Lines and planes

1. Consider the points $(1, 2, -1)$ and $(2, 0, 3)$.

   (a) Find a vector equation of the line through these points in parametric form.

   (b) Find the distance between this line and the point $(1, 0, 1)$.

   (c) Find the point on the line closest to the point $(1, 0, 1)$.

2. Find an equation of the plane that passes through the points $(1, 2, -1)$, $(2, 0, -1)$ and $(-1, -1, 0)$.

3. Consider a plane defined by the equation $3x + 4y - z = 2$ and a line defined by the following vector equation (in parametric form) $(2 - 2t, -1 + 3t, -t)$.

   (a) Find the point where the line intersects the plane.

   (b) Find a normal vector to the plane.

   (c) Find the angle at which the line intersects the plane.

4. Find the distance between the parallel planes defined by the equations $2x-y+3z = -4$ and $2x - y + 3z = 24$.

5. Consider two planes defined by the equations $3x + 4y - z = 2$ and $-2x + y + 2z = 6$.

   (a) Find where the planes intersect the x, y and z axes.

   (b) Find normal vectors for the planes.

   (c) Find an equation of the line defined by the intersection of these planes.

   (d) Find the angle between these two planes.

6. Here are the equations of two lines $(1 + t, 1 - 3t, 2 + 2t)$ and $(3s, 1 - 2s, 2 - s)$.

   (a) Find the distance d between the two lines.

   (b) Find the uniquely determined points on the two lines that are this distance d apart.

7. Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three points/vectors in $\mathbf{R}^3$. Prove that the distance between the line through $\mathbf{u}$ and $\mathbf{v}$ to the point $\mathbf{w}$ is

$$\frac{|(\mathbf{u}-\mathbf{w}) \times (\mathbf{v}-\mathbf{w})|}{|\mathbf{v}-\mathbf{u}|}.$$

(Hint: What does the numerator and denominator of this fraction mean geometrically?) Double check your answer in 1b) using this formula.

8. Calculate the distance in 1b) one more time using calculus. So, if $\mathbf{r}(t)$ is the equation of the line and $\mathbf{v}$ is the point in question, calculate the minimum of the function

$$|\mathbf{r}(t) - \mathbf{v}|^2$$

in the variable t using the usual calculus tricks.[2]

SOME TEST QUESTIONS

9. What does it mean for a coordinate system to be right-handed?

10. Explain *based only* on the geometric definition of the vector product why for any number a and vectors $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$$

$$(a\mathbf{u}) \times \mathbf{v} = a \cdot (\mathbf{u} \times \mathbf{v})$$

$$\mathbf{u} \times \mathbf{u} = \mathbf{0}$$

11. Given three points, is it possible that there is more than one plane that contains all three of them?

12. Given four random points in space what are the chances that all four are contained in: (a) a line, (b) a plane?

13. Two solid cubes are hovering in space. How is the distance between them defined?

14. A line intersects a plane. How is the angle between them defined?

15. Two planes intersect. What is the angle between them?

16. Explain how you would find the distance in $\mathbf{R}^3$ between: (a) two lines, (b) a point and a line, (c) two parallel planes, (d) a point and a plane? (Without using a ready made formula!)

17. Using the formula $\cos(\theta) = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$ show that the scalar projection of the vector $\mathbf{v}$ onto the vector $\mathbf{u}$ is $\frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|}$. What is its vector projection?

[2] You are really interested in minimizing $|\mathbf{r}(t) - \mathbf{v}|$ but the square of this expression takes on its minimum at the same t and is easier to manipulate.

18. How can you use the direction vectors of lines and the normal vectors of planes to quickly determine whether: (a) two planes are parallel; (b) a line intersects a plane; (c) a line is parallel to a plane?

19. Looking at the equation of a plane how do you tell at a glance: (a) whether the plane is parallel to one of the coordinate planes; (b) passes through the origin $(0, 0, 0)$?

20. Does it make sense to speak of *the* vector equation of a line or of *the* equation of a plane?