MTH1030 A1

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Part 1 The weirdest 3d Pythagoras yet

A:

Set u and v which are span the parallelogram

```
In[•]:= u = {u1, u2, u3}
        v = {v1, v2, v3}
```

Plane P

We can calculate the area of P and we can get P^2 which is the right hand side

```
In[•]:= AreaP = Norm[Cross[u, v]]
```

```
In[•]:= AreaP^2
```

Out[•]= $\text{Abs}[-u2\ v1 + u1\ v2]^2 + \text{Abs}[u3\ v1 - u1\ v3]^2 + \text{Abs}[-u3\ v2 + u2\ v3]^2$

Plane O

Due to the plane O is the orthogonal projection of P on yz plane, we can know the u and v projection on yz is:

```
In[•]:= uo = {0, u2, u3}
        vo = {0, v2, v3}
```

Then we use the cross product of uo and vo calculate the area of O, then we can know the area O^2

```
In[•]:= AreaO = Norm[Cross[uo, vo]]
        AreaO^2
```

Out[•]= $\text{Abs}[-u3\ v2 + u2\ v3]$

Out[•]= $\text{Abs}[-u3\ v2 + u2\ v3]^2$

Plane N

Due to the plane N is the orthogonal projection of P on zx plane, we can know the u and v projection on zx is:

```
In[•]:= un = {u1, 0, u3}
        vn = {v1, 0, v3}
```

Then we use the cross product of un and vn calculate the area of N, then we can know the area N^2

In[ ]:=
```
AreaN = Norm[Cross[un, vn]]
AreaN^2
```

Out[ ]= Abs[u3 v1 − u1 v3]

Out[ ]= $\text{Abs}[u3\ v1 - u1\ v3]^2$

Plane M

Due to the plane M is the orthogonal projection of P on xy plane, we can know the u and v projection on xy is:

In[ ]:=
```
um = {u1, u2, 0}
vm = {v1, v2, 0}
```

Then we use the cross product of um and vm calculate the area of M, then we can know the area M^2

In[ ]:=
```
AreaM = Norm[Cross[um, vm]]
AreaM^2
```

Out[ ]= Abs[−u2 v1 + u1 v2]

Out[ ]= $\text{Abs}[-u2\ v1 + u1\ v2]^2$

Final result

We can make a equation and and let Mathematica to verify it is True

In[ ]:=
```
AreaM^2 + AreaN^2 + AreaO^2 == AreaP^2
```

Out[ ]= True

B:

Set the highest point of tetrahedron is point D, and other point is point A, point B and point C
Then we can get vectors: DA, DB, DC
The bottom triangle side vectors: CA, CB ,AB

Area of ADC, ADB and CDB

We can using the cross product to calculate the area, which is the left hand side

In[ ]:=
```
TopArea = 1/2 * (Norm[Cross[DA, DC]] + Norm[Cross[DB, DC]] + Norm[Cross[DA, DB]])
```

Out[ ]= $\frac{1}{2}\left(\text{Norm}[DA \times DB] + \text{Norm}[DA \times DC] + \text{Norm}[DB \times DC]\right)$

Area of ABC

We also use cross product of two vectors to calculate the area of bottom

$\frac{1}{2}\text{Norm}[CA \times CB] = \frac{1}{2}\text{Norm}[(DA\text{-}DC)x(DB\text{-}DC)]$

$= \frac{1}{2}\text{Norm}[(DA\text{-}DC)x(DB\text{-}DC)]$

$= \frac{1}{2}\text{Norm}[(DAxDB)+ -(DAxDC)+ -(DCxDB)+(DCxDC)]$

$= \frac{1}{2}\text{Norm}[(DAxDB)+ -(DAxDC)+ -(DCxDB)]$

$= \frac{1}{2}\text{Norm}[(DAxDB)]+\text{Norm}[(DAxDC)]+\text{Norm}[(DCxDB)]$ = Left hand side

C:

We can firstly get the area of plane, then get the area of bottom plane, and add up

```
In[ ]:= upper = 1 / 2 * (2.5 * 3 + 3 * 4 + 2.5 * 4) == 3.75 + 6 + 5
Out[ ]= True

In[ ]:= upper = 1 / 2 * (2.5 * 3 + 3 * 4 + 2.5 * 4)
Out[ ]= 14.75

In[ ]:= bottom = (3.75^2 + 6^2 + 5^2) ^ (1 / 2)
Out[ ]= 8.66386

In[ ]:= theSum = upper + bottom
Out[ ]= 23.4139
```

Part 2 The Big Cube

Find Aa length and it vector

Let’s tell *Mathematica* about **A** and **a, b**. (a is A’)

```
In[ ]:= A = {43.1162, 29.9541, 7.72383}
        a = {42.3208, 27.8349, 0}
        b = {25.7441, 19.8898, 0}
```

and we make vector Aa

```
In[ ]:= Aa = a - A
Out[ ]= {-0.7954, -2.1192, -7.72383}
```

and we figure out the length of Aa

```
In[ ]:= Aa_length = Norm[Aa]
Out[ ]= 8.04868
```

Find Plane ABCD equation

aA is vertical with plane ABCD, so we can **set aA as normal vector of ABCD**, and we put the coordinate of A in it to get the equation(A is on the plane)

```
In[ ]:= ABCD = 0.7954 x + 2.1192 y + 7.72383 z == 157.431
```

Find the distance of Bb

We start with finding the distance of Bb
Find point p on the plane setting y and z =0

```
In[ ]:= p = {157.431 / 0.7954, 0, 0}
Out[ ]= {197.927, 0, 0}

In[ ]:= v = b - p
```

Out[]= {-172.183, 19.8898, 0}

In[]:= distanceBb = Norm[Projection[v, n]]

Out[]= 11.7788

Find point B

We set B as {Bx,By,Bz}

In[]:= B = {Bx, By, Bz}

In[]:= Bb = {25.7441 - Bx, 19.8898 - By, -Bz}

In[]:= Norm[Bb] == distanceBb

$$\sqrt{[25.7441` - Bx]^2 + [19.8898` - By]^2 + \text{Abs}[Bz]^2} == 11.778788659188484`$$

We find AB and AB vertical Bb

In[]:= AB = B - A

Dot[AB, Bb] == 0

Out[]= -16.2081 (25.7441 - Bx) - 6.96297 (19.8898 - By) - 3.57956 Bz == 0

Then we set up 3 equations for solving the coordinate of B

1. The B is on the plane ABCD
2. Norm[Bb]==11.77
3. Dot[AB,Bb]==0

In[]:= Solve$\Big[$0.7954 Bx + 2.1192 By + 7.72383 Bz == 157.431 &&

$\sqrt{(25.7441 - Bx)^2 + (19.8898 - By)^2 + (Bz)^2}$ == 11.7788 && (25.7441 - Bx) (-43.1162 + Bx) +

(19.8898 - By) (-29.9541 + By) - (-7.72383 + Bz) Bz == 0, {Bx, By, Bz}$\Big]$

Out[]= {{Bx → 26.9081, By → 22.9911, Bz → 11.3034}}

In[]:= B = {26.90813027581234`, 22.991133262610784`, 11.303390101606857`}

In[]:= Bb = b - B

Out[]= {-1.16403, -3.10133, -11.3034}

Find point D (Due to character D is protected, we set variable as pointD)

In[]:= pointD = {Dx, Dy, 11.303390101606857`}

Out[]= {Dx, Dy, 11.3034}

In[]:= AD = pointD - A

Out[]= {-43.1162 + Dx, -29.9541 + Dy, 3.57956}

We set up 2 equations for solving the coordinate of D

1. AB vertical AD
2. D on plane ABCD

In[]:= Dot[AB, AD] == 0

Out[]= 12.8133 - 16.2081 (-43.1162 + Dx) - 6.96297 (-29.9541 + Dy) == 0

In[]:= 0.7954 Dx + 2.1192 Dy + 7.72383 * 11.3034 == 157.431

Out[•]= 87.3055 + 0.7954 Dx + 2.1192 Dy == 157.431

```
In[•]:= Solve[0.7954 Dx + 2.1192 Dy + 7.72383 * 11.3034 == 157.431 && Dot[AB, AD] == 0, {Dx, Dy}]
Out[•]= {{Dx → 50.7409, Dy → 14.0459}}
```

```
In[•]:= pointD = {50.740870303284865`, 14.045947404099287`, 11.303390101606857`}
Out[•]= {50.7409, 14.0459, 11.3034}
```

Find point d

The point d in plane abcd, that mean dz = 0

```
In[•]:= pointd = {dx, dy, 0}
```

```
In[•]:= Dd = pointd - pointD
Out[•]= {-50.7409 + dx, -14.0459 + dy, -11.3034}
```

We set up 2 equations for solving the coordinate of d

1. Aa//Dd
2. Dd vertical AD

```
In[•]:= AD = pointD - A
        Solve[Norm[Cross[Dd, Aa]] == 0 && Dot[Dd, AD] == 0, {dx, dy}]
Out[•]= {7.62467, -15.9082, 3.57956}
Out[•]= {{dx → 49.5768, dy → 10.9446}}
```

```
In[•]:= pointd = {49.57684784855689`, 10.944616098272855`, 0}
Out[•]= {49.5768, 10.9446, 0}
```

Find point C

```
In[•]:= pointC = {Cx, Cy, Cz}
```

```
In[•]:= CD = pointD - pointC
        CB = B - pointC
Out[•]= {50.7409 - Cx, 14.0459 - Cy, 11.3034 - Cz}
Out[•]= {26.9081 - Cx, 22.9911 - Cy, 11.3034 - Cz}
```

We set up 3 equations for solving the coordinate of C

1. CD vertical CB
2. The C is on the plane ABCD
3. CD//AB

```
In[•]:= Solve[Dot[CD, CB] == 0 && 0.7954 Cx + 2.1192 Cy + 7.72383 Cz == 157.431 &&
          Norm[Cross[CD, AB]] == 0, {Cx, Cy, Cz}]
Out[•]= {{Cx → 50.7409, Cy → 14.0459, Cz → 11.3034}, {Cx → 34.5328, Cy → 7.08298, Cz → 14.8829}}
```

Due to the height which is **Cz must larger than Az (7.72383)**

```
In[•]:= pointC = {34.532800339947386`, 7.082980029533567`, 14.882947880911194`}
Out[•]= {34.5328, 7.08298, 14.8829}
```

Find point c

```
In[ ]:= c = {cx, cy, 0}

In[ ]:= cd = d - c
        cb = b - c
        Cc = c - pointC
        CD = pointD - pointC
        Dd = d - pointD

Out[ ]= {49.5768 - cx, 10.9446 - cy, 0}

Out[ ]= {25.7441 - cx, 19.8898 - cy, 0}

Out[ ]= {-34.5328 + cx, -7.08298 + cy, -14.8829}

Out[ ]= {16.2081, 6.96297, -3.57956}

Out[ ]= {-1.16402, -3.10133, -11.3034}
```

We set up 2 equations for solving the coordinate of c

1. Cc vertical CD
2. Dd //Cc

```
In[ ]:= Solve[Norm[Cross[Dd, Cc]] == 0 && Dot[Cc, CD] == 0, {cx, cy}]

Out[ ]= {{cx → 33.0001, cy → 2.99952}}

In[ ]:= c = {33.00014769626712`, 2.9995154578492857`, 0}

Out[ ]= {33.0001, 2.99952, 0}
```

Find side length

```
In[ ]:= Cc = c - pointC
        cd = d - c

Out[ ]= {-1.53265, -4.08346, -14.8829}

Out[ ]= {16.5767, 7.9451, 0}

In[ ]:= sidelength = Norm[AB] * 4 + Norm[Aa] + Norm[Cc] + Norm[Bb] * 2 + Norm[ab] * 2 + Norm[cd] * 2

Out[ ]= 192.644
```

Find size of plane abcd

abcd is not a square, **but ab = ad, bc = dc**

so we can draw lines **ac and bd**

Two lines separate abcd into two triangles

```
In[ ]:= ac = c - a
        bd = d - b

Out[ ]= {-9.32065, -24.8354, 0}

Out[ ]= {23.8327, -8.94518, 0}

In[ ]:= abcd = 1 / 2 * (Norm[ac] * (1 / 2 * Norm[bd])) * 2

Out[ ]= 337.635
```

Find volume

If we make a rotate of this cube and set it as cube 2, the combination of cube and cube 2 will look like a cuboid, and we just need to divide the volume of combination by 2.

In[ ]:=
```
areaOfABCD = Norm[AB] * Norm[BC] / 2
```
Out[ ]=
```
168.532
```

In[ ]:=
```
Height = Norm[Bb] * 2
```
Out[ ]=
```
23.5576
```

In[ ]:=
```
volume = areaOfABCD * Height / 2
```
Out[ ]=
```
1985.1
```

Final Result

In[ ]:=
```
A
B
C = pointC
D = pointD
a
b
c
d
Norm[Aa]
Norm[Bb]
Norm[Cc]
Norm[Dd]
sidelength
abcd
volume
```

Out[ ]= {43.1162, 29.9541, 7.72383}

Out[ ]= {26.9081, 22.9911, 11.3034}

Out[ ]= {34.5328, 7.08298, 14.8829}

Out[ ]= {50.7409, 14.0459, 11.3034}

Out[ ]= {42.3208, 27.8349, 0}

Out[ ]= {25.7441, 19.8898, 0}

Out[ ]= {33.0001, 2.99952, 0}

Out[ ]= {49.5768, 10.9446, 0}

Out[ ]= 8.04868

Out[ ]= 11.7788

Out[ ]= 15.5089

Out[ ]= 11.7788

Out[ ]= 192.644

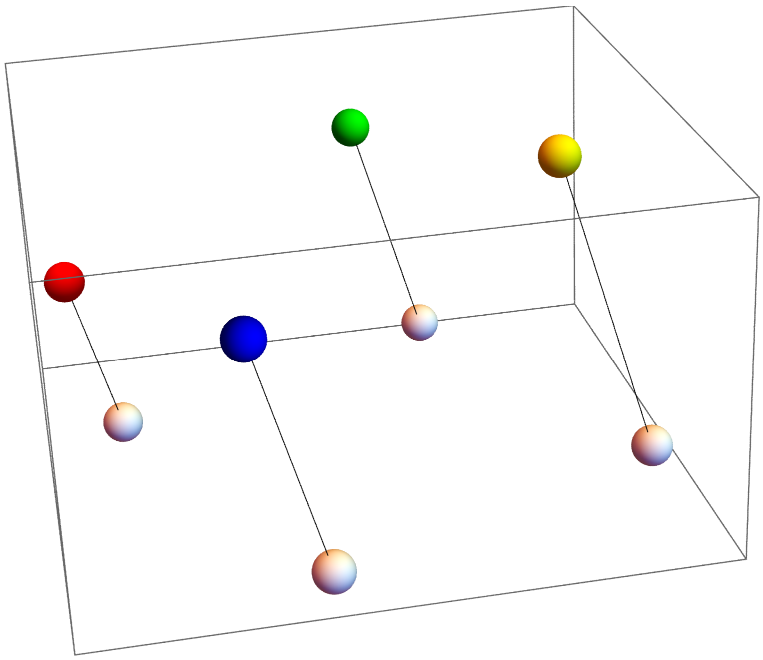
Out[ ]= 337.635

Out[ ]= 1985.1

In[ ]:=
```
r = 1
Graphics3D[{
  {Red, Sphere[A, r]},
  {Blue, Sphere[B, r]},
  {Green, Sphere[pointD, r]},
  {Yellow, Sphere[pointC, r]},
  Sphere[b, r],
  Sphere[a, r],
  Sphere[d, r],
  Sphere[c, r],
  Line[{A, a}],
  Line[{B, b}],
  Line[{pointD, d}],
  Line[{pointC, c}]
 }]
```

Out[ ]= 1

Out[ ]=



Part 3 An amazing property of unit cubes

A:

First we set up the vectors (xyw, xyv and xyu means the orthogonal projection of vector on xy-plane)

In[ ]:=
```
xyw = {w1, w2, 0}
xyv = {v1, v2, 0}
xyu = {u1, u2, 0}
w = {w1, w2, w3}
v = {v1, v2, v3}
u = {u1, u2, u3}
```

and we can find the area of orthogonal projection of the cube onto the xy-plane

```
In[ ]:= area = Norm[Cross[xyw, xyv]] + Norm[Cross[xyw, xyu]] + Norm[Cross[xyu, xyv]]
Out[ ]= Abs[-u2 v1 + u1 v2] + Abs[u2 w1 - u1 w2] + Abs[v2 w1 - v1 w2]
```

Then we start calculate with the vector SN
Due to the vector can be replace by the cross product of other two vectors
so we replace w, v and u vector with the cross product

```
In[ ]:= SN = Cross[v, u] + Cross[w, v] + Cross[w, u]
Out[ ]= {u3 v2 - u2 v3 + u3 w2 + v3 w2 - u2 w3 - v2 w3,
  -u3 v1 + u1 v3 - u3 w1 - v3 w1 + u1 w3 + v1 w3, u2 v1 - u1 v2 + u2 w1 + v2 w1 - u1 w2 - v1 w2}
```

Find the length of its orthogonal projection onto the z-axis

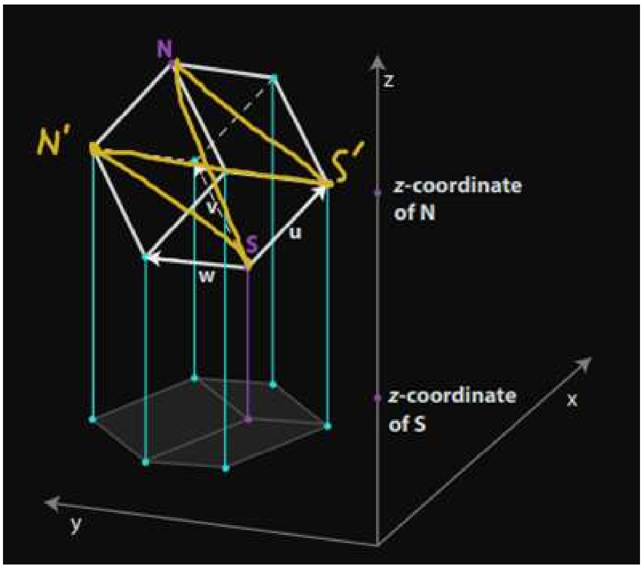
```
In[ ]:= z = {0, 0, 1}
Out[ ]= {0, 0, 1}

In[ ]:= Norm[Projection[SN, z]]
Out[ ]= Abs[u2 v1 - u1 v2 + u2 w1 + v2 w1 - u1 w2 - v1 w2]
```

Then we find Norm[Projection[SN,z]] can be in form:
=Abs[u2 v1-u1 v2]+Abs[u2 w1-u1 w2]+Abs[v2 w1-v1 w2]=area
So Norm[SN] = area

B:

Set the start of vector u is S and the end of u is S'
Set the other side of N is N'

The minimum area of projection in xy-plane can deduced by the projection of SS'NN'plane on z-coordinates



The minimum length on z-coordinates is while SS' // z, this is because after the cube roll over SS'// z, the Norm[SN] is not equal to the area on xy-plane anymore, it turn to Norm[S'N'] = area.

While the SS' // z, the shape on xy-plane is a square, it is the minimum area

In[ ]:=
```
minimum = 1 * 1
```
Out[ ]=
```
1
```

Maximum happen while SN // z, that means SN show it longest length on z-coordinates
we can use Pythagorean theorem to find the Norm[SN]

In[ ]:=
```
maximum = Square[Square[1^2 + 1^2] + 1^2]
```
Out[ ]=
```
□ (1 + □2)
```

C:

If we change the side length, the relationship will not be change