Power series

1. Find the Maclaurin series of
$$x^3 + x^2 + 2x + 1$$

2. Find the Maclaurin series of $f(x) = \sin(x)$. Also find the 4th remainder term $R_4(x)$ of $\sin(x)$ at 0 (this is the remainder term used to estimate how well the 4th partial sum of the Maclaurin series approximates $\sin(x)$.)

3. Find the Maclaurin series of $f(x) = \ln(1 + x)$. Also find the 4th remainder term $R_4(x)$ of $\ln(1 + x)$ at 0 (this is the remainder term used to estimate how well the 4th partial sum of the Maclaurin series approximates $\ln(1 + x)$.)

4. Find the Maclaurin series of the following functions by making suitable substitutions into one of the Maclaurin series for $e^x, \sin(x), \frac{1}{1-x}$.
$$(i)\ e^{-x}, \quad (ii)\ \sin(2x), \quad (iii)\ \frac{1}{1+x^3}.$$
And also figure out for which values of x these Maclaurin series are equal to the functions.

5. Differentiate the Maclaurin series of e^x, $\sin(x)$ and $\cos(x)$ to double-check that $(e^x)' = e^x$, that $(\sin(x))' = \cos(x)$ and that $(\cos(x))' = -\sin(x)$.

6. Double-check that the Maclaurin series
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$
$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
all have an infinite radius of convergence using the formula
$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

7. Find the *interval* of convergence of the following power series.

$$(i)\sum_{n=1}^{\infty} nx^n, \quad (ii)\sum_{n=1}^{\infty}\frac{(-1)^n}{5^n\sqrt{n}}x^n, \quad (iii)\sum_{n=1}^{\infty}\frac{(-1)^n n}{2^n(n+1)^3}x^n, \quad (iv)\sum_{n=1}^{\infty}\frac{(2n)!}{n!}x^n$$

Do this by first figuring out the radius of convergence using the following limit

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

(if it exists) and then testing the series for convergence at $x = R$ and $x = -R$.

8. Power series expansions can also be very useful for calculating indeterminate forms. Just for fun, try to calculate the indeterminate forms

$$\lim_{x\to 0}\frac{\sin(x)}{x}, \text{ and } \lim_{x\to 0}\frac{e^x - x - 1}{x^2}$$

using our power series representation of $\sin(x)$ and e^x.

$$\sin(x) = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

9. Compute the Taylor series of each of the following functions at the given points from scratch.

$$(i)\ \frac{1}{x} \text{ at } x = 1, \quad (ii)\ e^x \text{ at } x = -1, \quad (iii)\ \ln(x) \text{ at } x = 2.$$

10. (Completely optional and definitely not examinable!) Here is another boredom killer. Define

$$F(x) = \sum_{n=1}^{\infty} f_n x^n,$$

where f_n is the nth Fibonacci number. So,

$$F(x) = 0 + 1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots.$$

Prove that $f_n < 2^n$ for all n and use this to show that $F(x)$ converges if $|x| < \frac{1}{2}$. Also show that

$$(1 - x - x^2)F(x) = x$$

and that therefore

$$F(x) = \frac{x}{1 - x - x^2}.$$

$F(x)$ is an example of a so-called *generating function*.

If you have a close look at questions 3 and 4 of the blackboard in *Goodwill Hunting* (we considered questions 1 and 2 in the lectures) then you will find that they are asking about the generating functions for the numbers $w_n(i,j)$ of n step walks from vertex i to vertex j. On the right side of the blackboard you can see that these generating functions can be expressed in terms of determinants.

Generating functions are silver bullets when it comes to finding closed formulas for sequences given in a recurrence relation, for solving enumeration problems, etc. Very nice stuff!!

TEST QUESTIONS

11. Prove Euler's formula from scratch using the Maclaurin series of $\sin(x)$ and $\cos(x)$. Derive Euler's identity from Euler's formula.

12. Be able to reproduce the proof in the lecture notes that e is an irrational number.

13. How does one define e^A if A is a diagonalizable matrix?

14. Is the Maclaurin series of a function equal to the function?

15. What is the radius of convergence of a power series? What is the interval of convergence of a power series?

16. Give examples of two different functions having the same Maclaurin series.

17. If you integrate a power series with radius of convergence R, what is the radius of convergence of the new series? How do the intervals of convergence of the original and the new series compare?

18. If you differentiate a power series with radius of convergence R, what is the radius of convergence of the new series? How do the intervals of convergence of the original and the new series compare?

19. What is the main use of Taylor's formula?

20. What is the difference between a Taylor series and a Maclaurin series?