Linear Algebra 1

Module 1 – Rank of Matrix, Solution of linear System of Equations, etc

Module 2 – Linear Algebra 2 – Vector space, Subspace, etc

Module 3 – Fourier analysis: Periodic function, Fourier Series etc

Module 4 – Laplace Transforms: Gamma functions, Beta functions, Laplace transforms, Inverse Laplace transform etc.

~~Module 3~~

Module ~~3~~ 4 Laplace Transforms

The method of Laplace Transforms has the advantage of directly giving the Solution of differential Equations with given boundary Values without the necessity of first finding the general Solution and then evaluating from it, the arbitrary Constants.

Ready Tables of Laplace Transforms helps to reduce the difficulty level of solving differential Equations.

Definition

Let $f(t)$ be a function of t defined for all positive Values of t. Then the Laplace transforms of $f(t)$, denoted by $L\{f(t)\}$ is defined by $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$, provided that the integral Exists. Here 's' is a parameter which may be a real or Complex number.

<u>$L\{f(t)\}$ being clearly a function of 's' is briefly written as $\bar{f}(s)$</u>. ie $L\{f(t)\} = \bar{f}(s)$ which Can also written as $f(t) = L^{-1}\{\bar{f}(s)\}$

Then <u>$f(t)$ is Called the inverse Laplace transform of $\bar{f}(s)$</u>.

The Symbol L which transforms $f(t)$ into $\bar{f}(s)$ is called Laplace Transformation operator

Some Results of Gamma Function

① $\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$

So $\int_0^\infty e^{-p} p^n dp = \Gamma n+1 = n!$

② $\Gamma 1 = 1$, ③ $\Gamma 1/2 = \sqrt{\pi}$, ④ $\Gamma n+1 = n \Gamma n$

Transforms of Elementary Functions

① $L(1) = \frac{1}{s}$ $(s>0)$

② $L(t^n) = \frac{n!}{s^{n+1}}$, when $n = 0, 1, 2, 3$ ---

$= \frac{\Gamma n+1}{s^{n+1}}$

③ $L(e^{at}) = \frac{1}{s-a}$

④ $L(\sin at) = \frac{a}{s^2+a^2}$ $(s>0)$

⑤ $L(\cos at) = \frac{s}{s^2+a^2}$ $(s>0)$

⑥ $L(\sinh at) = \frac{a}{s^2-a^2}$ $(s > |a|)$

⑦ $L(\cosh at) = \frac{s}{s^2-a^2}$ $(s > |a|)$

Proofs

① $$L(1) = \int_0^\infty e^{-st} \cdot 1\, dt$$

$$= -\frac{e^{-st}}{s}\Bigg]_0^\infty = \frac{1}{s} \quad (s>0)$$

$\underline{L(1) = \frac{1}{s}}$

② $$L(t^n) = \int_0^\infty e^{-st} t^n dt = \int_0^\infty e^{-p}\left(\frac{p}{s}\right)^n \frac{dp}{s}$$

Put $st = p$
$dt = \frac{dp}{s}$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-p} p^n dp = \frac{\Gamma(n+1)}{s^{n+1}}, \quad n > -1 \text{ and } s > 0$$

$$\therefore L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}} \text{ or } \frac{n!}{s^{n+1}}$$

$\boxed{\because \Gamma(n+1) = n!}$

Another results

$$L\left(t^{-\frac{1}{2}}\right) = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}$$

$$L\left(t^{\frac{1}{2}}\right) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{1/2\,\Gamma(1/2)}{s^{3/2}} = \frac{\sqrt{\pi}}{2 s^{3/2}}$$

$$\left[\because \Gamma(n+1) = n\Gamma(n) \therefore \Gamma(3/2) = \frac{1}{2}\Gamma(1/2)\right]$$

③ $$L(e^{at}) = \int_0^\infty e^{-st} \cdot e^{at} dt$$

$$= \int_0^\infty e^{-(s-a)t} dt = \frac{e^{-(s-a)}}{-(s-a)}\Bigg]_0^\infty$$

$$\therefore L(e^{at}) = \frac{1}{s-a} \quad \text{Provided } s > a.$$

④ $L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt$

Now $\int e^{-st} \sin at \, dt = \sin at \int e^{-st} dt - \int a\cos at . \int e^{-st} dt$ use ILATE

$$= \frac{\sin at \, e^{-st}}{-s} - \int a\cos at \frac{e^{-st}}{-s} dt$$

$$= \frac{\sin at \, e^{-st}}{-s} + \frac{a}{s} \int \cos at \, e^{-st} dt$$

$$= \frac{\sin at \, e^{-st}}{-s} + \frac{a}{s} \left[\cos at \int e^{-st} dt - \int -a\sin at \int e^{-st} dt \right]$$

$$= \frac{\sin at \, e^{-st}}{-s} + \frac{a}{s} \left[\cos at . \frac{e^{-st}}{-s} + a \int \sin at \frac{e^{-st}}{-s} dt \right]$$

$$\therefore \int e^{-st} \sin at \, dt = \frac{\sin at \, e^{-st}}{-s} + \frac{-a}{s^2} \cos at . e^{-st} - \frac{a^2}{s^2} \int e^{-st} \sin at \, dt$$

$$\text{i.e. } \left(1 + \frac{a^2}{s^2}\right) \int e^{-st} \sin at \, dt = \frac{\sin at \, e^{-st}}{-s} - \frac{a}{s^2} \cos at \, e^{-st}$$

$$\text{i.e. } \int e^{-st} \sin at \, dt = \frac{s^2}{s^2+a^2} \left(\frac{\sin at \, e^{-st}}{-s} - \frac{a}{s^2} \cos at \, e^{-st} \right)$$

$$\int e^{-st} \sin at \, dt = \frac{-s\sin at \, e^{-st} - a\cos at \, e^{-st}}{s^2+a^2}$$

$$\therefore L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt = \left. \frac{-s\sin at \, e^{-st} - a\cos at \, e^{-st}}{s^2+a^2} \right]_0^\infty$$

$$\therefore L(\sin at) = \frac{a}{s^2+a^2}$$

⑤ $$L(\sinh at) = \int_0^\infty e^{-st}\sinh at\,dt = \int_0^\infty e^{-st}\left(\frac{e^{at}-e^{-at}}{2}\right)dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(s-a)t}\,dt - \int_0^\infty e^{-(s+a)t}\,dt\right]$$

$$= \frac{1}{2}\left\{\left[\frac{1}{-(s-a)}e^{-(s-a)t}\right]_0^\infty - \left[\frac{1}{-(s+a)}e^{-(s+a)t}\right]_0^\infty\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2}, \text{ for } s>|a|$$

Properties of Laplace Transforms

I **Linearity Property**: If a, b, c are any constants and f, g, h any function of t, then

$$L\left[a f(t) + b g(t) - c h(t)\right] = a L\{f(t)\} + b L\{g(t)\} - c L\{h(t)\}$$

For by definition

$$\int_0^\infty e^{-st}\{a f(t) + b g(t) - c h(t)\}\,dt =$$

$$a\int_0^\infty e^{-st} f(t)\,dt + b\int_0^\infty e^{-st} g(t)\,dt - c\int_0^\infty e^{-st} h(t)\,dt$$

$$a L\{f(t)\} + b L\{g(t)\} - c L\{h(t)\}.$$

~~First~~

Problems

① Find the Laplace Transforms of the following

$$L\{6\sin 2t + 2e^{4t}\} = 6L\{\sin 2t\} + 2L\{e^{4t}\}$$
$$= 6\cdot\frac{2}{s^2+4} + 2\cdot\frac{1}{s-4}.$$
$$= \frac{12}{s^2+4} + \frac{2}{s-4}$$

Some Trigonometric Formulas.

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$
$$\cos^2 A = \frac{1}{2}[1+\cos 2A]$$
$$\sin^3 A = \frac{3\sin A - \sin 3A}{4}$$

② ~~L sin2t * sin3t~~ $L\{\sin 2t\sin 3t\}$

$$\therefore L\{\sin 2t\sin 3t\} = L\left\{\frac{1}{2}[\cos t - \cos 5t]\right\}$$
$$= \frac{1}{2}L(\cos t) - L(\cos 5t)$$
$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right] = \frac{12s}{(s^2+1)(s^2+25)}$$

③ $$L\{\cos^2 2t\} = L\left\{\frac{1}{2}(1+\cos 4t)\right\}$$
$$= \frac{1}{2}\{L(1) + L(\cos 4t)\}$$
$$= \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\left(\frac{s}{s^2+16}\right)$$