CS 161 Intro. To Artificial Intelligence

Week 7, Discussion 1A

Review from last time

A “Model” in first order logic:

- An instantiation for functions and predicates
    - The objects a function returns given each possible object
    - The true/false values a function returns given each possible object
- Not an instantiation over objects or variables
    - All variables must be quantified over or the expression is not well formed
- Includes the available objects in the system as well as the signatures for predicates and functions

Model Example

Objects

$Orange$

$Apple$

Predicates

$IsRed(\cdot)$

$HasVitaminC(\cdot)$

Functions

$OppositeOf(\cdot)$

Example model:

Predicate	Argument	Value
$IsRed$	$Orange$	$False$
$IsRed$	$Apple$	$True$
$HasVitaminC$	$Orange$	$True$
$HasVitaminC$	$Apple$	$True$

Function	Argument	Return
$OppositeOf$	$Orange$	$Apple$
$Opposite$	$Apple$	$Orange$

FOL Inference –Instantiation

- A Ground term (literal)
  - A ground term is a term without variables
  - E.g., Apple, Color(Apple)
- Variable
  - Can be any specific object in a certain domain
  - E.g. x, y □ we often write variable in lower-case letters

If we have only ground terms, we can infer any sentence obtained by substituting a ground term (a term without variables) for the variable.

Universal Instantiation

Universal Instantiation (UI):

For every object in the KB, just write out the rule with the variables substituted.

- UI can be applied **several times** to add new sentences;
  - The new KB is logically **equivalent** to the old
  - Can produce infinite instances if we have a function

$$\frac{\forall x \, \alpha}{subst(\{x/g\}, \alpha)}$$

g: ground term

Example: The objects in our KB include Apple, Orange, MyCar, TheSky

$$\forall x, Fruit(x) \implies Tasty(x) \qquad Fruit(Apple) \implies Tasty(Apple)$$

$$Fruit(Orange) \implies Tasty(Orange)$$

$$Fruit(MyCar) \implies Tasty(MyCar)$$

$$Fruit(TheSky) \implies Tasty(TheSky)$$

This isn't propositional logic - no variables. But Fruit(Apple) can be mapped to a variable x_1

Existential Instantiation

Existential Instantiation (EI):

Assign a new constant (**<u>Skolem constant</u>**) to the variable.

$$\frac{\exists x\, \alpha}{Subst(\{x/k\}, \alpha)}$$

- Constant name cannot be one we've already used.
- Quantifier can then be discarded.
- EI can be **applied only once** to replace the existential sentence;
    - The new KB is **not equivalent** to the old,
    - But it's satisfiable iff the old KB was satisfiable

Example:

$$\exists x, Car(x) \wedge ParkedIn(x, E23)$$

$$Car(C) \wedge ParkedIn(C, E23)$$

□ C is the Skolem constant in this example

Reduce FOL to Propositional Logic

How to determine if $\forall x\ P$ is True?

- We can reduce it to propositional inference, which is also called **grounding**
- Instantiating all quantified sentences allows us to <u>ground</u> the KB, that is, to make KB propositional
- Then we can apply the inference rules on the propositional KB to show $\Delta \vDash \alpha$

Entailment in FOL is semidecidable: Can show entailment, but not not-entailment

Herbrand (1930): If sentence α is entailed by an FOL KB, it's entailed by a finite subset of the propositional KB

Δ could be infinite!

- If $\Delta \vDash \alpha$, then $\Delta' \vDash \alpha$ where Δ' is a subset of Δ
- For nesting situation (when we have a function), such as Father(Father(John))), we increase the nesting index by one each time to see if α is entailed by this KB.

Definite Clauses

Definite clauses: has exactly one positive literal

- E.g. $A \vee \neg B$, A,$\neg A \vee \neg B \vee C$ → can also be written as $(A \wedge B \Rightarrow C)$
- Similar to if-then rule, with positive literals on each side

If we have definite clause KB, we can do **forward chaining** and **backward chaining!**

Remember horn clauses have “at most one positive literal”

Definite Clause KB

Example:
The law says that it is a crime for an American to sell weapons to hostile nations.
The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

Prove that Colonel West is a criminal.

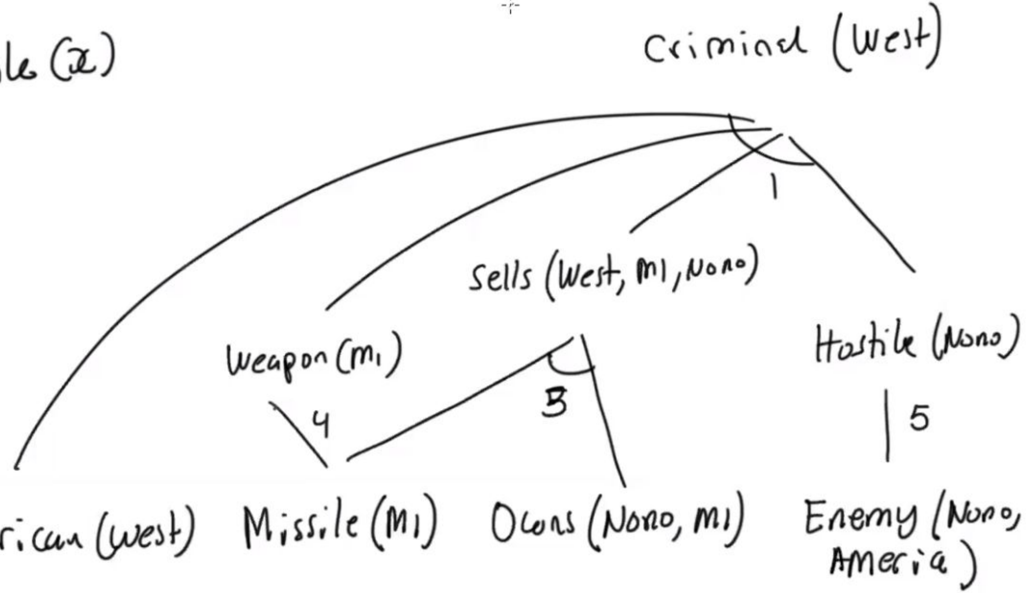
Definite Clause KB – Forward Chaining

1. $American(x) \land Weapon(y) \land Sells(x, y, z) \land Hostile(z) \Rightarrow Criminal(x)$
2. $\exists x\ Owns(Nono, x) \land Missile(x)$ ✓
   $\rightarrow Owns(Nono, M_1) \land Missile(M_1)$ ✓
3. $Missile(x) \land Owns(Nono, x) \Rightarrow Sells(West, x, Nono)$
4. $Missile(x) \Rightarrow Weapon(x)$
5. $Enemy(x, America) \Rightarrow Hostile(x)$
6. $American(West)$
7. $Enemy(Nono, America)$

$Criminal(West)$

"Datalog"

Forward Chaining

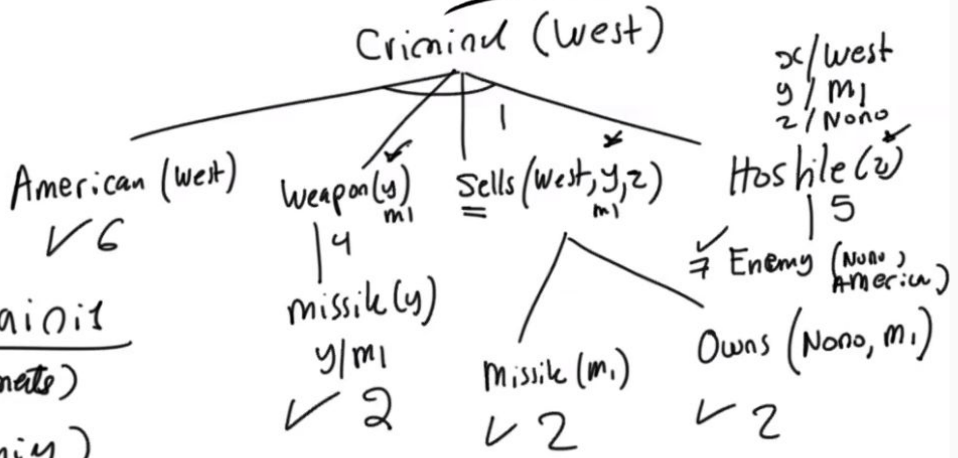


start with -> Ground Terms:

Definite Clause KB – Backward Chaining

1. $American(x) \wedge Weapon(y) \wedge Sells(x, y, z) \wedge Hostile(z) \Rightarrow Criminal(x)$
2. $\exists x\; Owns(Nono, x) \wedge Missile(x)$
   $\rightarrow Owns(Nono, M_1) \wedge Missile(M_1)$ ✓
3. $Missile(x) \wedge Owns(Nono, x) \Rightarrow Sells(West, x, Nono)$
4. $Missile(x) \Rightarrow Weapon(x)$
5. $Enemy(x, America) \Rightarrow Hostile(x)$
6. $American(West)$
7. $Enemy(Nono, America)$

Backward Chaining



Criminal (West)

"Datalog"

Backward chaining
(with improvements)
(logic programming)
(Prolog)

Resolution in FOL - Unification

Idea: $\Delta \models \alpha$ **iff** $\Delta \wedge \neg\alpha$ **is unsatisfiable**

Unification: finding substitutions that make different logical expressions look identical.

- i.e. Unify(α, β) = θ if SUBST(θ, α) = SUBST(θ, β)
- takes two **atomic** (i.e. single predicates) sentences α, β
- returns a substitution θ that makes α, β identical

unifiers ↓

$$\begin{aligned}
&\textsc{Unify}(Knows(John, x),\ Knows(John, Jane)) = \{x/Jane\}\\
&\textsc{Unify}(Knows(John, x),\ Knows(y, Bill)) = \{x/Bill, y/John\}\\
&\textsc{Unify}(Knows(John, x),\ Knows(y, Mother(y))) = \{y/John, x/Mother(John)\}\\
&\textsc{Unify}(Knows(John, x),\ Knows(x, Elizabeth)) = fail\ .
\end{aligned}$$

Resolution in FOL - Unification

Standardizing apart: eliminates overlap of variables

$$\text{UNIFY}(Knows(John, x),\ Knows(John, Jane)) = \{x/Jane\}$$
$$\text{UNIFY}(Knows(John, x),\ Knows(y, Bill)) = \{x/Bill, y/John\}$$
$$\text{UNIFY}(Knows(John, x),\ Knows(y, Mother(y))) = \{y/John, x/Mother(John)\}$$
$$\boxed{\text{UNIFY}(Knows(John, x),\ Knows(x, Elizabeth)) = fail\ .}$$

This fails because the two atomic sentences use the same variable name.
If we use Knows(z, Elizabeth) instead of Knows(x, Elizabeth), it should be fine

Most General Unifier

Most General Unifier (MGU):

For every unifiable pair of expressions, there is a single most general unifier (MGU) that is unique up to renaming and substitution of variables

- Unify(Knows(John, x), Knows(y, z)) could return the following two:
  - {y/John, x/z}
    - Then both sentences become Knows{John,z}
    - This is **more general!**
    - This is MGU for this pair of sentences
  - {y/John, x/John, z/John}
    - Then both sentences become Knows(John, John)

Conversion to CNF

Every FOL sentence can be converted into an inferentially equivalent CNF sentence.

- Inferentially equivalent: it is satisfiable exactly when the original sentence is satisfiable

We usually do the following things:

- Eliminate $\Rightarrow$, $\Leftrightarrow$
- Move $\neg$ down to the atomic formulas
- Eliminate $\forall$, $\exists$
    - Eliminate $\exists$ is **skolemize**
- Rename the variables, if necessary
    - This is called **standardize**
- Distribute $\wedge$ over $\vee$
    - Move $\vee$ down to the literals

Conversion to CNF

Example: Everyone who loves all animals is loved by someone:

$\forall x\ [\forall y\ Animal(y) \Rightarrow Loves(x,y)] \Rightarrow [\exists y\ Loves(y,x)]$

1. Eliminate biconditionals and implications

$$\forall x\ [\neg\forall y\ \neg Animal(y) \lor Loves(x,y)] \lor [\exists y\ Loves(y,x)]$$

2. Move $\neg$ inwards: $\neg\forall x, p\ \equiv \exists x\ \neg p$, $\neg\exists x, p\ \equiv \forall x\ \neg p$:

$$\forall x\ [\exists y\ \neg(\neg Animal(y) \lor Loves(x,y))] \lor [\exists y\ Loves(y,x)]$$
$$\forall x\ [\exists y\ \neg\neg Animal(y) \land \neg Loves(x,y)] \lor [\exists y\ Loves(y,x)]$$
$$\forall x\ [\exists y\ Animal(y) \land \neg Loves(x,y)] \lor [\exists y\ Loves(y,x)]$$

Conversion to CNF

3. Standardize apart variables: each quantifier should use a different one

$$\forall x \; [\exists y \; Animal(y) \wedge \neg Loves(x,y)] \vee [\exists z \; Loves(z,x)]$$

4. Skolemize: a more general form of existential instantiation. Each existential variable is replaced by a **Skolem function** of the enclosing universally quantified variables:

$$\forall x \;\; [Animal(\underline{F(x)}) \wedge \neg Loves(x, \underline{F(x)})] \vee Loves(\underline{G(x)}, x)$$

Equivalently just map to F or G!

5. Drop universal quantifiers:

$$[Animal(F(x)) \wedge \neg Loves(x, F(x))] \vee Loves(G(x), x)$$

6. Distribute $\wedge$ over $\vee$:

$$[Animal(F(x)) \vee Loves(G(x), x)] \wedge [\neg Loves(x, F(x)) \vee Loves(G(x), x)]$$

Resolution in FOL

- Resolution in Propositional Logic:

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{\ell_1 \vee \cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee \ell_k \vee m_1 \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots \vee m_n}$$

  - each l and each m is a literal and exist one li and mj are complementary literals (i.e. one is the negation of the other)

- **Resolution in FOL:**

Unify then resolve!

Resolvent depends on unifier

Full first-order version:

$$\frac{\ell_1 \vee \cdots \vee \ell_k, \qquad m_1 \vee \cdots \vee m_n}{\mathrm{SUBST}(\theta, (\cdots \vee \ell_{i-1} \vee \ell_{i+1} \vee \cdots \vee m_{j-1} \vee m_{j+1} \vee \cdots))}$$

where $\textsc{Unify}(\ell_i, \neg m_j) = \theta$.$(Unify(\alpha, \beta) = \theta$ if $SUBST(\theta, \alpha) = SUBST(\theta, \beta))$

Resolution in FOL – Example 1

For example,

- $[Animal(F(x)) \vee Loves(G(x), x)]$ and $[\neg Loves(u, v) \vee \neg Kills(u, v)]$
- We could eliminate $Loves(G(x), x)$ and $\neg Loves(u, v)$ with unifier $\theta = \{u/G(x), v/x\}$ to produce the resolvent clause $Animal(F(x)) \vee \neg Kills(G(x), x)$

Resolution in FOL – Example 2

Same example as before:
The law says that it is a crime for an American to sell weapons to hostile nations.
The country Nono, an enemy of America, has some missiles, and all of its missiles were sold to it by Colonel West, who is American.

Prove that Colonel West is a criminal.

Resolution in FOL – Example 2

Convert sentences to FOL:

... it is a crime for an American to sell weapons to hostile nations:

$American(x) \land Weapon(y) \land Sells(x, y, z) \land Hostile(z) \Rightarrow Criminal(x)$

Nono ... has some missiles, i.e., $\exists x\ Owns(Nono, x) \land Missile(x)$:

$Owns(Nono, M_1)$ and $Missile(M_1)$

... all of its missiles were sold to it by Colonel West

$\forall x\ \ Missile(x) \land Owns(Nono, x) \Rightarrow Sells(West, x, Nono)$

Missiles are weapons:

$Missile(x) \Rightarrow Weapon(x)$

An enemy of America counts as "hostile":

$Enemy(x, America) \Rightarrow Hostile(x)$

West, who is American ...

$American(West)$

The country Nono, an enemy of America ...

$Enemy(Nono, America)$

Resolution in FOL – Example 2

Transfer FOL into CNF:

- $American(x) \wedge Weapon(y) \wedge Sells(x,y,z) \wedge Hostile(z) \Rightarrow Criminal(x)$
  - $\neg American(x) \vee \neg Weapon(y) \vee \neg Sells(x,y,z) \vee \neg Hostile(z) \vee Criminal(x)$
- $\forall x \;\; Missile(x) \wedge Owns(Nono,x) \Rightarrow Sells(West,x,Nono)$
  - $\neg Missile(x) \vee \neg Owns(Nono,x) \vee Sells(West,x,Nono)$
- $Missile(x) \Rightarrow Weapon(x)$
  - $\neg Missile(x) \vee Weapon(x)$
- ...

Resolution in FOL – Example 2

Now we have KB:

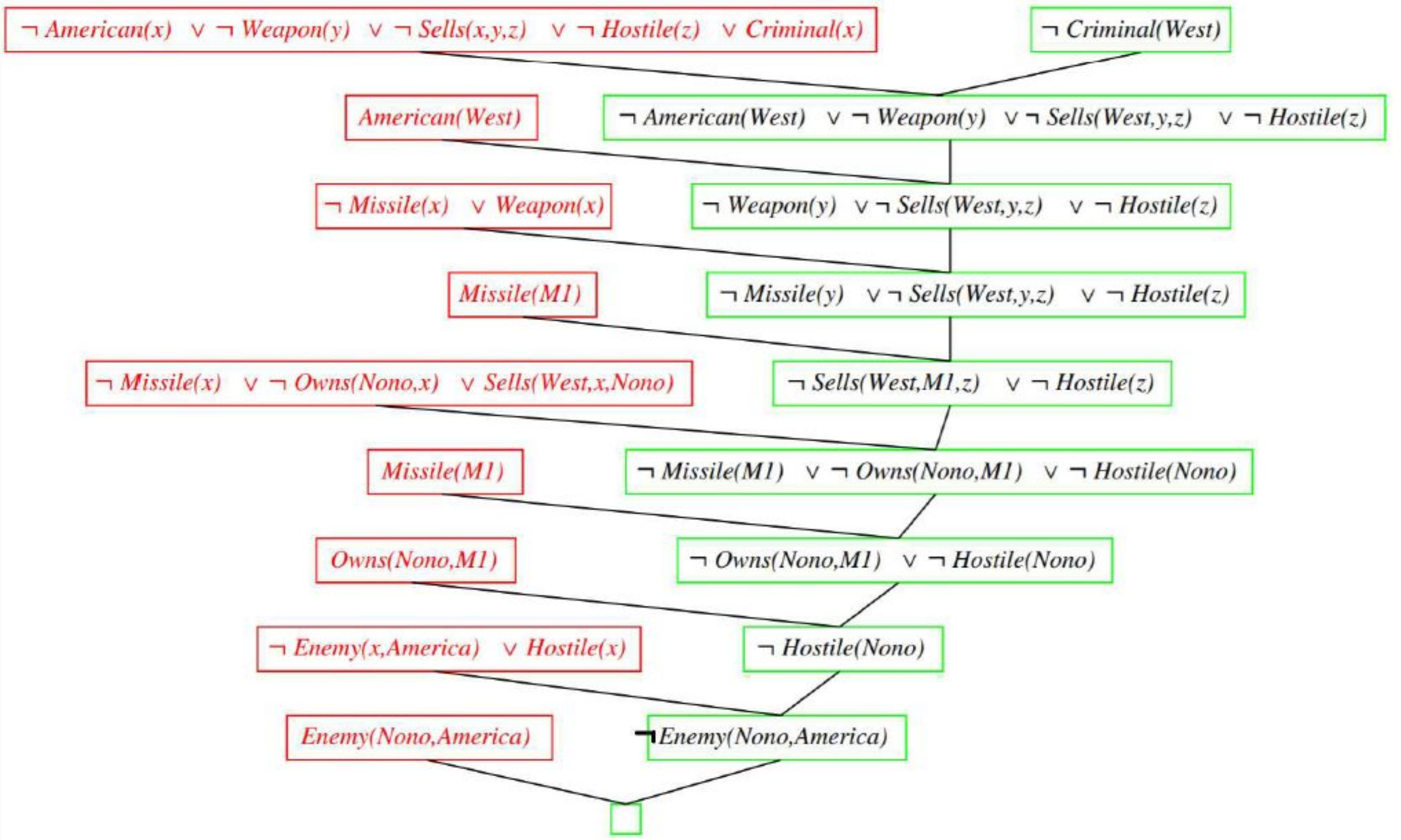
- $\neg American(x) \lor \neg Weapon(y) \lor \neg Sells(x, y, z) \lor \neg Hostile(z) \lor Criminal(x)$
- $\neg Missile(x) \lor \neg Owns(Nono, x) \lor Sells(West, x, Nono)$
- $\neg Enemy(x, America) \lor Hostile(x)$
- $\neg Missile(x) \lor Weapon(x)$
- $Owns(Nono, M_1)$
- $American(West)$
- $Missile(M_1)$
- $Enemy(Nono, America)$

We want to prove $Criminal(West)$

- Apply resolution steps to $CNF(KB \land \neg\alpha)$
- Show $KB \land \neg Criminal(West)$ is unsatisfiable!

Resolution in FOL – Example 2

Resolution Steps:



Summary of Propositional Logic & FOL

- Propositional Logic
  - Inference: *Depth-first enumeration, Deduction Theorem, Modus Ponens, And Introduction, Or Introduction, And Elimination, Resolution*
  - Inference and Proof (of entailment): *Model Checking, Use Inference Rules such as Resolution (proof with refutation + resolution + CNF), SAT solvers (search), tractable NNF circuits*
- First-order Logic
  - Syntax and Semantics: *basic element, sentences, model, KB*
  - Quantifier: $\forall$ *and* $\exists$*, nesting quantifier, duality*
  - Translating Between English Sentences and FOL
- Inference First-order Logic
  - Reduce FOL to PL: instantiation, grounding
  - Definite clause KB: *Forward and Backward Chaining (proof with modus ponens)*
  - Directly apply rules:
    - Unification
    - Resolution: turn text into KB, unification, conversion to CNF, resolution + refutation

Monotonicity

If $\Delta \vDash \alpha$

Then $\Delta \wedge \beta \vDash \alpha$

If

$M(\Delta) \subseteq (\alpha)$

then

By $M(\Delta \wedge \beta) \subseteq M(\alpha)$

$M(\Delta) \cap M(\beta)$

Additional information makes previous conclusions (assumptions) INCORRECT

Problems of Logical Inference

Consider the scenario:

Q: Let action A_t = leave for airport t minutes before flight, will A_t get me there on time?

A: If we use a purely logical approach:

- "A_{25} will get me there on time", may be wrong!
- "A_{25} will get me there on time if there's no accident on the bridge and it doesn't rain and my tires remain intact, etc.",

→ There exists uncertainty! Logical inference can't handle uncertainty!

How to handle uncertainty?

Problems of Logical Inference

Probability helps to handle uncertainty!

Probabilistic assertions summarize effects of

Changing the probability of an existing assertion is a type of Belief Revision

- Laziness: failure to enumerate exceptions, qualifications, etc.
- Ignorance: lack of relevant facts, initial conditions, etc.

- **Subjective** or **Bayesian** probability:
  - Probabilities relate propositions to one's own state of knowledge
    - E.g., P(A25|no reported accidents) = 0.06
  - Probabilities of propositions <u>can change with new evidence</u>:
    - E.g., P(A25|no reported accidents, 5 a.m.) = 0.15

Term and Properties

- Sample Space: $\boldsymbol{\Omega}$
    - E.g. 6 possible rolls of a die (1, 2, 3, 4, 5, 6)
- Sample Point/Possible World/Atomic Event: $\omega \in \boldsymbol{\Omega}$
    - E.g. 1 is a sample point
- Probability Space/Probability Model:
    - A sample space with an assignment $P(\omega)$ for every $\omega \in \boldsymbol{\Omega}$ s.t. $\Sigma P(\omega)=1$ and $0=<P(\omega)<=1$
        - E.g. $P(1) = P(2) = \ldots = P(6) = \frac{1}{6}$
- Event:
    - a subset $A \in \boldsymbol{\Omega}$, $P(A) = \Sigma\_\{\omega \in A\} P(\omega)$
        - E.g. $P(\text{die roll}<4) = P(1)+P(2)+P(3)=\frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{1}{2}$

$$\boxed{Pr(\alpha) = \sum_{\omega \models \alpha} Pr(\omega)}$$

- Random Variable:
    - Variable in probability theory
    - Each random variable has a domain (set of possible values it can take on), and is associate with a probability distribution

Probability Math

- sentence α

$$0 \leq Pr(\alpha) \leq 1$$

- α is inconsistent (unsatisfiable)

$$Pr(\alpha) = \emptyset$$

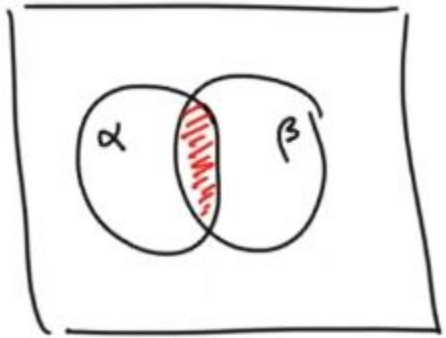
- α is valid

$$Pr(\alpha) = 1$$

- $Pr(\alpha) + Pr(\neg\alpha) = 1$

- $Pr(\alpha \vee \beta) = Pr(\alpha) + Pr(\beta) - Pr(\alpha \wedge \beta)$

- If α, β are mutually exclusive then $Pr(\alpha \vee \beta) = Pr(\alpha) + Pr(\beta)$



$$M(\alpha) \cap M(\beta) = \emptyset$$

Term and Properties

- Prior/Unconditional Probability
  - Belief prior to arrival of any evidence
    - E.g. P(weather = sunny) = 0.8
- Probability distribution
  - Gives values for all possible assignments **for one variable**:
    - E.g. D(Weather) = {0.8, 0.09, 0.1, 0.01} (**normalized**, i.e., sums to 1)
- Joint Probability Distribution
  - Gives the probability of every atomic event **for multiple variables**
    - E.g. D(weather, event) = 4 x 2 matrix of values:

Weather =	sunny	rainy	cloudy	snow
Event = true	0.7	0.04	0.09	0.002
Event = false	0.1	0.05	0.01	0.008

- Conditional/Posterior Probability
  - A measure of the probability of an event occurring given that another event has (by assumption, presumption, assertion or evidence) occurred

Conditional Probability

Conditional Probability:

$$P(a|b) = \frac{P(a \wedge b)}{P(b)} \text{ if } P(b) \neq 0$$

- Product rule:

$$P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$$

- Chain Rule (general product rule):

$$\begin{aligned} \mathbf{P}(X_1, \ldots, X_n) &= \mathbf{P}(X_1, \ldots, X_{n-1})\ \mathbf{P}(X_n | X_1, \ldots, X_{n-1}) \\ &= \mathbf{P}(X_1, \ldots, X_{n-2})\ \mathbf{P}(X_{n-1} | X_1, \ldots, X_{n-2})\ \mathbf{P}(X_n | X_1, \ldots, X_{n-1}) \\ &= \ldots \\ &= \prod_{i=1}^{n} \mathbf{P}(X_i | X_1, \ldots, X_{i-1}) \end{aligned}$$

← Note that:
$P(X_1,\ldots,X_n) = P(X_1 \wedge X_2 \wedge \cdots \wedge X_n)$

Conditional Probability - Example

Given the probability of each world, what's the probability of P(A,B|A)

A	B	P(A,B)
T	T	0.2
T	F	0.1
F	T	0.5
F	F	0.2

A:

A	B	P(A,B | A)
T	T	
T	F	
F	T	
F	F	

Conditional Probability - Example

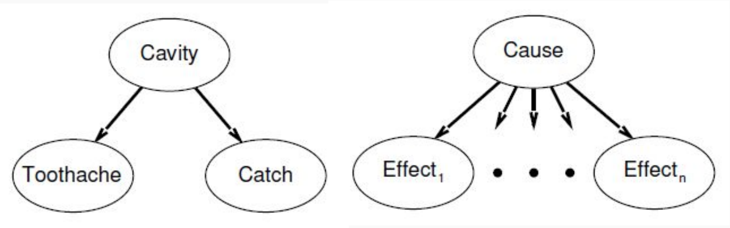
Given the probability of each world, what's the probability of P(A,B|A)

A	B	P(A,B)
T	T	0.2
T	F	0.1
F	T	0.5
F	F	0.2

A:

A	B	P(A,B | A)
T	T	(0.2)/(0.2+0.1)
T	F	(0.1)/(0.2+0.1)
F	T	(0.5)/(0.5+0.2)
F	F	(0.2)/(0.5+0.2)

Independence



Independence:

- Independence: $A \perp B$
  - P(A|B) = P(A), <u>or</u> P(B|A) = P(B), <u>or</u> P(A,B) = P(A)P(B)
  - E.g.

$$\mathbf{P}(Toothache, Catch, Cavity, Weather) \\ = \mathbf{P}(Toothache, Catch, Cavity)\mathbf{P}(Weather)$$

- Conditional independence: $(A \perp B)|C$
  - P(A,B|C) = P(A|C)P(B|C), <u>or</u> P(A|B,C) = P(A|C)
  - This indicates: A and B are independent when the value of C is known and fixed
  - E.g.

$Catch$ is **conditionally independent** of $Toothache$ given $Cavity$:

$$\mathbf{P}(Catch|Toothache, Cavity) = \mathbf{P}(Catch|Cavity)$$

Pick catches on tooth

$$\begin{aligned} &\mathbf{P}(Toothache, Catch, Cavity) \\ &= \mathbf{P}(Toothache|Catch, Cavity)\mathbf{P}(Catch, Cavity) \\ &= \mathbf{P}(Toothache|Catch, Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity) \\ &= \mathbf{P}(Toothache|Cavity)\mathbf{P}(Catch|Cavity)\mathbf{P}(Cavity) \end{aligned}$$

Case Analysis

Case analysis (law of total probability)

$$\Pr(\alpha) = \sum_{i=1}^{n} \Pr(\alpha \land \beta_i)$$

where $\beta_1, \beta_2, \ldots \beta_n$ are mutually exclusive and exhaustive

$\boxed{\beta, \neg\beta}$ $\Pr(\alpha) = \Pr(\alpha \land \beta) + \Pr(\alpha \land \neg\beta)$

Probability Inference – Bayes Rule

Probability inference: inference the probability of one event

How to do it?

- Inference by enumeration
- Inference rule:
    - Bayes' Rule:

probability of **cause** given **effect**, not the other way around

Bayes' Rule can be used in probablity inference when we have $P(b|a)$ but not $P(a|b)$.

Product rule $P(a \wedge b) = P(a|b)P(b) = P(b|a)P(a)$

$$\Rightarrow \textbf{Bayes' rule } P(a|b) = \frac{P(b|a)P(a)}{P(b)}$$

or in distribution form

$$\mathbf{P}(Y|X) = \frac{\mathbf{P}(X|Y)\mathbf{P}(Y)}{\mathbf{P}(X)} = \alpha \mathbf{P}(X|Y)\mathbf{P}(Y)$$

Bayes Rule

Useful for assessing **diagnostic** probability from **causal** probability:

$$P(Cause|Effect) = \frac{P(Effect|Cause)P(Cause)}{P(Effect)}$$

- **Naïve Bayes** model:

$$\mathbf{P}(Cause, Effect_1, \ldots, Effect_n) = \mathbf{P}(Cause)\prod_i \mathbf{P}(Effect_i|Cause)$$

Example:

$$\begin{aligned}
&\mathbf{P}(Cavity|toothache \wedge catch) \\
&= \alpha\,\mathbf{P}(toothache \wedge catch|Cavity)\mathbf{P}(Cavity) \\
&= \alpha\,\mathbf{P}(toothache|Cavity)\mathbf{P}(catch|Cavity)\mathbf{P}(Cavity)
\end{aligned}$$