MTH1030: Assignment 2, 2021: Solution

Serious series

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Up to 20 marks overall were deducted for messy/incomplete/etc.  presentation in
increments of 5 marks
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1 Calculating π for real [35 marks]

a) The general term of this series is

$$(-1)^{n+1}\frac{4}{2n-1}$$

This means that the nth partial sum is

```
Sum[(-1)^(i + 1) 4/(2 i - 1), {i,1, n}]
```

So we just play around with the

```
N[Pi-4*Sum[(-1)^(i + 1) 4/(2 i - 1), {i,1, n}]
```

until what we get just dips below 0.0001. This happens for $n = 10,000$ (quite a coincidence!)

`3 marks for getting the general term right, 4 for the complete Mathematica expression, and the remaining 3 for getting the correct` n.

b)

$$|a_{n+1}| = \frac{4}{2(n+1)-1} = \frac{4}{2n+1}.$$

This means we can be sure that

$$|\pi - s_n| < 0.0001$$

if

$$\frac{4}{2n+1} < 0.0001.$$

Solving for n gives $n > 19,999.5$.

`3 marks for setting up the right inequality and 2 marks for solving it.`

c) $\alpha = \tan^{-1}(1/5)$. Using the addition formula we get

$$\tan(2\alpha) = \frac{\frac{1}{5}+\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2} = \frac{5}{12}$$

`Award 3 marks for getting this right.`

Using the addition formula again we get

$$\tan(4\alpha) = \frac{\frac{5}{12}+\frac{5}{12}}{1-\left(\frac{5}{12}\right)^2} = \frac{120}{119}.$$

2 marks for getting this right.

And using the addition formula one last time we get

$$\tan\left(\frac{\pi}{4}-4\alpha\right)=\frac{\tan\left(\frac{\pi}{4}\right)+\tan(-4\alpha)}{1-\tan\left(\frac{\pi}{4}\right)\tan(-4\alpha)}=\frac{1-\frac{120}{119}}{1+\frac{120}{119}}=-\frac{1}{239}.$$

2 marks for getting this right.

This means that

$$\frac{\pi}{4}=4\alpha+\tan^{-1}\left(-\frac{1}{239}\right)=4\tan^{-1}\left(\frac{1}{5}\right)+\tan^{-1}\left(-\frac{1}{239}\right)$$

Since $\tan^{-1}(x)=-\tan^{-1}(-x)$ this can also be written as

$$\pi=16\tan^{-1}\left(\frac{1}{5}\right)-4\tan^{-1}\left(\frac{1}{239}\right).$$

3 marks for getting this right.

d) We need

$$\tan^{-1}(1/5)\approx(1/5)-\frac{(1/5)^3}{3}+\frac{(1/5)^5}{5}$$

and

$$\tan^{-1}(1/239)\approx(1/239)-\frac{(1/239)^3}{3}+\frac{(1/239)^5}{5}$$

and so

$$16\left((1/5)-\frac{(1/5)^3}{3}+\frac{(1/5)^5}{5}\right)-4\left((1/239)-\frac{(1/239)^3}{3}+\frac{(1/239)^5}{5}\right)=3.14162...$$

This number coincides in the first four digits with π and overall the approximation just using these few terms is about as good as using 10,000 terms of the Leibniz series.

3 marks for the right interpretation of what is asked for here. 2 more marks for getting the right answer.

e) We know that

$$|\tan^{-1}(1/5)-T_n|\leq\frac{1}{5^{2n+1}(2n+1)}.$$

and therefore

$$|16\tan^{-1}(1/5) - 16T_n| \le \frac{16}{5^{2n+1}(2n+1)}.$$

Now let's find the smallest n such that the right side is smaller than 10^{-102}.

Some poking around using *Mathematica* shows that $n = 72$ will do.

`3 marks for the right interpretation of what is asked for here. 2 more marks for getting the right answer.`

2 Ramanujan's impossible sum [40 marks]

a) [**20 marks**] Following the instructions in the video we calculate the array of term-by-term products.

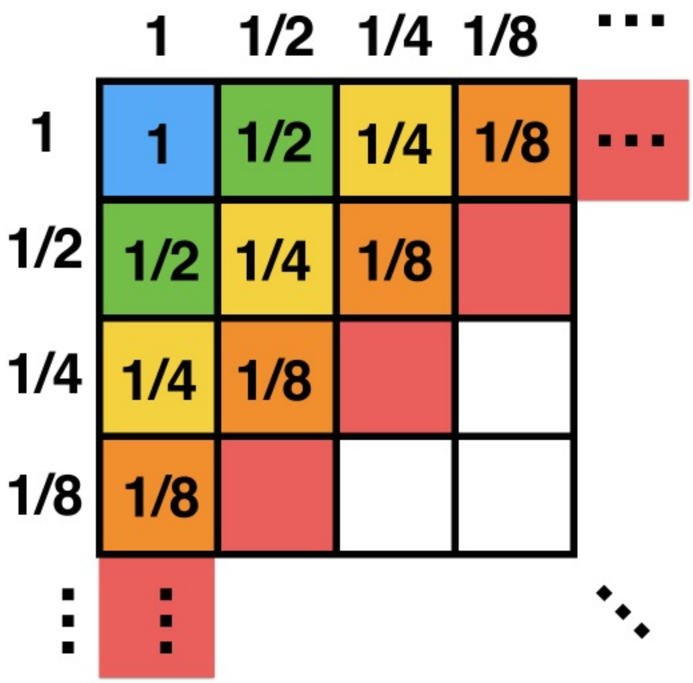

Figure 1: Grid for calculating the Cauchy product.

`4 marks for this grid.`

Then summing along coloured diagonals we arrive at the following series

$$\sum_{n=0}^{\infty}(n+1)\frac{1}{2^n} = 1\cdot\frac{1}{2^0} + 2\cdot\frac{1}{2^1} + 3\cdot\frac{1}{2^2} + 4\cdot\frac{1}{2^3} + \cdots$$

`4 marks for this series.`

Using the information given in the problem, we conclude that the sum of this series is the square of the sum of the original series, that is

$$1\cdot\frac{1}{2^0} + 2\cdot\frac{1}{2^1} + 3\cdot\frac{1}{2^2} + 4\cdot\frac{1}{2^3} + \cdots = 2^2 = 4$$

`4 marks for this sum.`

Calculating the first 100 terms of this series, for example using the following code

```
Partialsum = 0;

For[i = 0, i < 100, i++, Partialsum = Partialsum + (i + 1) 1/2^i]

N[Partialsum, 40]
```

gives

3.999999999999999999999999999839072375335

which, as expected, is very close to 4 the actual sum of the Cauchy product.

`8 marks for reasonably looking Mathematica code (or anything else reasonable) and getting roughly the 100th partial sum right.`

b) [**20 marks**] The sequence of partial sums of

$$1 - 1 + 0 + 1 - 1 + 0 + 1 - 1 + 0 + 1 - 1 + 0 + \cdots$$

is

$$1, 0, 0, 1, 0, 0, 1, 0, 0, \ldots$$

`3 marks for this sequence.`

This means that the sequence of averages is

$$1, \frac{1}{2}, \frac{1}{3}, \frac{2}{4}, \frac{2}{5}, \frac{2}{6}, \ldots, \frac{n}{3n-2}, \frac{n}{3n-1}, \frac{n}{3n}, \ldots$$

for $n = 1, 2, 3, \ldots$ Since all three expressions in the variable n have limit $\frac{1}{3}$ as n goes to infinity, this sequence converges to $\frac{1}{3}$. This means that the Cesaro sum of the series is $\frac{1}{3}$.

`2 marks for explanation of convergence and`

`2 marks for this 1/3.`

(This is intuitively clear because large chunks of the sequence of partial sums contain roughly 1/3 1s.)

The sequence of partial sums of

$$1 - 1 + 0 + 0 + 1 - 1 + 0 + 0 + 1 - 1 + 0 + 0 + \cdots$$

is

$$1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, \ldots$$

`3 marks for this sequence.`

The sequence of averages of this sequence converges to $\frac{1}{4}$. This means that the Cesaro sum of the series is $\frac{1}{4}$.

```
2 marks for explanation of convergence and

2 marks for this 1/4.
```

Given a convergent (in the standard way) series, its sum won't change if you insert 0s into it. Reason: Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots$ with corresponding sequence of partial sums $s_1, s_2, s_3, \ldots$, inserting a 0 after the nth summand changes the sequence of partial sums by replacing s_n in the sequence by s_n, s_n. This means that inserting 0s results in a sequence of partial sums that consists of the same elements in the same order, just with copies of certain elements inserted multiple times in a row. It then follows that the sequence of partial sums has the same limit as the original sequence and that therefore the original series has the same sum as the new series.

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6 marks for explanation
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3 ... an opportunity to challenge yourself with something crazy ...

Just change the (i+1)^0 in the second line of the original code to (i+1)^2 and it will first calculate the first 5000 partial sums and then the first 5000 elements of the sequence of averages of the partial sums. Duplicating the code for the latter part two more times will result in code that calculates the first 5000 elements of sequence of averages of averages of averages that we are after. This sequence seems to converge to 0 which also happens to be the correct answer. The 5000th element of this sequences is 0.00182976. Here is the extended code

```
ClearAll

PartialSum[0] = 0;

For[i = 0, i < 5000, i++,

PartialSum[i + 1] = PartialSum[i] + (i + 1)^2 (-1)^i]

SumOfPartialSum[0] = 0;

For[i = 0, i < 5000, i++,

SumOfPartialSum[i + 1] = SumOfPartialSum[i] + PartialSum[i + 1];
```

```
Average[i + 1] = SumOfPartialSum[i + 1]/(i + 1)]

SumOfAverage[0] = 0;

For[i = 0, i < 5000, i++,

SumOfAverage[i + 1] = SumOfAverage[i] + Average[i + 1];

AveOfAverage[i + 1] = SumOfAverage[i + 1]/(i + 1)]

SumOfAveOfAverage[0] = 0;

For[i = 0, i < 5000, i++,

SumOfAveOfAverage[i + 1] =

SumOfAveOfAverage[i] + AveOfAverage[i + 1];

AveOfAveOfAverage[i + 1] = SumOfAveOfAverage[i + 1]/(i + 1);

Print[N[AveOfAveOfAverage[i + 1]]]]
```