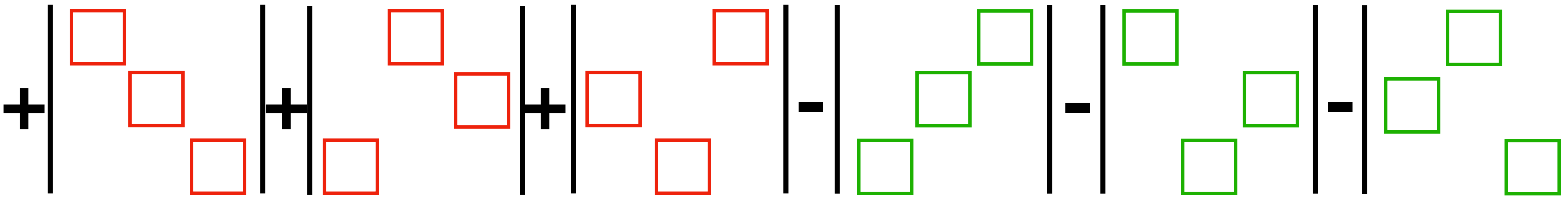
13&14

 MAGIC BULLET NO 2 

determinants of square matrices

Determinant = signed volume of parallelotope

geometrical interpretation	**det(A) = 0**
system of equations	**det(A) $\neq$ 0**
reduced row echelon form	
existence of inverse	



zero matrix & identity

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

diagonal & triangular matrices

$$\begin{vmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{vmatrix}$$

zero rows and columns

$$\begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} \qquad \begin{vmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \\ 0 & 5 & 6 \end{vmatrix}$$

swap two rows

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 0 \end{pmatrix}$$

$$\det(B) = (-1)\det(A)$$

multiply a row by a number

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} a & 2a & 3a \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}$$

$$\det(B) = a\,\det(A)$$

add multiple of one row to another row

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 3 \end{pmatrix}$$

$$\det(B) = \det(A)$$

effect of elementary operations on the determinant

swap two rows

$$A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&0\end{pmatrix}\qquad B=\begin{pmatrix}4&5&6\\1&2&3\\7&8&0\end{pmatrix}$$

$$\det(B) = (-1)\det(A)$$

multiply a row by a number

$$A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&0\end{pmatrix}\qquad B=\begin{pmatrix}a&2a&3a\\4&5&6\\7&8&0\end{pmatrix}$$

$$\det(B) = a\det(A)$$

add multiple of one row to another row

$$A=\begin{pmatrix}1&2&3\\4&5&6\\7&8&0\end{pmatrix}\qquad B=\begin{pmatrix}1&2&3\\4&5&6\\8&10&3\end{pmatrix}$$

$$\det(B) = \det(A)$$

$$\begin{pmatrix}1&4&3&2\\2&3&2&4\\3&3&0&6\\1&4&3&2\end{pmatrix}$$

$$\begin{pmatrix}1&2&3&4\\2&4&6&8\\1&0&1&0\\3&4&3&2\end{pmatrix}$$

$$\begin{pmatrix}1&2&0&3\\2&1&1&2\\3&0&1&2\\0&1&2&-1\end{pmatrix}\longrightarrow\begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&-1\\0&0&0&0\end{pmatrix}$$

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Question 1

swap two rows

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \qquad B=\begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 0 \end{pmatrix}$$

$$\det(B) = (-1)\det(A)$$

multiply a row by a number

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \qquad B=\begin{pmatrix} a & 2a & 3a \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}$$

$$\det(B) = a\det(A)$$

add multiple of one row to another row

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \qquad B=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 3 \end{pmatrix}$$

$$\det(B) = \quad \det(A)$$

What's the determinant of this matrix?

$$\begin{pmatrix} 1 & 2 & 0 & 3 \\ 2 & 1 & 1 & 2 \\ 3 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{rref}$$

swap two rows

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \qquad B=\begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 0 \end{pmatrix}$$

$$\det(B) = (-1)\det(A)$$

multiply a row by a number

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \qquad B=\begin{pmatrix} a & 2a & 3a \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}$$

$$\det(B) = a\det(A)$$

add multiple of one row to another row

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix} \qquad B=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 10 & 3 \end{pmatrix}$$

$$\det(B) = \quad \det(A)$$

A	det(A)=0	det(A)$\neq$0
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
rref	det(rref)=0	det(rref)$\neq$0
	$\Updownarrow$	$\Updownarrow$
	rref $\neq$ I	rref = I
	$\Updownarrow$	$\Updownarrow$
system of equations	no, infinite	unique
existence of inverse	does not exist	exists

$$\det(AB) = \det(A)\det(B)$$

$$\det(A^T) = \det(A)$$

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Question 2

If is invertible and $\det(M)=5$

what is $\det(M^{-1})$?

cofactors and cofactor matrix

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A)=\begin{pmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

cofactors and cofactor matrix

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A)=\begin{pmatrix} \_ & \_ & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

cofactors and cofactor matrix

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A)=\begin{pmatrix} \underline{-1} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

cofactors and cofactor matrix

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A)=\begin{pmatrix} -1 & \_ & \_ \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

cofactors and cofactor matrix

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A)=\begin{pmatrix} \underline{-1} & \underline{4} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

cofactors and cofactor matrix

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A)=\begin{pmatrix} \underline{-1} & \underline{4} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

cofactors and cofactor matrix

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A)=\begin{pmatrix} \underline{-1} & \underline{4} & \underline{-4} \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

cofactors and cofactor matrix

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A)=\begin{pmatrix} \underline{-1} & \underline{4} & \underline{-4} \\ \_ & \_ & \_ \\ \_ & \_ & \_ \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

cofactors and cofactor matrix

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A)=\begin{pmatrix} \underline{-1} & \underline{4} & \underline{-4} \\ \_ & \_ & \underline{6} \\ \_ & \_ & \_ \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

cofactors and cofactor matrix

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A)=\begin{pmatrix} \underline{-1} & \underline{4} & \underline{-4} \\ \_\_ & \_\_ & \underline{6} \\ \_\_ & \_\_ & \_\_ \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

cofactors and cofactor matrix

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A)=\begin{pmatrix} -1 & 4 & -4 \\ \underline{\quad} & -11 & 6 \\ \underline{\quad} & \underline{\quad} & \underline{\quad} \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

cofactors and cofactor matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A) = \begin{pmatrix} -1 & 4 & -4 \\ 4 & -11 & 6 \\ -5 & 10 & -5 \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

cofactors and cofactor matrix

$$A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad cof(A)=\begin{pmatrix} -\underline{1} & \underline{4} & -\underline{4} \\ \underline{4} & -\underline{11} & \underline{6} \\ -\underline{5} & \underline{10} & -\underline{5} \end{pmatrix}$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 0 & 4 & 5 \\ 3 & 0 & 3 & 2 \end{pmatrix}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

$$= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}$$

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

$$= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}$$

$$= a_{1,1}(a_{2,2}a_{3,3} - a_{2,3}a_{3,2}) - a_{1,2}(a_{2,1}a_{3,3} - a_{2,3}a_{3,1}) + a_{1,3}(a_{2,1}a_{3,2} - a_{2,2}a_{3,1})$$

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

$$= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}$$

$$= a_{1,1}(a_{2,2}a_{3,3} - a_{2,3}a_{3,2}) - a_{1,2}(a_{2,1}a_{3,3} - a_{2,3}a_{3,1}) + a_{1,3}(a_{2,1}a_{3,2} - a_{2,2}a_{3,1})$$

$$= a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2}\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3}\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

expanding along rows and columns

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix}$$

$$= a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}$$

$$= a_{1,1}(a_{2,2}a_{3,3} - a_{2,3}a_{3,2}) - a_{1,2}(a_{2,1}a_{3,3} - a_{2,3}a_{3,1}) + a_{1,3}(a_{2,1}a_{3,2} - a_{2,2}a_{3,1})$$

$$= a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2}\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3}\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

expanding along rows and columns

cofactors and cofactor matrix

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} \qquad A=\begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad cof(A)=\begin{pmatrix} -1 & 4 & -4 \\ 4 & -11 & 6 \\ -5 & 10 & -5 \end{pmatrix}$$

$$= a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

expanding along rows and columns

cofactors and cofactor matrix

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} \qquad A=\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{array}\right) \qquad \mathrm{cof}(A)=\begin{pmatrix} \underline{-1} & \underline{4} & \underline{-4} \\ \underline{4} & \underline{-11} & \underline{6} \\ \underline{-5} & \underline{10} & \underline{-5} \end{pmatrix}$$

$$= a_{1,1}\begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2}\begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3}\begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

expanding along rows and columns

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 0 & 4 & 5 \\ 3 & 0 & 3 & 2 \end{pmatrix} \qquad 5\,(-1) \begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 3 & 2 \end{vmatrix}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

expanding along rows and columns

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 0 & 4 & 5 \\ 3 & 0 & 3 & 2 \end{pmatrix} \qquad 5\,(-1) \begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 3 & 2 \end{vmatrix}$$

$$\begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{array}$$

expanding along rows and columns

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 5 & 1 & 2 & 3 \\ 1 & 0 & 4 & 5 \\ 3 & 0 & 3 & 2 \end{pmatrix} \qquad 5\,(-1) \begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 3 & 2 \end{vmatrix} + 4\,(+1) \begin{vmatrix} 1 & 3 & 4 \\ 1 & 4 & 5 \\ 3 & 3 & 2 \end{vmatrix}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

expanding along rows and columns

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 0 & 4 & 5 \\ 3 & 0 & 3 & 2 \end{pmatrix} \qquad 5\,(-1) \begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 3 & 2 \end{vmatrix} + 4\,(+1) \begin{vmatrix} 1 & 3 & 4 \\ 1 & 4 & 5 \\ 3 & 3 & 2 \end{vmatrix} + 2\,(-1) \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix}$$

$$\begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{array}$$

expanding along rows and columns

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

$$\begin{pmatrix} 1 & 0 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 0 & 4 & 5 \\ 3 & 0 & 3 & 2 \end{pmatrix}$$

$$5\,(-1) \begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 3 & 2 \end{vmatrix} + 4\,(+1) \begin{vmatrix} 1 & 3 & 4 \\ 1 & 4 & 5 \\ 3 & 3 & 2 \end{vmatrix} + 2\,(-1) \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} + 3\,(+1) \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 3 \end{vmatrix}$$

$$\begin{array}{cccc} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{array}$$

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1} \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 4 & 2 & 3 \\ 1 & 3 & 4 & 5 \\ 3 & 4 & 3 & 2 \end{pmatrix} \qquad 5\,(-1) \begin{vmatrix} 2 & 3 & 4 \\ 3 & 4 & 5 \\ 4 & 3 & 2 \end{vmatrix} + 4\,(+1) \begin{vmatrix} 1 & 3 & 4 \\ 1 & 4 & 5 \\ 3 & 3 & 2 \end{vmatrix} + 2\,(-1) \begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 3 & 4 & 2 \end{vmatrix} + 3\,(+1) \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 3 & 4 & 3 \end{vmatrix}$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

expanding along rows and columns

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{a} \times \mathbf{b}$$

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1}(a_{2,2}a_{3,3} - a_{2,3}a_{3,2}) - a_{1,2}(a_{2,1}a_{3,3} - a_{2,3}a_{3,1}) + a_{1,3}(a_{2,1}a_{3,2} - a_{2,2}a_{3,1})$$

mnemonic for cross product

expanding by row or column implies...

- **formula for inverse in terms of determinants**

$$\boxed{A^{-1} = \frac{1}{|A|}\mathbf{cof}(A)^T}$$

expanding by row or column implies...

- **formula for inverse in terms of determinants**

$$\boxed{A^{-1} = \frac{1}{|A|}\mathbf{cof}(A)^T}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A) = \begin{pmatrix} \underline{-1} & \underline{4} & \underline{-4} \\ \underline{4} & \underline{-11} & \underline{6} \\ \underline{-5} & \underline{10} & \underline{-5} \end{pmatrix}$$

$$A^{-1} = \frac{1}{}$$

expanding by row or column implies...

- **formula for inverse in terms of determinants**

$$\boxed{A^{-1} = \frac{1}{|A|}\mathbf{cof}(A)^T}$$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 4 & 2 & 1 \end{pmatrix} \qquad \mathrm{cof}(A) = \begin{pmatrix} \underline{-1} & \underline{4} & \underline{-4} \\ \underline{4} & \underline{-11} & \underline{6} \\ \underline{-5} & \underline{10} & \underline{-5} \end{pmatrix}$$

$$A^{-1} = \frac{1}{-5}\begin{pmatrix} -1 & 4 & -5 \\ 4 & -11 & 10 \\ -4 & 6 & -5 \end{pmatrix}$$

expanding by row
vs.
inverse

expanding by row or column implies...

- **formula for inverse in terms of determinants**

$$\boxed{A^{-1} = \frac{1}{|A|}\mathbf{cof}(A)^T}$$

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$$

$$\begin{matrix} + & - \\ - & + \end{matrix}$$

expanding by row or column implies...

- **formula for inverse in terms of determinants**

$$\boxed{A^{-1} = \frac{1}{|A|}\mathbf{cof}(A)^T}$$

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{|A|}\begin{pmatrix} +d & -c \\ -b & +a \end{pmatrix}^T$$

$$\begin{matrix} + & - \\ - & + \end{matrix}$$

expanding by row or column implies...

- **formula for inverse in terms of determinants**

$$\boxed{A^{-1} = \frac{1}{|A|}\mathbf{cof}(A)^T}$$

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{|A|}\begin{pmatrix} +d & -c \\ -b & +a \end{pmatrix}^T = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{matrix} + & - \\ - & + \end{matrix}$$

expanding by row or column implies...

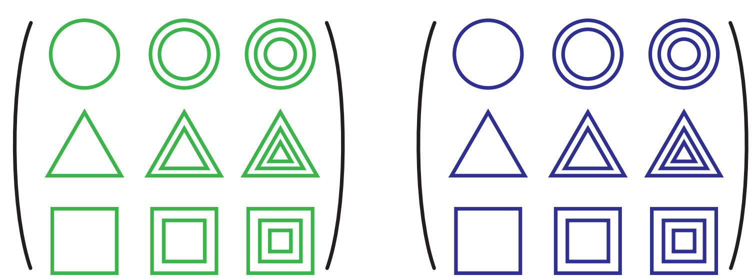
- **formula for inverse in terms of determinants**

$$\boxed{A^{-1} = \frac{1}{|A|}\mathbf{cof}(A)^T}$$

$$|A|\ I = A\ \mathbf{cof}(A)^T$$

$$\boxed{A^{-1} = \frac{1}{|A|}\mathbf{cof}(A)^T}$$

$$|A|\ I = A\ \mathbf{cof}(A)^T$$

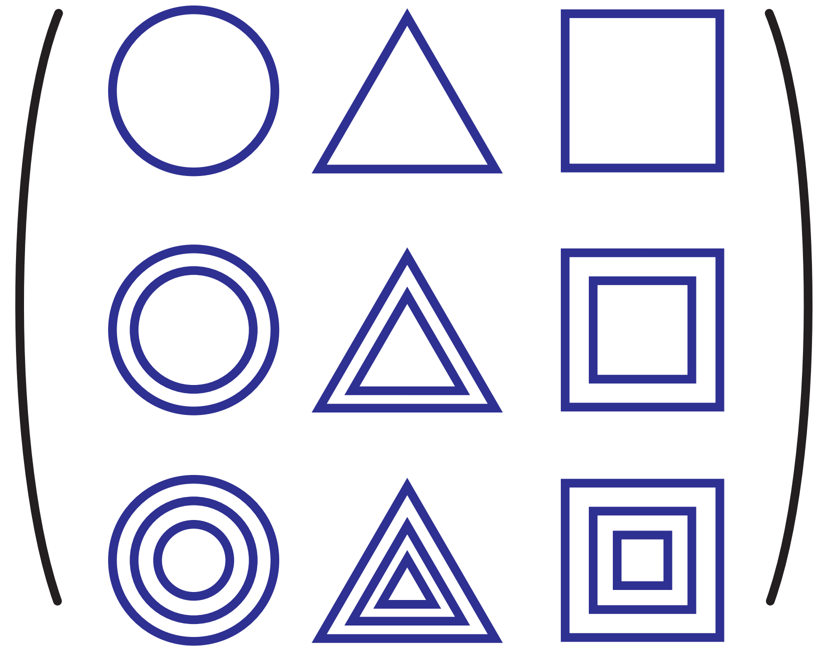


matrix **A** cofactor matrix of **A**

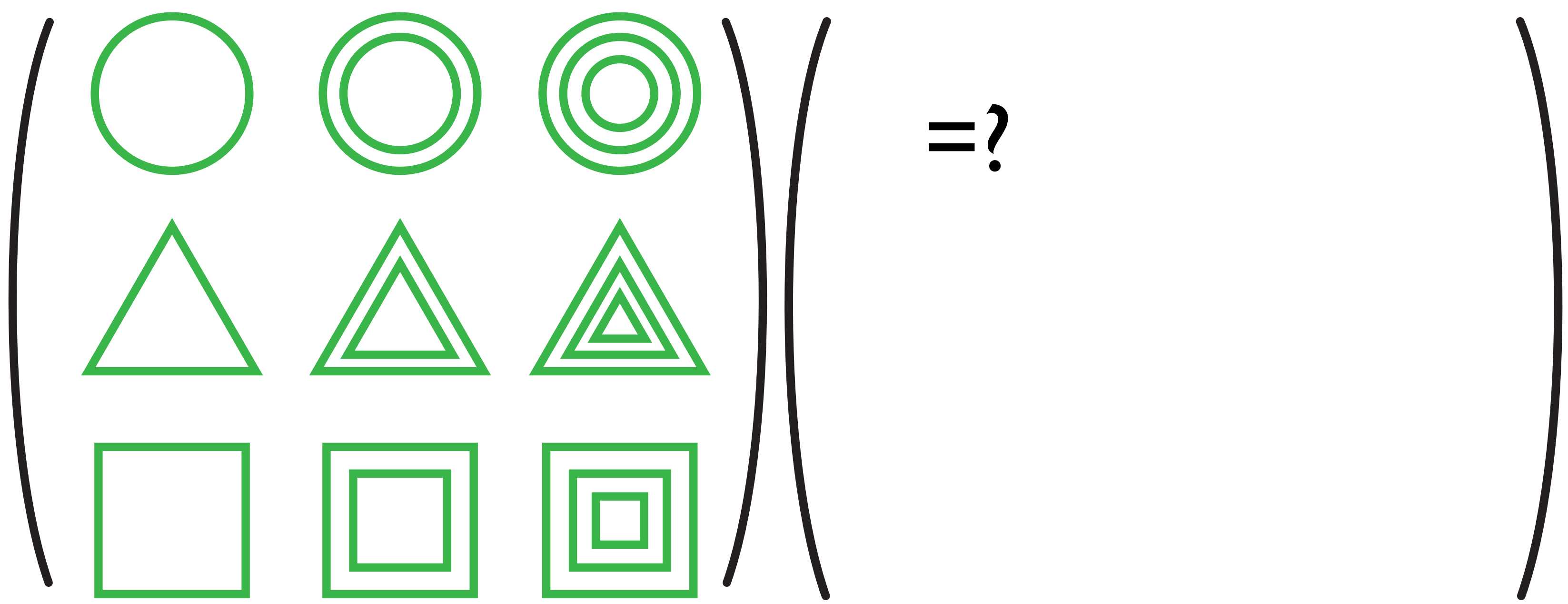
det(A) = 

idea for proof that

$$|A|\ I = A\ \mathbf{cof}(A)^T$$



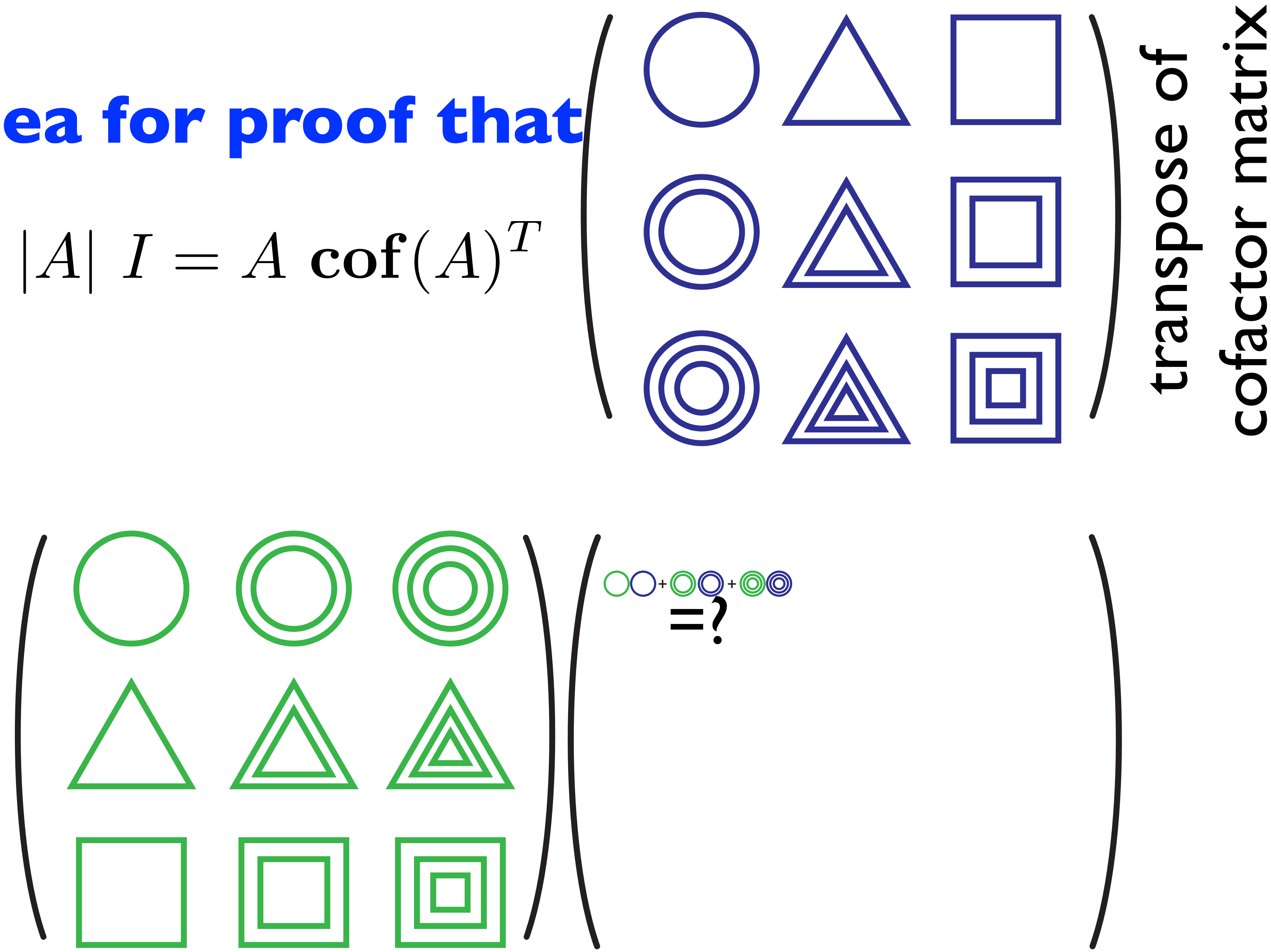
transpose of cofactor matrix



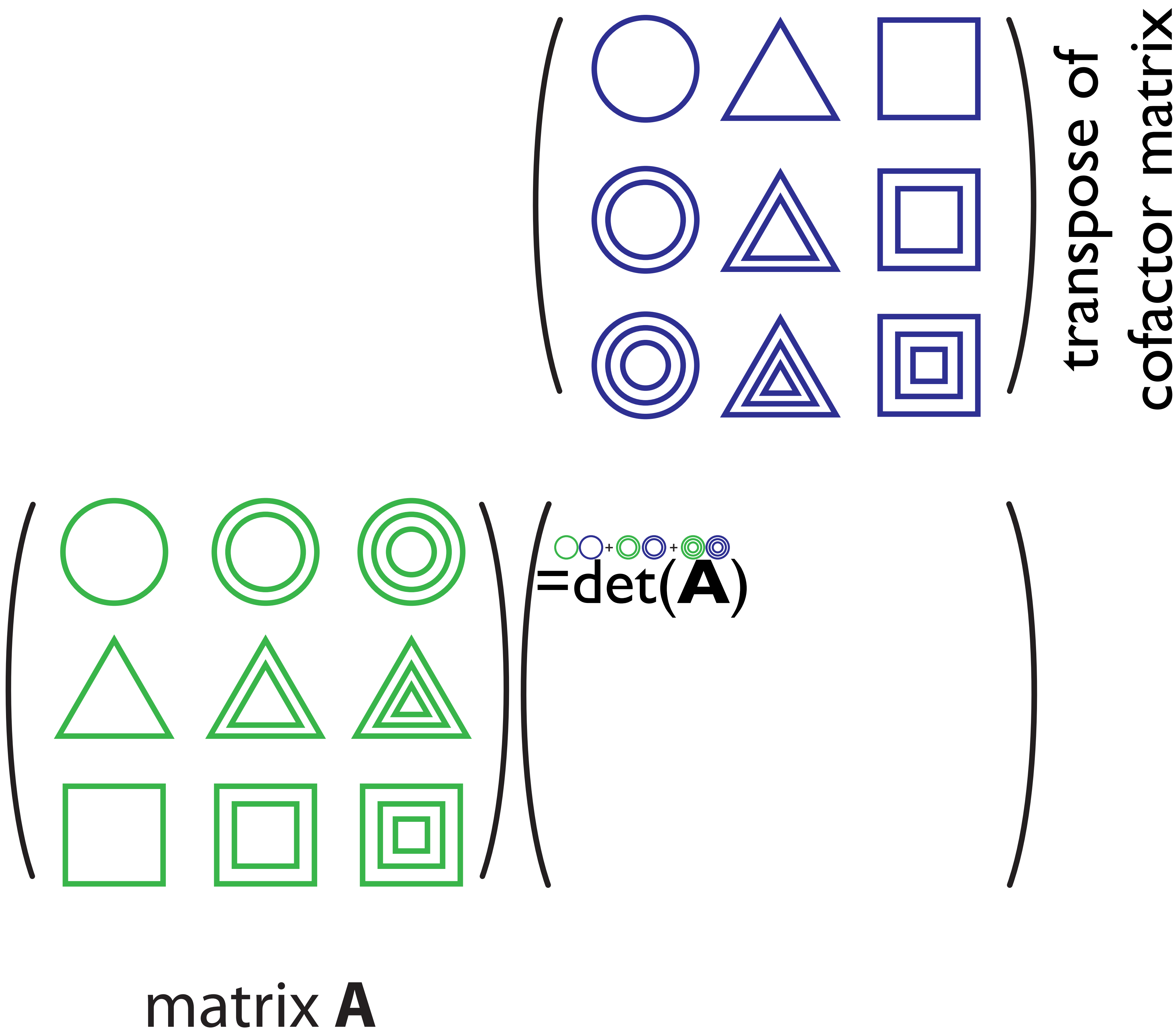
matrix **A**

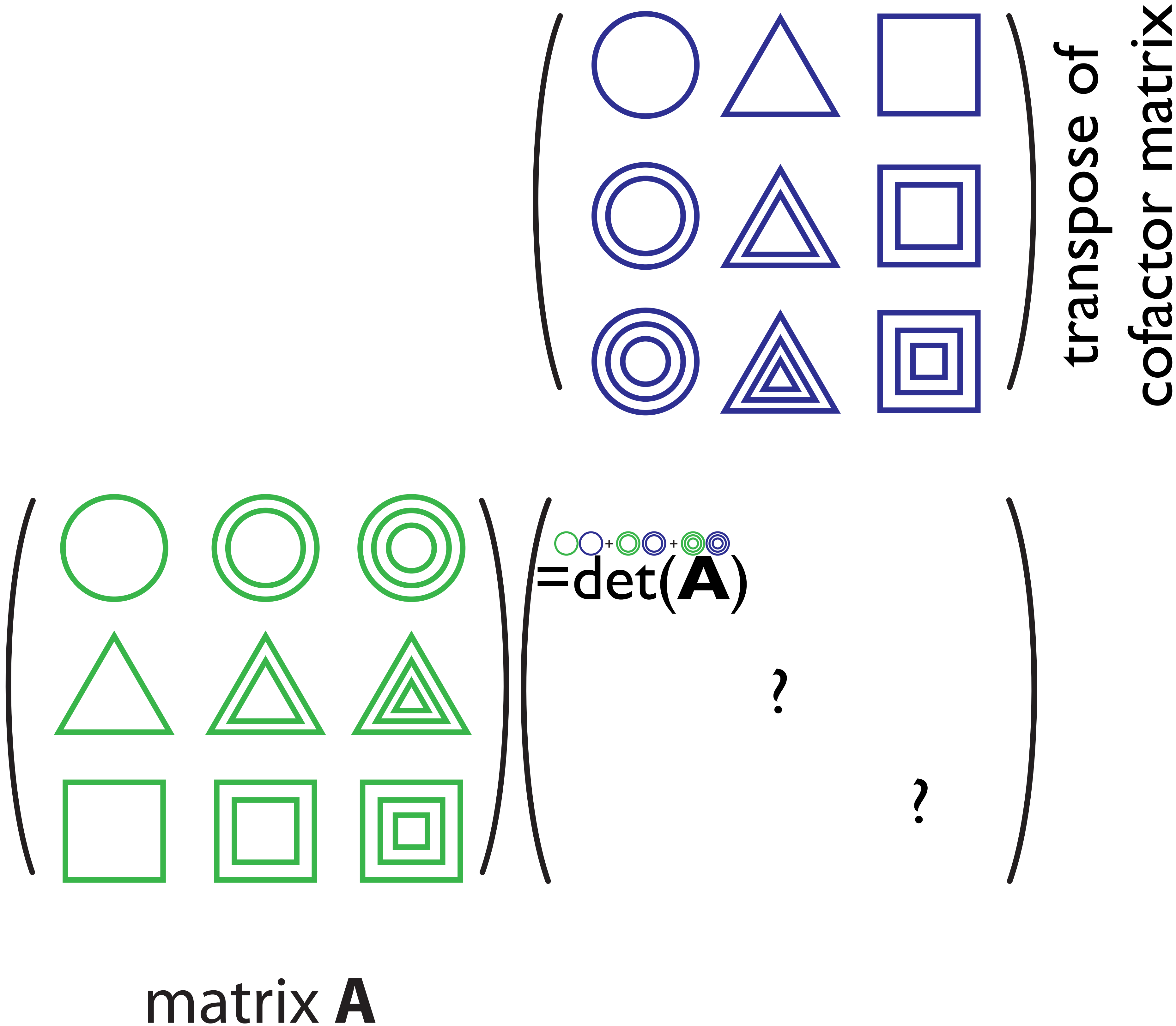
idea for proof that

$$|A|\ I = A\ \mathbf{cof}(A)^T$$

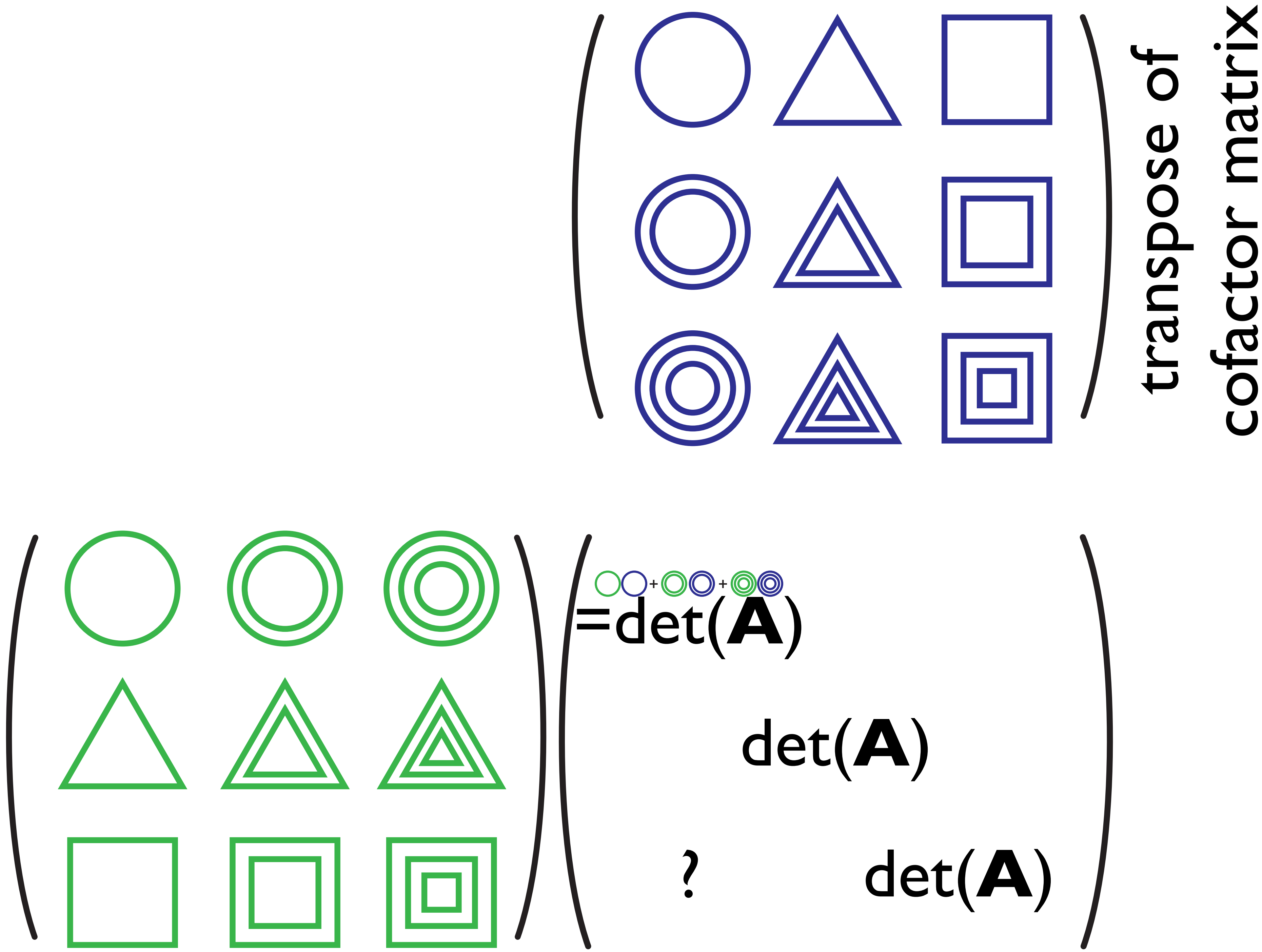


matrix **A**

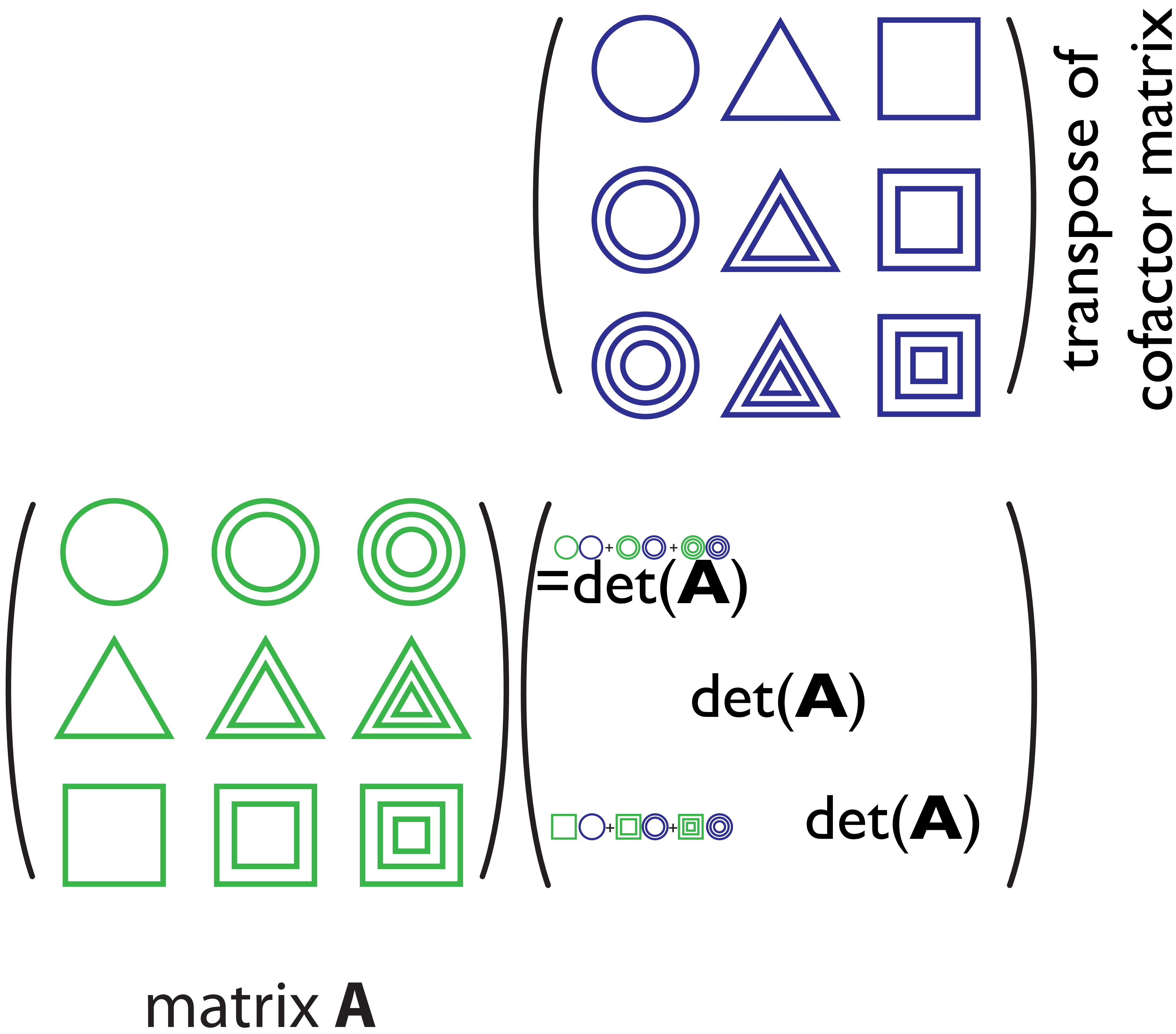

transpose of
cofactor matrix
=det(A)
matrix A

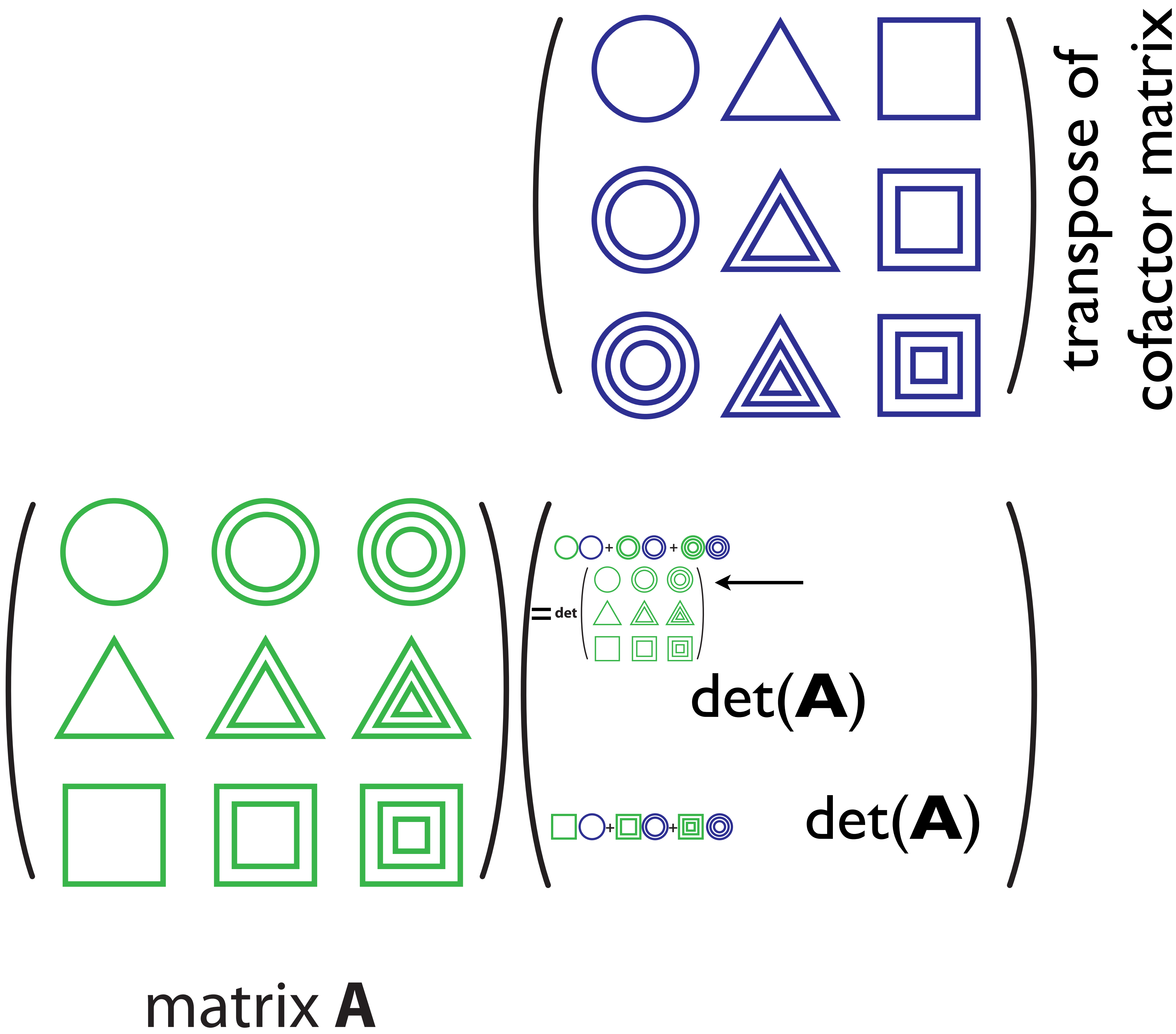


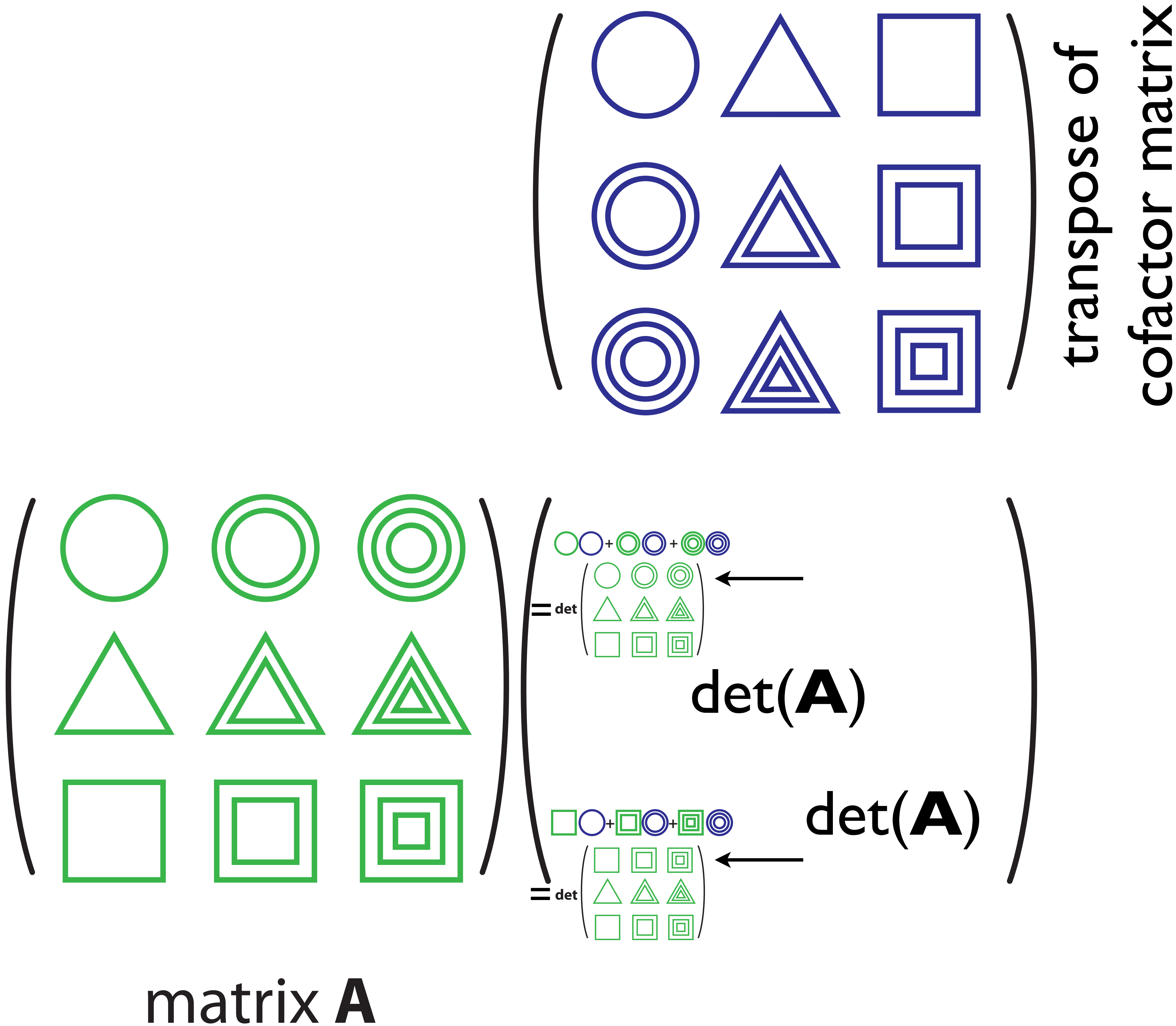
transpose of
cofactor matrix
=det(A)
?
?
matrix A



matrix **A**


transpose of
cofactor matrix
=det(A)
det(A)
det(A)
matrix A

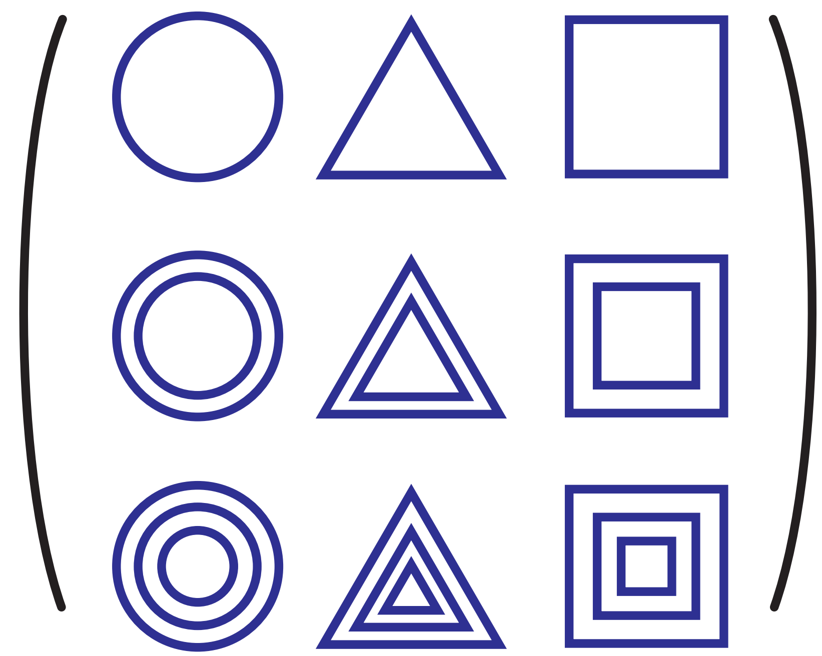

transpose of
cofactor matrix
det
det(A)
det(A)
matrix A



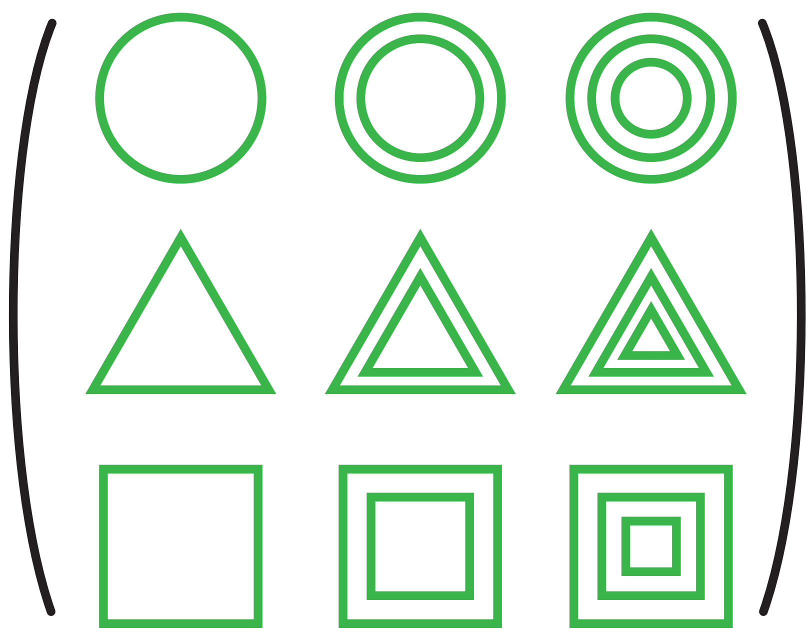
transpose of
cofactor matrix
= det
det(**A**)
= det
det(**A**)
matrix **A**

and therefore...

$$|A|\ I = A\ \mathbf{cof}(A)^T$$



transpose of cofactor matrix



$$\begin{pmatrix} \det(\mathbf{A}) & 0 & 0 \\ 0 & \det(\mathbf{A}) & 0 \\ 0 & 0 & \det(\mathbf{A}) \end{pmatrix}$$

matrix **A**

The most FAMOUS Mathematician?

the most famous mathematician?

I	*often*	*wondered*	*when*	*I*	*cursed*
Often	*feared*	*where*	*I*	*would*	*be*
Wondered	*where*	*she'd*	*yield*	*her*	*love*
When	*I*	*yield*	*so*	*will*	*she*
I	*would*	*her*	*will*	*be*	*pitied*
Cursed	*be*	*love*	*she*	*pitied*	*me*

?

 1866

IV. "Condensation of Determinants, being a new and brief Method for computing their arithmetical values." By the Rev. C. L. Dodgson, M.A., Student of Christ Church, Oxford. Communicated by the Rev. Bartholomew Price, M.A., F.R.S. Received May 15, 1866.

If it be proposed to solve a set of n simultaneous linear equations, not being all homogeneous, involving n unknowns, or to test their compatibility when all are homogeneous, by the method of determinants, in these, as well as in other cases of common occurrence, it is necessary to compute the arithmetical values of one or more determinants—such, for example, as

$$\begin{vmatrix} 1, & 3, & -2 \\ 2, & 1, & 4 \\ 3, & 5, & -1 \end{vmatrix}.$$

Now the only method, so far as I am aware, that has been hitherto employed for such a purpose, is that of multiplying each term of the first row or column by the determinant of its complemental minor, and affecting the products with the signs $+$ and $-$ alternately, the determinants required in the process being, in their turn, broken up in the same manner until determinants are finally arrived at sufficiently small for mental computation.

This process, in the above instance, would run thus:—

$$\begin{vmatrix} 1, & 3, & -2 \\ 2, & 1, & 4 \\ 3, & 5, & -1 \end{vmatrix} = 1 \times \begin{vmatrix} 1, & 4 \\ 5, & -1 \end{vmatrix} - 2 \times \begin{vmatrix} 3, & -2 \\ 5, & -1 \end{vmatrix} + 3 \times \begin{vmatrix} 3, & -2 \\ 1, & 4 \end{vmatrix}$$
$$= -21 - 14 + 42 = 7.$$

But such a process, when the block consists of 16, 25, or more terms, is so tedious that the old method of elimination is much to be preferred for solving simultaneous equations; so that the new method, excepting for equations containing 2 or 3 unknowns, is practically useless.

The new method of computation, which I now proceed to explain, and for which "Condensation" appears to be an appropriate name, will be found, I believe, to be far shorter and simpler than any hitherto employed.

In the following remarks I shall use the word "Block" to denote any number of terms arranged in rows and columns, and "interior of a block" to denote the block which remains when the first and last rows and columns are erased.

The process of "Condensation" is exhibited in the following rules, in which the given block is supposed to consist of n rows and n columns:—

(1) Arrange the given block, if necessary, so that no ciphers occur in its interior. This may be done either by transposing rows or columns, or by adding to certain rows the several terms of other rows multiplied by certain multipliers.

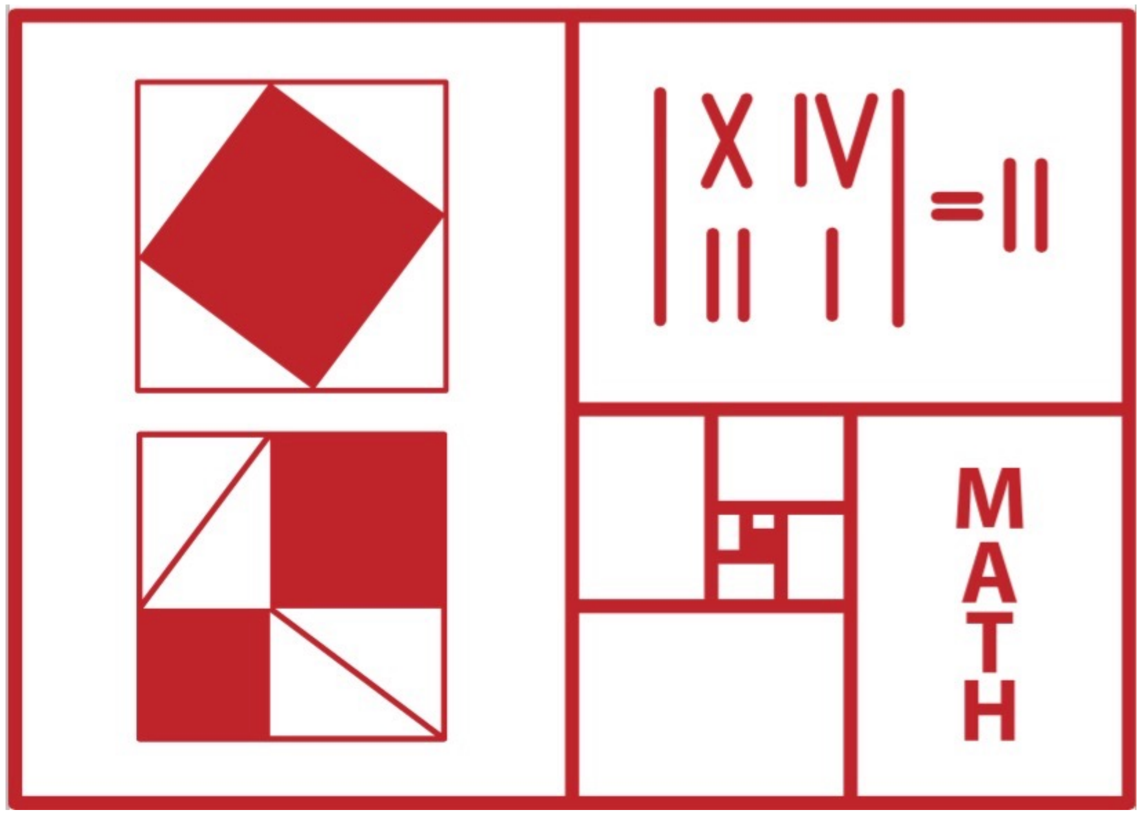
(2) Compute the determinant of every minor consisting of four adjacent

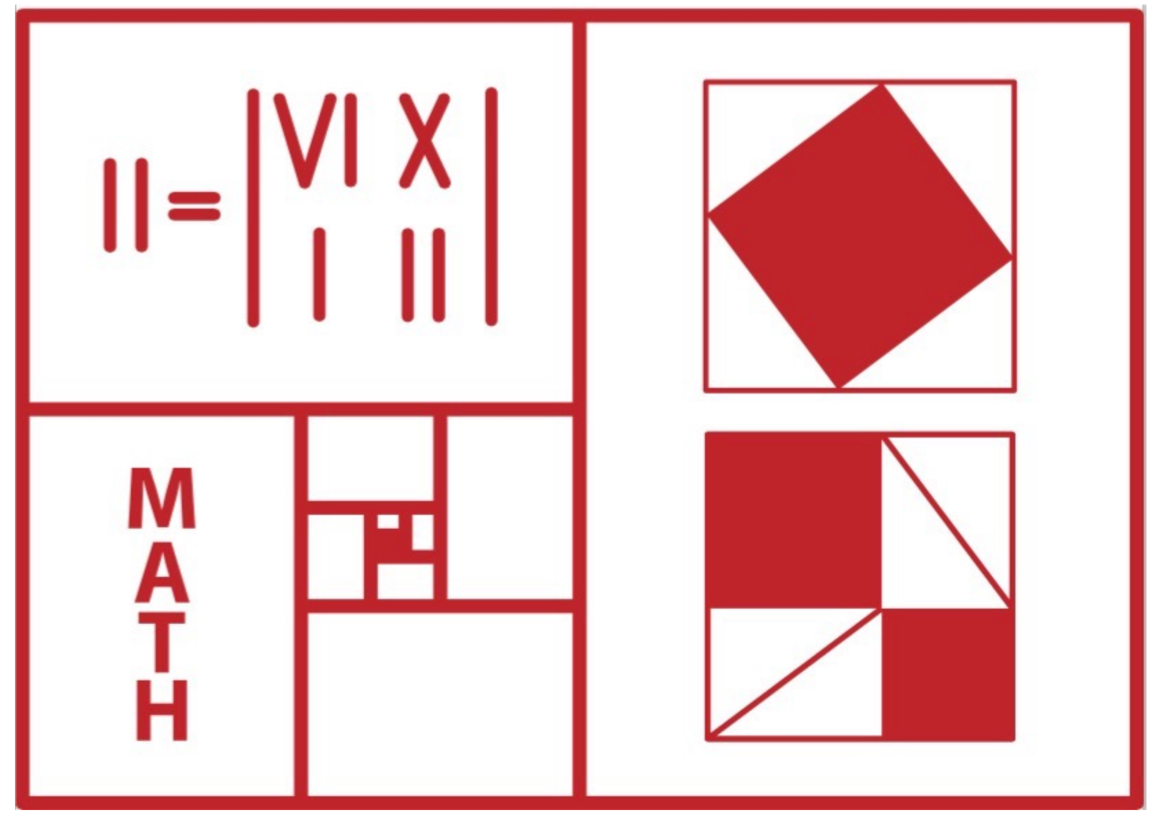
Lewis Carrol

I	*often*	*wondered*	*when*	*I*	*cursed*
Often	*feared*	*where*	*I*	*would*	*be*
Wondered	*where*	*she'd*	*yield*	*her*	*love*
When	*I*	*yield*	*so*	*will*	*she*
I	*would*	*her*	*will*	*be*	*pitied*
Cursed	*be*	*love*	*she*	*pitied*	*me*

- $$\begin{pmatrix} 2 & -3 & 1 & 2 & 5 \\ 4 & 1 & -2 & -3 & 2 \\ 5 & -4 & 2 & 2 & -3 \\ 3 & -1 & 5 & 2 & 1 \\ -4 & 1 & 5 & -1 & 2 \end{pmatrix}$$
- $$\begin{pmatrix} 14 & 5 & 1 & 19 \\ -21 & -6 & 2 & 5 \\ 7 & -18 & -6 & 8 \\ -1 & -10 & -15 & 5 \end{pmatrix}$$
- $$\begin{pmatrix} 21 & 16 & -33 \\ 420 & 72 & 46 \\ -88 & 210 & 90 \end{pmatrix}.$$
- $$\begin{pmatrix} 21 & -8 & 11 \\ -105 & 36 & 23 \\ 88 & 42 & 45 \end{pmatrix}$$
- $$\begin{pmatrix} -84 & -580 \\ -7578 & 654 \end{pmatrix}.$$
- $$\begin{pmatrix} 14 & -290 \\ 421 & -109 \end{pmatrix}.$$
- 120,564
- 3349

MATH

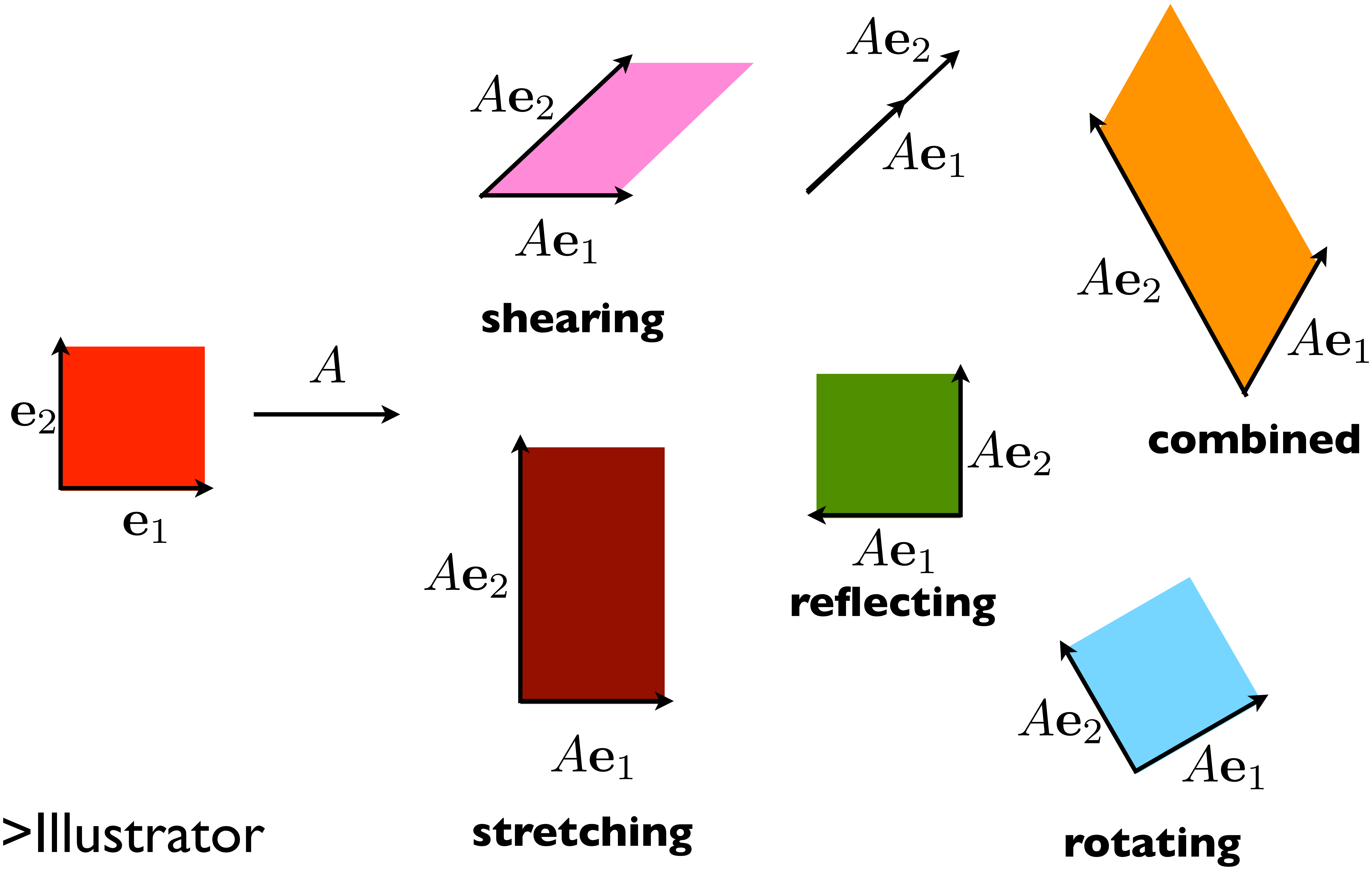

X IV
II I
=II
MATH


II=
VI X
I II
MATH

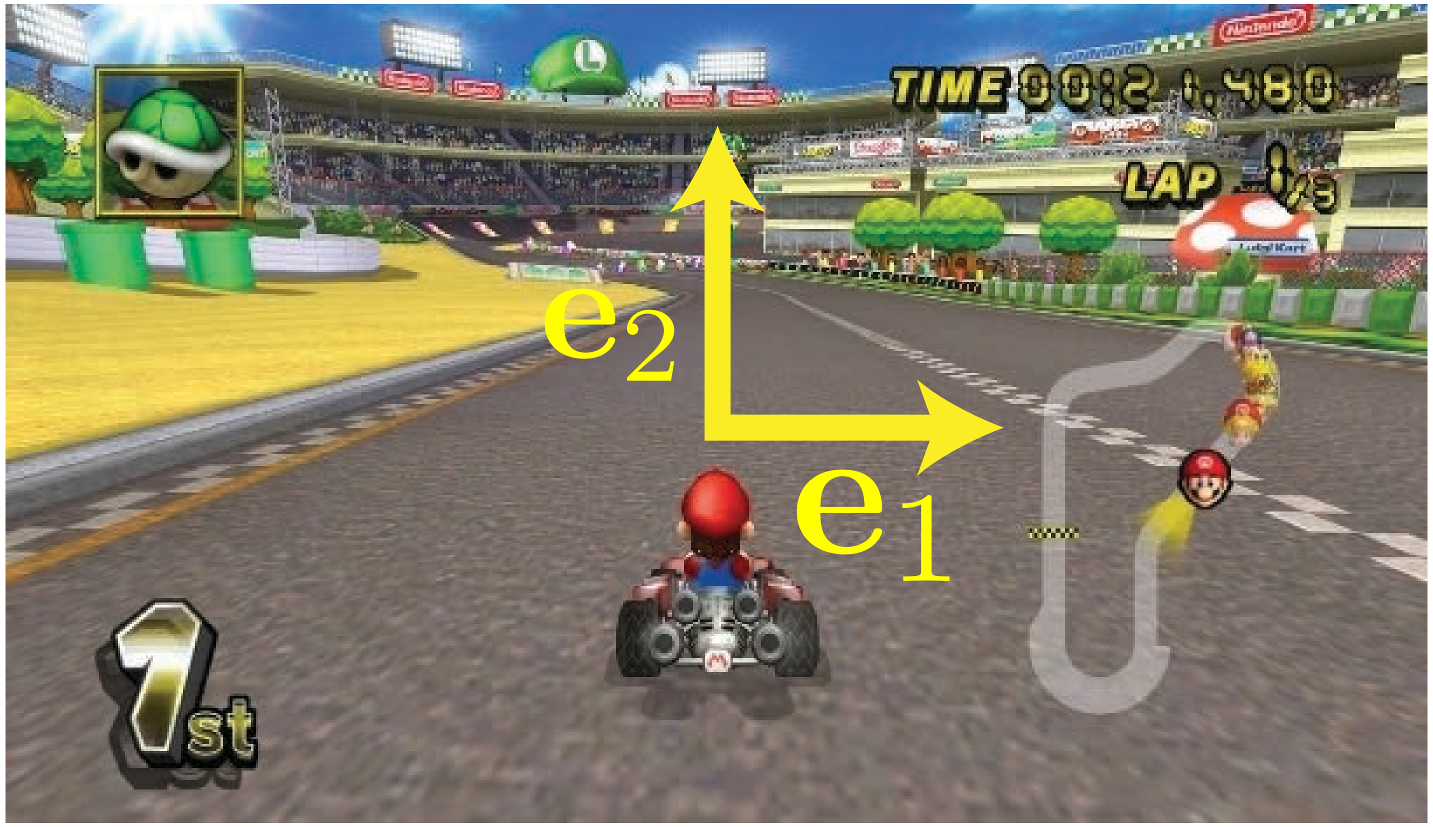
linear transformations of space

The images of the standard basis vectors under a linear transformation are the columns of the matrix.

The images of the standard basis vectors under a linear transformation are the columns of the matrix.



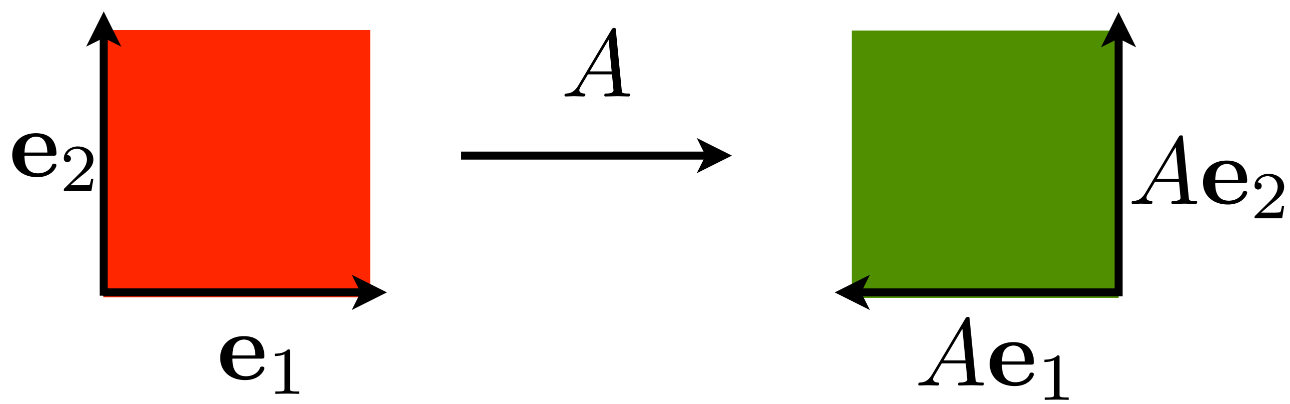
>Illustrator



>Illustrator

Lectures 13&14

Question 3



Where does the matrix A map the point $\begin{pmatrix} 666 \\ 0 \end{pmatrix}$? (Answer: 1st , 2nd coordinate)

Examples of functions that can be described with matrices

2D rotations

3D rotations

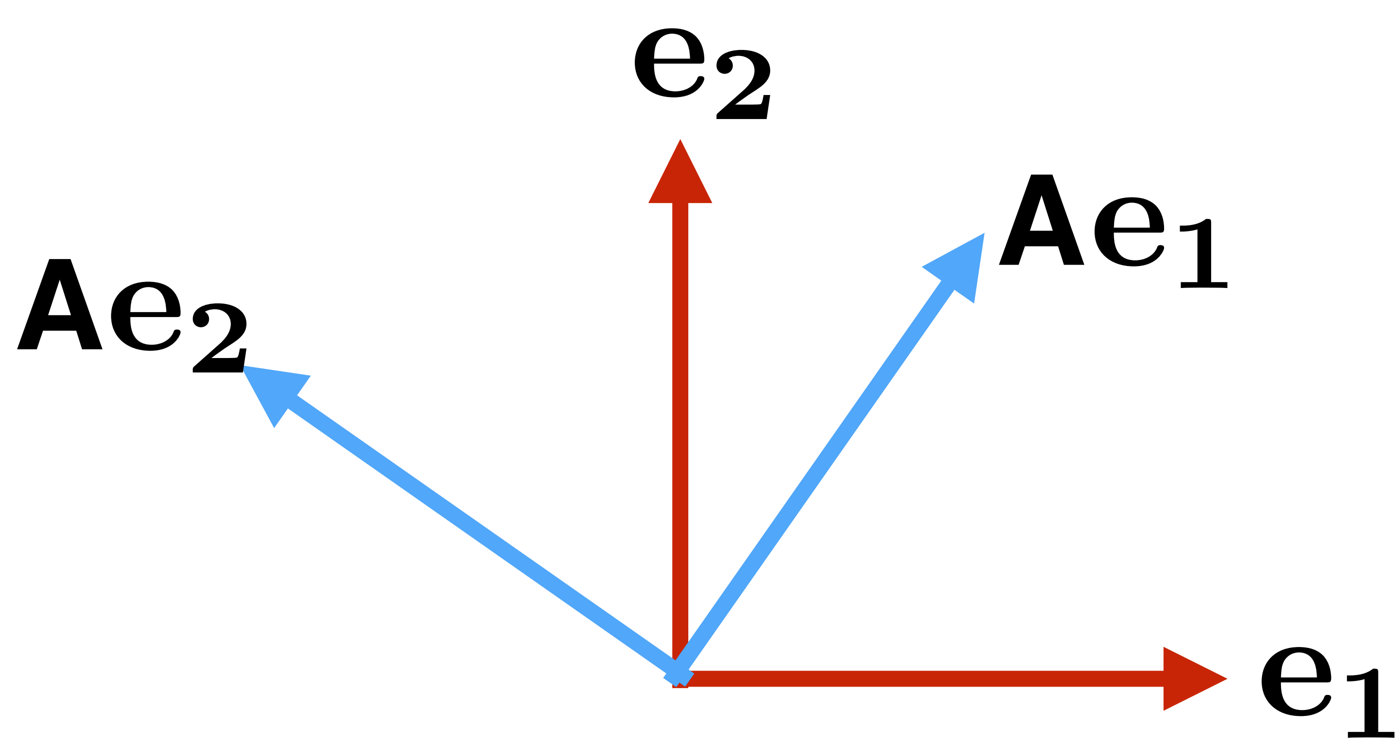
Projections

Reflections

The images of the standard basis vectors under a linear transformation are the columns of the matrix.

Example:

counterclockwise 2D rotation around the origin



$A_\theta \mathbf{e_1} = \begin{pmatrix} \quad \end{pmatrix}$ $A_\theta \mathbf{e_2} = \begin{pmatrix} \quad \end{pmatrix}$ $A_\theta = \begin{pmatrix} \quad \end{pmatrix}$

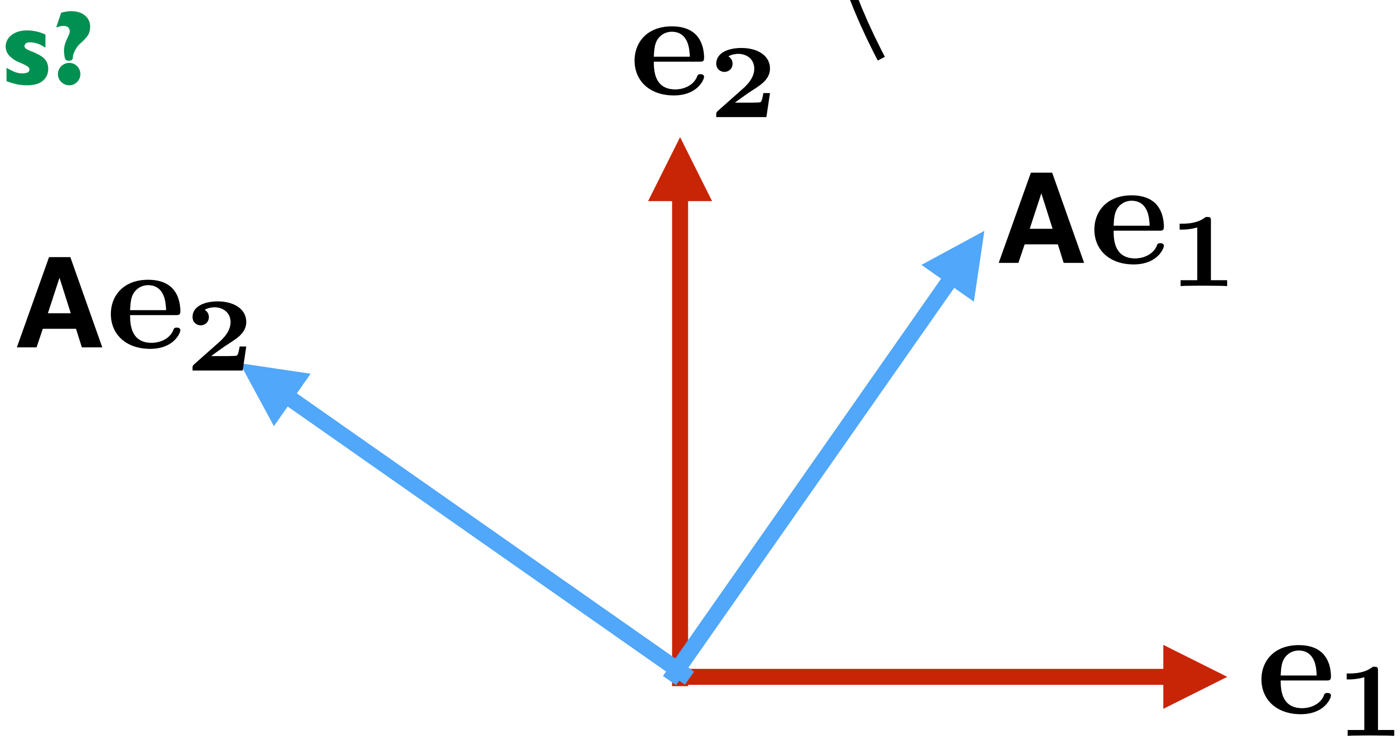
The images of the standard basis vectors under a linear transformation are the columns of the matrix.

$$\left(\quad \right)$$

counterclockwise 3D rotation around the z-axis?

Example:

counterclockwise 2D rotation around the origin



$$A_\theta \mathbf{e}_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} A_\theta \mathbf{e}_2 = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \qquad A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

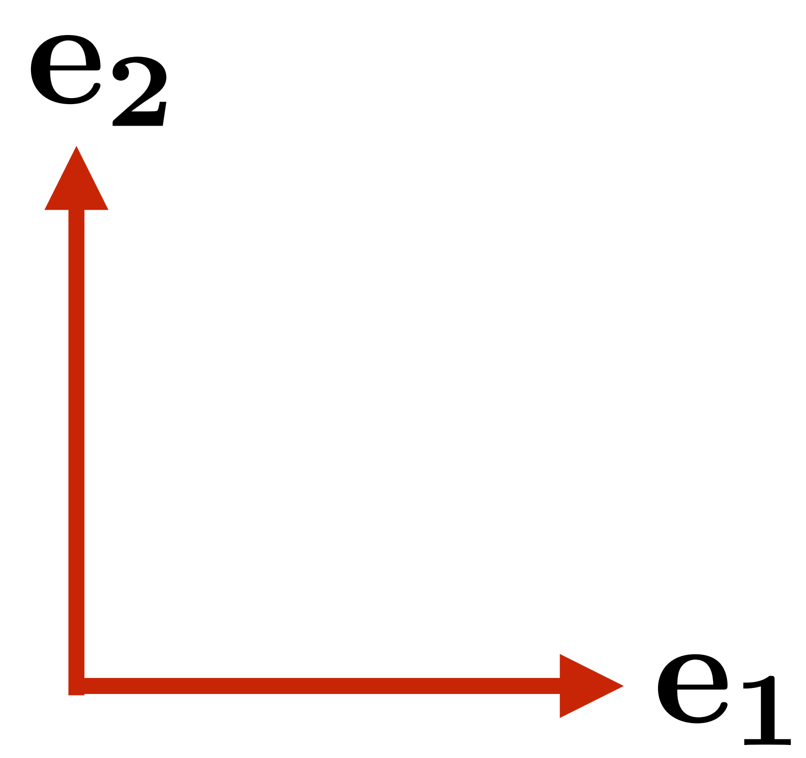
Lectures 13&14

Question 4

What's the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of a clockwise quarter-turn rotation?

(Answer: a, b, c, d)

The images of the standard basis vectors under a linear transformation are the columns of the matrix.

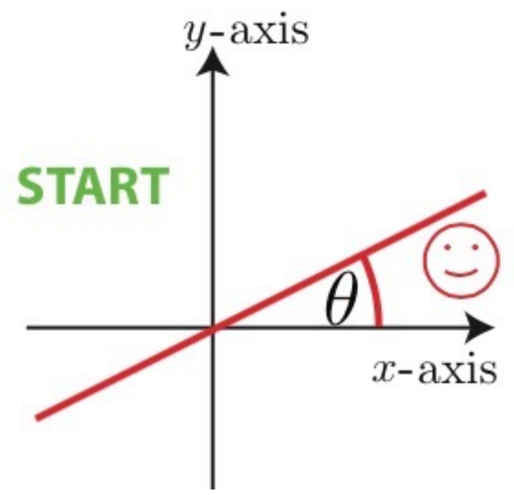


Example:

2D reflection through x-axis

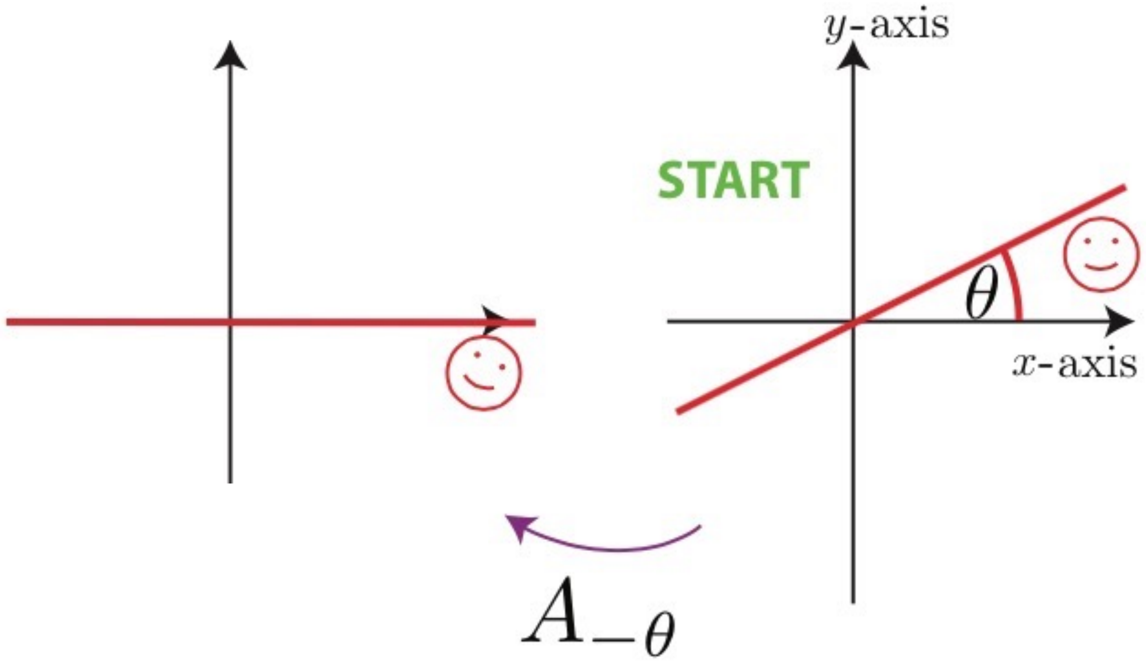
$$R_x\mathbf{e_1} = \begin{pmatrix} \quad \end{pmatrix} \quad R_x\mathbf{e_2} = \begin{pmatrix} \quad \end{pmatrix} \qquad R_x = \begin{pmatrix} \quad \end{pmatrix}$$

The images of the standard basis vectors under a linear transformation are the columns of the matrix.



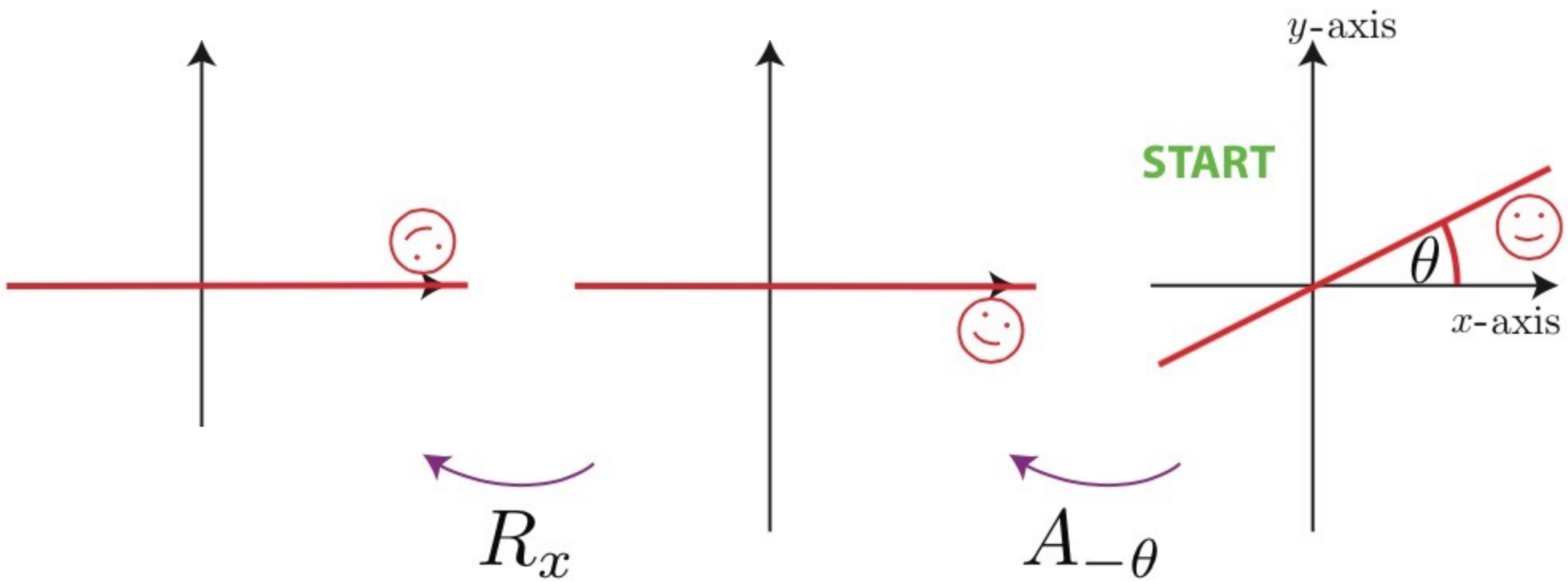
general 2D reflection

The images of the standard basis vectors under a linear transformation are the columns of the matrix.



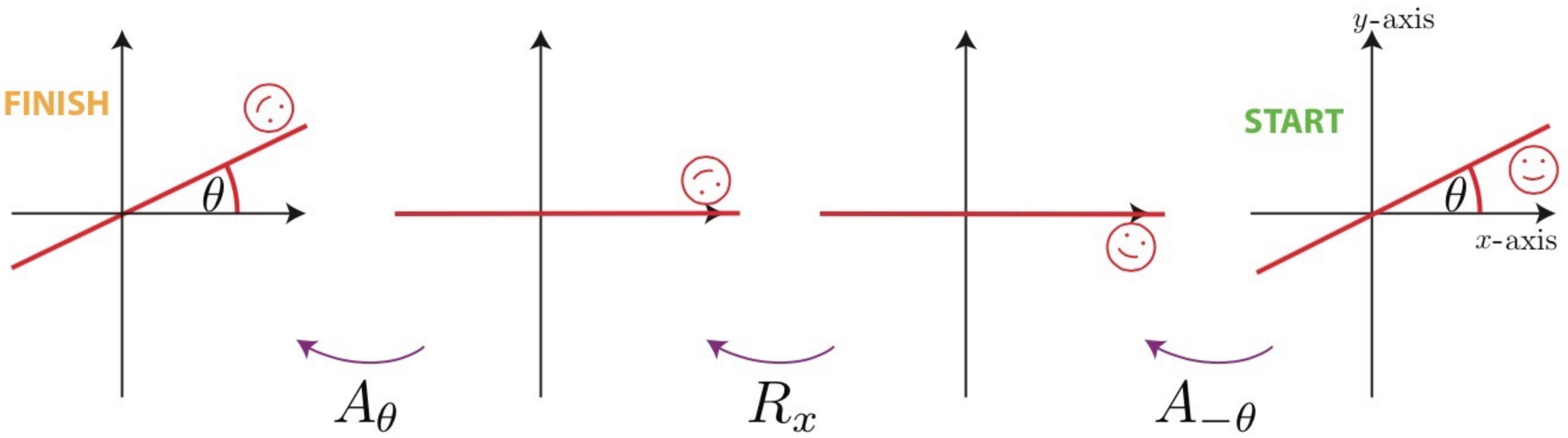
general 2D reflection

The images of the standard basis vectors under a linear transformation are the columns of the matrix.



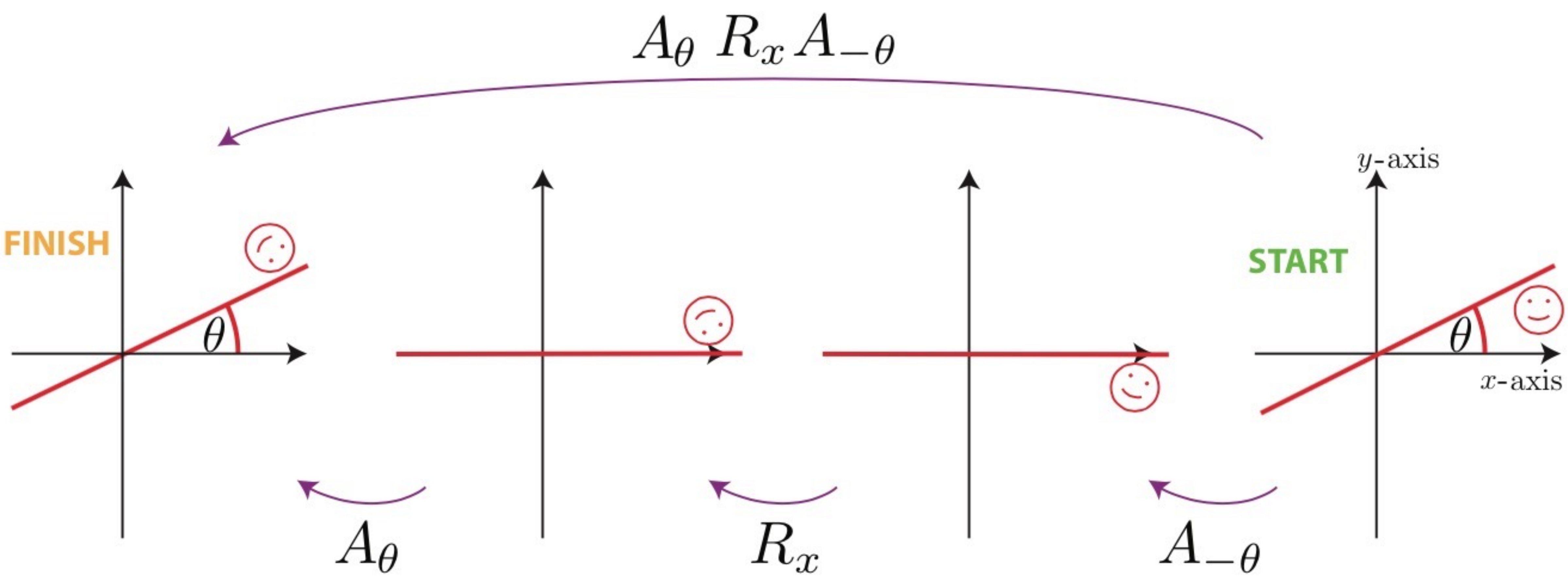
general 2D reflection

The images of the standard basis vectors under a linear transformation are the columns of the matrix.



general 2D reflection

The images of the standard basis vectors under a linear transformation are the columns of the matrix.



general 2D reflection

$$= \begin{pmatrix} \cos^2(\theta) - \sin^2(\theta) & 2\cos(\theta)\sin(\theta) \\ 2\cos(\theta)\sin(\theta) & \sin^2(\theta) - \cos^2(\theta) \end{pmatrix}$$

The images of the standard basis vectors under a linear transformation are the columns of the matrix.

general 2D reflection $= \begin{pmatrix} \cos^2(\theta) - \sin^2(\theta) & 2\cos(\theta)\sin(\theta) \\ 2\cos(\theta)\sin(\theta) & \sin^2(\theta) - \cos^2(\theta) \end{pmatrix}$

? ??!! Mathematica ???!!!

$$\mathbf{R}^n \to \mathbf{R}^n : \mathbf{x} \mapsto A\mathbf{x}$$

Examples:

$$A_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

2D rotation

$$R_x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

2D reflection through x-axis
2D reflection through arbitrary lir

$$\begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & 1 \end{pmatrix}$$

3D rotation

3D reflection through xy-plane
3D reflection through plane

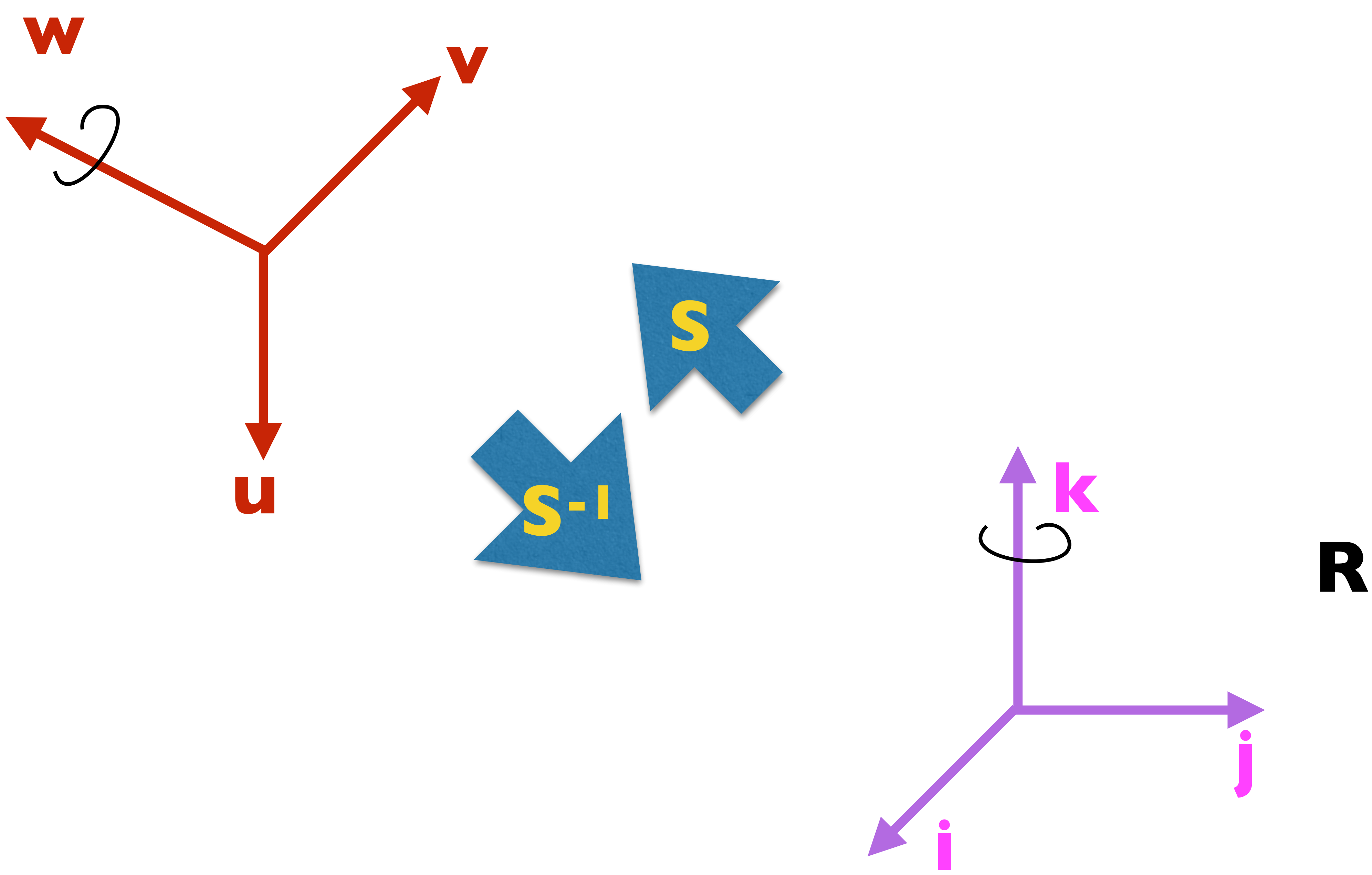
3D reflection through origin
3D reflection through line

Projections

Example:

3D cc rotation about unit vector w by a certain angle θ

$$\mathbf{w} = (a, b, c)\,, \ \sqrt{a^2 + b^2 + c^2} = 1$$



$\mathbf{w} = (a, b, c)\,, \ \sqrt{a^2 + b^2 + c^2} = 1$

Strategy:

Find unit vectors **u,v** such that

u x **v** = **w**

$\mathbf{w} = (a, b, c), \ \sqrt{a^2 + b^2 + c^2} = 1$

$\mathbf{u} = \left(-\frac{b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}, 0\right)$

Strategy:

Find unit vectors **u,v** such that

u x **v** = **w**

Find matrix **S** that sends

i, j, k to **u, v, w**, respectively.

$$\mathbf{w} = (a,b,c)\,,\ \sqrt{a^2+b^2+c^2} = 1$$

$$\mathbf{u} = \left(-\frac{b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}, 0\right)$$

$$\mathbf{S} \quad \begin{pmatrix} -\frac{b}{\sqrt{a^2+b^2}} & -\frac{c}{\sqrt{(a^2+b^2)}}a & a \\ \frac{a}{\sqrt{a^2+b^2}} & -\frac{c}{\sqrt{(a^2+b^2)}}b & b \\ 0 & \frac{a^2+b^2}{\sqrt{a^2+b^2}} & c \end{pmatrix}$$

Strategy:

Find unit vectors **u,v** such that **u** x **v** = **w**

Find matrix **S** that sends **i, j, k** to **u, v, w**, respectively.

Let **R** be the 3D extension of the 2D rotation matrix that describes a cc rotation by θ

$$\mathbf{w} = (a, b, c), \ \sqrt{a^2 + b^2 + c^2} = 1$$

$$\mathbf{u} = \left(-\frac{b}{\sqrt{a^2+b^2}}, \frac{a}{\sqrt{a^2+b^2}}, 0\right)$$

$$\mathbf{S} = \begin{pmatrix} -\frac{b}{\sqrt{a^2+b^2}} & -\frac{c}{\sqrt{(a^2+b^2)}}a & a \\ \frac{a}{\sqrt{a^2+b^2}} & -\frac{c}{\sqrt{(a^2+b^2)}}b & b \\ 0 & \frac{a^2+b^2}{\sqrt{a^2+b^2}} & c \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Strategy:

Find unit vectors **u,v** such that **u** x **v** = **w**

Find matrix **S** that sends **i, j, k** to **u, v, w**, respectively.

Let **R** be the 3D extension of the 2D rotation matrix that describes a cc rotation by θ

then the rotation matrix we are looking for is **SRS^{-1}**.

$$\mathbf{w} = (a, b, c), \ \sqrt{a^2 + b^2 + c^2} = 1$$

$$\mathbf{u} = \left(-\frac{b}{\sqrt{a^2 + b^2}}, \frac{a}{\sqrt{a^2 + b^2}}, 0 \right)$$

$$\mathbf{S} = \begin{pmatrix} -\frac{b}{\sqrt{a^2+b^2}} & -\frac{c}{\sqrt{(a^2+b^2)}}a & a \\ \frac{a}{\sqrt{a^2+b^2}} & -\frac{c}{\sqrt{(a^2+b^2)}}b & b \\ 0 & \frac{a^2+b^2}{\sqrt{a^2+b^2}} & c \end{pmatrix}$$

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

SRS⁻¹=

$$\begin{pmatrix} \cos\theta - a^2\cos\theta + a^2 & -ba\cos\theta + ba - c\sin\theta & (\sin\theta)\, b - (\cos\theta)\, ca + ca \\ -ba\cos\theta + ba + c\sin\theta & -b^2\cos\theta + b^2 + \cos\theta & -(\sin\theta)\, a - (\cos\theta)\, cb + cb \\ -(\sin\theta)\, b - (\cos\theta)\, ca + ca & (\sin\theta)\, a - (\cos\theta)\, cb + cb & (1 - c^2)\cos\theta + c^2 \end{pmatrix}$$

Strategy:

Find unit vectors **u,v** such that **u** x **v** = **w**

Find matrix **S** that sends **i, j, k** to **u, v, w**, respectively.

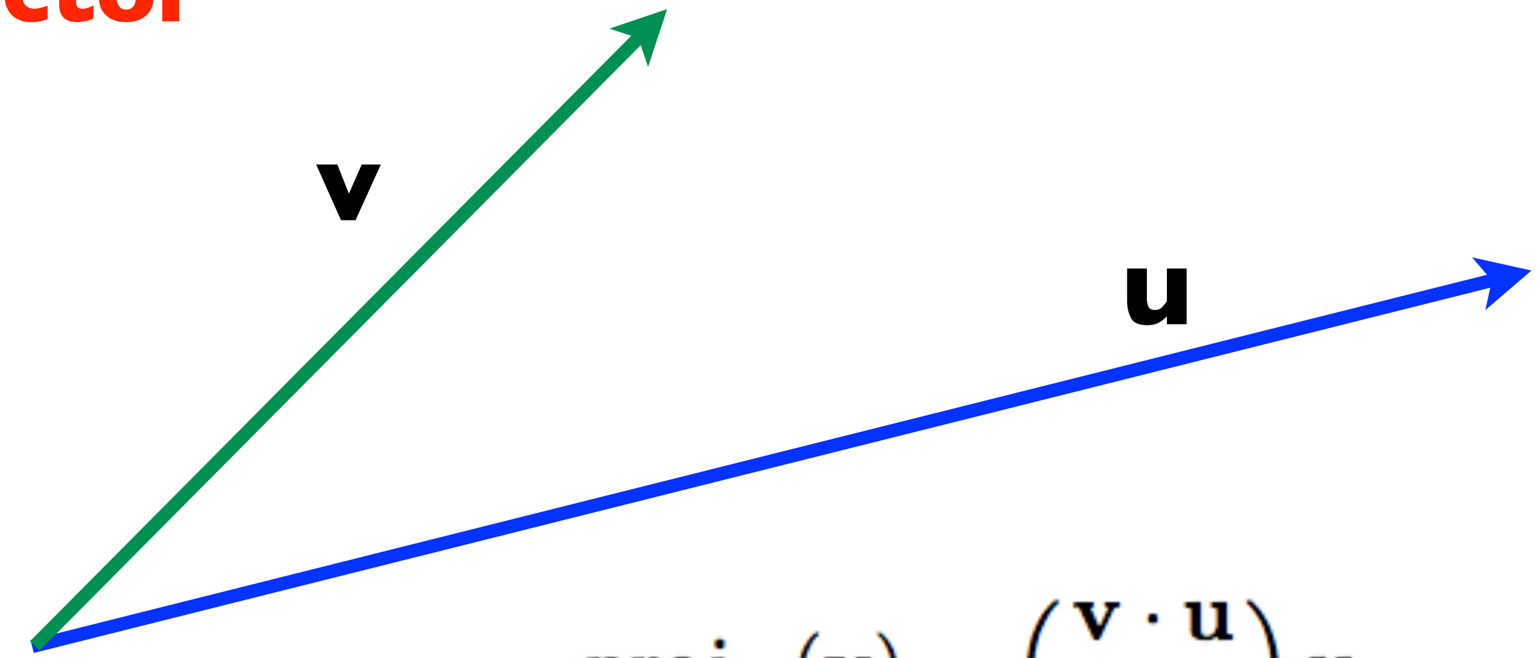
Let **R** be the 3D extension of the 2D rotation matrix that describes a cc rotation by θ

then the rotation matrix we are looking for is **SRS**$^{-1}$.

reflect in a plane

projection onto a vector

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$



$$\text{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

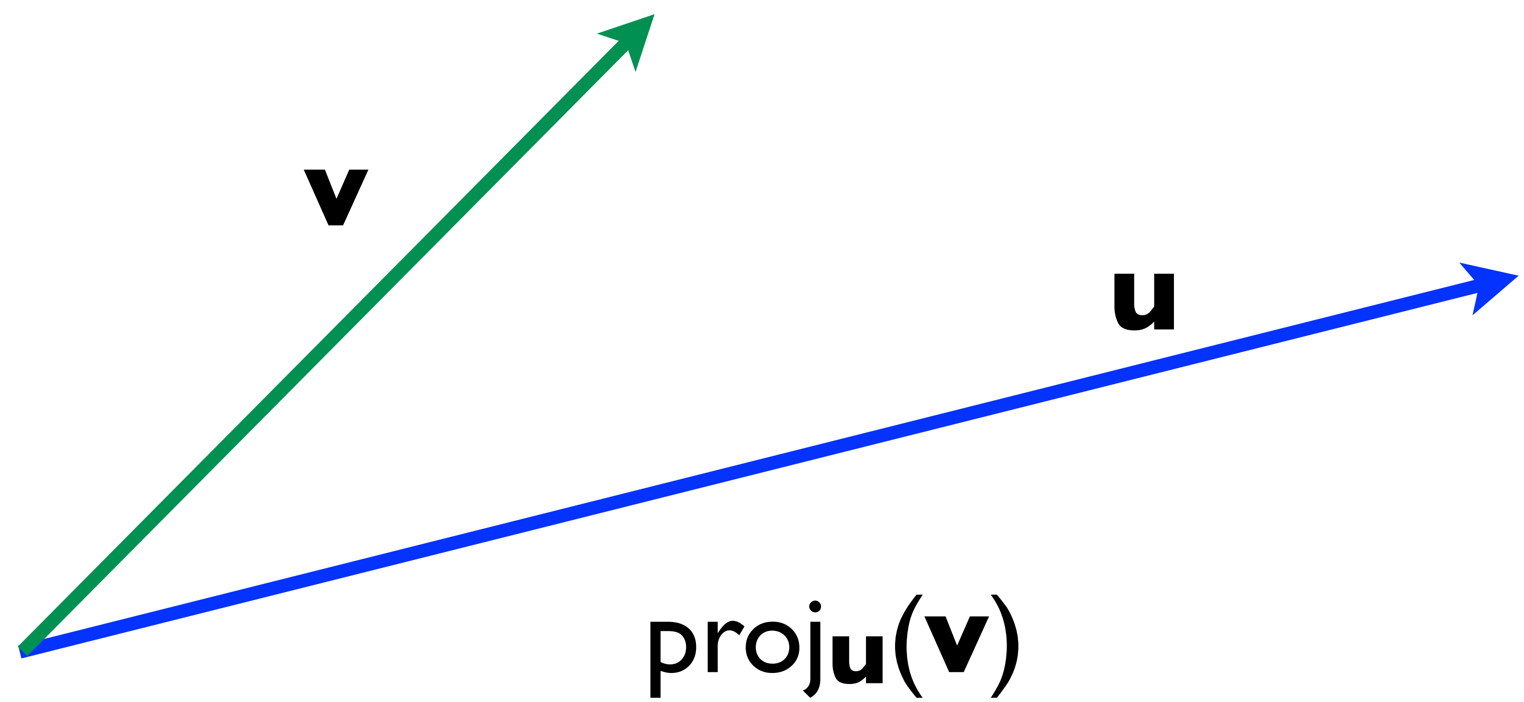
$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Example:

projections onto a vector

$$\mathrm{proj}_{\mathbf{u}}(\mathbf{v}) = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

$$\frac{1}{14}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$



n x m matrices describe
linear transformations

$$\mathbf{R}^n \to \mathbf{R}^m$$

These are exactly the functions

$$T : \mathbf{R}^n \to \mathbf{R}^m$$

such that

$T(a\mathbf{u}+b\mathbf{v}) = aT(\mathbf{u})+bT(\mathbf{v})$

for all

$\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$

$a,b \in \mathbf{R}$