

Maths for Donut.c

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1 Introduction

Before anything I would advise checking <https://www.alk0n.net/2011/07/20/donut-math.html> to understand better this article.

Let's note the rotation matrices useful for the following:

$$R_y(\phi) = \begin{pmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{pmatrix}$$

$$R_x(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos A & \sin A \\ 0 & -\sin A & \cos A \end{pmatrix}$$

$$R_z(B) = \begin{pmatrix} \cos B & \sin B & 0 \\ -\sin B & \cos B & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A torus is defined by two radii and two rotation angles:

R_2 = distance from the tube center to the torus center, R_1 = tube radius.

A point of the torus in 3D coordinates (before rotations via A and B) is given by:

$$(x, y, z) = ((R_2 + R_1 \cos \theta) \cos \phi, R_1 \sin \theta, (R_2 + R_1 \cos \theta) \sin \phi).$$

where:

- θ traverses the circular section of the tube, - ϕ goes around the torus.

Then, to animate it, we apply two successive rotations: around the z axis (angle B) and rotation around the x axis (angle A).

The final position of a point of the torus after 3D transformation is obtained by applying these two rotations to (x, y, z) (in the following order):

$$R_y(\phi)R_x(A)R_z(B)$$

$$\begin{bmatrix} x1 \\ y1 \\ z1 \end{bmatrix} = \begin{pmatrix} (R_2 + R_1 \cos \theta) \cos \phi \cos B + (R_1 \sin \theta \cos A + (R_2 + R_1 \cos \theta) \sin \phi \sin A) \sin B \\ -(R_2 + R_1 \cos \theta) \cos \phi \sin B + (R_1 \sin \theta \cos A + (R_2 + R_1 \cos \theta) \sin \phi \sin A) \cos B \\ (R_2 + R_1 \cos \theta) \sin \phi \cos A - R_1 \sin \theta \sin A \end{pmatrix}$$

Rigorous demonstration of the torus normal

We are going to determine the normal to the torus.

1. Torus parametrization

As said before, a torus is parameterized by two angles: Without rotation, the 3D coordinates of a point of the torus are:

$$r(\theta, \phi) = ((R_2 + R_1 \cos \theta) \cos \phi, R_1 \sin \theta, (R_2 + R_1 \cos \theta) \sin \phi)$$

2. Tangent vectors of the surface

The surface is parameterized in (θ, ϕ) . Its tangent vectors are:

$$\frac{\partial \vec{r}}{\partial \theta} = \begin{pmatrix} -R_1 \sin \theta \cos \phi \\ R_1 \cos \theta \\ -R_1 \sin \theta \sin \phi \end{pmatrix}, \quad \frac{\partial \vec{r}}{\partial \phi} = \begin{pmatrix} -(R_2 + R_1 \cos \theta) \sin \phi \\ 0 \\ (R_2 + R_1 \cos \theta) \cos \phi \end{pmatrix}.$$

3. Normal = cross product

The normal is given by:

$$\vec{N} = \frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \phi}$$

Which is:

$$\vec{N} = R_1(R_2 + R_1 \cos \theta)(\cos \theta \cos \phi, \sin \theta, \cos \theta \sin \phi)$$

The normal vectors are proportional to \vec{N} , that is:

$$(\cos \theta \cos \phi, \sin \theta, \cos \theta \sin \phi) \in \text{Vect}(\vec{N})$$

Dot product of the normal with light

The normal vector after torus parametrization (before rotation) is therefore:

$$\tilde{\mathbf{N}}_0 = (\cos \theta \cos \phi, \sin \theta, \cos \theta \sin \phi)$$

Then after rotation:

$$\tilde{\mathbf{N}} = \tilde{\mathbf{N}}_0 R_z(B) R_x(A)$$

$$\tilde{\mathbf{N}} = \begin{pmatrix} \cos \theta \cos \phi \cos B - \sin \theta \sin B \\ \cos A(\cos \theta \cos \phi \sin B + \sin \theta \cos B) - \cos \theta \sin \phi \sin A \\ \sin A(\cos \theta \cos \phi \sin B + \sin \theta \cos B) + \cos \theta \sin \phi \cos A \end{pmatrix}$$

The light is directed by:

$$\vec{L} = (0, 1, -1)$$

The dot product gives the luminance Lu :

$$\begin{aligned} Lu &= \tilde{\mathbf{N}} \cdot \vec{L} \\ &= (\cos A - \sin A)(\cos \theta \cos \phi \sin B + \sin \theta \cos B) - \cos \theta \sin \phi (\cos A + \sin A) \end{aligned}$$