Vectors, dot product, cross product

1. Find all the vectors whose tips and tails are among the three points with coordinates $(2,-2,3)$, $(3,2,1)$ and $(0,-1,-4)$.

   Answer. Let $A=(2,-2,3)$, $B=(3,2,1)$, and $C=(0,-1,4)$.
   $\overrightarrow{AB}=B-A=(1,4,-2)=-\overrightarrow{BA}$,
   $\overrightarrow{AC}=C-A=(-2,1,-7)=-\overrightarrow{CA}$,
   $\overrightarrow{BC}=C-B=(-3,-3,-5)=-\overrightarrow{CB}$,
   $\overrightarrow{AA}=\overrightarrow{BB}=\overrightarrow{CC}=(0,0,0)$.

2. Let $\mathbf{v}=(3,2,-2)$. How long is $-2\mathbf{v}$. Find a unit vector (a vector of length 1) in the direction of $\mathbf{v}$.

   Answer.

   $$\|-2\mathbf{v}\|=2\,\|\mathbf{v}\|=2\sqrt{3^2+2^2+(-2)^2}=2\sqrt{17}$$

   Unit vector obtained by dividing $\mathbf{v}$ by its length. Length of $\mathbf{v}$ is $\sqrt{17}$, not $2\sqrt{17}$. You might get confused and accidentally use the length of $-2\mathbf{v}$.
   $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}=\dfrac{1}{\sqrt{17}}(3,2,-2)$

3. For each pair of vectors given below, calculate the dot product and the angle θ between the vectors.

   (a) $\mathbf{v}=(3,2,-2)$ and $\mathbf{w}=(1,-2,-1)$
   (b) $\mathbf{v}=(0,-1,4)$ and $\mathbf{w}=(4,2,-2)$
   (c) $\mathbf{v}=(2,0,2)$ and $\mathbf{w}=(-3,-2,0)$

   Answer.

   (a) $\mathbf{v}\cdot\mathbf{w}=3-4+2=1$,
   $\|\mathbf{v}\|=\sqrt{3^2+2^2+2^2}=\sqrt{17}$,
   $\|\mathbf{w}\|=\sqrt{1^2+2^2+1^2}=\sqrt{6}$,
   $\theta=\arccos\left(\frac{1}{\sqrt{6\cdot 17}}\right)\approx 1.4716$ radians.

(b) $\mathbf{v} \cdot \mathbf{w} = 0 - 2 - 8 = -10$,
$\|\mathbf{v}\| = \sqrt{0^2 + 1^2 + 4^2} = \sqrt{17}$,
$\|\mathbf{w}\| = \sqrt{4^2 + 2^2 + 2^2} = \sqrt{24}$,
$\theta = \arccos\left(\frac{-10}{\sqrt{17 \cdot 24}}\right) \approx 2.0887$ radians.

(c) $\mathbf{v} \cdot \mathbf{w} = -6 + 0 + 0 = -6$,
$\|\mathbf{v}\| = \sqrt{2^2 + 0^2 + 2^2} = \sqrt{8}$,
$\|\mathbf{w}\| = \sqrt{3^2 + 2^2 + 0^2} = \sqrt{13}$,
$\theta = \arccos\left(\frac{-6}{\sqrt{8 \cdot 13}}\right) \approx 2.1998$ radians.

4. Given the two vectors $\mathbf{v} = (\cos(\theta), \sin(\theta), 0)$ and $\mathbf{w} = (\cos(\phi), \sin(\phi), 0)$, use the dot product to derive the trigonometric identity $\cos(\theta - \phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)$.

Answer.
$\mathbf{v} \cdot \mathbf{w} = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)$,
$\|\mathbf{v}\| = \sqrt{\cos(\theta)^2 + \sin(\theta)^2} = 1$,
$\|\mathbf{w}\| = \sqrt{\cos(\phi)^2 + \sin(\phi)^2} = 1$,
$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos(\theta - \phi) = 1 \cdot 1 \cdot \cos(\theta - \phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)$.

5. Use the dot product to determine which of the following two vectors are perpendicular to one another: $\mathbf{u} = (3, 2, -2), \mathbf{v} = (1, 2, -2), \mathbf{w} = (2, -1, 2)$.

Answer.
The two vectors are perpendicular if their dot product is equal to zero.
$\mathbf{u} \cdot \mathbf{v} = 3 + 4 + 4 = 11$,
$\mathbf{u} \cdot \mathbf{w} = 6 - 2 - 4 = 0$,
$\mathbf{v} \cdot \mathbf{w} = 2 - 2 - 4 = -4$.
Therefore $\mathbf{u}$ and $\mathbf{w}$ are the only vectors perpendicular to each other.

6. For each pair of vectors given below, calculate the cross product by hand. Calculate the area of the parallelogram spanned by the two vectors.

(a) $\mathbf{v} = (3, 2, -2), \mathbf{w} = (1, -2, -1)$

(b) $\mathbf{v} = (0, -1, 4), \mathbf{w} = (4, 2, -2)$

(c) $\mathbf{v} = (2, 0, 2), \mathbf{w} = (-3, -2, 0)$

Answer.
The area of a parallelogram is equal to the length of the cross product.

(a) $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -2 \\ 1 & -2 & -1 \end{vmatrix} = (-6, 1, -8), \qquad \|\mathbf{v} \times \mathbf{w}\| = \sqrt{101}$

(b) $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & 4 \\ 4 & 2 & -2 \end{vmatrix} = (-6, 16, 4), \qquad \|\mathbf{v} \times \mathbf{w}\| = 2\sqrt{77}$

(c) $\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 2 \\ -3 & -2 & 0 \end{vmatrix} = (4,-6,-4), \qquad \|\mathbf{v}\times\mathbf{w}\| = 2\sqrt{17}$

7. Find the area of the triangle determined by the three points, $(1,2,3)\,,(4,2,0)$ and $(-3,2,1)$.

   Answer.
   The area of a triangle is half the area the parallelogram which is equal to the length of the cross product of the vectors formed by the three points.
   Let $A = (1,2,3)$, $B = (4,2,0)$, and $C = (-3,2,1)$.
   $\overrightarrow{AB} = B - A = (3,0,-3)$,
   $\overrightarrow{AC} = C - A = (-4,0,-2)$,
   $\mathbf{v}\times\mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & -3 \\ -4 & 0 & 2 \end{vmatrix} = (0,18,0), \qquad \|\mathbf{v}\times\mathbf{w}\| = 18.$
   The area of a triangle is half the area of the parallelogram which in this case, we just half the length of $\mathbf{v}\times\mathbf{w}$, which gives us a final answer of 9.

8. Calculate the volume of the parallelepiped defined by the three vectors $\mathbf{u} = (3,2,-2)$, $\mathbf{v} = (1,2,-2)$, $\mathbf{w} = (2,-1,2)$.

   Answer.
   The area of a parallelogram is equal to the length of $\mathbf{u}\times\mathbf{v}$. If we multiply this by a height vector $\mathbf{w}$, then we get a volume of a parallelepiped, which we calculate by $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}$.
   $(\mathbf{u}\times\mathbf{v}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -2 \\ 1 & 2 & -2 \end{vmatrix} = (0,4,4),$
   $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = (0,4,4)\cdot(2,-1,2) = 0-4+8 = 4.$
   Therefore the volume of the parallelepiped is 4.

EXTRA QUESTIONS

9. Prove that, given any two vectors $\mathbf{u},\mathbf{v}\in\mathbf{R}^n$, the following holds

$$||\mathbf{u}| - |\mathbf{v}|| \le |\mathbf{u}-\mathbf{v}|.$$

   (Hint: Start by writing $\mathbf{u} = \mathbf{u}-\mathbf{v}+\mathbf{v}$ and then apply the triangle inequality.)

   Proof:

$$\mathbf{u} = \mathbf{u}-\mathbf{v}+\mathbf{v}.$$

   Therefore

$$|\mathbf{u}| = |(\mathbf{u}-\mathbf{v})+\mathbf{v}|$$

   Applying the triangle inequality to the right side of this equation gives

$$|\mathbf{u}| \le |\mathbf{u}-\mathbf{v}| + |\mathbf{v}|.$$

Therefore

$$|\mathbf{u}| - |\mathbf{v}| \leq |\mathbf{u} - \mathbf{v}|.$$

Similarly,

$$|\mathbf{v}| - |\mathbf{u}| \leq |\mathbf{v} - \mathbf{u}| = |\mathbf{u} - \mathbf{v}|.$$

Finally, combining the two inequalities gives the result that we are after

$$||\mathbf{u}| - |\mathbf{v}|| \leq |\mathbf{u} - \mathbf{v}|.$$

$\blacksquare$

10. Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$. Define as in the lecture notes

$$\mathbf{proj_u}(\mathbf{v}) = \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2}\right) \mathbf{u} \text{ (the projection of } \mathbf{v} \text{ onto } \mathbf{u})$$

and

$$\mathbf{perp_u}(\mathbf{v}) = \mathbf{v} - \mathbf{proj_u}(\mathbf{v}).$$

Prove that $\mathbf{perp_u}(\mathbf{v}) \cdot \mathbf{proj_u}(\mathbf{v}) = 0$, that is, $\mathbf{perp_u}(\mathbf{v})$ is perpendicular to $\mathbf{proj_u}(\mathbf{v})$.

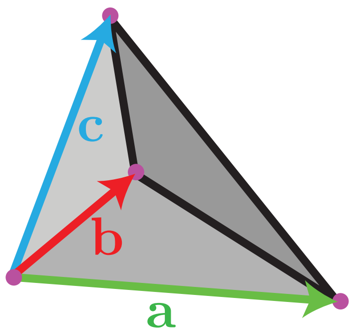
Proof:

$$\mathbf{perp_u}(\mathbf{v}) \cdot \mathbf{proj_u}(\mathbf{v}) = \left(\mathbf{v} - \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2}\right) \mathbf{u}\right) \cdot \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2}\right) \mathbf{u}$$

$$= \mathbf{v} \cdot \mathbf{u} \frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2} - \left(\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|^2}\right)^2 \mathbf{u} \cdot \mathbf{u} = \frac{(\mathbf{v} \cdot \mathbf{u})^2}{|\mathbf{u}|^2} - \frac{(\mathbf{v} \cdot \mathbf{u})^2}{|\mathbf{u}|^4} |\mathbf{u}|^2 = 0.$$

$\blacksquare$

11. Prove that the tetrahedron determined by the three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ (shown in the following diagram) is equal to one sixth of the volume of the parallelepiped spanned by the same vectors.



Proof. The volume of a pyramid is

$$\frac{1}{3} \text{ area base pyramid } \times \text{ height pyramid}$$

and the volume of the parallelepiped is

$$\text{area base pyramid } \times \text{ height pyramid.}$$

If we let the base of both shapes be spanned by $\mathbf{a}$ and $\mathbf{b}$, then

$$\text{area base pyramid} = \frac{1}{2}\text{area base parallelepiped}$$

and both have the same height. This means that the volume of the pyramid is equal to $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ the volume of the parallelepiped. ■

12. If $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 5$ what is the value of $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$?

    Answer. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 5$ as well. To see this note that

    $$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$$

    This means that we are dealing with a box product that involves the same vectors as $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Hence the absolute value of both expressions is the same. On the other hand, both $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{b}, \mathbf{c}, \mathbf{a}$ form right-handed systems, so both box products also have the same sign.

THE FOLLOWING PROBLEMS ARE FROM KUTTLER'S BOOK

13. $(1, 2) + (3, 4, 5)$ does not make any sense since only vectors that are elements of the same $\mathbf{R}^n$ can be added.

14. Three forces are applied to a point which does not move. Two of the forces are $2\mathbf{i}+\mathbf{j}+3\mathbf{k}$ Newtons and $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ Newtons. Find the third force.

    Answer. $(2,\ 1,\ 3) + (1,\ -3, 2) + (x,\ y,\ z) = (0, 0, 0)$. Consequently the third force is $(-3, 2,\ -5)$.

15. Does it make sense to speak of $\mathbf{proj_0}(\mathbf{v})$?

    Answer. No, it does not. The zero vector has no direction and the formula doesn't make sense either since you'd be dividing by 0.

16. Show that

    $$\mathbf{a} \cdot \mathbf{b} = \frac{1}{4}[|\mathbf{a} + \mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2].$$

    Proof.

    $$\frac{1}{4}[|\mathbf{a} + \mathbf{b}|^2 - |\mathbf{a} - \mathbf{b}|^2] = \frac{1}{4}(\mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} + 2(\mathbf{a} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2(\mathbf{a} \cdot \mathbf{b}))) = \mathbf{a} \cdot \mathbf{b}$$

    ■

17. Prove the parallelogram identity,

$$|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

Proof. Start with the left side.

$$|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 =$$

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 + 2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2(\mathbf{a}\cdot\mathbf{b})$$
$$= 2|\mathbf{a}|^2 + 2|\mathbf{b}|^2$$

■

18. Show that if $\mathbf{a}\times\mathbf{u} = \mathbf{0}$ for all unit vectors $\mathbf{u}$, then $\mathbf{a} = \mathbf{0}$.

Proof. If $\mathbf{a} \neq \mathbf{0}$, then there is a vector $\mathbf{b}$ that is not parallel to $\mathbf{a}$. If $\mathbf{b}'$ is the unit vector in the direction of $\mathbf{b}$, then $\mathbf{a}$ and $\mathbf{b}'$ span a parallelogram with non-zero area. Hence, $\mathbf{a}\times\mathbf{b}' \neq \mathbf{0}$. This means that if $\mathbf{a}\times\mathbf{u} = \mathbf{0}$ for all unit vectors $\mathbf{u}$, then $\mathbf{a} = \mathbf{0}$. ■

19. Suppose $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are three vectors whose components are all integers. Can you conclude that the volume of the parallelepiped determined by these three vectors will always be an integer?

Answer. Yes, the volume will be an integer since it is the sum of products of integers.

20. Is $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$? (Hint: Try $(\mathbf{i}\times\mathbf{j})\times\mathbf{j}$.)

Answer. $(\mathbf{i}\times\mathbf{j})\times\mathbf{j} = \mathbf{k}\times\mathbf{j} = -\mathbf{i}$. On the other hand, $\mathbf{i}\times(\mathbf{j}\times\mathbf{k}) = \mathbf{i}\times\mathbf{i} = \mathbf{0}$.

This means that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$ is not true in general. In other words, the cross product is not associative.

SOME MORE SHORT QUESTIONS TO TEST YOUR UNDERSTANDING OF THIS MATERIAL. No answers are provided for these questions. You should be able to answer these questions if you've studied the lecture notes.

21. Two vectors $(1,2)$ and $(2,3)$ are added by adding corresponding components. Interpreted as arrows in the plane, the two vectors get added by arranging them tip to tail. Explain why this gives the same result.

22. Give an algebraic and a geometric argument that shows why $\mathbf{a}+\mathbf{b} = \mathbf{b}+\mathbf{a}$.

23. What has the distance between two points in $\mathbf{R}^n$ to do with Pythagoras' theorem?

24. Can you justify the individual steps in the proof of the Cauchy-Schwarz inequality $|\mathbf{a}\cdot\mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$ given in the lecture notes?

25. Prove the triangle inequality $|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$ using the Cauchy-Schwarz inequality.

26. Why is it called the triangle inequality?

27. Use the Cosine Rule

$$a^2 + b^2 - 2ab\cos(\theta) = c^2$$

to show that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta)$ for vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ and the angle θ between them.