COM SCI 260B HW 3 Solution

Ashish Kumar Singh (UID:105479019)

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Problem 1. Show that $\max_{v:||v||=1} ||Xv|| \leq \sigma_1$

Solution 1. Given $X = U\Sigma V^T$, we can write,

$$X = \sum_i \sigma_i u_i v_i^T \tag{1}$$

We can write vector v in the basis of columns of V, i.e.

$$v = \sum_i \alpha_i v_i \tag{2}$$

$$||v||^2 = \sum_i \alpha_i^2 ||v_i||^2$$

Since, v_i are orthonormal and $||v|| = 1$,

$$||v||^2 = \sum_i \alpha_i^2 = 1 \tag{3}$$

Combining eqn (1) and (2),

$$Xv = \sum_i \sigma_i u_i v_i^T . \sum_i \alpha_i v_i$$

Since v_i and u_i are orthonormal,

$$Xv = \sum_i \sigma_i \alpha_i u_i v_i^T v_i$$

$$Xv = \sum_i \sigma_i \alpha_i u_i$$

$$||Xv||^2 = \sum_i \sigma_i^2 \alpha_i^2 ||u_i||^2$$

$$||Xv||^2 = \sum_i \sigma_i^2 \alpha_i^2$$

By definition, σ_1 is greatest of all the singular values, hence $\sigma_i \leq \sigma_1$

$$||Xv||^2 \leq \sigma_1^2 \sum_i \alpha_i^2$$

$$||Xv||^2 \leq \sigma_1^2$$

$$||Xv|| \leq \sigma_1$$

Problem 2. Best-fit subspace dimension k

Solution 2. Lets assume that first $k-1$ singular vectors gives a best-fit subspace of dimension $k-1$, which means if $v_1, v_2, ..., v_{k-1}$ are the first $k-1$ singular vectors then for any vectors $w_1, w_2, ..., w_{k-1}$, we have:

$$\sum_{i=1}^{k-1} ||Xw_i||^2 \leq \sum_{i=1}^{k-1} ||Xv_i||^2 \tag{4}$$

Now, lets say that S^* is the best fit subspace of dimension k, we can choose the orthonormal basis for S^* such that $w_k \perp v_i$ for $i \in \{1, k-1\}$

$$S^* = Span(w_1, w_2, ..., w_k)$$

Lets say that v_k is the kth singular vector, then by definition

$$v_k = argmax_{||v||=1, v \perp v_i, i \in \{1,k-1\}} ||Xv||$$

$$||Xv_k|| \geq ||Xw_k|| \tag{5}$$

Lets say S is the span of first k singular vectors,

$$S = Span(v_1, v_2, ..., v_k)$$

On combining eqn 4 and 5 we get,

$$\sum_{i=1}^{k-1} ||Xw_i||^2 + ||Xw_k||^2 \leq \sum_{i=1}^{k-1} ||Xv_i||^2 + ||Xv_k||^2$$

$$\sum_{i=1}^{k} ||Xw_i||^2 \leq \sum_{i=1}^{k} ||Xv_i||^2$$

$$Var(S^*; X) \leq Var(S; X)$$

which shows that S maximizes var in dimension k, hence the span of first k right singular vectors gives the best-fit subspace of dimension k. We know that for $k = 2$ it is true, thus by induction it is true for every k.

Problem 3. Smallest Singular Vector

Solution 3. We first find the largest singular vector and corresponding largest singular value σ_1 of X using Power Iteration method.
Now, we construct new matrix $Y = X^t X$, If $X = U\Sigma V^T$, then $Y = V\Sigma^2 V^T$, we can see that $YV = V\Sigma^2$ which means eigenvalues of Y are square of the singular values of X.
Since Y is symmetric, we can shift its singular value by shifting the matrix by scaled identity matrix.
We can form another matrix $Z = Y - \sigma_1^2 I$. Thus making the magnitude of smallest singular value the highest. Thus, on running Power Iteration method on Z, we will get the smallest right singular vector of X.

Problem 4. Singular Value Projection

Solution 4a.

$$L = \sum_{(i,j)\in O} (X_{ij} - Y_{ij})^2$$

$$\frac{\partial L}{\partial Y_{ij}} = \begin{cases} 0 & \text{for } i, j \notin O \\ 2(Y_{ij} - X_{ij}) & \text{for } i, j \in O \end{cases}$$

$$\frac{\partial L}{\partial Y_{ij}} = 2(Y_{ij} - X_{ij}).O_{ij}$$

$$\frac{\partial L}{\partial Y} = 2(Y - X).O$$

Solution 4b. code and plots attached at the end

Problem 5. Singular Value Projection

Solution 5. code and plots attached at the end

Mode	Number of Iteration	Time
Scipy SVD	-	16.17
PI	10	0.52
PI	20	1.07
PI	30	1.59
PI	40	2.13
PI	50	2.67
PI	60	3.20
PI	70	3.72
PI	80	4.25
PI	90	4.77
PI	100	5.30

Table 1: Time comparison for Scipy vs PI

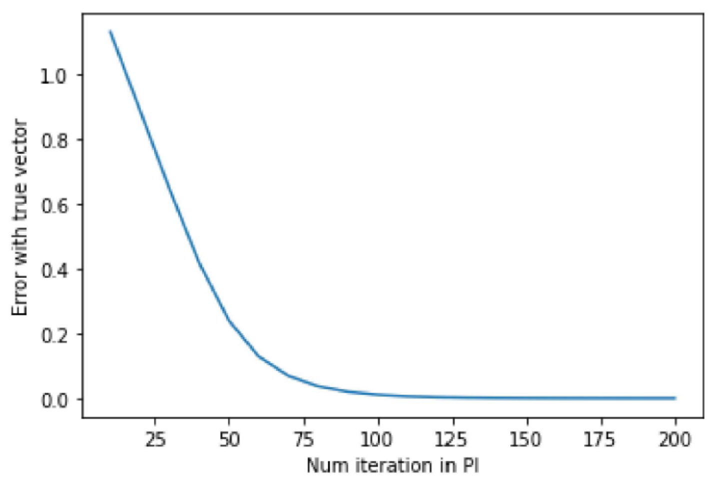

Figure 1: Plot of Error vs number of iteration in PI