Evaluation of Real Definite integrals

An integral of the type $\int f(\sin\theta, \cos\theta)\,d\theta$

where $f(\sin\theta, \cos\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$

put $z = e^{i\theta}$ $\quad dz = ie^{i\theta}\,d\theta$ $\quad \boxed{e^{i\theta}\,d\theta} = dz/iz$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = 1/2(z+1/z)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = 1/2i(z-1/z)$$

θ varies from 0 to 2Πi

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta = \int_C f\left(\frac{z^{-1}+z}{2}, \frac{z^{-1}-z}{2i}\right) \frac{dz}{iz}$$ where C is the unit circle

1)Evaluate $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

put $z = e^{i\theta}$

$\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$dz = ie^{i\theta}d\theta$

$dz = iz\,d\theta$

$\frac{dz}{iz} = d\theta$

$$\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_0^{2\pi} \frac{dz/iz}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}$$

$$= \oint_C \frac{1}{\frac{4z+z^2+1}{2z}} \frac{dz}{iz}$$

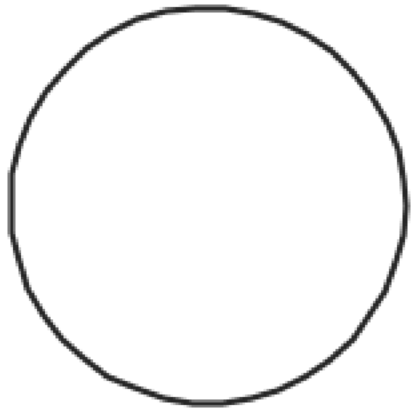
$$= \frac{2}{i}\oint_C \frac{1}{z^2+4z+1}\,dz$$

where C is the circle $|z| = 1$

To obtain poles equate denominator = 0

$$z^2+4z+1=0$$

$$z = \frac{-4 \pm \sqrt{4^2 - 4\times 1 \times}}{2\times 1} = \frac{-4 \pm \sqrt{16-4}}{2}$$



$$= -2 \pm \sqrt{3}$$

only $z = -2+\sqrt{3}$ lies inside the circle $|z|=1$

Evaluate residue at $z=-2+\sqrt{3}$ let it be α

$$\underset{z=\alpha}{\mathrm{Res}} = \lim_{z\to\alpha} \frac{z-\alpha}{(z-\alpha)(z-(-2-\sqrt{3}))} = \lim_{z\to -2+\sqrt{3}} \frac{1}{z+2+\sqrt{3}}$$

$$= \lim_{z \to -2+\sqrt{3}} \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

$\therefore$ by Residue thm $\displaystyle\int_C \frac{dz}{z^2+4z+1} = 2\pi i \left[\frac{1}{2\sqrt{3}}\right. = \frac{\pi i}{\sqrt{3}}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2}{i} \oint_C \frac{dz}{z^2+4z+1} = \frac{2}{i} \times \frac{\pi i}{\sqrt{3}} = \underline{\underline{\frac{2\pi}{\sqrt{3}}}}$$

18|5|21

UQ

2)Evaluate $\int_C \frac{\cos 3\theta \, d\theta}{5-4\cos\theta}$

Ans:

put $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{1}{2}\left(e^{i\theta}+\frac{1}{e^{i\theta}}\right)$

$$\cos 3\theta = \frac{1}{2}\left(e^{i3\theta}+\frac{1}{e^{i3\theta}}\right) = \frac{1}{2}\left(z^3+\frac{1}{z^3}\right)$$

$$\int_C \frac{\cos 3\theta \, d\theta}{5-4\cos\theta} = \int \frac{\frac{1}{2}\left(z^3+1/z^3\right)}{5-4\frac{\left(z+\frac{1}{z}\right)}{2}} \frac{dz}{iz}$$

$$= \frac{\frac{1}{2}\int \frac{z^6+1}{z^3} \frac{dz}{iz}}{5 - 2\frac{(z^2+1)}{z}} = \int \frac{z^6+1}{2iz^4\left(5z-2z^2-2\right)} \times z\,dz$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1}{2i}\int \frac{z^6+1}{z^3\left(2z^2-5z+2\right)} dz$$

$$\int \frac{\cos 3\theta}{5-4\cos\theta} = \frac{-1}{2i}\int_C \frac{z^6+1}{z^3(2z-1)(z-2)} dz$$

where C is the unit circle $|z|=1$

$$= \frac{-1}{2i}\int f(z)\,dz$$

$f(z)$ has pole of order 3 at $z=0$,

and simple pole at $z = 1/2$ & $z = 2$

Only $z = 0$ & $z = \frac{1}{2}$ lies inside the circle

$$\operatorname{Res} f(a) = \lim_{z \to a} (z-a) f(z)$$

$z = a$ simple pole

$$\operatorname{Res} f(1/2) = \lim_{z \to 1/2} \frac{\left(z - \frac{1}{2}\right)(z^6+1)}{(2z-1)(z-2)z^3}$$

$$= \lim_{z \to 1/2} \frac{z^6+1}{2z^3(z-2)} = \frac{\frac{1}{2^6}+1}{\frac{1}{2^2} \times \frac{-3}{2}} = \frac{-65}{24}$$

$$\boxed{\operatorname{Res} f(a) = \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{dz^{n-1}} \left\{ (z-a)^n f(z) \right\} \right\}_{z=a}}$$

$z = a$ is a pole of order n

here $n = 3$, $a = 0$

$$\mathrm{Res}\,f(0) = \frac{1}{(3-1)!}\left\{\frac{d^{3-1}}{dz^{3-1}}\frac{(z-0)^3(z^6+1)}{z^3(2z^2-5z+2)}\right\}_{z=0}$$

$$= \frac{1}{2}\left\{\frac{d^2}{dz^2}\frac{z^6+1}{2z^2-5z+2}\right\}_{z=0}$$

$$= \frac{1}{2}\left\{\frac{d}{dz}\left\{\frac{(2z^2-5z+2)6z^5-(z^6+1)(4z-5)}{(2z^2-5z+2)^2}\right\}\right\}_{z=0}$$

$$= \frac{1}{2}\left\{\frac{d}{dz}\left\{\frac{8z^7-25z^6+12z^5-4z+5}{(2z^2-5z+2)^2}\right\}\right\}_{z=0}$$

$$= \frac{1}{2}\left\{ \frac{(2z^2-5z+2)^2(56z^6-150z^5+60z^4-4) - (8z^7-25z^6+12z^5-4z+5)\,2(2z^2-5z+2)(4z-5)}{(2z^2-5z+2)^4} \right\}_{z=0}$$

$$= \frac{1}{2}\left\{ \frac{(0-0+4)(0-0+0-4) - 5\times 2\times 2\times -5}{16} \right\}$$

$$\operatorname{Res} f(0) = \frac{1}{2}\left\{ \frac{-16+100}{16} \right\} = \frac{84}{32} = \frac{21}{8}$$

By residue theorem

$$\int f(z)\,dz = 2\pi i \left\{ \operatorname{Res} f(0) + \operatorname{Res} f(1/2) \right\}$$

$$\int f(z) = 2\pi i\left\{-\frac{65}{24}+\frac{21}{8}\right\} = 2\pi i\left\{\frac{-65+63}{24}\right\} = \pi i \times \frac{-1}{6}$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta = \frac{-1}{2i}\int f(z)\,dz$$

$$= \frac{-1}{2i} \times \frac{-\pi i}{6} = \frac{\pi}{12}$$

3) Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ where $a > |b|$ 18/5

Ans

$$\boxed{\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \quad \text{if } f(2a-x) = f(x)}$$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = 2\int_0^{\pi} \frac{d\theta}{a+b\cos\theta}$$

$\cos(2\pi-\theta)$
$= \cos\theta$

put $z = e^{i\theta}$ so that $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$ $d\theta = \frac{dz}{iz}$

$$\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \oint_C \frac{1}{a+\frac{b}{2}\left(z+\frac{1}{z}\right)} \frac{dz}{iz}$$

C is the unit circle

$$= \frac{2}{i}\oint_C \frac{dz}{bz^2+2az+b} = \frac{2}{ib}\int \frac{dz}{z^2+\frac{2a}{b}z+1}$$

$$z^2+\frac{2az}{b}+1=0$$

Only $\frac{-a \pm \sqrt{a^2-b^2}}{b}$ lies inside circle

$$z = \frac{-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4}}{2}$$

$$\operatorname{Res}\left(\frac{-a+\sqrt{a^2-b^2}}{b}\right) = \lim_{z\to\alpha} \frac{(z-\alpha)\,2}{ib(z-\alpha)(z-\beta)}$$

$$= \frac{2}{ib(\alpha-\beta)}$$

$$z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b^2}$$

$$= \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

$$= \lim_{z \to \alpha} \frac{1}{(z-\beta)} = \frac{2}{ib}\left[\frac{1}{\frac{-a+\sqrt{a^2-b^2}}{b} - \frac{\left(-a-\sqrt{a^2-b^2}\right)}{b}}\right]$$

$$= \frac{2}{ib}\left[\frac{1}{\frac{2\sqrt{a^2-b^2}}{b}}\right] = \frac{1}{i\sqrt{a^2-b^2}}$$

$$\frac{2}{i}\oint \frac{dz}{(bz^2+2az+b)} = \frac{2}{ib}\int \frac{dz}{\left(z^2+\frac{2a}{b}z+1\right)}$$

$$= 2\pi i \times \frac{1}{i\sqrt{a^2-b^2}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

(By snesfel