# Lines and planes

1. Consider the points $(1, 2, -1)$ and $(2, 0, 3)$.

   (a) Find a vector equation of the line through these points in parametric form.

   (b) Find the distance between this line and the point $(1, 0, 1)$.

   (c) Find the point on the line closest to the point $(1, 0, 1)$.

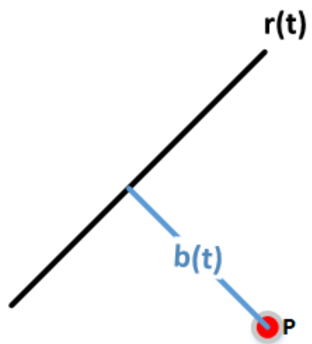
*Answer.*

(a) The vector equation is given by $\mathbf{r}(t) = \mathbf{v} + t\mathbf{d}$.
Let $\mathbf{v} = (1, 2, -1)$ or $\mathbf{v} = (2, 0, 3)$.
$\mathbf{d}$ is the direction of the line and is equal to $\mathbf{d} = (2, 0, 3) - (1, 2, -1) = (1, -2, 4)$.
Two possible solutions are

- $\mathbf{r}(t) = (1 + t, 2 - 2t, -1 + 4t)$
- $\mathbf{r}(t) = (2 + t, -2t, 3 + 4t)$

Remember, there are lots of possible answers.

(b) It will be easier to answer this question if we visualise it using the following picture.




We define the point $P = (1, 0, 1)$ and $b(t)$ as the line between $r(t)$ and the point $P$.

Let $\mathbf{b}(t) = \mathbf{r}(t) - (1, 0, 1)$. Our goal is to minimise the distance function $\mathbf{b}(t)$. The vector method to do this is to recognise that the shortest distance between a line and a point hits the line at a right angle. Hence $\mathbf{b}(t)$ is perpendicular to the direction $\mathbf{d}$ of line $\mathbf{r}(t)$. Using the dot product and making it equal zero ($\mathbf{a}\cdot\mathbf{b} = 0$ if the vectors are at right angles to each other),

$$\begin{aligned}
\mathbf{b}(t)\cdot\mathbf{d} &= 0,\\
(1+t, -2t, 2+4t)\cdot(1,-2,4) &= 0,\\
1+t+4t+8+16t &= 0,\\
21t &= -9,\\
t &= \frac{-3}{7},\\
\mathbf{b}\left(\frac{-3}{7}\right) &= \left(\frac{4}{7}, \frac{6}{7}, \frac{2}{7}\right),\\
\|\mathbf{b}\left(\frac{-3}{7}\right)\| &= \frac{\sqrt{4^2+6^2+2^2}}{7} = \frac{2\sqrt{14}}{7} \approx 1.06904.
\end{aligned}$$

A common mistake here is to dot product $\mathbf{b}(t)$ with $\mathbf{r}(t)$ instead of the direction of the line $\mathbf{d}$. The function $\mathbf{r}(t)$ is a vector from the origin to some point on the line whilst $\mathbf{d}$ is a vector from one point on the line to another point on the line.

(c) The point on the line closest to the point $(1,0,1)$ has the same $t$ value derived in part b).
$\mathbf{r}\left(\frac{-3}{7}\right) = \left(\frac{11}{7}, \frac{6}{7}, \frac{9}{7}\right)$

2. Find an equation of the plane that passes through the points $(1,2,-1)$, $(2,0,-1)$ and $(-1,-1,0)$.

*Answer.* Let $A = (1,2,-1)$, $B = (2,0,-1)$, $C = (-1,-1,0)$.
Let $\mathbf{v} = A - B = (-1,2,0)$
Let $\mathbf{u} = A - C = (2,2,-1)$
The normal vector $\mathbf{N}$ is calculated by the cross product, $\mathbf{N} = \mathbf{v}\times\mathbf{u}$,

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 2 & 0\\ 2 & 3 & -1 \end{vmatrix} = (-2-0, -(1-0), -3-4)$$
$$\mathbf{N} = (-2,-1,-7)$$

Vector equation of a plane is $(\mathbf{r}-\mathbf{r}_0)\cdot\mathbf{N} = 0$, where $\mathbf{r} = (x,y,z)$, $\mathbf{r}_0$ is some point on the plane. Let $\mathbf{r}_0 = C$.

$$\begin{aligned}
(x+1, y+1, z)\cdot(-2,-1,-7) &= 0\\
-2x-2-y-1-7z &= 0\\
2x+y+7z &= -3
\end{aligned}$$

3. Consider a plane defined by the equation $3x+4y-z = 2$ and a line defined by the following vector equation (in parametric form) $(2-2t, -1+3t, -t)$.

(a) Find the point where the line intersects the plane.

(b) Find a normal vector to the plane.

(c) Find the angle at which the line intersects the plane.

*Answer.*

(a) The intersection occurs when the equation of the line is equal to the equation of the plane. Writing out all the equations we have gives,

$$\begin{aligned} x &= 2 - 2t, && (1)\\ y &= -1 + 3t, && (2)\\ z &= -t, && (3)\\ 2 &= 3x + 4y - z. && (4) \end{aligned}$$

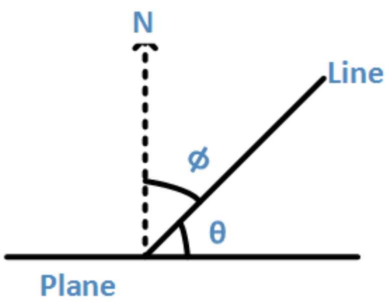
Substitution of (1), (2) and (3) into (4) yields

$$\begin{aligned} 6 - 6t - 4 + 12t + t &= 2,\\ 7t &= 0,\\ t &= 0. \end{aligned}$$

Therefore substitution of $t = 0$ into the line yields the answer $(2, -1, 0)$.

(b) The normal vector is simply the coefficients of $x$, $y$, $z$ in the equation of the plane. Therefore $\mathbf{N} = (3, 4, -1)$.

(c) Let $\theta$ be the angle between the line and the plane, and $\phi$ be the angle between the line and the normal to the plane as shown.



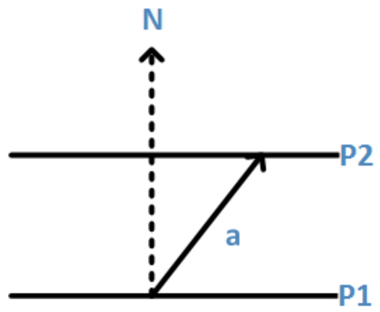

Finding the angle $\theta$ directly is a bit tricky, so it is easier to find the angle $\phi$, then calculate $\theta = \frac{\pi}{2} - \phi$.

From the dot product,

$$\phi = \arccos\left(\frac{\mathbf{N}\cdot\mathbf{d}}{\|\mathbf{N}\|\|\mathbf{d}\|}\right)$$
$$\mathbf{N}\cdot\mathbf{d} = -6+12+1 = 7$$
$$\|\mathbf{N}\| = \sqrt{2^2+4^2+1^2} = \sqrt{26}$$
$$\|\mathbf{d}\| = \sqrt{2^2+3^2+1^2} = \sqrt{14}$$
$$\phi = \arccos\left(\frac{7}{\sqrt{26}\sqrt{14}}\right)$$
$$\theta = \frac{\pi}{2} - \phi$$
$$\theta = \frac{\pi}{2} - \arccos\left(\frac{\sqrt{91}}{26}\right) \approx 0.37567 \text{ radians}$$

4. Find the distance between the parallel planes defined by the equations $2x-y+3z=-4$ and $2x-y+3z=24$.

   *Answer.* By now you may see that it is useful to draw pictures for these types of problems. For this particular problem, we have,




   Our goal is to modify the length of the normal vector $\mathbf{N}$ such that it is exactly the distance between the two planes.
   The normal vector for both planes is $\mathbf{N} = (2,-1,3)$. Create the vector $\mathbf{a}$ which joins a point on one plane to another point on the other plane. Letting $y=0$, $z=0$, in both equations gives the two points $P_1 = (-2,0,0)$ and $P_2 = (12,0,0)$. Therefore $\mathbf{a} = P_2 - P_1 = (14,0,0)$.
   The distance between the planes is calculated by the scalar projection of vector $\mathbf{a}$ onto vector $\mathbf{N}$.

$$\frac{\mathbf{a}\cdot\mathbf{N}}{\|\mathbf{N}\|} = \frac{28}{\sqrt{2^2+1^2+3^2}} = 2\sqrt{14} \approx 7.48331.$$

5. Consider two planes defined by the equations $3x+4y-z=2$ and $-2x+y+2z=6$.

   (a) Find where the planes intersect the $x, y$ and $z$ axes.

   (b) Find normal vectors for the planes.

(c) Find an equation of the line defined by the intersection of these planes.

(d) Find the angle between these two planes.

*Answer.*

(a) This is equivalent in asking to find the $x$, $y$, and $z$ intercept of the planes. To find the $x$ intercept, let $y = 0$ and $z = 0$. A similar approach is taken for the other intercepts.
The answers are $(2/3, 0, 0)$, $(0, 1/2, 0)$, $(0, 0, -2)$ and $(-3, 0, 0)$, $(0, 6, 0)$, $(0, 0, 3)$

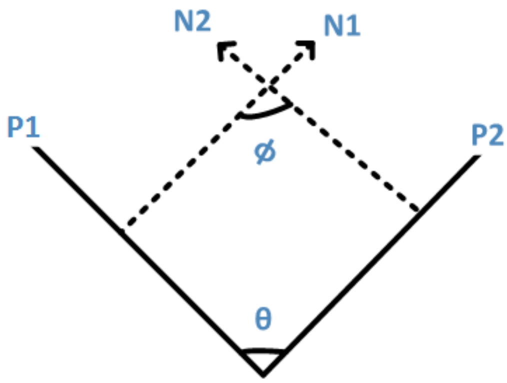
(b) The normal vector is simply the vector of coefficients of $x$, $y$, and $z$.
The answers are $\mathbf{N}_1 = (3, 4, -1)$ and $\mathbf{N}_2 = (-2, 1, 2)$

(c) The direction of this line is in the same direction as the normal to the two normals of the planes, $\mathbf{N} = \mathbf{N}_1 \times \mathbf{N}_2$.

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & -1 \\ -2 & 1 & 2 \end{vmatrix} = (9, -4, 11)$$

To find a point that intersects the two planes, first let $z = 0$, then solve the two equations simultaneously for $x$ and $y$. This gives the point $(-2, 2, 0)$.
The answer is $\mathbf{r}(t) = (-2, 2, 0) + t(9, -4, 11) = (-2 + 9t, 2 - 4t, 11t)$.

(d) Let $\theta$ be the angle between the two planes and $\phi$ the angle between the two normals as shown.




As the intersection of the planes and normals make a quadrilateral, the sum of all the angles add up to $2\pi$. Therefore $\theta = \pi - \phi$. Using the dot product to calculate $\phi$,

$$\phi = \arccos\left(\frac{\mathbf{N}_1 \cdot \mathbf{N}_2}{\|\mathbf{N}_1\|\|\mathbf{N}_2\|}\right),$$

$$\phi = \arccos\left(\frac{-6 + 4 - 2}{\sqrt{3^2 + 4^2 = 1^2}\sqrt{2^2 + 2^2 + 1}}\right),$$

$$\phi = \arccos\left(\frac{-2\sqrt{26}}{39}\right) \approx 1.835 \text{ radians},$$

$$\theta = \pi - \theta \approx 1.306.$$

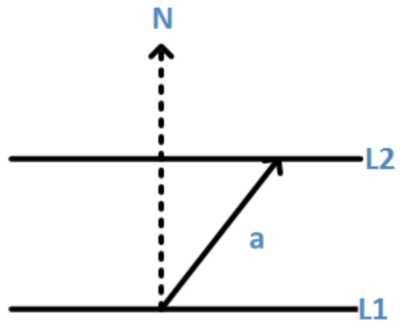
6. Here are the equations of two lines $(1+t, 1-3t, 2+2t)$ and $(3s, 1-2s, 2-s)$.

   (a) Find the distance $d$ between the two lines.

   (b) Find the uniquely determined points on the two lines that are this distance $d$ apart.

*Answer.*

(a) The distance $d$ between the two lines will be along the normal $\mathbf{N}$ to both lines like so.




Let $\mathbf{N} = \mathbf{d}_1 \times \mathbf{d}_2$, where $\mathbf{d}_1$ and $\mathbf{d}_2$ are the direction vectors for the two lines.

$$\mathbf{N} = (1, -3, 2) \times (3, -2, -1)\mathbf{N} \qquad = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 3 & -2 & -1 \end{vmatrix}$$

$$\mathbf{N} = (7, 7, 7)$$

We need to scale the normal vector by using scalar projection to get the distance $d$. Let $\mathbf{a}$ be a vector joining a point on line 1 to a point on line 2. Let $\mathbf{a} = (1, 1, 2) - (0, 1, 2) = (1, 0, 0)$.

$$\frac{\mathbf{a} \cdot \mathbf{N}}{\|\mathbf{N}\|} = \frac{7}{7\sqrt{3}} = \frac{1}{\sqrt{3}}$$

The distance $d$ between the two lines is equal to $\frac{1}{\sqrt{3}} \approx 0.57735$.

(b) To find the two points that are a distance $d$ from each other, we try to solve the equation,

$$\mathbf{r}_1(t) = \mathbf{r}_2(s) + d\frac{\mathbf{N}}{\|\mathbf{N}\|}$$

We can expand this into 3 equations, one for $x$, one for $y$, and one for $z$, then solve these 3 equations simultaneously. Note that we have two unknowns $s$ and $t$, yet 3 equations. Hence we would use the first 2 equations to solve for $s$ and $t$, then use the third equation to check that it works for all equations. If it does, awesome, if not, it could mean our normal is pointing in the wrong direction, so

multiply $\mathbf{N}$ by a negative, and solve the equations again.
The three equations to solve are,

$$1+t = 3s + \frac{1}{3}$$
$$1-3t = 1-2s+\frac{1}{3}$$
$$2+2t = 2-s+\frac{1}{3}$$

Solving the first two equations gives $t=\frac{1}{21}$ and $s=\frac{5}{21}$. If you substitute these into the third equation, you should get the same values on both sides. Substitution of $t=\frac{1}{21}$ and $s=\frac{5}{21}$ into $\mathbf{r}_1(t)$ and $\mathbf{r}_2(s)$ gives the two solutions,

$$(22/21, 6/7, 44/21) \approx (1.04762, 0.857143, 2.09524) \ (t \text{ line})$$

and

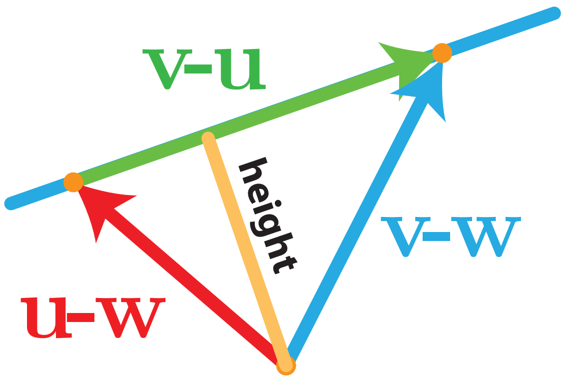
$$(5/7, 11/21, 37/21) \approx (0.71427, 0.52382, 1.76191) \ (s \text{ line})$$

7. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three points/vectors in $\mathbf{R}^3$. Prove that the distance between the line through $\mathbf{u}$ and $\mathbf{v}$ to the point $\mathbf{w}$ is

$$\frac{|(\mathbf{u}-\mathbf{w}) \times (\mathbf{v}-\mathbf{w})|}{|\mathbf{v}-\mathbf{u}|}.$$

(Hint: What does the numerator and denominator of this fraction mean geometrically?)

*Prove.* Here is the picture to keep in mind.




The numerator is equal to two times the area of the triangle and the denominator is equal to the length of the green vector = the green base of the triangle. We are interested in the orange height of the triangle. Then the area formula for triangles applied to the triangle in the picture is

$$\text{area triangle} = \frac{1}{2} \text{ green base } \times \text{ orange height}$$

This translates into

$$\frac{1}{2}|(\mathbf{u}-\mathbf{w}) \times (\mathbf{v}-\mathbf{w})| = \frac{1}{2}|\mathbf{v}-\mathbf{u}| \cdot \text{distance point-line}$$

Solving for the distance gives the formula. ■

8. Calculate the distance in 1b) one more time using calculus. So, if $\mathbf{r}(t)$ is the equation of the line and $\mathbf{v}$ is the point in question, calculate the minimum of the function

$$|\mathbf{r}(t) - \mathbf{v}|^2$$

in the variable $t$ using the usual calculus tricks.

*Answer.* Line equation minus point:

$$(1 + t, 2 - 2t, -1 + 4t) - (1, 0, 1) = (t, 2 - 2t, -2 + 4t)$$

This means we want to minimize the function

$$f(t) = t^2 + (2 - 2t)^2 + (-2 + 4t)^2 = 8 - 24t + 21t^2$$

Then

$$f'(t) = -24 + 42t.$$

Solving $-24 + 42t = 0$ we get $t = 4/7$. Then the distance is

$$\sqrt{8 - 24(4/7) + 21(4/7)^2} = \text{ same as under 1 b.}$$

**SOME TEST QUESTIONS**

9. What does it mean for a coordinate system to be right-handed?

10. Explain *based only* on the geometric definition of the vector product why for any number $a$ and vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

$$(a\mathbf{u}) \times \mathbf{v} = a \cdot (\mathbf{u} \times \mathbf{v})$$

$$\mathbf{u} \times \mathbf{u} = \mathbf{0}$$

11. Given three points, is it possible that there is more than one plane that contains all three of them?

12. Given four random points in space what are the chances that all four are contained in: (a) a line, (b) a plane?

13. Two solid cubes are hovering in space. How is the distance between them defined?

14. A line intersects a plane. How is the angle between them defined?

15. Two planes intersect. What is the angle between them?

16. Explain how you would find the distance in $\mathbf{R}^3$ between: (a) two lines, (b) a point and a line, (c) two parallel planes, (d) a point and a plane? (Without using a ready made formula!)

17. Using the formula $\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$ show that the scalar projection of the vector $\mathbf{v}$ onto the vector $\mathbf{u}$ is $\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|}$. What is its vector projection?

18. How can you use the direction vectors of lines and the normal vectors of planes to quickly determine whether: (a) two planes are parallel; (b) a line intersects a plane; (c) a line is parallel to a plane?

19. Looking at the equation of a plane how do you tell at a glance: (a) whether the plane is parallel to one of the coordinate planes; (b) passes through the origin $(0, 0, 0)$?

20. Does it make sense to speak of *the* vector equation of a line or of *the* equation of a plane?