Linear transformations

A LOT OF THE FOLLOWING ARE ADAPTED FROM KUTTLER'S BOOK

1. Find the matrix of the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\pi/3$ (=60 degrees) in the counterclockwise direction.

   Answer.
   $$A_{\frac{\pi}{3}} = \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

2. Find the matrix of the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\pi/4$ (=45 degrees) in the *clockwise* direction.

   Answer.
   $$A_{-\frac{\pi}{4}} = \begin{pmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

3. Find the matrix for the linear transformation which rotates every vector in $\mathbf{R}^2$ through an angle of $\pi/3$ in the counterclockwise direction and then reflects across the x-axis.

   Answer.
   $$R_x A_{\frac{\pi}{3}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

4. Find the matrix of the linear transformation which reflects every vector in $\mathbf{R}^2$ through the x-axis and then rotates every vector through an angle of $\pi/3$ in the counterclockwise direction.

   Answer.
   $$A_{\frac{\pi}{3}} R_x = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

5. Find the matrix of the linear transformation which reflects every vector in $\mathbf{R}^2$ through the line containing the origin and making an angle of 45 degrees with the x-axis.

   Answer.
   $$A_{\frac{\pi}{4}} R_x A_{-\frac{\pi}{4}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

6. Find the matrix of the linear transformation $\mathbf{R}^2 \to \mathbf{R}^2$ that first rotates through an angle α in the clockwise direction and then rotates through an angle β in the counterclockwise direction.

   Answer. This is the same as rotating through an angle of $\beta - \alpha$ in the counterclockwise direction. So the matrix is

$$A_\beta A_{-\alpha} = A_{\beta-\alpha} = \begin{pmatrix} \cos(\beta-\alpha) & -\sin(\beta-\alpha) \\ \sin(\beta-\alpha) & \cos(\beta-\alpha) \end{pmatrix}.$$

7. Find the matrix of the linear transformation which rotates every vector in $\mathbf{R}^3$ counterclockwise around the z-axis (when viewed from the positive z-axis) through an angle of $\pi/3$ and then reflects through the xy-plane.

   Answer.

$$R_{xy}A_{\frac{\pi}{3}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

8. Find the matrix for $\mathbf{proj_u}(\mathbf{v})$ where $\mathbf{u} = (1, 5, 3)^T$.

   Answer.

$$\frac{\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}} = \frac{1}{35}\begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}, \mathbf{e}_1 \cdot \mathbf{u} = 1, \mathbf{e}_2 \cdot \mathbf{u} = 5, \mathbf{e}_3 \cdot \mathbf{u} = 3,$$

$$\therefore \mathbf{proj_u}(\mathbf{v}) = \frac{1}{35}\begin{pmatrix} 1 & 5 & 3 \\ 5 & 25 & 15 \\ 3 & 15 & 9 \end{pmatrix}$$

9. Find the matrix for $\mathbf{proj_u}(\mathbf{v})$ where $\mathbf{u} = (1, 0, 3)^T$

   Answer.

$$\frac{\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}} = \frac{1}{10}\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \mathbf{e}_1 \cdot \mathbf{u} = 1, \mathbf{e}_2 \cdot \mathbf{u} = 0, \mathbf{e}_3 \cdot \mathbf{u} = 3,$$

$$\frac{1}{10}\begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & 9 \end{pmatrix}$$

10. What is the matrix that describes the orthogonal projection onto the xy-plane in $\mathbf{R}^3$.

    Answer.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

11. How would you construct the matrix that describes the orthogonal projection onto a plane through the origin given by a normal vector $\mathbf{u}$ in $\mathbf{R}^3$?

    Answer. The same way we constructed the matrix that describes a rotation around $\mathbf{u}$ (in the lecture notes).

12. What is the determinant of a rotation of $\mathbf{R}^2$ and what is the determinant of a rotation of $\mathbf{R}^3$.

    Answer. For the 2d rotations we get

$$\begin{vmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{vmatrix} = \sin^2(\theta) + \cos^2(\theta) = 1.$$

The matrix of every 3d rotation is of the form SAS^{-1}, where A is the 3d extension of the 2d rotation above. It is easy to see that $\det(A) = 1$. Furthermore

$$\det(SAS^{-1}) = \det(S)\det(A)\det(S^{-1}) = \det(SS^{-1})\det(A) = \det(I)\det(A) = 1. \quad \blacksquare$$

13. The columns of the matrix S that we used to construct the 3d rotation matrices are three mutually orthogonal unit vectors.

    (a) Prove that

$$S^T S = I,$$

    that is, the inverse of S is just its transpose.

    Answer. The (i, j)th entry of $S^T S$ is simply the dot product of the ith column vector with the jth column vector of S. Since these vectors are mutually orthogonal, this product is equal to zero if $i \neq j$. Because we are dealing with unit vectors, the dot product is equal to 1 if $i = j$. We conclude that $S^T S = I$. $\blacksquare$

    (b) A matrix B with the property that $B^T B = I$ is called an **orthogonal** matrix. Prove that the columns of a matrix are mutually orthogonal unit vectors if and only if the matrix is orthogonal.

    Proof. Same idea as under (a).

    (c) For an orthogonal matrix B what are the possible values for $\det(B)$.

    Answer.

$$1 = \det(I) = \det(B^T B) = \det(B^T)\det(B) = \det(B)\det(B) = \det(B)^2.$$

    And, of course, $\det(B)^2 = 1$ implies that $\det(B) = 1$ or -1. $\blacksquare$

    (d) Prove that the rows of an orthogonal matrix also form a set of mutually orthogonal unit vectors.

    Proof. Since $B^T B = I$ we also have $BB^T = I$. This means that B^T is also an orthogonal matrix. So, the columns of B^T which are the rows of B also form a set of mutually orthogonal unit vectors. $\blacksquare$

(e) Prove that the product of two orthogonal matrices A and B is an orthogonal matrix.

Proof. Let's check this

$$(AB)^T(AB) = (B^TA^T)(AB) = B^T(A^TA)B = B^TIB = B^TB = I. \quad \blacksquare$$

(f) Prove that all rotations are othogonal matrices. (In fact, it turns out that in $\mathbf{R}^2$ and $\mathbf{R}^3$ the orthogonal matrices with determinant 1 are exactly the rotation matrices. All other orthogonal matrices describe rotations followed by some reflection through a line in $\mathbf{R}^2$ or a plane in $\mathbf{R}^3$ (containing the origin).)

Proof. It is easy to see that the rows of a 2d rotation matrix are mutually orthogonal.

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

As we've seen, the 3d rotation matrices are of the form SBS^{-1}, where B is the 3d extension of a 2d rotation matrix. It is easily checked that B is orthogonal. But then since both S and S^{-1} are orthogonal, the product SBS^{-1} is orthogonal, as well. $\blacksquare$

(g) Let $\mathbf{u}$ be a unit column vector. Prove that the matrix $I - 2\mathbf{u}\mathbf{u}^T$ is an orthogonal matrix. (This matrix is called a **Householder matrix**. In $\mathbf{R}^2$ and $\mathbf{R}^3$ these matrices are the matrices of reflections through lines and planes, respectively, containing the origin and orthogonal to $\mathbf{u}$. If you are keen try to prove this, too!)

Proof.

$$\begin{aligned} &(I - 2\mathbf{u}\mathbf{u}^T)^T(I - 2\mathbf{u}\mathbf{u}^T) \\ &= (I - 2\mathbf{u}\mathbf{u}^T)(I - 2\mathbf{u}\mathbf{u}^T) \\ &= I - 2\mathbf{u}\mathbf{u}^T - 2\mathbf{u}\mathbf{u}^T + 4\mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T = I. \end{aligned}$$

(Note that the red product is equal to the dot product of $\mathbf{u}$ with itself. And since this vector is a unit vector the product is equal to 1. $\blacksquare$

(h) Prove that an orthogonal matrix preserves distances, that is,

$$|A\mathbf{x}| = |\mathbf{x}|$$

for all possible vectors $\mathbf{x}$.

Proof.

$$|\mathbf{x}|^2 = \mathbf{x}\cdot\mathbf{x} = \mathbf{x}^T\mathbf{x} = \mathbf{x}^TA^TA\mathbf{x} = (A\mathbf{x})^TA\mathbf{x} = (A\mathbf{x})\cdot(A\mathbf{x}) = |A\mathbf{x}|^2. \quad \blacksquare$$

14. You have a linear transformation T and

$$T\begin{pmatrix} 1 \\ 1 \\ -8 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} -1 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \\ -1 \end{pmatrix}.$$

Find the matrix of T.

Answer. So far it was always clear what a linear transformation does to the unit coordinate vectors and it was then straightforward to assemble the matrix of the linear transformations from the images of the unit coordinate vectors. Here we are given the images of three vectors that are different from the unit coordinate vectors,

$$\mathbf{u_1} = \begin{pmatrix} 1 \\ 1 \\ -8 \end{pmatrix}, \quad \mathbf{u_2} = \begin{pmatrix} -1 \\ 0 \\ 6 \end{pmatrix}, \quad \mathbf{u_3} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix},$$

under a mystery linear transformation and we are supposed to figure out what the matrix is. Tricky!

Here is the matrix whose columns are the three vectors.

$$C = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -8 & 6 & 3 \end{pmatrix}.$$

Interpreted as a linear transformation it maps $\mathbf{e_1}$ to $\mathbf{u_1}$, $\mathbf{e_2}$ to $\mathbf{u_2}$, and $\mathbf{e_3}$ to $\mathbf{u_3}$. This means that the inverse of this matrix (which happens to exist) maps $\mathbf{u_1}$ to $\mathbf{e_1}$, $\mathbf{u_2}$ to $\mathbf{e_2}$, and $\mathbf{u_3}$ to $\mathbf{e_3}$.

So, if we want to know what happens to a given vector $\mathbf{x}$ under the linear transformation T, we first multiply it by

$$C^{-1} = \begin{pmatrix} 6 & 3 & 1 \\ 5 & 3 & 1 \\ 6 & 2 & 1 \end{pmatrix}.$$

For example, let

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Then

$$C^{-1}\mathbf{x} = \begin{pmatrix} 15 \\ 14 \\ 13 \end{pmatrix}.$$

What this means is that

$$\mathbf{x} = 15\mathbf{u_1} + 14\mathbf{u_2} + 13\mathbf{u_3}.$$

But then the image of $\mathbf{x}$ under the linear transformation is simply

$$T(\mathbf{x}) = 15T(\mathbf{u_1}) + 14T(\mathbf{u_2}) + 13T(\mathbf{u_3}).$$

So, if D is the matrix with columns $T(\mathbf{u_1}), T(\mathbf{u_2}), T(\mathbf{u_3})$, then

$$T(\mathbf{x}) = DC^{-1}\mathbf{x}.$$

This means that the matrix we are after is simply DC^{-1}.

So the answer to this problem is

$$\begin{pmatrix} 1 & 2 & 6 \\ 3 & 4 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -8 & 6 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 2 & 6 \\ 3 & 4 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 6 & 3 & 1 \\ 5 & 3 & 1 \\ 6 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 52 & 21 & 9 \\ 44 & 23 & 8 \\ 5 & 4 & 1 \end{pmatrix}.$$

15. Prove that the function $T_{\mathbf{u}}$ defined by $T_{\mathbf{u}}(\mathbf{x}) = \mathbf{v} - \mathbf{proj_u}(\mathbf{x})$ is a linear transformation.

    Proof.

$$T_{\mathbf{u}}(\mathbf{v}+\mathbf{w}) = (\mathbf{v}+\mathbf{w}) - \frac{(\mathbf{v}+\mathbf{w})\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \left(\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) + \left(\mathbf{w} - \frac{\mathbf{w}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) = T_{\mathbf{u}}(\mathbf{v}) + T_{\mathbf{u}}(\mathbf{w}).$$

    and

$$T_{\mathbf{u}}(a\mathbf{v}) = \left(a\mathbf{v} - \frac{(a\mathbf{v})\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) = a\left(\mathbf{v} - \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) = aT_{\mathbf{u}}(\mathbf{v}). \quad \blacksquare$$

16. Here are some descriptions of functions $\mathbf{R}^n \to \mathbf{R}^n$.

    (a) T multiplies the j^{th} component of the vector $\mathbf{x}$ by a non-zero number b.

    (b) T replaces the i^{th} component of the vector $\mathbf{x}$ with b times the j^{th} component added to the i^{th} component.

    (c) T switches two components of the vector.

    Show that these functions are linear transformations and describe their matrices.

    Answer. Each of these functions corresponds to an elementary matrix. The first is the elementary matrix which multiplies the j^{th} diagonal entry of the identity matrix by b. The second is the elementary matrix which takes b times the j^{th} row and adds to the i^{th} row and the third is just the elementary matrix which switches the i^{th} and the j^{th} rows where the two components are in the i^{th} and j^{th} positions.

SOME TEST QUESTIONS

17. Construct from scratch a 2×2 matrix that describes a counterclockwise rotation around the origin by an angle α.

18. How would you go about constructing a 3×3 rotation matrix from scratch that rotates in the counterclockwise direction around a given unit vector by a certain angle α?