COM SCI 161A HW 7 Solution

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Problem 1. Proof by induction

Solution 1.

$$Pr(\alpha_1, ..., \alpha_n|\beta) = Pr(\alpha_1|\alpha_2, ...\alpha_n, \beta)Pr(\alpha_2|\alpha_3, ...\alpha_n, \beta)....Pr(\alpha_n|\beta) \tag{1}$$

Base case, $n = 1$ is trivial,

$$Pr(\alpha_1|\beta) = Pr(\alpha_1|\beta)$$

Base case, $n = 2$ can be proved by bayes rule,

$$Pr(\alpha_1, \alpha_2|\beta) = \frac{Pr(\alpha_1, \alpha_2, \beta)}{Pr(\beta)}$$

$$Pr(\alpha_1, \alpha_2|\beta) = \frac{Pr(\alpha_1, \alpha_2, \beta)}{Pr(\alpha_2, \beta)}\frac{Pr(\alpha_2, \beta)}{Pr(\beta)}$$

$$Pr(\alpha_1, \alpha_2|\beta) = Pr(\alpha_1|\alpha_2, \beta)Pr(\alpha_2|\beta)$$

Now lets assume for $n = k$, the eqn (1) is true,

$$Pr(\alpha_1, ..., \alpha_k|\beta) = Pr(\alpha_1|\alpha_2, ...\alpha_k, \beta)Pr(\alpha_2|\alpha_3, ...\alpha_k, \beta)....Pr(\alpha_k|\beta)$$

Using Bayes rule, we can write:

$$Pr(\alpha_1, ..., \alpha_k, \alpha_{k+1}|\beta) = Pr(\alpha_1, ..., \alpha_k|\alpha_{k+1}, \beta)Pr(\alpha_{k+1}|\beta)$$

$$Pr(\alpha_1, ..., \alpha_k, \alpha_{k+1}|\beta) = Pr(\alpha_1|\alpha_2, ...\alpha_k, \alpha_{k+1}, \beta)Pr(\alpha_2|\alpha_3, ...\alpha_k, \alpha_{k+1}, \beta)...Pr(\alpha_{k+1}|\beta)$$

$$Pr(\alpha_1, ..., \alpha_{k+1}|\beta) = Pr(\alpha_1|\alpha_2, ..., \alpha_{k+1}, \beta)Pr(\alpha_2|\alpha_3, ..., \alpha_{k+1}, \beta)....Pr(\alpha_{k+1}|\beta)$$

which proves that eqn 1 is true for $n = k + 1$ if we assume it is true for $n = k$, we also know that for bases cases $n = 2$ it is true thus by induction eqn (1) is true for all n.

Problem 2. What's the probability that oil is present?

Solution 2. Let O, N, T denotes presence of oil, natural gas and positive test respectively Using Bayes rule,

$$Pr(O|T) = \frac{Pr(T|O)Pr(O)}{Pr(T)}$$

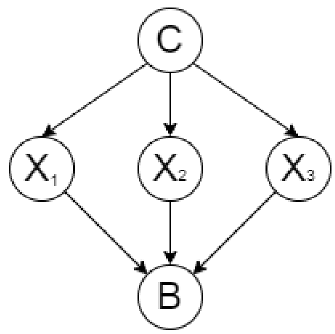
$$Pr(O|T) = \frac{Pr(T|O)Pr(O)}{Pr(T|O)Pr(O) + Pr(T|N)Pr(N) + Pr(T|\neg O, \neg N)Pr(\neg O, \neg N)}$$

$$Pr(O|T) = \frac{0.9 * 0.5}{0.9 * 0.5 + 0.3 * 0.2 + 0.1 * 0.3}$$

$$Pr(O|T) = \frac{45}{54} = 0.8333$$

Problem 3. Bayesian network and CPTs

Solution 3. Bayesian network can be seen below:



where $C = \{a, b, c\}$, $X_i = \{head, tail\}$ and $B = \{on, off\}$. CPTs are as follows:

C	$Pr(C)$
a	1/3
b	1/3
c	1/3

All X_i have same CPT:

C	X_i	$Pr(X_i\|C)$
a	head	0.2
a	tail	0.8
b	head	0.4
b	tail	0.6
c	head	0.8
c	tail	0.2

X_1	X_2	X_3	B	$Pr(B\|X_1, X_2, X_3)$
head	head	head	on	1
head	head	head	off	0
head	head	tail	on	0
head	head	tail	off	1
head	tail	head	on	0
head	tail	head	off	1
head	tail	tail	on	0
head	tail	tail	off	1
tail	head	head	on	0
tail	head	head	off	1
tail	head	tail	on	0
tail	head	tail	off	1
tail	tail	head	on	0
tail	tail	head	off	1
tail	tail	tail	on	1
tail	tail	tail	off	0

Problem 4. DAG

Solution 4a. Markovian assumptions are:
$I(A, \phi, BE), I(B, \phi, AC), I(C, A, DBE), I(D, AB, CE), I(E, B, ACDFG), I(F, CD, ABE),$
$I(G, F, ABCDEH)$ and $I(H, FE, ABCDG)$

Solution 4b. i) d-separated(A,F,E) False because path $ADBE$ is open. A and E are not d-separated if F is known.
ii) d-separated(G,B,E) True because all the paths $GFHE$, $GFDBE$ and $GFCADBE$ are closed. G and E are d-separated if B is known.
iii) d-separated(AB,CDE,GH) True because all paths are between AB and GH are closed. C, E valves are closed as they are known and lie on sequential. D is closed when the path is sequential through D. This completely cuts A,B from G,H. Hence AB is d-separated from GH when CDE is known.

Solution 4c. On multiplying CPT parameter for all variables:

$$Pr(a, b, c, d, e, f, g, h) = Pr(a)Pr(b)Pr(c|a)Pr(d|a, b)Pr(e|b)Pr(f|c, d)Pr(g|f)Pr(h|e, f)$$

Solution 4d. From Markovian assumption $I(A, \phi, BE)$, A and B are independent, hence,

$$Pr(A = 1, B = 1) = Pr(A = 1) * Pr(B = 1)$$

$$Pr(A = 1, B = 1) = 0.2 * 0.7 = 0.14$$

From the same Markovian assumption above, A and E are independent, hence,

$$Pr(E = 0|A = 0) = Pr(E = 0)$$

$$Pr(E = 0|A = 0) = Pr(E = 0|B = 0) * Pr(B = 0) + Pr(E = 0|B = 1) * Pr(B = 1)$$
$$Pr(E = 0|A = 0) = 0.1 * 0.3 + 0.9 * 0.7 = 0.66$$

Problem 5. Joint Probability

Solution 5a. We can form the truth table for α, as follows:

Worlds	A	B	Pr(A,B)	$\alpha : A \implies B$
w_0	T	T	0.3	T
w_1	T	F	0.2	F
w_2	F	T	0.1	T
w_3	F	F	0.4	T

$$M(\alpha) = \{w_0, w_2, w_3\}$$

Solution 5b.

$$Pr(\alpha) = Pr(w_0) + Pr(w_2) + Pr(w_3)$$

$$Pr(\alpha) = 0.3 + 0.1 + 0.4$$

$$Pr(\alpha) = 0.8$$

Solution 5c. Figure 4
Below table shows probability values:

Worlds	A	B	Pr(A,B)	$\alpha : A \implies B$	$Pr(A, B\vert\alpha)$
w_0	T	T	0.3	T	$0.3/0.8 = 0.375$
w_1	T	F	0.2	F	0
w_2	F	T	0.1	T	$0.1/0.8 = 0.125$
w_3	F	F	0.4	T	$0.4/0.8 = 0.5$

Solution 5d. Using Bayes rule,

$$Pr((A \implies \neg B)|\alpha) = \frac{Pr((A \implies \neg B) \wedge \alpha)}{Pr(\alpha)}$$

$$Pr((A \implies \neg B)|\alpha) = \frac{Pr(\{w_1, w_2, w_3\} \cap \{w_0, w_2, w_3\})}{Pr(\alpha)}$$

$$Pr((A \implies \neg B)|\alpha) = \frac{Pr(\{w_2, w_3\})}{Pr(\alpha)}$$

$$Pr((A \implies \neg B)|\alpha) = \frac{0.5}{0.8}$$

$$Pr(\alpha) = 0.625$$