MTH1030 A3

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1. Sequences

1.1

The infinite expression:

```
a0 = 2^(1/2); a2 = 2^(3/4); a3 = 2^(7/8);
```

Then we found the power number always changing, we can present the power number with n (n>=1)

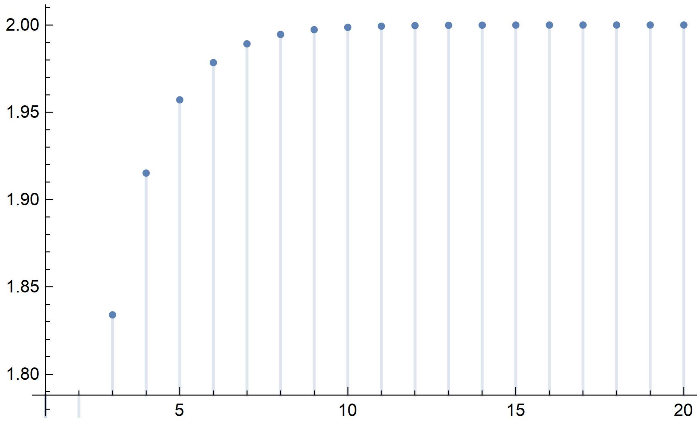
In[ ]:= $a_n = 2^{\frac{(2^n-1)}{2^n}}$

Out[ ]= $2^{2^{-n}(-1+2^n)}$

In[ ]:= Simplify$\left[2^{2^{-n}(-1+2^n)}\right]$

Out[ ]= $2^{1-2^{-n}}$

In[ ]:= DiscretePlot$\left[2^{1-2^{-n}}, \{n, 1, 20\}\right]$

Out[ ]= 

While n goes infinity, $2^{1-2^{-n}}$ close to a number, so the sequence has limit

Then we calculate the limit

In[ ]:= Limit$\left[2^{1-2^{-n}}, n \to \infty\right]$

Out[ ]= 2

1.2

i)

f($\texttt{Limit}[\frac{1}{n+1}, n \to \infty]$) = **1** has exactly 1 solution

Sequence converges

In[ ]:= $\texttt{Limit}\left[\frac{1}{n+1}, n \to \infty\right]$

Out[ ]= 0

ii)

f($\texttt{Limit}[\frac{1}{n}, n \to \infty]$) = 1/0, which is invalid

Sequence converges

In[ ]:= $\texttt{Limit}\left[\frac{1}{n}, n \to \infty\right]$

Out[ ]= 0

iii)

f(Limit[Sin[n],n→∞]) no solution because limit does not have solution

In[ ]:= `Limit[Sin[n], n → ∞]`

Out[ ]= `Indeterminate`

iv)

f($\texttt{Limit}\left[2^{\frac{(2^n-1)}{2^n}}, n \to \infty\right]$) has solution $2^{3/4}$ but sequence diverge

In[ ]:= $\texttt{Limit}\left[2^{\frac{(2^n-1)}{2^n}}, n \to \infty\right]$

Out[ ]= 2

In[ ]:= $\sum_{n=1}^{\infty} 2^{\frac{(2^n-1)}{2^n}}$

··· Sum: Sum does not converge.

Out[ ]= $\sum_{n=1}^{\infty} 2^{2^{-n}\left(-1+2^n\right)}$

In[ ]:= $2^{\frac{(2^2-1)}{2^2}}$

Out[ ]= $2^{3/4}$

2. Serious series

2.1

A)

$$f[\_n] = \frac{\text{Sin}[n]}{n}$$

We notice a_n>0

$$\text{Table}\left[\frac{\text{Sin}[n]}{n}, \{n, 1, 10\}\right]$$

Out[ ]= $\left\{\text{Sin}[1], \frac{\text{Sin}[2]}{2}, \frac{\text{Sin}[3]}{3}, \frac{\text{Sin}[4]}{4}, \frac{\text{Sin}[5]}{5}, \frac{\text{Sin}[6]}{6}, \frac{\text{Sin}[7]}{7}, \frac{\text{Sin}[8]}{8}, \frac{\text{Sin}[9]}{9}, \frac{\text{Sin}[10]}{10}, \frac{\text{Sin}[11]}{11}, \frac{\text{Sin}[12]}{12}, \frac{\text{Sin}[13]}{13}, \frac{\text{Sin}[14]}{14}, \frac{\text{Sin}[15]}{15}, \frac{\text{Sin}[16]}{16}, \frac{\text{Sin}[17]}{17}, \frac{\text{Sin}[18]}{18}, \frac{\text{Sin}[19]}{19}, \frac{\text{Sin}[20]}{20}\right\}$

We test (2) the output should be false

In[ ]:= $\frac{\text{Sin}[19]}{19} > \frac{\text{Sin}[20]}{20}$

Out[ ]= False

We test (3) should be 0

In[ ]:= $\text{Limit}\left[\frac{\text{Sin}[n]}{n}, n \to \infty\right]$

Out[ ]= 0

B)

$$f[\_n] = 1 + \frac{1}{n}$$

We notice a_n>0

In[ ]:= $\text{Table}\left[1 + \frac{1}{n}, \{n, 1, 20\}\right]$

Out[ ]= $\left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \frac{7}{6}, \frac{8}{7}, \frac{9}{8}, \frac{10}{9}, \frac{11}{10}, \frac{12}{11}, \frac{13}{12}, \frac{14}{13}, \frac{15}{14}, \frac{16}{15}, \frac{17}{16}, \frac{18}{17}, \frac{19}{18}, \frac{20}{19}, \frac{21}{20}\right\}$

We test (2) the output should be True

In[ ]:= $\frac{20}{19} > \frac{21}{20}$

Out[ ]= True

We test (3) should not be 0

In[•]:= $\text{Limit}\left[1+\frac{1}{n}, n\to\infty\right]$

Out[•]= 1

2.2

A)

In this case, I would like to use a converge function and it converge in Abs[]

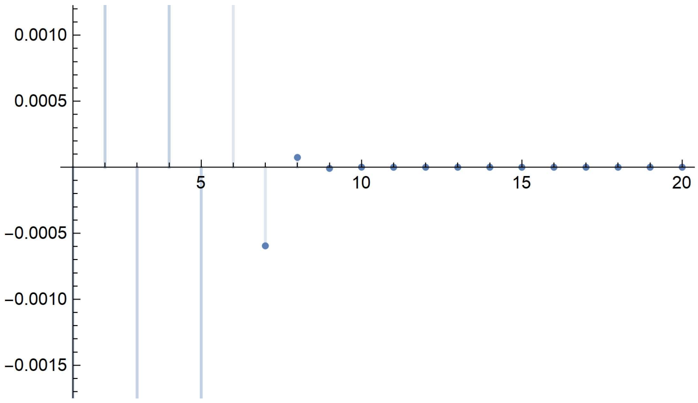
In[•]:= `3 (-1) ^n / n !`

Out[•]= $\frac{3(-1)^n}{n!}$

Now we make a graph to show it is converge or not

In[•]:= $\text{DiscretePlot}\left[\frac{3(-1)^n}{n!}, \{n, 1, 20\}\right]$

Out[•]=



After we confirm it is converge we calculate the sum

In[•]:= $\sum_{n=1}^{\infty}\frac{3(-1)^n}{n!}$

Out[•]= $\frac{3(1-e)}{e}$

In[•]:= $N\left[\frac{3(1-e)}{e}\right]$

Out[•]= -1.89636

Then we start calculate the Abs[sum]

In[•]:= $\sum_{n=1}^{\infty}\text{Abs}\left[\frac{3(-1)^n}{n!}\right]$

Out[•]= $3(-1+e)$

In[•]:= `N[3 (-1 + e)]`

Out[•]= 5.15485

In this case, with the absolute, the property of series does not change, that means the absolute value of a_n is continue get closer to a number, however their sum is not equal.

B)

We start making a series base on $\frac{1}{n}$ which is not converge.
In order to make the series bouncing in positive and negative range
we put $(-1)^{n-1}$ on the top of the $\frac{1}{n}$:

In[ ]:= $\frac{(-1)^{n-1}}{n}$

Then we calculate does it has sum or not

In[ ]:= $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

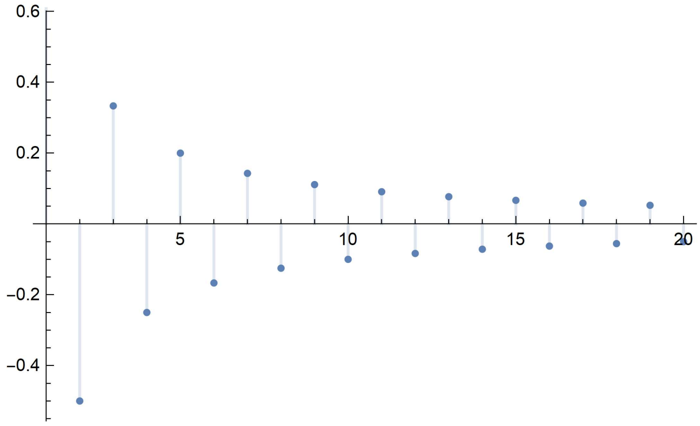
Out[ ]= Log[2]

In[ ]:= N[Log[2]]

Out[ ]= 0.693147

Surprisedly, it has sum, and we make a plot of it

In[ ]:= DiscretePlot[$\frac{(-1)^{n-1}}{n}$, {n, 1, 20}]

Out[ ]=


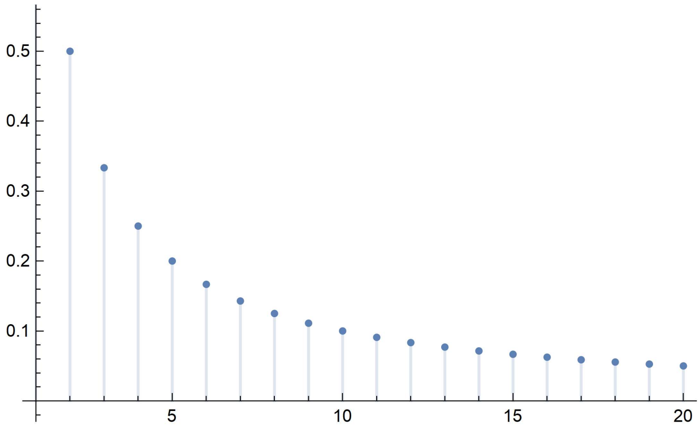
Here is the absolute version of example

In[ ]:= Abs[$\frac{(-1)^{n-1}}{n}$]

Out[ ]= $\frac{e^{-\pi \text{Im}[n]}}{\text{Abs}[n]}$

In[ ]:= $\texttt{DiscretePlot}\left[\dfrac{e^{-\pi\,\texttt{Im}[n]}}{\texttt{Abs}[n]},\ \{n, 1, 20\}\right]$

Out[ ]=



We can let Mathematica calculate the sum

In[ ]:= $$\sum_{n=1}^{\infty} \frac{e^{-\pi\,\texttt{Im}[n]}}{\texttt{Abs}[n]}$$

Sum: Sum does not converge.

Out[ ]= $$\sum_{n=1}^{\infty} \frac{e^{-\pi\,\texttt{Im}[n]}}{\texttt{Abs}[n]}$$

We notice on the top of simplify $\texttt{Abs}\left[\frac{(-1)^{n-1}}{n}\right]$, it contain Euler's identity

In[ ]:= $\texttt{Abs}\left[(-1)^{n}\right]$

Out[ ]= $e^{-\pi\,\texttt{Im}[n]}$

We can conclude that if the absolute of series contain the product of Euler's identity, like $\texttt{Abs}\left[(-1)^{n}\right]$, and the series is convergent, then the absolute of series is divergent

2.3

Here is the example two diverge series can be add up to converge

In the example, we choose two bouncing series but an+bn can be 0, then the whole series converge to 0

In[ ]:= `Limit[Sin[n], n → ∞]`

Out[ ]= `Indeterminate`

In[ ]:= `Limit[Sin[n - Pi], n → ∞]`

Out[ ]= `Indeterminate`

In[ ]:= `Limit[Sin[n] + Sin[n - Pi], n → ∞]`

Out[ ]= `0`

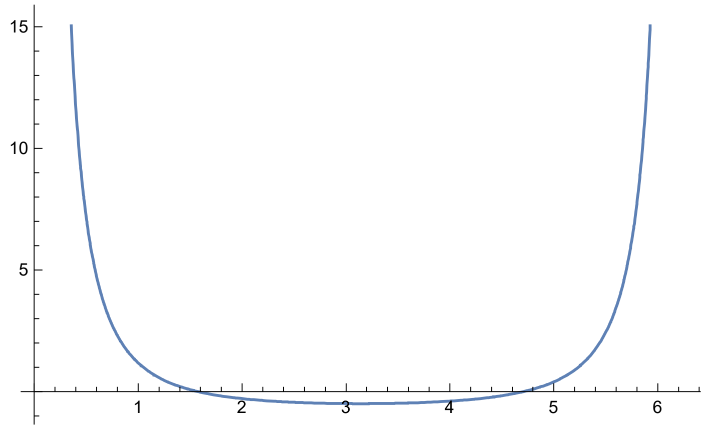
2.4

Let's make a plot of Cos[x]^n

In[ ]:= $\sum_{n=1}^{\infty}$ Cos[x]^n

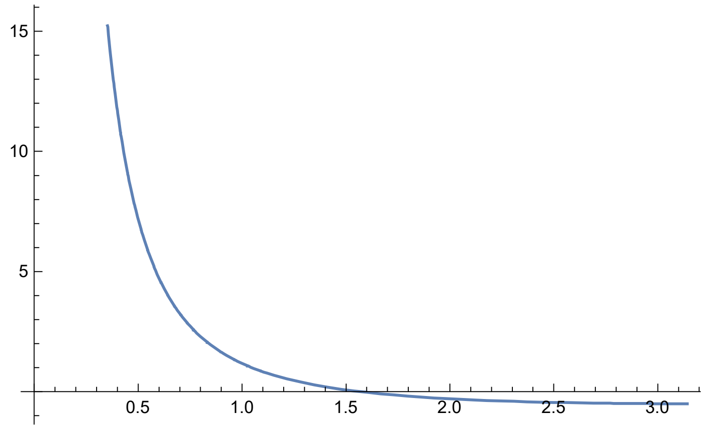
Out[ ]= $-\frac{\text{Cos}[x]}{-1+\text{Cos}[x]}$

In[ ]:= Plot[$-\frac{\text{Cos}[x]}{-1+\text{Cos}[x]}$, {x, 0, 2 Pi}]

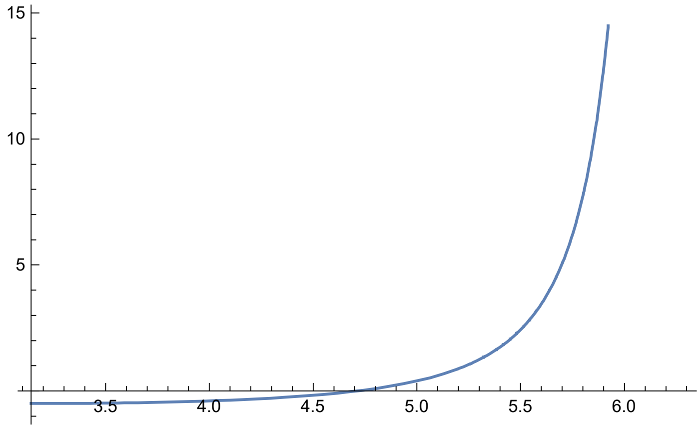
Out[ ]=


It seems like in {x,0,π}, Cos[x]^n converge, and in {x,π,2π}, Cos[x]^n diverge

In[ ]:= Plot[$-\frac{\text{Cos}[x]}{-1+\text{Cos}[x]}$, {x, 0, Pi}]

Out[ ]=


In[ ]:=
```
Plot[-Cos[x]/(-1 + Cos[x]), {x, Pi, 2 Pi}]
```

Out[ ]=


Let just calculate it to prove assumption is true or not

In[ ]:= $D\left[-\frac{\text{Cos}[x]}{-1+\text{Cos}[x]}, x\right]$

Out[ ]= $\frac{\text{Sin}[x]}{-1+\text{Cos}[x]} - \frac{\text{Cos}[x]\,\text{Sin}[x]}{(-1+\text{Cos}[x])^2}$

In[ ]:= $\frac{\text{Sin}[x]}{-1+\text{Cos}[x]} - \frac{\text{Cos}[x]\,\text{Sin}[x]}{(-1+\text{Cos}[x])^2} == 0$

Out[ ]= $\frac{\text{Sin}[x]}{-1+\text{Cos}[x]} - \frac{\text{Cos}[x]\,\text{Sin}[x]}{(-1+\text{Cos}[x])^2} == 0$

In[ ]:= $\text{Solve}\left[\frac{\text{Sin}[x]}{-1+\text{Cos}[x]} - \frac{\text{Cos}[x]\,\text{Sin}[x]}{(-1+\text{Cos}[x])^2} == 0, \{x\}, \mathbb{R}\right]$

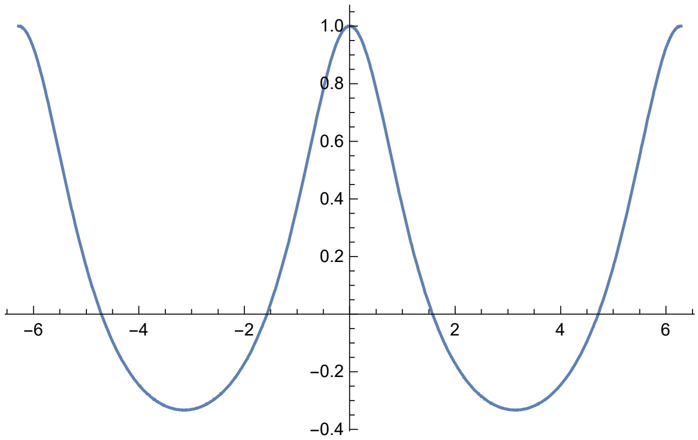
Out[ ]= $\left\{\left\{x \to \boxed{2\left(-\frac{\pi}{2} + 2\pi c_1\right) \text{ if } c_1 \in \mathbb{Z}}\right\}, \left\{x \to \boxed{2\left(\frac{\pi}{2} + 2\pi c_1\right) \text{ if } c_1 \in \mathbb{Z}}\right\}\right\}$

Therefore, π is the boundary of Cos[x]^n converge and diverge,
In range [(2x)π, (2x+1)π],
Let try to figure out does it also work in $\sum_{n=1}^{\infty}$ Cos[x]^n / 2^n

In[ ]:= $\sum_{n=1}^{\infty} \text{Cos}[x]\text{^}n / 2\text{^}n$

Out[ ]= $-\frac{\text{Cos}[x]}{-2+\text{Cos}[x]}$

In[∘]:= Plot$\left[-\frac{\text{Cos}[x]}{-2+\text{Cos}[x]}, \{x, -2\,\text{Pi}, 2\,\text{Pi}\}\right]$

Out[∘]=


In[∘]:= D$\left[-\frac{\text{Cos}[x]}{-2+\text{Cos}[x]}, x\right]$

In[∘]:= $\frac{\text{Sin}[x]}{-2+\text{Cos}[x]} - \frac{\text{Cos}[x]\,\text{Sin}[x]}{(-2+\text{Cos}[x])^2} == 0$

Out[∘]= $\frac{\text{Sin}[x]}{-2+\text{Cos}[x]} - \frac{\text{Cos}[x]\,\text{Sin}[x]}{(-2+\text{Cos}[x])^2} == 0$

In[∘]:= Solve$\left[\frac{\text{Sin}[x]}{-2+\text{Cos}[x]} - \frac{\text{Cos}[x]\,\text{Sin}[x]}{(-2+\text{Cos}[x])^2} == 0, \{x\}, \mathbb{R}\right]$

Out[∘]= $\left\{\left\{x \to \boxed{2\pi c_1 \text{ if } c_1 \in \mathbb{Z}}\right\}, \left\{x \to \boxed{\pi + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}}\right\}\right\}$

It seems like the $\sum_{n=1}^{\infty}$ Cos[x]^n / 2^n is move away and become converge in range [0,π]

$$\sum_{n=1}^{\infty} \text{Cos}[x]\text{^}n / 2\text{^}n$$

In[∘]:= Sum[Cos[x]^n / 2^n, {n, 1, ∞}, {x, 2 π , 3 π}]

Out[∘]= $-\frac{2\,(4 - \text{Cos}[1]\,\text{Cos}[2] - \text{Cos}[1]\,\text{Cos}[3] - \text{Cos}[2]\,\text{Cos}[3] + \text{Cos}[1]\,\text{Cos}[2]\,\text{Cos}[3])}{(-2+\text{Cos}[1])\,(-2+\text{Cos}[2])\,(-2+\text{Cos}[3])}$

In[∘]:= N$\left[-\frac{2\,(4 - \text{Cos}[1]\,\text{Cos}[2] - \text{Cos}[1]\,\text{Cos}[3] - \text{Cos}[2]\,\text{Cos}[3] + \text{Cos}[1]\,\text{Cos}[2]\,\text{Cos}[3])}{(-2+\text{Cos}[1])\,(-2+\text{Cos}[2])\,(-2+\text{Cos}[3])}\right]$

Out[∘]= 0.866809

2.5

The best example is the point of the corner never be removed, because we cannot remove the point outside of the square

We set up the S of whole square is 1

S1 = (1/3)^2

S2 = (1/3)^2+8*(1/9)^2

S3 = (1/3)^2+8*(1/9)^2+8^2*(1/27)^2

Sn = 8^(n-1)/9^n

Then we can calculate the sum of this series

In[ ]:= $\sum_{n=1}^{\infty} 8^{\wedge}(n-1)/9^{\wedge}n$

Out[ ]= 1

1-1=0, therefore while the removing point is close to infinity, the whole block will be remove, even there are still points cannot be removed

2.6

i) First four partial sums

In[ ]:= $\sum_{n=1}^{1} n/(n+1)!$

Out[ ]= $\frac{1}{2}$

In[ ]:= $\sum_{n=1}^{2} n/(n+1)!$

Out[ ]= $\frac{5}{6}$

In[ ]:= $\sum_{n=1}^{3} n/(n+1)!$

Out[ ]= $\frac{23}{24}$

In[ ]:= $\sum_{n=1}^{4} n/(n+1)!$

Out[ ]= $\frac{119}{120}$

ii) Show that our series is convergent and evaluate its sum

In[ ]:= $\sum_{n=1}^{\infty} n/(n+1)!$

Out[ ]= 1

In this case, we know the whole sum is getting to 1,

Now we start prove it

We calculate the S[3]

In[•]:= $S[3] = \sum_{n=1}^{3} n/(n+1)!$

Out[•]= $\frac{23}{24}$

Then we calculate $\int_{3}^{\infty} \frac{n}{(1+n)!} \, dn$

In[•]:=
```
NIntegrate[n / (n + 1) !, {n, 3, ∞}]
```

Out[•]= 0.0920188

Then we calculate $\int_{3+1}^{\infty} \frac{n}{(1+n)!} \, dn$

In[•]:=
```
NIntegrate[n / (n + 1) !, {n, 4, ∞}]
```

Out[•]= 0.0210469

Now we use:

$S[3] + \int_{3+1}^{\infty} \frac{n}{(1+n)!} \, dn < S < S[3] + \int_{3}^{\infty} \frac{n}{(1+n)!} \, dn$

For $S[3] + \int_{3}^{\infty} \frac{n}{(1+n)!} \, dn$

In[•]:= $\frac{23}{24} + 0.0920187627024872`$

Out[•]= 1.05035

For $S[3] + \int_{3+1}^{\infty} \frac{n}{(1+n)!} \, dn$

In[•]:= $\frac{23}{24} + 0.021046908746266267`$

Out[•]= 0.97938

Therefore:

$0.97938 < S < 1.05035$

Which means the S(sum) ≈ 1

3. A cat and dog game

A)

In[•]:= $cat[x] = \sum_{n=0}^{\infty} C_n x^n$

Out[•]= $\sum_{n=0}^{\infty} x^n C_n$

In[•]:=
```
cat'[x_]
```

Out[•]= $\sum_{n=0}^{\infty} n\, x\_^{-1+n} C_n$

Due to the first term n!=0, the whole summation start in n=1

$$\sum_{n=1}^{\infty} n\, x\_^{-1+n}\, C_n;$$

And we can make it turn to start with n=0

$$\sum_{n=0}^{\infty} (n+1)\; x\_^{n}\, C_{n+1};$$

Now we calculate cat''(x)

In[∘]:= $D\left[\sum_{n=0}^{\infty} (n+1)\; x\_^{n}\, C_{n+1},\; x\_\right]$

Out[∘]= $\sum_{n=0}^{\infty} n\; (1+n)\; x\_^{-1+n}\, C_{1+n}$

In the same step we make it start with 0, then we get cat''(x)

$$\sum_{n=0}^{\infty} (n+1)\; (2+n)\; x\_^{n}\, C_{2+n};$$

Due to cat''(x) = cat(x), we can know

$(n+1)\; (2+n)\; C_{2+n} \;=\; C_n;$

Due to cat'(x) = dog(x), we can know

$d_n = \;(n+1)\; C_{n+1};$

$d_1 \;=\; 1 * C_1 == 1;$

Then we know C1 = 1
As we already know cat(0) = 0, which means we know C_0 = 0
We can try to figure out could we put C_0 and C_1 inside of $C_n \;=\; (n+1)\; (2+n)\; C_{2+n};$
We can try to find C2

$C_n \;=\; (n+1)\; (2+n)\; C_{2+n};$
$C_n \,/\, (n+1)\; (2+n) \;=\; C_{2+n};$
$C_{2+n} \;=\; C_n \,/\, (n+1)\; (2+n);$
$C_2 \;=\; C_0 \;/\, 2 == 0;$

The number is 0, So right here I guess there maybe a relationship between the function and odd-/even
Let try let n = 2m (which mean let n is even), and we create a $C_{-2\,m+n}$ with C_0 and get the result

$C_n \,/\, (n+1)\; (2+n) \;=\; C_{2+n};$
$C_n \;=\; C_{n-2} \,/\, (n-1)\; (n);$

Then we find the fourth term step by step:

The third term

In[∘]:= $D\left[\sum_{n=0}^{\infty} (n+1)\; (2+n)\; x\_^{n}\, C_{2+n},\; x\_\right]$

Out[∘]= $\sum_{n=0}^{\infty} n\; (1+n)\; (2+n)\; x\_^{-1+n}\, C_{2+n}$

$$\sum_{n=0}^{\infty} (1+n)\ (2+n)\ (3+n)\ x\_^{n}\ C_{3+n}$$

The fourth term:

In[ ]:= $D\left[\sum_{n=0}^{\infty} (1+n)\ (2+n)\ (3+n)\ x\_^{n}\ C_{3+n},\ x\_\right]$

Out[ ]= $\sum_{n=0}^{\infty} n\ (1+n)\ (2+n)\ (3+n)\ x\_^{-1+n}\ C_{3+n}$

$$\sum_{n=0}^{\infty} (1+n)\ (2+n)\ (3+n)\ (4+n)\ x\_^{n}\ C_{4+n}$$

Then we find the order of even term:

$C_n = C_{n-4} / n\ (n-1)\ (n-2)\ (n-3);$

...

$C_n = C_{n-2m} / n\ (n-1)\ (n-2)\ ...\ (n-2m+1)$

$= C_0 / n!$

$= 0;$

While n is even, Cn = 0.
Now with n = 2m+1 we know it going to be C1 on the top and n on the bottom and the process is going to be the same so I directly show it:

$C_n = C_1 / n!$

We already know C1 = 1:

$C_n = 1 / n!$

After we got cat, we can find dog base on $d_n = (n+1)\ C_{n+1}$
While n = 2m

$d_{2m} = (2m+1)\ C_{2m+1} = (2m+1) * 1 / (2m+1)! = 1 / (2m)!$

Therefore we know while n is even

$d_n = 1 / n!$

In the same way, we can know while n is odd

$d_n = 0$

In conclusion:
$C_n = 0$ (n is even)
$C_n = 1 / n!$ (n is odd)
$d_n = 0$ (n is odd)
$d_n = 1 / n!$ (n is even)

B)

Find e^x and e^{-x}

In[ ]:=
```
Series[Exp[x], {x, 0, 5}]
```

Out[ ]= $1+x+\frac{x^2}{2}+\frac{x^3}{6}+\frac{x^4}{24}+\frac{x^5}{120}+O[x]^6$

In[ ]:=
```
Series[Exp[-x], {x, 0, 5}]
```

Out[ ]= $1-x+\frac{x^2}{2}-\frac{x^3}{6}+\frac{x^4}{24}-\frac{x^5}{120}+O[x]^6$

And we get sequence of cat and dog base on a)

cat(x) = $(\frac{x^1}{1!}+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots)$

dog(x)= $(1+\frac{x^2}{2}+\frac{x^4}{4!}+\ldots)$

And then we can know:

e^x = cat(x) +dog(x)

e^{-x} = -cat(x) +dog(x)

C)

Dog(x)

e^{-x} = -cat(x) +dog(x)

Then:

dog(x) = e^{-x}-cat(x)

And:

dog(x) = cat(x) - e^x

So:

2dog(x) =e^{-x}- (- e^x)

Result:

dog(x) = (e^{-x}+ e^x)/2

Cat(x)

e^{-x} = -cat(x) +dog(x)

Then:

cat(x) = dog(x) - e^{-x}

cat(x) = e^x -dog(x)

So:

2 cat(x) = e^x - e^{-x}

Result:

cat(x) = (e^x - e^{-x})/2

D)

Plot cat(x) and first four different partial sums of its power series

In[ ]:= Plot$\left[\left\{\frac{1}{2}\ (-\text{Exp}[-x]+\text{Exp}[x]),\ x,\ x+\frac{x^3}{3\,!},\ x+\frac{x^3}{3\,!}+\frac{x^5}{5\,!},\ x+\frac{x^3}{3\,!}+\frac{x^5}{5\,!}+\frac{x^7}{7\,!}\right\},\ \{x,\ -2,\ 2\}\right]$

Out[ ]=