

# Teoria Sterowania w Robotyce

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May 26, 2020

## 1 Introduction

Throughout the rest of the course, we will be considering a single dynamic object – 2 Degrees of Freedom (2DoF) Planar Manipulator. As you can see in Figure 1 it consists of 2 links, 2 joints, and an object attached at the tip.

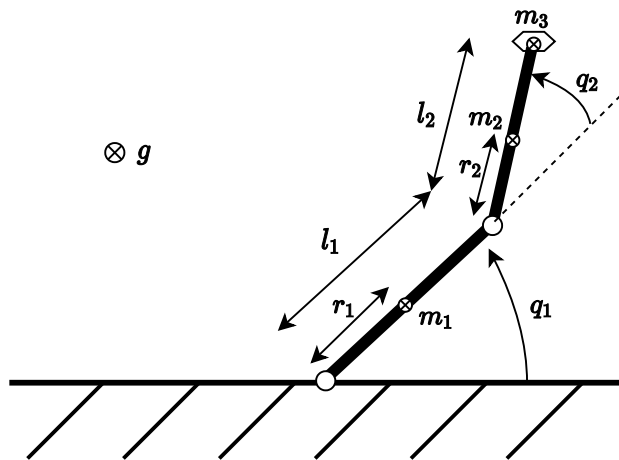


Figure 1: Scheme of the 2DoF manipulator

## 2 Derivation of dynamics of 2 DoF planar manipulator

In order to derive the dynamics of 2 DoF planar manipulator we will use a Lagrangian dynamic formulation – an energy based approach to dynamics. Namely, we will obtain dynamics from the formula

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}} \right) - \frac{\partial \mathcal{L}}{\partial q} = \tau, \quad (1)$$

where  $q$  is the robot state (configuration),  $\tau$  are the actuator torques and

$$\mathcal{L} = T - V, \quad (2)$$

where  $T$  is a kinetic energy of the manipulator and  $V$  is its potential energy. For the sake of this considerations we assume that manipulator work space lies in a plane such that vector of the gravity is orthogonal to it, thus  $V = 0$ .

Lets consider kinetic energy  $T$  of the 2 DoF planar manipulator as a sum of the energies of its links (for simplicity, we are assuming that there is no object at the tip of the second link, thus  $m_3 = 0$  and  $I_{z3} = 0$ )

$$T = T_1 + T_2, \quad (3)$$

where

$$T_i = \frac{1}{2} m_i v_i^2 + \frac{1}{2} I_{zi} \left( \sum_{k=1}^i \omega_k \right)^2, \quad (4)$$

which can be also reformulated to the form

$$T_i = \frac{1}{2}m_i (\dot{x}_i^2 + \dot{y}_i^2) + \frac{1}{2}I_{zi} \left( \sum_{k=1}^i \omega_k \right)^2, \quad (5)$$

where  $x_i$  and  $y_i$  are the coordinates of the center of mass,  $\omega_i$  is the angular velocity,  $I_{zi}$  is the inertia around  $z$  axis and  $m_i$  is the mass of the  $i$ -th link.

At the very end we expect to obtain a dynamics in the following form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} = \tau, \quad (6)$$

where  $M(q)$  is the mass matrix,  $C(q, \dot{q})$  is the matrix of Coriolis and centrifugal forces and  $\tau$  is the vector of actuator torques. To do so, we have to express the  $\mathcal{L}$  and thus  $T$  in terms of robot state  $q$  elements and its derivatives.

Let first define the coordinates of the links centers of masses in the global coordinate system. For first link they will be following

$$\begin{cases} x_1 = r_1 c_1 \\ y_1 = r_1 s_1 \end{cases}, \quad (7)$$

where  $s_1 = \sin q_1$  and  $c_1 = \cos q_1$  for brevity. Similarly, for second link we obtain

$$\begin{cases} x_2 = l_1 c_1 + r_2 c_{12} \\ y_2 = l_1 s_1 + r_2 s_{12} \end{cases}, \quad (8)$$

where  $s_{12} = \sin(q_1 + q_2)$ .

Then, take time derivative of (7) and (8) and obtain

$$\begin{cases} \dot{x}_1 = -r_1 s_1 \dot{q}_1 \\ \dot{y}_1 = r_1 c_1 \dot{q}_1 \end{cases} \quad (9)$$

and

$$\begin{cases} \dot{x}_2 = -l_1 s_1 \dot{q}_1 - r_2 s_{12}(\dot{q}_1 + \dot{q}_2) \\ \dot{y}_2 = l_1 c_1 \dot{q}_1 + r_2 c_{12}(\dot{q}_1 + \dot{q}_2) \end{cases}. \quad (10)$$

Next, substitute (5) with the time derivatives from (9) for first link

$$T_1 \stackrel{(5)}{=} \frac{1}{2}m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}I_{z1}\dot{q}_1^2 \stackrel{(9)}{=} \frac{1}{2}(m_1 r_1^2 + I_{z1})\dot{q}_1^2 \quad (11)$$

and the second link

$$\begin{aligned} T_2 &\stackrel{(5)}{=} \frac{1}{2}m_2 (\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2}I_{z2}(\dot{q}_1 + \dot{q}_2)^2 \stackrel{(10)}{=} \\ &\stackrel{(10)}{=} \frac{1}{2}m_2 \left( (l_1 \dot{q}_1 s_1 + r_2(\dot{q}_1 + \dot{q}_2)s_{12})^2 + (l_1 \dot{q}_1 c_1 + r_2(\dot{q}_1 + \dot{q}_2)c_{12})^2 \right) \\ &\quad + \frac{1}{2}I_{z2}(\dot{q}_1 + \dot{q}_2)^2 = \\ &= \frac{1}{2}m_2 (l_1^2 \dot{q}_1^2 s_1^2 + 2l_1 \dot{q}_1 s_1 r_2(\dot{q}_1 + \dot{q}_2)s_{12} + r_2^2(\dot{q}_1 + \dot{q}_2)^2 s_{12}^2) + \\ &\quad + \frac{1}{2}m_2 (l_1^2 \dot{q}_1^2 c_1^2 + 2l_1 \dot{q}_1 c_1 r_2(\dot{q}_1 + \dot{q}_2)c_{12} + r_2^2(\dot{q}_1 + \dot{q}_2)^2 c_{12}^2) \\ &\quad + \frac{1}{2}I_{z2}(\dot{q}_1 + \dot{q}_2)^2 = \\ &\quad \frac{1}{2}m_2 (l_1^2 \dot{q}_1^2 + r_2^2(\dot{q}_1 + \dot{q}_2)^2 + 2l_1(\dot{q}_1 + \dot{q}_2)r_2\dot{q}_1(s_1 s_{12} + c_1 c_{12})) + \\ &\quad + \frac{1}{2}I_{z2}(\dot{q}_1 + \dot{q}_2)^2 = \\ &= \frac{1}{2}m_2 (l_1^2 \dot{q}_1^2 + r_2^2(\dot{q}_1 + \dot{q}_2)^2 + 2l_1(\dot{q}_1 + \dot{q}_2)r_2\dot{q}_1 c_2) + \frac{1}{2}I_{z2}(\dot{q}_1 + \dot{q}_2)^2 \end{aligned} \quad (12)$$

Then, let's find the derivatives of  $\mathcal{L}$  with respect to  $\dot{q}$  as they are necessary to obtain (1)

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} &= (m_1 r_1^2 + I_{z1} + m_2(l_1^2 + r_2^2) + I_{z2} + 2m_2 l_1 r_2 c_2) \dot{q}_1 \\ &\quad + (m_2 l_1 r_2 c_2 + m_2 r_2^2 + I_{z2}) \dot{q}_2 = \\ &= \alpha \dot{q}_1 + \gamma \dot{q}_2 + \beta c_2 (2\dot{q}_1 + \dot{q}_2)\end{aligned}\quad (13)$$

where for simplicity we used following substitution

$$\begin{cases} \alpha = m_1 r_1^2 + I_{z1} + m_2(l_1^2 + r_2^2) + I_{z2} \\ \beta = m_2 l_1 r_2 \\ \gamma = m_2 r_2^2 + I_{z2} \end{cases} \quad (14)$$

Similarly, we get

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = \gamma(\dot{q}_1 + \dot{q}_2) + \beta c_2 \dot{q}_1, \quad (15)$$

and subsequently derivatives with respect to  $q$

$$\frac{\partial \mathcal{L}}{\partial q_1} = 0, \quad (16)$$

and

$$\frac{\partial \mathcal{L}}{\partial q_2} = -\beta s_2(\dot{q}_1^2 + \dot{q}_1 \dot{q}_2). \quad (17)$$

Last step to perform is to take the time derivative of (13)

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_1} \right) = (\alpha + 2\beta c_2) \ddot{q}_1 + (\gamma + \beta c_2) \ddot{q}_2 - \beta s_2 \dot{q}_2 (\dot{q}_1 + \dot{q}_2), \quad (18)$$

and (15)

$$\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_2} \right) = (\gamma + \beta c_2) \ddot{q}_1 + \gamma \ddot{q}_2 + \beta s_2 \dot{q}_1^2. \quad (19)$$

Then, we can form a matrix form of (1) with the formulas derived in (16), (17), (18) and (19), obtaining

$$\begin{bmatrix} \alpha + 2\beta c_2 & \gamma + \beta c_2 \\ \gamma + \beta c_2 & \gamma \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -\beta s_2 \dot{q}_2 & -\beta s_2 (\dot{q}_1 + \dot{q}_2) \\ \beta s_2 \dot{q}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}. \quad (20)$$

*Notice:* The form of the  $C(q, \dot{q})$  matrix is determined with the use of Christoffel symbols. However, for our purposes it is not obligatory.

## 2.1 Exercise

Derive dynamics of 2 DoF planar manipulator with a point mass  $m_3$  at the tip of the second link.

## 3 Feedback linearization

As you can see, the system described with (20) is nonlinear. There are some ways of dealing with this class of systems. The one which we will cover here – feedback linearization is one of those.

Form engineering studies you already familiar with linear dynamical systems of the form

$$\dot{x} = Ax + Bu, \quad (21)$$

however, the real world is usually nonlinear and only sometimes can be approximated well with the linear model.

Lets consider in general nonlinear system of the form

$$\dot{x} = f(x, \cdot) + g(x, \cdot)u, \quad (22)$$

where  $f(x, \cdot)$  and  $g(x, \cdot)$  are in general nonlinear functions of the state  $x$  and some other parameters. If we assume that  $g(x, \cdot) \neq 0$  for all  $t > 0$ . We can propose a feedback

$$u(x, \cdot) = \frac{1}{g(x, \cdot)} (v - f(x, \cdot)), \quad (23)$$

where  $v$  is the new desired input. Then, system (22) simplifies to the following form

$$\dot{x} = f(x, \cdot) + g(x, \cdot)u \stackrel{(23)}{=} f(x, \cdot) + g(x, \cdot) \left( \frac{1}{g(x, \cdot)} (v - f(x, \cdot)) \right) = v. \quad (24)$$

In effect, we got a linear system, where  $A = 0$  and  $B = \mathbb{I}$ , which is called an *integrator*.

An attentive student can spot that it is also possible to linearize (6) in that manner. We can propose following control law

$$\tau = M(q)v + C(q, \dot{q})\dot{q}, \quad (25)$$

which applied to (6) results in following dynamics

$$\ddot{q} = v. \quad (26)$$

Such a system is called the *double integrator*.

## 4 Polynomial trajectory

Polynomials are widely used in engineering, as a representation of paths and trajectories. They are simple and guarantee some desired level of smoothness. In our laboratory we will use following parametrization of 3rd degree polynomials

$$q(t) = \sum_{i=0}^3 a_i t^i (1-t)^{3-i} \quad \text{for } t \in [0; 1], \quad (27)$$

where  $a_i$  are the parameters of the polynomials, which has to be determined using initial and desired states  $q_0$  and  $q_d$ , and velocities  $\dot{q}_0$  and  $\dot{q}_d$ .

To determine parameters  $a_i$  one need to create a system of 4 equations

$$\begin{cases} q(0) = q_0 \\ \dot{q}(0) = \dot{q}_0 \\ q(1) = q_d \\ \dot{q}(1) = \dot{q}_d \end{cases}. \quad (28)$$

To do so, we have to compute derivatives of (27) with respect to the parameter  $t$ , namely

$$\dot{q} = \frac{dq}{dt}(t) = \frac{d}{dt} \left( \sum_{i=0}^3 a_i t^i (1-t)^{3-i} \right). \quad (29)$$

Next, we have to evaluate (27) and (29) at 0 and 1 and plug in into (28), and solve for all parameters  $a_i$ .

*Notice* In the laboratory we will be assuming that  $\dot{q}_0 = \dot{q}_d = 0$ .

Finally, having the polynomial parameters one can calculate the second derivative

$$\ddot{q} = \frac{d^2 q}{dt^2}(t) = \frac{d^2}{dt^2} \left( \sum_{i=0}^3 a_i t^i (1-t)^{3-i} \right), \quad (30)$$

which will be our new control.

## 5 Tasks - Laboratory 1

1. Implement model of the 2DoF manipulator, using the parameters from the manipulators/planar\_2dof.py (remember to set self.m\_3 to 0) and equation (20).
2. Implement Feedback Linearization Controller, based on the previously implemented model and (25).
3. Change the mass m\_3 in the manipulator object to nonzero value and observe how the performance of the controller degrades (structural uncertainty).
4. Use the derivation from exercise 2.1 to implement model which takes mass m\_3 into account.
5. Observe how the parametric uncertainty affects the performance of the controller.
6. Implement the polynomial trajectory generator, according to point 4.
7. Check out how the polynomial trajectory generator works together with the controller.

## 6 Adaptive Control

In this section we will consider an adaptive control. In our last considerations, as well as throughout the engineering studies, you were analyzing stationary processes. There was some manipulator, with constant properties (such as mass, links lengths etc.), however in real-life we meet a lot of processes where that is not a case! For example: rocket, which burns the fuel during the flight and modifies its mass during the flight, or manipulator, which picks and moves different objects. In both cases, simple stationary controller, which do not adapt to the changes in the model, can be not enough to control the system. However, in such cases, we can use Adaptive Control. It is a vast branch of control theory, which considers the situations, when it is necessary to adapt the behavior of the controller to the changing dynamics of the process being controlled.

In this class we will focus on the method called Multi-model Adaptive Control, which is the topic of the next Section, and another technique called Active Disturbance Rejection Control.

## 7 Multi-model Adaptive Control

Multi-model Adaptive Control is a term which unifies many different approaches to model-based adaptive control. General scheme of this approach is depicted in Figure 2. Basically the idea of MMAC is the following: based on the output of the object  $y$ , and the output of the models, models selector decide, which model approximates the object dynamics the best at given moment and use that model to produce the control signal. Simply, try to predict the future with some bunch of models, observe how the reality works, and choose the most suitable one for control.

Seems straightforward and easy, but actually it works. However, as every method this one has also its drawbacks. Can you name one?

It is important to notice how big impact has the switching behavior! For example: it can destabilize the system, even each of individual controllers is stable (see Figure 3), or it can decrease the control signal quality by making it discontinuous.

There is a lot of research in the area of Adaptive Control and MMAC, some basic view on the field can be found in <http://www.gipsa-lab.fr/>

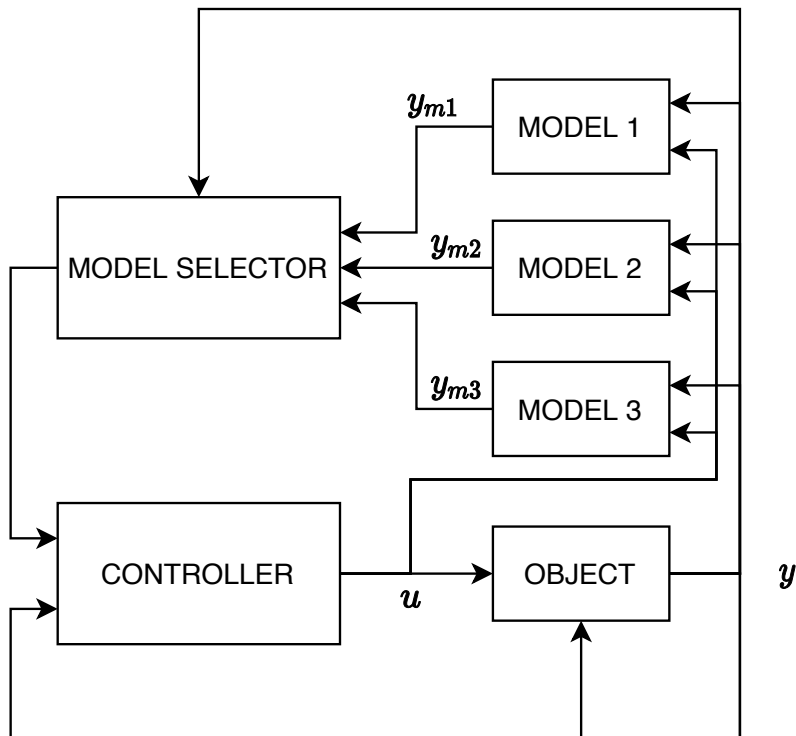


Figure 2: General scheme of MMAC.

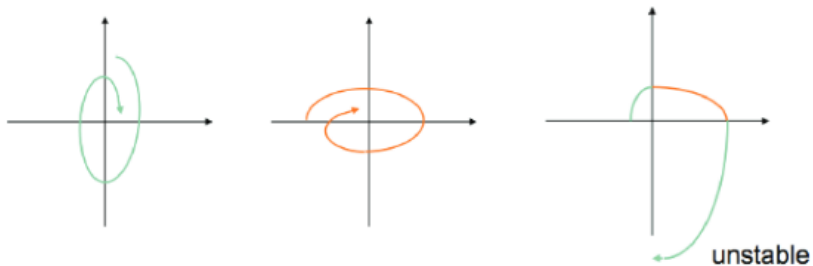


Figure 3: MMAC can be unstable, even each of controllers alone is stable.

[~ioandore.landau/adaptivecontrol/Transparent/Courses/AdaptiveCourse5GRK.pdf](https://ioandore.landau/adaptivecontrol/Transparent/Courses/AdaptiveCourse5GRK.pdf).

In our classes we consider only very basic version of Multi-model Adaptive Control (MMAC), which can be described with the scheme in Figure 4.

We will use the Feedback Linearization Controller (FLC) you implemented previously, and we will only modify its model by choosing the best one out of three model candidates. Model selection will be made, based on the comparison of the  $\dot{x}$  obtained from the object, and predicted  $\dot{x}_{mi}$  by the models. The idea is to choose the model with the lowest error

$$\arg \min_i e_i = \arg \min_i (\dot{x} - \dot{x}_{mi}). \quad (31)$$

And then, to use this model to produce the matrices  $M(q)$  and  $C(q, \dot{q})$ , which will be used by the FLC.

In our setup we will assume that there are 3 states, in which mass  $m_3$  and radius  $R_3$  of the object at the tip is different. States are put in the circular queue and the change of states is modeled as the Bernoulli process, where drawing 1 means that we change the model for the next one in the queue.

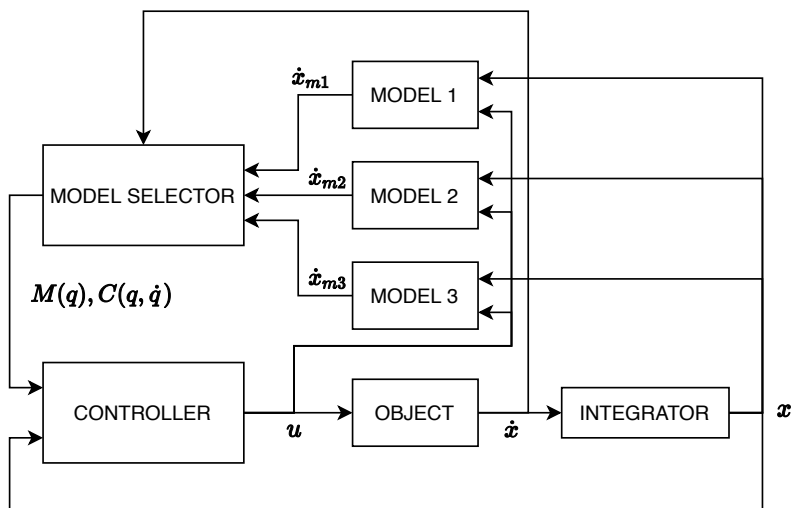


Figure 4: General scheme of the MMAC version we will implement in the lab.

## 8 Tasks - Laboratory 2

1. Implement the list of 3 models (you can use the class created in the previous tasks), which parameters will be same as in the object class (*manipulators/mm\_planar\_2dof.py*)
2. Implement the model selector, which will choose the most suitable model
3. Implement the FLC with the use of selected model
4. Observe how your model is doing

## 9 Bibliography

Richard M. Murray, S. Shankar Sastry, and Li Zexiang. 1994. A Mathematical Introduction to Robotic Manipulation (1st. ed.). CRC Press, Inc., USA.