Formulae Sheet for MTH1030/MTH1035

Vectors

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$.

Length	$\lvert\mathbf{u}\rvert$	$=$	$\sqrt{u_1^2 + u_2^2 + \ldots + u_n^2}$
Dot product	$\mathbf{u} \cdot \mathbf{v}$	$=$	$u_1v_1 + u_2v_2 + \ldots + u_nv_n = \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\cos\theta$
Cross product ($n = 3$)	$\mathbf{u} \times \mathbf{v}$	$=$	$(u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)$
Scalar projection			$\frac{\mathbf{u}\cdot\mathbf{v}}{\lvert\mathbf{u}\rvert}$
Vector projection	$\mathrm{proj}_{\mathbf{u}}(\mathbf{v})$	$=$	$\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$
Cauchy-Schwarz inequality	$\lvert\mathbf{u} \cdot \mathbf{v}\rvert$	$\leq$	$\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert$ (with equality iff ....)

Matrices

Not so obvious matrix and determinant identities

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$(AB)^T = B^T A^T$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$(AB)C = A(BC)$$

$$\det(AB) = \det(A)\det(B)$$

$$\det(A^T) = \det(A)$$

Calculating determinants of 2×2 and 3×3 matrices

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$= 1\cdot4 - 2\cdot3$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

$$=(1\cdot5\cdot9 + 2\cdot6\cdot7 + 3\cdot4\cdot8)-(7\cdot5\cdot3+ 6\cdot8\cdot1 + 4\cdot2\cdot9)$$

Calculating the determinant of a matrix A by expanding along row i

$$\det(A) = \sum_{j=1}^{n} a_{i,j}\mathrm{cof}(A)_{i,j},$$

where $\mathrm{cof}(A)_{i,j}$ is the (i, j)th cofactor of A.

The inverse of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

Calculating the inverse of a matrix via the cofactor matrix

$$A^{-1} = \frac{1}{\det(A)} \operatorname{cof}(A)^T,$$

where $\operatorname{cof}(A)$ is the cofactor matrix of A.

Basic 2D rotation

The matrix

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

rotates points/vectors in $\mathbf{R}^2$ counterclockwise around the origin through an angle of θ.

Characteristic equation of a square matrix A

$$\det(\lambda I - A) = 0$$

Sequences

Squeeze Law

If $a_n \leq b_n \leq c_n$ for all n and

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n,$$

then

$$\lim_{n\to\infty} b_n = L$$

as well.

Series

The sum of a geometric series

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r},$$

for $|r| < 1$. If $|r| \geq 1$ and $a \neq 0$, then the geometric series diverges.

The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots$$

converges for $p > 1$ and diverges for $0 < p \leq 1$.

nth-term test for divergence

If either

$$\lim_{n\to\infty} a_n \neq 0$$

or this limit does not exist, then the infinite series

$$\sum_{n=0}^{\infty} a_n$$

diverges.

Comparison test

Given a convergent series $\sum_{n=1}^{\infty} a_n$, and a second series $\sum_{n=1}^{\infty} b_n$ such that $0 \leq b_n \leq a_n$, then the second series also converges.

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Integral test and remainder estimate

Let $f(x)$ be a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$ for all integers $n \geq 1$. Then the series and the improper integral

$$\sum_{n=1}^{\infty} a_n \text{ and } \int_1^{\infty} f(x)dx$$

either both converge or both diverge.
If both converge, then we have the following estimate for R_n, the difference between the sum of the series and its n-th partial sum:

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_n^{\infty} f(x)dx$$

Ratio test

Suppose that for all terms of the infinite series $\sum_{n=0}^{\infty} a_n$ we have $a_n \neq 0$ and the limit

$$p = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

exists (finite or infinite). Then the series converges absolutely if $p < 1$ and diverges if $p > 1$. If $p = 1$ the ratio test is inconclusive.

Power series

Taylor series at a of a function $f(x)$ that is infinitely often differentiable at a.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots$$

Maclaurin series of a function $f(x)$ that is infinitely often differentiable at 0.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \dots$$

Maclaurin series of some of our favourite functions

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, x \in \mathbf{R}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots, x \in \mathbf{R}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots, x \in \mathbf{R}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, |x| < 1$$

Euler's formula

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

$$e^{i\pi} + 1 = 0$$

Radius of convergence of a power series

If $\sum_{n=0}^{\infty} a_n x^n$ is a power series, then either

1. The series converges absolutely for all x, or
2. The series converges only when $x = 0$, or
3. There exists a number $R > 0$ such that the series converges absolutely if $|x| < R$ and diverges if $|x| > R$.

In particular, if

$$R = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

exists, then we are dealing with case 1, 2, or 3 depending on whether R is infinite, $R = 0$ or R is finite ($\neq 0$), respectively.

Integration by Parts

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx.$$

Differential equations

Separation of variables

If a differentiable equation can be written in the form

$$\frac{dy}{dx} = g(x)h(y),$$

then the differentiable equation is said to be *separable* and (most of) its solutions may be found from

$$\int \frac{dy}{h(y)} = \int g(x)dx.$$

General solution of linear 1st-order differential equations

The general solution of

$$y' + P(x)y = Q(x)$$

is

$$y(x) = \frac{1}{I(x)}\left[\int I(x)Q(x)dx + C\right],$$

where

$$I(x) = e^{\int P(x)\ dx}.$$

If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point x_0, then the initial value problem

$$y' + P(x)y = Q(x), \quad y(x_0) = y_0$$

has a unique solution $y(x)$ on I, given by the above formula with an appropriate value of C.

General solution of homogeneous linear 2nd-order differential equations with constant coefficients

Given the equation

$$ay'' + by' + cy = 0$$

with constant coefficients a, b and c, first solve the characteristic equation

$$a\lambda^2 + b\lambda + c = 0$$

for λ. Let the two roots be λ_1 and λ_2. The general solutions of the differential equation are as follows:

- If λ_1, λ_2 are real and $\lambda_1 \neq \lambda_2$, then $y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$.
- If $\lambda_1, \lambda_2 = \alpha \pm i\beta$, then $y(x) = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$.
- If $\lambda_1 = \lambda_2 = \lambda$, then $y(x) = (A + Bx)e^{\lambda x}$.