MTH1030
Techniques for Modelling

Lecture 27

Taylor series and Power series (part 2)

Monash University

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Warm welcoming words

We now know a lot about Taylor series. A Taylor series is a power series representation of a function (if it's nice). We also know a power series may not converge for all x values...but we never actually figured out how to determine this set of x values! So let's step back and figure this out.

Power series

Recall a power series is the series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

We noted that

- A power series is a function.
- For every chosen x value, we get a different series.
- The power series may not converge for all $x \in \mathbb{R}$.
- It will definitely converge for $x = 0$, in which case $f(0) = a_0$.

Power series

So, when does a power series converge? That is, what is the set of x values such that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges? Well, let's apply the ratio test!

Power series

Theorem

Consider the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$. *Define*

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

The power series converges absolutely if $x \in (-R, R)$. *It diverges for* $x \in (-\infty, R)$ *or* $x \in (R, \infty)$. *It may converge absolutely, conditionally converge or diverge for* $x = R$ *or* $x = -R$.

Remark

The number R is the called the *radius of convergence* of the power series. Be very careful, R is defined as sort of the opposite of what it looks like in the ratio test.

Power series

A couple of notes:

- The *interval of convergence* of a power series is the set of all x such that the power series converges. It can be any one of the following:

$$(-R, R), [-R, R), (-R, R], [-R, R].$$

- R is determined via the ratio test. As a consequence of the ratio test, we must use other methods to determine if convergence occurs at $x = -R$ or $x = R$.
- For a power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we always have $f(0) = a_0$. Hence the worst case scenario is that convergence occurs only at $x = 0$. Hence, the radius of convergence is always $R \geq 0$ (and can be ∞).

Power series

Example

We found that the Maclaurin series for e^x was

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \cdots .$$

So this is a power series with $a_n = 1/n!$. To find its interval of convergence, we should first compute its radius of convergence R.

Power series

Example

We found that the Maclaurin series for $\ln(1+x)$ was

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots$$

So this is a power series with $a_n = (-1)^{n+1}/n$. To find its interval of convergence, we should first compute its radius of convergence R.

Question 1

Question (1)

Consider the power series

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{4^n} = 1 - \frac{x}{4} + \frac{x^2}{4^2} \cdots$$

The interval of convergence of it is

1. $(-4, 4)$.
2. $[-4, 4)$.
3. $(-4, 4]$.
4. $[-4, 4]$.

Power series

A power series can be differentiated term by term or integrated term by term.

Theorem (Termwise differentiation and integration)

Consider the power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

with radius of convergence $R \neq 0$. Then f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

And also for $x \in (-R, R)$

$$\int_0^x f(t)\mathrm{d}t = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Power series

Seems obvious right...? Well it's not! Series are not actually sums, so properties of sums will not always apply. Luckily, termwise differentiation and integration of power series does hold.

Power series

Example

Consider the inverse tangent function $\arctan(x)$ (or $\tan^{-1}(x)$). It is known (via implicit differentiation) that

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = \frac{1}{1+x^2}.$$

We can use the previous theorem to find a power series representation of $\arctan(x)$ real easy.

Power series

This is a cool exercise!

Exercise

Let $f(x)$ and $g(x)$ be two power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

$$g(x) = \sum_{n=0}^{\infty} b_n x^n.$$

Show that $h(x) = f(x)g(x)$ is also a power series

$$h(x) = \sum_{n=0}^{\infty} c_n x^n,$$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

General power series

We didn't really talk about this but the function

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

for some $c \in \mathbb{R}$ is also called a power series. The mathematical theory of these more general power series is basically the same as the case of $c = 0$. E.g., the radius of converge R is still defined as

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|.$$

But instead

- Absolute convergence of $f(x)$ will occur for $x \in (-R + a, a + R)$.
- Divergence will occur for $x \in (-\infty, -R + a)$ and $x \in (a + R, \infty)$.
- We don't know what will happen when $x = -R + a$ or $x = R + a$ (well, we will need to manually investigate these cases).