$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$$

$$= \frac{1}{2}\left\{-\log(s^2+a^2) + \log(s^2+b^2)\right\}$$

$$L\left(\frac{\cos at - \cos bt}{t}\right) = \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2}$$

③ Evaluate $\int_0^\infty t e^{-2t} \sin 2t \, dt$

By definition $\int_0^\infty e^{-st} f(t) = \int_0^\infty e^{-2t} t \sin 2t \, dt$

$= L\{t \sin 2t\}$ where $s = 2$

we know $L\{t f(t)\} = (-1)^1 \frac{d}{ds} \bar{f}(s)$

$$L\{f(t)\} = L\{\sin 2t\} = \frac{2}{s^2+4} = \bar{f}(s)$$

$$L\{t \sin 2t\} = (-1)^1 \frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= -1\left[\frac{(s^2+4)\cdot 0 - 2\cdot 2s}{(s^2+4)^2}\right]$$

$$L(t \sin 2t) = \frac{4s}{(s^2+4)^2}$$

Now $L(t \sin 2t) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-2t}(t \sin 2t)\,dt$

By definition

and here $s = -2$.

$$\therefore L(t \sin 2t) = \frac{4 \cdot -2}{\left[(-2)^2+4\right]^2} = \frac{-8}{64} = \frac{-1}{8}$$

④ Evaluate $\int_0^\infty t e^{-2t} \sin t \, dt$

(3) Evaluate $\int_0^\infty \frac{\sin mt}{t}\, dt$

~~This is of the form~~

Since $L\{\sin mt\} = \frac{m}{s^2+m^2} = f(s)$

$\therefore$ we have the formula $L\left\{\frac{\sin mt}{t}\right\} = \int_s^\infty f(s)\,ds$

$$= \int_s^\infty \frac{m}{s^2+m^2}\,ds$$

Now $\int \frac{m}{s^2+m^2}\,ds = m\int \frac{1}{m^2\left[1+\frac{s^2}{m^2}\right]}\,ds$

$\therefore \frac{1}{m}\int \frac{1}{1+\frac{s^2}{m^2}}\,ds = \int \frac{du}{1+u^2}$ Now Put $\frac{s}{m} =$ ds $=$

$= \tan^{-1}(u)$

$= \tan^{-1}\frac{s}{m}\Big]_s^\infty$

$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{m}\right)$

$= \cot^{-1}\left(\frac{s}{m}\right) = \tan^{-1}\left(\frac{m}{s}\right)$

Now $\cot^{-1}\left(\frac{s}{m}\right) = y \Rightarrow \frac{s}{m} = \cot y \Rightarrow \frac{s}{m} = \frac{1}{\tan y}$

$\therefore \tan y = \frac{m}{s} \quad \therefore y = \tan^{-1}\left(\frac{m}{s}\right)$

$\therefore \cot^{-1}\frac{s}{m} = \tan^{-1}\left(\frac{m}{s}\right)$

Now $L\left\{\frac{\sin mt}{t}\right\} = \tan^{-1}\left(\frac{m}{s}\right) = \int_0^\infty e^{-st}\,\frac{\sin mt}{t}\,dt$

Thus taking the lt $s \to 0$ $\quad s=0$

$$\therefore \int_0^\infty \frac{\sin mt}{t}\,dt = \tan^{-1}\left(\frac{m}{0}\right)$$

(Consider $m>0$ and $m<0$)

$$\int_0^\infty \frac{\sin mt}{t} = \frac{\pi}{2} \text{ if } m>0$$

$$\qquad\quad -\frac{\pi}{2} \text{ if } m<0$$

Find the Laplace Transforms of

① $L\left\{\frac{e^{-t}\sin t}{t}\right\}$ ② $L\left\{\frac{1-\cos 2t}{t}\right\}$

Inverse Laplace Transforms-Table

① $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$	$L[e^{at}] = \frac{1}{s-a}$
② $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$	$L[e^{-at}] = \frac{1}{s+a}$
③ $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$	$L[\sin at] = \frac{a}{s^2+a^2}$
④ $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$	$L[\cos at] = \frac{s}{s^2+a^2}$
⑤ $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$	$L[\sinh at] = \frac{a}{s^2-a^2}$

(6) $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

$L[\cosh at] = \frac{s}{s^2-a^2}$

(7) $L^{-1}\left[\frac{1}{s}\right] = 1$

$L\{1\} = \frac{1}{s}$

(8) $L^{-1}\left[\frac{1}{s^2}\right] = t$

$L\{t\} = \frac{1}{s^2}$

(9) $L^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$

$L\{t^n\} = \frac{n!}{s^{n+1}}$

(10) $L^{-1}\left[\frac{1}{(s-a)^2}\right] = te^{at}$

$L\{te^{at}\} = \frac{1}{(s-a)^2}$

(11) $L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$

(12) $L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{1}{b}e^{at}\sin bt$

(13) $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\} = \frac{1}{2a^3}\left[\sin at - at\cos at\right]$

(14) $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a}\sin at$

FINDING INVERSE TRANSFOMS — METHOD OF PARTIAL FRACTIONS

Note on Partial Fractions.

① Case To a non-repeated linear factor $s-a$ in the denominator corresponds a partial fraction of the form $\frac{A}{s-a}$

eg $\frac{s}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$

② Case to a repeated linear factor $(s-a)^r$ in the denominator corresponds the the partial fractions of the sum of r partial fractions of the form $\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \frac{A_3}{(s-a)^3} + \cdots + \frac{A_r}{(s-a)^r}$

eg $\frac{s}{(s-1)^4} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{(s-1)^3} + \frac{D}{(s-1)^4}$

③ Case To a non-repeated quadratic factor in the denominator, s^2+as+b in the denominator, corresponds to a partial Fraction of the Form $\frac{As+B}{s^2+as+b}$

eg $\frac{s}{s^2+2s+9} = \frac{As+B}{s^2+2s+9}$

④ Case To a repeated quadratic Factor $(s^2+as+b)^2$ in the denominator corresponds the sum of partial fractions of the form $\frac{A_1s+B_1}{s^2+as+b} + \frac{A_2s+B_2}{(s^2+as+b)^2} + \cdots \frac{A_rs+B_r}{(s^2+as+b)^r}$

eg $\frac{s}{(s^2+3s+2)^3} = \frac{A_1s+B_1}{s^2+3s+2} + \frac{A_2s+B_2}{(s^2+3s+2)^2} + \frac{A_3s+B_3}{(s^2+3s+2)^3}$