MTH1030
Techniques for Modelling

Lecture 24

Series (part 3), Intro to Power series and Taylor series

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Warm welcoming words

Now we know a lot about series. We know it is defined as a limit of partial sums $S_1, S_2, S_3 \dots$. To determine whether convergence occurs, we can compute the N-th partial sum S_N. But if this is not possible, then we can instead use convergence tests to help us determine if the series converges or not!

Absolute convergence

A series can converge in a rather 'strong' manner, which is called absolute convergence.

Definition (Absolute convergence)

Consider the series

$$\sum_{n=1}^{\infty} a_n.$$

Then we say this series converges *absolutely* if the series

$$\sum_{n=1}^{\infty} |a_n|$$

converges.

Seems arbitrary...so why do we care about this?

Absolute convergence

Why is this convergence ‘strong’? Well first of all, absolute convergence implies convergence of the series.

Theorem

If a series converges absolutely, it must converge. Mathematically, if

$$\sum_{n=1}^{\infty} |a_n|$$

converges, then

$$\sum_{n=1}^{\infty} a_n$$

converges.

Absolute convergence

This theorem can be quite useful to determine convergence of a series!

Example

Consider the series

$$\sum_{n=1}^{\infty} \frac{\cos(n)}{n^2}.$$

It is hard to determine if this converges due to cosine being either positive or negative. But it turns out that it converges absolutely, which is easy to show!

Absolute convergence

The following example is surprising. Consider the so-called alternating Harmonic series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \cdots$$

This series does not converge absolutely (why?). But it does converge (which we won't prove!). The value of the series is in fact $\ln(2)$.

However...if we rearrange the terms on the RHS in the following way:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} \cdots$$

Now the value of the series is $\frac{1}{2}\ln(2)$. Weird!

Absolute convergence

The previous series converged but did not converge absolutely. Such a series is called *conditionally convergent*. In fact we get the very surprising theorem.

Theorem (Riemann series)

1. *A series which conditionally converges can have its elements rearranged in order to obtain any other value, or even in diverge.*
2. *The value of a series which absolutely converges will not change when its elements are rearranged.*

We won't prove this, but it is good to know! So absolutely convergent series behave 'more' like finite sums than conditionally convergent series.

Absolute convergence

So now we have the following facts:

- A series either: diverges, conditionally converges, or converges absolutely.
- A series which converges absolutely must also converge.
- A conditionally convergent series can have its terms rearranged to obtain any value or even diverge.
- An absolutely convergent series can have its elements rearranged and not change the value of the series.

Question 1

Question (1)

Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

The series

1. Converges absolutely.
2. Conditionally converges.
3. Diverges to positive or negative infinity.
4. Diverges and does not settle down.

Ratio test

To determine absolute convergence, you can just use the usual tests, but there are others! Probably the most useful one is the following:

Theorem (Ratio test)

Let a_n be a sequence. Let

$$\rho_n = \left| \frac{a_{n+1}}{a_n} \right|$$

and $\rho = \lim_{n\to\infty} \rho_n$. Then the series

$$\sum_{n=1}^{\infty} a_n$$

- *converges absolutely if $\rho < 1$.*
- *diverges if $\rho > 1$.*
- *inconclusive if $\rho = 1$.*

Ratio test

We can use the ratio test to prove something that we already knew...

Example

Consider the Geometric series

$$\sum_{n=0}^{\infty} ar^n$$

Then the sequence being summed is $b_n = ar^n$.

Ratio test

Example

Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{n!}.$$

We can apply the ratio test quite easily to this!

Ratio test

Example

Consider the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}.$$

Ratio test

Example

Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Question 2

Question (2)

Consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^3}{4^n}.$$

The series

1. Converges absolutely.
2. Conditionally converges.
3. Diverges to infinity.
4. Diverges and does not settle down.

Two parts of series

If we have the series

$$S = \lim_{N\to\infty} S_N = \sum_{n=1}^{\infty} a_n$$

we can decompose S into its 'positive part' and 'negative part'. How?

E.g., consider $a_n = \frac{(-1)^{n+1}}{n}$, then define a_n^+ and a_n^- by

$$a_1^+ = 1, \;\; a_2^+ = 0, \;\; a_3^+ = \frac{1}{3}, \;\; a_4^+ = 0, \ldots$$

and

$$a_1^- = 0, \;\; a_2^- = -\frac{1}{2}, \;\; a_3^- = 0, \;\; a_4^- = -\frac{1}{4} \ldots$$

Then $a_n = a_n^+ + a_n^-$.

Two parts of series

In general, this gives us the positive series and negative series

$$S^+ = \lim_{N\to\infty} S_N^+ = \sum_{n=1}^{\infty} a_n^+,$$

$$S^- = \lim_{N\to\infty} S_N^- = \sum_{n=1}^{\infty} a_n^-.$$

Then $S = S^+ + S^-$. This brings us to the following neat theorem.

Theorem (Two parts of series)

Let S^+ and S^- denote the positive and negative series. Then

1. *If S^+ and S^- both converge, then S converges absolutely.*
2. *If only one of S^+ or S^- converge, then S diverges.*
3. *If both S^+ and S^- diverge, then S either conditionally converges or diverges.*

Power series

Definition

Let a_n be a sequence. Then the function

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is called a *power series*.

Note it is traditional to start a power series at $n = 0$, so our sequence a_n is actually $\{a_n\}_{n=0}^{\infty}$.

Power series

So a power series is just a usual series, except we are summing up the sequence $b_n = a_n x^n$. Meaning

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} a_n x^n = f(x).$$

Notice that this means $f(0) = \sum_{n=0}^{\infty} a_n 0^n = a_0$ always!

Power series

For example, if we choose the sequence $a_n = 1/(n+1)^2$, then we have

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)^2}.$$

This is a function! For a fixed x, we get a different series. For example

- $f(0) = \sum_{n=0}^{\infty} \frac{0^n}{(n+1)^2} = \frac{0^0}{1^2} = 1.$
- $f(1) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{\pi^2}{6}.$
- $f(2) = \sum_{n=0}^{\infty} \frac{2^n}{(n+1)^2} = \infty.$

Power series

Okay so if we have a power series, it may not converge for all x. We will talk more about this later, but for now we will move onto something else.

Taylor series

A power series is a function. For different sequences a_n you get different power series. We could ask the reverse question: if we have some arbitrary function f, can we represent it as a power series? The answer is often yes.

Definition (Taylor series)

Let $f : I \to \mathbb{R}$ be an infinitely differentiable function at a point $a \in I$. The Taylor series of f centred at a is given by

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots$$

Why are Taylor series important? Well for most 'nice' functions we have that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Taylor series

Note that if a Taylor series is centred at $a = 0$, it is often called a *Maclaurin series*. But, often people just call this a Taylor series.

Precisely, the Maclaurin series of an infinitely differentiable function $f : I \to \mathbb{R}$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Taylor series

Terminology wise:

- Power series: Any series in the form $f(x) = \sum_{n=0}^{\infty} a_n x^n$.
- Taylor series: A power series associated with an arbitrary function, often a representation of it if it is a 'nice' function, specifically

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

- Maclaurin series: A Taylor series of an arbitrary function centred at $a = 0$. Specifically,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Taylor series

We have actually already encountered a Taylor series. If we have the function $f(x) = 1/(1-x)$ for $|x| < 1$, then we know its power series representation. It is the Geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

for $|x| < 1$.

In other words, the Taylor series for the function is its power series representation. What else is there?

Common Maclaurin series

- $$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

  for $x \in \mathbb{R}$.

- $$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

  for $x \in \mathbb{R}$.

- $$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

  for $x \in \mathbb{R}$.

Common Maclaurin series

- $$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$$

  for $x \in (-1, 1]$.

- $$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots$$

  for $x \in [-1, 1)$.

Common Maclaurin series

- $$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$

  for $|x| < 1$.

- $$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

  for $|x| < 1$.

- $$\frac{1}{(1-x)^3} = \sum_{n=2}^{\infty} \frac{1}{2} n(n-1)x^{n-2} = \frac{1}{2} + 3x + 6x^2 + 10x^3 + \cdots$$

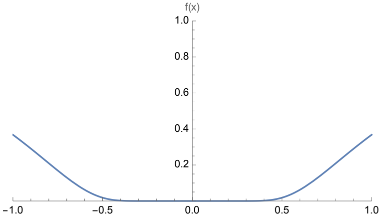
  for $|x| < 1$.

Taylor series

A Taylor series of a function does not always equal the function! For example,

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

It turns out that the Taylor series of this function centred at 0 (so Maclaurin series) is just 0 (specifically, $f^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots$).



Leaving time

A lot more on power series and Taylor series next time!