MTH1030/35: Assignment 2, 2022

Applications of matrices

Due online Thursday, 28 April, 11:55 pm

The Rules of the Game

Before starting to work on this assignment, please make sure that you understand the instructions on Moodle that go with our assignments (Assignment 1 (=Assignment 2) rules and F.A.Q. plus Academic integrity, plagiarism, and collusion policy).

Your submission for this assignment should be ONE pdf file.

This assignment is worth 100 marks.

Now, let's have some fun!

1 Funny numbers (25 Marks)

Let's call a 2×2 matrix a *funny number* if it is of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

a) Show that: i) the identity matrix is a funny number; ii) sums and products of funny numbers are funny numbers; iii) all funny numbers except for one (which one?) have an inverse; iv) multiplication of funny numbers is commutative (that is, given any two funny numbers A and B, it is always true that $AB = BA$). **(10 Marks)**

Together with the fact that matrix multiplication and addition are associative, distributive, and so on, this means that funny numbers really do behave like real numbers. This means, that you can do linear algebra with funny numbers instead of real numbers. Let's do that in the next question.

b) What is the solution of the following system of two linear equations with funny number coefficients in the funny number unknowns X and Y?

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} X + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hint: Start by setting $X = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ and $Y = \begin{pmatrix} v & -w \\ w & v \end{pmatrix}$, and then find v, w, x, y such that the above equations are satisfied. **(5 Marks)**

There are also *funny funny numbers*. These are all funny number 2×2 matrices of the following form:

$$\left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \right)$$
$$\left(\begin{pmatrix} -c & d \\ -d & -c \end{pmatrix} \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right)$$

c) Which funny funny number plays the role of the number 1? That is, which funny funny number I has the property that $IF = FI = F$ for all funny funny numbers F. What is the determinant of a funny funny number. Which funny funny numbers have an inverse? **(5 Marks)**

d) Find two funny funny numbers A and B such that $AB \neq BA$.

What is all this good for? Well, believe it or not, but these funny and funny funny numbers are some of the most important objects in the whole of mathematics, just heavily disguised.

2   What's next? (25 Marks)

0, 1, 5, 19, 65, 211,....

are the first few elements of an infinite sequence of numbers that satisfies the following *recurrence relation*:
$$x_n = 5x_{n-1} - 6x_{n-2},$$
where $x_0 = 0, x_1 = 1, x_2 = 5, x_3 = 19$, and so on.

This means that every element of the sequence is equal to 5 times the previous element minus 6 times the element before that. So, $5 = 5 \cdot 1 - 6 \cdot 0, 19 = 5 \cdot 5 - 6 \cdot 1$, etc. Using this recurrence relation over and over, we can generate as many of the elements of this sequence as we wish. Of course, calculating the millionth element in this way would take quite a while.

Wouldn't it be nice if, instead, we could find a formula for x_n, that is, express x_n as some explicit function in the variable n?

Using some linear algebra this is indeed possible (we did this for the Fibonacci sequence in the lectures). Here is what you have to do to find such a magical formula.

a) Find a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that for all natural numbers $n \geq 2$

$$\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = M \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix}.$$

Consequently, for all natural number $n \geq 2$

$$\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = M^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This means that you can also use powers of the matrix M to calculate the elements of the sequence.x **(5 Marks)**

b) Diagonalize M, i.e., write M in the form $M = NDN^{-1}$, where D is a diagonal matrix. To do this, first find the eigenvalues e and f of M, as well as two corresponding eigenvectors (it does not matter which). Then we know that the following matrices will do the trick:

$$D = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix},$$

$N =$ the 2×2 matrix the first and second columns of which are the eigenvectors of e and f, respectively. **(10 Marks)**

c) Using b) derive a formula for x_n. **(10 Marks)**

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3 Matrices of Graphs (50 marks, 6 marks for each of the seven questions plus two bonus points for really smart reasoning.)
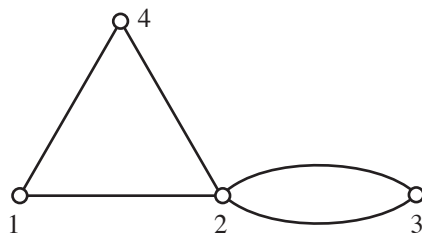
This is an example of a *graph*. It consists of four *vertices* (the four circles), labeled 1, 2, 3, and 4 and a number of *edges* connecting these vertices. Its *adjacency matrix A* is the following 4×4 matrix.

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

The number in the i'th row and j'th column is equal to the number of edges connecting vertex i with vertex j. For example, the 2s stand for the two edges connecting vertex 2 with 3 (or, equivalently, vertex 3 with vertex 2).

If n is any natural number, an n-step walk connecting two vertices is a journey from one of the vertices to the other that includes exactly n edges; here an edge counts as many times as you travel along it. In particular, this means that we could also call the matrix A 'the matrix which summarizes the numbers of 1-step walks between the different vertices of the graph.'

One neat result in graph theory says that, in general, the number of n-step walks between the different vertices of the graph is summarized by the nth power of the matrix A. For example, if we are looking for the numbers of 3-step walks for our graph we'd be looking at the matrix

$$A^3 = \begin{pmatrix} 2 & 7 & 2 & 3 \\ 7 & 2 & 12 & 7 \\ 2 & 12 & 0 & 2 \\ 3 & 7 & 2 & 2 \end{pmatrix}.$$

So, the 2 in the upper left corner of this matrix means that there are exactly two 3-step walks that start and end at vertex 1. These two walks consist of traveling clockwise and counterclockwise around the triangle visible in our graph.

In the following, let G be a graph in which two vertices are connected by at most one edge[1]

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[1] unlike our sample graph in which vertex 2 and vertex 3 are connected by two edges.

and no edge starts and ends at the same vertex. This means that the entries of its adjacency matrix are all either 0 or 1, and that all the entries on the main diagonal of A are 0.

a) The trace of a matrix, $tr(A)$, is the sum of all the entries on the main diagonal. Explain why

- $tr(A) = 0$;

- $tr(A^2) = 2 \times$ the number of edges of the graph;

- $tr(A^3) = 6 \times$ the number of triangles in the graph. Here a *triangle* in the graph consists of three vertices and three edges connecting any two of these vertices.

b) Our graph can be *two-coloured* if it is possible to colour its vertices with two colours such that any two vertices that are connected by an edge have different colours. Show that a graph cannot be two-coloured if the trace of any of the odd powers of A differs from 0.

In fact, it turns out that the converse is also true, that is, if the traces of all odd powers of A are 0, then the graph can be two-coloured. You don't need to show this.[2]

c) A graph is *connected* if it is possible to walk along the edges from any vertex to any other vertex inside the graph. If this is not possible the graph is called *disconnected*. The *distance* between two vertices inside a graph is the length of the shortest walk between them. The *diameter* of a graph is the maximum distance between two of its vertices.[3] For example, our sample graph is connected and its diameter is 2.

Let $P_1 = A$, $P_2 = A + A^2$, $P_3 = A + A^2 + A^3$ and in general

$$P_m = A + A^2 + A^3 + A^4 + \cdots + A^m.$$

How would you measure the distance between two given vertices and the diameter of a graph using the matrices P_m? How could you decide, again using these matrices, whether or not a graph is connected or disconnected?

d) Here is an adjacency matrix of a mystery graph. Use the results in a), b) and c) to answer the following questions: How many triangles does this graph have? Is the graph connected?

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[2]On the other hand, you definitely get the bonus point if you manage to find a proof of this last fact.

[3]For disconnected graphs the diameter is not defined. Similarly, it does not make sense to speak of the distance between vertices in different components of a disconnected graph.

What is its diameter? Is it two-colourable?

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

You should use *Mathematica* for these calculations. But just in case you cannot bring yourself to doing that you can also use online tools like the WIMS matrix calculator at `https://tinyurl.com/55nvf2br` to calculate powers of A, the matrices P_m, and in general analyze matrices to your heart's content. Again, it is a good idea to prepare your matrix in a word processor and then cut and paste it into the calculator. This way you can use it over and over.

e) Note that the entries in all the rows and columns of this matrix add up to 3. What does this mean for the graph? When you calculate the eigenvalues of this matrix (again using the online calculator) you will find that 3 is also among these eigenvectors. Show that this is not a coincidence, i.e., show that if all the rows of a matrix add up to the same number k, then k is an eigenvalue of this matrix. Describe one possible eigenvector corresponding to this special eigenvalue.

f) Calculating the matrices

$$P_m = A + A^2 + A^3 + A^4 + \cdots + A^m$$

is "expensive" if m is a large number, when cost is measured by the numbers of calculations necessary. If you replace the matrix A by a number a, this sum turns into

$$a + a^2 + a^3 + a^4 + \cdots + a^m$$

and this sum is known to be equal to

$$\frac{a - a^{m+1}}{1 - a}.$$

For example, check out "Geometric series" on Wikipedia to learn how this formula is derived. What would the corresponding formula for $A + A^2 + A^3 + A^4 + \cdots + A^m$ look like and for which matrices A would it be valid? Would it be valid for the 8×8 matrix of our mystery graph?

What is all this good for? Well, it's all really beautiful mathematics. But apart from that graphs are also very applicable. For example, any maze is really a graph with the junctions at which paths meet playing the part of the vertices and the paths themselves playing the

part of the edges. Using our tools it is possible to figure out whether a maze actually has a solution and how complex it is (using the graph distance between start and finish as a measure for complexity.) The world wide web can be considered as a huge graph with the individual webpages playing the role of the vertices and the links between the pages playing the role of the edges. A special eigenvector of the corresponding adjacency matrix is calculated once a week by Google to rank the webpages of the entire net.

Our sample graph actually played an important role in the movie *Good Will Hunting* as I already discussed in the lectures. Also have a look at Oliver Knill's page (google "knill good will hunting problem") for some screenshots, clips from the movie and some more neat linear algebra that pops up in the movie.