MTH1030
Techniques for Modelling

Lecture 22 & 23

Series (part 2)

Monash University

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Warm welcoming words

Last time we defined an object called an infinite series, which serves the purpose of an infinite sum. However, an infinite series is a limit, and thus it is not straightforward to determine whether convergence occurs or not. Moreover, if convergence does occur, it is not always straightforward to determine the value of the infinite series!

Warm welcoming words

When dealing with an infinite series we have three questions we can ask ourselves:

1. Does the infinite series converge or diverge?
2. If it converges, what is the value of the infinite series?
3. If it diverges, how does it diverge (to infinity or does not settle?)

We always care about 1. We often (but not always!) care about 2. We sometimes care about 3.

Telescoping sum

The following type of sum is incredibly useful in mathematics.

Definition (Telescoping sum)

Let a_n be a sequence. Then

$$\sum_{k=1}^{n}(a_k - a_{k-1}) = a_n - a_0$$

is called a telescoping sum.

If we instead had an infinite series rather than a finite sum, we would call this a *Telescoping series*.

Telescoping sum

Telescoping sums are weirdly powerful in mathematics!

Example

Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}.$$

To determine whether convergence occurs, and what the limit would be in this case, we can try and find the partial sum S_N.

Question 1

Question (Question 1)

Consider the series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right).$$

The series

1. Converges.
2. Diverges to positive infinity.
3. Diverges to negative infinity.
4. Diverges and does not settle down.

Harmonic series

The harmonic series is the infinite series of $a_n = 1/n$, specifically,

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

It turns out

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

However, the harmonic series passes the n-th term test, as the limit of the sequence being summed (i.e. $a_n = 1/n$) is 0. So how can we show this series diverges?

Harmonic series

Lemma

The harmonic series diverges to positive infinity. Specifically,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Convergence tests

There are many (many!) tests to determine whether a series converges or diverges. We will slowly present a few over the next few lectures.

Integral test

The previous proof regarding divergence of the harmonic series alludes to the following theorem:

Theorem (Integral test)

Let f be a non-negative monotone decreasing function on the interval $[1, \infty)$. Let $a_n = f(n)$. Then:

1. $\sum_{n=1}^{\infty} a_n$ ***converges*** *if and only if* $\int_1^{\infty} f(x)\mathrm{d}x$ *converges.*
2. $\sum_{n=1}^{\infty} a_n$ ***diverges to infinity*** *if and only if* $\int_1^{\infty} f(x)\mathrm{d}x$ *diverges.*

Essentially, the convergence/divergence of the series is the same as the convergence/divergence of the integral. Perhaps surprising this is an if and only if...

Question 2

Question (2)

Let $f(x) = e^{-x}$ and $a_n = f(n) = e^{-n}$. Does the infinite series

$$\sum_{n=1}^{\infty} e^{-n}$$

converge or diverge?

p-series

A *p-series* is the following infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

for $p > 0$. Clearly $p = 1$ gives the Harmonic series.

Proposition (p series)

1. The p-series converges if $p > 1$.
2. The p-series diverges if $p \leq 1$.

p-series

Okay, so p-series converge for $p > 1$. Great! What does it converge to? Well...definitely something. But it's usually much harder to determine the value of a convergent series! It turns out:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

However, it is not known what the exact values are when $p > 1$ and p is odd. For example, $\sum_{n=1}^{\infty} \frac{1}{n^3} = ?$.

Integral test remainder

The integral test alludes to a way to bound the value of our infinite series. How?

Integral test remainder

Which brings us to the following theorem.

Theorem (Integral test remainder)

Let f be a non-negative monotone decreasing function on the interval $[1, \infty)$. Let $a_n = f(n)$. Suppose that

$$\int_1^\infty f(x)\mathrm{d}x$$

converges. Then

$$\int_{n+1}^\infty f(x)\mathrm{d}x \leq S - S_n \leq \int_n^\infty f(x)\mathrm{d}x,$$

where $S_n = \sum_{k=1}^n a_k$ and $S = \lim_{n\to\infty} S_n$.

Integral test remainder

So although we can't find out what $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is exactly, we can approximate it!

Example

Using the integral test remainder with a sufficiently large N, we can see that

$$\sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.20206.$$

Riemann zeta function

What if we looked at

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots$$

where $z = a + ib$ is a complex number? This is called the *Riemann zeta function*.

Exercise (Riemann hypothesis)

Show that $\zeta(z) = 0$ if and only if z is a negative even integer (i.e., $z = -2, -4, -6, \dots$) or z has real part equal to $1/2$ (i.e., $z = \frac{1}{2} + ib$).

I'm kidding here. This is a Millennium problem. It has not been proven, and if you do prove this, you get 1 million US dollars.

Tail of series

Whilst we're at it, we should appreciate the following fact. The convergence/divergence of your series has NOTHING to do with the terms at the start. For example,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=10}^{\infty} \frac{1}{n} = \sum_{n=10^{100}}^{\infty} \frac{1}{n} = \infty.$$

And also

$$\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty, \quad \sum_{n=10}^{\infty} \frac{1}{n^2} < \infty, \quad \sum_{n=10^{100}}^{\infty} \frac{1}{n^2} < \infty.$$

So if your series does converge, starting it at a further point in the series will change the value, but it will still converge!

Tail of sequences

This is also true for sequences. Clearly

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

And if we define a sequence

$$b_n = \begin{cases} n, & n = 1, 2, \dots, 10^{100}, \\ \frac{1}{n}, & n = 10^{100} + 1, \dots. \end{cases}$$

Then $\lim_{n\to\infty} b_n = 0$ despite b_n increasing for the first 10^{100} elements!

It's the tail that matters!

To put this succinctly:

Remark

When determining convergence/divergence of a sequence or series, the ONLY thing that matters is the tail. Of course, starting a convergent series further down, or removing finitely many values from it will change its value, but it will always converge.

Comparison test

Another test. The idea: if you have a series whose convergence is hard to determine...'compare' it (somehow) to another series whose convergence you can easily determine!

Theorem (Comparison test)

Suppose that a_n and b_n are non-negative sequences and suppose $a_n \leq b_n$ for all n beyond some positive integer N. Then

1. *Suppose that $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ must converge.*
2. *Suppose that $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.*

For 1., basically if the sequence b_n eventually 'dominates' a_n, and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ must also converge. Similar idea for 2.

Comparison test

Example

Using the comparison test, we can show that the following series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$$

converges.

Comparison test

Example

Using the comparison test, we can give an alternative proof to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

Question 3

Question (3)

Does the infinite series

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$$

converge or diverge?

Properties of series

Series are not (finite) sums, but they do behave a bit like them...

Theorem

Suppose $\sum_{n=1}^{\infty} a_n = A$ *and* $\sum_{n=1}^{\infty} b_n = B$. *Then*

$$\sum_{n=1}^{\infty} a_n + b_n = A + B.$$

If $c \in \mathbb{R}$, *then*

$$\sum_{n=1}^{\infty} ca_n = cA.$$

This also makes sense if $A = B = \infty$ (both diverge to positive infinity) or $A = B = -\infty$ (both diverge to negative infinity). It DOES NOT make sense if $A = \infty$ and $B = -\infty$ (or vice versa).