Strain induced quantum Hall effect of excitons in graphene

Oleg L. Berman[1], Roman Ya. Kezerashvili[1], Yurii E. Lozovik[2], and Klaus G. Ziegler[3]
[1]*Physics Department, New York City College of Technology, The City University of New York, Brooklyn, NY 11201, USA*
[2]*Institute of Spectroscopy, Russian Academy of Sciences, 142190 Troitsk, Moscow, Russia*
[3]*Institut für Physik, Universität Augsburg, D-86135 Augsburg, Germany*
(Dated: April 23, 2020)

We study the effect of a uniform pseudomagnetic field, induced by a strain in a monolayer and double layer of gapped graphene, acting on excitons. For our analysis it is crucial that the pseudomagnetic field acts on the charges of the constituent particles of the excitons, i.e., the electrons and holes, the same way in contrast to a magnetic field. Moreover, using a circularly polarized laser field, the electrons and the holes can be excited only in one valley of the honeycomb lattice of gapped graphene. This breaks the time-reversal symmetry and provides the possibility to observe the various Quantum Hall phenomena in this pseudomagnetoexciton system. Our study poses a fundamental problem of the quantum Hall effect for composite particles and paves the way for quantum Hall physics of pseudomagnetoexcitons.

In this study we focus on the influence of strain on the properties of excitons in mono and double layers of gapped graphene. It was predicted that an in-plane distortion of the graphene lattice due to non-uniform strain can create large, nearly uniform pseudomagnetic fields, acting on electrons, which lead to the formation of Landau levels and zero magnetic field Quantum Hall effect for electrons [1]. Note that the Quantum Hall phenomena in graphene and graphene based structures in a high magnetic field attracted the great interest [2–4]. It may also create quantum Hall plateaus in the Hall conductivity when the time-reversal symmetry is broken. Landau quantization of the electronic spectrum for highly strained nanobubbles was experimentally observed, and pseudomagnetic fields in excess of 300 T have been measured [5]. An elastic shear strain can produce an effective vector potential [6, 7] that arises from changes in the electron-hopping amplitude between carbon atoms in graphene. The strain induced effective vector potential $\mathbf{A}$ appears in the electronic Hamiltonian like a gauge field as $i\hbar\nabla+\mathbf{A}$, in result a pseudomagnetic field $\mathbf{B}=\nabla\times\mathbf{A}$ is arose [8]. This strain induced pseudomagnetic field mimics the influence of a magnetic field applied perpendicular to the graphene sheet [1]. Since the strain induced pseudomagnetic field in contrast to a magnetic field acts on the electric charge the same way, its effect on electrons and holes is the same and this occurs to be essential for effects predicted below.

In this Letter we calculate the Landau levels of direct and indirect excitons in mono and double layers of gapped graphene, respectively, in the presence of a strain induced pseudomagnetic field. Such excitons we refer as pseudomagnetic excitons (PME). We predict the existence of Quantum Hall plateaus for the PME.

Let us consider the formation of excitons in the strain induced pseudomagnetic field and obtain their eigenfunctions and eigenenergies. Starting from the effective Dirac Hamiltonian for a single valley for gapped graphene in the presence of a strain field with components A_x and A_y [7], we consider here the Dirac equation for a pair of an electron at position $\mathbf{r}_1$ and a hole at position $\mathbf{r}_2$ as

$$(H_0 + \ V\left(|\mathbf{r}_1-\mathbf{r}_2|\right))\,\Psi=\mathcal{E}\Psi, \tag{1}$$

with

$$H_0=v_F\sum_{j=1}^{2}\begin{pmatrix} 2\Delta/v_F & i\hbar\partial_{x_j}+A_x(\mathbf{r}_j)+\hbar\partial_{y_j}-iA_y(\mathbf{r}_j) \\ i\hbar\partial_{x_j}+A_x(\mathbf{r}_j)-\hbar\partial_{y_j}+iA_y(\mathbf{r}_j) & -2\Delta/v_F \end{pmatrix},\ \ \Psi=\begin{pmatrix}\psi_1(\mathbf{r}_1,\mathbf{r}_2)\\ \psi_2(\mathbf{r}_1,\mathbf{r}_2)\end{pmatrix}. \tag{2}$$

In Eq. (2) H_0 is the Hamiltonian of the non-interacting electron and hole in gapped graphene in the presence of the strain field, where ∂_{x_j}, ∂_{y_j} are the components of the 2D gradient $\nabla_{\mathbf{r}_j}=(\partial_{x_j},\partial_{y_j})$ and v_F is the Fermi velocity, while the wave function Ψ can be understood as a two-component spinor. The potential energy of electron-hole attraction $V\left(|\mathbf{r}_1-\mathbf{r}_2|\right)$ in Eq. (1) can be described either by the Coulomb potential or, taking into account screening, by the Rytova-Keldysh (RK) potential [9, 10]. Here it should be noticed that, in contrast to the electromagnetic field, the strain induced gauge field $\mathbf{A}=(A_x,A_y)$ does not carry a charge-dependent prefactor in the Hamiltonian H_0. Therefore, in stark contrast to the vector potential of the electromagnetic field, the strain induced effective vector potentials $\mathbf{A}(\mathbf{r}_1)$ and $\mathbf{A}(\mathbf{r}_2)$, acting on an electron and a hole, forming PME, are not coupled to the charges of the particles and have the same sign in the Hamiltonian (2), and these potentials act on an electron and a hole the same way, and the quasi-Lorentz force due to strain induced pseudomagnetic field can be exerted on a moving PME. Note that the units of $\mathbf{A}(\mathbf{r}_{1(2)})$ and $\mathbf{B}$ in SI system are kg $\times$ m/s and kg/s, correspondingly, while the units of $\mathbf{B}/e$, where e is an electron charge, are Teslas. As the first step, we consider a non-interacting an electron and a hole in the stain-induced field and find their eigenfunctions and eigenenergies: $H_0\Psi_0 = \mathcal{E}_0\Psi_0$. Since the two particles are independent, the Hamiltonian H_0 allows to write their wave function as a product $\psi_{0s}(\mathbf{r}_1, \mathbf{r}_2) = \psi_{s,1}(\mathbf{r}_1)\psi_{s,2}(\mathbf{r}_2)$ and their energy as $\mathcal{E}_0 = \mathcal{E}_1 + \mathcal{E}_2$. We assume that the strain field is uniform. In the following we are interested in the spectral properties, which allows us to consider the corresponding Schrödinger equation. The latter can be rewritten with the two-component strain field $\mathbf{A} = (A_x, A_y)$ in the case of a uniform pseudomagnetic field $B_z = (\nabla \times \mathbf{A})_z$ as (see Supplementary Material S1)

$$v_F^2[i\hbar\nabla_{\mathbf{r}_j} + \mathbf{A}(\mathbf{r}_j)]^2\psi_{1,j} = E_j\psi_{1,j} \ , \quad E_j = \mathcal{E}_j^2 + \hbar v_F^2 B_z - 4\Delta^2 \ . \tag{3}$$

The energy $\mathcal{E}_0$ is related to the eigenvalue E_1 (E_2) of the electron (hole) from Eq. (3) via

$$\mathcal{E}_0 = \sum_{j=1}^{2}\sqrt{E_j - \hbar v_F^2 B_z + 4\Delta^2} \ . \tag{4}$$

By considering the non-interacting electron and hole the homogeneous B_z field [11] Eq. (4) becomes

$$\mathcal{E}_{0n_1,n_2} = \sqrt{2\hbar v_F^2 B_z n_1 + 4\Delta^2} + \sqrt{2\hbar v_F^2 B_z n_2 + 4\Delta^2}, \tag{5}$$

where n_1, $n_2 = 0, 1, 2, ...$ are the quantum numbers, corresponding to the Landau levels for the electron and the hole.

In monolayer gapped graphene the effective electron and hole masses are defined as $m_0 = \Delta/v_F^2$ [12] and two Schrödinger equations Eq. (3), one for the electron and one for the hole, can be combined as

$$\sum_{j=1}^{2}\left[i\hbar\nabla_{\mathbf{r}_j} + \mathbf{A}(\mathbf{r}_j)\right]^2\psi_{1,1}(\mathbf{r}_1)\psi_{1,2}(\mathbf{r}_2) = \frac{1}{v_F^2}\left(E_1 + E_2\right)\psi_{1,1}(\mathbf{r}_1)\psi_{1,2}(\mathbf{r}_2) \ . \tag{6}$$

However, if one considers the formation of indirect [13, 14] PMEs in a graphene double layer, and applies different doping to the two graphene monolayers, this can lead to the formation of two different gaps in these graphene monolayers. Note that an electron and a hole, forming an indirect PME, are located in different parallel graphene layers. In this case, the electron and hole effective masses can be unequal: $m_1 \neq m_2$. Therefore, one can consider more general case for electrons and holes with masses $m_j = \Delta_j/v_F^2$, $j = 1, 2$ in double layer system assuming that the induced by strain pseudomagnetic field is the same in both layers. Moreover, the energy of an indirect PME in a double layer of gapped graphene can be obtained by substituting the Coulomb or RK potentials into Eq. (1) in which the corresponding interparticle distance should be replaced by the expression $\sqrt{r^2 + D^2}$ [15, 16], where D is the separation between graphene monolayers.

In general, we can always map the single-particle wave function of the Schrödinger equation for a given particle mass and a given vector potential $\mathbf{A}$ to the first component of the corresponding Dirac equation with the same vector potential in the $x - y$-plane and the mass adjusted to the A_z component in the Dirac equation. From now on we focus solely on the Schrödinger equation (6) and take as the only result of the corresponding Dirac equation the Landau levels $\mathcal{E}_{0n_1,n_2}$ into account by the mapping onto the Dirac spectrum. The Schrödinger equation (6) defines the eigenfunctions and the eigenenergies of the non-interacting electron and hole with different effective masses in the strain induced pseudomagnetic field. There are essential differences between the properties of a magnetoexciton and a PME induced in a high magnetic and high strain induced pseudomagnetic fields, respectively. The Schrödinger equations for a magnetoexciton in a magnetic field and for the PME in a pseudomagnetic field are invariant with respect to the translation and the gauge transformations. The invariance for a magnetic field results in the conservation of the operator of the magnetic momentum of the magnetoexciton $\hat{\tilde{\mathbf{P}}} = -i\hbar\nabla_{\mathbf{r}_1} - i\hbar\nabla_{\mathbf{r}_2} - \frac{e\mathbf{B}_0\times(\mathbf{r}_1-\mathbf{r}_2)}{2}$ [13, 14, 17] and in the conservation of the operator of pseudomagnetic momentum $\hat{\mathbf{P}} = -i\hbar\nabla_{\mathbf{r}_1} - i\hbar\nabla_{\mathbf{r}_2} - \frac{\mathbf{B}\times(\mathbf{r}_1+\mathbf{r}_2)}{2}$ of the PME in for the pseudomagnetic field. Since the operators $\hat{\tilde{\mathbf{P}}}$ and the Hamiltonian of a magnetoexciton, as well as $\hat{\mathbf{P}}$ and the Hamiltonian of a PME commute, these pairs of operators have the same eigenfunctions, respectively. The action of the magnetic field on the particles depends on their charges, while the action of the strain induced pseudomagnetic field does not depend on their charges. The latter leads to the fact that the spectrum of a PME in a high strain-induced pseudomagnetic field is discrete (see Supplementary Material), in contrast to the spectrum of a magnetoexciton in a high magnetic field which is continuous in the representation of magnetic momentum [13, 17]. Moreover, the collective properties of a many-PME system and a many-magnetoexciton system in high strain induced pseudomagnetic and high magnetic fields, respectively, are essentially different.

Expressing the left side of Eq. (6) in terms of the operator $\hat{\mathbf{P}}$, after lengthly calculations presented in Supplementary Material S2 and S3 for more general case for unequal $m_1 \neq m_2$ masses, we obtain

$$H = \frac{1}{4m_0}\left[\hat{\mathbf{P}}^2 - \frac{1}{m_0^2}\left(2\hbar m_0\nabla_{\mathbf{r}} - i\mathbf{S}(\mathbf{r})\right)^2\right], \quad \mathbf{S}(\mathbf{r}) = \frac{m_0}{2}\mathbf{B}\times\mathbf{r}, \tag{7}$$

The corresponding eigenfunctions and eigenenergies of Hamiltonian (7) for the non-interacting electron and hole with different effective masses in the strain induced pseudomagnetic field are obtained in Supplementary Material S3. Using the Hamiltonian (7) for the non-interacting electron and hole one can write the Hamiltonian $\hat{H}_{eh}$ for the PME in the Schrödinger picture as

$$\hat{H}_{eh} = \hat{H} + V\left(|\mathbf{r}_1 - \mathbf{r}_2|\right). \tag{8}$$

Assuming that contribution of $V\left(|\mathbf{r}_1 - \mathbf{r}_2|\right)$ to the energy of the PME (the binding energy of the PME) is small compared with the difference between the eigenvalues of $\hat{H}$, we start with the zeroth order of the perturbation theory with respect to $V\left(|\mathbf{r}_1 - \mathbf{r}_2|\right)$ and find the eigenvalues and eigenfunctions for (7). Later we will take into account the electron-hole interaction $V\left(|\mathbf{r}_1 - \mathbf{r}_2|\right)$ within the first order of the perturbation theory.

The wavefunction of the electron-hole pair in the strain induced pseudomagnetic field, neglecting the electron-hole attraction, can be written as $\Psi_{n,m,\tilde{n},\tilde{m}}(\mathbf{R},\mathbf{r}) = \psi^{(0)}_{n,m}(\mathbf{R})\tilde{\varphi}^{(0)}_{\tilde{n},\tilde{m}}(\mathbf{r})$, where $\psi^{(0)}_{n,m}(\mathbf{R})$ and $\tilde{\varphi}^{(0)}_{\tilde{n},\tilde{m}}(\mathbf{r})$ are the wavefunctions for a free particle in the pseudomagnetic field $2\mathbf{B}$ and $\tilde{\mathbf{B}} = \mathbf{B}/2$, respectively, in the cylindrical gauge [11, 13, 14]. The functions $\tilde{\varphi}^{(0)}_{\tilde{n},\tilde{m}}(\mathbf{r})$ are defined in Supplementary Material S3. Using the pseudomagnetic fields B_z and $\tilde{B}_z = B_z/2$ in the Schrödinger equation with center-of-mass and relative coordinates (cf. Supplementary Material), and $m_0 = \Delta/v_F^2$ for electrons and holes masses, we obtain a simple elegant expression:

$$\mathcal{E}_{0n,\tilde{n}} = \sqrt{4\hbar v_F^2 B_z n + 16\Delta^2} + \sqrt{\hbar v_F^2 B_z \tilde{n} + \Delta^2}, \tag{9}$$

where $n = 0, 1, 2, \ldots$ and $\tilde{n} = \min(n_1, n_2)$ are the quantum numbers for the motion of the center-of-mass and the relative motion of an electron and a hole [14] in the PME, respectively. In the case of the double layer when it is also possible that $m_1 \neq m_2$, Eq. (9) can be written in the form shown in Supplementary Material S3. Therefore, Eq. (9) presents the quantized eigenenergy of the non-interacting electron and hole in the strain induced pseudomagnetic field. Thus, the energy levels of a non-interacting two-dimensional electron-hole system in a pseudomagnetic field are quantized into a discrete set of Landau levels with degeneracy proportional to the area of the system [18].

Now let us find the energies of a PME in a mono and double layer of gapped graphene in the presence of the electron-hole attraction. The attractive electron-hole interaction is treated in the framework of the perturbation theory. Neglecting the transitions between different Landau levels, the first order perturbation with respect to electron-hole attraction results in the following expression for the energy $E_{\tilde{n},\tilde{m}}$ of a PME:

$$E'_{\tilde{n},\tilde{m}} = \left\langle \Psi_{n,m,\tilde{n},\tilde{m}}(\mathbf{R},\mathbf{r}) \left| V(r) \right| \Psi_{n,m,\tilde{n},\tilde{m}}(\mathbf{R},\mathbf{r}) \right\rangle = \left\langle \tilde{\varphi}^{(0)}_{\tilde{n},\tilde{m}}(\mathbf{r}) \left| V(r) \right| \tilde{\varphi}^{(0)}_{\tilde{n},\tilde{m}}(\mathbf{r}) \right\rangle. \tag{10}$$

One can calculate the energies of a direct and an indirect PME using the Coulomb and Rytova-Keldysh potentials. By substituting these potentials into Eq. (10), and using the wavefunctions for the corresponding state given in Supplementary Material S4, we obtain the analytical expressions for the eigenenergies

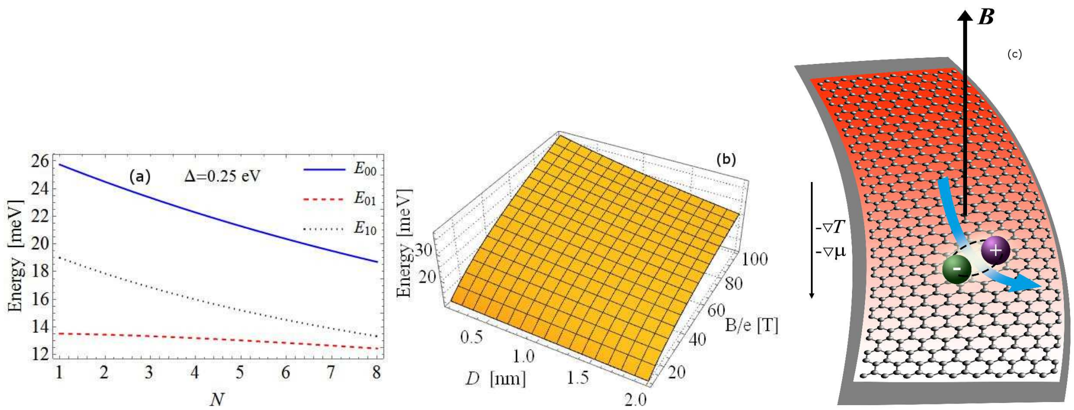

FIG. 1: (Color online) (a) The dependence of energies of indirect PMEs $E'_{\tilde{n},\tilde{m}}$ on the separation D between gapped graphene layers. Calculations performed for the value of magnetic length l that corresponds to $B/e = 50$ T. (b) The dependence of the energies of indirect PMEs $E'_{\tilde{n},\tilde{m}}$ on the separation D between gapped graphene layers and pseudomagnetic field B/e. (c) Conceptual picture of the quantum Hall effect of PMEs in graphene.

$E_{0,0}$, $E_{0,1}$ and $E_{1,0}$. The total energy $E^{(tot)}_{n,\tilde{n},\tilde{m}}$ of a direct PME in gapped graphene monolayer, taking into account electron-hole attraction, is

$$\mathcal{E}_{n,\tilde{n},\tilde{m}} = \mathcal{E}_{0n,\tilde{n}} + E'_{\tilde{n},\tilde{m}}, \tag{11}$$

where $\mathcal{E}_{0n,\tilde{n}}$ is given by Eq. (9), and analytical expressions for the energy $E'_{\tilde{n},\tilde{m}}$ of direct PMEs for the Coulomb and the RK potentials, are given in Supplementary Material S5. The energy of an indirect PME in a double layer of gapped graphene can be obtained by substituting the Coulomb or RK potentials into Eq. (10). In the case of the Coulomb potential the corresponding matrix elements can be evaluated analytically and the results are given in Supplementary Material S6. However, in the case of the Rytova-Keldysh potential the energy of an indirect PME in a double layer system could be found only numerically.

In our calculations the uniform pseudomagnetic field B, acting on an electron or a hole, is related to the strain as $B = \frac{8\hbar\beta c}{a}$ [1], where $a = 2.566$ $\mathring{A}$ is the lattice constant [19], and the parameter $\beta \approx 2$ and the constant c are defined in Ref. [1]. The effective electron and hole masses are defined as $m_1 = m_2 = \Delta/v_F^2$ [12], where $v_F \approx 10^6$ m/s. For the gaps $\Delta = 0.25$ eV and $\Delta = 0.5$ eV, one has $m_1 = m_2 = 0.044m_e$ and $m_1 = m_2 = 0.088m_e$, correspondingly, where m_e is the free electron mass. Following Ref. [20], for the gaps $\Delta = 0.25$ eV and $\Delta = 0.5$ eV, one obtains the 2D polarizability of gapped graphene $\zeta = 1.25$ Å and $\zeta = 0.63$ Å, correspondingly. The corresponding Landau levels are $\mathcal{E}_{00,0} = 1.25$ eV, $\mathcal{E}_{00,1} = 1.30$ eV, and $\mathcal{E}_{01,0} = 1.32$ eV. It can be seen from Fig. 1a that the energies $E'_{\tilde{n},\tilde{m}}$ of PME's are decreasing with the increase of the separation between gpaphene layers. Interesting enough, the comparison of results for the binding energies of the direct and indirect PME energies, calculated using the RK and Coulomb potential, for the parameters used are very close and almost the same. Also, the magnitudes of PME energies, calculated using the RK potential are a little bit smaller than ones, calculated using the Coulomb potential, because the RK potential implies the screening effects. In addition, the energies for the direct PME are greater than ones for the indirect PME. The corresponding values are presented in Table SI, Supplementary Material S6. The energies of indirect PMEs $E_{\tilde{n},\tilde{m}}$ as a function of separations between gapped graphene layers D and the 3D plot of the dependence of $E'_{\tilde{n},\tilde{m}}$ on the separation D and pseudomagnetic field B are presented in Figs. 1a and 1b, respectively. It can be seen, the energies of indirect PMEs decrease with the increase of D and increase with the increase of B.

A strain induced pseudomagnetic field leads to quantization of the PME spectrum which is discrete. Scattering effects due to random potentials and/or interaction between the PMEs would lead to a broadening of the Landau levels. The resulting multiband structure can be characterized by the Chern numbers of these PME bands, which can be observed in the transport properties. When graphene is under the strain, shear strain induces a pseudomagnetic field, while the dilatation gives rise to an effective scalar potential which results in the pseudoelectric field, acting on the charge carriers similarly to an effective electric field along the plane of the monolayer [6]. The dilatation-induced pseudoelectric field can be chosen to be normal to the pseudomagnetic field. The other option is to use statistical forces originating from the temperature and/or chemical potential which will drive electrons and holes in the same direction. Therefore, laser illumination on the edge of samples, which creates gradients of the temperature T and/or chemical potential μ, can trigger the diffusion transport of PMEs without breaking the bound states. These fields cause a flow of the PMEs inside the graphene layers. Together with the pseudomagnetic field this gives rise to the Hall effect, whose Hall conductivity is quantized according to the Chern numbers of the multiband structure. This can also be formulated in terms of an effective Ginzburg-Landau approach by coupling an additional statistical Chern-Simons gauge field to the bosonic PMEs [21, 22], where the resulting Hall conductivity is related to the Chern-Simons constant. Thus, analogously to the standard Integer Quantum Hall effect (IQHE) for the 2D electron gas (2DEG) in a magnetic field [18], we obtain for the system of the PMEs in the presence of impurities the occurrence of the set of plateaus in the Hall resistivity ϱ_{xy} and conductivity σ_{xy} with quantized values: $\varrho_{xy} = -1/\sigma_{xy} = h/n$ ($n = 1, 2, 3, \ldots$ is the Landau level for the motion of the center-of-mass of the PME). In the region of the plateau we have $\varrho_{xx} = \sigma_{xx} = 0$. The conceptual picture of the quantum Hall effect of PME's is presented in Fig. 1c.

The degeneracy d of the Landau levels n is given by $d = 2BS/\Phi_0$, where $\Phi_0 = h/2$ is the quantum of pseudomagnetic flux, S is the area of the system, and the factor 2 appears due to the same action of the pseudomagnetic field an electron and a hole. Thus, one can control the filling factor of the Landau level $\nu = N/d$ (N is the number of the PMEs) either by changing the strain, inducing the pseudomagnetic field, or by laser pumping changing the number N of the PMEs. So one can observe not only the IQHE but also the Fractional Quantum Hall effect (FQHE) for PMEs. For example, to observe the FQHE at the filling factor $\nu = 1/3$, it occurs that one needs the pseudomagnetic field B corresponding to four (but not three as for the 2DEG in a magnetic field) quanta of pseudomagnetic flux accounting for one bosonic PME. In this case, a composite fermion can be formed via attachment of one pseudomagnetic flux quantum to one PME, and these composite fermions with three remaining pseudomagnetic flux quanta form the FQHE state at $\nu = 1/3$ analogously to the FQHE for the 2DEG in a magnetic field [26]. Note that we also can achieve the state of the PME system analogous to the state of composite fermions at the filling factor $\nu = 1/2$ [24, 25], when the filling factor of PMEs $\nu = 1/3$. Really the latter corresponds to three pseudomagnetic flux quanta on one PME. When one pseudomagnetic flux quantum is attached to each PME, one obtains the system of the composite fermions at the filling factor $\nu = 1/2$ (experimentally observable composite fermions, analogously to electrons with two attached magnetic flux quanta [24], correspond to the PMEs with three attached pseudomagnetic flux quanta). Thus, for the neutral PMEs (bosons) one can observe the IQHE, FQHE, and $\nu = 1/2$ phenomena similar to the ones for charged electrons, forming 2DEG in the high magnetic field [26].

It should be mentioned that by breaking the isotropy in the honeycomb lattice strain creates a pseudomagnetic field for each of the two Dirac cones, but it does not break the time-reversal symmetry. By reversing the time the momentum changes the sign, and two valleys are switched. Therefore, a strain induced pseudomagnetic field induces the exciton Hall valley flows in opposite directions in two valleys of graphene and the total Hall exciton flow is absent due to the time-reversal symmetry, if the excitons in both valleys are excited. However, the Hamiltonian (6) for an electron-hole pair corresponds to creation of excitons in only one valley of gapped graphene by the circularly polarized pumping beam [27, 28]. This circularly polarized pumping beam breaks the time-reversal symmetry, and a pseudomagnetic field induces the Hall exciton flows in a monolayer or the Hall electron and hole currents in two layers, forming a double layer, in only one valley of gapped graphene. Thus, Hall valley flows of direct and indirect excitons similar to Hall currents of charged particles can be excited in a mono or double layer of the gapped graphene, respectively. These exciton Hall valley flows, excited in two valleys of graphene are characterized by opposite directions, corresponding to the excitons in each valley. However, we consider the scenario, when the charge carriers are excited in only one valley of gapped graphene by the circularly polarized pumping beam [27, 28]. In this case, the Hall exciton flows in a monolayer or the Hall electron and hole currents in double layer, are excited in only one valley of graphene. Therefore, the PME in a high strain induced pseudomagnetic field can demonstrate the phenomena observed and studied for the charged particles in high magnetic fields, namely, IQHE, FQHE [18], and the state of composite fermions at the filling factor $\nu = 1/2$ [24, 25].

In conclusion, we predict the zero magnetic field quantum Hall effect of excitons in a high strain induced pseudomagnetic field. The Hall valley flows of direct and indirect PME's, similar to Hall currents of charged particles, can be excited in a mono or double layer of the gapped graphene, respectively. In order to observe the quantum Hall effect for PMEs, one has to measure the PME flows. For a double layer with spatially separated electrons and holes, one can measure the electron and hole currents in each layer, forming a double layer. Electrons and holes forming an indirect PME contribute to these currents.

These measurements can be performed, since electrons and holes carry an electric charge.

There are also two optical methods to observe the PME flows in mono and double layers: (i) by measuring locally the photoluminescence; (ii) by measuring the shift of the angular distribution of the photons emitted due to PME recombination as it was suggested for polaritons in Ref. [29].

The results of our study could provide a novel route to quantum Hall physics of PMEs and PME-based valleytronics in graphene.

---

[1] F. Guinea, M. I. Katsnelson, and A. K. Geim, Nat. Phys. **6**, 30 (2010).
[2] G. A. Csáthy and J. K. Jain, Nat. Phys. **15**, 884 (2019).
[3] X. Liu, Z. Hao, K. Watanabe, T. Taniguchi, B. I. Halperin, and P. Kim, Nat. Phys. **15**, 893 (2019).
[4] J. I. A. Li, Q. Shi, Y. Zeng, K. Watanabe, T. Taniguchi, J. Hone, and C. R. Dean, Nat. Phys. **15**, 898 (2019).
[5] N. Levy, S. A. Burke, K. L. Meaker, M. Panlasigui, A. Zettl, F. Guinea, A. H. Castro Neto, and M. F. Crommie, Science **329**, 544 (2010).
[6] F. Guinea, A. K. Geim, M. I. Katsnelson, and K. S. Novoselov, Phys. Rev. B **81**, 035408 (2010).
[7] B. Amorim, A. Cortijo, F. de Juan, A. G. Grushin, F. Guinea, A. Gutiérrez-Rubio, H. Ochoa, V. Parente, R. Roldán, P. San-Jose, J. Schiefele, M. Sturla, and M. A. H. Vozmediano, Physics Reports **617**, 1 (2016).
[8] F. Guinea, M. I. Katsnelson, and M. A. H. Vozmediano, Phys. Rev. B **77**, 075422 (2008).
[9] N. S. Rytova, Proc. Moscow Stare University, Phys. Astron. **3**, 30 (1967).
[10] L. V. Keldysh, JETP Lett. **29**, 658 (1979).
[11] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Pergamon, Oxford, 1977).
[12] T. G. Pedersen, A.-P. Jauho, and K. Pedersen, Phys. Rev. B **84**, 085406 (2011).
[13] I. V. Lerner and Yu. E. Lozovik, Sov. Phys. JETP **51**, 588 (1980).
[14] Yu. E. Lozovik and A. M. Ruvinsky, Phys. Lett. A **227**, 271 (1997); JETP **85**, 979 (1997).
[15] O. L. Berman, G. Gumbs, and R. Ya. Kezerashvili, Phys. Rev. B **96**, 014505 (2017).
[16] O. L. Berman and R. Ya. Kezerashvili, Phys. Rev. B **96**, 094502 (2017).
[17] L. P. Gorkov and I. E. Dzyaloshinskii, Sov. Phys. JETP **26**, 449 (1967).
[18] *The Quantum Hall Effect*, edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1987).
[19] V. Lukose, R. Shankar, and G. Baskaran, Phys. Rev. Lett. **98**, 116802 (2007).
[20] P. K. Pyatkovskiy, J. Phys.: Condens. Matter **21**, 025506 (2009).
[21] S. C. Zhang, T. H. Hansson, S. Kivelson, Phys. Rev. Lett. **62**, 82 (1989).
[22] E. Fradkin, *Field theories of Condensed Matter Physics*, Cambridge University Press (Cambridge 2013).
[23] T. Chakraborty and P. Pietilainen, *The Quantum Hall Effects*, Springer Series in Solid State Sciences No. 85 (Springer, New York 1995).
[24] B. I. Halperin, P. A. Lee, and N. Read, Phys. Rev. B **47**, 7312 (1993).
[25] J. P. Eisenstein and A. H. MacDonald, Nature (London) **432**, 691 (2004).
[26] J. Jain, *Composite Fermions*, Cambridge University Press (Cambridge 2007).
[27] D. Xiao, W. Yao, and Q. Niu, Phys. Rev. Lett. **99**, 236809 (2007)
[28] W. Yao, D. Xiao, and Q. Niu, Phys. Rev. B **77**, 235406 (2008).
[29] O. L. Berman, R. Ya. Kezerashvili, and Yu. E. Lozovik, Phys. Rev. B **82**, 125307 (2010).

Supplementary material: Strain induced quantum Hall effect of excitons in graphene

Oleg L. Berman[1], Roman Ya. Kezerashvili[1], Yurii E. Lozovik[2], and Klaus G. Ziegler[3]
[1] *Physics Department, New York City College of Technology, The City University of New York, Brooklyn, NY 11201, USA*
[2] *Institute of Spectroscopy, Russian Academy of Sciences, 142190 Troitsk, Moscow, Russia*
[3] *Institut für Physik, Universität Augsburg, D-86135 Augsburg, Germany*
(Dated: April 23, 2020)

S1. DIRAC PARTICLES

In the presence of a strain field the Dirac equation was discussed in Ref. [1]. When we apply this idea to independent particles we obtain product states with single particle wave function $\psi_j(\mathbf{r}_j)$ of energy $\mathcal{E}_j$

$$v_F\begin{pmatrix} 2\Delta/v_F & i\hbar\partial_{x_j}+A_x(\mathbf{r}_j)+\hbar\partial_{y_j}-iA_y(\mathbf{r}_j) \\ i\hbar\partial_{x_j}+A_x(\mathbf{r}_j)-\hbar\partial_{y_j}+iA_y(\mathbf{r}_j) & -2\Delta/v_F \end{pmatrix}\begin{pmatrix}\psi_{1,j}(\mathbf{r}_j)\\ \psi_{2,j}(\mathbf{r}_j)\end{pmatrix}=\mathcal{E}_j\begin{pmatrix}\psi_{1,j}(\mathbf{r}_j)\\ \psi_{2,j}(\mathbf{r}_j)\end{pmatrix}. \tag{S1}$$

In general, Δ can be caused not only by a strain field but also by some other symmetry breaking source of the underlying lattice model.

The component $\psi_{1,j}$ of the Dirac spinor satisfies the Schrödinger equation

$$\frac{1}{2m_j}[(i\hbar\partial_{x_j}+A_{x,j})^2+(i\hbar\partial_{y_j}+A_{y,j})^2]\psi_{1,j}=\frac{\mathcal{E}_j^2+\hbar v_F^2B_z-4\Delta^2}{2v_F^2m_j}\psi_{1,j} \tag{S2}$$

and the second component is related to the first as

$$\psi_{2,j}(\mathbf{r}_j)=\frac{v_F}{\mathcal{E}_j+v_F\Delta}\left[i\hbar\partial_{x_j}+A_x(\mathbf{r}_j)-\hbar\partial_{y_j}+iA_y(\mathbf{r}_j)\right]\psi_{1,j}(\mathbf{r}_j)\ . \tag{S3}$$

This implies that the eigenvalue of the Dirac equation reads $\mathcal{E}_j=\sqrt{2m_jv_F^2E_j-\hbar v_F^2B_z+4\Delta^2}$, where E_j is the eigenvalue of the corresponding Schrödinger equation (S2). In the case of a homogeneous pseudomagnetic field we have $E_j=\hbar(B_z/m_j)(n_j+1/2)$, which implies the relation

$$\mathcal{E}_{j,n_j}=\sqrt{2\hbar v_F^2B_zn_j+4\Delta^2}\ . \tag{S4}$$

S2. COORDINATE TRANSFORMATION

We introduce the vectors of the center-of-mass $\mathbf{R}$ and relative motion $\mathbf{r}$ coordinates as

$$\mathbf{R}=\frac{m_1\mathbf{r}_1+m_2\mathbf{r}_2}{m_1+m_2},\qquad \mathbf{r}=\mathbf{r}_1-\mathbf{r}_2. \tag{S5}$$

Using these coordinates one can rewrite the operator of pseudomagnetic momentum $\hat{\mathbf{P}}$ as

$$\hat{\mathbf{P}}=-i\hbar\nabla_{\mathbf{R}}-\mathbf{B}\times\mathbf{R}-\frac{\gamma\mathbf{B}\times\mathbf{r}}{2}, \tag{S6}$$

where

$$\gamma=\frac{m_2-m_1}{m_2+m_1}. \tag{S7}$$

Since in the case of equal electron and hole effective masses at $m_1=m_2$ one has $\gamma=0$, the third term in the r.h.s. of Eq. (S6) vanishes, and the gauge pseudomagnetic field acts only on the center-of-mass of an electron and a hole and does not affect on their relative motion.

The operator $\hat{\mathbf{P}}^2$ is given by

$$\hat{\mathbf{P}}^2 = -\hbar^2\nabla_{\mathbf{R}}^2 + 2i\hbar\left(\mathbf{B}\times\mathbf{R}\right)\cdot\nabla_{\mathbf{R}} + B^2R^2 + i\hbar\gamma\left(\mathbf{B}\times\mathbf{r}\right)\cdot\nabla_{\mathbf{R}} + \gamma\left(\mathbf{B}\times\mathbf{R}\right)\cdot\left(\mathbf{B}\times\mathbf{r}\right) + \frac{\gamma^2B^2r^2}{4}. \quad \text{(S8)}$$

Using Eq. (S5), the Hamiltonian (3) can be written in the following form:

$$\hat{H} = \hat{D}_1 + \hat{D}_2 + \hat{D}_3, \quad \text{(S9)}$$

where

$$\hat{D}_1 = -\frac{\hbar^2\nabla_{\mathbf{R}}^2}{2M} - \frac{\hbar^2\nabla_{\mathbf{r}}^2}{2\mu}, \quad \text{(S10)}$$

$$\hat{D}_2 = \frac{i\hbar}{2}\left[\frac{2\left(\mathbf{B}\times\mathbf{R}\right)\cdot\nabla_{\mathbf{R}}}{M} + \frac{\left(m_2-m_1\right)\left(\mathbf{B}\times\mathbf{R}\right)\cdot\nabla_{\mathbf{r}}}{m_1m_2} + \frac{\gamma\left(\mathbf{B}\times\mathbf{r}\right)\cdot\nabla_{\mathbf{R}}}{M} + \frac{\left(m_1^2+m_2^2\right)\left(\mathbf{B}\times\mathbf{r}\right)\cdot\nabla_{\mathbf{r}}}{m_1m_2M}\right], \quad \text{(S11)}$$

$$\hat{D}_3 = \frac{B^2}{8}\left[\frac{R^2}{\mu} + \frac{2(m_2-m_1)\left(\mathbf{r}\cdot\mathbf{R}\right)}{m_1m_2} + \frac{\left(m_1^3+m_2^3\right)r^2}{m_1m_2(m_1+m_2)^2}\right]. \quad \text{(S12)}$$

and M and μ are the total and reduced exciton masses, respectively, given by

$$M = m_1 + m_2, \qquad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \quad \text{(S13)}$$

S3. DERIVATION OF THE WAVEFUNCTION AND THE ENERGY OF A PME

Here we find the eigenfunctions and eigenvalues of the Hamiltonian $\hat{H}$ (3) without the assumption $m_1 = m_2$. By introducing the coordinates for the center-of-mass $\mathbf{R}$ and relative motion $\mathbf{r}$ of the electron-hole system in the Hamiltonian (3) and expressing this Hamiltonian in terms of the operator $\hat{\mathbf{P}}$ after lengthly calculations presented in Supplementary Material S1 we obtain:

$$\hat{H} = \frac{1}{2M}\left[\hat{\mathbf{P}}^2 - \frac{1}{m_1m_2}\left(\hbar M\nabla_{\mathbf{r}} - i\mathbf{S}(\mathbf{R},\mathbf{r})\right)^2\right], \quad \text{(S14)}$$

where

$$\mathbf{S}(\mathbf{R},\mathbf{r}) = \frac{\mathbf{B}}{2}\times\left[(m_2-m_1)\mathbf{R} + \frac{\left(m_1^2+m_2^2\right)\mathbf{r}}{M}\right]. \quad \text{(S15)}$$

Since the Hamiltonian $\hat{H}$ commutes with the operator $\hat{\mathbf{P}}^2$, the eigenfunctions of $\hat{H}$ are also the eigenfunctions of $\hat{\mathbf{P}}^2$. While for an electron-hole pair in a magnetic field the eigenfunctions of the Hamiltonian are the eigenfunctions of the magnetoexciton momentum operator [2–4], in the strain induced pseudo-magnetic field we present the eigenfunctions of $\hat{H}$ as the eigenfunctions of $\hat{\mathbf{P}}^2$. Note that the x and y components of $\hat{\mathbf{P}}$ do not commute with one another. Therefore, the eigenfunctions of $\hat{H}$ cannot be presented as the eigenfunctions of $\hat{\mathbf{P}}$.

Let us find the eigenfunctions of the operator $\widetilde{H}_0$, defined as

$$\widetilde{H}_0 = \frac{\hat{\mathbf{P}}^2}{2M} = \frac{1}{2M}\left[-i\hbar\nabla_{\mathbf{R}} - \mathbf{B}\times\mathbf{R} - \mathbf{A}_0\right]^2, \quad \text{(S16)}$$

where the vector $\mathbf{A}_0$ is given by

$$\mathbf{A}_0 = \frac{\gamma\mathbf{B}\times\mathbf{r}}{2}. \quad \text{(S17)}$$

Let us find the eigenfunctions and eigenvalues of the Hamiltonian $\widetilde{H}_0$ for the both cases $\mathbf{A}_0 = 0$ and $\mathbf{A}_0 \neq 0$. If $\mathbf{A}_0 = 0$, the eigenfunction of $\widetilde{H}_0$ is given by $\psi^{(0)} = \psi^{(0)}_{n,m}(\mathbf{R})$, which is the wavefunction for a free particle of unit charge in the effective pseudomagnetic field $2\mathbf{B}$ in the cylindrical gauge in eigenvalue $E^{(0)}_n$ of $\widetilde{H}_0$ is defined as [5]

$$E^{(0)}_n = \frac{P^2_n}{2M} = \left(n + \frac{1}{2}\right)\hbar\omega_c, \tag{S18}$$

where $\omega_c = 2B/M$ is the cyclotron frequency for the motion of the center-of-mass of a PME. The quantum numbers $n = 0, 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$ for $\psi^{(0)} = \psi^{(0)}_{n,m}(\mathbf{R})$ and in Eq. (S18) are related to the motion of the center-of-mass of a PME.

If $\mathbf{A}_0 \neq 0$, we define the scalar function $f(\mathbf{R})$ so that $\mathbf{A}_0 \equiv \nabla_{\mathbf{R}} f(\mathbf{R})$ and we have $f(\mathbf{R}) = \mathbf{A}_0 \cdot \mathbf{R}$. In this case the eigenvalue of $\widetilde{H}_0$ is the same as the eigenvalue at $\mathbf{A}_0 = 0$ given by $E_n = E^{(0)}_n = P^2_n/(2M)$, and the eigenfunction of $\widetilde{H}_0$ denoted as ψ is given by

$$\psi \equiv \psi_{n,m}(\mathbf{R}) = \psi^{(0)}_{n,m}(\mathbf{R}) e^{if(\mathbf{R})/\hbar}. \tag{S19}$$

We can see that

$$e^{if(\mathbf{R})/\hbar} = e^{i\mathbf{A}_0\cdot\mathbf{R}/\hbar} = e^{i\gamma(\mathbf{B}\times\mathbf{r})\cdot\mathbf{R}/2\hbar} = e^{i\gamma(\mathbf{B}\times\mathbf{R})\cdot\mathbf{r}/2\hbar}. \tag{S20}$$

The eigenfunction Ψ of the Hamiltonian $\hat{H}$ is given by

$$\Psi = \psi^{(0)}_{n,m}(\mathbf{R}) e^{i\gamma(\mathbf{B}\times\mathbf{R})\cdot\mathbf{r}/2\hbar}\Phi(\mathbf{r}). \tag{S21}$$

The function $\Phi(\mathbf{r})$ can be obtained from the solution of the following equation:

$$\left[E_n - \frac{1}{2Mm_1m_2}\left(\hbar M\nabla_{\mathbf{r}} - i\mathbf{S}(\mathbf{R},\mathbf{r})\right)^2 + V(r)\right] e^{i\gamma(\mathbf{B}\times\mathbf{R})\cdot\mathbf{r}/2\hbar}\Phi(\mathbf{r}) = \mathcal{E} e^{i\gamma(\mathbf{B}\times\mathbf{R})\cdot\mathbf{r}/2\hbar}\Phi(\mathbf{r}), \tag{S22}$$

where $\mathcal{E} = E_n + \tilde{E}$ is the eigenvalue of the Hamiltonian $\hat{H}$ (S14). $\tilde{E}$ and $\Phi(\mathbf{r})$ can be obtained from the solution of the following equation:

$$\left[-\frac{1}{2Mm_1m_2}\left(\hbar M\nabla_{\mathbf{r}} - i\mathbf{S}(\mathbf{R},\mathbf{r})\right)^2 + V(r)\right] e^{i\gamma(\mathbf{B}\times\mathbf{R})\cdot\mathbf{r}/2\hbar}\Phi(\mathbf{r}) = \tilde{E} e^{i\gamma(\mathbf{B}\times\mathbf{R})\cdot\mathbf{r}/2\hbar}\Phi(\mathbf{r}). \tag{S23}$$

Eq. (S23) can be rewritten as

$$\left[\frac{1}{2\mu}\left(-i\hbar\nabla_{\mathbf{r}} - \frac{\left(m_1^2 + m_2^2\right)\mathbf{B}\times\mathbf{r}}{2M^2} - \frac{\gamma\mathbf{B}\times\mathbf{R}}{2}\right)^2 + V(r)\right]\tilde{\Phi}(\mathbf{R},\mathbf{r}) = \tilde{E}\tilde{\Phi}(\mathbf{R},\mathbf{r}), \tag{S24}$$

where $\tilde{\Phi}(\mathbf{R},\mathbf{r})$ is defined as

$$\tilde{\Phi}(\mathbf{R},\mathbf{r}) \equiv e^{i\gamma(\mathbf{B}\times\mathbf{R})\cdot\mathbf{r}/2\hbar}\Phi(\mathbf{r}). \tag{S25}$$

By applying to Eq. (S24) the procedure similar to one was used to find the eigenvalues and eigenfunctuions of the operator $\widetilde{H}_0$ given by Eq. (S16), we get

$$\tilde{\Phi}(\mathbf{R},\mathbf{r}) = \tilde{\varphi}^{(0)}(\mathbf{r}) e^{i\gamma(\mathbf{B}\times\mathbf{r})\cdot\mathbf{R}/2\hbar}. \tag{S26}$$

Neglecting the electron-hole attraction one obtains $\tilde{\varphi}^{(0)}(\mathbf{r}) = \tilde{\varphi}^{(0)}_{\tilde{n},\tilde{m}}(\mathbf{r})$, where $\tilde{\varphi}^{(0)}_{\tilde{n},\tilde{m}}(\mathbf{r})$ is the wavefunction for a free particle of unit charge the effective pseudomagnetic field $\tilde{\mathbf{B}} = \left(m_1^2 + m_2^2\right)\mathbf{B}/M^2$ in the cylindrical gauge in Refs. [3–5]:

$$\tilde{\varphi}^{(0)}_{\tilde{n},\tilde{m}}(\mathbf{r}) = \left[\frac{\tilde{n}!}{2\pi\left(\tilde{n} + |\tilde{m}|\right)!}\right]^{1/2} \frac{\exp\left(i\tilde{m}\phi\right)}{l}\left(\frac{r}{\sqrt{2}l}\right)^{|\tilde{m}|} L^{|\tilde{m}|}_{\tilde{n}}\left(\frac{r^2}{2l^2}\right)\exp\left(-\frac{r^2}{4l^2}\right)\ , \tag{S27}$$

where $l=\sqrt{\hbar/\tilde{B}}$ is the pseudomagnetic length. In Eq. (S27), $L_{\tilde{n}}^{|\tilde{m}|}$ denotes Laguerre polynomials. The quantum numbers $\tilde{n}=\min(n_1,n_2)$, $\tilde{m}=|n_1-n_2|$ are related to the relative motion of an electron and a hole in the PME. The indexes n_1 and n_2 represent the electron and hole quantum numbers, correspondingly. Let us mention that l is measured in m, since $\tilde{B}$ is measured in kg/s. Note that we consider a PME formed by an electron and a hole located in the same type of valley, e.g., in the point K (or K') of the Brillouin zone.

The value $\tilde{E}$ is the same as the energy of a free electron of mass μ in the effective pseudomagnetic field $\tilde{\mathbf{B}}$ in the cylindrical gauge [3, 5]

$$\tilde{E}_{\tilde{n}}=\left(\tilde{n}+\frac{1}{2}\right)\hbar\tilde{\omega}_c, \tag{S28}$$

where $\tilde{\omega}_c=\left(m_1^2+m_2^2\right)\mathbf{B}/\left(M^2\mu\right)$ is the cyclotron frequency for the relative motion of an electron and a hole in the PME.

Combining Eqs. (S21), (S25), and (S26), one can see that the wavefunction of the electron-hole pair in the strain induced gauge pseudomagnetic field, neglecting the electron-hole attraction, can be written as

$$\Psi_{n,m,\tilde{n},\tilde{m}}(\mathbf{R},\mathbf{r})=\psi_{n,m}^{(0)}(\mathbf{R})\tilde{\varphi}_{\tilde{n},\tilde{m}}^{(0)}(\mathbf{r})e^{i\gamma(\mathbf{B}\times\mathbf{r})\cdot\mathbf{R}/2\hbar}, \tag{S29}$$

where γ is defined by Eq. (S7), $\psi_{n,m}^{(0)}(\mathbf{R})$ is the wavefunction for a free particle in the effective pseudomagnetic field $2\mathbf{B}$ in the cylindrical gauge in Refs. [3–5], $\tilde{\varphi}_{\tilde{n},\tilde{m}}^{(0)}(\mathbf{r})$ is the wavefunction for a free particle the effective pseudomagnetic field $\tilde{\mathbf{B}}=\left(m_1^2+m_2^2\right)\mathbf{B}/M^2$ in the cylindrical gauge in Eq. (S27).

In the expressions for E_n and $\tilde{E}_{\tilde{n}}$ given by Eqs. (S18) and (S28), respectively, $\omega_c=2B/M$ and $\tilde{\omega}_c=\tilde{B}/\mu$ are the cyclotron frequencies for the motion of the center-of-mass and the relative motion of an electron and a hole in the PME, respectively, and $n=0,1,2,\ldots$ and $\tilde{n}=0,1,2,\ldots$ are the corresponding quantum numbers. In the case when $m_1=m_2\equiv m_0$ we have $\omega_c=\tilde{\omega}_c=B/m_0$. These expressions are used to find the spectrum of the corresponding Dirac equation for the non interacting electron-hole pair. Let us mention that for the PMEs in a high strain-induced pseudomagnetic field we obtain that the energy spectrum of both the motion of the center-of-mass and the relative motion of an electron and a hole are quantized, in contrast to magnetoexcitons in a high magnetic field, where the energy spectrum of the center-of-mass is continuous, and the energy spectrum of the relative motion of an electron and a hole is quantized [2–4].

In the case of the double layer when it is also possible that $m_1\neq m_2$, Eq. (9) from the main text can be written as:

$$\mathcal{E}_{0n,\tilde{n}}=\sqrt{4\hbar v_F^2B_zn+4(\Delta_1+\Delta_2)^2}+\sqrt{2\hbar v_F^2\tilde{B}_z\tilde{n}+4\left[\Delta_1\Delta_2/(\Delta_1+\Delta_2)\right]^2}, \tag{S30}$$

where Δ_1 and Δ_2 are the band gaps in the first and the second graphene layers, respectively. Therefore, Eq. (S30) presents the quantized eigenenergy of the non-interacting electron and hole in the strain induced pseudomagnetic field.

There are essential differences between the properties of a magnetoexciton and a PME in a high magnetic and high strain-induced pseudomagnetic fields, respectively. The Schrödinger equation for a magnetoexciton in a magnetic field $\mathbf{B}_0$ is are invariant with respect to the translation and the gauge transformations [4]. This invariance for a magnetic field results in the conservation of the operator of the magnetic momentum of the magnetoexciton $\hat{\tilde{\mathbf{P}}}=-i\hbar\nabla_{\mathbf{r}_1}-i\hbar\nabla_{\mathbf{r}_2}-\frac{e\mathbf{B}_0\times(\mathbf{r}_1-\mathbf{r}_2)}{2}$ [2–4]. Since the operators $\hat{\tilde{\mathbf{P}}}$ and the Hamiltonian of a magnetoexciton commute, they have the same eigenfunctions. If one acts by the Hamiltonian of the magnetoexciton on the eigenfunction of $\hat{\tilde{\mathbf{P}}}$ and employs the certain variable change, the dependence of the resulting Hamiltonian on the eigenvalue $\widetilde{\mathbf{P}}$ appears only in the term responsible for the electron-hole Coulomb attraction as the replacement of $\mathbf{r}$ by $\mathbf{r}+\mathbf{r}_0$, where the continuously changing parameter $\mathbf{r}_0$ is directly proportional to the eigenvalue $\widetilde{\mathbf{P}}$, which can vary continuously from 0 to infinity [3, 4]. Therefore, while the energy spectrum of a magnetoexciton is discrete in the zeroth order with respect to the electron-hole attraction, the energy spectrum of a magnetoexciton becomes a continuous function of the eigenvalue $\widetilde{\mathbf{P}}$ in the first order perturbation theory with respect to the electron-hole Coulomb attraction [3, 4]. The simultaneous invariance of the Schrödinger equation for a PME in the strain-induced pseudomagnetic field $\mathbf{B}$ with respect to the translation and the gauge transformations results in the conservation of the operator of pseudomagnetic momentum $\hat{\mathbf{P}} = -i\hbar\nabla_{\mathbf{r}_1} - i\hbar\nabla_{\mathbf{r}_2} - \frac{\mathbf{B}\times(\mathbf{r}_1+\mathbf{r}_2)}{2}$. The difference between the third terms of $\hat{\tilde{\mathbf{P}}}$ and $\hat{\mathbf{P}}$ is caused by the fact that while the action of the magnetic field on particles depends on the value and sign of charge of a particle, the action of the strain-induced pseudomagnetic field on particles does not depend on the value and sign of charge of a particle. Therefore, the strain-induced pseudomagnetic field acts on an electron and a hole the same way contrary to the magnetic field, which acts on an electron and a hole differently. In this case, the resulting Hamiltonian does not demonstrate the dependence on the continuously changing eigenvalue $\mathbf{P}$ only in the term responsible for the electron-hole Coulomb attraction as the replacement of $\mathbf{r}$ by $\mathbf{r} + \mathbf{r}_0$, where the continuously changing parameter $\mathbf{r}_0$ is directly proportional to the eigenvalue $\mathbf{P}$. The strain-induced pseudomagnetic field acts on a PME similar to the action of the magnetic field on two identical charged particles. In the present Letter we demonstrated, that the latter leads to the fact that the spectrum of a PME in a high strain-induced pseudomagnetic field is discrete, in contrast to the spectrum of a magnetoexciton in a high magnetic field which is continuous in the representation of magnetic momentum [2, 3]. Thus, Hall valley flows of direct and indirect PMEs similar to Hall currents of charged particles can be excited in a mono or double layer of the gapped graphene, respectively. These valley Hall flows can be excited by circularly polarized light. Note that the Hall valley flows of PMEs can be observed experimentally by studying the spatial and angular characteristics of exciton photolumenescence. For spatially indirect PMEs Hall flows can be measured in separated layers by analyzing the electric currents of electrons and opposite currents of holes by standard methodology.

S4. THE ENERGY OF A PME

To find the energy of a direct and indirect exciton PME one should evaluate the following matrix elements

$$E_{0,0} = 2\pi \int_0^{+\infty} \left[\tilde{\varphi}_{0,0}^{(0)}(\mathbf{r})\right]^2 V(r) r dr, \tag{S31}$$

$$E_{0,1} = \int_0^{2\pi} d\phi \int_0^{+\infty} r dr \left[\tilde{\varphi}_{0,1}^{(0)}(\mathbf{r})\right]^2 V(r), \tag{S32}$$

$$E_{1,0} = 2\pi \int_0^{+\infty} \left[\tilde{\varphi}_{1,0}^{(0)}(\mathbf{r})\right]^2 V(r) r dr, \tag{S33}$$

where

$$\tilde{\varphi}_{0,0}^{(0)}(\mathbf{r}) = \left[\frac{1}{2\pi}\right]^{1/2} \frac{1}{l} \exp\left(-\frac{r^2}{4l^2}\right), \tag{S34}$$

$$\tilde{\varphi}_{0,1}^{(0)}(\mathbf{r}) = \left[\frac{1}{2\pi}\right]^{1/2} \frac{\exp(i\phi)}{l} \left(\frac{r}{\sqrt{2}l}\right) \exp\left(-\frac{r^2}{4l^2}\right), \tag{S35}$$

$$\tilde{\varphi}_{1,0}^{(0)}(\mathbf{r}) = \left[\frac{1}{2\pi}\right]^{1/2} \frac{1}{l} \left(1 - \frac{r^2}{2l^2}\right) \exp\left(-\frac{r^2}{4l^2}\right) \tag{S36}$$

and $V(r)$ is the Coulomb or RK potential. In the case of an indirect PME in the potential $V(r)$ the corresponding interparticle distance should be replaced by the expression $\sqrt{r^2 + D^2}$ [6, 7], where D is the interlayer separation.

S5. THE ENERGY FOR DIRECT PME'S FOR THE COULOMB AND RYTOVA-KELDYSH POTENTIALS

The energy of a direct PME in a monolayer of gapped graphene double layer can be calculated by substituting the Coulomb potential into Eq. (6) and one obtains

$$E_{0,0} = -E_0; \quad E_{0,1} = -\frac{E_0}{2}; \quad E_{1,0} = -\frac{3E_0}{4}. \tag{S37}$$

In Eq. (S37) E_0 is given by

$$E_0 = \frac{ke^2}{\varepsilon_d l}\sqrt{\frac{\pi}{2}}, \tag{S38}$$

where $l = \sqrt{\hbar/\tilde{B}}$ is the pseudomagnetic length.

The analytical expressions for the energy of a direct PME obtained using the Rytova-Keldysh (RK) potential [8, 9] are the following:

$$E_{0,0} = \frac{\pi ke^2}{(\varepsilon_1+\varepsilon_2)\,\rho_0}\left[e^{-\frac{l^2}{2\rho_0^2}}\mathrm{Erfi}\left(\frac{l}{\sqrt{2}\rho_0}\right) - G\left(\left\{\{0\},\{-\frac{1}{2}\}\right\},\left\{\{0,0\},\{-\frac{1}{2}\}\right\};\frac{l^2}{2\rho_0^2}\right)\right]. \tag{S39}$$

$$E_{0,1} = \frac{\pi ke^2}{(\varepsilon_1+\varepsilon_2)\,\rho_0}\frac{1}{2\pi\rho_0^2}e^{-\frac{l^2}{2\rho_0^2}}\left[-e^{\frac{l^2}{2\rho_0^2}}(2\rho_0^2-\sqrt{2\pi}\rho l)\ -\pi(l^2-2\rho^2)\mathrm{Erfi}\left(\frac{l}{\sqrt{2}\rho_0}\right)+(l^2-2\rho_0^2)\mathrm{Ei}\left(\frac{l^2}{2\rho_0^2}\right)\right] \tag{S40}$$

$$\begin{aligned} E_{1,0} \ =\ & E_{0,0} - \frac{\pi ke^2}{(\varepsilon_1+\varepsilon_2)\,\rho_0}\frac{1}{4\pi\rho_0^4}e^{-\frac{l^2}{2\rho_0^2}}\left[l^4(\gamma-1)+l^3\rho\sqrt{2\pi}e^{\frac{l^2}{2\rho_0^2}}-2l^2\rho^2(1+\gamma)-e^{\frac{l^2}{2\rho_0^2}}(7\sqrt{2\pi}l\rho^3-12\rho^4)\right.\\ & -(6l^2\rho^2-8\rho^4)\mathrm{Ei}\left(\frac{l^2}{2\rho_0^2}\right)-\pi(l^4-8l^2\rho^2+8\rho^4)\mathrm{Erfi}\left(\frac{l}{\sqrt{2}\rho_0}\right)-(l^4+2l^2\rho^2)\ln 2+2(l^4-2l^2\rho^2)\ln\frac{l}{\rho}\\ & \left.+e^{\frac{l^2}{2\rho_0^2}}\left(\sqrt{2}l\rho_0-\rho_0^2l^2{}_1F_1\left(2,1;\frac{l^2}{2\rho_0^2}\right)\right)\right], \end{aligned} \tag{S41}$$

where γ is Euler constant, $\mathrm{Erfi}(x)$ is the imaginary error function, the Maijer $G-$function, $\mathrm{Ei}(x)$ is the exponential integral function, and ${}_1F_1\,(a,b;x)$ is the Kummer confluent hypergeometric function.

S6. THE ENERGY FOR INDIRECT PME'S FOR THE COULOMB POTENTIAL

$$E_{0,0}(D) \ =\ -E_0\exp\left[\frac{D^2}{2l^2}\right]\mathrm{Erfc}\left[\frac{D}{\sqrt{2}l}\right], \tag{S42}$$

$$E_{0,1}(D) \ =\ -E_0\left[\left(\frac{1}{2}-\frac{D^2}{2l^2}\right)\exp\left[\frac{D^2}{2l^2}\right]\mathrm{Erfc}\left[\frac{D}{\sqrt{2}l}\right]+\frac{D}{\sqrt{2\pi}l}\right], \tag{S43}$$

$$E_{1,0}(D) \ =\ -E_0\left[\left(\frac{3}{4}+\frac{D^2}{2l^2}+\frac{D^4}{4l^4}\right)\exp\left[\frac{D^2}{2l^2}\right]\mathrm{Erfc}\left[\frac{D}{\sqrt{2}l}\right]-\frac{D}{2\sqrt{2\pi}l}-\left(\frac{D}{\sqrt{2}l}\right)^3\frac{1}{\sqrt{\pi}}\right], \tag{S44}$$

where $\mathrm{Erfc}(x)$ is the complementary error function and E_0 is given by (S38). These expressions partially concise with the expressions obtained in the case of uniform magnetic field [4].

In Table (S1) are given results of calculations for the energies using Eqs. (S37) - (S43).

[1] B. Amorim, A. Cortijo, F. de Juan, A. G. Grushin, F. Guinea, A. Gutiérrez-Rubio, H. Ochoa, V. Parente, R. Roldán, P. San-Jose, J. Schiefele, M. Sturla, and M. A. H. Vozmediano, Physics Reports **617**, 1 (2016).
[2] L. P. Gorkov and I. E. Dzyaloshinskii, Sov. Phys. JETP **26**, 449 (1967).
[3] I. V. Lerner and Yu. E. Lozovik, Sov. Phys. JETP **51**, 588 (1980).
[4] Yu. E. Lozovik and A. M. Ruvinsky, Phys. Lett. A **227**, 271 (1997); JETP **85**, 979 (1997).
[5] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Pergamon, Oxford, 1977).
[6] O. L. Berman and R. Ya. Kezerashvili, Phys. Rev. B **96**, 094502 (2017).
[7] O. L. Berman, G. Gumbs, and R. Ya. Kezerashvili, Phys. Rev. B **96**, 014505 (2017).
[8] N. S. Rytova, Proc. Moscow Stare University, Phys. Astron. **3**, 30 (1967).
[9] L.V. Keldysh, JETP Lett. **29**, 658 (1979).

TABLE S1: Calculations performed for the gapped graphene, $\Delta = 0.25$ eV, $\varepsilon = 13$ and the value of magnetic length l that corresponds to $B/e = 50$ T. Two gapped graphene layers are separated by $D = 1.7$ nm.

Energy	Potential	Monolayer	2 Layers	Landau Level, eV
$E_{0,0}$, meV	RK	27.001	21.185	1.25
	Coulomb	27.097	21.187	
$E_{0,1}$, meV	RK	13.548	13.013	1.30
	Coulomb	13.550	13.014	
$E_{1,0}$, meV	RK	20.228	15.128	1.32
	Coulomb	20.322	15.130	