MTH1030
Techniques for Modelling

Lecture 34 & 35

Differential equations (part 3)

Monash University

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Warm welcoming words

Today...second-order DEs!

Second-order DEs

Consider the second-order DE of the form:

$$P(x)y'' + Q(x)y' + R(x)y = S(x).$$

This is called a linear second-order DE. How would we find $y(x)$?

Second-order DEs

We don't. Too hard! Look at instead the linear second-order DE of the form:

$$ay'' + by' + cy = S(x),$$

where a, b, c are constants. Now how would we find $y(x)$?

Second-order DEs

Well we can do this eventually...but first we have to consider the simpler situation where $S(x) = 0$, so

$$ay'' + by' + cy = 0.$$

Such a linear DE is called *homogeneous* (0 on the RHS). To solve this...we literally guess a solution and go from there.

Assume that $y(x) = e^{\lambda x}$ is a solution to this, then if that is true then

$$\left(a\lambda^2 + b\lambda + c\right) e^{\lambda x} = 0$$

which gives

$$a\lambda^2 + b\lambda + c = 0, \tag{1}$$

because $e^{\lambda x} \neq 0$. (1) is called the *characteristic equation*.

Second-order DEs

Solving the characteristic equation yields

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$
$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

What comes next greatly depends on the behaviour of λ_1 and λ_2.

Before we go on, an important remark:

Remark

If $y_1(x)$ and $y_2(x)$ are solutions to

$$ay'' + by' + cy = 0,$$

then so is

$$y(x) = Ay_1(x) + By_2(x).$$

Distinct roots

Let's look at the following second-order linear homogeneous DE

$$2y'' + y' - y = 0.$$

The characteristic equation is

$$2\lambda^2 + \lambda - 1 = 0$$

and this gives the solutions

$$\lambda_1 = -1,$$
$$\lambda_2 = 1/2.$$

Hence we have *at least* two solutions e^{-x} and $e^{x/2}$. Are there more?

Distinct roots

Yes, by the remark, any linear combination of the solutions e^{-x} and $e^{x/2}$ solve the DE. That is

$$y(x) = Ae^{-x} + Be^{x/2}$$

solves the DE

$$2y'' + y' - y = 0$$

for any $A, B \in \mathbb{R}$. In fact, these are all the solutions!

Complex roots

How about this DE?

$$y'' + 2y' + 5y = 0.$$

The characteristic equation is

$$\lambda^2 + 2\lambda + 5 = 0$$

and this gives the solutions

$$\lambda_1 = -1 - 2i,$$
$$\lambda_2 = -1 + 2i.$$

We now have complex solutions (also note $\lambda_1 = \bar{\lambda}_2$) to the characteristic equation...but we want real solutions for $y(x)$. What do we do?

Complex roots

Let's play along...so it seems that any linear combination of $e^{(-1-2i)x}$ and $e^{(-1+2i)x}$ should be a solution, that is

$$Ae^{(-1-2i)x} + Be^{(-1+2i)x}.$$

But if this were the solution, then $y(x)$ would be complex! So we need to determine for which A, B do we get real solutions.

Let's rewrite this as

$$e^{-x}\left(Ae^{-2ix} + Be^{2ix}\right) = e^{-x}\left((A+B)\cos(2x) + i(A-B)\sin(2x)\right).$$

Now we sub in various forms of A, B which will make the preceding expression real!

- Plug in $A = B = 1/2$, what happens?
- Plug in $A = -B = -i/2$, what happens?

Complex roots

So the general solution to

$$y'' + 2y' + 5y = 0,$$

is

$$y(x) = e^{-x}\left(A\cos(2x) + B\sin(2x)\right)$$

where we should recall that the solutions to the characteristic equation were $\lambda_1 = -1 - 2i$ and $\lambda_2 = -1 + 2i$.

Repeated roots

How about this DE?

$$y'' - 2y' + y = 0.$$

The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0$$

and this gives the solutions

$$\lambda_1 = 1,$$
$$\lambda_2 = 1.$$

We have the same solution (repeated roots). So we know e^x is a solution to the DE. It turns out that xe^x is also a solution to the DE.

Repeated roots

Now by the remark any linear combination of the solutions e^x and xe^x will solve the DE. That is

$$y(x) = Ae^x + Bxe^x$$

solves the DE

$$y'' - 2y' + y = 0,$$

for any $A, B \in \mathbb{R}$.

Second-order DEs

So to solve the linear second-order homogeneous DE

$$ay'' + by' + cy = 0,$$

we first guess a solution $e^{\lambda x}$ and then study the solutions λ_1 and λ_2 to

$$a\lambda^2 + b\lambda + c = 0.$$

We then have 3 cases:

1. λ_1 and λ_2 are real and distinct (i.e., $\lambda_1 \neq \lambda_2$). Then
$$y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}.$$
2. λ_1 and λ_2 are complex (i.e., $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$). Then
$$y(x) = e^{\alpha x}\left(A\cos(\beta x) + B\sin(\beta x)\right).$$
3. λ_1 and λ_2 are the same (i.e., $\lambda_1 = \lambda_2 \equiv \lambda$). Then
$$y(x) = Ae^{\lambda x} + Bxe^{\lambda x}.$$

Second-order DEs

Example

Let's find the general solution to the following linear second-order homogeneous DE:

$$2y'' + 4y' + 4y = 0.$$

Second-order DEs

Example

Let's find the general solution to the following linear second-order homogeneous DE:

$$y'' + 2\sqrt{2}y' + 2y = 0.$$

Question 1

Question (1)

The general solution to the following linear second-order homogeneous DE

$$y'' + 4y' - y = 0$$

is:

1. $y(x) = Ae^{(-2-\sqrt{5})x} + Bxe^{(-2+\sqrt{5})x}$.
2. $y(x) = e^{(-2-\sqrt{5})x} + e^{(-2+\sqrt{5})x}$.
3. $y(x) = e^{-2x}\left(A\cos(\sqrt{5}x) + B\cos(\sqrt{5}x)\right)$.
4. $y(x) = Ae^{(-2-\sqrt{5})x} + Be^{(-2+\sqrt{5})x}$.

Second-order DEs

As you may have guessed, a linear second-order DE is called non-homogeneous (or inhomogeneous) if it is of the form

$$ay'' + by' + cy = S(x).$$

We'll deal with these types next time!