MTH1030
Techniques for Modelling

Lecture 36

Differential equations (part 4)

Monash University

Semester 1, 2022

Warm welcoming words

Today...solving linear second-order non-homogeneous DEs!

Second-order DEs

We now know how to find solutions to linear second-order homogeneous DEs

$$ay'' + by' + cy = 0,$$

where a, b, c are constants.

However we also would like to know how to solve DEs of the form

$$ay'' + by' + cy = S(x),$$

which are called linear second-order *non-homogeneous* DEs. How?

Second-order DEs

Surprisingly, for a non-homogeneous DE

$$ay'' + by' + cy = S(x),$$

it turns out that its homogeneous version will help us out!

Specifically, first look at the homogeneous version of it:

$$ay_h'' + by_h' + cy_h = 0.$$

Then the general solution to the non-homogeneous version is

$$y(x) = y_h(x) + y_p(x)$$

where $y_p(x)$ is called a *particular solution*, which is guessed to 'look like' $S(x)$ and solves

$$ay_p'' + by_p' + cy_p = S(x).$$

Second-order DEs

Confused? Same. Let's look at an example:

$$y'' + y' - 6y = 1 + 2x$$

Here $a = 1, b = 1, c = -6$ and $S(x) = 1 + 2x$. We first consider the homogeneous DE

$$y_h'' + y_h' - 6y_h = 0.$$

The characteristic equation is

$$\lambda^2 + \lambda - 6 = 0$$

with solutions

$$\begin{aligned} \lambda_1 &= -3, \\ \lambda_2 &= 2. \end{aligned}$$

So

$$y_h(x) = Ae^{-3x} + Be^{2x}.$$

Second-order DEs

How do we find the particular solution $y_p(x)$? Well it needs to 'look like' $S(x) = 1 + 2x$ and also solve

$$y_p'' + y_p' - 6y_p = 1 + 2x.$$

So we guess that $y_p(x) = e_0 + e_1 x$ and then solve for e_0, e_1. Sub this into the DE!

$$(0) + (e_1) - 6(e_0 + e_1 x) = 1 + 2x$$

which yields

$$-6e_0 + e_1 + (-6e_1)x = 1 + 2x.$$

Hence $e_1 = -1/3$ and $e_0 = -2/9$. Thus

$$y_p(x) = -\frac{2}{9} - \frac{1}{3}x$$

is the particular solution.

Second-order DEs

So the general solution to

$$y'' + y' - 6y = 1 + 2x,$$

is

$$\begin{aligned} y(x) &= y_h(x) + y_p(x) \\ &= Ae^{-3x} + Be^{2x} - \frac{2}{9} - \frac{1}{3}x, \end{aligned}$$

for any $A, B \in \mathbb{R}$.

Second-order DEs

We should prove that thing we stated before, i.e., the sum of the homogeneous and particular solution is the non-homogeneous solution.

Proposition

Consider the linear second-order non-homogeneous DE

$$ay'' + by' + cy = S(x).$$

Let $y_h(x)$ be the homogeneous solution, i.e., $y_h(x)$ solves

$$ay_h'' + by_h' + cy_h = 0.$$

Let $y_p(x)$ be the particular solution, i.e., $y_p(x)$ solves

$$ay_p'' + by_p' + cy_p = S(x).$$

The general solution is then $y(x) = y_h(x) + y_p(x)$.

Second-order DEs

So to solve a linear second-order non-homogeneous DE

$$ay'' + by' + cy = S(x),$$

do the following:

1. First find the homogeneous solution $y_h(x)$ which solves

$$ay_h'' + by_h' + cy_h = 0.$$

2. Then find a particular solution $y_p(x)$ which 'looks like' $S(x)$ and which solves

$$ay_p'' + by_p' + cy_p = S(x).$$

   The general solution is then $y(x) = y_h(x) + y_p(x)$.

Second-order DEs

Sounds good! So how do we guess the form of the particular solution $y_p(x)$? Well it has to 'look like' $S(x)$. Here are some choices depending on $S(x)$:

- $$S(x) = d_0 + d_1 x + d_2 x^2 + \cdots + d_n x^n,$$
  $$y_p(x) = e_0 + e_1 x + e_2 x^2 + \cdots + e_n x^n.$$

- $$S(x) = \left(d_0 + d_1 x + d_2 x^2 + \cdots + d_n x^n\right) e^{kx},$$
  $$y_p(x) = \left(e_0 + e_1 x + e_2 x^2 + \cdots + e_n x^n\right) e^{kx}.$$

- $$S(x) = \left(d_1 \sin(bx) + d_2 \cos(bx)\right) e^{kx},$$
  $$y_p(x) = \left(e_1 \cos(bx) + e_2 \sin(bx)\right) e^{kx}.$$

Second-order DEs

Now we just do examples.

Example

Consider the following linear second-order non-homogeneous DE

$$y'' + y' - 6y = e^{3x}.$$

Second-order DEs

This one looks similar.

Example

Consider the following linear second-order non-homogeneous DE

$$y'' + 3y' - 4y = e^{3x}.$$

Question 1

Question (1)

Consider the following linear second-order non-homogeneous DE

$$y'' - 2y' + y = \sin(2x) + \cos(2x).$$

A guess for the particular solution $y_p(x)$ would be:

1. $\cos(2x) + \sin(2x)$.
2. $e_1 \cos(2x) + e_2 \sin(2x)$.
3. $(e_1 \cos(2x) + e_2 \sin(2x))e^x$.
4. $e_1 \cos(x) + e_2 \sin(x)$.

Second-order DEs

Example

Consider the following linear second-order non-homogeneous DE

$$y'' - 2y' + y = \sin(2x) + \cos(2x).$$

Second-order DEs

This one is a bit peculiar!

Exercise

Consider the following linear second-order non-homogeneous DE

$$y'' + y' - 6y = e^{2x}.$$

Show that the general solution is

$$y(x) = Ae^{-3x} + Be^{2x} + \frac{e^{2x}}{25}(5x - 1).$$

Leaving time

Thank you!