Extended Kalman Filter – Two Explicit Numeric Iterations

Contents

1 Problem set-up

- **State**: $\boldsymbol{x} = [x,\, y,\, \theta]^\top$ (metres, radians).
- **Landmark**: $\boldsymbol{m} = [2,\, 1]^\top$ m (known a priori).
- **Control**: $\boldsymbol{u} = [v,\, \omega]^\top$

$$v = 0.500\,00\,\mathrm{m/s}, \qquad \omega = 10°/s = 0.17453 \text{ rad/s}.$$

- **Timestep**: $\Delta t = 0.100\,00\,\mathrm{s}$.
- **Initial belief**

$$\boldsymbol{x}_{0|0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad P_{0|0} = (0.5^2,\, 0.5^2,\, (10°)^2).$$

- **Noise covariances**

$$Q = (0.1^2,\, 0.1^2,\, (1°)^2), \qquad R = (0.50^2,\, (5°)^2).$$

Motion model (discrete unicycle)

$$f(\boldsymbol{x}, \boldsymbol{u}) = \begin{pmatrix} x + v\cos\theta\,\Delta t \\ y + v\sin\theta\,\Delta t \\ \theta + \omega\,\Delta t \end{pmatrix}.$$

State-Jacobian

$$F_x = \frac{\partial f}{\partial \boldsymbol{x}} = \begin{pmatrix} 1 & 0 & -v\sin\theta\,\Delta t \\ 0 & 1 & v\cos\theta\,\Delta t \\ 0 & 0 & 1 \end{pmatrix}. \tag{$*$}$$

Control-Jacobian

$$B_x = \frac{\partial f}{\partial \boldsymbol{u}} = \begin{pmatrix} \cos\theta\,\Delta t & 0 \\ \sin\theta\,\Delta t & 0 \\ 0 & \Delta t \end{pmatrix}. \tag{$**$}$$

Observation model (range–bearing)

$$h(\boldsymbol{x}) = \begin{pmatrix} r \\ \phi \end{pmatrix}, \quad r = \|\boldsymbol{m} - \boldsymbol{p}\|,\ \phi = \operatorname{atan2}(y_m - y,\ x_m - x) - \theta.$$

Iteration 1

Prediction (k=1 | 0)

$$\boldsymbol{x}_{1|0} = f(\boldsymbol{x}_{0|0}, \boldsymbol{u}) = \begin{pmatrix} 0.05 \\ 0 \\ 0.017453 \end{pmatrix}.$$

With $\theta = 0$,

$$F_x^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0.05 \\ 0 & 0 & 1 \end{pmatrix}, \qquad B_x^{(1)} = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \\ 0 & 0.1 \end{pmatrix}.$$

$$P_{1|0} = F_x^{(1)} P_{0|0} F_x^{(1)\mathsf{T}} + B_x^{(1)} Q B_x^{(1)\mathsf{T}} = \begin{bmatrix} 0.250100 & 0 & 0 \\ 0 & 0.250176 & 0.001520 \\ 0 & 0.001520 & 0.030465 \end{bmatrix}.$$

Correction (k=1 | 1)

z_{pred} has been computed by a known landmarks spot

$$\boldsymbol{z}_{\text{pred}} = \begin{pmatrix} 2.19146 \\ 0.45640 \end{pmatrix}, \quad H^{(1)} = \begin{pmatrix} -0.8898 & -0.4563 & 0 \\ 0.2082 & -0.4060 & -1 \end{pmatrix}.$$

Camera-derived reading $\boldsymbol{z}_1 = \begin{pmatrix} 3.07349 \\ 0.49132 \end{pmatrix}$.

$$\boldsymbol{y}_1 = \boldsymbol{z}_1 - \boldsymbol{z}_{\text{pred}} = \begin{pmatrix} 0.88203 \\ 0.03492 \end{pmatrix}, \qquad S^{(1)} = H^{(1)} P_{1|0} H^{(1)\mathsf{T}} + R,$$

$$K^{(1)} = P_{1|0} H^{(1)\mathsf{T}} (S^{(1)})^{-1}.$$

$$\boxed{\boldsymbol{x}_{1|1} = \boldsymbol{x}_{1|0} + K^{(1)} \boldsymbol{y}_1 = \begin{pmatrix} -0.32319 \\ -0.23926 \\ 0.00478 \end{pmatrix}}, \quad P_{1|1} = (I - K^{(1)} H^{(1)}) P_{1|0}.$$

Iteration 2

Prediction $(k{=}2\,|\,1)$

Heading now $\theta_{1|1} = 0.00478$ rad:

$$F_x^{(2)} = \begin{pmatrix} 1 & 0 & -v\sin\theta\,\Delta t \\ 0 & 1 & v\cos\theta\,\Delta t \\ 0 & 0 & 1 \end{pmatrix}_{\theta=0.00478} = \begin{pmatrix} 1 & 0 & -0.00024 \\ 0 & 1 & 0.05000 \\ 0 & 0 & 1 \end{pmatrix},$$

$$B_x^{(2)} = \begin{pmatrix} 0.10000 & 0 \\ 0.00048 & 0 \\ 0 & 0.10000 \end{pmatrix}.$$

$$\boldsymbol{x}_{2|1} = f(\boldsymbol{x}_{1|1}, \boldsymbol{u}) = \begin{pmatrix} -0.27319 \\ -0.23902 \\ 0.02223 \end{pmatrix},$$

$$P_{2|1} = F_x^{(2)} P_{1|1} F_x^{(2)\mathsf{T}} + B_x^{(2)} Q B_x^{(2)\mathsf{T}}.$$

Correction $(k{=}2\,|\,2)$

$$\boldsymbol{z}_{\text{pred}} = \begin{pmatrix} 2.58893 \\ 0.47681 \end{pmatrix}, \quad \boldsymbol{z}_2 = \begin{pmatrix} 3.40110 \\ 0.42342 \end{pmatrix}, \quad \boldsymbol{y}_2 = \begin{pmatrix} 0.81217 \\ -0.05339 \end{pmatrix}.$$

Compute $H^{(2)}, S^{(2)}, K^{(2)}$ analogously; the final updated state:

$$\boxed{\boldsymbol{x}_{2|2} = \begin{pmatrix} -0.51849 \\ -0.35986 \\ 0.04329 \end{pmatrix}}, \qquad P_{2|2} = \begin{bmatrix} 0.088289 & -0.007937 & 0.018186 \\ -0.007937 & 0.095784 & -0.033176 \\ 0.018186 & -0.033176 & 0.017659 \end{bmatrix}.$$

2 Key points

- The theoretical Jacobian $(*)$ shows that only the $(0,2)$ and $(1,2)$ elements depend on heading.
- Numeric $F_x^{(2)}$ uses $-v\sin\theta\Delta t = -0.00024$, fixing the earlier factor-10 error.
- Subsequent matrices P, S, K must be recomputed whenever F_x is corrected.