MTH1030/35: Assignment 2, 2022 Solutions

Applications of linear systems

In general, if no serious attempt at solving one of the questions was made assign 0 marks to that question.

Overall, we are marking generously :)

1 Funny numbers (25 Marks)

Solution: The funny numbers are just the standard matrix representation of the complex numbers and the funny funny numbers correspond to the quaternions :)

a) **(10 marks)**

i) Setting $a = 1$ and $b = 0$ gives the 2×2 identity matrix and so this identity matrix is a funny number.

2 marks for this one

ii) We check that the sum and product of two funny numbers are again funny numbers:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} a+c & -(b+d) \\ b+d & a+c \end{pmatrix}$$

2 marks for this one

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -(bc+ad) \\ bc+ad & ac-bd \end{pmatrix}$$

2 marks for this one

iii) To check which funny numbers are invertible we calculate the determinant:

$$\begin{vmatrix} a & -b \\ b & a \end{vmatrix} = a^2 + b^2.$$

This means that only the 2×2 zero matrix is not invertible. Or, in other words, the only funny number you cannot divide by is the zero matrix. The inverse of a funny number is

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}^{-1} = \frac{1}{a^2+b^2} \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

This means the inverse is also a funny numbers.

2 marks for this one

(Students are supposed to show that "all funny numbers except for one (which one?) have an inverse". I should have been a bit more careful in phrasing this.

The way it is written students may interpret "inverse" as "matrix inverse only" not the stronger "matrix inverse that is a funny, funny number". Anyway don't mark them down if they don't explicitly state that the matrix inverse, if it exists, is also a funny, funny number.)

iv) Check that multiplication of the funny numbers is commutative:

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -(bc+ad) \\ bc+ad & ac-bd \end{pmatrix}$$

$$= \begin{pmatrix} ca-db & -(cb+da) \\ cb+da & ca-db \end{pmatrix} = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

2 marks for this one

b) **(5 marks)**

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} X + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Y = \begin{pmatrix} w+x & v-y \\ -v+y & w+x \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} X + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} Y = \begin{pmatrix} v+x+y & -w+x-y \\ w-x+y & v+x+y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

And so,

$$w+x=2, v-y=0, v+x+y=1, w-x+y=0.$$

Solving this system of linear equations in four unknowns gives the unique solution

$$x=1, y=0, v=0, w=1.$$

And so, the solution to our funny number system of equations is

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

2 marks for the correct result, 3 marks for some reasonable working out.

c) **(5 marks)**

Choosing $a=1, b=c=d=0$ gives the funny funny counterpart of the number 1

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

2 marks for this one. Students don't have to show that this funny funny number really commutes with all funny funny numbers.

The determinant of a funny funny number is the funny number

$$\begin{pmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{pmatrix}$$

This means that all funny funny numbers except for the funny funny zero ($a=b=c=d=0$) are invertible.

1 mark for the correct statement at the end, 2 marks for giving a valid reason.

d) **(5 marks)** No, not commutative. Choose

$$I = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \end{pmatrix}, (a=c=d=0, b=1)$$

$$J = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}, (a=b=d=0, c=1)$$

$$K = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}, (a=b=c=0, d=1)$$

Then

$$JK = I \neq -I = KJ.$$

(This corresponds to the fact that the vector product is not commutative—$jk = i = -kj$).

Pretty much any choice of two funny funny numbers will result in the two numbers not commuting. Start with giving 5 marks and deduct 2 or 4 or 5 marks if something is obviously not correct.

Feedback in the form

Marks for a+Marks for b+Marks for c+Marks for d–deductions for missing explanations, overall mess, etc.

Here "deductions=" = 0 or 5.

2 What's next? (25 Marks)

Solution: a) **(5 marks)**

$$\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = M \begin{pmatrix} x_{n-1} \\ x_{n-2} \end{pmatrix} = \begin{pmatrix} ax_{n-1} + bx_{n-2} \\ cx_{n-1} + dx_{n-2} \end{pmatrix}$$

This will work if we choose $a = 5, b = -6, c = 1, d = 0$. So,

$$M = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}.$$

Pretty much binary marking, either students get the matrix right or they don't. So, either 0 or 5 marks.

b) **(10 marks)** Characteristic polynomial of M is

$$(x-2)(x-3)$$

So, its eigenvalues are 2 and 3. Eigenvectors corresponding to 2 and 3 are

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Therefore, we can choose

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, N = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$$

Pretty much binary marking, either students get the matrices right or they don't (of course the eigenvectors in T can be scaled versions for what I've got). So, 5 marks each for the two matrices. Students don't really have to show any working out here since I told them that they can use Mathematica to diagonalise. Anyway, overall 5 easy marks.

c) **(10 marks)** We know that

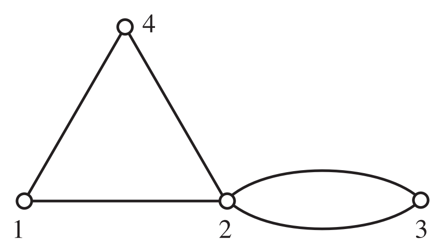
$$\begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix} = M^{n-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^{n-1} & 0 \\ 0 & 3^{n-1} \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3^n - 2^n \\ 3^{n-1} - 2^{n-1} \end{pmatrix}.$$

Therefore, $x_n = 3^n - 2^n$ is the formula that we are after.

5 marks for the correct formula, 5 marks for the working out.

Feedback in the form Marks for a + Marks for b + Marks for c - deductions for overall mess (0 or 5)

3 Matrices of Graphs (50 marks, 8 marks for each of the six questions plus two bonus points for really smart reasoning.)



Solution: a) **(8 marks)**

$tr(A) = 0$ because we required that all entries on the main diagonal are 0.

2 marks for this

The entries on the main diagonal of A^2 are the numbers of 2-step walks that start and end at a vertex. Well, the only way such a walk can be accomplished is by following an edge to a neighbouring vertex and then returning via the same edge. This implies that $tr(A^2)$ counts the total number of edges ending in the different vertices. However, since every edge ends in two vertices, every edge is counted twice in the process.

3 marks for this

An entry on the main diagonal of A^3 counts the number of 3-step walks starting and ending at the corresponding vertex. So, basically we are counting triangles. However, there are two different directions in which a triangle can be traversed plus three different starting points. This accounts for the factor 6.

3 marks for this

b) **(8 marks)** If a graph can be two-coloured, then on any circular path the colours of the vertices you come across alternate. This means that any such path has even length. This means that there are no circular paths of odd lengths. Consequently, all entries on the diagonal of any odd power of A are zero. Hence the traces of all these odd powers are zero.

This or something similar is worth the full 8 marks. Give 3 or 6 marks for reasonable attempts.

Conversely (students don't have to do this), the traces of all odd powers being zero implies that there are no circular walks of odd length. To show that such a graph can be two-coloured simply start by colouring one of the vertices black, all neighbours of this vertex white, all neighbours of these vertices that are not coloured yet black, etc. How can we be sure that this yields a two-colouring of the graph? Let's assume that this is not the case and that we end up with two neighbouring vertices with the same colour. Then there is a path from the first vertex that we colored to one of the conflicting vertices and another path to the other vertex. In addition the paths can be chosen such that the colours alternate along these paths. However, now we can join the two paths plus the edge between the conflicting vertices into a circular path which because of the coloring of the individual pieces has to be of odd length. This is impossible. Only caveat—the graph could have several connected components, in which case the colouring has to be constructed separately for every one of these components.

You can award the two bonus marks for any reasonable attempt to get this converse sorted or for particularly well presented solutions within this third problem set, formal proofs, etc.

c) **(8 marks)** An ijth entry in P_m is non-zero if there is a walk of length m or less between vertices i and j. This means that the distance between these two vertices is equal to the smallest m for which the ijth entry of P_m is not zero. **The diameter is the smallest m for which all entries of P_m are not zero.**

4 marks for this

A graph is connected if it has a diameter.

4 marks for this

You don't have to go on forever for this. Basically, if a graph has m vertices, then its diameter cannot be greater than m. This means that if P_m has zero entries that graph is disconnected, otherwise it is not.

d) **(8 marks)**

$$A^3 = \begin{pmatrix} 0 & 7 & 7 & 0 & 7 & 0 & 0 & 6 \\ 7 & 0 & 0 & 7 & 0 & 7 & 6 & 0 \\ 7 & 0 & 0 & 7 & 0 & 6 & 7 & 0 \\ 0 & 7 & 7 & 0 & 6 & 0 & 0 & 7 \\ 7 & 0 & 0 & 6 & 0 & 7 & 7 & 0 \\ 0 & 7 & 6 & 0 & 7 & 0 & 0 & 7 \\ 0 & 6 & 7 & 0 & 7 & 0 & 0 & 7 \\ 6 & 0 & 0 & 7 & 0 & 7 & 7 & 0 \end{pmatrix}$$

All entries on the diagonal of A^3 are 0. We conclude that **the graph has no triangles**.

2 marks for this

$$A + A^2 = \begin{pmatrix} 3 & 1 & 1 & 2 & 1 & 2 & 2 & 0 \\ 1 & 3 & 2 & 1 & 2 & 1 & 0 & 2 \\ 1 & 2 & 3 & 1 & 2 & 0 & 1 & 2 \\ 2 & 1 & 1 & 3 & 0 & 2 & 2 & 1 \\ 1 & 2 & 2 & 0 & 3 & 1 & 1 & 2 \\ 2 & 1 & 0 & 2 & 1 & 3 & 2 & 1 \\ 2 & 0 & 1 & 2 & 1 & 2 & 3 & 1 \\ 0 & 2 & 2 & 1 & 2 & 1 & 1 & 3 \end{pmatrix}$$

has 0 entries whereas

$$A + A^2 + A^3 = \begin{pmatrix} 3 & 8 & 8 & 2 & 8 & 2 & 2 & 6 \\ 8 & 3 & 2 & 8 & 2 & 8 & 6 & 2 \\ 8 & 2 & 3 & 8 & 2 & 6 & 8 & 2 \\ 2 & 8 & 8 & 3 & 6 & 2 & 2 & 8 \\ 8 & 2 & 2 & 6 & 3 & 8 & 8 & 2 \\ 2 & 8 & 6 & 2 & 8 & 3 & 2 & 8 \\ 2 & 6 & 8 & 2 & 8 & 2 & 3 & 8 \\ 6 & 2 & 2 & 8 & 2 & 8 & 8 & 3 \end{pmatrix}$$

doesn't. Hence **our graph is connected and has diameter 3**.

3 marks for this: 1 mark for diameter 3 and 2 marks for valid reason including the matrices.

$$A^3 = \begin{pmatrix} 0 & 7 & 7 & 0 & 7 & 0 & 0 & 6 \\ 7 & 0 & 0 & 7 & 0 & 7 & 6 & 0 \\ 7 & 0 & 0 & 7 & 0 & 6 & 7 & 0 \\ 0 & 7 & 7 & 0 & 6 & 0 & 0 & 7 \\ 7 & 0 & 0 & 6 & 0 & 7 & 7 & 0 \\ 0 & 7 & 6 & 0 & 7 & 0 & 0 & 7 \\ 0 & 6 & 7 & 0 & 7 & 0 & 0 & 7 \\ 6 & 0 & 0 & 7 & 0 & 7 & 7 & 0 \end{pmatrix},$$

$$A^5 = \begin{pmatrix} 0 & 61 & 61 & 0 & 61 & 0 & 0 & 60 \\ 61 & 0 & 0 & 61 & 0 & 61 & 60 & 0 \\ 61 & 0 & 0 & 61 & 0 & 60 & 61 & 0 \\ 0 & 61 & 61 & 0 & 60 & 0 & 0 & 61 \\ 61 & 0 & 0 & 60 & 0 & 61 & 61 & 0 \\ 0 & 61 & 60 & 0 & 61 & 0 & 0 & 61 \\ 0 & 60 & 61 & 0 & 61 & 0 & 0 & 61 \\ 60 & 0 & 0 & 61 & 0 & 61 & 61 & 0 \end{pmatrix},$$

$$A^7 = \begin{pmatrix} 0 & 547 & 547 & 0 & 547 & 0 & 0 & 546 \\ 547 & 0 & 0 & 547 & 0 & 547 & 546 & 0 \\ 547 & 0 & 0 & 547 & 0 & 546 & 547 & 0 \\ 0 & 547 & 547 & 0 & 546 & 0 & 0 & 547 \\ 547 & 0 & 0 & 546 & 0 & 547 & 547 & 0 \\ 0 & 547 & 546 & 0 & 547 & 0 & 0 & 547 \\ 0 & 546 & 547 & 0 & 547 & 0 & 0 & 547 \\ 546 & 0 & 0 & 547 & 0 & 547 & 547 & 0 \end{pmatrix}$$

etc. have only zero entries on the main diagonal, **hence the graph is two-colourable**. We don't really have to look at higher powers because the diameter of the graph is 3.

3 marks for this: 1 marks for saying "is 2-colorable" and 2 marks for giving a valid reason.

B.t.w., the mystery graph is the edge graph of a cube and the two colours correspond to the two tetrahedra inscribed in the cube.

e) **(8 marks)** If k is the common sum, then multiplying the matrix by the vector $\mathbf{u} = (1, 1, 1, \ldots 1)^T$ gives $k\mathbf{u}$. Hence k is an eigenvector of the matrix with eigenvector $\mathbf{u}$.

This one is pretty much a binary one. So 0, 6 or 8 marks

f) **(8 marks)** The same argument that is used to derive the formula for the number a also works for the matrix A. Of course, we cannot divide but have to use the inverse. The formula then pans out to be

$$(A - A^{m+1})(I - A)^{-1},$$

where I is the identity matrix of the same dimension as A. Now, the formula for the number a only works is $a \neq 1$, i.e. if you can divide by $1 - a$. Similarly, the formula for A only works if the matrix $I - A$ is invertible. This is not the case for our 8×8 matrix because the determinant of $I - A$ is 0.

4 marks for stating that the formula is basically the same, 4 marks for reasoning that this does not work for the matrix at hand.

Feedback in the form

Marks for a+ Marks for b+...+Marks for f-deductions for overall mess (0, 5, 10)+ up to 2 bonus marks