Maths for Donut.c

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1 Introduction

Before anything I would advise checking https://www.a1k0n.net/2011/07/20/donut-math.html to understand better this article.

Let's note the rotation matrices useful for the following:

$$R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$

$$R_x(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos A & \sin A \\ 0 & -\sin A & \cos A \end{pmatrix}$$

$$R_z(B) = \begin{pmatrix} \cos B & \sin B & 0 \\ -\sin B & \cos B & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A torus is defined by two radius and two rotation angles:

$$R_2 = \text{distance from the tube center to the torus center}, \quad R_1 = \text{tube radius}.$$

A point on the torus in 3D coordinates (before rotations by A and B) is given by (cf. A1kon website, and verify the calculations yourself with $R_y(\phi)$):

$$(x, y, z) = \big((R_2 + R_1\cos\theta)\cos\phi, R_1\sin\theta,\ (R_2 + R_1\cos\theta)\sin\phi\ \big).$$

where:
- θ traverses the circular section of the tube, - φ goes around the torus.

Then, to animate it, we apply two successive rotations: around the z axis (angle B) and rotation around the x axis (angle A).

The final position of a point of the torus after 3D transformation is obtained by applying these two rotations to (x, y, z) (in the following order):

$$R_y(\phi)R_x(A)R_z(B)$$

$$\begin{bmatrix} x1 \\ y1 \\ z1 \end{bmatrix} = \begin{pmatrix} (R_2 + R_1\cos\theta)\cos\phi\cos B + \big(R_1\sin\theta\cos A + (R_2 + R_1\cos\theta)\sin\phi\sin A\big)\sin B \\ -(R_2 + R_1\cos\theta)\cos\phi\sin B + \big(R_1\sin\theta\cos A + (R_2 + R_1\cos\theta)\sin\phi\sin A\big)\cos B \\ (R_2 + R_1\cos\theta)\sin\phi\cos A - R_1\sin\theta\sin A \end{pmatrix}$$

Rigorous demonstration of the torus normal

We are going to determine the normal to the torus.

1. Torus parametrization

As said before, a torus is parameterized by two angles: Without rotation, the 3D coordinates of a point of the torus are:

$$r(\theta,\phi) = \big((R_2 + R_1\cos\theta)\cos\phi, R_1\sin\theta,\ (R_2 + R_1\cos\theta)\sin\phi\ \big)$$

2. Tangent vectors of the surface

The surface is parameterized in (θ,ϕ). Its tangent vectors are:

$$\frac{\partial\vec{r}}{\partial\theta} = \begin{pmatrix} -R_1\sin\theta\cos\phi \\ R_1\cos\theta \\ -R_1\sin\theta\sin\phi \end{pmatrix}, \qquad \frac{\partial\vec{r}}{\partial\phi} = \begin{pmatrix} -(R_2 + R_1\cos\theta)\sin\phi \\ 0 \\ (R_2 + R_1\cos\theta)\cos\phi \end{pmatrix}.$$

3. Normal = cross product

The normal is given by:

$$\vec{N} = \frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\phi}$$

Which is:

$$\vec{N} = R_1(R_2 + R_1\cos\theta)\big(\cos\theta\cos\phi,\ \sin\theta,\ \cos\theta\sin\phi\big)$$

The normal vectors are proportional to $\vec{N}$, that is:

$$\big(\cos\theta\cos\phi,\ \sin\theta,\ \cos\theta\sin\phi\big) \in \mathrm{Vect}(\vec{N})$$

Dot product of the normal with light

The normal vector after torus parametrization (before rotation) is therefore:

$$\tilde{\mathbf{N}}_0 = \big(\cos\theta\cos\phi,\ \sin\theta,\ \cos\theta\sin\phi\big)$$

Then after rotation:

$$\tilde{\mathbf{N}} = \tilde{\mathbf{N}}_0\, R_z(B) R_x(A)$$

$$\tilde{\mathbf{N}} = \begin{pmatrix} \cos\theta\cos\phi\cos B - \sin\theta\sin B \\ \cos A(\cos\theta\cos\phi\sin B + \sin\theta\cos B) - \cos\theta\sin\phi\sin A \\ \sin A(\cos\theta\cos\phi\sin B + \sin\theta\cos B) + \cos\theta\sin\phi\cos A \end{pmatrix}$$

The light is directed by:

$$\vec{L} = (0, 1, -1)$$

The dot product gives the luminance Lu:

$$\begin{aligned} Lu &= \tilde{\mathbf{N}} \cdot \vec{L} \\ &= (\cos A - \sin A)(\cos\theta\cos\phi\sin B + \sin\theta\cos B) - \cos\theta\sin\phi(\cos A + \sin A) \end{aligned}$$