 \\ 1 \end{pmatrix}, t \in \mathbf{R} \setminus \{0\}$$

The reason for having $\mathbf{R} \setminus \{0\}$ instead of just $\mathbf{R}$ is that we need to exclude the zero vector. You would obtain the same collection of vectors if you replaced t with $4t$. Therefore a simpler description of the eigenvectors of A is

$$t \cdot \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}, t \in \mathbf{R} \setminus \{0\}.$$

To double-check that we have not made a mistake, let's multiply A by one of the vectors we just found,

$$\begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 25 \\ -10 \\ 20 \end{pmatrix} = 5 \begin{pmatrix} 5 \\ -2 \\ 4 \end{pmatrix}.$$

As you can see, we are really dealing with an eigenvector of A with eigenvalue 5.

Now let's find the eigenvectors corresponding to the eigenvector $\lambda = 10$. These vectors are solutions of the equation,

$$\left(10 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}\right) \mathbf{x} = \mathbf{0}.$$

or

$$\begin{pmatrix} 5 & 10 & 5 \\ -2 & -4 & -2 \\ 4 & 8 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The corresponding augmented matrix is

$$\left(\begin{array}{rrr|r} 5 & 10 & 5 & 0 \\ -2 & -4 & -2 & 0 \\ 4 & 8 & 4 & 0 \end{array}\right).$$

The reduced row echelon form for this matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

and so the eigenvectors are of the form

$$\begin{pmatrix} -2s-t \\ s \\ t \end{pmatrix} = s \cdot \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, s, t \in \mathbf{R} \text{ but not both equal to } 0.$$

9.3 A second example

Let

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Then

$$\det\left(\lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}\right) = 0$$

reduces to

$$\lambda^3 - 6\lambda^2 + 8\lambda = 0.$$

The solutions are $0, 2$ and 4. Now find the eigenvectors. For $\lambda = 0$ the augmented matrix for finding the solutions is

$$\left(\begin{array}{ccc|c} -2 & -2 & 2 & 0 \\ -1 & -3 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right)$$

and its reduced row echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, the eigenvectors are the vectors

$$t \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, t \in \mathbf{R} \setminus \{0\}.$$

Next, find the eigenvectors for $\lambda = 2$. The augmented matrix for the system of equations needed to find these eigenvectors is

$$\left(\begin{array}{ccc|c} 0 & -2 & 2 & 0 \\ -1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right).$$

Its reduced row echelon form is

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, the eigenvectors are the vectors

$$t \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, t \in \mathbf{R} \setminus \{0\}.$$

Finally, find the eigenvectors for $\lambda = 4$. The augmented matrix for the system of equations needed to find these eigenvectors is

$$\left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 3 & 0 \end{array}\right)$$

and its reduced row echelon form is

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, the eigenvectors are the vectors

$$t \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, t \in \mathbf{R} \setminus \{0\}.$$

9.4 The good news and the bad news

Well, finding eigenvalues and eigenvectors of a matrix may look a bit tedious but definitely doable, right? However, in general, for larger matrices this problem tends to become very tricky. The problem is that to find the eigenvalues we have to solve the characteristic equation. And for equations of orders greater than 4 no general solutions comparable to the quadratic formula for second order equations exist. Of course, we can still find approximations to these solutions which will always give us a good idea for what the eigenvalues will be. However, approximations don't cut it at all when it comes to finding the associated eigenvectors using our method. Why? Because if you use anything but an exact solution λ_0 to the characteristic equation in the linear system $(\lambda_0 I - A)\mathbf{x} = \mathbf{0}$ the only solution will be the (useless) zero vector.

Okay, that was the bad news. Here is some good news. Although it is usually extremely hard to solve the eigenvalue problem, the method we've described above always works for upper triangular and lower triangular matrices.

For example, let's find the eigenvalues of

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$

We need to solve

$$\begin{aligned} 0 = & \det\left(\lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}\right) \\ = \det & \begin{pmatrix} \lambda - 1 & -2 & -3 \\ 0 & \lambda - 4 & -5 \\ 0 & 0 & \lambda - 6 \end{pmatrix} = (\lambda - 1)(\lambda - 4)(\lambda - 6). \end{aligned}$$

Obviously, all this works in general and so we have the following proposition.

Proposition 9.4.1 (The eigenvalues of a triangular matrix) *The eigenvalues of an upper or lower diagonal matrix are the numbers on the main diagonal of the matrix.*

Note that a diagonal matrix is just a special case of a triangular matrix. Hence this result also applies to diagonal matrices. For example, the eigenvalues of

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

are 1, 2 and 3. In this case it is also clear what the corresponding eigenvectors are, namely all non-zero multiples of the coordinate vectors.

9.5 Diagonalization

Sometimes it is possible to **diagonalize** a square matrix using its eigenvalues and eigenvectors. To explain what we mean by this let's have another look at our second big example. Here the matrix under consideration was

$$A = \begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

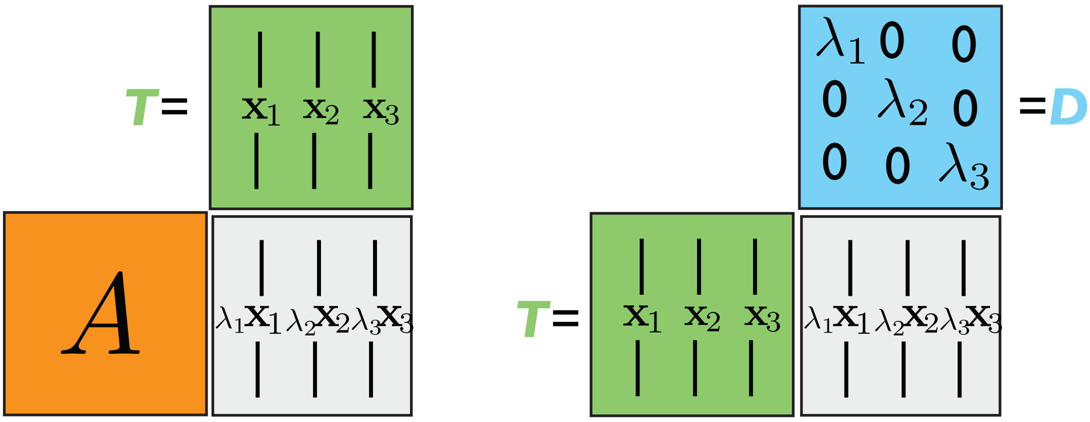
This matrix has the eigenvalues $\lambda_1 = 0$, $\lambda_2 = 2$ and $\lambda_3 = 4$ and some eigenvectors corresponding to λ_1, λ_2 and λ_3 are

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{x_2} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \mathbf{x_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Now make up a diagonal matrix which has the three eigenvalues on the diagonal (order does not matter) and another matrix T whose columns are the three eigenvectors, arranged in the same order in which we put their eigenvalues in the diagonal matrix,

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}, T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Now have a look at the following two diagrams.[2]



They show that the two products AT and TD are equal (to the grey matrix), that is,

$$AT = TD.$$

Multiplying this identity with the inverse of T (which does exist) from the right side then gives

$$A = TDT^{-1}.$$

All this generalizes in a straightforward way as follows.

Theorem 9.5.1 (Diagonalizing a matrix) *Let A be square matrix that has n eigenvectors that form a basis of $\mathbf{R}^n$. Form a diagonal matrix D and an additional matrix T using these eigenvectors and their corresponding eigenvalues, as described above. Then*

$$A = TDT^{-1}.$$

The converse of this theorem is also true.

Theorem 9.5.2 *If a matrix A is* **diagonalizable**, *that is, if A can be written in the form*

$$A = TDT^{-1},$$

[2]If you are having problems figuring out what the diagrams stand for, revisit the introduction on page 73.

where D is a diagonal matrix, then the entries on the diagonal of D are eigenvalues of A and the columns of T are corresponding eigenvectors.

The proof for this converse can again be read off directly from the two diagrams above.

In any case, what's the big deal? [3]

9.6 Interlude: Finding the nth Fibonacci number

As a lecturer it's always good to be able to point to some cool applications of the things I am getting excited about in mathematics. So, in this section I'd like to show you how one can derive a formula for the nth Fibonacci number. The material in this section is optional and I will gloss over some details and will only sketch the main steps.

Along the way we'll use the following neat insight which allows us to express the powers of a diagonalizable matrix $A = TDT^{-1}$ in terms of powers of D.

$$\begin{aligned} A^n &= (TDT^{-1})^n \\ &= (TDT^{-1})(TDT^{-1})(TDT^{-1})\cdots(TDT^{-1}) \text{ } n \text{ times} \\ &= TD(T^{-1}T)D(T^{-1}T)D\cdots(T^{-1}T)DT^{-1} \\ &= TD^nT^{-1}. \end{aligned}$$

Just as a reminder, the **Fibonacci sequence** is the sequence that starts out as follows:

$$1, 1, 2, 3, 5, 8, 13, 21, ...,$$

where two consecutive elements of the sequence always add up to the next element in the sequence. So, $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, and so on.

Let's call the nth Fibonacci number f_n. We'll also extend the Fibonacci sequence to the left by setting $f_0 = 0$. This makes sense since $f_0 + f_1 = 0 + 1 = 1 = f_2$.

What is f_{1000}? One way of figuring out the answer to this question is to "simply" calculate the first thousand elements of the sequence. However, there is a much less tedious way. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

[3]Diagonalization of matrices has a very important geometrical interpretation that we only mention at this point, but which you'll explore in depth if you ever take a unit dedicated to linear algebra. Let's say the matrix A can be diagonalized: $A = TDT^{-1}$. Geometrically this means the following. If instead of the standard coordinate system you make up a coordinate system using the column vectors of T as the unit coordinate vectors, then in terms of the new coordinate system the linear transformation described by A is simply the diagonal matrix D.

Then

$$A\begin{pmatrix} f_n \\ f_{n-1}\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix} f_n \\ f_{n-1}\end{pmatrix} = \begin{pmatrix} f_n + f_{n-1} \\ f_n\end{pmatrix} = \begin{pmatrix} f_{n+1} \\ f_n\end{pmatrix}.$$

What this means is that we can use matrix multiplication to calculate the elements of the Fibonacci sequence as follows:

$$\begin{aligned}
\begin{pmatrix}1\\1\end{pmatrix} &= A\begin{pmatrix}1\\0\end{pmatrix}\\
\begin{pmatrix}2\\1\end{pmatrix} &= A\begin{pmatrix}1\\1\end{pmatrix} = A^2\begin{pmatrix}1\\0\end{pmatrix}\\
\begin{pmatrix}3\\2\end{pmatrix} &= A\begin{pmatrix}2\\1\end{pmatrix} = A^3\begin{pmatrix}1\\0\end{pmatrix}\\
\begin{pmatrix}5\\3\end{pmatrix} &= A\begin{pmatrix}3\\2\end{pmatrix} = A^4\begin{pmatrix}1\\0\end{pmatrix}\\
&\vdots
\end{aligned}$$

And, in general,

$$\begin{pmatrix} f_{n+1} \\ f_n\end{pmatrix} = A^n\begin{pmatrix}1\\0\end{pmatrix}.$$

Now, A can be diagonalized as follows (by following the usual steps)

$$A = TDT^{-1}, T = \begin{pmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{pmatrix}, D = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix},$$

where

$$\lambda_1 = \frac{1+\sqrt{5}}{2} \text{ is the **golden ratio** and } \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

Therefore,

$$\begin{pmatrix} f_{n+1} \\ f_n\end{pmatrix} = TD^nT^{-1}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix}\lambda_1^{n+1} - \lambda_2^{n+1}\\ \lambda_1^n - \lambda_2^n\end{pmatrix}.$$

Hence we conclude that

$$f_n = \frac{\lambda_1^n - \lambda_2^n}{\sqrt{5}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

is a formula for the nth Fibonacci number, courtesy of our matrix diagonalization trick. Using this formula it is now very easy to calculate f_{1000}.

The formula is called **Binet's formula** and is a real gem worth pondering a little. Both λ_1, λ_2 are irrational numbers. This means that, written as decimal numbers, they have infinite non-repeating tails. Nevertheless all this irrationality cancels out in this formula and yields a whole number for all possible choices of n. Pretty marvellous, isn't it?

Also note that since $\frac{1-\sqrt{5}}{2} \approx -0.618...$ is less than 1, as n increases the expression

$$\left(\frac{1-\sqrt{5}}{2}\right)^n$$

gets very small very quickly. This means that

$$f_n \approx \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

In fact it turns out that you always get the nth Fibonacci number by rounding the expression on the right to the nearest natural number.

9.7 When is a matrix diagonalizable?

Being able to find closed forms of matrix powers can be of great use, as we've seen in the previous section. As well, diagonalizable matrices play an important role in many areas of science. We only mention the following result.

Theorem 9.7.1 (Matrices that are diagonalizable) *Let A be an $n \times n$ matrix. Then we can be sure that A is diagonalizable if it has one of the following two properties.*

1. *A has n different eigenvalues.*
2. *A is a symmetric matrix, that is, $A^T = A$.*

Square matrices that are diagonalizable are also often called **non-defective** and those that aren't are called **defective** matrices.

Here is an example of a defective matrix.

Let

$$A = \begin{pmatrix} 2 & -2 & -1 \\ -2 & -1 & -2 \\ 14 & 25 & 14 \end{pmatrix}$$

The characteristic equation of this matrix is

$$(\lambda - 3)(\lambda - 6)^2 = 0.$$

It turns out that all eigenvectors of corresponding to the eigenvalue 3 are multiples of one vector and the same is true for the eigenvectors corresponding to the eigenvalue 6. To diagonalise A we would need to build an invertible matrix T whose three columns are eigenvectors of A. However, this is impossible. Why? This means that the dimension of the eigenspace of the eigenvalue 6 is only of dimension 1. Hence the geometric multiplicity of this eigenvalue differs from its algebraic multiplicity and we conclude that the matrix is not diagonalizable, or defective.

Here is a nice quote from Kuttler's book:

> The word **defective** seems to suggest there is something wrong with the matrix. This is in fact the case. Defective matrices are a lot of trouble in applications and we may wish they never occurred. However, they do occur as the above example shows. When you study linear systems of differential equations, you will have to deal with the case of defective matrices and you will see how awful they are. The reason these matrices are so horrible to work with is that it is impossible to obtain a basis of eigenvectors. When you study differential equations, solutions to first order systems are expressed in terms of eigenvectors of a certain matrix times $e^{\lambda t}$ where λ is an eigenvalue. In order to obtain a general solution of this sort, you must have a basis of eigenvectors. For a defective matrix, such a basis does not exist and so you have to go to something called generalized eigenvectors. Unfortunately, it is never explained in beginning differential equations units why there are enough generalized eigenvectors and eigenvectors to represent the general solution. In fact, this reduces to a difficult question in linear algebra equivalent to the existence of something called the Jordan canonical form which is much more difficult than everything discussed in the entire differential equations unit.

9.8 Cayley-Hamilton

One of the most beautiful and unexpected theorems in linear algebra is the following.

Theorem 9.8.1 (The Cayley-Hamilton theorem) *Any square matrix satisfies its own characteristic equation.*

We'll illustrate what this means using our standard example

$$A = \begin{pmatrix} 5 & -10 & -5 \\ 2 & 14 & 2 \\ -4 & -8 & 6 \end{pmatrix}.$$

Its characteristic equation is

$$\lambda^3 - 25\lambda^2 + 200\lambda - 500 = 0.$$

Now, for A to satisfy its own characteristic equation means that

$$A^3 - 25A^2 + 200A - 500I = 0,$$

where the 0 on the right stands for the zero matrix with the same dimensions as A. You never saw that one coming, right?

Here is a simple proof that this theorem is true for diagonalizable matrices. Proofs that work for all square matrices need a bit more footwork than we have time for in this unit.

Proof (of Cayley-Hamilton for diagonalizable matrices). To avoid unnecessary notational gymnastics we'll use the above example as a stand-in for the general case. This also means that $\mathbf{R}^3$ will play the role of $\mathbf{R}^n$.

We have to check that the matrix $B = A^3 - 25A^2 + 200A - 500I$ is really the zero matrix. The idea is to show that B multiplied by any vector is equal to the zero vector and then to conclude that since B does the same in this respect as the zero matrix it actually is the zero matrix.

To start with, let's multiply B by an eigenvector $\mathbf{x}_0$ of A with eigenvalue λ_0. So

$$\begin{aligned}
B\mathbf{x}_0 &= (A^3 - 25A^2 + 200A - 500I)\mathbf{x}_0 \\
&= A^3\mathbf{x}_0 - 25A^2\mathbf{x}_0 + 200A\mathbf{x}_0 - 500I\mathbf{x}_0 \\
&= \lambda_0^3\mathbf{x}_0 - 25\lambda_0^2\mathbf{x}_0 + 200\lambda_0\mathbf{x}_0 - 500\mathbf{x}_0 \\
&= (\lambda_0^3 - 25\lambda_0^2 + 200\lambda_0 - 500)\mathbf{x}_0 \\
&= 0\mathbf{x}_0 \\
&= \mathbf{0}.
\end{aligned}$$

This shows that B wipes out every eigenvector of A. However, since A is diagonalizable, it has three eigenvectors $\mathbf{x_1}$, $\mathbf{x_2}$, and $\mathbf{x_3}$ that form a basis of $\mathbf{R}^3$. This means that an arbitrary vector $\mathbf{x}$ of $\mathbf{R}^3$ can be written as a linear combination of these three eigenvectors, that is,

$$\mathbf{x} = a\mathbf{x}_1 + b\mathbf{x}_2 + c\mathbf{x}_3$$

for some real numbers a, b and c. But then

$$\begin{aligned}
B\mathbf{x} &= B(a\mathbf{x}_1 + b\mathbf{x}_2 + c\mathbf{x}_3) \\
&= aB\mathbf{x}_1 + bB\mathbf{x}_2 + cB\mathbf{x}_3 \\
&= \mathbf{0}.
\end{aligned}$$

■

Finding the inverse: Take 3

Here is a nice application of the Cayley-Hamilton theorem.

So, we know that

$$A^3 - 25A^2 + 200A - 500I = 0.$$

We can also write this equation as

$$I = \frac{1}{500}(A^3 - 25A^2 + 200A).$$

Assuming that the inverse of A exists, we multiply this equation by A^{-1}. This gives

$$A^{-1} = \frac{1}{500}(A^2 - 25A + 200I).$$

Neat, the inverse of A in terms of (small) powers of A. In fact, it turns out that any power of A can be written in terms of the same small powers of A. Here is how.

Solve for A^3

$$A^3 = 25A^2 - 200A + 500I.$$

Multiply by A

$$A^4 = 25A^3 - 200A^2 + 500A.$$

Substitute the expression for A^3 on the right side and simplify

$$\begin{aligned} A^4 &= 25(25A^2 - 200A + 500I) - 200A^2 + 500A \\ &= 425A^2 - 4500A + 12500I \end{aligned}$$

There you go, A^4 in terms of I, A and A^2. To get a similar expression for A^5 multiply the last equation by A

$$A^5 = 425A^3 - 4500A^2 + 12500A.$$

Continue by substituting the expression for A^3 on the right side, etc. In this way, one by one, you express all powers of A in terms of I, A and A^2. The same is also possible for all negative powers.