Matrices

Matrix operations

1. Evaluate each of the following
$$2\begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & -4 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 1 \\ 1 & 2 \end{pmatrix}.$$
Answer.
   (a) $\begin{pmatrix} 2 & 2 \\ 2 & -8 \end{pmatrix} - \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 3 \\ -1 & -9 \end{pmatrix},$
   (b) $\begin{pmatrix} 1\times 2 + 1\times 3 & 1\times(-1) + 1\times 1 \\ 1\times 2 - 4\times 3 & 1\times(-1) - 4\times 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ -10 & -5 \end{pmatrix},$
   (c) $\begin{pmatrix} 1\times 2 + 1\times 3 + 3\times 1 & 1\times(-1) + 1\times 1 + 3\times 2 \\ 1\times 2 - 4\times 3 + 2\times 1 & 1\times(-1) - 4\times 1 + 2\times 2 \end{pmatrix} = \begin{pmatrix} 8 & 6 \\ -8 & -1 \end{pmatrix},$
   (d) Impossible as the number of columns of the first matrix does not equal the number of rows of the second matrix.

2. Given
$$A = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix},$$
compute A^2, A^3 and hence write down A^n for $n > 1$.
Answer.
$$A^2 = \begin{pmatrix} 1 & 2k \\ 0 & 1 \end{pmatrix}$$
$$A^2 = \begin{pmatrix} 1 & 3k \\ 0 & 1 \end{pmatrix}$$
This pattern suggests that
$$A^n = \begin{pmatrix} 1 & nk \\ 0 & 1 \end{pmatrix}$$

If you know how to, use proof by induction to prove that this is really the case.

Inverse

1. Compute the inverse A^{-1} of the following matrices by hand.

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}, \qquad A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$$

Verify that $A^{-1}A = I$ and $AA^{-1} = I$.

Answer.
Use full Gaussian elimination to get the inverse of each matrix.

(a)

$$\begin{aligned}
&\left[\begin{array}{rr|rr} 1 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \end{array} \\
\sim &\left[\begin{array}{rr|rr} 1 & 1 & 1 & 0 \\ 0 & 5 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 = R_1 - R_2 \end{array} \\
\sim &\left[\begin{array}{rr|rr} 1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & -\frac{1}{5} \end{array}\right] \begin{array}{l} R_1 \\ R_2 = \frac{R_2}{5} \end{array} \\
\sim &\left[\begin{array}{rr|rr} 1 & 0 & \frac{4}{5} & \frac{1}{5} \\ 0 & 1 & \frac{1}{5} & -\frac{1}{5} \end{array}\right] \begin{array}{l} R_1 = R_1 - R_2 \\ R_2 \end{array}
\end{aligned}$$

Therefore

$$A^{-1} = \frac{1}{5}\begin{pmatrix} 4 & 1 \\ 1 & -1 \end{pmatrix},$$

(b)

$$\left[\begin{array}{rrr|rrr} 2 & 3 & -1 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 1 & 2 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & -7 & 1 & -2 & 0 \\ 0 & -1 & 1 & 1 & 0 & -2 \end{array}\right] \begin{array}{l} R_1 \\ R_2 = R_1 - 2R_2 \\ R_3 = R_1 - 2R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & -7 & 1 & -2 & 0 \\ 0 & 0 & -6 & 2 & -2 & -2 \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 = R_3 + R_2 \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 1 & -7 & 1 & -2 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \begin{array}{l} R_1 \\ R_2 \\ R_3 = -\frac{R_3}{6} \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 2 & 3 & 0 & \frac{2}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{4}{3} & \frac{1}{3} & \frac{7}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \begin{array}{l} R_1 = R_1 + R_3 \\ R_2 = R_2 + 7R_3 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 2 & 0 & 0 & \frac{14}{3} & -\frac{2}{3} & -\frac{20}{3} \\ 0 & 1 & 0 & -\frac{4}{3} & \frac{1}{3} & \frac{7}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \begin{array}{l} R_1 = R_1 3R_2 \\ R_2 \\ R_3 \end{array}$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{7}{3} & -\frac{1}{3} & -\frac{10}{3} \\ 0 & 1 & 0 & -\frac{4}{3} & \frac{1}{3} & \frac{7}{3} \\ 0 & 0 & 1 & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right] \begin{array}{l} R_1 = \frac{R_1}{2} \\ R_2 \\ R_3 \end{array}$$

Therefore

$$A^{-1} = \frac{1}{3}\begin{pmatrix} 7 & -1 & -10 \\ -4 & 1 & 7 \\ -1 & 1 & 1 \end{pmatrix}$$

2. Consider the following pair of matrices

$$A = \begin{pmatrix} 11 & 18 & 7 \\ a & 6 & 3 \\ -3 & -5 & -2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & 1 & 12 \\ b & -1 & -5 \\ -2 & 1 & -6 \end{pmatrix}$$

Compute the values of a and b so that A is the inverse of B while B is the inverse of A.

Answer.

$$AB = \begin{pmatrix} 19 + 18b & 0 & 0 \\ -6 + 3a + 6b & -3 + a & -48 + 12a \\ -5 - 5b & 0 & 1 \end{pmatrix}$$

If this is supposed to be the identity matrix then on the main diagonal we must have $19 + 18b = 1$ and $-3 + a = 1$. Hence, $a = 4$ and $b = -1$. And, indeed, once you substitute these values of a and b into the original matrices we get $AB = I$ and $BA = I$.

3. Let
$$A = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.$$
Show that
$$A^2 - 6A + I = 0,$$
where I is the 2×2 identity matrix and 0 stands for the 2×2 zero matrix. Use this result to compute A^{-1}.

Answer.
$$\begin{aligned} A^2 &= \begin{pmatrix} 29 & 12 \\ 12 & 5 \end{pmatrix}, 6A = \begin{pmatrix} 30 & 12 \\ 12 & 6 \end{pmatrix} \\ A^2 - 6A &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ A^2 - 6A + I &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$
To find A^{-1},
$$\begin{aligned} I &= 6A - A^2, \\ A^{-1}I &= 6A^{-1}A - A^{-1}A^2 \\ A^{-1} &= 6I - A = \begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}. \end{aligned}$$

MORE PROBLEMS ON MATRIX OPERATIONS AND INVERSES FROM KUTTLER'S BOOK

1. Suppose A and B are square matrices of the same size. Which of the following are correct?

   (a) $(A - B)^2 = A^2 - 2AB + B^2$

   (b) $(AB)^2 = A^2B^2$

   (c) $(A + B)^2 = A^2 + 2AB + B^2$

   (d) $(A + B)^2 = A^2 + AB + BA + B^2$

   (e) $A^2B^2 = A\ (AB)\ B$

   (f) $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$

   (g) $(A + B)(A - B) = A^2 - B^2$

   Answer. d and e. The rest are not true because, usually, you don't have $AB = BA$.

2. $A = \begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix}$. Find ALL 2×2 matrices B such that AB is the 2×2 zero matrix.
Answer.

$$\begin{pmatrix} -1 & -1 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} -x-z & -w-y \\ 3x+3z & 3w+3y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This means that you have to solve the linear system consisting of the four equations corresponding to the four entries of the 2×2 matrix: $-x-z=0, -w-y=0, 3x+3z=0, 3w+3y=0$. We find that $w=-y, x=-z$. This means that it is exactly the matrices of the form

$$\begin{pmatrix} x & y \\ -x & -y \end{pmatrix},$$

$x, y \in \mathbf{R}$, that, when multiplied by A from the left, result in a zero matrix.

3. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ 1 & k \end{pmatrix}$. Is it possible to choose k such that $AB = BA$?

Answer.

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & k \end{pmatrix} = \begin{pmatrix} 3 & 2k+2 \\ 7 & 4k+6 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & k \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 10 \\ 3k+1 & 4k+2 \end{pmatrix}$$

However, the matrices on the right can never be equal since the entry in the upper left corner of the first matrix is 3 whereas that of the second matrix is 7. We conclude that we cannot choose k to make the two matrices commute.

4. Let A be a square matrix. Show that A equals the sum of a symmetric and a skew symmetric matrix. (A matrix M is skew symmetric if $M = -M^T$. M is symmetric if $M^T = M$.) Hint: Show that $\frac{1}{2}(A^T + A)$ is symmetric and then consider using this as one of the matrices.

Answer.

$$\left(\frac{1}{2}(A^T + A)\right)^T = \frac{1}{2}((A^T)^T + A^T) = \frac{1}{2}(A + A^T) = \frac{1}{2}(A^T + A).$$

This proves that $\frac{1}{2}(A^T + A)$ is a symmetric matrix. Taking the hint to heart we should test whether $\frac{1}{2}(A^T + A) - A = \frac{1}{2}(A^T - A)$ is skew-symmetric. This is indeed the case

$$\left(\frac{1}{2}(A^T - A)\right)^T = \frac{1}{2}((A^T)^T - A^T) = -\frac{1}{2}(A^T - A).$$

Consequently, A can be written as a sum of a symmetric and an skew symmetric matrix as follows

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}.$$

5. Prove that every skew symmetric matrix has all zeros down the main diagonal. The main diagonal consists of every entry of the matrix which is of the form a_{ii}.

   Proof. If $A = -A^T$, then $a_{ii} = -a_{ii}$ and so $a_{ii} = 0$. ■

6. Prove that if $AB = AC$ and A^{-1} exists, then $B = C$.

   Proof. Since $AB = AC$ we have $(A^{-1}A)B = (A^{-1}A)C$ and therefore $B = C$. ■

7. Give an example of matrices A and B such that neither A nor B is equal to a zero matrix and yet AB is equal to a zero matrix.

   Answer.
$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

8. Prove that if A^{-1} exists and $A\mathbf{x} = \mathbf{0}$ then $\mathbf{x} = \mathbf{0}$.

   Proof. $A\mathbf{x} = \mathbf{0}$ implies $(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{0}$. Hence $\mathbf{x} = \mathbf{0}$. ■

9. Prove that if A is an invertible $n \times n$ matrix, then so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

   Proof. We have to show that $A^T(A^{-1})^T = (A^{-1})^T A^T = I$
$$\begin{aligned} A^T(A^{-1})^T &= (A^{-1}A)^T = I^T = I \\ (A^{-1})^T A^T &= (AA^{-1})^T = I^T = I \end{aligned}$$
   Hence the inverse of A^T exists and is $(A^{-1})^T = (A^T)^{-1}$. ■

10. Prove that $(AB)^{-1} = B^{-1}A^{-1}$ by verifying that $AB(B^{-1}A^{-1}) = I$ and $B^{-1}A^{-1}(AB) = I$.

    Proof. $(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = AA^{-1} = I$

    $B^{-1}A^{-1}(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$. ■

11. Prove that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

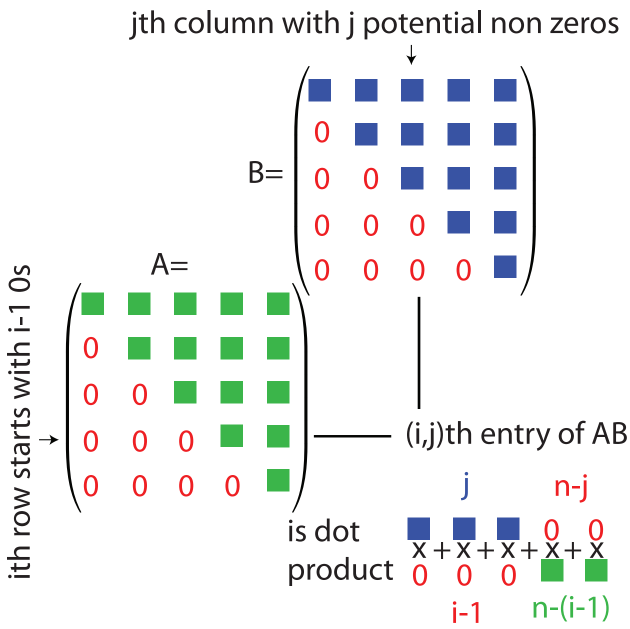
    Proof. Exactly the same as for the previous problem. ■

12. If A is invertible, show $(A^2)^{-1} = (A^{-1})^2$.

    Answer. Apply $(AB)^{-1} = B^{-1}A^{-1}$ by setting $A = B$.

13. Why is the product of two upper triangular $n \times n$ matrices A and B upper triangular?

    Answer. Consider the following diagram.



Then it is clear that the $(i, j)th$ entry of the product AB is definitely 0 if

$$j \leq i - 1$$

and

$$n - (i - 1) \leq n - j.$$

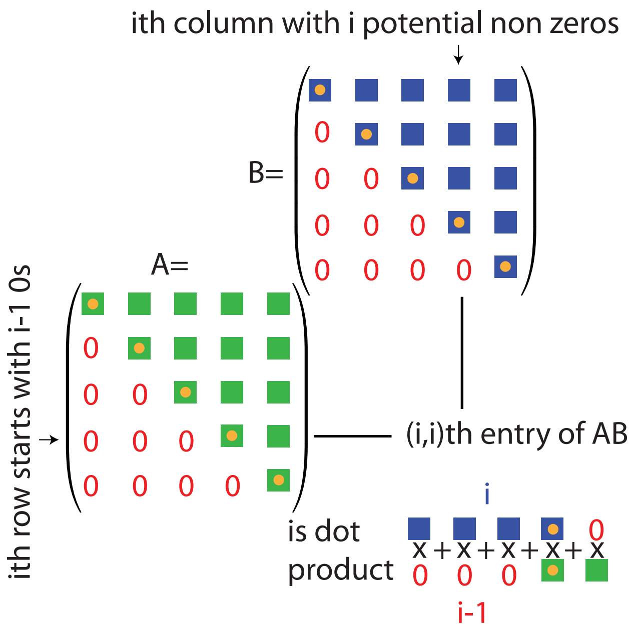
Both inequalities are equivalent to

$$j + 1 \leq i.$$

This means that all entries below the main diagonal are 0. Of course, this just says that the product is upper triangular.

14. Given two upper triangular matrices A and B of the same dimension, express the coefficients on the diagonal of AB in terms of the coefficients of A and B.

Answer. Consider the following diagram.



Then it is clear that the (i, i)th entry of AB is simply the product of the $a_{i,i}b_{i,i}$.

Determinants

1. For the matrix
$$A = \begin{pmatrix} 2 & 3 & -1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{pmatrix}$$
compute the determinant twice, first by expanding along the top row and second by expanding along the second column.

   Answer. Expanding along the top row gives
$$\det(A) = 2\begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} - 3\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2(-1-6) - 3(-1-3) - (2-1)$$
$$\therefore \det(A) = -3$$
   Expanding along the second column gives
$$\det(A) = -3\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix} = -3(-1-3) + (-2+1) - 2(6+1)$$
$$\therefore \det(A) = -3$$

2. Given
$$A = \begin{pmatrix} 1 & 1 \\ 1 & -4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix},$$
compute $\det(A)$, $\det(B)$ and $\det(AB)$. Verify that $\det(AB) = \det(A)\det(B)$.

   Answer.
   Calculating the determinant of A,
$$\det(A) = \begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix} = 1 \times (-4) - 1 \times 1 = -5$$
$$\therefore \det(A) = -5.$$
   Calculating the determinant of B,
$$\det(B) = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 \times 1 - (-1) \times 3 = 5$$
$$\therefore \det(B) = 5.$$

Multiplying A and B together gives,

$$AB = \begin{bmatrix} 5 & 0 \\ -10 & -5 \end{bmatrix}.$$

Calculating the determinant of AB,

$$\det(AB) = \begin{vmatrix} 5 & 0 \\ -10 & -5 \end{vmatrix} = 5 \times (-5) - 0 \times (-10) = -25$$
$$\therefore \det(AB) = -25 = -5 \times 5 = \det(A)\det(B).$$

3. Compute the following determinants using expansions along a suitable row or column.

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 2 \\ 0 & 9 & 8 \end{vmatrix}, \quad \begin{vmatrix} 4 & 3 & 2 \\ 1 & 7 & 8 \\ 3 & 9 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & 2 & 3 \\ 4 & 0 & 5 & 0 \\ 1 & 2 & 1 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 5 & 1 & 3 \\ 2 & 1 & 7 & 5 \\ 1 & 2 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{vmatrix}.$$

Answer.

(a) Expanding along the first column,

$$\det = 1\begin{vmatrix} 2 & 2 \\ 9 & 8 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 9 & 8 \end{vmatrix} + 0\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix}$$
$$\det = (16 - 18) - 3(16 - 27) + 0 = -2 + 33 = 31.$$

(b) Expanding along the first row,

$$\det = 4\begin{vmatrix} 7 & 8 \\ 9 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 8 \\ 3 & 3 \end{vmatrix} + 2\begin{vmatrix} 1 & 7 \\ 3 & 9 \end{vmatrix}$$
$$\det = 4(21 - 72) - 3(3 - 24) + 2(9 - 21) = -204 + 63 - 24 = -165$$

(c) Column 2 and 4 are the same, hence the determinant is equal to 0.

(d) Expanding along the third row,

$$\det = 1\begin{vmatrix} 5 & 1 & 3 \\ 1 & 7 & 5 \\ 1 & 0 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 & 3 \\ 2 & 7 & 5 \\ 3 & 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 5 & 3 \\ 2 & 1 & 5 \\ 3 & 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 5 & 1 \\ 2 & 1 & 7 \\ 3 & 1 & 0 \end{vmatrix}$$
$$\det = \left(\begin{vmatrix} 1 & 3 \\ 7 & 5 \end{vmatrix} + \begin{vmatrix} 5 & 1 \\ 1 & 7 \end{vmatrix}\right) - 2\left(3\begin{vmatrix} 1 & 3 \\ 7 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix}\right) + \left(\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix} - 5\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} + 3\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix}\right)$$
$$\det = (-16 + 34) - 2\left(3(-16) + 5\right) + (-4 - 5(-13) + 3(-1)) = 162$$

4. Compute the following determinants as fast as you can (under one second per determinant, if possible).

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 8 \end{vmatrix}, \quad \begin{vmatrix} 0 & 0 & 3 \\ 0 & 2 & 2 \\ 1 & 9 & 8 \end{vmatrix}, \quad \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix}, \quad \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}.$$

Answer.

(a) $\det = 1 \times 2 \times 8 = 16$

(b) Swap rows 1 and 3, then $\det = -1 \times 2 \times 8 = -16$

(c) All rows are the same therefore $\det = 0$

(d) Row 1 and 3 are the same, therefore $\det = 0$

(e) Swap rows 2 and 3, then $\det = -1 \times 8 \times 2 \times 2 = -32$

5. Which of the following statements are true?

   (a) If A is a 3×3 matrix with a zero determinant, then one row of A must be a multiple of some other row.

   (b) Even if any two rows of a square matrix are equal, the determinant of that matrix may be non-zero.

   (c) If any two columns of a square matrix are equal, then the determinant of that matrix is zero.

   (d) For any pair of $n \times n$ matrices, A and B, we always have $\det(A+B) = \det(A) + \det(B)$.

   (e) Let A be a 3×3 matrix. Then $\det(7A) = 7^3 \det(A)$.

   (f) If A^{-1} exists, then $\det(A^{-1}) = \det(A)$.

   Answer. Only c and e are true.

6. Assume that A is square matrix with inverse A^{-1}. Prove that $\det(A^{-1}) = 1/\det(A)$

   Proof.
$$1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$$
   Hence $\det(A^{-1}) = 1/\det(A)$. ■

MORE PROBLEMS ON DETERMINANTS FROM KUTTLER'S BOOK

7. An elementary operation is applied to transform the first matrix into the second. Which elementary operation is being used and how does it affect the value of the determinant.
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & b \\ a+c & b+d \end{pmatrix}$$
   Answer. The elementary operation consists of adding the first row to the second. Therefore the determinant is unchanged.

8. An elementary operation is applied to transform the first matrix into the second. Which elementary operation is being used and how does it affect the value of the determinant.
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & b \\ 2c & 2d \end{pmatrix}$$
   Answer. The second row was multiplied by 2 so the determinant of the second matrix is 2 times the determinant of the original matrix.

9. An elementary operation is applied to transform the first matrix into the second. Which elementary operation is being used and how does it affect the value of the determinant.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} b & a \\ d & c \end{pmatrix}$$

Answer. In this case the two columns were switched. Consequently, the determinant of the second matrix is -1 times the determinant of the first matrix.

10. Prove from scratch that $\det(AB) = \det(A)\det(B)$ if A and B are 2×2 matrices.

Proof. Set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

Then

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

and $\det(A) = ad - bc$, $\det(B) = eh - fg$. Finally,

$$\det(AB) = \det(A)\det(B) = bcfg - adfg - bceh + adeh. \qquad \blacksquare$$

11. An $n \times n$ matrix is called **nilpotent** if for some positive integer k we have $A^k = 0$. If A is a nilpotent matrix and k is the smallest possible integer such that $A^k = 0$, what are the possible values of $\det(A)$?

Answer. The determinant must be 0 because $0 = \det(0) = \det\left(A^k\right) = (\det(A))^k$.

12. A matrix is said to be **orthogonal** if $A^T A = I$. This means that the inverse of an orthogonal matrix is just its transpose. What are the possible values of $\det(A)$ if A is an orthogonal matrix?

Answer. $\det\left(AA^T\right) = \det(A)\det\left(A^T\right) = (\det(A))^2 = 1$. Therefore $\det(A) = 1$, or -1.

13. Fill in the missing entries to make the following matrix orthogonal.

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \_ & \_ \\ \_ & \frac{\sqrt{6}}{3} & \_ \end{pmatrix}.$$

Answer. Set

$$A = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{\sqrt{12}}{6} \\ \frac{1}{\sqrt{2}} & a & b \\ c & \frac{\sqrt{6}}{3} & d \end{pmatrix}$$

Then

$$AA^T = \begin{pmatrix} 1 & \frac{a}{\sqrt{6}} + \frac{b}{\sqrt{3}} - \frac{1}{2} & -\frac{c}{\sqrt{2}} + \frac{d}{\sqrt{3}} + \frac{1}{3} \\ \frac{a}{\sqrt{6}} + \frac{b}{\sqrt{3}} - \frac{1}{2} & a^2 + b^2 + \frac{1}{2} & \sqrt{\frac{2}{3}}a + \frac{c}{\sqrt{2}} + bd \\ -\frac{c}{\sqrt{2}} + \frac{d}{\sqrt{3}} + \frac{1}{3} & \sqrt{\frac{2}{3}}a + \frac{c}{\sqrt{2}} + bd & c^2 + d^2 + \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solving the corresponding system of equations in the unknowns a, b, c, d gives

$$a = \frac{1}{\sqrt{6}}, b = \frac{1}{\sqrt{3}}, c = 0, d = -\frac{1}{\sqrt{3}}.$$

So, the completed matrix is

$$\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{\sqrt{6}}{3} & -\frac{1}{\sqrt{3}} \end{pmatrix}.$$

14. Let A and B be two $n \times n$ matrices. $A \sim B$ (A is **similar** to B) means there exists an invertible matrix S such that $A = S^{-1}BS$. Prove that if $A \sim B$, then $B \sim A$. Show also that $A \sim A$ and that if $A \sim B$ and $B \sim C$, then $A \sim C$.

    Proof. If $A = S^{-1}BS$, then $SAS^{-1} = B$ and so if $A \sim B$, then $B \sim A$. It is obvious that $A \sim A$ because you can let $S = I$. Say $A \sim B$ and $B \sim C$. Then $A = P^{-1}BP$ and $B = Q^{-1}CQ$. Therefore,

    $$A = P^{-1}Q^{-1}CQP = (QP)^{-1} C (QP)$$

    and so $A \sim C$. ■

15. Continuing on with the last problem prove that if $A \sim B$, then $\det(A) = \det(B)$.

    Proof. $\det(A) = \det(S^{-1}BS) = \det(S^{-1})\det(B)\det(S) = \det(B)\det(S^{-1}S) = \det(B)$. ■

16. Tell whether the statement is true or false.

    (a) If A is a 3×3 matrix with a 0 determinant, then one column must be a multiple of some other column.

    Answer. False. Consider $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 5 & 4 \\ 0 & 3 & 3 \end{pmatrix}$.

    (b) If any two columns of a square matrix are equal, then the determinant of the matrix equals zero.

    Answer. True.

    (c) For A and B two $n \times n$ matrices, $\det(A + B) = \det(A) + \det(B)$.

    Answer. False.

(d) For A an $n \times n$ matrix, $\det(3A) = 3\det(A)$.

Answer. False (except for $n = 1$).

(e) If A^{-1} exists then $\det(A^{-1}) = \det(A)^{-1}$.

Answer. True.

(f) If B is obtained by multiplying a single row of A by 4, then $\det(B) = 4\det(A)$.

Answer. True.

(g) For A a square matrix, $\det(-A) = (-1)^n \det(A)$.

Answer. True.

(h) If A is a square matrix, then $\det\left(A^T A\right) \geq 0$.

Answer. True.

(i) Cramer's rule is useful for finding solutions to systems of linear equations in which there is an infinite set of solutions.

Answer. False.

(j) If $A^k = 0$ for some positive integer k, then $\det(A) = 0$.

Answer. True

(k) If $A\mathbf{x} = \mathbf{0}$ for some $\mathbf{x} \neq \mathbf{0}$, then $\det(A) = 0$.

Answer. True.

17. Use Cramer's rule to find the solution to

$$\begin{aligned} x + 2y &= 1 \\ 2x - y &= 2 \end{aligned}$$

Solution is: $x = 1, y = 0$.

18. Here is a matrix,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & -\sin(t) \\ 0 & \sin(t) & \cos(t) \end{pmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

Answer. No. This matrix has the non-zero determinant $\cos^2(t) + \sin^2(t) = 1$ for all t.

19. Here is a matrix,

$$\begin{pmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{pmatrix}$$

Does there exist a value of t for which this matrix fails to have an inverse? Explain.

Answer. $\begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ t & 0 & 2 \end{vmatrix} = t^3 + 2$. This means that the matrix has no inverse when $t = -\sqrt[3]{2}$.

20. Use the formula for the inverse in terms of the cofactor matrix to find the inverse of the matrix,

$$A = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t \cos(t) & e^t \sin(t) \\ 0 & e^t \cos(t) - e^t \sin(t) & e^t \cos(t) + e^t \sin(t) \end{pmatrix}.$$

Answer.

$$\det \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t \cos(t) & e^t \sin(t) \\ 0 & e^t \cos(t) - e^t \sin(t) & e^t \cos(t) + e^t \sin(t) \end{pmatrix} = e^{3t}.$$

Hence the inverse is

$$e^{-3t} \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} \cos(t) + e^{2t} \sin(t) & e^{2t} \sin(t) - e^{2t} \cos(t) \\ 0 & -e^{2t} \sin(t) & e^{2t} \cos(t) \end{pmatrix}^T$$

$$= \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{-t} (\cos(t) + \sin(t)) & -\sin(t) e^{-t} \\ 0 & -e^{-t} (\cos(t) - \sin(t)) & \cos(t) e^{-t} \end{pmatrix}$$

21. Suppose A is an upper triangular matrix. Show that A^{-1} exists if and only if all entries on the main diagonal are non-zero. Is it true that A^{-1} will also be upper triangular? Explain. Is everything the same for lower triangular matrices?

    Answer. The given condition is what it takes for the determinant to be non-zero. Recall that the determinant of an upper triangular matrix is just the product of the entries on the main diagonal. A^{-1} will also be upper triangular. You can use the formula for the inverse involving the cofactor matrix to show this. All these statements are also true for lower triangular matrices.

22. If A, B, and C are each $n \times n$ matrices and ABC is invertible, why are each of A, B, and C invertible?

    Answer. This is obvious because $\det(ABC) = \det(A)\det(B)\det(C)$ and if this product is non-zero, then each determinant in the product is non-zero and so each of these matrices is invertible.

SOME TEST QUESTIONS

23. Let A be an $n \times n$ matrix with zero determinant. Is it possible by looking at A alone to determine whether a system of linear equations $A\mathbf{x} = \mathbf{b}$ has no or infinitely many solutions.

24. State the two universal laws of swaps.

25. Describe an algorithm/a computer program that can turn any permutation into the identity permutation using only swaps.

26. Which of the following is a term of the general formula for the 5×5 determinant: (a) $a_{1,1}a_{2,3}a_{3,4}a_{4,2}a_{5,5}$; (b) $a_{1,1}a_{2,3}a_{1,4}a_{4,2}a_{5,5}$, (c) $a_{1,1}a_{2,2}a_{3,6}a_{4,4}a_{5,5}a_{6,3}$.

27. In the 4×4 determinant formula which sign precedes $a_{1,4}a_{2,3}a_{3,2}a_{4,1}$.

28. Why is the product of two upper triangular matrices of the same dimension upper triangular?

29. Show that that among the permutations of $123...n$ exactly half are odd and half are even.

30. Based only on our definition of the determinant in the lecture notes show that a square matrix with a row of zeros has determinant equal to zero.

31. Based only on our definition of the determinant in the lecture notes show that the determinant of an upper triangular matrix is equal to the product of the coefficients on the diagonal.

32. What is the minimum number of non-zero coefficients of an $n \times n$ matrix with non-zero determinant.

33. What are the elementary 5×5 matrices that: a) swap rows 1 and 5; b) multiply the 3rd row by 27; c) add -5 times the first row to the 3rd row.

34. Show that elementary matrices are invertible and that their inverses are also elementary matrices.

35. What are the inverses of the three elementary matrices above?

36. Find a 4×4 matrix P that inverts the rows of any other 4×4 matrix A in the sense that the 1st, 2nd, 3rd and 4th row of the matrix PA are the 4th, 3rd, 2nd, and 1st row, respectively, of the matrix A.

37. Someone tells you that the product of 666 6×6 matrices is invertible. How many of the 666 matrices do you expect to be invertible, too?

38. Describe three ways to calculate the inverse of a square matrix.