MTH1030
Techniques for Modelling

Lecture 30

Integration (part 2)

Monash University

Semester 1, 2022

Warm welcoming words

Today we will look at improper integration. That is all.

Integration

A Riemann integral of a function $f : I \to \mathbb{R}$ is only defined for bounded and piecewise continuous functions, and only for when the interval of integration is finite.

So what does

$$\int_0^1 \frac{1}{x^2} \mathrm{d}x$$

mean?

And what does

$$\int_0^\infty e^{-x} \mathrm{d}x$$

mean?

Improper integral

The previous integrals are called *improper integrals*.

Improper integral

The various types of improper integrals include:

- Improper integral to infinity:

$$\int_a^\infty f(x)\mathrm{d}x := \lim_{b\to\infty} \int_a^b f(x)\mathrm{d}x.$$

- Improper integral to negative infinity:

$$\int_{-\infty}^a f(x)\mathrm{d}x := \lim_{b\to-\infty} \int_b^a f(x)\mathrm{d}x.$$

- Improper integral to a point c where f becomes unbounded:

$$\int_a^c f(x)\mathrm{d}x := \lim_{b\uparrow c} \int_a^b f(x)\mathrm{d}x.$$

- Improper integral to a point a where f becomes unbounded:

$$\int_a^c f(x)\mathrm{d}x := \lim_{b\downarrow a} \int_b^c f(x)\mathrm{d}x.$$

Improper integral

A couple of notes:

1. An improper integral is not technically a Riemann integral! It is a limit of a Riemann integral.
2. Since they are limits, improper integrals can converge or diverge.
3. It's really easy to calculate them...

Improper integral

Improper integration is best illustrated with examples!

Example

Consider

$$\int_0^1 \frac{1}{\sqrt{x}} \mathrm{d}x.$$

This is an improper integral (why?).

Improper integral

Example

Consider

$$\int_0^\infty e^{-3x} \mathrm{d}x.$$

Improper integral

Example

Consider

$$\int_0^\infty \cos(x)\mathrm{d}x.$$

Improper integral

Great. So what would the integral

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x$$

mean?

Improper integral

Example

This one is rather dubious.

$$\int_{-\infty}^{\infty} x \mathrm{d}x.$$

Improper integral

Example

Is this familiar?

$$\int_{-\infty}^{\infty} xe^{-x^2}\,\mathrm{d}x.$$

Question 1

Question (1)

The improper integral

$$\int_1^\infty xe^{-x^2} \mathrm{d}x$$

is equal to:

1. ∞.
2. $\frac{1}{2e}$.
3. $2e$.
4. 1.

Improper integral

Exercise

We have that

$$\int_0^1 \frac{1}{x^p} \mathrm{d}x = \infty$$

if $p \in [1, \infty)$ and converges if $p \in (0, 1)$. Also,

$$\int_1^\infty \frac{1}{x^p} \mathrm{d}x = \infty$$

if $p \in (0, 1)$ and converges if $p \in [1, \infty)$.

We used this fact to prove the convergence and divergence of p-series.

Done

See ya.