Power series

1. Find the Maclaurin series of
$$x^3 + x^2 + 2x + 1$$
Answer. The Maclaurin series of any polynomial is the polynomial itself. So, in this case the Maclaurin series is
$$1 + 2x + x^2 + x^3.$$

2. Find the Maclaurin series of $f(x) = \sin(x)$. Also find the 4th remainder term $R_4(x)$ of $\sin(x)$ at 0 (this is the remainder term used to estimate how well the 4th partial sum of the Maclaurin series approximates $\sin(x)$.)

   Answer.
$$\begin{aligned} f(x) &= \sin(x), \text{ hence } f(0) = 0 \\ f'(x) &= \cos(x), \text{ hence } f'(0) = 1 \\ f''(x) &= -\sin(x), \text{ hence } f''(0) = 0 \\ f^{(3)}(x) &= -\cos(x), \text{ hence } f^{(3)}(0) = -1 \\ f^{(4)}(x) &= \sin(x), \text{ hence } f^{(4)}(0) = 0 \\ f^{(5)}(x) &= \cos(x), \text{ hence } f^{(5)}(0) = 1 \end{aligned}$$
From now on things repeat and we find, as already mentioned in the lecture notes, that the Maclaurin series of $\sin(x)$ is
$$\begin{aligned} &f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \end{aligned}$$
Furthermore we find that
$$\sin(x) = x - \frac{x^3}{3!} + R_4(x),$$
with
$$R_4(x) = \frac{f^{(4+1)}(z)}{(4+1)!}x^{4+1} = \frac{\cos(z)}{5!}x^5$$
for some number z between 0 and x.

3. Find the Maclaurin series of $f(x) = \ln(1+x)$. Also find the 4th remainder term $R_4(x)$ of $\ln(1+x)$ at 0 (this is the remainder term used to estimate how well the 4th partial sum of the Maclaurin series approximates $\ln(1+x)$.)

Answer.

$$\begin{aligned}
f(x) &= \ln(1+x), \text{ hence } f(0) = 0 \\
f'(x) &= \frac{1}{1+x}, \text{ hence } f'(0) = 1 \\
f''(x) &= -\frac{1}{(1+x)^2}, \text{ hence } f''(0) = -1 \\
f^{(3)}(x) &= \frac{2}{(1+x)^3}, \text{ hence } f^{(3)}(0) = 2! \\
f^{(4)}(x) &= -\frac{3!}{(1+x)^4}, \text{ hence } f^{(4)}(0) = -3! \\
f^{(5)}(x) &= \frac{4!}{(1+x)^5}, \text{ hence } f^{(5)}(0) = 4!
\end{aligned}$$

At this point the pattern is clear and we find that the Maclaurin series of $\ln(1+x)$ is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$$

$$= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + \frac{(-1)^{n-1}}{n}x^n + \cdots$$

Furthermore we find that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + R_4(x)$$

with

$$R_4(x) = \frac{f^{(4+1)}(z)}{(4+1)!}x^{4+1} = \frac{\frac{4!}{(1+z)^5}}{5!}x^5 = \frac{x^5}{5(1+z)^5}.$$

for some number z between 0 and x.

4. Find the Maclaurin series of the following functions by making suitable substitutions into one of the Maclaurin series of $e^x, \sin(x), \frac{1}{1-x}$.

$$(i)\ e^{-x}, \quad (ii)\ \sin(2x), \quad (iii)\ \frac{1}{1+x^3}.$$

And also figure out for which values of x these Maclaurin series are equal to the functions.

Answer. (i) Substitute $-x$ for x in the Maclaurin series of e^x.

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!}.$$

This identity is valid for all x.

(ii) Substitute $2x$ for x in the Maclaurin series of $\sin(x)$.

$$\sin(2x) = 2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \frac{512x^9}{9!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n+1}}{(2n+1)!}.$$

This identity is valid for all x.

(iii) Substitute $-x^3$ for x in the Maclaurin series of $\frac{1}{1-x}$.

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{3n}$$

This identity is valid for all x in the interval $(-1, 1)$.

5. Differentiate the Maclaurin series of e^x, $\sin(x)$ and $\cos(x)$ to double-check that $(e^x)' = e^x$, that $(\sin(x))' = \cos(x)$ and that $(\cos(x))' = -\sin(x)$.

   Answer.

$$(e^x)' = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)' = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x.$$

$$(\sin(x))' = \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)' = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \cos(x).$$

$$(\cos(x))' = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)' = -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \cdots = -\sin(x).$$

6. Double-check that the Maclaurin series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

   all have an infinite radius of convergence using the formula

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

Answer. For the Maclaurin series of e^x we find

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty.$$

In the case of the Maclaurin series of $\cos(x)$ we cannot apply our formula directly because every second term of this series is equal to zero and we don't want to divide by zero. So, what we'll do first is to substitute x for x^2. This gives the series

$$\sum_{n=0}^{\infty}\frac{(-1)^n x^n}{(2n)!} = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots$$

and our formula tells us that the radius of convergence of this new series is

$$R = \lim_{n\to\infty}\frac{(2n+2)!}{(2n)!} = \lim_{n\to\infty}(2n+2)(2n+1) = \infty.$$

But then it is also clear that the radius of convergence of the original series has to be infinite, as well.

In the case of the Maclaurin series of $\cos(x)$ we also cannot apply our formula directly because every second term of this series is equal to zero.

Here we first rewrite the series as follows

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)$$

and then substitute x for x^2 in the series in the brackets and calculate its radius of convergence, etc.

7. Find the *interval* of convergence of the following power series.

$$(i)\sum_{n=1}^{\infty} nx^n, \quad (ii)\sum_{n=1}^{\infty}\frac{(-1)^n}{5^n\sqrt{n}}x^n, \quad (iii)\sum_{n=1}^{\infty}\frac{(-1)^n n}{2^n(n+1)^3}x^n, \quad (iv)\sum_{n=1}^{\infty}\frac{(2n)!}{n!}x^n$$

$$\int_0^x f(t)dt$$

Do this by first figuring out the radius of convergence using the limit

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$$

(if it exists) and then testing the series for convergence at $x = R$ and $x = -R$.

Answer.

(i)

$$\sum_{n=1}^{\infty} nx^n$$

We calculate the radius of convergence

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$

So the radius of convergence of this series is 1. For $x = 1$ the series becomes

$$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + 4 + \cdots .$$

This series clearly diverges. For $x = -1$ the series becomes

$$\sum_{n=1}^{\infty} n(-1)^n = -1 + 2 - 3 + 4 - 5 + \cdots .$$

This series also diverges. This means that the interval of convergence of the power series $\sum_{n=1}^{\infty} nx^n$ is $(-1, 1)$.

(ii)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{5^n\sqrt{n}} x^n$$

We calculate the radius of convergence

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{5^{n+1}\sqrt{n+1}}{5^n\sqrt{n}} = \lim_{n\to\infty} 5\sqrt{\frac{n+1}{n}} = 5.$$

For $x = 5$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

This series converges by the alternating series test. For $x = -5$ the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}.$$

This is a diverging p-series ($p = \frac{1}{2}$). We conclude that the interval of convergence of our power series is $(-5, 5]$.

(iii)

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{2^n(n+1)^3} x^n$$

We calculate the radius of convergence

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{n2^{n+1}(n+2)^3}{2^n(n+1)^3(n+1)} = 2\lim_{n\to\infty} \frac{n(n+2)^3}{(n+1)^4} = 2.$$

For $x = 2$ the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n n}{(n+1)^3}$$

This series converges by the alternating series test. For $x = -2$ the series becomes

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)^3} = \sum_{n=1}^{\infty} \frac{n}{n^3 + 3n^2 + 3n + 1} = \sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 3 + \frac{1}{n}}$$

which converges because it is dominated by a converging p-series with $(p = 2)$:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 3 + \frac{1}{n}} \leq \sum_{n=1}^{\infty} \frac{1}{n^2},$$

Therefore the interval of convergence of our series is $[-2, 2]$.

(iv)

$$\sum_{n=1}^{\infty} \frac{(2n)!}{n!} x^n$$

We calculate the radius of convergence

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(2n)!(n+1)!}{n!(2n+2)!} = \lim_{n\to\infty} \frac{n+1}{(2n+1)(2n+2)} = 0.$$

This means that this power series converges only at $x = 0$.

8. Power series expansions can also be very useful for calculating indeterminate forms. Just for fun, try to calculate the indeterminate forms

$$\lim_{x\to 0} \frac{\sin(x)}{x}, \text{ and } \lim_{x\to 0} \frac{e^x - x - 1}{x^2}$$

using our power series representation of $\sin(x)$ and e^x.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Answer.

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) = 1.$$

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2} = \lim_{x\to 0} \left(\frac{1}{2!} + \frac{x}{3!} + \frac{x^2}{4!} + \cdots\right) = \frac{1}{2}.$$

Note that the series in the brackets are the Maclaurin series the functions

$$\begin{cases} \frac{\sin(x)}{x} & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$$

and

$$\begin{cases} \frac{e^x - x - 1}{x^2} & \text{if } x \neq 0; \\ \frac{1}{2} & \text{if } x = 0. \end{cases}$$

9. Compute the Taylor series of each of the following functions at the given points from scratch.

$$(i)\ \frac{1}{x} \text{ at } x = 1, \quad (ii)\ e^x \text{ at } x = -1, \quad (iii)\ \ln(x) \text{ at } x = 2.$$

(i)

$$f(x) = \frac{1}{x}, \text{ hence } f(1) = 1$$

$$f'(x) = -\frac{1}{x^2}, \text{ hence } f'(1) = -1$$

$$f''(x) = \frac{2}{x^3}, \text{ hence } f''(1) = 2!$$

$$f^{(3)}(x) = -\frac{3!}{x^4}, \text{ hence } f^{(3)}(1) = -3!$$

$$f^{(4)}(x) = \frac{4!}{x^5}, \text{ hence } f^{(4)}(1) = 4!$$

The pattern is clear. This means that the Taylor series of $\frac{1}{x}$ at 1 is

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \cdots + \frac{f^{(n)}(1)}{n!}(x-1)^n + \cdots$$

$$= 1 - (x-1) + (x-1)^2 - (x-1)^3 + \cdots + (-1)^n(x-1)^n + \cdots$$

Note that a simpler way to derive this Taylor series would be to simply substitute $-(x-1)$ for x in

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots.$$

(ii)

$$f(x) = e^x, \text{ hence } f(-1) = \frac{1}{e}.$$

In fact, the same is true for all derivatives of $f(x)$:

$$f^{(n)}(x) = e^x, \text{ hence } f^{(n)}(-1) = \frac{1}{e}.$$

This means that the Taylor series of e^x at $x = -1$ is

$$f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 + \cdots + \frac{f^{(n)}(-1)}{n!}(x+1)^n + \cdots$$

$$= \frac{1}{e}\left(1 + (x+1) + \frac{1}{2!}(x+1)^2 + \cdots + \frac{1}{n!}(x+1)^n + \cdots\right)$$

Note that a simpler way to derive this Taylor series would be to simply substitute $x+1$ for x in

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

and then to divide by e on both sides.

(iii)

$$\begin{aligned}
f(x) = \ln(x), &\text{ hence } f(2) = \ln(2) \\
f'(x) = \frac{1}{x}, &\text{ hence } f'(2) = \frac{1}{2} \\
f''(x) = -\frac{1}{x^2}, &\text{ hence } f''(1) = -\frac{1}{2^2} \\
f^{(3)}(x) = \frac{2}{x^3}, &\text{ hence } f^{(3)}(1) = \frac{2}{2^3} \\
f^{(4)}(x) = -\frac{3!}{x^4}, &\text{ hence } f(1)^{(4)} = -\frac{3!}{2^4} \\
f^{(5)}(x) = \frac{4!}{x^5}, &\text{ hence } f(1)^{(5)} = \frac{4!}{2^5}
\end{aligned}$$

The pattern is clear. This means that the Taylor series of $\ln(x)$ at 2 is

$$\ln(2)+\frac{1}{2}(x-2)-\frac{1}{2}\frac{1}{2^2}(x-2)^2+\frac{1}{3}\frac{1}{2^3}(x-2)^3-\frac{1}{4}\frac{1}{2^4}(x-2)^4+\cdots+\frac{(-1)^{n-1}}{n}\frac{1}{2^n}(x-2)^n+\cdots$$

10. (Completely optional and definitely not examinable!) Here is another boredom killer. Define

$$F(x) = \sum_{n=0}^{\infty} f_n x^n,$$

where f_n is the nth Fibonacci number. So,

$$F(x) = 0 + 1x + 1x^2 + 2x^3 + 3x^4 + 5x^5 + 8x^6 + \cdots .$$

Prove that $f_n < 2^n$ for all n and use this to show that $F(x)$ converges if $|x| < \frac{1}{2}$. Also show that

$$(1 - x - x^2)F(x) = x$$

and that therefore

$$F(x) = \frac{x}{1 - x - x^2}.$$

$F(x)$ is an example of a so-called *generating function*.

If you have a close look at questions 3 and 4 of the blackboard in *Good Will Hunting* (we considered questions 1 and 2 in the lectures) then you will find that they are asking about the generating functions for the numbers $w_n(i, j)$ of n step walks from vertex i to vertex j. On the right side of the blackboard you can see that these generating functions can be expressed in terms of determinants.

Generating functions are silver bullets when it comes to finding closed formulas for sequences given in a recurrence relation, for solving enumeration problems, etc. Very nice stuff!!

Answer. We can prove that $f_n < 2^n$ by induction. It is evident that $f_1 < 2$, so assume this holds for all $n \leq k$. We have

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} \\ &< 2^k + 2^{k-1} \text{ (by induction hypothesis)} \\ &< 2^k + 2^k \ \ (2^{k-1} < 2^k) \\ &= 2(2^k) \\ &= 2^{k+1} \end{aligned}$$

As it holds for $n = k + 1$ using the induction hypothesis, it holds for all n.
We have

$$F(x) = \sum_{n=0}^{\infty} f_n x^n$$

But we have just proven that $f_n < 2^n$. Therefore we get that

$$F(x) < \sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n$$

This is a geometric series that converges when $|2x| < 1$. This means that $F(x)$ converges if $|x| < \frac{1}{2}$.
We have

$$
\begin{aligned}
(1-x-x^2)F(x) &= F(x) - xF(x) - x^2F(x) \\
&= \sum_{n=0}^{\infty} f_n x^n - x\sum_{n=0}^{\infty} f_n x^n - x^2\sum_{n=0}^{\infty} f_n x^n \\
&= \sum_{n=0}^{\infty} f_n x^n - \sum_{n=0}^{\infty} f_n x^{n+1} - \sum_{n=0}^{\infty} f_n x^{n+2} \\
&= \sum_{n=0}^{\infty} f_n x^n - \sum_{n=1}^{\infty} f_{n-1} x^n - \sum_{n=2}^{\infty} f_{n-2} x^n \\
&= f_0x^0 + f_1x^1 + \sum_{n=2}^{\infty} f_n x^n - f_0x^1 - \sum_{n=2}^{\infty} f_{n-1} x^n - \sum_{n=2}^{\infty} f_{n-2} x^n \\
&= x + \sum_{n=2}^{\infty} (f_n - f_{n-1} - f_{n-2})x^n \qquad (f_0 = 0, f_1 = 1) \\
&= x
\end{aligned}
$$

Since we have this result, we can simply divide by $1 - x - x^2$ to get

$$F(x) = \frac{x}{1 - x - x^2}$$

TEST QUESTIONS

11. Prove Euler's formula from scratch using the Maclaurin series of $\sin(x)$ and $\cos(x)$. Derive Euler's identity from Euler's formula.

12. Be able to reproduce the proof in the lecture notes that e is an irrational number.

13. How does one define e^A if A is a diagonalizable matrix?

14. Is the Maclaurin series of a function equal to the function?

15. What is the radius of convergence of a power series? What is the interval of convergence of a power series?

16. Give examples of two different functions having the same Maclaurin series.

17. If you integrate a power series with radius of convergence R, what is the radius of convergence of the new series? How do the intervals of convergence of the original and the new series compare?

18. If you differentiate a power series with radius of convergence R, what is the radius of convergence of the new series? How do the intervals of convergence of the original and the new series compare?

19. What is the main use of Taylor's formula?

20. What is the difference between a Taylor series and a Maclaurin series?