(5) Find the Laplace Transform of $t \sin^2 t$

(6) $t^2 e^{-3t} \sin 2t$

(5) $L\{t \sin^2 t\} = ?$

First Find $f(s)$, $\therefore L\{\sin^2 t\} = L\{\frac{1}{2}(1-\cos 2$

$\therefore L\left(\frac{1}{2}\right) - L(\cos 2t) = \frac{1}{2s} - \frac{s}{2(s^2+4)} = f(s)$

Now $L\{t \sin^2 t\} = (-1)^1 \cdot \frac{d}{ds} f(s) = (-1)^1 \frac{d}{ds}\left\{\frac{1}{2s} - \frac{s}{2(s}\right.$

$\therefore L\{t \sin^2 t\} = -\frac{d}{ds}\left\{\frac{1}{2s} - \frac{s}{2(s^2+4)}\right\}$

$= -\frac{1}{2} \frac{d}{ds}\left\{\frac{1}{s} - \frac{s}{s^2+4}\right\}$

$= -\frac{1}{2}\left\{\frac{-1}{s^2} - \left[\frac{(s^2+4)\cdot 1 - s \cdot 2s}{(s^2+4)^2}\right]\right\}$

$= \frac{1}{2s^2} + \frac{1}{2} \frac{(-s^2+4)}{(s^2+4)^2}$

$= \frac{1}{2}\left[\frac{1}{s^2} + \frac{-s^2+4}{(s^2+4)^2}\right]$

$= \frac{1}{2}\left[\frac{(s^2+4)^2 + (4-s^2)s^2}{s^2(s^2+4)^2}\right]$

$= \frac{1}{2}\left\{\frac{\cancel{s^4} + 8s^2 + 16 + 4s^2 - \cancel{s^4}}{s^2(s^2+4)^2}\right\}$

$= \frac{1}{2} \frac{12s^2+16}{s^2(s^2+4)^2} = \frac{\cancel{4}^{2}(3s^2+4)}{\cancel{2}\, s^2(s^2+4)}$

$\therefore$ Ans is $\frac{2(3s^2+4)}{s^2(s^2+4)^2}$

Division by t

If $L\{f(t)\} = \bar{f}(s)$ then $L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty \bar{f}(s)\,ds$

Provided the Integral exists.

Problems Find the Laplace Transform of

(1) $\frac{(1-e^t)}{t}$

$$L\left(\frac{1-e^t}{t}\right) = ?$$

We know $L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty \bar{f}(s)\,ds$

$$\therefore L\{1-e^t\} = L(1) - L(e^t) = \frac{1}{s} - \frac{1}{s-1} = \bar{f}(s)$$

Now $L\left(\frac{1-e^t}{t}\right) = \int_s^\infty \frac{1}{s} - \frac{1}{s-1}\,ds = \int_s^\infty \frac{1}{s}\,ds - \int_s^\infty \frac{1}{s-1}\,ds$

$$= \log s\Big]_s^\infty - \log(s-1)\Big]_s^\infty$$

$$= \cancel{\frac{\log s}{\log(s-1)}}\Big]_s^\infty = \cancel{\log s}$$

$$= \log\left(\frac{s}{s-1}\right)\Big]_s^\infty = \log \frac{\cancel{s}}{\cancel{s}(1-\frac{1}{s})}\Big]_s^\infty$$

$$= \log \frac{1}{1-\frac{1}{s}}\Big]_s^\infty$$

$$= \log 1 - \log \frac{1}{1-\frac{1}{s}}$$

$$= -\log\left(\frac{s}{s-1}\right) = \log\left(\frac{s-1}{s}\right)$$

(2) Find $L\left\{\frac{\cos at - \cos bt}{t}\right\}$

$L\left\{\frac{\cos at - \cos bt}{t}\right\} = ?$

we know $L\left\{\frac{1}{t} f(t)\right\} = \int_s^\infty \bar{f}(s)\, ds$

$\therefore L\{\cos at - \cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = \bar{f}(s)$

Now $L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\, ds$

$\therefore \int_s^\infty \frac{s}{s^2+a^2}\, ds - \int_s^\infty \frac{s}{s^2+b^2}\, ds$

$\therefore \int_s^\infty \frac{2s\, ds}{2(s^2+a^2)} - \int_s^\infty \frac{2s\, ds}{2(s^2+b^2)}$

Now $s^2+a^2 = u$
$2s\,ds = du$
$s^2+b^2 = v$
$2s\,ds = dv$

$\therefore \int \frac{1}{2u}\, du - \int \frac{1}{2v}\, dv = \frac{1}{2}\log u - \log v$

$= \frac{1}{2}\log(s^2+a^2)\Big]_s^\infty - \frac{1}{2}\log(s^2+b^2)\Big]_s^\infty$

$= \frac{1}{2}\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\Bigg]_s^\infty$

$= \frac{1}{2}\log\left[\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right]_s^\infty$

$= \frac{1}{2}\left\{\log\left[\frac{1+0}{1+0}\right] - \log\left(\frac{1+\frac{a^2}{s^2}}{1+\frac{b^2}{s^2}}\right)\right.$

$$= \frac{1}{2}\left\{\log 1 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}$$

$$= \frac{1}{2}\left\{-\log(s^2+a^2) + \log(s^2+b^2)\right\}$$

$$L\left(\frac{\cos at - \cos bt}{t}\right) = \log\left(\frac{s^2+b^2}{s^2+a^2}\right)^{1/2}$$

③ <u>Evaluate</u> $\int_0^\infty t\, e^{-2t} \sin 2t \, dt$

we know $L\{t f(t)\} = (-1)^1 \frac{d}{ds} \bar{f}(s)$

$$L\{f(t)\} = L\{\sin 2t\} = \frac{2}{s^2+4} = \bar{f}(s)$$

$$L\{t \sin 2t\} = (-1)^1 \frac{d}{ds}\left(\frac{2}{s^2+4}\right)$$

$$= -1\left[\frac{(s^2+4)\cdot 0 - 2\cdot 2s}{(s^2+4)^2}\right]$$

$$L(t \sin 2t) = \frac{4s}{(s^2+4)^2}$$

Now $L(t \sin 2t) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-2t}(t \sin 2t)\, dt$

By definition

and here $s = -2$

$$\therefore L(t \sin 2t) = \frac{4 \cdot -2}{[(-2)^2+4]^2} = \frac{-8}{64} = \frac{-1}{8}$$

④ Evaluate $\int_0^\infty t\, e^{-2t} \sin t \, dt$