Eigenvalues and eigenvectors

1. Here is a matrix
$$A = \begin{pmatrix} -3 & -7 & 19 \\ -2 & -1 & 8 \\ -2 & -3 & 10 \end{pmatrix}.$$
Determine which of the following vectors are eigenvectors of A. If a vector is an eigenvector also figure out what the corresponding eigenvalue is.
$$(a) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad (b) \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \quad (c) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad (d) \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (e) \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

2. Here is a matrix
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}.$$
Find the eigenvalues and corresponding eigenvectors of A. Determine whether or not the matrix can be diagonalized. If it can be diagonalized do so.

3. Compute the eigenvalues and eigenvectors of the following matrices. If possible use the eigenvalues and eigenvectors to diagonalize the matrices.
$$(a) \begin{pmatrix} 4 & -2 \\ 5 & -3 \end{pmatrix}, \quad (b) \begin{pmatrix} 6 & 1 \\ -3 & 2 \end{pmatrix}, \quad (c) \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}.$$

4. Prove that a matrix A has 0 as an eigenvalue if and only if $A\mathbf{x} = \mathbf{0}$ has non-zero solutions.

5. The following are characteristic equations of some matrices.

   (a)
   $$t^5(t-1)^3(t+1)^4 = 0$$

   (b)
   $$t^3 + 3t^2 + 3t + 1 = 0$$

(c)

$$(t-2)(t-3)(t-4)(t-5)=0$$

(d)

$$t^2+1=0$$

For every one of these equations determine the dimensions of the matrices they come from and the eigenvalues of the matrices.

6. What are the eigenvalues and eigenspaces of the following diagonal matrix.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

7. If A is the matrix of a linear transformation which rotates all vectors in $\mathbf{R}^2$ through $30^{\mathbf{o}}$, explain why A cannot have any eigenvalues.

8. Let A be the matrix of a 3d rotation through 30° around the vector $(1,2,3)$. What are its eigenvalues and eigenvectors?

9. Let A be the matrix of the reflection through the x-axis of $\mathbf{R}^2$. Find A and its eigenvalues and corresponding eigenvectors.

10. If $char(\lambda) = \det(\lambda I - A)$ is the characteristic polynomial of an $n \times n$ matrix A what is $char(0)$?

11. If A is an $n \times n$ matrix and c is a non-zero constant, compare the eigenvalues of A and cA.

12. If A is an invertible $n \times n$ matrix, compare the eigenvalues of A and A^{-1}. More generally, for m an arbitrary integer, compare the eigenvalues of A and A^m.

13. Suppose A is a square matrix and it satisfies $A^m = A$ for some m a positive integer larger than 1. Show that if λ is an eigenvalue of A then $|\lambda|$ equals either 0 or 1.

SOME TEST QUESTIONS

14. Derive the determinant formula for the characteristic equation of a square matrix.

15. How do you figure out the geometric and algebraic multiplicities of an eigenvalue?

16. What does it mean for a matrix to be defective?

17. Sketch a proof of the Cayley-Hamilton theorem in the case of non-defective matrices.

18. Sketch a proof that non-defective matrices can be diagonalized.

19. What is the eigenspace corresponding to a given eigenvalue?
20. Why is finding the eigenvalues of a matrix a "hard" problem?
21. What are some practical applications of eigenvalues?