Integrals

NOTE: When you evaluate the following integrals you may sometimes come up with a correct answer that is different from mine. There are two possible reasons for this: 1) I messed up (in which case please let me know), and 2) both my and your solutions are correct but look very different. This happens quite often when the functions you are dealing with are trigonometric functions and the paths we took to arrive at our respective solutions are different.

One way to quickly test whether both solutions are the same, up to adding some constant, is to plot their difference in *Mathematica*. If this plot looks like that of a constant function, then there is a very good chance that both solutions just differ by a constant.

STEP-BY-STEP SOLUTIONS in *Mathematica*: It used to be possible to evaluate integrals in *Wolfram Alpha* and then have a step-by-step solution displayed. Now you have to pay for a Pro version to be able to access this features, at least when you use *Wolfram Alpha* on the web. One of you told me that this feature is still included for free in some mobile *Wolfram Alpha* app(s). Be that as it may, it is also possible to access this feature from within *Mathematica* by copying and pasting the following commands into *Mathematica* and then executing them.

```
WolframAlpha["e^x sin(x)", IncludePods -> {"Indefinite Integral"},
PodStates -> {"Step-by-step solution", "Show all steps"}]
```

Alternatively you can also execute the following

```
WolframAlpha["e^x sin(x)"]
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and then hit the *Step-by-step solution* button in the *Indefinite integral* box of the answer. Try it![1]

You have to be connected to the web for this to work as *Mathematica* is actually acting as

[1] *Wolfram Alpha* provides step-by-step solutions to a host of other problems; see http://blog.wolframalpha.com/2009/12/01/step-by-step-math/ To access them from within *Mathematica* simply adjust the commands that I've given you in the case of indefinite integrals, and differential equations (see the next chapter).

an interface to *Wolfram Alpha* when you execute these commands.

Evaluate each of the following integrals using integration by parts.

1.
$$\int \sin^2(x)\ dx$$

Answer. Let $f(x) = \sin(x)$ and $g'(x) = \sin(x)$. Then $f'(x) = \cos(x)$ and $g(x) = -\cos(x)$. Now

$$\begin{aligned} \int \sin^2(x)\ dx &= -\sin(x)\cos(x) + \int \cos^2(x)\ dx \\ &= -\sin(x)\cos(x) + \int (1 - \sin^2(x))\ dx \\ &= -\sin(x)\cos(x) + x - \int \sin^2(x)\ dx. \end{aligned}$$

Note that the integral highlighted in blue shows up on both sides of the equation. We now move both these integrals to the left side and divide by 2 to arrive at our answer.

$$\begin{aligned} 2\int \sin^2(x)\ dx &= x - \sin(x)\cos(x) + C; \\ \int \sin^2(x)\ dx &= \frac{1}{2}(x - \sin(x)\cos(x)) + C. \end{aligned}$$

It's okay to be sloppy in the last line and write C instead of $C/2$.

2.
$$\int x\ln(x)\ dx$$

Answer. Let $f(x) = \ln(x)$ and $g'(x) = x$. Then $f'(x) = \frac{1}{x}$ and $g(x) = \frac{1}{2}x^2$. Then

$$\int x\ln(x)\ dx = \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x\ dx = \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C.$$

3.
$$\int x(\ln(x))^2\ dx$$

Answer.

Let $f(x) = (\ln(x))^2$ and $g'(x) = x$. Then $f'(x) = \frac{2\ln(x)}{x}$ and $g(x) = \frac{1}{2}x^2$. Thus

$$\int x(\ln(x))^2\ dx = \frac{1}{2}(x\ln(x))^2 - \int x\ln(x)\ dx.$$

But the integral on the right is the one we evaluated in the previous problem and so we can continue our chain of equalities as follows

$$= \frac{1}{2}(x\ln(x))^2 - \frac{1}{2}x^2\ln(x) + \frac{1}{4}x^2 + C.$$

4.
$$\int x^2 \sin(x)\ dx$$

Answer. Let $f(x) = x^2$ and $g'(x) = \sin(x)$. Then $f'(x) = 2x$ and $g(x) = -\cos(x)$. Then
$$\int x^2 \sin(x)\ dx = -x^2\cos(x) + 2\int x\cos(x)\ dx.$$
We apply the integration by parts trick a second time by letting $f(x) = x$ and $g'(x) = \cos(x)$. Then $f'(x) = 1$ and $g(x) = \sin(x)$. Then
$$\int x^2 \sin(x) dx = -x^2\cos(x) + 2\left(x\sin(x) - \int \sin(x)\ dx\right) = -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C.$$

5.
$$\int x\sqrt{x+1}\ dx$$

Answer. Let $f(x) = x$ and $g'(x) = (x+1)^{1/2}$. Then $f'(x) = 1$ and $g(x) = \frac{2}{3}(x+1)^{3/2}$. Now
$$\begin{aligned}\int x(x+1)^{1/2}\ dx &= \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3}\int (x+1)^{3/2}\ dx \\ &= \frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2} + C.\end{aligned}$$

6.
$$\int e^x \sin(x)\ dx$$

Answer. Let $f(x) = e^x$ and $g'(x) = \sin(x)$. Then $f'(x) = e^x$ and $g'(x) = -\cos(x)$. Now
$$\int e^x \sin(x)\ dx = -e^x\cos(x) + \int e^x\cos(x)\ dx.$$
To evaluate the integral on the right let $f(x) = e^x$ and $g'(x) = \cos(x)$. Then $f'(x) = e^x$ and $g(x) = \sin(x)$. Then
$$\int e^x\cos(x)\ dx = e^x\sin(x) - \int e^x\sin(x)\ dx.$$
Therefore
$$\int e^x\sin(x)\ dx = -e^x\cos(x) + \left(e^x\sin(x) - \int e^x\sin(x)\ dx\right)$$
or
$$2\int e^x\sin(x)\ dx = -e^x\cos(x) + e^x\sin(x) + C.$$
Finally,
$$\int e^x\sin(x)\ dx = \frac{e^x}{2}(\sin(x) - \cos(x)) + C.$$

Note that you can use the same calculations to evaluate

$$\int e^x \cos(x)\ dx.$$

Use a substitution and integration by parts to evaluate each of the following integrals.

7.

$$\int e^{\sqrt{x}}\ dx$$

Answer. Let $t = \sqrt{x}$, so that $dt/dx = \frac{1}{2\sqrt{x}}$ and $dx = 2\sqrt{x}\ dt = 2t\ dt$. Thus

$$\int e^{\sqrt{x}} dx = \int 2te^t\ dt = 2\int te^t\ dt.$$

To evaluate the integral on the right let $f(t) = t$ and $g'(t) = e^t$. Then $f'(t) = 1$ and $g(t) = e^t$. Hence

$$\int e^{\sqrt{x}} dx = 2\int te^t dt = 2\left(te^t - \int e^t\ dt\right) = 2e^{\sqrt{x}}(\sqrt{x} - 1) + C.$$

8.

$$\int e^{2x} \cos(e^x)\ dx$$

Answer.

$$\cos(e^x) + e^x \sin(e^x) + C$$

(Start by substituting $t = e^x$.)

9.

$$\int (3x - 7)\sin(5x + 2)\ dx$$

Answer.

$$\frac{3}{25}\sin(5x + 2) + \frac{1}{5}(7 - 3x)\cos(5x + 2) + C$$

10.

$$\int \cos(x)\sin(x)e^{\cos(x)}\ dx$$

Answer.

$$e^{\cos(x)}(1 - \cos(x)) + C$$

11. Use integration by parts to derive the **reduction formula**,

$$\int x^n e^{-x^2} dx = -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} \int x^{n-2} e^{-x^2} dx \text{ for } n > 2.$$

Answer. Let $f(x) = x^{n-1}$ and $g'(x) = xe^{-x^2}$. Then $f'(x) = (n-1)x^{n-2}$ and $g(x) = -\frac{1}{2}e^{-x^2}$ and therefore

$$\int x^n e^{-x^2} dx = -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} \int x^{n-2} e^{-x^2} dx.$$

12. Spot the error in the following calculation.

    We wish to compute

$$\int \frac{1}{x}\, dx.$$

For this we will use integration by parts with $f(x) = \frac{1}{x}$ and $g'(x) = 1$. This gives us $f'(x) = -\frac{1}{x^2}$ and $g(x) = x$. And so

$$\int \frac{1}{x}\, dx = 1 + \int \frac{1}{x}\, dx$$

and we conclude that $0 = 1$. (If this answer does not cause you serious grief then a career in accountancy beckons).

Answer. Did we forget an integration constant? (And so with the natural order restored, fears of a career in accountancy fade from view.)

13. One of you asked a very good question in class. See whether you can reconstruct my answer. When we perform integration by parts we are dealing with the two functions $f(x)$ and $g'(x)$. We differentiate the first and integrate the second. However, the integral of $g'(x)$ is only determined up to the addition of a constant. So, what happens if instead of $g(x)$ you use, say, $g(x) + 666$ as the integral of $g'(x)$ in the formula for integration by parts?

    Answer. Instead of

$$\int f(x)g'(x)\, dx = f(x)g(x) - \int g(x)f'(x)\, dx.$$

we get

$$\begin{aligned}\int f(x)g'(x)\, dx &= f(x)(g(x) + 666) - \int (g(x) + 666)f'(x)\, dx \\ &= f(x)g(x) - \int g(x)f'(x)\, dx + 666f(x) - 666 \int f'(x)\, dx.\end{aligned}$$

Of course, the bits in black cancel out (up to an added constant that we don't have to worry about because there is still the integral in blue floating around). And so we can see that we get the same right sides no matter whether we use $g(x)$ or $g(x) + 666$ (or $g(x)+$ any other number).

14. (**Boredom killer**) Define

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt.$$

It can be shown that the function is well-defined for positive x, that is, the integral on the right converges for positive x. Find $\Gamma(n)$ for all natural numbers n.

Answer. Clearly,

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1.$$

Now, after choosing $f(t) = t^{x-1}$ and $g'(x) = e^{-t}$ we go on integration-by-parts autopilot.

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt$$

$$= \left[-t^{x-1}e^{-t}\right]_0^\infty + \int_0^\infty (x-1)t^{x-2}e^{-t}\, dt.$$

Now we can show that the first summand is equal to 0 and so

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt = (x-1)\int_0^\infty t^{x-2}e^{-t}\, dt = (x-1)\Gamma(x-1).$$

However, repeated application of this equation for a natural number $x = n$ gives

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = (n-1)(n-2)(n-3)\Gamma(n-3) = \cdots$$

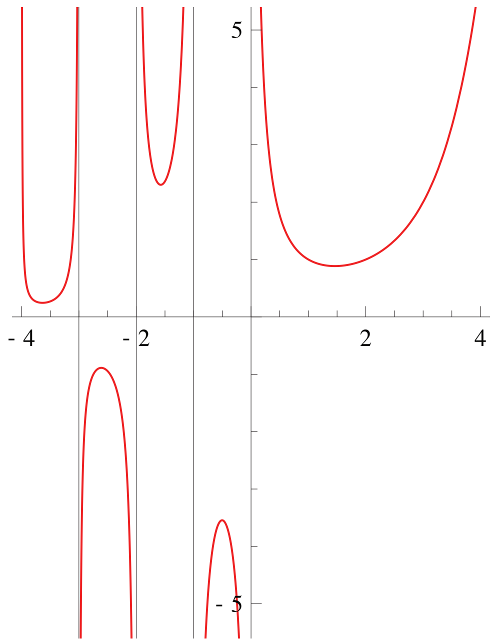
$$= (n-1)!\Gamma(1).$$

But since $\Gamma(1) = 1$ we find that

$$\Gamma(n) = (n-1)!$$

for all natural numbers n. Neat a "factorial" function! In fact, the **Gamma function** $\Gamma(x)$ is more than just neat, it is one of the most useful functions in mathematics. Google it!

Here is a plot of the Gamma function in the interval $(-4, 4)$. As suggested by this graph, the Gamma function is defined for all values of x except for the non-positive integers.



15. (**Bit of a joke**) A fun way to derive the formula for $\int x^n\,dx$ using integration by parts starts out as follows. Let $f(x) = x^n$ and $g'(x) = 1$, then ... Complete the argument!

Answer. ... then $f'(x) = nx^{n-1}$ and $g(x) = x$. Hence

$$\int x^n\,dx = x^n x - n\int x^{n-1}x\,dx = x^{n+1} - n\int x^n\,dx.$$

Note that the original integral shows up on both sides of this equation. Then solving for this integral gives

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}.$$

16. (**A very cute proof**) I want you to show using integration by parts that

$$\int_0^\pi \sin(mx)\sin(nx)\,dx = 0$$

if m and n are distinct natural numbers.

Proof. Let $f(x) = \sin(mx)$ and $g'(x) = \sin(nx)$ then $f'(x) = m\cos(mx)$ and $g(x) = -\frac{1}{n}\cos(nx)$ and therefore

$$\int_0^\pi \sin(mx)\sin(nx)\,dx = \left[-\frac{1}{n}\sin(mx)\cos(nx)\right]_0^\pi + \frac{m}{n}\int_0^\pi \cos(mx)\cos(nx)\,dx.$$

Clearly, the first summand vanishes and therefore

$$\int_0^\pi \sin(mx)\sin(nx)\,dx = \frac{m}{n}\int_0^\pi \cos(mx)\cos(nx)\,dx.$$

After performing a second integration by parts we get

$$\int_0^\pi \sin(mx)\sin(nx)\ dx = \frac{m^2}{n^2}\int_0^\pi \sin(mx)\sin(nx)\ dx.$$

Since $m \neq n$ we know that $\frac{m^2}{n^2} \neq 1$. Consequently

$$\int_0^\pi \sin(mx)\sin(mx)\ dx = 0. \qquad \blacksquare$$

TEST QUESTIONS

17. What is an elementary function?

18. Explain in what sense it is easy to differentiate elementary functions and in what sense it is in general very hard to integrate elementary functions.

19. Use the product rule for differentiating to derive the rule that we use to perform integration by parts.

20. Explain the connection between the chain rule for differentiating and integration by substitution.

21. In what sense is it not possible to evaluate the integral

$$\int e^{-t^2} dt.$$