COM SCI 260B HW 2 Solution

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Problem 1. Mistake Bounded Model

Solution 1. We take all indices initially in our S_0 and then whenever we make mistake we remove the indices where x_i was zero and in case of no mistake we keep the subset as it is. Following is the pseudocode for the algorithm:

Algorithm 1 Algorithm for mistake bounded model

```
S_0 ← {1, 2, ....d}
for day i: 1,2.... do
    if no mistake then
        S_i ← S_{i-1}
    else
        S_i ← S_{i-1} − {set of indices where x_i is zero}
    end if
end for
```

Run-time: When we make a mistake, we need to check indices where x_i is zero, this can be done trivially in $O(d)$ time. Thus the algorithm takes $O(d)$ time for each example.

Number of Mistakes: Let $W(i)$ denote the length of set S_i. Then $W(0) = d$ and $W(i) \leq W(i-1)$. When we make mistake at day t, we have $W(t) \leq W(t-1) - 1$.
Now, since S_* is non-empty, we have $W(i) > 0$, If M_i is the number of mistakes,

$$0 < W(i) \leq W(i-1)$$

$$0 < W(i) \leq W(0) - M_i$$

$$0 < d - M_i$$

$$M_i < d$$

So the algorithm will make atmost $d-1$ mistakes.

Problem 2. Learning with experts

Solution 2. Consider the base situation and general situation below:

2 experts: Consider the situation with two experts, when they always gives different prediction and no matter what we predict, we (and one of the expert) will get a loss. In this scenario, after t days, we will have loss t, while the best expert will have loss atmost $t/2$. This is because each day only one of the expert is getting a loss while, we are always incurring loss.

2*n experts: Consider the general situation with $2 * n$ experts, we can divide them in two groups Group A and Group B. All experts in their respective groups give same prediction. Now we can treat these groups as 2 experts situation above. We will again get that after t days, we will have loss t, while the best experts will have loss atmost $t/2$.

Thus, with a deterministic algorithm our loss is always worse than a factor two approximation to the loss of the best expert.

Problem 3. Randomized algorithm

Solution 3. We will make a small change in the MWM defined in class with $L(t,i)$ being the difference in prediction and actual value rather than {0,1}. Following is the pseudocode for updated algorithm:

Algorithm 2 Algorithm for updated MWM

$w(0,i) = 1$ for $i : 1, 2...n$
for each day t: 1,2....T **do**
  Pick and expert i with probability $\propto w(t-1,i)$
  Pr[expert i is picked] $= \dfrac{w(t-1,i)}{\sum_{j=1}^{n} w(t-1,j)}$
  Follow expert i
  $value_t \leftarrow$ true value
  for each expert $j : 1, 2, ...n$ **do**
    $L(t,j) \leftarrow |pred(t,j) - value_t|$
    $w(t,j) \leftarrow (1-\epsilon)^{L(t,j)} w(t-1,j)$
  end for
end for

$$L(t) = \mathbb{E}[\text{loss on day t}]$$

$$L(t) = \sum_{i=1}^{n} Pr[\text{pick expert i}] L(t,i)$$

$$L(t) * W(t-1) = \sum_{i=1}^{n} w(t-1,i) * L(t,i)$$

Where, $W(t)$ is defined as:

$$W(t) = \sum_{i=1}^{n} w(t,i)$$

$$W(t) = \sum_{i=1}^{n} w(t-1,i) * (1-\epsilon)^{L(t,i)}$$

Using binomial expansion, $(1-a)^x \leq (1-ax)$ for $0 < x < 1$

$$W(t) \leq \sum_{i=1}^{n} w(t-1,i) * (1 - \epsilon * L(t,i))$$

$$W(t) \leq W(t-1) * (1 - \epsilon * L(t))$$

Using $1 - x \leq e^{-x}$,

$$W(t) \leq W(t-1) * e^{-\epsilon L(t)}$$

Therefore,

$$W(T) \leq W(0) * e^{-\epsilon L(1)} * e^{-\epsilon L(2)} ..... * e^{-\epsilon L(T)}$$

$$W(T) \leq n * e^{-\epsilon A(T)}$$

Also, we have,

$$W(T) \geq (1-\epsilon)^{A_*(T)}$$

on combining above two equations, we get,

$$(1-\epsilon)^{A_*(T)} \leq n * e^{-\epsilon A(T)}$$

$$A_*(T) \log(1-\epsilon) \leq \log n - \epsilon A(T)$$

$$A(T) \leq \frac{-\log(1-\epsilon)}{\epsilon} A_*(T) + \frac{\log n}{\epsilon}$$

For $\epsilon < 1/2$, $\dfrac{-\log(1-\epsilon)}{\epsilon} \leq (1+\epsilon)$

$$A(T) \leq (1+\epsilon) A_*(T) + \frac{\log n}{\epsilon}$$

For $1/2 < \epsilon < 1$, we can use $\delta = \epsilon/2$, on doing the same derivation, we would get,

$$A(T) \leq (1+\delta) A_*(T) + \frac{\log n}{\delta}$$

$$A(T) \leq (1+\epsilon/2) A_*(T) + \frac{2\log n}{\epsilon}$$

$$A(T) \leq (1+\epsilon) A_*(T) + \frac{2\log n}{\epsilon}$$

Problem 4. MWM

Solution 4. We define the loss function as $L(t, i) = (-y_t * x(t, i))$. It is only added if the prediction from $w(t-1)$ was wrong. We can view each coordinate of x as giving us an estimate of what the real sign is. So now if y matches the sign, then we are rewarding (giving less loss to) that coordinate and penalizing (giving more loss to) those that don't match.

Now, using this loss we can perform MWM, using regret bound equation from MWM we get,

$$Regret \leq 2 * \sqrt{T \log d} \tag{1}$$

Now for getting regret we compute following,

$$\mathbb{E}[L(T)] = \sum_{t=1}^{T} \sum_{i=1}^{d} w(t, i)(-y_t * x(t, i))$$

$$\mathbb{E}[L(T)] = \sum_{t=1}^{T} -y_t * \langle w(t), x(t) \rangle$$

Since loss is only computed when the prediction is opposite to y_t, the above will always be positive

$$\mathbb{E}[L(T)] \geq 0 \tag{2}$$

The loss of the best expert $L_*(T)$ can be written as follows:

$$L_*(T) = min_i \sum_t -y_t * x(t, i)$$

Since w_* can be taken as a distribution over experts, we can write, minimum to be atmost the average,

$$L_*(T) \leq \sum_i w_i^* \sum_t -y_t * x(t, i)$$

$$L_*(T) \leq -\sum_t y_t \sum_i w_i^* * x(t, i)$$

$$L_*(T) \leq -\sum_t y_t \langle w^*, x(t) \rangle$$

$$L_*(T) \leq -\sum_t \gamma$$

Let M_T be the number of mistakes, then

$$L_*(T) \leq -\gamma M_T \tag{3}$$

Substituting equation (2) and (3) in (1), using $Regret = \mathbb{E}[L(T)] - L_*(T)$, we get,

$$0 - (-\gamma M_T) \leq 2 * \sqrt{T \log d}$$

$$\gamma M_T \leq 2 * \sqrt{T \log d}$$

$$M_T \leq \frac{2 * \sqrt{T \log d}}{\gamma}$$

Thus the number of mistakes after T days is $O\left(\frac{\sqrt{T \log d}}{\gamma}\right)$