# MTH1030
# Techniques for Modelling

## Lecture 33

## Differential equations (part 2)

Monash University

Semester 1, 2022

# Warm welcoming words

Verifying a solution to a DE is easy, but how do we determine the solution in the first place? We'll look at two very standard methods for first-order DEs.

# Separation of variables

Let's look at a first-order DE of the form

$$y' = g(x).$$

We can just integrate both sides using the fundamental theorem of calculus to get

$$y(x) = \int g(x)\mathrm{d}x + C.$$

Easy. Of course requires $g$ to have an antiderivative to work out well...e.g., if $g(x) = e^{-x^2}$, then we're in a bit of trouble!

# Separation of variables

### Example

Consider the IVP

$$\begin{aligned} y' &= \cos(x), \\ y(0) &= 3. \end{aligned}$$

# Separation of variables

How about the following type?

$$y' = h(y).$$

We could try solve it by doing the following very questionable method:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= h(y) \\ \implies \frac{1}{h(y)}\mathrm{d}y &= \mathrm{d}x \\ \implies \int \frac{1}{h(y)}\mathrm{d}y &= \int \mathrm{d}x \\ \implies \int \frac{1}{h(y)}\mathrm{d}y &= x + C. \end{aligned}$$

Call $F(y) = \int \frac{1}{h(y)}\mathrm{d}y$. Then we get

$$F(y) = x + C.$$

# Separation of variables

Does that actually work out? Let's consider the following IVP

$$\begin{aligned} y' &= y, \\ y(0) &= 1. \end{aligned}$$

Then

$$\frac{1}{y}\mathrm{d}y = \mathrm{d}x.$$

So

$$\begin{aligned} \int \frac{1}{y}\mathrm{d}y &= x + C, \\ \implies \ln(|y|) &= x + C. \end{aligned}$$

Hence

$$|y(x)| = e^C e^x.$$

But $y(0) = 1 = e^C$.

# Separation of variables

So the solution to the IVP

$$\begin{aligned} y' &= y, \\ y(0) &= 1, \end{aligned}$$

is $y(x) = e^x$.

# Separation of variables

## Exercise

Show the solution to the IVP

$$\begin{aligned} y' &= y, \\ y(0) &= -1, \end{aligned}$$

is $y(x) = -e^x$.

# Definitions of $e^x$

Actually, we now have three alternative definitions for $e^x$.

1.
$$e^x := \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n .$$

2.
$$e^x := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

3. $e^x := y(x)$ where $y(x)$ is the solution to the IVP
$$\begin{aligned} y' &= y, \\ y(0) &= 1. \end{aligned}$$

# Separation of variables

What we're doing here is called *separation of variables*. We 'separate variables' to get $y$ on the LHS and $x$ on the RHS.

Let's consider a more general case

$$y' = g(x)h(y).$$

Writing this as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)h(y)$$

and then separating variables we get

$$\int \frac{1}{h(y)}\mathrm{d}y = \int g(x)\mathrm{d}x.$$

# Separation of variables

We can summarise our findings via the following proposition:

## Proposition (Separation of variables)

Consider the DE

$$y' = g(x)h(y).$$

Define $F(y) = \int \frac{1}{h(y)} \mathrm{d}y$. Then

$$F(y) = \int g(x) \mathrm{d}x.$$

## Remark

$F(y)$ is called an *implicit solution* for $y$. To use this we need to compute all integrals that turn up, and ideally we want to get rid of $F$ somehow to get $y$ on its own,

# Separation of variables

The previous proposition is indeed true, but it easier to just derive the separation of variables formula from scratch each time!

## Example

Consider the IVP

$$\begin{aligned} y' &= -6xy, \\ y(0) &= 7. \end{aligned}$$

# Separation of variables

## Example

Consider the IVP

$$\begin{aligned} y' &= y^2, \\ y(0) &= 1. \end{aligned}$$

# Separation of variables

### Remark

Warning! Sometimes $y(x) = 0$ will be a solution to such DEs. However, we may 'lose' it in the separation of variables method as we often divide by $y$. So you must also check if $y(x) = 0$ is a solution!

# Integrating factor

We can also deal with first-order DEs of the form

$$y' + P(x)y = Q(x)$$

rather magically. These are called *linear* first-order DEs.

First multiply both sides by a mystery function $I(x)$ to get

$$I(x)y' + I(x)P(x)y = I(x)Q(x).$$

Now we will *assume* that we can rewrite the LHS in order to get this:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(I(x)y\right) = I(x)Q(x).$$

If so, then integrating we get

$$I(x)y = \int I(x)Q(x)\mathrm{d}x.$$

And so

$$y(x) = \frac{1}{I(x)}\int I(x)Q(x)\mathrm{d}x.$$

# Integrating factor

Great! So what's $I(x)$? Well at one point we assumed

$$I(x)y' + I(x)P(x)y = \frac{\mathrm{d}}{\mathrm{d}x}\left(I(x)y\right).$$

And by the product rule this implies

$$I(x)y' + I(x)P(x)y = I(x)y' + I'(x)y.$$

So this suggests that $I(x)$ is a solution to the DE

$$I'(x) = P(x)I(x).$$

By separation of variables we get that a solution to this DE is

$$I(x) = e^{\int P(x)\mathrm{d}x}.$$

# Integrating factor

## Proposition (Integrating factor)

Consider the first-order linear DE

$$y' + P(x)y = Q(x).$$

Then the solution is given by

$$y(x) = \frac{1}{I(x)} \int I(x)Q(x)\mathrm{d}x$$

where

$$I(x) = e^{\int P(x)\mathrm{d}x}.$$

## Remark

This function $I(x)$ is called an *integrating factor*. When finding it, the constant of integration can be ignored (why?).

# Integrating factor

Best to do some examples...here's one from before:

**Example**

Consider the IVP

$$y' + 2xy = 6x,$$
$$y(0) = 4.$$

# Integrating factor

## Example

Consider the IVP

$$x^3 y' + x^2 y = 2x^3 + 1,$$
$$y(1) = 3.$$