MTH1030
Techniques for Modelling

Lecture 21

Series (part 1)

Monash University

Semester 1, 2022

Warm welcoming words

Giving rigorous meaning to the concept of an ‘infinite sum’ in mathematics is not straightforward! We can define an object which acts like how we believe an infinite sum should (mostly), and this is called an infinite series.

Finite sums

Consider the following. If we have a sequence of real numbers $a_1, a_2, \ldots,$ then it makes sense to do the following:

- $a_1 + a_2$.
- $a_1 + a_2 + a_3$.
- $a_1 + a_2 + a_3 + a_4$.
- $a_1 + a_2 + a_3 + a_4 + a_5$.

And so forth!

Finite sums

So it makes perfect sense to add together finitely many values. In fact we will write

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n$$

which is the sum of the first n terms in the sequence a_n.

Naive notion of infinite sum

What if we want to add up infinitely many elements of $a_1, a_2, \ldots$? How would we do that?

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + \cdots$$

If we apply the same procedure that we did for finite sums, then the process here will never terminate! Hence our naive notion of infinite sum doesn't make sense!

Infinite series

It turns out that an infinite sum would be very useful in mathematics. So we would like an object which serves the purpose of an infinite sum. Can we define one? Of course.

Definition (Infinite series)

Let a_n be a sequence. Define another sequence

$$S_N = \sum_{n=1}^{N} a_n$$

which is called the N-th *partial sum* of a_n. Then the infinite series of a_n is defined as

$$\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N$$

provided this limit exists.

If the infinite series exists, we say that it converges. If it does not exist, we say it diverges (same terminology as sequences!).

Infinite series

A note on terminology:

- Although an infinite series of a_n is written using sum notation, and 'models' the notion of an infinite sum, it is not actually a sum. It is a limit of a sequence! Precisely, it is the limit of the sequence of partial sums of a_n.
- Sometimes we will just call an infinite series a 'series'.

Remark

Confusingly, it has become common to actually call infinite series as 'infinite sums'. So from now on if I ever say infinite sum, I actually mean infinite series. There should be no confusion here, as the naive notion of infinite sum never made sense in the first place.

Infinite series

Example

Consider the sequence $a_n = n$. We want to find its infinite series

$$\sum_{n=1}^{\infty} n$$

(if it exists!).

Question 1

Question (1)

What does the infinite series

$$\sum_{n=1}^{\infty} (-1)^n$$

converge to? Or does it not converge? Write 'DNC' if you think it does not converge.

Geometric series

Possibly the first infinite series people learn about is the Geometric series, which is defined as

$$\sum_{n=0}^{\infty} ar^n$$

for some $r \in \mathbb{R}$ and $a \in \mathbb{R}\backslash\{0\}$. So the sequence being summed is $b_n = ar^n$.

Theorem

The Geometric series converges for $|r| < 1$, in which case its value is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

and diverges for $|r| \geq 1$.

Infinite series

Before we move on let me plug two of the Mathologer's (aka Burkard's) videos.

$9.999999\ldots = 10.$

$1 + 2 + 3 + 4 + 5 + \cdots = -1/12?$

Infinite series

Obviously we want to know whether our series converges or diverges. The FIRST THING you should when determining convergence of your series is appeal to the following theorem.

Theorem (n-th term test)

Let a_n be a sequence. Suppose

$$\lim_{n\to\infty} a_n \neq 0$$

or does not exist. Then $\sum_{n=1}^{\infty} a_n$ does not exist.

Sometimes the above theorem is equivalently stated as: Suppose that

$$\sum_{n=1}^{\infty} a_n$$

exists. Then $\lim_{n\to\infty} a_n = 0$. The fact these two theorems are the same is due to the so-called *contrapositive*.

Infinite series

So the first thing you should do when checking if your series converges is to check whether the sequence being summed (say a_n) converges to 0. If a_n converges to ANYTHING but 0, then $\sum_{n=1}^{\infty} a_n$ diverges!

Example

Consider the following series:

1. $\sum_{n=1}^{\infty}(-1)^n$
2. $\sum_{n=1}^{\infty} n$
3. $\sum_{n=1}^{\infty} \frac{1}{\pi^n}$
4. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Let's use the nth term test to determine whether they diverge, and also which *could* possibly converge.

Question 2

Question (2)

Does the infinite series

$$\sum_{n=1}^{\infty} \frac{n}{e^n}$$

pass the *n*-th term test?

Infinite series

The *n*-th term test gives us a *necessary* but not *sufficient* condition for convergence of an infinite series.

Precisely, for $\sum_{n=1}^{\infty} a_n$ to converge, it is necessary that $\lim_{n\to\infty} a_n = 0$, but this is not sufficient!

Infinite series

For example if we look at the series

$$\sum_{n=1}^{\infty} \frac{1}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

then these both pass the n-th term test.

However,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

yet

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

More on this next time...