MATH 138 Personal Notes

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Common Integrals:

Function	Integral
x^n	$\frac{x^{n+1}}{n+1} + C$
$\frac{1}{x}$	$\ln\|x\| + C$
e^x	$e^x + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\frac{1}{1+x^2}$	$\arctan(x) + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C$
$\frac{-1}{\sqrt{1-x^2}}$	$\arccos(x) + C$
$\sec(x)\tan(x)$	$\sec(x) + C$
a^x where $a \in \mathbb{R} > 0, a \neq 1$	$\frac{a^x}{\ln(a)} + C$

Inverse Trigonometric Substitutions:

Class of Integrand	Integral	Trig Substitution	Trig Identity
$\sqrt{a^2 - b^2x^2}$	$\int \sqrt{a^2 - b^2x^2}\mathrm{d}x$	$bx = a\sin(u)$	$\sin^2(x) + \cos^2(x) = 1$
$\sqrt{a^2 + b^2x^2}$	$\int \sqrt{a^2 + b^2x^2}\mathrm{d}x$	$bx = a\tan(u)$	$\sec^2(x) - 1 = \tan^2(x)$
$\sqrt{b^2x^2 - a^2}$	$\int \sqrt{b^2x^2 - a^2}\mathrm{d}x$	$bx = a\sec(u)$	$\sec^2(x) - 1 = \tan^2(x)$

Contents

Integration

Def Riemann Sum: Given a bounded function f on $[a, b]$, a partition P where $a = t_0 < t_1 < \dots t_{n-1} < t_n = b$, and a set $\{c_1, c_2, \dots c_n\}$ where $c_i \in [t_{i-1}, t_i]$, then the Riemann sum for f is of the form

$$S = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

Note: The norm of the partition P is the length of the widest subinterval denoted by

$$\|P\| = \max\{\Delta t_1, \Delta t_2, \dots \Delta t_n\}$$

- **Def** Right-hand Riemann Sum: The Riemann sum R obtained by choosing $c_i = t_i$

$$R = \sum_{i=1}^{n} f(t_i)\Delta t_i$$

- **Def** Left-hand Riemann Sum: The Riemann sum R obtained by choosing $c_i = t_{i-1}$

$$R = \sum_{i=1}^{n} f(t_{i-1})\Delta t_i$$

Def Regular n-Partition: Given an interval $[a, b]$ and an $n \in \mathbb{N}$, a regular n-partition is the partition P^n where each subinterval has the same length, thus $\Delta t_i = \frac{b-a}{n}$

Right-hand Regular Sum: $$R_n = \sum_{i=1}^{n} f(t_i)\frac{b-a}{n} = \sum_{i=1}^{n} f\left(a + i\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right)$$

Left-hand Regular Sum: $$R_n = \sum_{i=1}^{n} f(t_{i-1})\frac{b-a}{n} = \sum_{i=1}^{n} f\left(a + (i-1)\left(\frac{b-a}{n}\right)\right)\left(\frac{b-a}{n}\right)$$

Def Definite Integral: A bounded function f is integrable on $[a, b]$ if there exists a unique $I \in \mathbb{R}$ such that if $\{P_n\}$ is a sequence of partitions with $\lim_{n\to\infty}\|P_n\| = 0$ and $\{S_n\}$ is a sequence of Riemann sums, we have

$$\lim_{n\to\infty} S_n = I$$

We call I the integral of f over $[a, b]$ and denote it by

$$\int_a^b f(t)\mathrm{d}t$$

where a and b are the limits of integration, $f(t)$ is the integrand, and t is the variable of integration

1.1 Integrability Theorem for Continuous Functions

Let f be continuous on $[a, b]$. Then f is integrable on $[a, b]$. Moreover, Let

$$S_n = \sum_{i=1}^{n} f(c_i)\Delta t_i$$

with regular n-partitions, then

$$\int_a^b f(t)\mathrm{d}t = \lim_{n\to\infty} S_n$$

1.2 Properties of Integrals

Assuming that f and g are integrable on the interval $[a, b]$:

(i) For any $c \in \mathbb{R}$, $\int_a^b cf(t)\mathrm{d}t = c\int_a^b f(t)\mathrm{d}t$

(ii) $\int_a^b (f+g)(t)\mathrm{d}t = \int_a^b f(t)\mathrm{d}t + \int_a^b g(t)\mathrm{d}t$

(iii) If $m \leq f(t) \leq M$ for all $t \in [a, b]$, then $m(b-a) \leq \int_a^b f(t)\mathrm{d}t \leq M(b-a)$

(iv) If $0 \leq f(t)$ for all $t \in [a, b]$, then $0 \leq \int_a^b f(t)\mathrm{d}t$

(v) If $g(t) \leq f(t)$ for all $t \in [a, b]$, then $\int_a^b g(t)\mathrm{d}t \leq \int_a^b f(t)\mathrm{d}t$

(vi) The function $|f|$ is integrable on $[a, b]$ and $|\int_a^b f(t)\mathrm{d}t| \leq \int_a^b |f(t)|\mathrm{d}t$

Def Identical Limits of Integration: Let $f(t)$ be defined at $t = a$, then

$$\int_a^a f(t)\mathrm{d}t = 0$$

Def Switching the Limits of Integration: Let f be integrable on the interval $[a, b]$ where $a < b$, then

$$\int_b^a f(t)\mathrm{d}t = -\int_a^b f(t)\mathrm{d}t$$

1.3 Integrals over Subintervals

Assume that f is integrable on an interval I containing a, b, and c. Then

$$\int_a^b f(t)\mathrm{d}t = \int_a^c f(t)\mathrm{d}t + \int_c^b f(t)\mathrm{d}t$$

Def Average Value of f: If f is continuous on $[a, b]$, the average value of f is

$$\frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$$

1.4 Average Value Theorem

Assume that f is continuous on $[a, b]$, then there exists $a \leq c \leq b$ such that

$$f(c) = \frac{1}{b-a}\int_a^b f(t)\mathrm{d}t$$

1.5 Fundamental Theorem of Calculus (Part 1)

Assume that f is continuous on an open interval I that contains a point a. Let

$$G(x) = \int_a^x f(t)\mathrm{d}t$$

Then $G(x)$ is differentiable at each $x \in I$, and

$$G'(x) = f(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t$$

1.6 Extended Version of the Fundamental Theorem of Calculus

Assume that f is continuous and that g and h are differentiable. Let

$$H(x) = \int_{g(x)}^{h(x)} f(t)\mathrm{d}t$$

Then $H(x)$ is differentiable and

$$H'(x) = f(h(x))h'(x) - f(g(x))g'(x)$$

Def Antiderivative: For a function f, its antiderivative is the function F such that

$$F'(x) = f(x)$$

Note: if $F'(x) = f(x)$ for all $x \in I$, then F is an antiderivative for f on I
Note: The family of all antiderivatives is denoted by

$$\int f(x)\mathrm{d}x$$

1.7 Power Rule for Antiderivatives

Assume $\alpha \neq -1$, then

$$\int x^\alpha \mathrm{d}x = \frac{x^{\alpha+1}}{\alpha+1} + C$$

1.8 Fundamental Theorem of Calculus (Part 2)

Assume that f is continuous and that F is any antiderivative of f. Then

$$\int_a^b f(t)\mathrm{d}t = F(b) - F(a)$$

Def Integrals: For an antiderivative F,

$$F(x)|_a^b = F(b) - F(a)$$

1.9 Change of Variables Theorem

Assume that $g'(x)$ is continuous on $[a, b]$ and $f(u)$ is continuous on $g([a, b])$. Then

$$\int_{x=a}^{x=b} f(g(x))g'(x)\mathrm{d}x = \int_{u=g(a)}^{u=g(b)} f(u)\mathrm{d}u$$

Techniques of Integration

Def Integration by Parts:

$$\int f(x)g'(x)\mathrm{d}x = f(x)g(x) - \int f'(x)g(x)\mathrm{d}x$$

2.1 Integration by Parts Theorem

Assume f', g' are continuous on an interval containing a, b. Then

$$\int_a^b f(x)g'(x)\mathrm{d}x = f(x)g(x)\mid_a^b - \int_a^b f'(x)g(x)\mathrm{d}x$$

Def Rational Functions: Are of the form

$$f(x) = \frac{p(x)}{q(x)}$$

Def Type I Partial Fraction Decomposition: Assume that $f(x) = \frac{p(x)}{q(x)}$ where p, q are polynomials such that $degree(p) < degree(q) = k$ and q can be factored into linear terms $q = a(x-a_1)(x-a_2)\dots(x-a_k)$ with distinct roots. Then there exists $A_1, A_2, \dots A_k$ such that

$$f(x) = \frac{1}{a}\left(\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \dots + \frac{A_k}{x-a_k}\right)$$

thus f admits a Type I Partial Fraction Decomposition

2.2 Integration of Partial Fractions

Assume that $f(x) = \frac{p(x)}{q(x)}$ admits a Type I Partial Fraction Decomposition. Then

$$\int f(x)\mathrm{d}x = \frac{1}{a}\left(\int \frac{A_1}{x-a_1}\mathrm{d}x + \dots + \int \frac{A_k}{x-a_k}\mathrm{d}x\right) = \frac{1}{a}\left(A_1 \ln|x-a_1| + \dots + A_k \ln|x-a_k|\right) + C$$

Def Type II Partial Fraction Decomposition: Assume that $f(x) = \frac{p(x)}{q(x)}$ where p, q are polynomials such that $degree(p) < degree(q) = k$ and q can be factored into linear terms $q = a(x-a_1)^{m_1}(x-a_2)^{m_2}\dots(x-a_k)^{m_k}$ with non-distinct roots. Then the partial fraction decomposition is

$$f(x) = \sum_{j=1}^{k}\left(\frac{A_{j,1}}{x-a_j} + \frac{A_{j,2}}{(x-a_j)^{m_2}} + \dots + \frac{A_{j,m_j}}{(x-a_j)^{m_j}}\right)$$

thus f admits a Type II Partial Fraction Decomposition

Note: m_j is the multiplicity of root a_j

Def Type III Partial Fraction Decomposition: Assume that $f(x) = \frac{p(x)}{q(x)}$ where p, q are polynomials such that $degree(p) < degree(q) = k$ but q cannot be factored into linear terms. Suppose q has an irreducible factor $x^2 + bx + c$, then this factor contributes as

$$\frac{B_1 x + C_1}{x^2+bx+c} + \frac{B_2 x + C_2}{(x^2+bx+c)^2} + \dots + \frac{B_m x + C_m}{(x^2+bx+c)^m}$$

thus f admits a Type III Partial Fraction Decomposition

Def Type I Improper Integrals:

1. Let f be integrable on $[a, b]$ where $a \leq b$, then the integral

   $$\int_a^{\infty} f(x)\mathrm{d}x$$

   converges if

   $$\lim_{b\to\infty}\int_a^b f(x)\mathrm{d}x$$

   otherwise it diverges

2. Let f be integrable on $[b, a]$ where $b \leq a$, then the integral

   $$\int_{-\infty}^{a} f(x)\mathrm{d}x$$

   converges if

   $$\lim_{b\to-\infty}\int_b^a f(x)\mathrm{d}x$$

   otherwise it diverges

3. Let f be integrable on $[a, b]$ where $a < b$, then the integral

   $$\int_{-\infty}^{\infty} f(x)\mathrm{d}x$$

   converges for $c \in \mathbb{R}$ if both

   $$\int_{-\infty}^{c} f(x)\mathrm{d}x \text{ and } \int_c^{\infty} f(x)\mathrm{d}x$$

   converge, otherwise it diverges

2.3 p-Test for Type I Improper Integrals

The improper integral

$$\int_1^\infty \frac{1}{x^p}\mathrm{d}x$$

converges if and only if $p > 1$. Then

$$\int_1^\infty \frac{1}{x^p}\mathrm{d}x = \frac{1}{p-1}$$

2.4 Properties of Type I Improper Integrals

Assume that $\int_a^\infty f(x)\mathrm{d}x$ and $\int_a^\infty g(x)\mathrm{d}x$ converge

1. $\int_a^\infty cf(x)\mathrm{d}x$ converges for all $c \in \mathbb{R}$

$$\int_a^\infty cf(x)\mathrm{d}x = c\int_a^\infty f(x)\mathrm{d}x$$

2. $\int_a^\infty f(x) + g(x)\mathrm{d}x$ converges

$$\int_a^\infty f(x) + g(x)\mathrm{d}x = \int_a^\infty f(x)\mathrm{d}x + \int_a^\infty g(x)\mathrm{d}x$$

3. If $f(x) \leq g(x)$ for all $a \leq x$

$$\int_a^\infty f(x)\mathrm{d}x \leq \int_a^\infty g(x)\mathrm{d}x$$

4. If $a < c < \infty$ then $\int_c^\infty f(x)\mathrm{d}x$ converges

$$\int_a^\infty f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^\infty f(x)\mathrm{d}x$$

2.5 The Monotone Convergence Theorem for Functions

Assume that f is non-decreasing on $[a, \infty)$

1. if $\{f(x) \mid x \in [a, \infty)\}$ is bounded above, then

$$\lim_{x\to\infty} f(x) = L = lub(\{f(x) \mid x \in [a, \infty)\})$$

2. If $\{f(x) \mid x \in [a, \infty)\}$ is not bounded above, then $\lim_{x\to\infty} f(x) = \infty$

2.6 Comparison Test for Type I Improper Integrals

Assume $0 \leq g(x) \leq f(x)$ for all $x \geq a$ and that f and g are continuous on $[a, \infty)$

1. if $\int_a^\infty f(x)\mathrm{d}x$ converges, then so does $\int_a^\infty g(x)\mathrm{d}x$
2. if $\int_a^\infty g(x)\mathrm{d}x$ diverges, then so does $\int_a^\infty f(x)\mathrm{d}x$

Fact: If f is integrable on $[a, b)$ for every $b \geq a$ and $f(x) \geq 0$ on $[a, \infty)$, then $\int_a^\infty f(x)\mathrm{d}x$ converges if and only if $\exists M$ such that

$$\int_a^b f(x)\mathrm{d}x \leq M$$

for all $b > a$

Def Absolute Convergence for Type I Improper Integrals: Let f be integrable on $[a, b)$ for all $b \geq a$. Then $\int_a^\infty$ converges absolutely if

$$\int_a^\infty |f(x)|\mathrm{d}x$$

converges

2.7 Absolute Convergence Theorem for Improper Integrals

Let f be integrable on $[a, b]$ for all $b > a$. Then $|f|$ is integrable on $[a, b]$ for all $b > a$. Moreover if

$$\int_a^\infty |f(x)|\mathrm{d}x$$

converges then so does

$$\int_a^\infty f(x)\mathrm{d}x$$

Def Gamma Function: For all $x \in \mathbb{R}$, the gamma function is defined as

$$\Gamma(x) = \int_a^\infty t^{x-1}e^{-1}\mathrm{d}x$$

Def Type II Improper Integrals:

1. Let f be integrable on $[t, b]$ for every $t \in (a, b]$ with $lim_{x \to a^+} = \infty$ or $lim_{x \to a^+} = -\infty$, then the integral
$$\int_a^b f(x) \mathrm{d}x$$
converges if
$$\lim_{t \to a^+} \int_t^b f(x) \mathrm{d}x$$
exists, otherwise it diverges

2. Let f be integrable on $[a, t]$ for every $t \in [a, b)$ with $lim_{x \to b^-} = \infty$ or $lim_{x \to b^-} = -\infty$, then the integral
$$\int_a^b f(x) \mathrm{d}x$$
converges if
$$\lim_{t \to b^-} \int_a^t f(x) \mathrm{d}x$$
exists, otherwise it diverges

3. If f has an infinite discontinuity at $x = c$ where $a < c < b$, then the integral
$$\int_a^b f(x) \mathrm{d}x$$
converges for $c \in \mathbb{R}$ if both
$$\int_a^c f(x) \mathrm{d}x \text{ and } \int_c^b f(x) \mathrm{d}x$$
converge, otherwise it diverges

2.8 p-Test for Type II Improper Integrals

The improper integral
$$\int_0^1 \frac{1}{x^p}$$
converges if and only if $p < 1$, Then
$$\int_0^1 \frac{1}{x^p} = \frac{1}{1-p}$$

Applications of Integration

Def Area Between Curves: Let f, g be continuous on $[a, b]$. The area of a region bounded by f, g, a line at $t = a$ and a line at $t = b$ is

$$A = \int_a^b |g(t) - f(t)| \mathrm{d}t$$

Def Volume of Revolution, Disk Method I: Let f be continuous on $[a, b]$ with $f(x) \geq 0$ for all $x \in [a, b]$. The volume V of the solid of revolution obtained by rotating the region bounded by $f(x), x = a, x = b$ around the x-axis is

$$V = \int_a^b \pi f(x)^2 \mathrm{d}x$$

Def Volume of Revolution, Disk Method II: Let f, g be continuous on $[a, b]$ with $0 \leq f(x) \leq g(x)$ for all $x \in [a, b]$. The volume V of the solid of revolution obtained by rotating the region bounded by $f(x), g(x), x = a, x = b$ around the x-axis is

$$V = \int_a^b \pi(g(x)^2 - f(x)^2) \mathrm{d}x$$

Def Volume of Revolution, Shell Method: Let f, g be continuous on $[a, b]$ with $f(x) \leq g(x)$ for all $x \in [a, b]$. The volume V of the solid of revolution obtained by rotating the region bounded by $f(x), g(x), x = a, x = b$ around the y-axis is

$$V = \int_a^b 2\pi x(g(x) - f(x)) \mathrm{d}x$$

Def Arc Length: Let f be continuously differentiable on $[a, b]$. The arc length S of f is

$$S = \int_a^b \sqrt{1 + (f'(x))^2} \mathrm{d}x$$

Differential Equations

Def Differential Equation: An equation involving an independent variable such as x, a function $y = y(x)$, and various derivatives of y

$$F(x, y, y', y'', \dots y^{(n)}) = 0$$

where a solution is a function φ such that

$$F(x, \varphi(x), \varphi'(x), \varphi''(x), \dots \varphi^{(n)}(x)) = 0$$

Note: The highest order of a derivative is the order of the equation

Def Separable Differential Equation: A first-order differentiable equation is separable if there exists $f = f(x)$ and $g = g(y)$ such that

$$y' = f(x)g(y)$$

1. Identify $f(x)$ and $g(x)$
2. Find Constant (Equilibrium) Solutions
3. Find Implicit Solution
4. Find Explicit Solutions

Def Constant (Equilibrium) Solution to a Separable Differential Equation: If $y' = f(x)g(y)$ is a separable differential equation and $\exists y_0 \in \mathbb{R}$ such that $g(y_0) = 0$ then there is a constant (equilibrium) solution

$$\phi(x) = y_0$$

Method Finding Implicit Solution to a Separable Differential Equation: If $y' = f(x)g(y)$ is a separable differential equation, then when $g(y) \neq 0$ it means $\frac{y'}{g(y)} = f(x)$, thus

$$\int \frac{y'}{g(y)} \mathrm{d}x = \int f(x) \mathrm{d}y \text{ or } \int \frac{1}{g(y)} \mathrm{d}x = \int f(x) \mathrm{d}x$$

which gives the implicit solution

$$F(y) = F(x) + C$$

Def First-Order Linear Differentiable Equations: A first-order differentiable equation is linear if it can be written as

$$y' = f(x)y + g(x)$$

4.1 Solving First-Order Linear Differential Equations Theorem

Let f, g be continuous and $y' = f(x)y + g(x)$ be a first-order linear differential equation. Then the solutions are of the form

$$y = \frac{\int g(x)I(x)\mathrm{d}x}{I(x)}$$

where $I(x) = e^{-\int f(x)\mathrm{d}x}$

4.2 Existence and Uniqueness Theorem for First-Order Linear Differential Equations

Let f, g be continuous on the interval I. Then $\forall x_0 \in I, \forall y_0 \in \mathbb{R}$ the initial value problem

$$y' = f(x)y + g(x) \text{ and } y(x_0) = y_0$$

has exactly one solution $y = \varphi(x)$ on the interval I

Def Exponential Growth and Decay: The solution to a differential equation $P' = kP$ that models unlimited resource growth is

$$P(t) = Ce^{kt}$$

Def Half-life formula: The time it takes for half of a substance to decay is model by

$$t_h = \frac{-\ln(2)}{k}$$

Def Newton's Law of Cooling: If $T(t)$ denotes the temperature of an object and T_a denotes the ambient temperature, then the solution to the differential equation $T' = k(T - T_a)$ is

$$T(t) = Ce^{kt} + T_a$$

Def Logistic Growth: The solution to a differential logistic equation $P' = kP(M - P)$ that models growth with maximum population M is

$$Ce^{Mkt} = \frac{|P(t)|}{|M - P(t)|}$$

thus

1. If $0 < P(0) < M$, then
$$P(t) = M\frac{Ce^{Mkt}}{1 + Ce^{Mkt}}$$
2. If $P(0) > M$, then
$$P(t) = M\frac{Ce^{Mkt}}{Ce^{Mkt} - 1}$$

Numerical Series

Def Series: Given a sequence $\{a_n\}$ the formal sum of the terms (A_i), with indexes i, is the series

$$\sum_{n=1}^{\infty} a_n$$

Def Convergence of a Series: Given a series as defined above, for each $k \in \mathbb{N}$ the k-th partial sum is

$$s_k = \sum_{n=1}^{k} a_n$$

The series converges if the sequence $\{S_k\}$ converges. If $L = \lim_{k\to\infty} S_k$, then

$$\sum_{n=1}^{\infty} a_n = L$$

otherwise it diverges

Def Geometric Series: A geometric series is of the form

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 \dots$$

where r is the ratio of the series

5.1 Geometric Series Test

The geometric series $\sum_{n=0}^{\infty} r^n$ converges if and only if $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

5.2 Divergence Test

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$, thus

$$\lim_{n\to\infty} a_n \neq 0 \implies \sum_{n=1}^{\infty} a_n \text{ diverges}$$

5.3 Arithmetic for Series I

Assume $\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} b_n$ converge, then

1. $\forall c \in \mathbb{R}$,
$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$
2. $$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

5.4 Arithmetic for Series II

Assume $\sum_{n=1}^{\infty} a_n$ converges, then

1. $\forall j \in \mathbb{Z}$, if
$$\sum_{n=1}^{\infty} a_n \text{ converges} \implies \sum_{n=j}^{\infty} a_n \text{ converges}$$
2. If $\exists j \in \mathbb{Z}$ where
$$\sum_{n=j}^{\infty} a_n \text{ converges} \implies \sum_{n=1}^{\infty} a_n \text{ converges}$$

Def Monotonic Sequence: A sequence $\{a_n\}$ is monotonic if it is

1. non-decreasing, that is $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$
2. increasing, that is $a_{n+1} > a_n$ for all $n \in \mathbb{N}$
3. non-increasing, that is $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$
4. decreasing, that is $a_{n+1} < a_n$ for all $n \in \mathbb{N}$

5.5 Monotone Convergence Theorem

Let $\{a_n\}$ be a non-decreasing sequence, then $\{a_n\}$ converges if and only if it is bounded above, that is

1. If $\{a_n\}$ is bounded above, then $\{a_n\}$ converges to $L = lub(\{a_n\})$
2. If $\{a_n\}$ is not bounded above, then $\{a_n\}$ diverges to ∞

Def Positive Series: A series $\sum_{n=1}^{\infty} a_n$ is positive if $a_n \geq 0$ for all $n \in \mathbb{N}$

5.6 Comparison Test for Series

Assume that $0 \leq a_n \leq b_n$ for each $n \in \mathbb{N}$,

1. If $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges
2. If $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges

5.7 Limit Comparison Test

Assume that $a_n > 0, b_n > 0$ for each $n \in \mathbb{N}$, and that

$$\lim_{n \to \infty} \frac{a_n}{b_n} = L$$

1. If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges
2. If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges, equivalently if $\sum_{n=1}^{\infty} a_n$ diverges then $\sum_{n=1}^{\infty} b_n$ diverges
3. If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges, equivalently if $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges

5.8 Integral Test for Convergence

Assume that $f(x) > 0$ is continuous and decreasing on $[a, \infty)$, and the $a_k = f(k)$. For each $n \in \mathbb{N}$, let $S_n = \sum_{k=1}^{n} a_k$, then

1. For all $n \in \mathbb{N}$,
$$\int_1^{n+1} f(x)\mathrm{d}x \leq S_n \leq a_1 + \int_1^n f(x)\mathrm{d}x$$
2. $\sum_{k=1}^{\infty} a_k$ converges if and only if $\int_1^{\infty} f(x)\mathrm{d}x$ converges
3. If $\sum_{k=1}^{\infty} a_k$ converges and $S = \sum_{k=1}^{\infty} a_k$, then
$$\int_1^{\infty} f(x)\mathrm{d}x \leq \sum_{k=1}^{\infty} a_k \leq a_1 + \int_1^{\infty} f(x)\mathrm{d}x$$
and
$$\int_{n+1}^{\infty} f(x)\mathrm{d}x \leq S - S_n \leq \int_n^{\infty} f(x)\mathrm{d}x$$

5.9 p-Series Test

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$

Def Alternating Series: If $a_n > 0$ for all n, then an alternating series is of the form

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n \text{ or } \sum_{n=1}^{\infty}(-1)^n a_n$$

5.10 Alternating Series Test

Assume that $a_n > 0$ and $a_{n+1} \leq a_n$ for all n, and that $\lim_{n\to\infty} = 0$, then

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

converges. If $S_k = \sum_{n=1}^{k}(-1)^{n-1}a_n$, then S_k approximates the alternating series with an error

$$|S_k - S| \leq a_{k+1}$$

Def Absolute vs Conditional Convergence: A series converges absolutely if

$$\sum_{n=1}^{\infty} |a_n|$$

converges, it converges conditionally if

$$\sum_{n=1}^{\infty} |a_n|$$

diverges but

$$\sum_{n=1}^{\infty} a_n$$

converges

5.11 Absolute Convergence Theorem

Assume that $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges

Note: $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n$ if and only if $a_n \geq 0$ for all n

Def Rearrangement of a Series: Given a series $\sum_{n=1}^{\infty}$ and a one-to-one and onto function $\phi : \mathbb{N} \to \mathbb{N}$ where $b_n = a_{\phi(n)}$, then the series

$$\sum_{n=1}^{\infty} b_n$$

is called a rearrangement

5.12 Rearrangement Theorem

Assume that $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series, if $\sum_{n=1}^{\infty} b_n$ is a rearrangement then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$$

Assume that $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series, if $\alpha \in \mathbb{R}$ or $\alpha = \pm\infty$ then there exists a rearrangement $\sum_{n=1}^{\infty} b_n$ such that

$$\sum_{n=1}^{\infty} b_n = \alpha$$

5.13 Ratio Test

Assume that for $\sum_{n=0}^{\infty} a_n$,

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

where $L \in \mathbb{R}$ or $L = \infty$

1. If $0 \leq L < 1$, then the series converges absolutely
2. If $L > 1$, then the series diverges

5.14 Polynomial vs Factorial Growth

For any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

Def Order of Magnitude: For $|x| > 1$,

$$\ln(n) \ll n^p \ll x^n \ll n! \ll n^n$$

5.15 Root Test

Assume that for $\sum_{n=1}^{\infty} a_n$,

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

where $L \in \mathbb{R}$ or $L = \infty$

1. If $0 \leq L < 1$, then the series converges absolutely
2. If $L > 1$, then the series diverges

Power Series

Def Power Series: A power series centered at the variable $x = a$ is of the form

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

where a_n is the coefficient of $(x-a)^n$

Def Interval of Convergence: For a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, the interval of convergence is the interval centered at $x = a$

$$I = \left\{ x_0 \mid \sum_{n=0}^{\infty} \mid a_n(x_0 - a)^n \text{ converges} \right\}$$

Def Radius of Convergence: For a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, the radius of convergence is

$$R := \begin{cases} lub(\{|x_0 - a| \mid x_0 \in I\}) & \text{if } I \text{ is bounded} \\ \infty & \text{if } I \text{ is not bounded} \end{cases}$$

6.1 Fundamental Convergence Theorem for Power Series

Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ centered at $x = a$

1. If $R = 0$, then the series converges for $x = a$ but diverges for all other x
2. If $0 < R < \infty$, then the series converges absolutely for every $x \in (a - R, a + R)$ and diverges if $|x - a| > R$
3. If $R = \infty$, then the series converges absolutely for every $x \in \mathbb{R}$

6.2 Test for the Radius of Convergence

Let R be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ such that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$$

where $0 \le L < \infty$ or $L = \infty$

1. If $0 < L < \infty$, then $R = \frac{1}{L}$
2. If $L = 0$, then $R = \infty$
3. If $L = \infty$, then $R = 0$

6.3 Equivalence of Radius of Convergence

Let $p, q \neq 0$ be polynomials where $q(n) \neq 0$ for $n \geq k$. Then the following series have the same radius of convergence

1. $$\sum_{n=k}^{\infty} a_n(x-a)^n$$

2. $$\sum_{n=k}^{\infty} \frac{a_n p(n)(x-a)^n}{q(n)}$$

Def Functions Represented by a Power Series: For a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ with a radius of convergence $R > 0$ and interval of convergence I. The function represented by the power series on I is

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

6.4 Abel's Theorem: Continuity of Power Series

Assume $\sum_{n=0}^{\infty} a_n(x-a)^n$ has interval of convergence I. Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for each $x \in I$, then $f(x)$ is continuous on I

6.5 Addition of Power Series

Assume the radii of convergence of

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

$$g(x) = \sum_{n=0}^{\infty} b_n (x-a)^n$$

are R_f, R_g with intervals of convergence I_f, I_g. Then

$$(f+g)(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-a)^n$$

Moreover if $R_f = R_G$ then $R \geq R_f$, if $R_f \neq R_g$ then $R = \min\{R_f, R_g\}$ and $I = I_f \cap I_g$

6.6 Multiplication of a Power Series by $(x-a)^m$

Assume that

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

has radius of convergence R_f with interval of convergence I_f. Let $m \in \mathbb{N}, h(x) = (x-a)^m f(x)$, then

$$h(X) = \sum_{n=0}^{\infty} a_n (x-a)^{n+m}$$

Moreover the series has the same radius and interval of convergence

6.7 Power Series of Composite Functions

Assume the power series centered at $u = 0$

$$f(u) = \sum_{n=0}^{\infty} a_n u^n$$

has radius of convergence R_f with interval of convergence I_f. Let $c \neq 0 \in \mathbb{R}, h(x) = f(cx^m)$, then

$$h(X) = \sum_{n=0}^{\infty} (a_n c^n) x^{nm}$$

with interval of convergence $I_h = \{x \in \mathbb{R} \mid cx^m \in I_f\}$
and if $R_f < \infty$, then radius of convergence $R_h = \sqrt[m]{\frac{R_f}{|c|}}$, otherwise $R_h = \infty$

Def Formal Derivative of a Power Series: Given a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, the formal derivative is

$$\sum_{n=0}^{\infty} na_n(x-a)^{n-1} = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$$

6.8 Term-by-Term Differentiation of Power Series

Assume that $\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $R > 0$. Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ for all $x \in (a-R, a+R)$, then f is differentiable on $(a-R, a+R)$ and

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$$

for all $x \in (a-R, a+R)$

6.9 Uniqueness of Power Series Representations

Assume that $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ for all $x \in (a-R, a+R)$ where $R > 0$, then

$$a_n = \frac{f^{(n)}(a)}{n!}$$

That is if $f(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$ then $a_n = b_n$ for each n

Def Formal Antiderivative of a Power Series: Given a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, the formal antiderivative is

$$\sum_{n=0}^{\infty} \int a_n(x-a)^n \mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

where C is an arbitrary constant

6.10 Term-by-Term Integration of Power Series

Assume that $\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $R > 0$. Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ for all $x \in (a-R, a+R)$, then

$$F(x) = \sum_{n=0}^{\infty} \int a_n(x-a)^n \mathrm{d}x = \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

also has radius of convergence R, and $F'(x) = f(x)$

Furthermore, if $[c, b] \in (a-R, a+R)$, then

$$\begin{aligned}\int_c^b f(x)\mathrm{d}x &= \int_c^b \sum_{n=0}^{\infty} a_n(x-a)^n \mathrm{d}x \\ &= \sum_{n=0}^{\infty} \int_c^b a_n(x-a)^n \mathrm{d}x \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n+1}((b-a)^{n+1} - (c-a)^{n+1})\end{aligned}$$

Def Taylor Polynomials: Assume f is n-times differentiable at $x = a$, the $n-th$ degree Taylor polynomial for f centered at $x = a$ is

$$T_{n,a} = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Def Taylor Remainder: Assume f is n-times differentiable at $x = a$, the $n-th$ degree Taylor remainder function for f centered at $x = a$ is

$$R_{n,a}(x) = f(x) - T_{n,a}(x)$$

6.11 Taylor's Theorem

Assume that f is $n+1$ times differentiable on an interval I containing a, let $x \in I$, then $\exists c \in (x, a)$ where

$$R_{n,a}(x) = f(x) - T_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

6.12 Taylor's Approximation Theorem I

Assume that $f^{(k+2)}$ is continuous on $[-1, 1]$, then there exists $M > 0$ such that

$$|f(x) - T_{k,0}(x)| \leq M|x|^{k+1}$$

or equivalently, for each $x \in [-1, 1]$

$$-M|x|^{k+1} \leq f(x) - T_{k,0}(x) \leq M|x|^{k+1}$$

Def Taylor Series: Assume f has derivatives of all orders at $a \in \mathbb{R}$, the Taylor Series centered at $x = a$ is

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Note: The special case of $a = 0$ is the MacLaurin Series

6.13 Convergence Theorem for Taylor Series

Assume that $f(x)$ has derivatives of all orders on an interval I containing a and that there exists $M \geq |f^{(k)}(x)|$ for all k and $x \in I$, then for all $x \in I$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

6.14 Binomial Theorem

Let $a \in \mathbb{R}, n \in \mathbb{N}$, then for each $x \in \mathbb{R}$,

$$(a+x)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} x^k$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

that is if $a = 1$ then

$$(1+x)^n = \sum_{k=0}^{n} \frac{n(n-1)\dots(n-k+1)}{k!} x^k$$

Def Generalized Binomial Coefficients and Binomial Series: Let $\alpha \in \mathbb{R}, k \in \mathbb{Z} \geq 0$, then

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}$$

and if $k \neq 0$ and $\binom{\alpha}{0} = 1$, then

$$1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

Generalized Binomial Theorem

Let $a \in \mathbb{R}$, then for each $x \in (-1, 1)$,

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$