Results

Show that ① $L(t\sin at) = \frac{2as}{(s^2+a^2)^2}$

② $L(t\cos at) = \frac{s^2-a^2}{(s^2+a^2)^2}$

We know $L(t) = \frac{1}{s^2}$

Now $L(te^{iat}) = \frac{1}{(s-ia)^2} = \frac{(s+ia)^2}{(s-ia)^2(s+ia)^2}$

$$= \frac{s^2+2ias-a^2}{[s^2-(ia)^2]^2}$$

$$= \frac{s^2-a^2+i2as}{(s^2+a^2)^2}$$

$\boxed{\text{Now } e^{iat} = \cos at + i\sin at}$

$\therefore L(t\{\cos at + i\sin at\}) = \frac{s^2-a^2+i2as}{(s^2+a^2)^2}$

$\therefore L(t\cos at) = \frac{s^2-a^2}{(s^2+a^2)^2}$

$L(t\sin at) = \frac{2as}{(s^2+a^2)^2}$

Transforms of Derivatives

① If $f'(t)$ be Continious and $L\{f(t)\} = \bar{f}(s)$ then

$$\boxed{L\{f'(t)\} = s\bar{f}(s) - f(0)}$$

~~Proof~~ $\cancel{L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt}$

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② Generalisation

If $f'(t)$ and its first $(n-1)$ derivatives be Continious, then

$$L\{f^n(t)\} = s^n\bar{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - s^{n-3}f''(0) - \cdots\cdots - s^{n-1}f^{n-3}(0) - s f^{n-2}(0) - f^{n-1}(0).$$

Then $\boxed{L f''(t) = s^2\bar{f}(s) - s f(0) - f'(0)}$

and $\boxed{L\{f'''(t)\} = s^3\bar{f}(s) - s^2 f(0) - s f'(0) - f''(0)}$

Transforms of Integrals

If $L\{f(t)\} = \bar{f}(s)$, then

$$\boxed{L\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\bar{f}(s)}$$

Multiplication by t^n

If $L\{f(t)\} = \bar{f}(s)$ then

$$\boxed{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{\bar{f}(s)\}, \text{ where } n = 1, 2, 3 \ldots}$$

Problems: Find the Laplace transform of

① $t\cos at$ ② $t^2 \sin at$ ③ $t^3 e^{-3t}$ ④ $te^{-t}\sin 3t$

① $L\{t\cos at\} = (-1)\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = -\left[\frac{(s^2+a^2)\cdot 1 - s\cdot 2s}{(s^2+a^2)^2}\right]$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

[Here $L\{\cos at\} = \frac{s}{s^2+a^2} = \bar{f}(s)$]

② $L\{t^2 \sin at\} = ?$

Here $f(t) = \sin at$ and $L\{\sin at\} = \frac{a}{s^2+a^2} = \bar{f}(s)$

$\therefore L\{t^2 \sin at\} = (-1)^2 \frac{d^2}{ds^2}\left[\frac{a}{s^2+a^2}\right]$

$$\therefore (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) = \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{d}{ds} \frac{(s^2+a^2)\cdot 0 - a\cdot 2s}{(s^2+a^2)^2}$$

$$= \frac{d}{ds}\left(\frac{-2as}{(s^2+a^2)^2}\right)$$

$$= \frac{(s^2+a^2)^2\cdot -2a + 2as\cdot 2(s^2+a^2)\cdot 2s}{(s^2+a^2)^4}$$

$$= \frac{\cancel{(s^2+a^2)}\left[(s^2+a^2)\cdot -2a + 8as^2\right]}{(s^2+a^2)^{\cancel{4}\,3}}$$

$$= \frac{-2as^2 - 2a^3 + 8as^2}{(s^2+a^2)^3}$$

$$\therefore L\{t^2 \sin at\} = \frac{6as^2 - 2a^3}{(s^2+a^2)^3}$$

(4) $L\{te^{-t}\sin 3t\} =$

Now $L\{t \sin 3t\} = (-1)^1 \frac{d}{ds}\bar{f}(s)$

Here $L\{\sin 3t\} = \frac{3}{s^2+9} = \bar{f}(s)$

Now $L\{t \sin 3t\} = (-1)^1 \frac{d}{ds}\left(\frac{3}{s^2+9}\right)$

$$= -1\left[\frac{(s^2+9)\cdot 0 - 3\cdot 2s}{(s^2+9)^2}\right]$$

$$= \frac{6s}{(s^2+9)^2}$$

$\therefore L\{e^{-t} t \sin 3t\} = \frac{6(s+1)}{\left[(s+1)^2+9\right]^2}$

OR

$L\{e^{-t}\sin 3t\} = \frac{3}{s^2+9} = \bar{f}(s)$ and use first shifting Property

$\therefore L\{e^{-t}\sin 3t\} = \frac{3}{(s+1)^2+9} = \bar{f}(s)$

Now $L\{t e^{-t}\sin 3t\} = (-1)^1 \frac{d}{ds}\left[\frac{3}{(s+1)^2+9}\right]$

$$= -1\left[\frac{\left((s+1)^2+9\right)\cdot 0 - 3\cdot 2(s+1)\cdot 1}{\left[(s+1)^2+9\right]^2}\right]$$

$$= \frac{6(s+1)}{\left[(s+1)^2+9\right]^2}.$$