# Sequences and series

## Sequences

For the following sequences determine whether they converge or diverge. If possible also determine the limits for those among the sequences that converge.

1.
$$\left\{\frac{n-1}{n^2-1}\right\}$$

*Answer.* Rewrite as
$$\left\{\frac{1-\frac{1}{n}}{n-\frac{1}{n}}\right\}.$$
As $n$ goes to infinity the numerator goes to 1 and the denominator to $+\infty$. We conclude that the limit of this sequence is 0. Formally this can be written as
$$\lim_{n\to\infty}\frac{n-1}{n^2-1}=\lim_{n\to\infty}\frac{1-\frac{1}{n}}{n-\frac{1}{n}}=\frac{1-\lim_{n\to\infty}\frac{1}{n}}{\lim_{n\to\infty}(n-\frac{1}{n})}=\frac{1}{\lim_{n\to\infty}n}=0.$$

2.
$$\left\{2-\left(-\frac{1}{2}\right)^n\right\}$$

*Answer.* Converges to 2 as
$$\lim_{n\to\infty}\left(-\frac{1}{2}\right)^n=0.$$

3.
$$\left\{\sin\left(\frac{1}{n}\right)\right\}$$

*Answer.* Since $\lim_{n\to\infty}\frac{1}{n}=0$ and $\sin(x)$ is continuous at 0, we have
$$\lim_{n\to\infty}\sin\left(\frac{1}{n}\right)=\sin(0)=0.$$

4.
$$\left\{\frac{1+(-1)^n}{\sqrt{n}}\right\}$$

*Answer.* Clearly $1+(-1)^n = 0$ for odd $n$ and $1+(-1)^n = 2$ for even $n$. Therefore
$$0 \leqq \frac{1+(-1)^n}{\sqrt{n}} \leqq \frac{2}{\sqrt{n}}$$
for all $n \geqq 1$. Now, since $\lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$ we can use squeeze law for sequences to conclude that
$$\lim_{n\to\infty} \frac{1+(-1)^n}{\sqrt{n}} = 0,$$
as well.

5.
$$\left\{\frac{2+\cos(n)}{n}\right\}$$

*Answer.* Adding 2 to the basic inequality $-1 \le \cos(n) \le 1$ gives $1 \leqq 2+\cos(n) \leqq 3$ and further
$$\frac{1}{n} \leqq \frac{2+\cos(n)}{n} \leqq \frac{3}{n}$$
for $n \geqq 1$. At this point we apply the squeeze law again to conclude that
$$\lim_{n\to\infty} \frac{2+\cos(n)}{n} = 0.$$

6.
$$\left\{(0.001)^{-\frac{1}{n}}\right\}$$

*Answer.* Note that
$$(0.001)^{-\frac{1}{n}} = 1000^{\frac{1}{n}} = \sqrt[n]{1000}.$$
Then it is clear that
$$\lim_{n\to\infty} (0.001)^{-\frac{1}{n}} = \lim_{n\to\infty} \sqrt[n]{1000} = 1.$$

7.
$$\left\{\frac{f_{n+1}}{f_n}\right\}, \text{ where } f_n \text{ denotes the } n\text{th Fibonacci number.}$$

*Answer.* In the linear algebra part of the course we proved that
$$f_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}.$$

Now $\lim_{n\to\infty}\left(\frac{1-\sqrt{5}}{2}\right)^n = 0$ because $|\frac{1-\sqrt{5}}{2}| < 1$. On the other hand, $\lim_{n\to\infty}\left(\frac{1+\sqrt{5}}{2}\right)^n = \infty$ because $\frac{1+\sqrt{5}}{2} > 1$. We conclude that

$$\lim_{n\to\infty}\frac{f_{n+1}}{f_n} = \lim_{n\to\infty}\frac{\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}}{\sqrt{5}}}{\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n}}{\sqrt{5}}} = \frac{1+\sqrt{5}}{2}$$

and, of course, this last number is equal to the golden ratio.

8. Turn the following infinite expression into a sequence.

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+\ldots}}}}$$

If this sequence converges, what is its limit?

*Answer.* A natural interpretation of this infinite expression as a sequence is

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

or

$$a_0 = \sqrt{2}, \quad a_i = \sqrt{2+a_{i-1}}, i \geq 1.$$

This means that if a limit $L$ exists it should satisfy the equation

$$L = \sqrt{2+L},$$

or

$$L^2 - L - 2 = 0.$$

The two solution of this equation are $-1$ and $2$. Clearly, $-1$ is impossible. This means that if the limit exists it should be $2$ (which is indeed the case, and you can prove this using an argument similar to the one I used in the lecture notes).

9. Turn the following infinite expression into two different sequences

$$\ldots + \sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2}}}}$$

If these sequences converge, what are their limits?

*Answer.* The first sequence could be the same as in the previous exercise namely

$$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

The second sequence could be

$$2, 2+\sqrt{2}, 2+\sqrt{2+\sqrt{2}}, 2+\sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$$

which is basically the first sequence added to the constant sequence $2, 2, 2, 2, \ldots$. Then, if the limit of the first sequence is 2 the limit of this second sequence would simply be 4, the sum of the limits of the first sequence and the limit of the constant sequence.

So, with these sort of infinite expressions there can be different interpretations in terms of sequences and then also different limits depending on which interpretation was chosen.

10. What is the limit of the sequence

$$3, 3.1, 3.14, 3.141, 3.1415, \ldots$$

*Answer.* It may appear that the answer is $\pi$. However, strictly speaking this sequence is not well-defined, just like any other sequence whose definition relies on you recognizing some pattern supposedly established by the first couple of terms like

$$1, 2, 3, 4, 5, \ldots$$

or

$$2, 4, 8, 16, \ldots$$

Strictly speaking sequences that are given like this are not well-defined because they can be continued in many different ways. In fact, there are usually many very reasonable ways to do so—check out the *Online Encyclopedia of Integer Sequences* and type in the first few terms of any “obvious” sequence like this to find out about zillions of natural patterns that yield different continuations of these finite sequences into infinite sequences.

Having said this, as long as you realize that this is the case it is not unreasonable to interpret all these sequences as you have been taught to interpret them since kindergarden. So, $1, 2, 3, 4, 5, \ldots$ “is” the sequence of natural numbers and $2, 4, 8, 16, \ldots$ is the sequence of the powers of 2 and $3, 3.1, 3.14, 3.141, 3.1415, \ldots$ does converge to $\pi$.

11. Prove from first principles, that is, based solely on Definition 3.1.1. in the lecture notes, that the sequence

$$\left\{ \frac{1}{n^2} \right\}$$

has the limit $L = 0$.

*Proof.* Because $L = 0$, we need only convince ourselves that to each positive number $\epsilon$ there corresponds an integer $N$ such that

$$\left| \frac{1}{n^2} \right| = \frac{1}{n^2} < \epsilon$$

if $n \geqq N$. Obviously, any fixed integer $N > \frac{1}{\sqrt{\epsilon}}$ will do, because $n \geqq N$ implies that

$$\frac{1}{n^2} \leqq \frac{1}{N^2} < \epsilon,$$

which then completes the proof. ■

12. What is the limit of the sequence

$$\left\{ \frac{\sin(5+\frac{1}{n}) - \sin(5)}{\frac{1}{n}} \right\}$$

*Answer.* Remember that the derivative of a differentiable function is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{n \to \infty} \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}$$

Now just let $f(x) = \sin(x)$ and $x = 5$ and you immediately see that the limit we are interested in here exists and is equal to $\cos(5)$, the derivative of $\sin(x)$ at $x = 5$.

13. If you are bored by all this and/or if you are up for a real challenge, try your hand at finding reasonable interpretations/limits of the crazy infinite expression that I listed in the lecture notes.

Or, what about this one: For which $x$ do the infinite expressions

$$\cdots (((x^x)^x)^x)^x \cdots \text{ and } \cdots x^{\left(x^{(x^x)}\right)}$$

make sense/converge? If they converge what are their values?

No solutions given for these problems! If you are interested in discussing your attempts at solving these problems, come and have a chat with me sometime.

# Series

For the following infinite series determine whether they converge or diverge. If possible also determine the sum for those among the series that converge.

1.
$$1 + \frac{1}{\pi} + \frac{1}{\pi^2} + \frac{1}{\pi^3} + \frac{1}{\pi^4} + \frac{1}{\pi^5} + \cdots = \sum_{n=0}^{\infty} \frac{1}{\pi^n}$$

*Answer.* This is a geometric series with $a = 1$ and $r = \frac{1}{\pi}$. Since $r < 1$ it converges and its sum is

$$\frac{1}{1 - \frac{1}{\pi}} = \frac{\pi}{\pi - 1}.$$

2.
$$1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \frac{1}{3^8} + \frac{1}{3^{10}} + \cdots = \sum_{n=0}^{\infty} \frac{1}{3^{2n}}$$

*Answer.* This series can also be interpreted as the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{(3^2)^n}.$$

In this geometric series $a = 1$ and $r = \frac{1}{3^2}$. Since $r < 1$ it converges and its sum is

$$\frac{1}{1 - \frac{1}{3^2}} = \frac{9}{8}.$$

3.
$$1 - 2 + 4 - 8 + 16 - 32 + \cdots = \sum_{n=0}^{\infty} (-2)^n$$

*Answer.* This is a geometric series with $a = 1$ and $r = -2$. Since $|r| > 1$ this series diverges.

4.
$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots = \sum_{n=1}^{\infty} \frac{n}{n+1}$$

*Answer.* This series diverges by the $n$th-term test since

$$\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0.$$

5.
$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[4]{4}} + \frac{1}{\sqrt[5]{5}} \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt[n]{n}}$$

*Answer.* This series diverges by the $n$th-term test since

$$\lim_{n\to\infty} \frac{1}{\sqrt[n]{n}} = 1 \neq 0.$$

6.
$$1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{5}} \cdots = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}}$$

*Answer.* This is a $p$-series with $p = 1/3$ and since $p < 1$ it diverges.

7.
$$1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} \cdots = \sum_{n=1}^{\infty} \frac{1}{n^5}$$

*Answer.* This is a $p$-series with $p = 5$ and since $p > 1$ it converges.

8.
$$\sum_{n=0}^{\infty} \frac{3^n - 2^n}{4^n}$$

*Answer.* This infinite series is the difference of two convergent geometric series. Its sum is

$$\sum_{n=0}^{\infty} \frac{3^n - 2^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n - \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1 - \frac{3}{4}} - \frac{1}{1 - \frac{1}{2}} = 4 - 2 = 2.$$

9.
$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{2^n}\right)$$
*Answer.* This series is divergent since it is the termwise difference between the divergent harmonic series
$$\sum_{n=1}^{\infty}\frac{1}{n}$$
and the convergent geometric series
$$\sum_{n=1}^{\infty}\frac{1}{2^n}.$$

10. Write the rational number that is equal to the following infinite repeating decimal as a fraction.
$$0.\overline{123}\ldots$$
*Answer.* This all boils down to calculating the sum of a certain geometric series
$$0.\overline{123}\ldots=\frac{123}{1000}+\frac{123}{1000^2}+\frac{123}{1000^3}+\cdots$$
$$=\frac{123}{1000}\sum_{n=0}^{\infty}\left(\frac{1}{1000}\right)^n=\frac{123}{1000}\frac{1}{1-\frac{1}{1000}}=\frac{123}{999}=\frac{41}{333}.$$
What's really interesting here is that
$$0.\overline{123}\ldots=\frac{123}{999}.$$
Similarly,
$$0.\overline{56789}\ldots=\frac{56789}{99999}$$
$$0.\overline{47}\ldots=\frac{47}{99}$$
I am sure you see what the general rule is. Let's apply it to some old friends
$$0.\overline{6}\ldots=\frac{6}{9}=\frac{2}{3}$$
$$0.\overline{9}\ldots=\frac{9}{9}=1.$$
Another streamlined way of translating a repeating decimal like $0.\overline{123}\ldots$ into a fraction runs like this: Set
$$x=0.\overline{123}\ldots$$
Multiplying this equation by 1000 gives
$$1000x=123.\overline{123}\ldots$$

Subtracting the first equation from the second gives

$$999x = 123$$

and therefore

$$x = \frac{123}{999}.$$

Finally, I should point out that you can use the same idea to quickly derive the sum of a geometric series.

$$S = 1 + r + r^2 + r^3 + r^4 + \cdots$$

Multiplying this equation by $r$ gives

$$Sr = r + r^2 + r^3 + r^4 + \cdots$$

Subtracting the second equation from the first gives

$$S(1 - r) = 1$$

and therefore

$$S = \frac{1}{1 - r}.$$

11. Write the rational number that is equal to the following infinite repeating decimal as a fraction.

$$4.34\overline{123}\ldots$$

*Answer.*

$$4.34\overline{123}\ldots = \frac{434}{100} + 0.00\overline{123}\ldots =$$
$$\frac{434}{100} + \frac{1}{100}0.\overline{123}\ldots$$

Now we use the result of the previous problem.

$$= \frac{434}{100} + \frac{1}{100}\frac{41}{333} = \frac{144563}{33300}.$$

12. For which values of $x$ is the following series a convergent infinite geometric series. For those values for which this is the case find the sum of the series.

$$\sum_{n=0}^{\infty} \left(\frac{x-1}{2}\right)^n$$

*Answer.* We are dealing with a geometric series with $a = 1$ and $r = \frac{x-1}{2}$. The series converges to

$$\frac{1}{1 - \frac{x-1}{2}} = \frac{2}{3 - x}$$

for all those values of $x$ for which

$$\left|\frac{x-1}{2}\right| < 1.$$

We conclude that the series is convergent for $-1 < x < 3$.

13. Find a closed formula for the $k$th partial sum of the following infinite series and, based on this formula, establish whether or not the series converges. (Hint: This is an example of a telescoping sum in action.)

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$$

*Answer.* The trick is to note that $\ln\left(\frac{n+1}{n}\right) = \ln(n+1) - \ln(n)$. Then the $k$th partial sum of our series is

$$S_k = (\ln(2) \underline{- \ln(1)}) + (\ln(3) - \ln(2)) + (\ln(4) - \ln(3)) + \cdots + (\underline{\ln(k+1)} - \ln(k))$$

All but the underlined terms cancel and we are left with

$$S_k = \ln(k+1) - \ln(1) = \ln(k+1).$$

Since $\lim_{k\to\infty} S_k = \infty$ we conclude that our infinite series diverges.

14. Find a closed formula for the $k$th partial sum of the following infinite series and, based on this formula, establish whether or not the series converges. (Hint: This is another example of a telescoping sum in action.)

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1}$$

*Answer.* The trick is to note that

$$\frac{1}{n^2-1} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right).$$

Then the $k$th partial sum of our series is

$$S_k = \sum_{n=2}^{k} \frac{1}{n^2-1}$$

$$= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \cdots + \frac{1}{k-2} - \frac{1}{k} + \frac{1}{k-1} - \frac{1}{k+1}\right)$$

All but the terms highlighted in red cancel and so

$$S_k = \frac{1}{2}\left(1 + \frac{1}{2} - \frac{1}{k} - \frac{1}{k+1}\right).$$

Therefore

$$\sum_{n=2}^{\infty} \frac{1}{n^2-1} = \lim_{k\to\infty} S_k = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}$$

is the sum of our series.

15. Use the integral test to figure out whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

*Answer.* The way this question is phrased suggests that you can apply the integral test straight away, so let's just do that. Use a simple substitution to figure out that

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1).$$

Then

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\ln(x^2+1)\right]_1^t = \lim_{t\to\infty}\frac{1}{2}\big(\ln(t^2+1)-\ln(2)\big) = \infty.$$

Therefore, by the integral test, $\sum_{n=1}^{\infty}\frac{n}{n^2+1}$ diverges.

IMPORTANT: Usually, before applying the integral test you have to check that $\frac{x}{x^2+1}$, the function under consideration, is non-negative and decreasing in the interval $[1,\infty)$. It is clear that the function is positive. In the case of this function possibly the easiest way to check that it is decreasing is to consider its derivative

$$\frac{1-x^2}{x^2+1^2}.$$

Clearly, this derivative is negative in the interval $[1,\infty)$ which implies that the function is decreasing there.

16. Use the integral test to figure out whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$$

*Answer.* Again the way this question is phrased suggests that you can apply the integral test straight away, so let's just do that. Use a simple substitution to figure out that

$$\int_1^{\infty} \frac{x}{e^{x^2}}dx = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^t = \frac{1}{2e}.$$

Therefore, since the integral is finite we conclude that the infinite series converges.

IMPORTANT: Usually, before applying the integral test you have to check that the function under consideration is non-negative and decreasing in the interval $[1,\infty)$. It is clear that the function is positive. In the case of this function possibly the easiest way to check that it is decreasing is to consider its derivative

$$e^{-x^2}(1-2x^2).$$

Clearly, this derivative is negative in the interval $[1,\infty)$ which implies that the function is decreasing there.

17. Use the integral test to figure out whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

*Answer.* We've already established in the lecture notes that this integral converges (by identifying it as a telescoping sum).

Let's establish convergence again using the integral test. Again the way this question is phrased suggests that you can apply the integration test straight away, so let's just do that.

$$\int_1^{\infty} \frac{1}{x(x+1)} dx = \int_1^{\infty} \left(\frac{1}{x} - \frac{1}{x+1}\right) dx$$

$$= \lim_{x\to\infty} \ln\left(\frac{x}{x+1}\right) - \ln\left(\frac{1}{2}\right) = \ln(1) - \ln\left(\frac{1}{2}\right) = \ln(2).$$

Therefore the given series converges.

IMPORTANT: Usually, before applying the integral test you have to check that the function under consideration is non-negative and decreasing in the interval $[1, \infty)$. It is clear that the function is positive. In the case of this function possibly the easiest way to check that it is decreasing is to consider its derivative

$$-\frac{1+2x}{x^2(x+1)^2}.$$

Clearly, this derivative is negative in the interval $[1, \infty)$ which implies that the function is decreasing there.

18. Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^2}{n^2-1}$$

by finding a closed formula for the partial product

$$P_k = \prod_{n=2}^{k} \frac{n^2}{n^2-1}$$

and then letting $k$ go to infinity.

*Answer.*

$$P_k = \prod_{n=2}^{k} \frac{n^2}{n^2-1} = \prod_{n=2}^{k} \frac{n^2}{(n-1)(n+1)}$$

$$= \frac{2\cdot 2}{1\cdot 3} \cdot \frac{3\cdot 3}{2\cdot 4} \cdot \frac{4\cdot 4}{3\cdot 5} \cdot \frac{5\cdot 5}{4\cdot 6} \cdots \frac{(k-1)\cdot(k-1)}{(k-2)k} \cdot \frac{k\cdot k}{(k-1)\cdot(k+1)}$$

All but the blue numbers cancel so we find

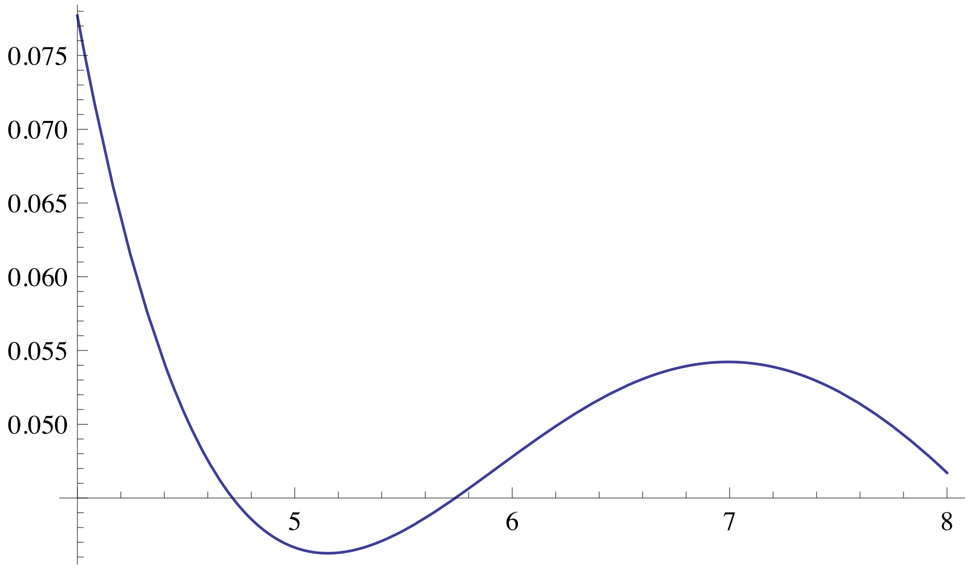
$$P_k = \frac{2k}{k+1}.$$

Therefore
$$\prod_{n=2}^{\infty} \frac{n^2}{n^2-1} = \lim_{k\to\infty} P_k = \lim_{k\to\infty} \frac{2k}{k+1} = 2.$$

19. Explain why the integral test does not apply to the following two series.
$$(i)\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \quad (ii)\ \sum_{n=1}^{\infty} \frac{2+\sin(n)}{n^2}$$

*Answer.* (i)
$$\frac{(-1)^x}{x}$$
(whatever this may be) is definitely not a positive function in the interval $[1, \infty)$. Hence the integral test does not apply to this function.

(ii)
$$\frac{2+\sin(x)}{x^2}$$
is a positive function but is not decreasing. Hence the integral test does not apply to this function. Here is a *Mathematica* plot of the function in the interval $[4, 8]$.




20. Determine the convergence or divergence of the following series using the suggested series for comparison.
$$(i)\ \sum_{n=0}^{\infty} \frac{n+2}{n+1} \text{ compare with } \sum_{n=0}^{\infty} 1$$
$$(ii)\ \sum_{n=0}^{\infty} \frac{1}{\left(2+\frac{1}{n+1}\right)^{n+1}} \text{ compare with } \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}$$

$$(iii) \quad \sum_{n=0}^{\infty} \frac{2+\sin(n)}{n+1} \text{ compare with } \sum_{n=0}^{\infty} \frac{1}{n+1}$$

*Answer.* (i) Since $n+2 > n+1$ we have $\frac{n+2}{n+1} > 1$. This means that the first sequence dominates the second series, i.e.,

$$\sum_{n=0}^{\infty} \frac{n+2}{n+1} \geq \sum_{n=0}^{\infty} 1$$

And since the second series clearly diverges, so does the first one.

Another quick way to see that the first series diverges is to observe that it diverges by the $n$th-term test.

(ii)

$$\sum_{n=0}^{\infty} \frac{1}{\left(2+\frac{1}{n+1}\right)^{n+1}} \text{ compare with } \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}$$

Clearly,

$$1 < 2 < 2 + \frac{1}{n+1}.$$

Therefore

$$2^{n+1} < \left(2 + \frac{1}{n+1}\right)^{n+1}$$

and

$$\frac{1}{\left(2+\frac{1}{n+1}\right)^{n+1}} < \frac{1}{2^{n+1}}.$$

Therefore

$$\sum_{n=0}^{\infty} \frac{1}{\left(2+\frac{1}{n+1}\right)^{n+1}} \leq \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}.$$

Finally, since the series on the right is a convergent geometric series, we can conclude that the series on the left is also convergent.

(iii)

$$\sum_{n=0}^{\infty} \frac{2+\sin(n)}{n+1} \text{ compare with } \sum_{n=0}^{\infty} \frac{1}{n+1}$$

We know that

$$-1 \leq \sin(n) \leq 1.$$

Add 2 and you get

$$1 \leq 2 + \sin(n) \leq 3.$$

Divide by $n+1$ and you get

$$\frac{1}{n+1} < \frac{2+\sin(n)}{n+1}.$$

This implies that

$$\sum_{n=0}^{\infty} \frac{1}{n+1} \leq \sum_{n=0}^{\infty} \frac{2+\sin(n)}{n+1}.$$

But now, since the series on the left is the diverging harmonic series, we can conclude that the series of the right is also divergent (to infinity).

**TEST QUESTIONS**

21. Define what it means for an infinite sequence to converge or diverge. Be able to reproduce the formal definitions in the lecture notes.

22. Define what it means for an infinite series to converge or diverge. Be able to reproduce the formal definitions in the lecture notes.

23. What does the squeeze law for sequences say?

24. What is $\lim_{x\to a} f(x)$ of a function that is continuous at $x = a$?

25. What is a monotone sequence, an increasing sequence, a decreasing sequence, a positive sequence, a non-negative sequence, a bounded sequence, an unbounded sequence, a constant sequence?

26. Can one determine whether a given infinite series converges or diverges merely by computing a sufficiently large number of partial sums?

27. Let $\{a_n\}$ be an increasing sequence with lower bound $B$. Does this sequence converge?

28. Let $\{a_n\}$ be an increasing sequence with upper bound $B$. Does this sequence converge?

29. Give a characterization of irrational numbers/rational numbers in terms of their decimal expansions.

30. How do we make sense of infinite expressions like

$$\cdots + \sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6}}}} \quad ?$$

31. What is a geometric series, the harmonic series, a $p$-series, a telescoping sum, a positive-term series, the Riemann zeta function?

32. Derive the formula for the partial sums of the infinite geometric series

$$\sum_{n=0}^{\infty} ar^n$$

from scratch and use it to decide for which combinations of $a$ and $r$ the infinite series converges and diverges and what the sum of a convergent infinite geometric series is.

33. Given a positive-term series, there are only two possibilities: (1) The series converges or (2) it diverges to infinity. Explain in your own words why other forms of divergence are not possible.

34. Prove from scratch that the harmonic series diverges, i.e. make sure you understand the proof given in the lecture notes and be able to reproduce it.

35. Say the integral test applied to $\sum_{n=0}^{\infty} f(n)$ tells you that the series converges. Describe in your own words how you can use an integral to estimate how well the $k$th partial sum of this series approximates the sum of the series.

36. What does it mean for one positive-term series to dominate another positive-term series?

37. How does the comparison test for positive-term series work?

38. You replace the first 666 terms of the harmonic series by 6s. Does the resulting series converge or diverge?