Math 33A Week 9
November 22, 2016

Constrained Optimization of Quadratic Forms

One of the most important applications of mathematics is **optimization**, and you have some experience with this from calculus. In these notes we're going to use some of our knowledge of quadratic forms to give linear-algebraic solutions to some optimization problems.

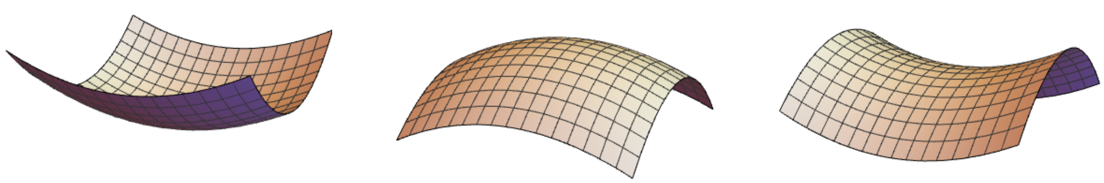
First let's suppose we have a quadratic form $f(x_1, x_2)$:

$$f(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2.$$

We can of course represent f by a symmetric matrix Q:

$$f(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

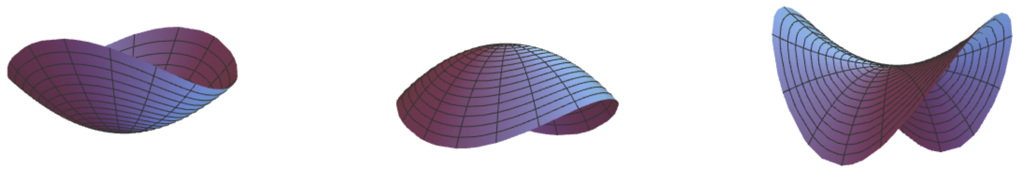
The graph $z = f(x, y)$ of f will then take one of three general shapes, depending on the definite-ness of the matrix Q. If Q is a positive-definite matrix, this graph will take the shape of an upward-opening paraboloid, as in Figure 1a. In this case, f will admit a minimum at $(0, 0)$. On the other hand, Q could be negative-definite, leaving the graph of f to be a downward-opening paraboloid, as in Figure 1b, and showing us that f has a maximum at $(0, 0)$. Finally, if Q is indefinite, the graph of f will have a saddle point at $(0, 0)$, decreasing in one direction and increasing in the other.



(a) A positive-definite form. (b) A negative-definite form. (c) An indefinite form.

Figure 1: Plots of quadratic forms.

Optimizing these quadratic forms isn't very interesting at this point; a positive definite form will have a global minimum at $(0, 0)$ and grow without bound, while a negative definite will have a global maximum at $(0, 0)$. A more interesting question would be *constrained* optimization, where we limit the behavior of our inputs x and y. For instance, we might demand that the point (x, y) lie inside some circle, and maximize $f(x, y)$ subject to this constraint. Examples of this can be seen in Figure 2, where we see that a quadratic form can have interesting behavior along the bounding circle.



(a) A positive-definite form. (b) A negative-definite form. (c) An indefinite form.

Figure 2: Constrained plots of quadratic forms.

Let's consider the problem of maximizing $f(x_1, x_2) = \vec{x}^T Q \vec{x}$ subject to the condition $x_1^2 + x_2^2 = 1$. That is, we want to maximize f along the unit circle in $\mathbf{R}^2$. On its face this seems to be a relatively difficult problem, but it will become much easier with the following fact in hand:

> **Fact.** Any symmetric matrix A is orthogonally diagonalizable. That is, there is a diagonal matrix S and a diagonal matrix D so that
>
> $$A = SDS^{-1},$$
>
> provided $A = A^T$.

Since the matrix Q representing our quadratic form is symmetric, we may diagonalize it orthogonally:

$$Q = SDS^{-1}.$$

Now recall that the columns of the diagonalizing matrix S are eigenvectors for Q, with their associated eigenvalues listed on the diagonal of D. Because the matrix S is orthogonal, these vectors in fact form an orthonormal basis for $\mathbf{R}^2$. That is, they're both of unit length (and thus lie on the unit circle), and they're orthogonal to each other. We call these vectors $\vec{u}_1$ and $\vec{u}_2$, with associated eigenvalues λ_1 and λ_2:

$$A\vec{u}_1 = \lambda_1 \vec{u}_1 \qquad \text{and} \qquad A\vec{u}_2 = \lambda_2 \vec{u}_2.$$

We also make the simplifying assumption that $\lambda_1 \geq \lambda_2$. Now consider the values of $f(\vec{u}_1)$ and $f(\vec{u}_2)$. We have

$$f(\vec{u}_i) = \vec{u}_i^T Q \vec{u}_i = \vec{u}_i^T (\lambda_i \vec{u}_i) = \lambda_i \vec{u}_i^T \vec{u}_i = \lambda_i$$

for $i = 1, 2$. Next we choose any vector $\vec{x} = (x_1, x_2)^T$ on the unit circle and consider $f(x_1, x_2) = \vec{x}^T Q \vec{x}$. Since $(\vec{u}_1, \vec{u}_2)$ forms an eigenbasis for $\mathbf{R}^2$ we may choose $\alpha, \beta \in \mathbf{R}$ so that

$$\vec{x} = \alpha \vec{u}_1 + \beta \vec{u}_2.$$

Then

$$\begin{aligned} f(x_1, x_2) &= \vec{x}^T Q \vec{x} = (\alpha \vec{u}_1 + \beta \vec{u}_2)^T Q (\alpha \vec{u}_1 + \beta \vec{u}_2) \\ &= \alpha^2 \vec{u}_1^T Q \vec{u}_1 + \alpha\beta \vec{u}_1^T Q \vec{u}_2 + \alpha\beta \vec{u}_2^T Q \vec{u}_1 + \beta^2 \vec{u}_2^T Q \vec{u}_2 \end{aligned}$$

$$
\begin{aligned}
&= \alpha^2\lambda_1 + \alpha\beta(\vec{u}_1^T(\lambda_2\vec{u}_2) + \vec{u}_2^T(\lambda_1\vec{u}_1)) + \beta^2\lambda_2 \\
&= \alpha^2\lambda_1 + \beta^2\lambda_2.
\end{aligned}
$$

The term

$$\alpha\beta(\vec{u}_1^T(\lambda_2\vec{u}_2) + \vec{u}_2^T(\lambda_1\vec{u}_1))$$

vanishes because of the orthogonality of $\vec{u}_1$ with $\vec{u}_2$. Now since $\vec{x}$ lies on the unit circle and $\vec{u}_1$ and $\vec{u}_2$ are orthonormal, we have

$$1 = \|\vec{x}\| = \alpha^2 + \beta^2.$$

This means that $f(x_1, x_2) = \alpha^2\lambda_1 + \beta^2\lambda_2$ lies between λ_1 and λ_2; since $\lambda_1 \geq \lambda_2$, we can maximize $f(x_1, x_2) = \alpha^2\lambda_1 + \beta^2\lambda_2$ by letting $\alpha^2 = 1$ and $\beta^2 = 0$. That is, $\vec{x} = \pm\vec{u}_1$ and $f(\vec{x}) = \lambda_1$. We can minimize this value by letting $\alpha^2 = 0$ and $\beta^2 = 1$, which gives us $\vec{x} = \pm\vec{u}_2$ and $f(\vec{x}) = \lambda_2$.

Example 1. Maximize and minimize the product $f(x, y) = xy$ subject to the condition that $x^2 + y^2 = 1$.

(*Solution*) Notice that this is a problem you've likely seen (at least half of) before: we're finding the largest rectangle that can be inscribed in the unit circle. For $x, y > 0$, the product xy gives the area of the rectangle with opposite corners at $(0, 0)$ and (x, y), and you may remember that this area is maximized when $x = y$, meaning that $x = 1/\sqrt{2}$ and $y = 1/\sqrt{2}$. We'll verify this now with our linear algebraic approach. First, we can represent f by a 2×2 matrix Q:

$$f(x, y) = xy = \begin{pmatrix} x \\ y \end{pmatrix}^T \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

According to our above work, the maximum and minimum values achieved by f are given by the eigenvalues of Q, so we compute

$$\det(Q - \lambda I) = \left| \begin{pmatrix} -\lambda & 1/2 \\ 1/2 & -\lambda \end{pmatrix} \right| = \lambda^2 - 1/4 = (\lambda - 1/2)(\lambda + 1/2).$$

So Q has two eigenvalues: $1/2$ and $-1/2$. We then have

$$Q - \frac{1}{2}I = \begin{pmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix},$$

so an eigenvector with associated eigenvalue $1/2$ is given by $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$. Similarly,

$$Q + \frac{1}{2}I = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

gives an eigenvector $\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}^T$ for the eigenvalue $\lambda = -1/2$. So $f(x, y)$ achieves its maximum value of $1/2$ when $x = 1/\sqrt{2}$ and $y = 1/\sqrt{2}$ (or when $x = -1/\sqrt{2}$ and $y = -1/\sqrt{2}$).

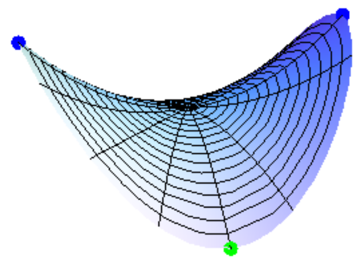


Figure 3: The extrema of $f(x, y)$ subject to the restriction $x^2 + y^2 = 1$.

On the other hand, f achieves its minimum value of $-1/2$ when $x = 1/\sqrt{2}$ and $y = -1/\sqrt{2}$ and also when $x = -1/\sqrt{2}$ and $y = 1/\sqrt{2}$. The extrema of this constrained optimization problem can be seen in Figure 3 (where the maxima are marked by the blue points and the minima by the green points). $\Diamond$

Next we consider the problem of maximizing $f(x_1, x_2) = \vec{x}^T Q\vec{x}$ subject to the slightly more exotic condition

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = 1,$$

for some constants $a, b > 0$. As before, we could diagonalize Q orthogonally to obtain orthonormal eigenvectors $\vec{u}_1$ and $\vec{u}_2$, but this won't be of much use to us because neither of these vectors necessarily satisfies our condition, and if we scale the vectors then we no longer have the nice algebraic properties that orthonormal vectors give us. Something we can do instead is to introduce a new variable — dependent on $\vec{x}$ — for which our condition will match our previous, easier condition. For instance, if we have x_1, x_2 satisfying

$$\left(\frac{x_1}{a}\right)^2 + \left(\frac{x_2}{b}\right)^2 = 1,$$

then the new variables $y_1 = x_1/a$ and $y_2 = x_2/b$ will satisfy

$$y_1^2 + y_2^2 = 1.$$

For this reason we introduce a change-of-variables matrix

$$P = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

and introduce the new variable

$$\vec{y} = P^{-1}\vec{x},$$

so that $\vec{x} = P\vec{y}$. (The matrix P is invertible, since $a, b > 0$.) Having made this change, our problem turns from maximizing $\vec{x}^T Q\vec{x}$ to maximizing $(P\vec{y})^T Q(P\vec{y})$. Notice that

$$(P^T QP)^T = P^T Q^T P = PQP$$

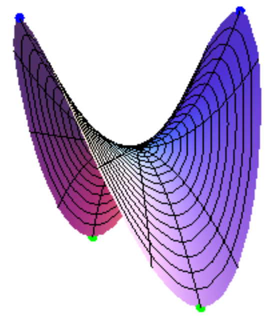


Figure 4: The extrema of $f(x, y)$ subject to the restriction $x^2 + 4y^2 = 16$.

so we want to maximize the quadratic form

$$\vec{y}^T PQP\vec{y}$$

subject to the condition $y_1^2 + y_2^2 = 1$ on $\vec{y}$. But we already know how to do this — we simply find the greatest eigenvalue of PQP. The associated eigenvector $\vec{u}_1$ will then be the input that allows us to achieve this maximum, so we let $\vec{y} = \vec{u}_1$. The maximizing input into our original function f will then be given by $\vec{x} = P\vec{y} = P\vec{u}_1$.

Example 2. Maximize and minimize the function $f(x_1, x_2) = x_1x_2$ subject to the condition that

$$\left(\frac{x_1}{4}\right)^2 + \left(\frac{x_2}{2}\right)^2 = 1.$$

(*Solution*) First we define $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and notice that

$$f(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \vec{x}^T Q\vec{x}.$$

Because our condition on $\vec{x}$ doesn't match the simpler condition we saw before, we introduce a change of variables, as suggested. First we define the matrix

$$P = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix},$$

and then we set $\vec{x} = P\vec{y}$. We now want to maximize

$$\vec{x}^T Q\vec{x} = (P\vec{y})^T Q(P\vec{y}) = \vec{y}^T PQP\vec{y}$$

subject to the condition that $y_1^2 + y_2^2 = 1$, so we'll need to compute the eigenvalues of

$$PQP = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 8 \\ 8 & 0 \end{pmatrix}.$$

The characteristic polynomial for this matrix is given by $\lambda^2 - 64 = (\lambda - 8)(\lambda + 8)$, so the eigenvalues are $\lambda_1 = 8$ and $\lambda_2 = -8$, respectively. We quickly compute

$$PQP - 8I = \begin{pmatrix} -8 & 8 \\ 8 & -8 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad PQP - (-8)I = \begin{pmatrix} 8 & 8 \\ 8 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

From here we see that

$$\vec{u}_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \qquad \text{and} \qquad \vec{u}_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

are eigenvectors of PQP with associated eigenvalues $\lambda_1 = 8$ and $\lambda_2 = -8$, respectively. So $\vec{y}^T PQP\vec{y}$ is maximized when $\vec{y} = \vec{u}_1$ and minimized when $\vec{y} = \vec{u}_2$. We then compute

$$\vec{x}_1 = P\vec{u}_1 = \begin{pmatrix} 2\sqrt{2} \\ \sqrt{2} \end{pmatrix} \qquad \text{and} \qquad \vec{x}_2 = P\vec{u}_2 = \begin{pmatrix} 2\sqrt{2} \\ -\sqrt{2} \end{pmatrix}.$$

These are the points where f will achieve its maximum value of $f(\vec{x}_1) = 4$ and minimum value of $f(\vec{x}_2) = -4$, respectively, subject to the given condition. Notice that f will also achieve its maximum at $-\vec{x}_1 = P(-\vec{u}_1)$ and its minimum at $-\vec{x}_2 = P(-\vec{u}_2)$, as can be seen in Figure 4. $\diamond$