Dynamic prefix-sums

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Web page: `https://github.com/rossanoventurini/CompetitiveProgramming`

These notes sketch the content of the lectures on static and dynamic prefix sums. These are rough and non-exhaustive notes that I used while lecturing. Please use them just to have a list of topics of each lecture and use the reported references to study these arguments.

1 Static prefix-sums

▶ **Problem 1** (Range sum). *Given an array $A[1, n]$ of integers, we would like to support the query* Sum(i)*, which returns the sum of the elements in $A[1, \ldots, i]$.*

The array does not change so the problem is trivial. Simply precompute the answers to all the possible (only n) queries. The operation Sum can be used to answer RangeSum queries. Given two indexes i and j, RangeSum(i, j) returns the sum of the elements in $A[i \ldots j]$. The latter query is very useful to solve other problems. The next one is an easy example.

▶ **Problem 2** (Ilya and Queries). *Given a string $s = s_1 s_2 \ldots s_n$ consisting only of characters a and b and m queries. Each query is described by a pair of integers l, r $(\leq l < r \leq n)$. The answer to the query l, r is the number of such integers $i \in [l, r]$ that $s_i = s_{i+1}$.*

The idea is that of computing the binary vector $B[1, n-1]$ such that $B[i] = 1$ if $s_i = s_{i+1}$, 0 otherwise. This way, the answer to the query l, r equals $\sum_{i=l}^{r} B[i]$. Thus, the problem is solved efficiently (constant time per query) by computing prefix-sums on vector B.

▶ **Problem 3** (Number of Ways). *Given an array $A[1, n]$, count the number of ways to split all the elements of the array into three contiguous parts so that the sum of elements in each part is the same.*

More formally, you need to find the number of such pairs of indices i and j $(2 \leq i \leq j \leq n-1)$ such that $\sum_{k=1}^{i-1} A[k] = \sum_{k=i}^{j} A[j] = \sum_{k=j+1}^{n} A[k]$

Let S be the sum of the values in the array. If 3 does not divide S, we conclude that the number of ways is zero. Otherwise, we compute the array C which stores in position i the number suffixes of $A[i \ldots n]$ that sum to $\frac{S}{3}$. We then compute the sum of the prefixes of A. Every time a prefix i sums to $\frac{S}{3}$, we add $C[i+2]$ to the result.

▶ **Problem 4** (Little Girl and Maximum). *We are given an array $A[1, n]$ and a set Q of q queries. Each query is a range sum query i, j which returns the sum of elements in $A[i \ldots j]$.*

The goal is to permute the element in A in order to maximize the sum of the results of the queries in Q.

We need to sort the entries of the array by their access frequencies and then assign the largest values to the most frequently accessed slots. We could use an array to keep track of the frequencies of the entries. However, let S be the sum of the length of queries size, just updating the above array would cost $\Theta(S)$. Note that S may be $\Theta(qn)$, which may be even worse than quadratic in n.

Thus, we need a cleaver way to compute those frequencies. The idea is to construct an array $F[1 \dots n]$, initially all the entries are set to 0. If we read the query $\langle l_i, r_i \rangle$, we add 1 to $F[l_i]$ and we subtract -1 to $F[r_i + 1]$. The prefix sum of F up to i equals the frequency of entry i. This algorithm costs only $O(q + n)$ time.

2 Dynamic prefix-sums: Binary indexed tree (BIT)

▶ **Problem 5** (Dynamic prefix-sums). *Given an array $A[1, n]$ of integers, we would like to support the following queries*

- $\mathsf{Sum}(i)$ *that returns the sum of the elements in $A[1, \dots, i]$;*
- $\mathsf{Add}(i, v)$ *that adds the value v to the entry $A[i]$.*

Binary indexed tree (BIT) (also known as Fenwick tree) solves the above queries in $\Theta(\log n)$ time by using linear space. Actually, BIT is an implicit data structure, which means that it uses only $O(1)$ space in addition to the space required to store the input data (the array A in our case).

Consider the following array A (note we are using one-based array).

	1	2	3	4	5	6	7	8
A	3	2	-1	5	7	-3	2	1

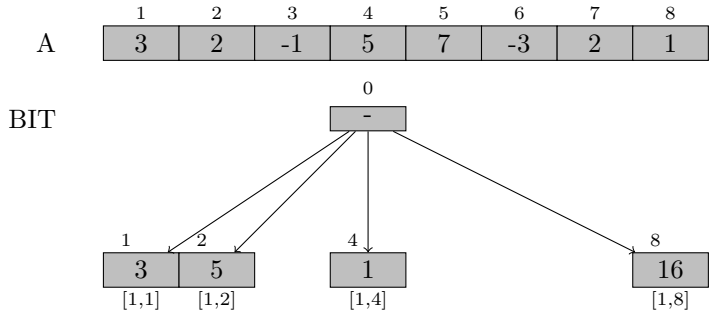
There are two possible trivial solutions to this problem.

1. Store A. $\mathsf{Sum}(i)$ is solved by scanning the array in $\Theta(n)$ time and $\mathsf{Add}(i, v)$ is solved in $O(1)$ time.
2. Store prefix-sums of A. $\mathsf{Sum}(i)$ is solved in $O(1)$ time and $\mathsf{Add}(i, v)$ is solved by modifying all the entries up to position i in $\Theta(n)$ time.

Both these solutions are clearly unsatisfactory.

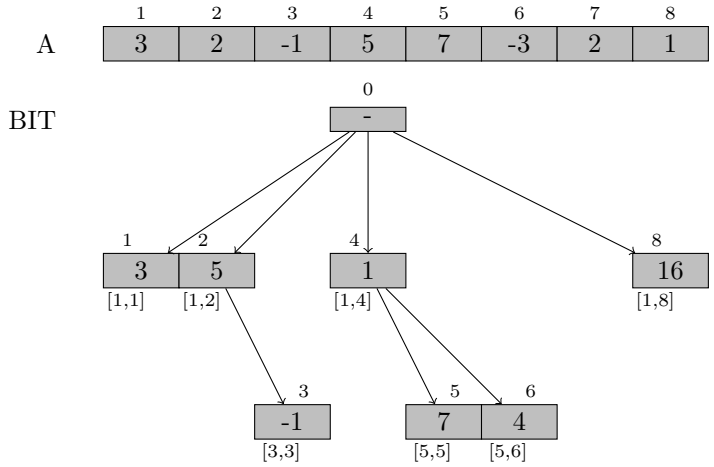
In order to approach this problem, let us assume that we only solve Sum queries on positions which are a power of 2 (i.e., positions $1, 2, 4, 8$ in our array A).

The idea for solving this relaxed version is to sparsify the latter trivial solution: we only store prefix-sums of positions that we could query. The figure below shows the solution for this particular case. For convenience we represent the solution as a tree. The root is a fake node named 0. Children of this node are nodes (named, $1, 2, 4, 8$) which store the sum up to the corresponding power of 2. Below every node we report the range covered by the corresponding value. For example, node 4 covers the range of positions in $[1, 4]$.



We solve the queries as follows. The query $\mathsf{Sum}(i)$ is easy. We simply access the node i and, thus, it takes constant time. Obviously, it works only for i which is a power of 2. The query $\mathsf{Add}(i, v)$ needs to add v to all the nodes covering ranges that contain the entry i. For example, for query $\mathsf{Add}(3, 10)$ we add the value 10 to nodes 4 and 8. In general, given i, we compute the smallest power of 2 which is larger than i, say j. Then, we add v to nodes $j, 2j, 2^2j, 2^3j, \ldots$. Thus, the query takes $\Theta(\log n)$ time.

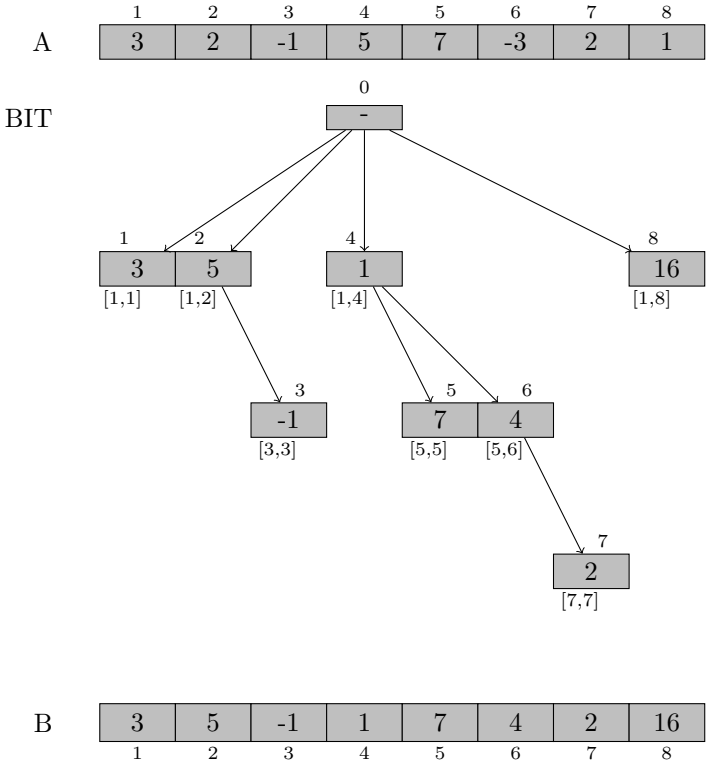
Now, is it possible to extend the above solution to support query Sum on more positions? Observe that we are not supporting the query on positions that belong to any range which is between to consecutive powers of 2. For example, positions in $[5, 7]$, which are the positions between 2^2 and 2^3. But wait, supporting the query on this subarray is just a smaller instance of our original problem. Thus, we can apply exactly the same strategy by adding a new level on our tree. If the subarray is $A[l, \ldots, r]$, the new level will support query $\mathsf{Sum}(i)$ for any i such that $i - l + 1$ is a power of 2.



Our two-level tree is now able to support $\mathsf{Sum}(i)$ queries also for any position i which is the sum of two powers of 2. Why? Well, let i be $2^{k'} + 2^k$ with $k' > k$. The range $[1, i]$ can be decomposed in two subranges $[1, 2^{k'}]$ and $[2^{k'} + 1, 2^{k'} + 2^k = i]$. Both these subranges are covered by nodes of the tree. Indeed, range $[1, 2^{k'}]$ is covered by node $2^{k'}$ at the first level, while $[2^{k'} + 1, 2^{k'} + 2^k = i]$ is covered by node i at the second level. As an example, consider the query $\mathsf{Sum}(5)$. This can be solved in our two-level tree because $5 = 2^2 + 2^0$. Thus, the range $[1, 5]$ is decomposed in $[1, 4]$ and $[5, 5]$ and the result (which is 6) is obtained by summing the values of nodes $2^2 = 4$ and $2^2 + 2^0 = 5$.

What are now the positions for which Sum is not supported? Any position i such that i is neither a power of 2 nor a sum of two powers of 2. In our example, as $n = 8$ only position

$7 = 2^2 + 2^1 + 2^0$ has such form. And now what? Let's add a new level on our tree to support the query on positions which are the sum of three powers of 2.



That's all. This is the binary indexed tree of the array A. Now some easy observations.

- Even if we gave a tree shape to our solution, it can be represented as an array B of size $n + 1$ as shown in the figure above.
- We no longer need the original array A as any of its entries i can be obtained as $A[i] = \mathsf{Sum}(i) - \mathsf{Sum}(i - 1)$.
- Let h be $\lfloor \log(n) + 1 \rfloor$, i.e., the length of the binary representation of any position in $[1, n]$. As any position is the sum of at most h powers of 2, the tree has no more than h levels. Actually, the number of levels is either h or $h - 1$ depending on the value of n.

Now let's see more precisely how to solve our queries Sum and Add on a BIT. As the solution is called binary indexed tree, it seems a good idea to consider the binary representation of the positions while solving a query. Let's start with $\mathsf{Sum}(i)$. For example, consider the case $i = 7$. The binary representation of 7 is $(\texttt{0111})_2$, so 7 is the sum of three powers of 2, and, thus, the range $[1, 7]$ is split in three subranges $[1, 4]$, $[5, 6]$, and $[7, 7]$. These subranges are covered by nodes 4, 6, and 7, one for each level of the tree. The sum of the values in these nodes is the result of the query. Not a surprise, we start from a node (7 in this case), we move to its parent (node 6), its grandparent (node 4), and so on until we reach the fake node 0.

It's clear that to solve query Sum we need the operation Parent. Let's take a look to the binary representations of the ids of nodes involved in answering this query.

7	$(\texttt{0111})_2$
6	$(\texttt{0110})_2$
4	$(\texttt{0100})_2$

Any pattern? Well, binary of the parent is obtained by removing rigthmost `1` from the binary representation of its children. "*Coincidence? I think NOT!*" cit.

Let's see why it works this way. Let i be a node. Its range is $[j, i]$ for some j. Its children will be nodes $i + 2^0$, $i + 2^1$, $i + 2^2$, and so on and will span ranges $[j + 1, i + 2^0]$, $[j + 1, i + 2^1]$, $[j + 1, i + 2^2]$, and so on. Thus, the binary representation of any of these children equal the one of i except for addition of the rigthmost `1` (becuase of the term 2^k).

Now we need a bittrick to easily obtain parent of a node. By discussion above, it is clear that we need a way to remove the rightmost `1` from the binary representation of some node i. The rightmost `1` in the binary representation of i can be isolated by computing $k = i\& - i$. Thus, $i - k$ is the parent of i.

Indeed, the two's complement of a N-bit numbers is its complement with respect to 2^N. For instance, 7 is $(\texttt{0111})_2$ in binary and its two's complement is $(\texttt{1001})_2 = -7$ and $(\texttt{0111})_2 + (\texttt{1001})_2$ is $(\texttt{10000})_2$ equals 2^5. If a binary number $(\texttt{0111})_2$ encodes the number 7, then its two's complement encodes its inverse -7. The binary encodings of a number and its inverse matches bits by bits only after their rightmost $(\texttt{1})_2$.

For example,

7	$(\texttt{0111})_2$	
-7	$(\texttt{1001})_2$	two's complement
7 & -7	$(\texttt{0001})_2$	k
7 - (7 & -7)	$(\texttt{0110})_2$	parent of 7

Let's now consider the query $\mathsf{Add}(i)$. We need to modify the values of nodes which range contains position i. For sure, node i is one of such nodes. Indeed, its range ends at i. Also right siblings of node i contain position i in their ranges. This is because ranges of siblings have the same starting position and right siblings have increasing sizes. Also the right siblings of the parent of node i contain position i. And the right siblings of the grand-grandparent of node i. And so on. Apparently, it seems that we have to change the value of a lot of nodes. However, a simple observation shows that this number is at most $\log n$. Indeed, every time we move from a node to its right sibling or to the right sibling of its parent, the size of the covered range at least double.

It remains to show how to find the list of nodes to modify given a position i. As we said above, node i is the first one. The next node can be obtained with a couple of easy bittricks.

Consider $i = 5$. The next node is its closest right sibling 6. Let's take a look to their binary representation.

5	$(\texttt{0101})_2$
6	$(\texttt{0110})_2$

Any pattern? Isolate the least significant bit in 5 ($(\texttt{0001})_2 = 1$) and sum it to 5 to get 6. *It could work!* cit. Let's try with another one. Right sibling of the parent of 6 (and, thus, of 5) is 8.

6	$(\texttt{0110})_2$
8	$(\texttt{1000})_2$

The least significant bit in 6 is $(\texttt{0010})_2 = 2$ and $6 + 2 = 8$. Cool!

Why is this correct? It is easy to see that the position of rightmost $(1)_2$ distinguishes among the siblings of a node. This means that we can move to the next sibling by moving the least significant bit one position to the left. That's exactly what happen when we sum 5 and its rightmost 1 bit. The rightmost and the second rightmost 1s in the last child of a node are consecutive. In this case, if we sum the rightmost 1, it disappears and the second rightmost 1 is moved one position to the left. Thus, we are moving to the right sibling of the parent.

Possible implementations of Sum and Add are reported below.

```
int sum(int idx) {
  int s = 0;
  while(idx != 0) {
      s += B[idx];
      idx -= idx & -idx; // parent(idx)
  }
  return s;
}

void add(int idx, int v){
  while(idx <= N) {
    B[idx] += v;
    idx += idx & -idx; // get_next(idx)
  }
}
```

What are the time complexities of Sum and Add?

Clearly, Sum takes time proportional to the height of the tree and, thus, takes $\Theta(\log n)$ time. This is also the time completity of Add. To see this we observe that every time we move to the right sibling of the current node or the right sibling of its parent, the rightmost $(1)_2$ of its binary representation is moved at least one position to the left. This is possible at most $\log n$ times.

3 Problem set 1

▶ **Problem 6** (Inversions count). *We are given an array $A[0 \dots n-1]$ of n distinct positive integers. If $i < j$ and $A[i] > A[j]$, then the pair (i, j) is called an inversion of A. The goal is to count the number of inversions of A.*

Assume that the integers are smaller than $n+1$. If not, sort A and replace each element with its rank in the sorted array. Then, we use a BIT to index an array B of n elements, initially all set to 0. Now, we scan the array from left to right. When processing $A[i]$, we set $B[i]$ to 1. The number of element larger than $A[i]$ that we have already processed is $i - \mathsf{Sum}(i)$.

▶ **Problem 7** (Update the array). *We are given an array $A[1, n]$, initially all the entries are set to 0. we need to perform a sequence of updates and accesses on it. Each update specifies l, r and v which are the starting index, ending index and value to be added. The update adds v to all the entries in $A[l, r]$. An access i requires to return the current value of $A[i]$.*

We use a BIT B. For the update l, r, v we add v to $B[l]$ and we subtract v to $B[r+1]$. This way, the value at position i is simply the prefix sum up to $B[i]$.

4 Range Updates

The updates of the last problem are different than the more difficult range updates. In the latter case we are both asked to add a value to all the positions in a range but, differently from the problem above, we are asked to support prefix-sums queries on the resulting array (and not just reporting a value at a certain position).

▶ **Problem 8** (Dynamic prefix-sums with range update)**.** *Given an array $A[1,n]$ of integers, we would like to support the following queries*

- $\mathsf{Sum}(i)$ *that returns the sum of the elements in $A[1,\ldots,i]$;*
- $\mathsf{Add}(i,v)$ *that adds the value v to the entry $A[i]$;*
- $\mathsf{RangeUpdate}(i,j,v)$ *that adds the value v to the entries $A[i],A[i+1],\ldots,A[j]$.*

Of course, we can solve a $\mathsf{RangeUpdate}(i,j,v)$ with $j-i+1$ Add queries in $\Theta((j-i+1)\log n)$ time.

BIT can be extended to solve any $\mathsf{RangeUpdate}$ in $\Theta(\log n)$ time, thus, in time independent on the size of the updated range. The cost for the efficient implementation of this extra operation is the need of using 2 BITS instead of one, i.e., we store $2n$ entries instead of n. Below we will only consider Sum and $\mathsf{RangeUpdate}$ queries as, clearly, we can support $\mathsf{Add}(i,v)$ by performing $\mathsf{RangeUpdate}(i,i,v)$.

Assume we are keeping a BIT B_1 that solves updates as in problem *Update the array*, i.e., a $\mathsf{RangeUpdate}(i,i,v)$ is solved by adding v to entry at position i and $-v$ to entry at position $j+1$. In this case, $\mathsf{Sum}(p)$ reports the prefix sum up to p with B_1 multiplied by p.

What is wrong with this approach?

Consider an easy case. Assume we start from all entries set to 0 and we perform $\mathsf{RangeUpdate}(i,j,v)$. The correct result for $\mathsf{Sum}(p)$ after the update is the following.

1. if $1 \le p < i$, $\mathsf{Sum}(p) = 0$
2. if $i \le p \le j$, $\mathsf{Sum}(p) = v(p-i+1)$
3. if $j < p \le j$, $\mathsf{Sum}(p) = v(j-i+1)$

Instead, the result returned by our implementation of $\mathsf{Sum}(p)$ is the following.

1. if $1 \le p < i$, $\mathsf{Sum}(p) = 0$
2. if $i \le p \le j$, $\mathsf{Sum}(p) = vp = v(i-1) + v(p-i+1)$
3. if $j < p \le j$, $\mathsf{Sum}(p) = (v-v)(p) = 0$

Observe that the result is correct if $1 \le p < i$. Instead, if $i \le p \le j$, we have the additional term $v(i-1)$ while we report 0 instead of $v(j-i+1)$ in the latter case. Is there any way to correct the result?

The idea is to use an other BIT B_2. Given an update $\mathsf{RangeUpdate}(i,j,v)$, we add $-v(i-1)$ to position i and vj to position $j+1$.

This way, the result of $\mathsf{Sum}(p)$ is $a*p+b$ where a is the sum up to p in B_1 and b is the sum up to p in B_2. The value of b is as follows.

1. if $1 \le p < i$, $b = 0$
2. if $i \le p \le j$, $b = -v(i-1)$
3. if $j < p \le j$, $b = vj - v(i-1) = v(j-i+1)$

Thus, the value of b corrects the errors in the value of ap.

5 Problem set 2

▶ **Problem 9** (Nested segments). *We are given n segments $\langle l_1, r_1\rangle, \langle l_2, r_2\rangle, \ldots, \langle l_n, r_n\rangle$ on a line. There are no ends of some segments that coincide. For each segment find the number of segments it contains.*

We can restate the problem as follows. For each i, the goal is to count the number of indices j such that the following conditions hold: $l_i < l_j$ and $r_j < r_i$. This is solve by maintaining a BIT B which stores the right ends of the processed segments. The segments are processed in decreasing order of their left ends (i.e., we need to sort the segments by their left ends). Once we process the segment $\langle l_i, r_i\rangle$, we add 1 to position r_i in the BIT B. Consider now the number of segments that are contained in the current segment $\langle l_i, r_i\rangle$. Obverse that the already processed segments $\langle l_j, r_j\rangle$ are all and the only segments such that $l_i < l_j$, i.e., they start at the right of the current segments. How many of them are contained in $\langle l_i, r_i\rangle$? Clearly, the answer is the number of them that ends before r_i, which can be computed with a query $\mathsf{Sum}(r_i)$ in $O(\log n)$ time.

▶ **Problem 10** (Pashmak and Parmida's problem). *There is a sequence A that consists of n integers $A[1], A[2], \ldots, A[n]$. Let $f(l, r, x)$ denote the number of indices k such that: $l \le k \le r$ and $A[k] = x$. The goal is to compute the number of pairs of indicies i and j ($1 \le i < j \le n$) such that $f(1, i, A[i]) > f(j, n, A[j])$.*

We start by remapping values in A so that they range in $[0, n-1]$. This is done by replacing each value with its rank in the sorted array. Equal values get the same rank.

We now compute the array $\mathsf{SuffixCounter}[1, n]$ which is such that its entry $\mathsf{SuffixCounter}[i]$ stores the number of occurrences of $A[i]$ in the suffix $A[i+1, n]$ (i.e., $\mathsf{SuffixCounter}[i] = f(i, n, A[i])$). This can be easily done by scanning A from right to left and by keeping an array to count the occurrences of each value in the already processed suffix of A.

We build a BIT B on values of $\mathsf{SuffixCounter}$ such that $B[i]$ equals to the number of entries in $\mathsf{SuffixCounter}$ that equals i. This means that $\mathsf{Sum}(k)$ equals the number of times the function $f(i, n, A[i])$ was smaller than of equal to k.

We now process the array A from left to right. Assume we are processing element $A[i]$. We can easily compute the value of $f(1, i, A[i])$ by keeping an array of counters as done before. Assume $f(1, i, A[i]) = k$, we need to count the number of indices j ($j > i$) are such that $f(j, n, A[j]) < k$. Observe that this would be $\mathsf{Sum}(k-1)$ on B if we had no restriction of j. We note that in order to consider the restriction on j we should have removed the contribute of entries in $\mathsf{SuffixCounter}[1, i]$. This can be done step-by-step by subtracting 1 to the entry $\mathsf{SuffixCounter}[i]$ of B.

The time complexity of this solution is $\Theta(n \log n)$.