Least square problem for polynomial regression

import library

In [ ]:

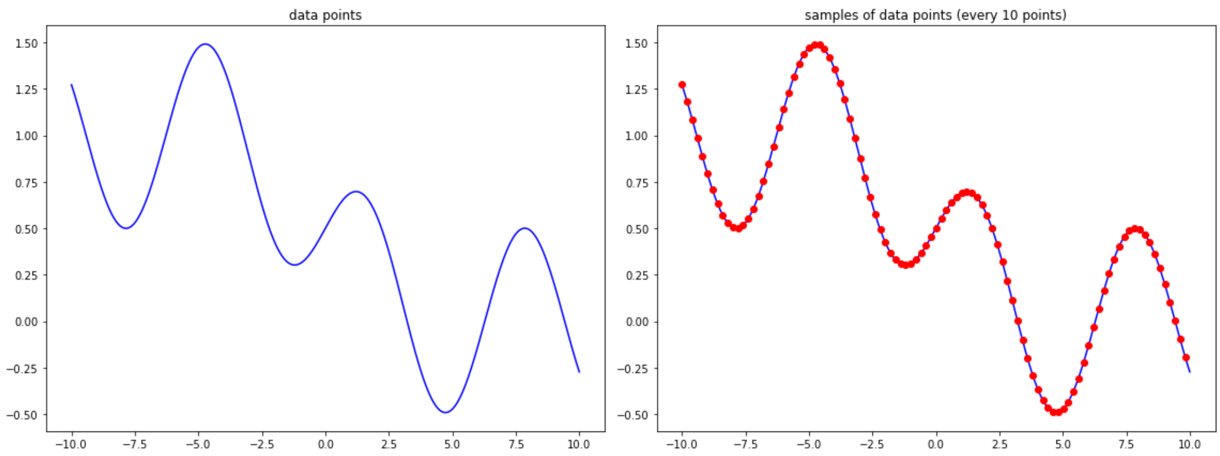
```
import numpy as np
import matplotlib.image as img
import matplotlib.pyplot as plt
import matplotlib.colors as colors
```

load data points

- $\{(x_i, y_i)\}_{i=1}^{n}$

In [ ]:

```
filename    = 'assignment_05_data.csv'
data        = np.loadtxt(filename, delimiter = ',')

x           = data[0, :]    # independent variable
y           = data[1, :]    # dependent variable

x_sample    = x[::10]
y_sample    = y[::10]

plt.figure(figsize=(16,6))

plt.subplot(121)
plt.plot(x, y, '-', color = 'blue')
plt.title('data points')

plt.subplot(122)
plt.plot(x, y, '-', color = 'blue')
plt.plot(x_sample, y_sample, 'o', color = 'red')
plt.title('samples of data points (every 10 points)')

plt.tight_layout()
plt.show()
```



solve a linear system of equation $Az = b$

$$A = \begin{bmatrix} x_1^0 & x_1^1 & \cdots & x_1^{p-1} \\ x_2^0 & x_2^1 & \cdots & x_2^{p-1} \\ \vdots & \vdots & \vdots & \vdots \\ x_n^0 & x_n^1 & \cdots & x_n^{p-1} \end{bmatrix}, \quad z = \begin{bmatrix} \theta_0 \\ \theta_1 \\ \vdots \\ \theta_{p-1} \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

construct matrix A for the polynomial regression with power $p-1$

- useful functions : `np.power`

In [ ]:

```
def construct_matrix_A(x, p):

    n = len(x)
    A = np.zeros([n, p])
    # ++++++++++++++++++++++++++++++++++++++++++++++++++
    # complete the blanks
    #
    A = np.zeros((n, 1))
    for i in range(p):
        PW = np.ones((1, n))*i
        temp = np.power(x,PW).reshape(n, 1)
        A = np.hstack((A, temp))

    A = np.delete(A, 0, axis =1)
    #
    # ++++++++++++++++++++++++++++++++++++++++++++++++++

    return A
```

construct vector b

In [ ]:

```
def construct_vector_b(y):

    n = len(y)
    b = np.zeros([n, 1])

    # ++++++++++++++++++++++++++++++++++++++++++++++++++
    # complete the blanks
    #
    b = y

    #
    # ++++++++++++++++++++++++++++++++++++++++++++++++++

    return b
```

solve the linear system of equation $Az = b$

- without regularization : $\min \frac{1}{2n} \|Az - b\|^2, \quad z = \left(A^T A\right)^{-1} A^T b$
- useful functions : `np.matmul, np.linalg.inv, np.sum`

In [ ]:

```
def solve_regression(x, y, p):
```

```
    z        = np.zeros([p, 1])
    loss     = 0

    # ++++++++++++++++++++++++++++++++++++++++++++++++++
    # complete the blanks
    #
    b = construct_vector_b(y)
    A = construct_matrix_A(x, p)
    coeff_matrix = np.matmul(A.T, A)
    coeff_matrix_inv = np.linalg.inv(coeff_matrix)
    temp = np.matmul(coeff_matrix_inv, A.T)
    z = np.matmul(temp, b)

    n = len(x)

    loss = np.sum(np.power(np.matmul(A,z) - b, 2)) / (2*n)
    #
    # ++++++++++++++++++++++++++++++++++++++++++++++++++

    return z, loss
```

- with regularization : $\min \frac{1}{2n}\|Az - b\|^2 + \frac{\alpha}{2}\|z\|^2, \quad z = \left(A^T A + n\alpha I\right)^{-1} A^T b$ where I denotes identity matrix
- useful functions : `np.matmul, np.linalg.inv, np.sum`

In [ ]:

```
def solve_regression_with_regularization(x, y, p, alpha):

    z        = np.zeros([p, 1])
    loss     = 0

    # ++++++++++++++++++++++++++++++++++++++++++++++++++
    # complete the blanks
    #
    n = len(x)
    b = construct_vector_b(y)
    A = construct_matrix_A(x, p)

    coeff_matrix = np.matmul(A.T, A)
    alpha_temp = np.identity(p)*alpha*n
    temp = coeff_matrix +alpha_temp
    temp = np.matmul(np.linalg.inv(temp), A.T)
    z = np.matmul(temp, construct_vector_b(y))



    loss = np.sum(np.power(np.matmul(A,z) - b, 2)) / (2*n) + np.sum(np.power(z,2)) *

    #
    # ++++++++++++++++++++++++++++++++++++++++++++++++++

    return z, loss
```

approximate by polynomial regression

- $\hat{y} = Az^*$
- useful functions : `np.matmul`

In [ ]:
```
def approximate(x, y, p):

    n       = len(y)
    y_hat   = np.zeros([n, 1])
    loss    = 0

    # ++++++++++++++++++++++++++++++++++++++++++++++++++
    # complete the blanks
    #
    A = construct_matrix_A(x, p)
    (z, loss) =  solve_regression(x,y,p)
    y_hat = np.matmul(A, z)

    #
    # ++++++++++++++++++++++++++++++++++++++++++++++++++

    return y_hat, loss
```

In [ ]:
```
def approximate_with_regularization(x, y, p, alpha):

    n       = len(y)
    y_hat   = np.zeros([n, 1])
    loss    = 0

    # ++++++++++++++++++++++++++++++++++++++++++++++++++
    # complete the blanks
    #
    A = construct_matrix_A(x, p)
    (z, loss) =  solve_regression_with_regularization(x,y,p,alpha)
    y_hat = np.matmul(A, z)

    #
    # ++++++++++++++++++++++++++++++++++++++++++++++++++

    return y_hat, loss
```

---

functions for presenting the results

In [ ]:
```
def function_result_01():

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.title('data points')
    plt.show()
```

In [ ]:
```
def function_result_02():

    p           = 2
    (y_hat, _)  = approximate(x, y, p)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
```

```
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:

```
def function_result_03():

    p           = 4
    (y_hat, _)  = approximate(x, y, p)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:

```
def function_result_04():

    p           = 8
    (y_hat, _)  = approximate(x, y, p)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:

```
def function_result_05():

    p           = 16
    (y_hat, _)  = approximate(x, y, p)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:

```
def function_result_06():

    p           = 32
    (y_hat, _)  = approximate(x, y, p)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:

```
def function_result_07():

    p           = 2
    alpha       = 0.1
    (y_hat, _)  = approximate_with_regularization(x, y, p, alpha)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:

```
def function_result_08():
```

```
    p            = 4
    alpha        = 0.1
    (y_hat, _)   = approximate_with_regularization(x, y, p, alpha)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:

```
def function_result_09():

    p            = 8
    alpha        = 0.1
    (y_hat, _)   = approximate_with_regularization(x, y, p, alpha)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:

```
def function_result_10():

    p            = 16
    alpha        = 0.1
    (y_hat, _)   = approximate_with_regularization(x, y, p, alpha)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:

```
def function_result_11():

    p            = 32
    alpha        = 0.1
    (y_hat, _)   = approximate_with_regularization(x, y, p, alpha)

    plt.figure(figsize=(8,6))
    plt.plot(x, y, '-', color='blue')
    plt.plot(x, y_hat, '-', color='red')
    plt.show()
```

In [ ]:

```
def function_result_12():

    p            = 4
    (_, loss)    = approximate(x, y, p)

    print('loss = ', loss)
```

In [ ]:

```
def function_result_13():

    p            = 16
    (_, loss)    = approximate(x, y, p)

    print('loss = ', loss)
```

In [ ]:
```
def function_result_14():

    p           = 4
    alpha       = 0.1
    (_, loss)   = approximate_with_regularization(x, y, p, alpha)

    print('loss = ', loss)
```

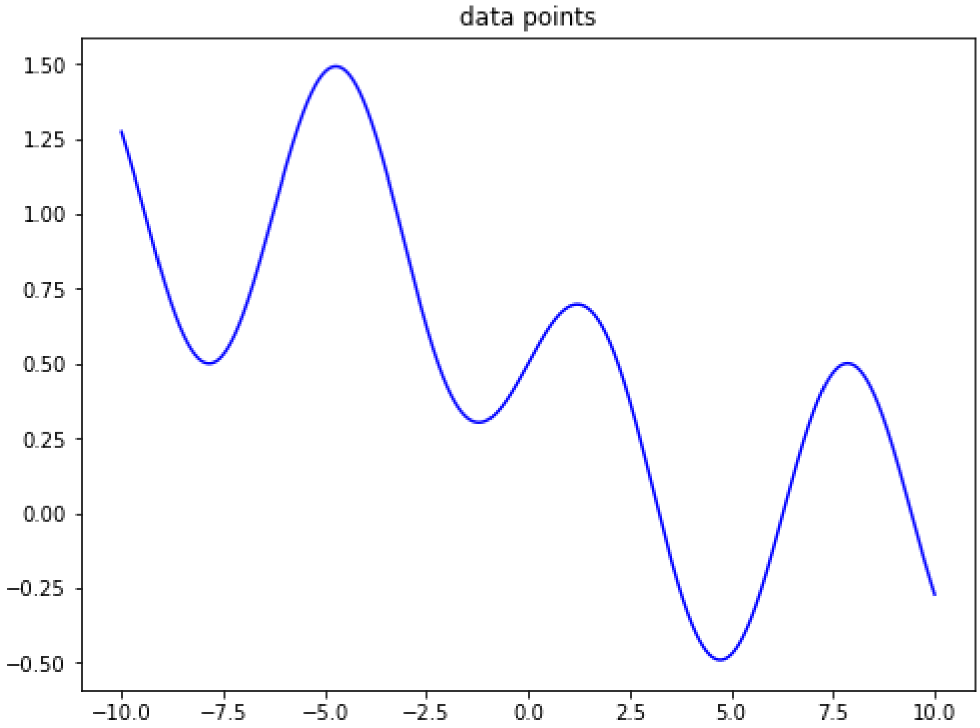
In [ ]:
```
def function_result_15():

    p           = 16
    alpha       = 0.1
    (_, loss)   = approximate_with_regularization(x, y, p, alpha)

    print('loss = ', loss)
```

---

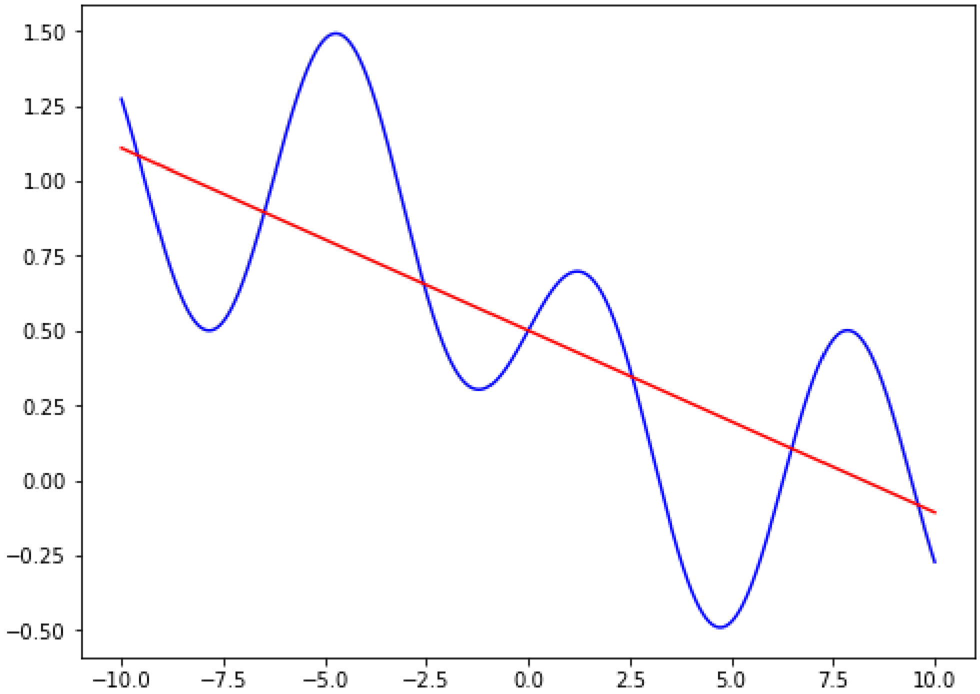
results

In [ ]:
```
number_result = 15

for i in range(number_result):
    title = '## [RESULT {:02d}]'.format(i+1)
    name_function = 'function_result_{:02d}()'.format(i+1)

    print('**************************************************')
    print(title)
    print('**************************************************')
    eval(name_function)
```

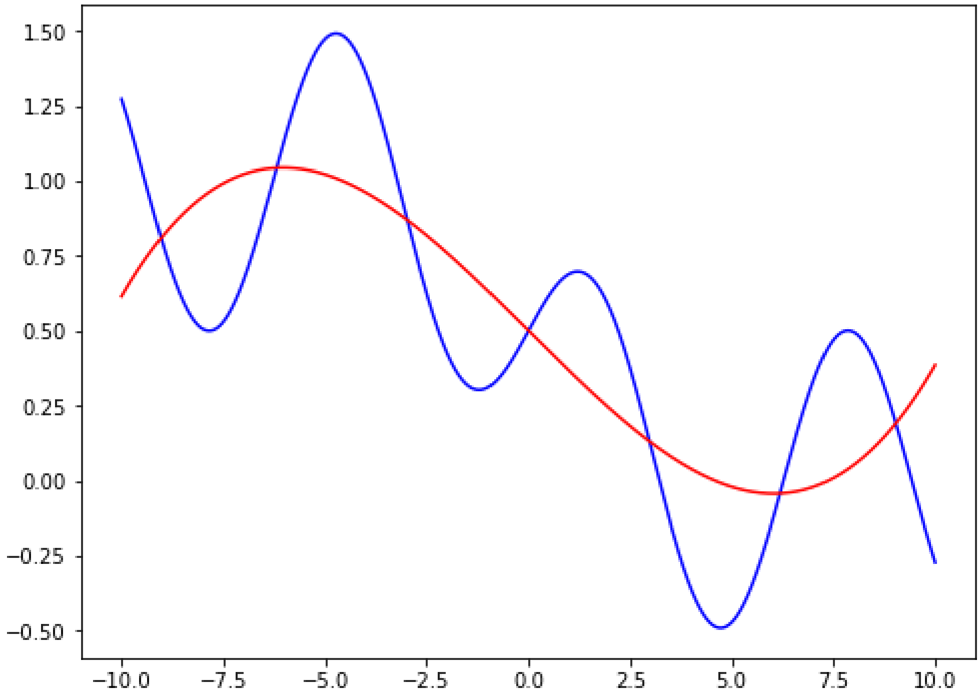
```
**************************************************
## [RESULT 01]
**************************************************
```



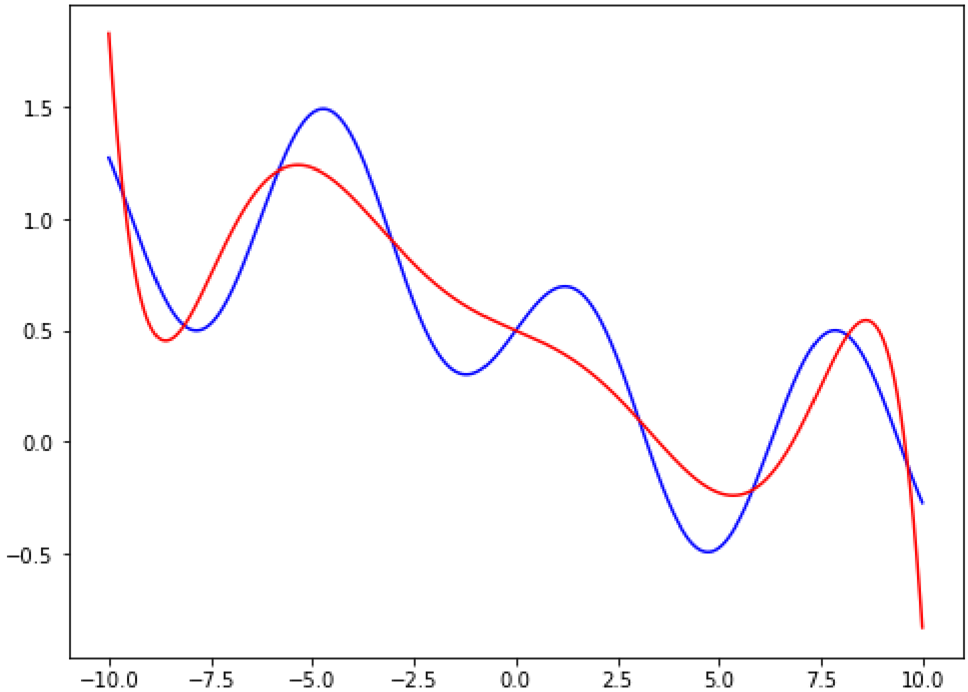

```
**************************************************
## [RESULT 02]
**************************************************
```



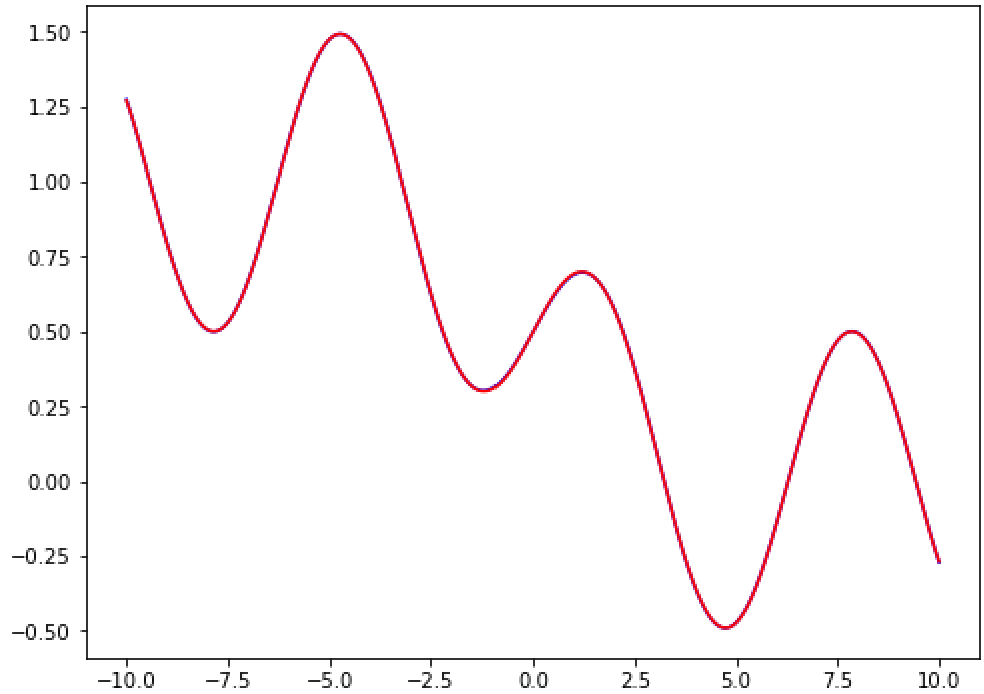

```
**************************************************
## [RESULT 03]
**************************************************
```



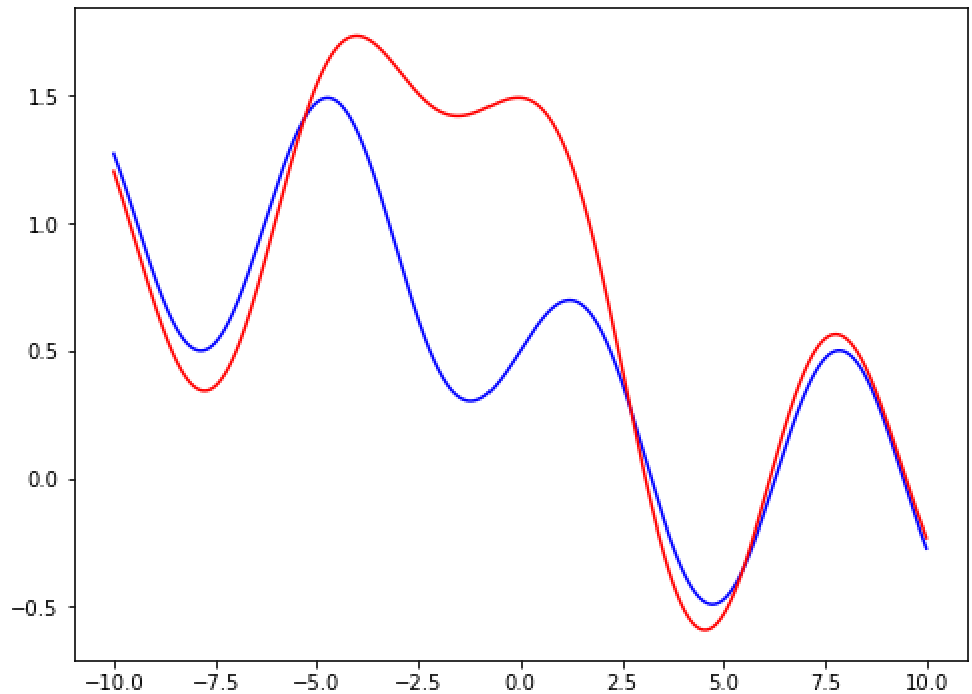

```
**************************************************
## [RESULT 04]
**************************************************
```



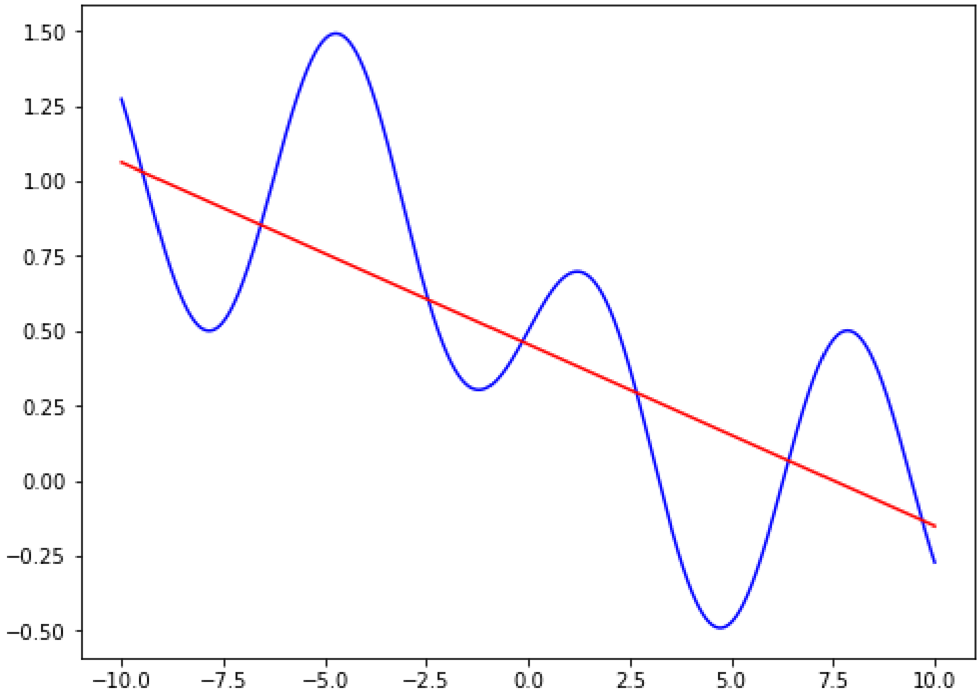

```
**************************************************
## [RESULT 05]
**************************************************
```



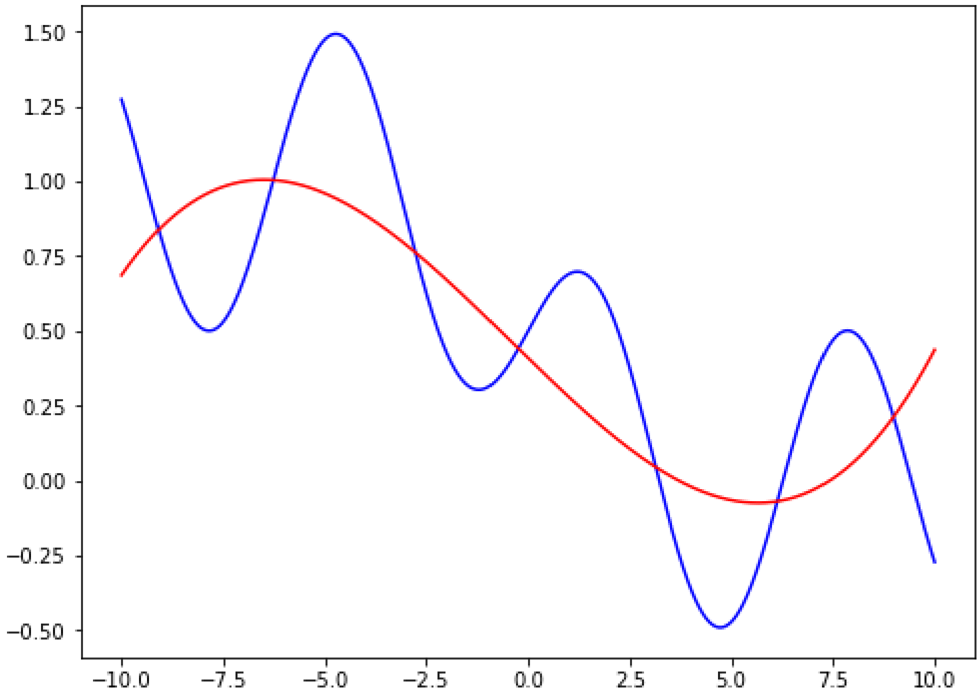

```
**************************************************
## [RESULT 06]
**************************************************
```



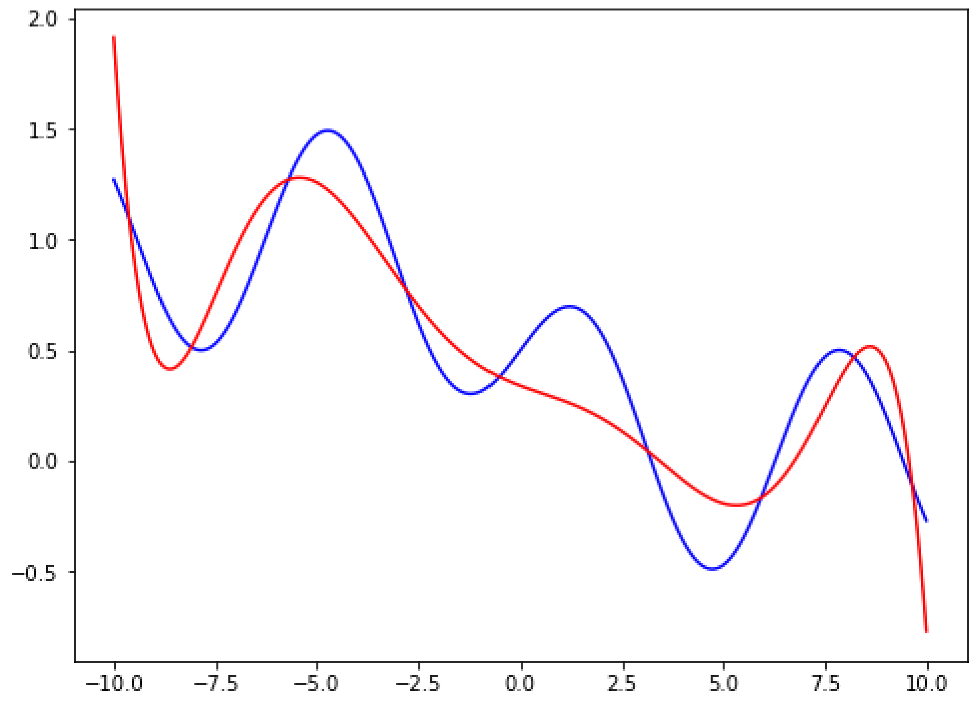

```
**************************************************
## [RESULT 07]
**************************************************
```



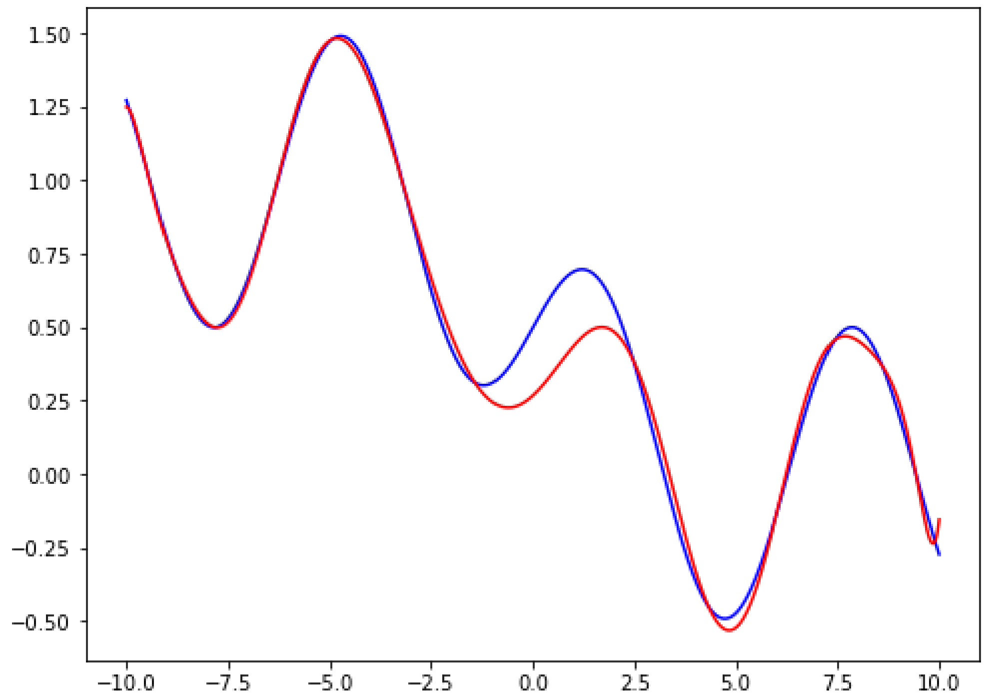

```
**************************************************
## [RESULT 08]
**************************************************
```



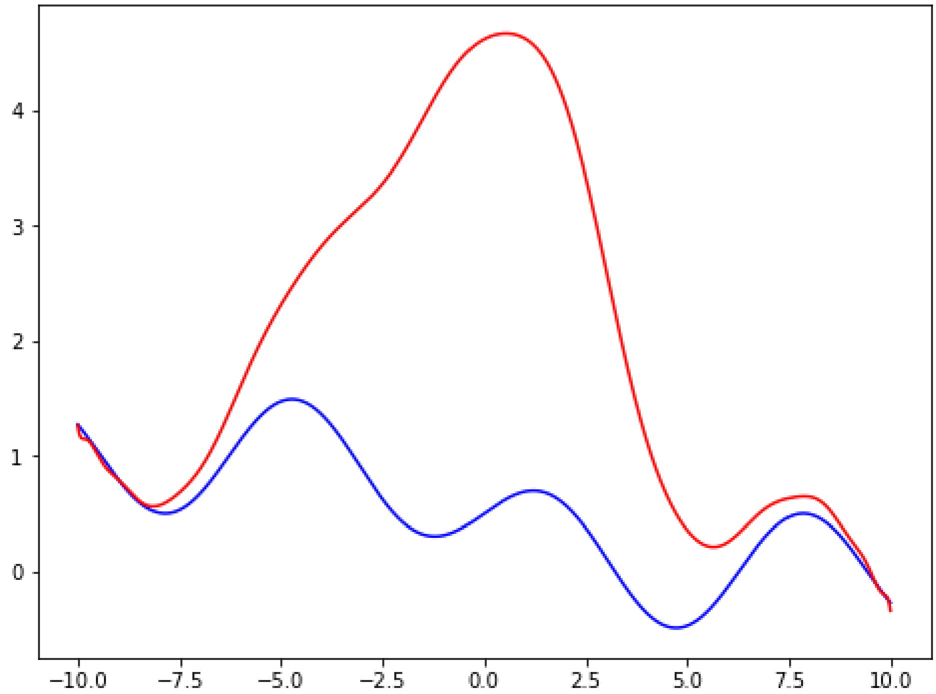

```
**************************************************
## [RESULT 09]
**************************************************
```




```
**************************************************
## [RESULT 10]
**************************************************
```




```
**************************************************
## [RESULT 11]
**************************************************
```




```
**************************************************
## [RESULT 12]
**************************************************
loss =  0.05269780686325942
**************************************************
## [RESULT 13]
**************************************************
loss =  1.208205906835679e-06
**************************************************
## [RESULT 14]
**************************************************
loss =  0.06379867384617091
**************************************************
## [RESULT 15]
**************************************************
loss =  0.00844492935699587
```