SINGULAR POINTS ,RESIDUE

Zero of Analytic function: is that value of f(z) for which f(z)=0

A point at which a function f(z)ceases to be analytic is called singular point or singularity of f(z).

eg z=2 is a singular point of f(z) = 1/z-2

i)isolated singularity

A singular point z =a of a function f(z) is called an isolated singular point if there exist a circle with centre a which contain no other singular points of f(z).

eg z =1, -1 are two isolated singular points of the function f(z) = z/z^{2} -1

f(z) =1/sinIIz has an infinite number of isolated singular points at z =+ 1,+2,............

When z =a is an isolated singular points of f(z) we can expand f(z) in a Laurents series about z =a

$f(z) = a0 + a1(z-a) + a2(z-a)^2 + ... + a-1(z-a)^{-1} + a-2(z-a)^{-2} + ...$ ----------------(1)

ii) Removable singularity

if all the negative powers of (z-a) in (1) zero then f(z) = an(z-a)^n.Here singularity can be removed by redefining f(z)at z=a in such a way that it becomes analytic at z =a .Such a singularity is called removable singularity.

ie if $\lim_{z \to a} f(z)$ exists finitely then z= a is a removable singularity

iii) Poles : If all negative powers of (z-a) in (1)after the nth are missing ,then the singularity at z=a is called a pole of order n

A pole of first order is called simple pole

iv) Essential singularity:

If the number of negative powers (z-a)in (1)is infinite then z =a is called an essential singularity

ie $\lim_{z \to a} f(z)$ does not exist

1) Find the nature and location of singularities for the following functions

i) $\dfrac{z - \sin z}{z^2}$ ii) $(z+1)\sin\dfrac{1}{z-2}$ iii) $\dfrac{1}{\cos z - \sin z}$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

Ans:

i) $z = 0$ is a singularity

$$\frac{z - \sin z}{z^2} = \frac{z - \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!}\right]}{z^2}$$

$$= \frac{z^3}{3!z^2} - \frac{z^5}{5!z^2} + \frac{z^7}{z^2 \times 7!}$$

$$= \frac{z}{3!} - \frac{z^3}{5!} + \frac{z^5}{7!}$$

Since no negative powers of z in the expansion, $z = 0$ is a removable singularity

ii) $(z+1) \sin \frac{1}{z-2}$

put $z=2$

$(2+1)\sin\frac{1}{0}$ $\therefore$ $z=2$ is a singularity

$$(z+1)\sin\frac{1}{z-2} =$$

put $z-2=t \Rightarrow z=t+2$

$$(t+2+1)\sin\frac{1}{t} = (t+3)\left\{\frac{1}{t}-\frac{1}{t^3 3!}+\frac{1}{5!t^5}-\cdots\right\}$$

$$t\left\{\frac{1}{t}-\frac{1}{3!t^3}+\frac{1}{5!t^5}-\cdots\cdots\right\}+3\left\{\frac{1}{t}-\frac{1}{3!t^3}+\frac{1}{5!t^5}\right\}$$

$$\left\{1-\frac{1}{3!\,t^2}+\frac{1}{5!}\times\frac{1}{t^4}+\cdots\cdots\right\}+\left[\frac{3}{t}-\frac{1}{2t^3}+\frac{3}{5!\,t^5}\cdots\cdots\right]$$

$$=1+\frac{3}{t}-\frac{1}{6t^2}-\frac{1}{2t^3}+\frac{1}{5!\,t^4}+\cdots\cdots$$

$$=1+\frac{3}{z-2}-\frac{1}{6(z-2)^2}-\frac{1}{2(z-2)^3}+\frac{1}{5!\,(z-2)^4}-\cdots\cdots$$

∞ no: of terms in the negative powers of $z-2$ $\therefore$ 2 is essential singularity

iii) $\dfrac{1}{\cos z - \sin z}$

To obtain poles equate Denominator to 0

$\sin z - \cos z = 0 \Rightarrow \sin z = \cos z$

$\tan z = 1 \Rightarrow z = \dfrac{\pi}{4}$

$z = \dfrac{\pi}{4}$ Simple pole

2) What type of singularities have the following function

H.W

i) $\frac{1}{1-e^{z}}$ ii) $\frac{e^{2z}}{(z-1)^{4}}$ iii) $\frac{e^{1/z}}{z^{2}}$

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Residues

The coefficient of $(z-a)^{-1}$ in the expansion around an isolated singularity is called residue of f(z) at that point .thus in Laurents series expansion of f(z)around z =a

$$f(z) = a_0+a_1(z-a)+a_2(z-a)^2+......+a_{-1}(z-a)^{-1}+a_{-2}(z-a)^{-2}+ .....$$

the residue of f(z)at z= a is

$$\boxed{\text{Res } f(a)= \frac{1}{2\pi i}\int f(z)dz}$$

$$\int f(z)dz = 2\pi i \text{ Res } f(a)$$

Residue theorem

If f(z)is analytic at all points inside and on asimple closed curve C ,except on a finite number of isolated sigular pointswithin C ,then

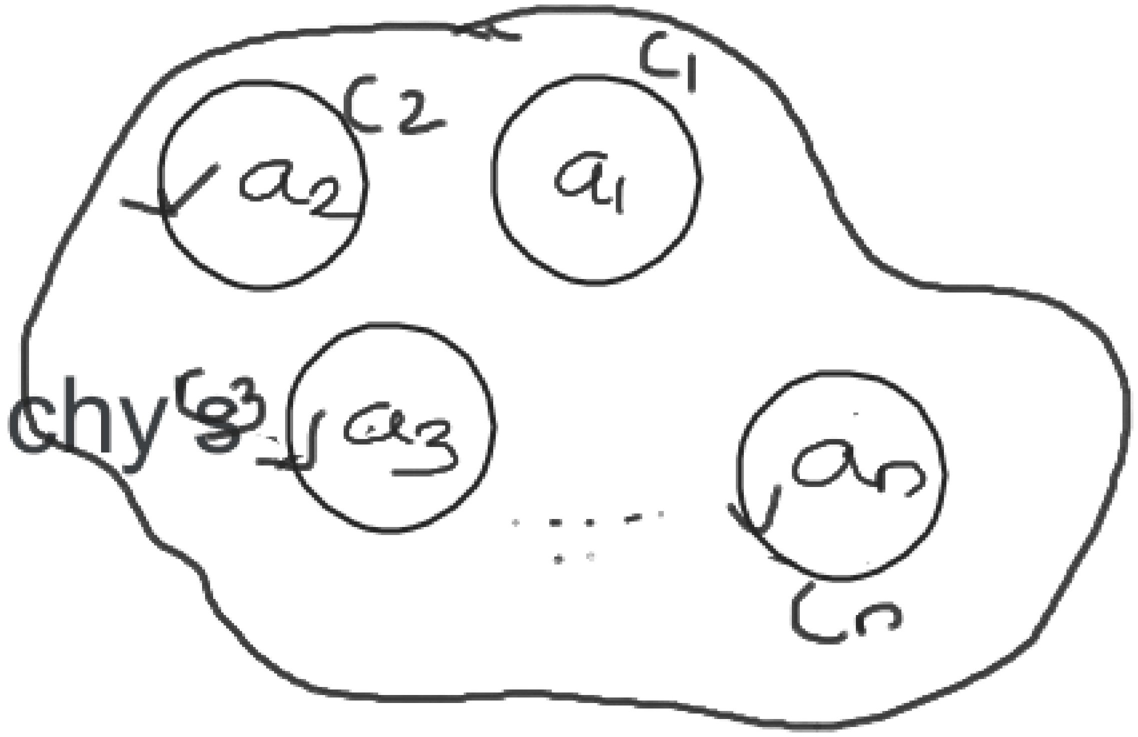
$$\int_C f(z)dz = 2\pi i \text{ (sum of residues at singular points within C)}$$

Proof

Around each of the isolated singular points a_1 ,a_2 ,a_3 ,a_4 ,a_5 .... a_n ,

draw non intersecting circles C1,C2,C3,..................Cn lying wholly inside C with centers at z = a1,a2,a3,........an respectively

f(z) is analytic in the multiply connected region bounded by C,C1,C2,C3,......Cn, we have by Cauchy's theorem for multiply connected region



$$\oint_C f(z)dz = \int_{C1} f(z)dz + \int_{C2} f(z)dz + \ldots\ldots + \int_{Cn} f(z)dz$$

$$= 2\Pi i\{\text{ Res } f(a1) + \text{ Res } f(a2) + \ldots\ldots + \text{Res } f(an)\ \}$$

CALCULATION OF RESIDUES

1)If f(z)has simple pole at z=a ,then

$$\boxed{\text{Res } f(a) = \underset{z\to a}{\text{Lt}} \ [(z-a)f(z)]}$$

2)f(z) = $\Phi(z)/\psi(z)$ where $\psi(z) = (z-a)F(z)$, $F(a) \neq 0$

$$\boxed{\text{Res } f(a) = \frac{\Phi(a)}{\psi'(a)}}$$

3)If f(z) has a pole of order n at z =a ,then

$$\boxed{\text{Res } f(a) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \{[(z-a)^n f(z)]\}_{z=a}}$$

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1) Find the sum of residues of $f(z) = \frac{\sin z}{z\cos z}$ at its poles inside the circle $|z| = 2$

Ans

poles of f(z) is obtained by equating denominator =0

$$z\cos z = 0$$

ie $z = 0$ and $\cos z = 0$

implies $z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\ldots$

only poles $z = \pm\pi/2$ lies inside the circle $|z| = 2$

$$\text{Res } f(0) = \lim_{z\to 0} [(z-0)f(z)] = \lim_{z\to 0} \frac{z\sin z}{z\cos z} = \lim_{z\to 0} \frac{\sin z}{\cos z} = 0$$

$$\text{Res } f(\pi/2) = \lim_{z\to \pi/2} (z-\pi/2)f(z) = \lim_{z\to \pi/2} \frac{(z-\pi/2)\sin z}{z\cos z} \qquad \frac{0}{0}$$

$f(z) = (z - \pi/2) \sin z$, $g(z) = z \cos z$

$f(\pi/2) = 0$ $\quad g\left(\frac{\pi}{2}\right) = 0$

$\therefore$ indeterminate forms

$$\lim_{z \to \frac{\pi}{2}} \frac{f(z)}{g(z)} = \lim_{z \to \frac{\pi}{2}} \frac{f'(z)}{g'(z)}$$

$$= \lim_{z \to \frac{\pi}{2}} \frac{\left(z - \frac{\pi}{2}\right) \cos z + \sin z \cdot 1}{\cos z - z \sin z}$$

$$= \frac{0 + 1}{0 - \frac{\pi}{2}} = \frac{-2}{\pi}$$

$\lim_{z \to a} \frac{f(z)}{g(z)}$ indeterminate forms

$$\lim_{z \to a} \frac{f(z)}{g(z)} = \lim_{z \to a} \frac{f'(z)}{g'(z)}$$

$$\operatorname{Res} f\left(-\frac{\pi}{2}\right) = \underset{z \to -\frac{\pi}{2}}{Lt} \left\{ \frac{\left(z+\frac{\pi}{2}\right)\sin z}{z\cos z} \right\} \qquad \frac{0}{0} \qquad \frac{\sin z}{z\cos z} = f(z)$$

$$= \underset{z \to \frac{\pi}{2}}{Lt} \frac{f'(z)}{g'(z)}$$

$$= \lim_{z \to -\frac{\pi}{2}} \left[\frac{\left(z+\frac{\pi}{2}\right)\cos z + 1 \cdot \sin z}{z\,\text{-}\sin z + \cos z \cdot 1} \right] = \frac{\sin -\frac{\pi}{2}}{-\frac{\pi}{2} \times -\sin -\frac{\pi}{2}}$$

$$= \frac{2}{\pi}$$

Sum of res $= 0 - \frac{2}{\pi} + \frac{2}{\pi} = \underline{0}$

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2)Determine the poles of the function

$f(z) = z^2/(z-1)^2(z+2)$ and the residue at each pole .Hence evaluate

$\int_C f(z)dz$ where C is the circle $|z| = 2.5$

Ans :

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$

For poles equate Denominator = 0

$$(z-1)^2(z+2) = 0 \Rightarrow z = 1, \ z = -2$$

$$\lim_{z \to a} (z-a) f(z) = \lim_{z \to -2} (z - -2) f(z)$$

$$= \lim_{z \to -2} (z+2) \frac{z^2}{(z-1)^2(z+2)} = \lim_{z \to -2} \frac{z^2}{(z-1)^2}$$

$$= \frac{(-2)^2}{(-2-1)^2} = \frac{4}{9}$$

$\lim_{z \to -2} (z - -2) f(z)$ finite

$\Rightarrow z = -2$ is a simple pole

$\text{Res} f(-2) = \frac{4}{9}$ //

$$\lim_{z \to 1} (z-1)^2 f(z) = \lim_{z \to 1} (z-1)^2 \times \frac{z^2}{(z-1)^2(z+2)}$$

$$= \lim_{z \to 1} \frac{z^2}{z+2} = \frac{1}{1+2} = \frac{1}{3} \text{ non zero finite}$$

$\therefore f(z)$ has a pole of order 2 at $z=1$

$$\operatorname{Res} f(a) = \frac{1}{(n-1)!} \left\{ \frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z) \right\}_{z=a}$$

$$\operatorname{Res} f(1) = \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}} (z-1)^2 f(z)$$

$$= \left. \frac{d}{dz} (z-1)^2 f(z) \right|_{z=1} = \frac{d}{dz} \frac{(z-1)^2 z^2}{(z-1)^2 (z+2)}$$

$$= \left\{ \frac{d}{dz} \left[\frac{z^2}{z+2} \right] \right\}_{z=1}$$

$$= \left. \frac{(z+2)2z - z^2 \times 1}{(z+2)^2} \right\}_{z=1}$$

$$\text{Res} f(1) = \left. \frac{2z^2 + 4z - z^2}{(z+2)^2} \right\}_{z=1} = \left. \frac{z^2+4z}{(z+2)^2} \right\}_{z=1} = \frac{1+4}{(1+2)^2}$$

$$= \frac{5}{9}$$

$$\oint_C f(z)\,dz = 2\pi i \left[\text{Res} f(1) + \text{Res} f(-2) \right] = 2\pi i \left[\frac{5}{9} + \frac{4}{9} \right] = 2\pi i$$

Another

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$

put $z = 1+t$

$$f(z) = \frac{(1+t)^2}{t^2(3+t)}$$

$$= \left[\frac{1+t}{t}\right]^2 \frac{1}{3\left(1+\frac{t}{3}\right)}$$

$$= \left(\frac{(1+t)^2}{t^2}\right) \frac{1}{3}\left[1+\frac{t}{3}\right]^{-1}$$

$$= \left(\frac{(1+t)^2}{t^2}\right) \frac{1}{3}\left(1 - \frac{t}{3} + \left(\frac{t}{3}\right)^2 - \cdots\right)$$

$$\frac{1+2t+t^2}{t^2} \times \frac{1}{3}\left(1 - \frac{t}{3} + \frac{t^2}{9} - \frac{t^3}{27} + \cdots\right)$$

$$= \left[\frac{1}{3t^2} - \frac{t}{9t^2} + \frac{t^2}{9t^2} + \frac{2t}{3t^2} - \frac{2t^2}{9t^2}\right]$$

$$\frac{1}{3} - \frac{t}{9} + \cdots$$

$$\text{Coeff } \frac{1}{t} = \frac{-1}{9} + \frac{2}{3} = \frac{-1}{9} + \frac{6}{9} = \frac{5}{9}$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$