MTH1030
Techniques for Modelling

Lecture 25 & 26

Taylor series and Power series (part 1)

Monash University

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Warm welcoming words

A power series is a series $f(x) = \sum_{n=0}^{\infty} a_n x^n$. For every fixed x you get a different series!

We learnt last time that sufficiently nice functions can be expressed as power series, and this representation is called its Taylor series.

Taylor series

If a function is 'nice', it is equal to its Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \cdots$$

In particular it is equal to its Maclaurin series ($a = 0$), meaning

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \cdots$$

So why would this be true?

Taylor series

Proof is easy. Assume f can be represented as a power series, so that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

We just need to convince ourselves that $a_n = f^{(n)}(0)/n!$.

Taylor series

Maclaurin series are usually the default Taylor series (they're the simplest!). So why would one centre the Taylor series at a point $a \neq 0$?

1. To obtain a good approximation to the function by 'truncating' the Taylor series.
2. To obtain a different interval of convergence.

Point 1. will be explained this lecture. Point 2. will be explained next lecture!

Taylor polynomials

To explain 1. we need the following definition:

Definition (Taylor polynomial)

Let $f : I \to \mathbb{R}$ be n times differentiable at a point $a \in I$. The Taylor polynomial of f centred at a of degree n is given by

$$\begin{aligned} S_n(x;a) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \cdots + \frac{1}{n!}f^{(n)}(a)(x-a)^n. \end{aligned}$$

We will write $S_n(x) \equiv S_n(x;0)$.

Hey this is just the n-th partial sum of the Taylor series! So, if all is nice, then $\lim_{n\to\infty} S_n(x;a) = f(x)$.

Taylor polynomials

So for example, if $f(x) = e^x$, then we have the Maclaurin series representation

$$e^x = 1 + x + \frac{x^2}{2} + \cdots$$

So this means

$$S_2(x) = 1 + x + \frac{x^2}{2}.$$

And thus apparently $S_2(x) \approx e^x$ when x is near 0.

What if we wanted to approximate e^x away from zero, maybe $a = 3$? Then we should use this

$$S_2(x; 3) = e^3 + e^3(x - 3) + \frac{e^3}{2}(x - 3)^2.$$

Taylor polynomials

Mathematica examples...

Taylor polynomials

So to gain an approximation to a function $f(x)$ you can approximate it by a Taylor polynomial $S_n(x; a)$. So how would you ensure the approximation is good? Well we only have control over the centring value a and degree n, so what should we do with them?

1. **Centring value a:** If you know you are are interested in say, $f(500.4)$, you should use a Taylor polynomial centred at let's say $a = 500$. So $S_2(500.4; 500)$ should be a good approximation to $f(500.4)$. $S_2(500.4; 0)$ would be very bad!!!
2. **Degree** n: e.g., S_{10} will be a better approximation than S_2.

Unfortunately point 2. is really quite costly. So in application, one often utilises a low-degree Taylor polynomial (2 or 3) centred at an optimal value.

Question 1

Question (1)

Consider the Maclaurin series of $\sin(x)$,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \cdots$$

Which of the following is its Taylor polynomial of degree 2, centred at $a = 0$?

1. x.
2. $x - \frac{x^3}{3!}$.
3. $x - \frac{x^3}{3!} + \frac{x^5}{5!}$.
4. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$.

Taylor series

So we sort of realised that using a high-order Taylor polynomial and/or centring at the right value gives a good approximation to $f(x)$. The error in our approximation of $f(x)$ by $S_n(x; a)$ is

$$R_n(x; a) = f(x) - S_n(x; a)$$

which is called the remainder.

If f equals its Taylor series, then the remainder goes to 0 as $n \to \infty$. Moreover, we expect that when a is near x, the remainder is smaller. But is this really true? Is there mathematical justification to this? Ans: yes.

Taylor series

Theorem (Taylor's theorem)

Let $f : I \to \mathbb{R}$ be $k+1$ times differentiable in an open interval around $a \in I$. Then there exists a real number $\xi \in (a, x)$ such that

$$R_n(x; a) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

In other words, we have

$$\begin{aligned} f(x) &= S_n(x; a) + R_n(x; a) \\ &= f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}. \end{aligned}$$

This theorem is only meant for MTH1035 students. Basically it tells you the error in the approximation of f by its Taylor polynomial of degree n, centred at a.

e is irrational

We have the Taylor series for e^x given by

$$e^x := \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$$

Well, what actually is e? We all know the definition:

$$e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n .$$

which naturally arises in finance, and gives you the value of your bank account if you continuously compound interest.

But the Maclaurin series of e^x gives us another way to define e, simply by plugging $x = 1$ into the Maclaurin series:

$$e := \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \cdots$$

e is irrational

We can use this to prove the following well known result.

Lemma

e is irrational. In other words, there does not exist integers a, b such that $e = a/b$.

Euler's formula

Taylor series have some unexpected applications too.

Theorem (Euler's formula)

Let $i = \sqrt{-1}$. Then

$$e^{ix} = \cos(x) + i\sin(x).$$

First of all, what actually is e^{ix}? We know e^x, but what happens when we put complex arguments? Well, we're going to *define* e^{ix} to be the power series:

$$e^{ix} := \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} \cdots .$$

De Moivre's theorem

And Euler's formula leads to the following theorem trivially!

Theorem (De Moivre's theorem)

Let $i = \sqrt{-1}$. *Then*

$$\cos(nx) + i\sin(nx) = (\cos(x) + i\sin(x))^n.$$

New Taylor series from old ones

We now know the Taylor series associated with the elementary functions. E.g., $e^x, \sin(x), \cos(x), \ln(1+x)$ and so on.

What if we want to look at something more complicated? How about

$$f(x) = e^{-x^2}?$$

Well to get the Taylor series you could differentiate it repeatedly, evaluate the derivative at 0 and hope to find a pattern...but there's a better way!

Question 2

Question (2)

The Taylor series of $1/(1+x^3)$ is

1. $$1 + x + x^2 + x^3 + \cdots$$

2. $$1 + x^3 + x^6 + x^9 + \cdots$$

3. $$1 - x^3 + x^6 - x^9 + \cdots$$

4. $$1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \cdots$$

New Taylor series from old ones

So this seems to suggest that if we have a function $f(x)$ with known Maclaurin series, and we want to find the Maclaurin series of $h(x) = f(g(x))$, that we just need to go

$$h(x) = f(g(x)) = f(0) + f'(0)g(x) + \frac{1}{2!}f''(0)g^2(x) + \cdots .$$

And this is pretty much correct but we have to be careful...

E.g., take $f(x) = 1/(1-x)$ and $g(x) = e^x$, then it is true that

$$\frac{1}{1-e^x} = 1 + e^x + e^{2x} + e^{3x} + \cdots$$

but what is the interval of convergence? In fact, how do you even determine the interval of convergence of a power series in general?

New Taylor series from old ones

What we're doing is haphazardly playing with power series...so we should really go back to power series and figure out how they actually work! But next time.