Teoria Sterowania w Robotyce

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1 Introduction

Throughout the rest of the course, we will be considering a single dynamic object – 2 Degrees of Freedom (2DoF) Planar Manipulator. As you can see in Figure 1 it consists of 2 links, 2 joints, and an object attached at the tip.

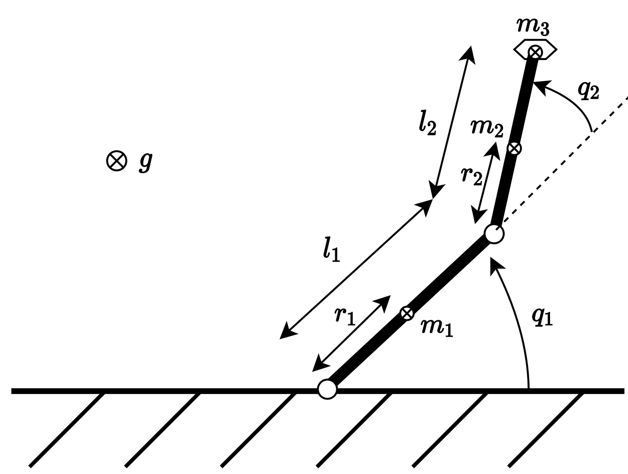

Figure 1: Scheme of the 2DoF manipulator

2 Derivation of dynamics of 2 DoF planar manipulator

In order to derive the dynamics of 2 DoF planar manipulator we will use a Lagrangian dynamic formulation – an energy based approach to dynamics. Namely, we will obtain dynamics from the formula

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = \tau, \tag{1}$$

where q is the robot state (configuration), τ are the actuator torques and

$$\mathcal{L} = T - V, \tag{2}$$

where T is a kinetic energy of the manipulator and V is its potential energy. For the sake of this considerations we assume that manipulator work space lies in a plane such that vector of the gravity is orthogonal to it, thus $V = 0$.

Lets consider kinetic energy T of the 2 DoF planar manipulator as a sum of the energies of its links (for simplicity, we are assuming that there is no object at the tip of the second link, thus $m_3 = 0$ and $I_{z3} = 0$)

$$T = T_1 + T_2, \tag{3}$$

where

$$T_i = \frac{1}{2}m_i v_i^2 + \frac{1}{2}I_{zi}\left(\sum_{k=1}^{i}\omega_k\right)^2, \tag{4}$$

which can be also reformulated to the form

$$T_i = \frac{1}{2} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 \right) + \frac{1}{2} I_{zi} \left(\sum_{k=1}^{i} \omega_k \right)^2 , \tag{5}$$

where x_i and y_i are the coordinates of the center of mass, ω_i is the angular velocity, I_{zi} is the inertia around z axis and m_i is the mass of the i-th link.

At the very end we expect to obtain a dynamics in the following form

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} = \tau, \tag{6}$$

where $M(q)$ is the mass matrix, $C(q, \dot{q})$ is the matrix of Coriolis and centrifugal forces and τ is the vector of actuator torques. To do so, we have to express the $\mathcal{L}$ and thus T in terms of robot state q elements and its derivatives.

Let first define the coordinates of the links centers of masses in the global coordinate system. For first link they will be following

$$\begin{cases} x_1 = r_1 c_1 \\ y_1 = r_1 s_1 \end{cases} , \tag{7}$$

where $s_1 = \sin q_1$ and $c_1 = \cos q_1$ for brevity. Similarly, for second link we obtain

$$\begin{cases} x_2 = l_1 c_1 + r_2 c_{12} \\ y_2 = l_1 s_1 + r_2 s_{12} \end{cases} , \tag{8}$$

where $s_{12} = \sin(q_1 + q_2)$.

Then, take time derivative of (7) and (8) and obtain

$$\begin{cases} \dot{x}_1 = -r_1 s_1 \dot{q}_1 \\ \dot{y}_1 = r_1 c_1 \dot{q}_1 \end{cases} \tag{9}$$

and

$$\begin{cases} \dot{x}_2 = -l_1 s_1 \dot{q}_1 - r_2 s_{12}(\dot{q}_1 + \dot{q}_2) \\ \dot{y}_2 = l_1 c_1 \dot{q}_1 + r_2 c_{12}(\dot{q}_1 + \dot{q}_2) \end{cases} . \tag{10}$$

Next, substitute (5) with the time derivatives from (9) for first link

$$T_1 \overset{(5)}{=} \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2 \right) + \frac{1}{2} I_{z1} \dot{q}_1^2 \overset{(9)}{=} \frac{1}{2} \left(m_1 r_1^2 + I_{z1} \right) \dot{q}_1^2 \tag{11}$$

and the second link

$$\begin{aligned}
T_2 &\overset{(5)}{=} \frac{1}{2} m_2 \left(\dot{x}_2^2 + \dot{y}_2^2 \right) + \frac{1}{2} I_{z2} (\dot{q}_1 + \dot{q}_2)^2 \overset{(10)}{=} \\
&\overset{(10)}{=} \frac{1}{2} m_2 \left((l_1 \dot{q}_1 s_1 + r_2 (\dot{q}_1 + \dot{q}_2) s_{12})^2 + (l_1 \dot{q}_1 c_1 + r_2 (\dot{q}_1 + \dot{q}_2) c_{12})^2 \right) \\
&\quad + \frac{1}{2} I_{z2} (\dot{q}_1 + \dot{q}_2)^2 = \\
&= \frac{1}{2} m_2 \left(l_1^2 \dot{q}_1^2 s_1^2 + 2 l_1 \dot{q}_1 s_1 r_2 (\dot{q}_1 + \dot{q}_2) s_{12} + r_2^2 (\dot{q}_1 + \dot{q}_2)^2 s_{12}^2 \right) + \\
&\quad + \frac{1}{2} m_2 \left(l_1^2 \dot{q}_1^2 c_1^2 + 2 l_1 \dot{q}_1 c_1 r_2 (\dot{q}_1 + \dot{q}_2) c_{12} + r_2^2 (\dot{q}_1 + \dot{q}_2)^2 c_{12}^2 \right) \\
&\quad + \frac{1}{2} I_{z2} (\dot{q}_1 + \dot{q}_2)^2 = \\
&\frac{1}{2} m_2 \left(l_1^2 \dot{q}_1^2 + r_2^2 (\dot{q}_1 + \dot{q}_2)^2 + 2 l_1 (\dot{q}_1 + \dot{q}_2) r_2 \dot{q}_1 (s_1 s_{12} + c_1 c_{12}) \right) + \\
&\quad + \frac{1}{2} I_{z2} (\dot{q}_1 + \dot{q}_2)^2 = \\
&= \frac{1}{2} m_2 \left(l_1^2 \dot{q}_1^2 + r_2^2 (\dot{q}_1 + \dot{q}_2)^2 + 2 l_1 (\dot{q}_1 + \dot{q}_2) r_2 \dot{q}_1 c_2 \right) + \frac{1}{2} I_{z2} (\dot{q}_1 + \dot{q}_2)^2
\end{aligned} . \tag{12}$$

Then, lets find the derivatives of $\mathcal{L}$ with respect to $\dot{q}$ as they are necessary to obtain (1)

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \dot{q}_1} &= \left(m_1 r_1^2 + I_{z1} + m_2(l_1^2 + r_2^2) + I_{z2} + 2m_2 l_1 r_2 c_2\right)\dot{q}_1 \\ &+ \left(m_2 l_1 r_2 c_2 + m_2 r_2^2 + I_{z2}\right)\dot{q}_2 = \\ &= \alpha\dot{q}_1 + \gamma\dot{q}_2 + \beta c_2(2\dot{q}_1 + \dot{q}_2)\end{aligned}, \tag{13}$$

where for simplicity we used following substitiution

$$\begin{cases}\alpha = m_1 r_1^2 + I_{z1} + m_2(l_1^2 + r_2^2) + I_{z2} \\ \beta = m_2 l_1 r_2 \\ \gamma = m_2 r_2^2 + I_{z2}\end{cases}. \tag{14}$$

Similarly, we get

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_2} = \gamma(\dot{q}_1 + \dot{q}_2) + \beta c_2 \dot{q}_1, \tag{15}$$

and subsequently derivatives with respect to q

$$\frac{\partial \mathcal{L}}{\partial q_1} = 0, \tag{16}$$

and

$$\frac{\partial \mathcal{L}}{\partial q_2} = -\beta s_2(\dot{q}_1^2 + \dot{q}_1\dot{q}_2). \tag{17}$$

Last step to perform is to take the time derivative of (13)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_1}\right) = (\alpha + 2\beta c_2)\ddot{q}_1 + (\gamma + \beta c_2)\ddot{q}_2 - \beta s_2 \dot{q}_2(\dot{q}_1 + \dot{q}_2), \tag{18}$$

and (15)

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_2}\right) = (\gamma + \beta c_2)\ddot{q}_1 + \gamma\ddot{q}_2 + \beta s_2 \dot{q}_1^2. \tag{19}$$

Then, we can form a matrix form of (1) with the formulas derived in (16), (17), (18) and (19), obtaining

$$\begin{bmatrix}\alpha + 2\beta c_2 & \gamma + \beta c_2 \\ \gamma + \beta c_2 & \gamma\end{bmatrix}\begin{bmatrix}\ddot{q}_1 \\ \ddot{q}_2\end{bmatrix} + \begin{bmatrix}-\beta s_2 \dot{q}_2 & -\beta s_2(\dot{q}_1 + \dot{q}_2) \\ \beta s_2 \dot{q}_1 & 0\end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix} = \begin{bmatrix}\tau_1 \\ \tau_2\end{bmatrix}. \tag{20}$$

Notice: The form of the $C(q, \dot{q})$ matrix is determined with the use of Christoffel symbols. However, for our purposes it is not obligatory.

2.1 Exercise

Derive dynamics of 2 DoF planar manipulator with a point mass m_3 at the tip of the second link.

3 Feedback linearization

As you can see, the system described with (20) is nonlinear. There are some ways of dealing with this class of systems. The one which we will cover here – feedback linearization is one of those.

Form engineering studies you already familiar with linear dynamical systems of the form

$$\dot{x} = Ax + Bu, \tag{21}$$

however, the real world is usually nonlinear and only sometimes can be approximated well with the linear model.

Lets consider in general nonlinear system of the form

$$\dot{x} = f(x, \cdot) + g(x, \cdot)u, \tag{22}$$

where $f(x, \cdot)$ and $g(x, \cdot)$ are in general nonlinear functions of the state x and some other parameters. If we assume that $g(x, \cdot) \neq 0$ for all $t > 0$. We can propose a feedback

$$u(x, \cdot) = \frac{1}{g(x, \cdot)} \left(v - f(x, \cdot)\right), \tag{23}$$

where v is the new desired input. Then, system (22) simplifies to the following form

$$\dot{x} = f(x, \cdot) + g(x, \cdot)u \overset{(23)}{=} f(x, \cdot) + g(x, \cdot) \left(\frac{1}{g(x, \cdot)} \left(v - f(x, \cdot)\right) \right) = v. \tag{24}$$

In effect, we got a linear system, where $A = 0$ and $B = \mathbb{I}$, which is called an *integrator*.

An attentive student can spot that it is also possible to linearize (6) in that manner. We can propose following control law

$$\tau = M(q)v + C(q, \dot{q})\dot{q}, \tag{25}$$

which applied to (6) results in following dynamics

$$\ddot{q} = v. \tag{26}$$

Such a system is called the *double integrator*.

4 Polynomial trajectory

Polynomials are widely used in engineering, as a representation of paths and trajectories. They are simple and guarantee some desired level of smoothness. In our laboratory we will use following parametrization of 3rd degree polynomials

$$q(t) = \sum_{i=0}^{3} a_i t^i (1-t)^{3-i} \quad \text{for} \quad t \in [0; 1], \tag{27}$$

where a_i are the parameters of the polynomials, which has to be determined using initial and desired states q_0 and q_d, and velocities $\dot{q}_0$ and $\dot{q}_d$.

To determine parameters a_i one need to create a system of 4 equations

$$\begin{cases} q(0) = q_0 \\ \dot{q}(0) = \dot{q}_0 \\ q(1) = q_d \\ \dot{q}(1) = \dot{q}_d 0 \end{cases}. \tag{28}$$

To do so, we have to compute derivatives of (27) with respect to the parameter t, namely

$$\dot{q} = \frac{dq}{dt}(t) = \frac{d}{dt} \left(\sum_{i=0}^{3} a_i t^i (1-t)^{3-i} \right). \tag{29}$$

Next, we have to evaluate (27) and (29) at 0 and 1 and plug in into (28), and solve for all parameters a_i.

Notice In the laboratory we will be assuming that $\dot{q}_0 = \dot{q}_d 0 = 0$.

Finally, having the polynomial parameters one can calculate the second derivative

$$\ddot{q} = \frac{d^2 q}{dt^2}(t) = \frac{d^2}{dt^2} \left(\sum_{i=0}^{3} a_i t^i (1-t)^{3-i} \right), \tag{30}$$

which will be our new control.

5 Bibliography

Richard M. Murray, S. Shankar Sastry, and Li Zexiang. 1994. A Mathematical Introduction to Robotic Manipulation (1st. ed.). CRC Press, Inc., USA.