MATH 136 Personal Notes

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Textbook: Elementary Linear Algebra by L Spence, A.J. Insel, A.H. Friedberg
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The Invertible Matrix Theorem Let $A \in M_{n\times n}(\mathbb{F})$, A is invertible if and only if

- A^{-1} is invertible (Def Invertibility)
- A^T is invertible (Lemma 13.13i)
- $\forall c \neq 0 \in \mathbb{F}, cA$ is invertible (Lemma 13.13ii)
- $\exists B \in M_{n\times n}(\mathbb{F})$ such that $AB = BA = I_n$ (Lemma 14.1)
- A is the product of elementary matrices (Lemma 13.14)
- $A\mathbf{x} = \mathbf{b}$ has a unique solution $\forall \mathbf{b} \in \mathbb{F}^n$ (Lemma 13.12)
- $A\mathbf{x} = \mathbf{0}$ has only the trivial solution (Lemma 13.12)
- $RREF(A) = I_n$ (Corollary 14.1)
- $nullity(A) = 0$ (Corollary 13.3)
- $Rank(A) = n$ (Lemma 14.2)
- $Col(A) = \mathbb{F}^n$ (Corollary 13.1)
- A has n pivots (Def Rank)
- $dim(Row(A)) = n$ (Corollary 22.1)
- $dim(Col(A)) = n$ (Lemma 13.13i, Corollary 22.1)
- $N(A) = \{\mathbf{0}\}$ (Lemma 13.5, Lemma 13.6)
- Columns of A are linearly dependent (Lemma 17.5)
- Columns of A form a basis for $\mathbb{F}^n$ (Lemma 17.11, Lemma 17.5)
- Columns of A span $\mathbb{F}^n$ (Lemma 17.9)
- Rows of A are linearly dependent (Def Rowspace)
- Rows of A span $M_{1\times n}(\mathbb{F})$ (Def Rowspace)
- $\det(A) \neq 0$ (Corollary 15.7)
- 0 is not an eigenvalue of A (Corollary 16.1)
- 0 is not root of Δ_A (Def Characteristic Polynomial)
- T_A is an invertible linear transformation (Lemma 13.16)
- $[T_A]_B$ is invertible for all basis B (Lemma 16.1, Lemma 18.2)
- T_A is onto (Def Matrix Representation)
- T_A is one-to-one (Def Matrix Representation)
- $N(T_A) = \{\mathbf{0}\}$ (Lemma 13.6)
- $R(T_A) = \mathbb{F}^n$ (Def Onto)

1 Vectors in $\mathbb{R}^n$

Def Vector: Has both magnitude and direction, notation may be $\mathbf{v}$, $\underline{v}, \overline{v}, \overrightarrow{v}$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

Note: The failure to include the ^{T} to indicate the transpose is incorrect

Def Addition: For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, their sum is

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}$$

Def Zero Vector: For a vector $\in \mathbb{R}^n$, it is the zero vector $\mathbf{0}$ if it has the property

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Lemma 1: Addition Rules. Let $\mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{R}^n$

(i) $\mathbf{w} + \mathbf{v} = \mathbf{v} + \mathbf{w}$

(ii) $\mathbf{z} + \mathbf{v} + \mathbf{w} = \mathbf{z} + (\mathbf{v} + \mathbf{w}) = (\mathbf{z} + \mathbf{v}) + \mathbf{w}$

(iii) $\mathbf{v} + \mathbf{0} = \mathbf{v}$

Def Subtraction: For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, subtraction is defined by

$$\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$$

Lemma 2: Cancellation Identity. Let $\mathbf{z} \in \mathbb{R}^n$

$$\mathbf{v} - \mathbf{v} = \mathbf{0}$$

Note: $-\mathbf{v}$ is called the additive inverse

Def Scalar Multiplication: For a vector $\mathbf{z} \in \mathbb{R}^n$ and scalar $p \in \mathbb{R}$, scalar multiplication is defined as

$$p\mathbf{v} = \begin{bmatrix} pv_1 \\ pv_2 \\ \vdots \\ pv_n \end{bmatrix}$$

Lemma 3: Properties of Scalar Multiplication. Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, $p, q \in \mathbb{R}$

(i) (p+q)$\mathbf{v}$ = p$\mathbf{v}$ + q$\mathbf{v}$

(ii) (qp)$\mathbf{v}$ = q(p$\mathbf{v}$)

(iii) p($\mathbf{v}$+$\mathbf{w}$) = p$\mathbf{v}$ + p$\mathbf{w}$

(iv) 0$\mathbf{v}$= $\mathbf{0}$

Lemma 4: Properties of Zero. Let $\mathbf{v} \in \mathbb{R}^n$, $a \in \mathbb{R}$

$$a\mathbf{v} = 0 \quad \Longrightarrow \quad a = 0 \vee \mathbf{v} = \mathbf{0}$$

2 Dot Product

Def Dot Product:

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \bullet \begin{bmatrix} 00-38w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = v_1w_1 + v_2w_2 + \cdots + v_nw_n$$

Lemma 1: Properties of the dot product. Let $\mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{R}^n$, $a \in \mathbb{R}$

(i) Symmetry: $\mathbf{v} \bullet \mathbf{w} = \mathbf{w} \bullet \mathbf{v}$

(ii) Linearity: $(\mathbf{v} + \mathbf{w}) \bullet \mathbf{z} = \mathbf{v} \bullet \mathbf{z} + \mathbf{w} \bullet \mathbf{z}$

(iii) Linearity: $(a\mathbf{w}) \bullet \mathbf{v} = a(\mathbf{w} \bullet \mathbf{v})$

(iv) Non-negativity: $\mathbf{v} \bullet \mathbf{v} >= 0$ thus $\mathbf{v} \bullet \mathbf{v} = 0 \Longleftrightarrow \mathbf{v} = \mathbf{0}$

Def Norm (Length): of $\mathbf{v} \in \mathbb{R}^n$ is

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \bullet \mathbf{v}}$$

Lemma 2: Let $\mathbf{v} \in \mathbb{R}^n$, $a \in \mathbb{R}$

$$\|a\mathbf{v}\| = |a|\|\mathbf{v}\|$$

Def Unit Vector: $\mathbf{v} \in \mathbb{R}^n$ is a unit vector if

$$\|\mathbf{v}\| = 1$$

Def Normalization: For a $\mathbf{z} \in \mathbb{R}^n$, produce a unit vector in the direction of $\mathbf{z}$ ($\hat{\mathbf{z}}$) by scaling it.

$$\hat{\mathbf{z}} = \frac{\mathbf{z}}{\|\mathbf{z}\|}$$

Def Orthogonal: The vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are orthogonal if $\mathbf{v} \bullet \mathbf{w} = 0$,

Note: $\mathbf{v}, \mathbf{0}$ are always orthogonal as $\mathbf{v} \bullet \mathbf{0} = 0$

Def Angle: The angle θ between vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ is

$$\mathbf{v} \bullet \mathbf{w} = \|\mathbf{v}\|\|\mathbf{w}\| \cos\theta \quad \text{or} \quad \theta = \arccos\left(\frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|}\right)$$

Def Projection: For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ where $\mathbf{w} \neq \mathbf{0}$, the projection of $\mathbf{v}$ along $\mathbf{w}$, or the projection of $\mathbf{v}$ in the $\mathbf{w}$ direction is

$$Proj_{\mathbf{w}}(\mathbf{v}) = \mathbf{w}\frac{\mathbf{v} \bullet \mathbf{w}}{\|\mathbf{w}\|^2} \quad \text{or} \quad Proj_{\mathbf{w}}(\mathbf{v}) = (\mathbf{v} \bullet \hat{\mathbf{w}})\hat{\mathbf{w}} \quad \text{or} \quad Proj_{\mathbf{w}}(\mathbf{v}) = \hat{\mathbf{w}}(\|\mathbf{v}\| \cos\theta)$$

Def Component: For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ where $\mathbf{w} \neq \mathbf{0}$, the component of $\mathbf{v}$ along $\mathbf{w}$, or the scalar component of $\mathbf{v}$ in the $\mathbf{w}$ direction is

$$Comp_{\mathbf{w}}(\mathbf{v}) = \|\mathbf{v}\| \cos\theta$$

Def Remainder: For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ where $w \neq 0$, the remainder r is

$$Perp_{\mathbf{w}}(\mathbf{v}) = \mathbf{v} - Proj_{\mathbf{w}}(\mathbf{v})$$

Lemma 3: Let $\mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{R}^n$, $a \in \mathbb{R}$

The projection of a vector $\mathbf{v}$ along $\mathbf{w}$ and the remainder are orthogonal to each other

3 Inner Product on $\mathbb{C}^n$

Def Standard Inner Product on $\mathbb{C}^n$: For vectors $\mathbf{w}, \mathbf{z} \in \mathbb{C}^n$, the standard inner product is

$$\langle \mathbf{w}, \mathbf{z} \rangle = w_1 \overline{z}_1 + w_2 \overline{z}_2 + \cdots + w_n \overline{z}_n$$

Lemma 1: Properties of the standard inner product. Let $\mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{C}^n$, $a \in \mathbb{C}$

(i) Conjugate Symmetry: $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$

(ii) Linearity: $\langle (\mathbf{v} + \mathbf{w}), \mathbf{z} \rangle = \langle \mathbf{v}, \mathbf{z} \rangle + \langle \mathbf{w}, \mathbf{z} \rangle$

(iii) Linearity: $\langle a\mathbf{v}, \mathbf{w} \rangle = a \langle \mathbf{v}, \mathbf{w} \rangle$

(iv) Non-negativity: $\langle \mathbf{v}, \mathbf{v} \rangle >= 0 \quad \text{thus} \quad \langle \mathbf{v}, \mathbf{v} \rangle = 0 \Longleftrightarrow \mathbf{v} = \mathbf{0}$

Def Length: of $\mathbf{v} \in \mathbb{C}^n$ is

$$\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$$

Lemma 2: Properties of the length. Let $\mathbf{v} \in \mathbb{C}^n$, $c \in \mathbb{C}$

(i) $\|c\mathbf{v}\| = |c| \|\mathbf{v}\|$

(ii) $\|c\mathbf{v}\| >= 0 \quad \text{thus} \quad \|\mathbf{v}\| = 0 \Longleftrightarrow \mathbf{v} = \mathbf{0}$

Def Orthogonality in $\mathbb{C}^n$: The vectors $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ are orthogonal if

$$\langle \mathbf{v}, \mathbf{w} \rangle = 0$$

Def Projection in $\mathbb{C}^n$: For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$, the projection of $\mathbf{v}$ in the $\mathbf{w}$ direction is defined as

$$Proj_{\mathbf{w}}(\mathbf{v}) = \mathbf{w} \frac{\langle \mathbf{v}, \mathbf{w} \rangle}{\|\mathbf{w}\|^2} \quad \text{or} \quad Proj_{\mathbf{w}}(\mathbf{v}) = \langle \mathbf{v}, \hat{\mathbf{w}} \rangle \hat{\mathbf{w}}$$

Def Field: The field $\mathbb{F}$ can cause different solutions to an equation depending on if $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$

Def Standard Inner Product on $\mathbb{F}^n$: For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n$, the standard inner product is

$$\langle \mathbf{w}, \mathbf{z} \rangle = w_1 \overline{z}_1 + w_2 \overline{z}_2 + \cdots + w_n \overline{z}_n$$

Note: if $\mathbb{F} = \mathbb{R}$, this is the dot product on $\mathbb{R}^n$
Note: if $\mathbb{F} = \mathbb{C}$, this is the Standard Inner Product on $\mathbb{C}^n$

4 The Cross Product

Def Cross Product: For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$,

$$\mathbf{u} \times \mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ -(u_1v_3 - u_3v_1) \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

Note: Defined only in $\mathbb{R}^3$

Lemma 1: Properties of the cross product. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, with $\mathbf{z} = \mathbf{u} \times \mathbf{v}$

(i) $\mathbf{z}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$, thus $\mathbf{z} \bullet \mathbf{u} = 0$ and $\mathbf{z} \bullet \mathbf{v} = 0$

(ii) Skew-symmetric: $\mathbf{v} \times \mathbf{u} = -\mathbf{z} = -(\mathbf{u} \times \mathbf{v})$

(iii) The length of $\mathbf{z}$ is $\|\mathbf{z}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin(\theta)$

(iv) Right-hand Rule: If the pointer finger of your right hand points in the direction of $\mathbf{u}$, and the middle finger of your right hand points in the direction of $\mathbf{v}$, then your thumb points in the direction of $\mathbf{z}$:

Lemma 2: Linearity of the cross product. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3, a \in \mathbb{R}$, them cross product is linear in both arguments.

$$\text{First argument: } \begin{cases} (\mathbf{x} + \mathbf{z}) \times \mathbf{y} = (\mathbf{x} \times \mathbf{y}) + (\mathbf{z} \times \mathbf{y}) \\ a\mathbf{x} \times \mathbf{y} = a(\mathbf{x} \times \mathbf{y}) \end{cases}$$

$$\text{Second argument: } \begin{cases} (\mathbf{x} \times (\mathbf{z} + \mathbf{y}) = (\mathbf{x} \times \mathbf{z}) + (\mathbf{x} \times \mathbf{y}) \\ \mathbf{x} \times a\mathbf{y} = a(\mathbf{x} \times \mathbf{y}) \end{cases}$$

5 An Introduction to Linear Combinations and Span

Def Linear Combination: For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{F}^n$, and scalars $a, b \in \mathbb{F}$. A linear combination is of the form

$$a\mathbf{v} + b\mathbf{w}$$

Note: $0\mathbf{v} + 0\mathbf{w} = \mathbf{0}$ is always a linear combination of $\mathbf{v}, \mathbf{w}$
Note: Linear Combinations can be extended to an arbitrary number of vectors in $\mathbb{F}^n$

Def Span: For vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p \in \mathbb{F}^n$. The span of the vectors is the set of all linear combination of the vectors

$$\text{Span}(\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}) = \{a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_p\mathbf{v}_p : a_1, a_2, \ldots, a_p \in \mathbb{F}\}$$

6 Lines and Planes in $\mathbb{R}^n$

There are 4 ways to create an equation of a straight line in $\mathbb{R}^n$

1. Slope (m) and y-intercept (b)
$$y = mx + b$$

2. A point (x_1, y_1) and slope (m)
$$y - y_1 = m(x - x_1)$$

3. Two points (x_1, y_1), (x_2, y_2)
$$\frac{y - y_2}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$$

4. A point (x_1, y_1), slope ($\frac{q}{p}$, $p \neq 0$) and a parameter (t)
$$x = x_1 + pt \text{ and } y = y_1 + qt$$

Def Parametric equations of a line in $\mathbb{R}^2$: For constants p, q, as t changes the point on the line shifts to all real numbers

$$x = x_1 + pt \text{ and } y = y_1 + qt, \text{ for } t \in \mathbb{R}$$

Note: If $p = 0$, then the line is vertical

Def Vector equation of a line in $\mathbb{R}^2$: The terminal point of the vector gives the coordinates for points on the line $(x_1 + tp, y_1 + tq)$

$$x = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_2 \end{bmatrix} + t \begin{bmatrix} p \\ q \end{bmatrix} = \mathbf{v} + t\mathbf{w} \text{ for } t \in \mathbb{R}$$

Note: $\mathbf{w}$ is parallel to the line, but is a point on the line iff $\mathbf{v}$ is a multiple of $\mathbf{w}$

Def Vector equation of a line in $\mathbb{R}^n$: For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \mathbf{w} \neq \mathbf{0}$, the line through $\mathbf{v}$ with direction $\mathbf{w}$ is
$$L = \{\mathbf{v} + t\mathbf{w} : t \in \mathbb{R}\}$$

Note: The are many other vectors which can produce the same line from a different $\mathbf{v}$

Def Parametric equations of a line in $\mathbb{R}^n$: Given an equation of a line in $\mathbb{R}^n$ in vector form, the parametric form of the equation is

$$\begin{cases} x = v_1 + tw_1 \\ y = v_2 + tw_2 \\ \vdots \\ z = v_n + tw_n \end{cases}$$

Def Line in $\mathbb{R}^n$: For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \mathbf{w} \neq \mathbf{0}$, the L is a set of vectors with associated terminal points

$$L = \{\mathbf{v} + t\mathbf{w} : t \in \mathbb{R}\}$$

Def Line through the Origin in $\mathbb{R}^n$ with Span: For vector $\mathbf{w} \in \mathbb{R}^n, \mathbf{w} \neq \mathbf{0}$, the line through the Origin with direction $\mathbf{w}$ is

$$\text{Span}(\{\mathbf{w}\}) = \{\mathbf{0} + t\mathbf{w} : t \in \mathbb{R}\}$$

Note: The line is unique, but it can be created in other ways

Def Plane through the Origin in $\mathbb{R}^n$: For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \mathbf{v}, \mathbf{w} \neq \mathbf{0}, \mathbf{w} \neq m\mathbf{v}$, the plane through the Origin is defined as

$$P = \text{Span}(\{\mathbf{v}, \mathbf{w}\}) = \{s\mathbf{v} + t\mathbf{w} : s, t \in \mathbb{R}\}$$

Def Vector equation of a plane in $\mathbb{R}^n$: For vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \mathbf{v}, \mathbf{w} \neq \mathbf{0}, \mathbf{w} \neq m\mathbf{v}, s, t \in \mathbb{R}$, any vector with a terminal point on the plane has the form

$$\mathbf{x} = s\mathbf{v} + t\mathbf{w}$$

Note: The vectors $\mathbf{v}$, $\mathbf{w}$ are tangent to the plane

Def Plane in $\mathbb{R}^n$: For vectors $\mathbf{p}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n, \mathbf{v}, \mathbf{w} \neq \mathbf{0}, \mathbf{w} \neq m\mathbf{v}$, a plane is defined as

$$P = \{\mathbf{p} + s\mathbf{v} + t\mathbf{w} : s, t \in \mathbb{R}\}$$

Note: This is not a Span
Note: $\mathbf{v}$ and $\mathbf{w}$ are on the line iff $\mathbf{p} \in \text{Span}(\{\mathbf{v}, \mathbf{w}\})$

Technique Given vectors $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$. A unique plane with these three points can be created by using the fact that $\mathbf{v} = \mathbf{q} - \mathbf{p}$ and $\mathbf{w} = \mathbf{r} - \mathbf{p}$ will always be tangential to the plane

$$\prod = \{\mathbf{p} + s(\mathbf{q} - \mathbf{p}) + t(\mathbf{r} - \mathbf{p}) : s, t \in \mathbb{R}\}$$

Def Scalar equation of a plane in $\mathbb{R}^3$: For vectors $\mathbf{p}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3, \mathbf{v}, \mathbf{w} \neq \mathbf{0}, \mathbf{w} \neq m\mathbf{v}$, the scalar equation of the plane passing through $\mathbf{p}$ with $\mathbf{v}$ and $\mathbf{w}$ tangential to it is

$$\mathbf{n} \bullet (\mathbf{x} - \mathbf{p}) = (\mathbf{v} \times \mathbf{w}) \bullet (\mathbf{x} - \mathbf{p}) = 0$$

Note: The plane goes through the origin iff the vector $\mathbf{0}$ satisfies this equation for $\mathbf{x}$

7 Systems of Linear Equations

Def Linear Equation: Each unknown $x_!, x_2, \dots x_n$ is either to the exponent 0 or 1

Def Linear System of m Equations with n unknowns: $\begin{cases} a_{11}x_1 + x_{12}x_2 + \cdots + a_{1n}x_n & = b_1 \\ a_{21}x_1 + x_{22}x_2 + \cdots + a_{2n}x_n & = b_2 \\ \vdots \\ a_{m1}x_1 + x_{m2}x_2 + \cdots + a_{mn}x_n & = b_m \end{cases}$

Note: The scalars $a_{ij} \in \mathbb{F}$ are known cofficients
Note: The variables $x_1, x_2, \ldots x_n \in \mathbb{F}$ are unknowns
Note: The variables $b_1, b_2, \ldots b_m \in \mathbb{F}$ are collectively the right-hand side

Def Solution Set: The scalars $y_1, y_2, \ldots y_n \in \mathbb{F}$ solve the equations if $x_i = y_i$ satisfies

$$\mathbf{x} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

Note: The solution set is the set of all solutions

Theorem 1 The solution set to a system of linear equations is either empty, contains 1 element, or contains infinite elements

Def Inconsistent and Consistent Systems: If a solution set is empty, the system is inconsistent, if the solution set is non-empty, it is consistent
Note: $0 = a$ where $a \neq 0$ is always inconsistent

Def Equivalent systems: Two linear systems are equivalent if they have the same solution set

Def Elementary Operations: Basic operations that can be performed on linear systems to produce an equivalent system

Type I: Interchange two equations

Type II: Multiply one equation by a non-zero scalar

Type III: Add one equation to the multiple of another equation

Note: Combinations of elementary operations are valid, but will not be used in this course

Def Trivial equation: The equation $0 = 0$ is always true and means nothing

Def Gaussian Elimination:

- Forward elimination: Create an equivalent solution with each first x_i having scalar 1
- Back substitution: Setting the above x_is to 0 with lowest x_i
- Backward elimination: Setting them all to scalar 1?.

Def Free variable: An unknown is a free variable when it can be assigned any real value in the solution set

Def Basic variable: An unknown is a basic variable if not a free variable

8 Gauss-Jordan

Def Coefficient Matrix: A linear system of equation can be represented by a matrix of its coefficients

$$\begin{bmatrix} a_{11} & a_{11} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Note: The $(i,j)^{th}$ entry of the matrix or c_{ij}, is row i, column j

Def Augmented Matrix: The coefficient matrix including the values of b, $B = (A \mid \mathbf{b})$

$$\begin{bmatrix} a_{11} & a_{11} & \dots & a_{1n} & \mid & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & \mid & b_2 \\ \vdots & & & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & \mid & b_m \end{bmatrix}$$

Def Zero Row: A row where all its entries are zeros, thus $0 = 0$
Note: If the coefficient matrix has less zero rows than an augmented matrix, the system of equations is inconsistent

Def Leading Entry: The first non-zero entry in a row Note: Leading 1 is a leading entry that has been scaled to 1

Def Leading Variable: The variable located at the leading entry position x_k

Def Pivot Column: The j column of a position of a leading entry

Def Pivot Position: The (i,j) position of a leading entry

Def Pivot: The Pivot Position if it is non-zero

Technique Gauss Procedure:

- Isolate a row with a non-zero tern in its first column, and Type I to first row
- Use Type III to reduce the i position of all lower rows to 0
- Repeat

Def Row Echelon Form: The $REF(A)$ matrix is created after Gauss Procedure is completed, of the form

$$\begin{bmatrix} a_{11} & a_{11} & \dots & a_{1n} & | & b_1 \\ 0 & a_{22} & \dots & a_{2n} & | & b_2 \\ \vdots & & & & & \\ 0 & 0 & \dots & a_{mn} & | & b_m \end{bmatrix}$$

Technique Jordan Procedure:

- Scale bottom pivot row to have pivot position 1 with Type II
- Use Type III to reduce the i position of all higher rows to 0
- Repeat

Note: Called backward-elimination

Def Reduced Row Echelon Form: The $RREF(A)$ matrix is created after Jordan Procedure is completed, of the form

$$\begin{bmatrix} 1 & 0 & a_{13} & \dots & 0 & | & b_1 \\ 0 & 1 & a_{23} & \dots & 0 & | & b_2 \\ 0 & 0 & 0 & \dots & 0 & | & b_3 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \dots & 1 & | & b_m \end{bmatrix}$$

Lemma 1 If A is a matrix, then there is a unique $RREF(A)$

Technique Canonical Gauss-Jordan:

- Isolate the first row with a non-zero tern in its first column, and Type I to first row
- Scale bottom pivot row to have pivot position 1 with Type II
- Use Type III to reduce the i position of all lower rows to 0
- Repeat
- Repeat: Use Type III to reduce the i position of all higher rows to 0

9 Systems of Linear Equations

Def Notation: The set of matrices with p rows and q columns is $M_{p\times q}(\mathbb{R}), M_{p\times q}(\mathbb{C}), M_{p\times q}$

Def Rank: The number of pivots when a matrix A is in RREF

$$rank(A) \leq p \text{ and } rank(A) \leq q$$

Note: If $rank(A) = p$, then $rank(A) = rank(A \mid \mathbf{b})$ is consistent

Lemma 1 The system of linear equations is consistent iff $rank(A) = rank(A \mid \mathbf{b})$

Def Nullity: The nullity of a matrix A is

$$nullity(A) = q - rank(A)$$

Lemma 2 If the system of linear equations is consistent, then the solution set contains $nullity(A)$ parameters

10 Real and Complex Examples

Def Homogeneous System: A system is homogeneous if in the augmented matrix $\mathbf{b} = 0$
Note: A homogeneous system is always consistent as the trivial solution is always satisfied $A\mathbf{0} = \mathbf{0}$

Def Null Space: The nullspace of a matrix A, is the solution set of the matrix denoted by $N(A)$
Note: The nullspace of a homogeneous system is a span

11 Matrix Multiplication

Def Row Vector: The vector $\mathbf{G} \in M_{1\times n}$ is a row, distinguished from column vectors by capitalization
Note: $\mathbf{G}_j$ is the entry in the j^{th} column

Def Decomposition of a Matrix:

$$\begin{bmatrix} a_{11} & a_{11} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^m \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$$

Def Matrix Multiplication: Let $A \in M_{m\times n}, \mathbf{x} \in \mathbb{F}^n$, then $A\mathbf{x} =$

$$\begin{bmatrix} a_{11} & a_{11} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix} \mathbf{x} = \begin{bmatrix} a_{11}x_1 + a_{11}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^m \end{bmatrix} \mathbf{x}$$

Lemma 1 Linearity of Matrix Multiplication: Let $A \in M_{m\times n}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$, then

$$\begin{cases} A(\mathbf{x}+\mathbf{y}) & = A\mathbf{x} + A\mathbf{y} \\ A(c\mathbf{x}) & = cA\mathbf{x} \end{cases}$$

Remark: for $A \in M_{m\times n}, \mathbf{w} \in \mathbb{F}^n$,

$$(A\mathbf{w})_i = \langle (\mathbf{A}^i)^T, \overline{\mathbf{w}} \rangle$$

thus if $A \in M_{m\times n}(\mathbb{R})$ then

$$(A\mathbf{w})_i = (\mathbf{A}^i)^T \bullet \mathbf{w}$$

Def Associated Homogeneous System: For an inhomogeneous system $C\mathbf{x} = \mathbf{d} \neq \mathbf{0}$, the associated homogeneous system is $D\mathbf{x} = \mathbf{0}$

Lemma 2 If $\mathbf{x}_1, \mathbf{x}_2 \in S, a_1 \in \mathbb{F}$, then $(\mathbf{x}_1 + \mathbf{x}_2) \in S$ and $a_1\mathbf{x}_1 \in S$

Lemma 3 Relation between $\tilde{S}$ and S I: If $\mathbf{y}_1, \mathbf{y}_2 \in$ an inhomogeneous system $\tilde{S}$, then $(\mathbf{y}_1 - \mathbf{y}_2) \in$ the associated homogeneous system S

Def Particular Solution: A particular solution to $A\mathbf{x} = \mathbf{b}$ is referred to as $\mathbf{x}_p$

Lemma 4 Relation between $\tilde{S}$ and S II: The solution set of an inhomogeneous system $\tilde{S}$ can be constructed from the associated homogeneous system S and a single particular solution

$$\tilde{S} = \{\mathbf{y}_p + \mathbf{x} : \mathbf{x} \in S\}$$

Lemma 5 Relation Between Inhomogeneous Systems with Matching Coefficient Matrices: Let $\tilde{S}_1$ be the solution set to $A\mathbf{x} = \mathbf{b}$ and $\tilde{S}_2$ be the solution set to $A\mathbf{x} = \mathbf{c}$. Then

$$\tilde{S}_2 = \{\mathbf{p}_2 + (\mathbf{z} - \mathbf{p}_1) : \mathbf{z} \in \tilde{S}_1\}$$

that is if

$$\tilde{S}_1 = \{\mathbf{p}_1 + a_1\mathbf{w}_1 + \dots + a_q\mathbf{w}_q : a_1, a_2, \dots, a_1 \in \mathbb{F}\}$$

then

$$\tilde{S}_2 = \{\mathbf{p}_2 + a_1\mathbf{w}_1 + \dots + a_q\mathbf{w}_q : a_1, a_2, \dots, a_1 \in \mathbb{F}\}$$

Def Matrix Multiplication: Let $A \in M_{m\times n}, B \in M_{n\times p}$, then

$$AB = C = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix}$$

where $C \in M_{m\times p}$

Note: The j^{th} column of C, $\mathbf{c}_j = A\mathbf{b}_j$

Note: The $(i,j)^{th}$ entry of C is $\mathbf{A}^i\mathbf{b}_j$

Def Column Span: The span of the columns of A

$$Col(A) = Span(\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}) = \{\alpha_1\mathbf{a}_1 + \alpha_2\mathbf{a}_2 + \cdots + \alpha_n\mathbf{a}_n : \alpha_1, \ldots, \alpha_n \in \mathbb{F}\}$$

Note: If $C = AB$, then $\mathbf{c}_k \in Col(A)$ for $k = 1, \ldots, p$

Lemma 6 Solution of a linear system: The system of linear equations $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b} \in Col(A)$

12 Properties of Matrices

Def Equality: Let $A \in M_{m\times n}, B \in M_{p\times q}$, A and B are equal means that

(i) $m = p$ and $n = q$ (same size)

(ii) $a_{ij} = b_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots n$ (entries are equal)

Note: Holds for $\mathbb{R}^n$ and $\mathbb{C}^n$

Def Addition: Let $A, B \in M_{m\times n}$, then

(i) $A + B = D \in M_{m\times n}$

(ii) $d_{ij} = a_{ij} + b_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots n$

Note: Addition of different sizes is not defined

Lemma 1 Properties of Matrix Addition: Let $A, B, C \in M_{m\times n}$, then

(i) $A + B = B + A$

(ii) $A + B + C = (A + B) + C = A + (B + C)$

(iii) $\exists \mathbb{O} \in M_{m\times n}, \mathbb{O} + A = A$

(iv) $-A + A = \mathbb{O}$

Note: The Zero Matrix is defined as $\mathbb{O}$, and sometimes includes size $\mathbb{O}_{m\times n}$

Def Multiplication by a Scalar: Let $A \in M_{m\times n}, c \in \mathbb{F}$, then

(i) $cA = F \in M_{m\times n}$

(ii) $f_{ij} = ca_{ij}$ for all $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots n$

Lemma 2 Properties of Matrix Multiplication by a Scalar: Let $A, B \in M_{m\times n}, C \in M_{n\times p}, c, d \in \mathbb{F}$, then

(i) $cA = Ac$

(ii) $c(A + B) = cA + cB$

(iii) $(c + d)A = cA + dA$

(iv) $c(dA) = (cd)A$

(v) $c(AC) = (cA)C = A(cC) = cAC$

Def Transpose of a Matrix: Let $A \in M_{m\times n}$, then the transpose is

$$(A^T)_{mn} = (A)_{nm}$$

Note: The rows are made into columns i nthe order in which they appear

Lemma 3 Properties of Transpose: Let $A, B \in M_{m\times n}, c \in \mathbb{F}$, then

(i) $(A + B)^T = A^T + B^T$

(ii) $(cA)^T = cA^T$

(iii) $(A^T)^T = A$

Lemma 4 Properties of Matrix Multiplication: Let $A, G \in M_{m\times n}, B, D \in M_{n\times p}, C \in M_{p\times q}$, then

(i) $(A + G)B = AB + GB$

(ii) $A(B + D) = AB + AD$

(iii) $(AB)C = A(BC) = ABC$

(iv) $(AB)^T = B^T A^T$

Def Square Matrix: Let $A \in M_{m\times n}$, then A is a square matrix iff $n = m$

Def Symmetric: Let $A \in M_{n\times n}$, then B is a symmetric iff $A = A^T$

Def Skew-symmetric: Let $A \in M_{n\times n}$, then A is a skew-symmetric iff $A = -A^T$

Def Upper Triangular: Let $A \in M_{n \times n}$, then A is a upper triangular ($U\Delta$) iff $a_{ij} = 0$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots n$ where $i > j$

Note: The product of $U\Delta$ matrices is $U\Delta$

*example

Def Lower Triangular: Let $A \in M_{n \times n}$, then A is a lower triangular ($L\Delta$) iff $a_{ij} = 0$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots n$ where $i < j$

Note: The transpose of $U\Delta$ is $L\Delta$

Note: The product of $L\Delta$ matrices is $L\Delta$

*example

Def Diagonal: Let $A \in M_{n \times n}$, then A is diagonal iff $c_{ij} = 0$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots n$ where $i \neq j$

Note: Is both $L\Delta$ and $U\Delta$

*example

Def Diagonal Entries: Let $A \in M_{n \times n}$, then a_{ii} are the diagonal entries of A, and $(a_{11}, a_{22}, \ldots a_{nn})$ is the main diagonal of A

Note: $C = diag(c_{11}, c_{22}, \ldots, c_{nn})$ is the diagonal matrix $C \in M_{n \times n}$

Def Identity Matrix: The matrix $diag(1, 1, \ldots 1)$ is called the identity matrix I where $I_n \in M_{n \times n}$

Note: For $A \in M_{m \times n}$, $I_m A = A$ and $AI_n = A$

Def Elementary Matrix: A matrix created by performing a single ERO on the identity matrix

Note: Elementary matrices can be classified as Type I, Type II, Type III

Lemma 5 Let $C \in M_{m \times n}$, if the same ERO is performed on $C \to B$ and $I_m \to E$, then

$$B = EC$$

Lemma 6 Let $C \in M_{m \times n}$, if a finite number q of EROs are performed on $C \to D$ and each is represented by $I_m \to E_1, E_2, \ldots E_q$, then

$$D = E_q \ldots E_2 E_1 C$$

13 Linear Transformations

Def Function Definition: Let $A \in M_{m\times n}(\mathbb{F})$, then the function determined by the matrix A is

$$T_A : \mathbb{F}^n \to \mathbb{F}^m, T_A(\mathbf{x}) = A\mathbf{x}$$

Lemma 1: Let $A \in M_{m\times n}(\mathbb{F}), \mathbf{x}, \mathbf{y} \in \mathbb{F}^n, c \in \mathbb{F}$, then T_A is linear, that is

$$\begin{cases} T_A(\mathbf{x}+\mathbf{y}) & = T_A(\mathbf{x}) + T_A(\mathbf{y}) \\ T_A(c\mathbf{x}) & = cT_A(\mathbf{x}) \end{cases}$$

Def Linear Transformation: Let $T : \mathbb{F}^n \to \mathbb{F}^m$, T is a linear transformation if and only if $\forall \mathbf{x}, \mathbf{y} \in \mathbb{F}^n, c \in \mathbb{F}$

$$\begin{cases} T(\mathbf{x}+\mathbf{y}) & = T(\mathbf{x}) + T(\mathbf{y}) \\ T(c\mathbf{x}) & = cT(\mathbf{x}) \end{cases}$$

Lemma 2: Let $T : \mathbb{F}^n \to \mathbb{F}^m$, T is a linear transformation if and only if $\forall \mathbf{x}, \mathbf{y} \in \mathbb{F}^n, \forall c_1, c_2 \in \mathbb{F}$

$$T(c_1\mathbf{x} + c_2\mathbf{y}) = c_1T(\mathbf{x}) + c_2T(\mathbf{y})$$

Lemma 3: Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation, then

$$T(\mathbf{0}_{\mathbb{F}^n}) = \mathbf{0}_{\mathbb{F}^m}$$

Def Range: Let $T : \mathbb{F}^n \to \mathbb{F}^m$, The range of T is the set of image points of T, that is

$$R(T) = \{T(\mathbf{x}) : \mathbf{x} \in \mathbb{F}^n\}$$

Note: $R(T)$ is a subset of $\mathbb{F}^m$

Lemma 4: Let $A \in M_{m\times n}(\mathbb{F})$ and $T_A : \mathbb{F}^n \to \mathbb{F}^m$, then

$$R(T_A) = Col(A)$$

Def Onto: The function $T : \mathbb{F}^n \to \mathbb{F}^m$ is onto if and only if the range of T is the entire codomain of T, that is

$$R(T) = \mathbb{F}^m$$

Note: If $S : \mathbb{F}^n \to \mathbb{F}^m$ is a linear transformation, then S is onto means that $R(S) = \mathbb{F}^m$

Corollary 1 from Lemma 4: Let $A \in M_{m\times n}(\mathbb{F})$ and $T_A : \mathbb{F}^n \to \mathbb{F}^m$, then T_A is onto if and only if $Col(A) = \mathbb{F}^m$

Corollary 2 from Lemma 4: Let $A \in M_{m\times n}(\mathbb{F})$ and $T_A : \mathbb{F}^n \to \mathbb{F}^m$, then T_A is onto if and only if $rank(A) = m$

Def Nullspace: Let $T : \mathbb{F}^n \to \mathbb{F}^m$, The nullspace of T is the set of vectors such that their image under T is the zero vector

$$N(T) = \{\mathbf{x} \in \mathbb{F}^n : T(\mathbf{x}) = \mathbf{0}_{\mathbb{F}^m}\}$$

Note: If T is a linear transformation, $\mathbf{0}_{\mathbb{F}^n} \in N(T)$ thus the nullspace is never empty

Lemma 5: Let $A \in M_{m\times n}(\mathbb{F})$ and $T_A : \mathbb{F}^n \to \mathbb{F}^m$, then

$$N(T_A) = N(A)$$

Def One-to-one: The function $T : \mathbb{F}^n \to \mathbb{F}^m$ is one-to-one if and only if distinct points have distinct images, that is $\forall \mathbf{x}, \mathbf{y} \in \mathbb{F}^n$

$$\mathbf{x} \neq \mathbf{y} \Longrightarrow T(\mathbf{x}) \neq T(\mathbf{y})$$

Lemma 6: Let $A \in M_{m\times n}(\mathbb{F})$ and $T_A : \mathbb{F}^n \to \mathbb{F}^m$, then T_A is one-to-one if and only if

$$N(T_A) = \{\mathbf{0}_{\mathbb{F}^n}\}$$

Corollary 3: Let $A \in M_{m\times n}(\mathbb{F})$ and $T_A : \mathbb{F}^n \to \mathbb{F}^m$, then T_A is onto if and only if $nullity(A) = 0$ if and only if $rank(A) = n$

Def Matrix representation of a linear transformation: Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation. The matrix representation of T is the $(m \times n)$ matrix whose columns are the images of the basic vectors in the standard basis in $\mathbb{F}^n$, that is

$$[T]_S = \begin{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} & \dots & \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} (T(\mathbf{e}_1) & T(\mathbf{e}_2) & \dots & T(\mathbf{e}_n)) \end{bmatrix}$$

Note: The $_S$ indicates that the standard basis is being used as the domain/codomain
Note: $[T_A]_S = A$
Note: $T_{[T]_S} = T$
Note: T is onto if and only if $rank([T]_S) = m$
Note: T is one-to-one if and only if $rank([T]_S) = n$

Lemma 7: Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation, if $\mathbf{x} \in \mathbb{F}^n$ then

$$T(\mathbf{x}) = [T]_S\mathbf{x}$$

Lemma 8: Let $f : \mathbb{R} \to \mathbb{R}$ be a linear transformation, if $p \in \mathbb{R}$ with $f(p) = \alpha \in \mathbb{R}$, then

$$f(x) = \frac{\alpha}{p}x$$

Def Composite Functions: For functions $T_1 : \mathbb{F}^n \to \mathbb{F}^m, T_2 : \mathbb{F}^m \to \mathbb{F}^p$, The composite function $T_2 \circ T_1 : \mathbb{F}^n \to \mathbb{F}^p$ is

$$T(\mathbf{x}) = (T_2 \circ T_1)(\mathbf{x}) = T_2(T_1(\mathbf{x}))$$

Lemma 9: Let $T_1 : \mathbb{F}^n \to \mathbb{F}^m, T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be a linear transformations, then $(T_2 \circ T_1)(\mathbf{x})$ is also a linear transformation

Lemma 10: Let $T_1 : \mathbb{F}^n \to \mathbb{F}^m, T_2 : \mathbb{F}^m \to \mathbb{F}^p$ be a linear transformations, then

$$[T_2 \circ T_1]_S = [T_2]_S[T_1]_S$$

Def T^p: For the function $T : \mathbb{F}^n \to \mathbb{F}^n$, we define

$$T^p = T \circ T^{p-1}$$

Note: $T^0 = T_I$, the identity transformation ($T_I(\mathbf{x}) = \mathbf{x}$)

Corollary 4 of Lemma 10: Let $T : \mathbb{F}^n \to \mathbb{F}^n, p \in \mathbb{N}$, then

$$[T^p]_S = ([T]_S)^p$$

Def Invertibility of a Matrix: For $A \in M_{n\times n}$, A is invertible if $\exists B \in M_{n\times n}$ where

$$AB = BA = I_n$$

Note: B or A^{-1}, the inverse, is also invertible

Def Singularity of a Matrix: For $A \in M_{n\times n}$, A is singular if it is not invertible

Lemma 11 Unique Inverses: Let $A \in M_{n\times n}$ be invertible, then B is unique

Lemma 12: Let $A \in M_{n\times n}$ be invertible, then

$$A\mathbf{x} = \mathbf{b} \text{ has a unique solution } \mathbf{z} = A^{-1}\mathbf{b}, \forall \mathbf{b} \in \mathbb{F}^n$$

Lemma 13 Properties of the Inverse: Let $A, B \in M_{n\times n}$ be invertible, $C, D \in M_{n\times m}$ be invertible, and $c \neq 0 \in \mathbb{F}$, then

(i) A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

(ii) cA is invertible and $(cA)^{-1} = c^{-1}A^{-1}$

(iii) AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

(iv) if $AC = AD$, then $C = D$

(v) if $AC = \mathbb{O}_{n\times m}$, then $C = \mathbb{O}_{n\times m}$

Lemma 14 Inverses of Elementary Matrices: All elementary matrices are invertible and their inverses are of the same type

(I) The inverse of a type I is itself

(II) The inverse of type II are from scaling by m^{-1} instead of m

(III) The inverse of type III are from subtracting instead of adding row i

Def Invertible functions: For the function $T_1 : \mathbb{F}^n \to \mathbb{F}^m$, it is invertible if and only if $\exists T_2 : \mathbb{F}^m \to \mathbb{F}^n$ such that

$$T_2 \circ T_1 = T_{I_{\mathbb{F}^n}} \text{ and } T_1 \circ T_2 = T_{I_{\mathbb{F}^m}}$$

Note: If and only if it is one-to-one and onto

Lemma 15: Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear transformation, if T is invertible then its inverse T^{-1} is unique and linear

Lemma 16: Let $T : \mathbb{F}^n \to \mathbb{F}^n$ be a linear transformation, then T is invertible if and only if $[T]_S$ is an invertible matrix, then

$$[T^{-1}]_S = ([T]_S)^{-1}$$

Corollary 5 of Lemma 16: Let $A \in M_{n\times n}(\mathbb{F})$, if $A\mathbf{x} = \mathbf{b}$ has a unique solution $\forall \mathbf{b} \in \mathbb{F}^n$ then A is an invertible matrix

Def Isomorphism: An invertible linear transformation is called an isomorphism

14 Matrix Inverse

Lemma 1: Let $A \in M_{n\times n}(\mathbb{F})$, if $\exists B \in M_{n\times n}(\mathbb{F})$ such that $AB = I_n$ then A is invertible

Lemma 2 Invertibility of a Matrix: Let $A \in M_{n\times n}(\mathbb{F})$, then A is invertible if and only if $Rank(A) = n$

Corollary 1 of Lemma 2: Let $A \in M_{n\times n}(\mathbb{F})$, then A is invertible if and only if $RREF(A) = I_n$

Lemma 3 Algorithm for Matrix Inversion: Let $A \in M_{n\times n}(\mathbb{F})$, then

- Construct $(A \mid I_n)$
- Reduce until A is in REF, if $rank(A) \neq n$, A is not invertible
- Reduce until A is in RREF, in $(I_n \mid B)$, $B = A^{-1}$

Lemma 4 Invertibility of a Matrix $M_{2\times 2}(\mathbb{F})$: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if and only if $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note: $ad - bc$ is the determinant of the matrix

Def from Lecture Rotation in $\mathbb{R}^2$: $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is the linear transformation from rotating θ radians around the origin

Notice that

$$T_\theta\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$$T_\theta\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

then

$$T_\theta(\mathbf{x}) = T_\theta\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1\cos(\theta) - x_2\sin(\theta) \\ x_1\sin(\theta) + x_2\cos(\theta) \end{bmatrix}$$

thus

$$[T_\theta(\mathbf{x})]_S = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

and

$$[T_{\alpha+\beta}]_S = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}\begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} = \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix}$$

15 The Determinant

Def Submatrix: The $(i,j)^{th}$ submatrix of $A \in M_{n\times n}$, $M_{ij}(A)$ is the $(n-1)\times(n-1)$ matrix obtained from removing the i^{th} row and j^{th} column

Def Determinant of 1×1, 2×2 matrices: If $A \in M_{1\times 1}(\mathbb{F})$ then

$$\det(A) = a_{11}$$

If $A \in M_{2\times 2}(\mathbb{F})$ then

$$\det(A) = a_{11}a_{22} - a_{12}a_{21}$$

Def First Row Expansion of the Determinant: If $A \in M_{n\times n}(\mathbb{F})$ with $n \geq 2$ then for $\det : M_{n\times n} \to \mathbb{B}$

$$\det(A) = \sum_{j=1}^{j=n} a_{1j}(-1)^{1+j}\det(M_{1j}(A))$$

Def I^{th} Row Expansion of the Determinant: If $A \in M_{n\times n}(\mathbb{F})$ with $n \geq 2$ for any $I \leq n$ then

$$\det(A) = \sum_{j=1}^{j=n} a_{Ij}(-1)^{I+j}\det(M_{Ij}(A))$$

Def J^{th} Column Expansion of the Determinant: If $A \in M_{n\times n}(\mathbb{F})$ with $n \geq 2$ for any $J \leq n$ then

$$\det(A) = \sum_{i=1}^{i=n} a_{iJ}(-1)^{i+J}\det(M_{iJ}(A))$$

Def Cofactor: If $A \in M_{n\times n}(\mathbb{F})$ the $(i,j)^{th}$ cofactor of A is

$$C_{ij}(A) = (-1)^{i+j}\det(M_{ij}(A))$$

Lemma 1: Let $A \in M_{n\times n}(\mathbb{F})$, then

$$\det(A) = \det(A^T)$$

Lemma 2: Let $A \in M_{n\times n}(\mathbb{F})$ be a upper (lower) triangle, then

$$\det(A) = a_{11}a_{22}\dots a_{nn} = \prod_{i=1}^{n} a_{ii}$$

Corollary 1 of Lemma 2: Let $A \in M_{n\times n}(\mathbb{F})$ be a diagonal matrix, then Lemma 2 holds and

$$\det(I_n) = 1$$

Theorem 1: Let $A \in M_{n\times n}(\mathbb{F}) = \begin{bmatrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^n \end{bmatrix}$, then

a) The determinant is skew-symmetric under the interchange of rows

$$\det\left(\begin{bmatrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^k \\ \vdots \\ \mathbf{A}^i \\ \vdots \\ \mathbf{A}^n \end{bmatrix}\right) = -\det\left(\begin{bmatrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^i \\ \vdots \\ \mathbf{A}^k \\ \vdots \\ \mathbf{A}^n \end{bmatrix}\right)$$

b) The determinant is a linear operation on rows, that is for $\mathbf{B}^i \in M_{1\times n}(\mathbb{F}), c_1, c_2 \in \mathbb{F}$

$$\det\left(\begin{bmatrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ c_1\mathbf{A}^i + c_2\mathbf{B}^i \\ \vdots \\ \mathbf{A}^n \end{bmatrix}\right) = c_1 \det\left(\begin{bmatrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^i \\ \vdots \\ \mathbf{A}^n \end{bmatrix}\right) + c_2 \det\left(\begin{bmatrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{B}^i \\ \vdots \\ \mathbf{A}^n \end{bmatrix}\right)$$

Note: The same statement is true if rows are replaced with columns throughout

Corollary 2 of Theorem 1: Let $A \in M_{n\times n}(\mathbb{F})$ have two identical rows (columns), then

$$\det(A) = 0$$

Corollary 3 of Theorem 1 Determinants of elementary matrices: Let E_k be an elementary matrix of type k, then

i) When E_1 is obtained from I_n by interchanging 2 rows then

$$\det(E_1) = -1$$

ii) When E_2 is obtained from I_n by scaling a row by $m \neq 0 \in \mathbb{R}$ then

$$\det(E_2) = m$$

iii) When E_3 is obtained from I_n by adding a multiple of a row to another row then

$$\det(E_3) = 1$$

Corollary 4 of Theorem 1 EROs and the determinant: Let $B \in M_{n\times n}(\mathbb{F})$ be a single ERO from $A \in M_{n\times n}(\mathbb{F})$, then

i) If ERO is type I, then $\det(B) = -\det(A)$

ii) If ERO is type II by $m \neq 0 \in \mathbb{R}$, then $\det(B) = m\det(A)$

iii) If ERO is type III, then $\det(B) = \det(A)$

Corollary 5: Let $B \in M_{n\times n}(\mathbb{F})$ be a single ERO with elementary matrix E from $A \in M_{n\times n}(\mathbb{F})$, then

$$\det(B) = \det(E)\det(A)$$

Corollary 6: Let $B \in M_{n\times n}(\mathbb{F})$ be a series of EROs $E_1E_2\dots E_q$ from $A \in M_{n\times n}(\mathbb{F})$, then

$$\det(B) = \det(E_1E_{q-1}\dots E_1A) = \det(E_q)\det(E_{q-1})\dots\det(E_1)\det(A)$$

Corollary 7 Invertibility iff the determinant is non-zero.: Let $A \in M_{n\times n}(\mathbb{F})$, then A is invertible if and only if

$$\det(A) \neq 0$$

def I think: A singular matrix must be if $\det(a) = 0$?

Corollary 8 Determinant of a product: Let $A, B \in M_{n\times n}(\mathbb{F})$, then

$$\det(AB) = \det(A)\det(B)$$

Corollary 9: Let $A \in M_{n\times n}(\mathbb{F})$ be invertible, then

$$\det(A^{-1}) = (\det(A))^{-1}$$

Def Adjoint (adjunct) of a Matrix: If $A \in M_{n\times n}(\mathbb{F})$ the adjoint of A is the transpose of the matrix of cofactors of A, that is $\forall i, j = 1, 2, \dots n$

$$(adj(A))_{ij} = C_{ji}(A)$$

Note: For $(I_n)_{ij}$, if $i = j$ then $(I_n)_{ij} = 1$, else $(I_n)_{ij} = 0$

Lemma 3: Let $A \in M_{n\times n}(\mathbb{F})$, then

$$Aadj(A) = adj(A)A = det(A)I_n$$

Corollary 10: Let $A \in M_{n\times n}(\mathbb{F})$, if $\det(A) \neq 0$ then

$$A^{-1} = \left(\frac{1}{\det(A)}\right) adj(A)$$

Lemma 4 Cramer's Rule: Let $A \in M_{n\times n}(\mathbb{F}), A\mathbf{x} = \mathbf{b} \in \mathbb{F}^n$ where $\det(A) \neq 0$, if B_j is A with the j^{th} column replaced by $\mathbf{b}$, then

$$A\mathbf{x} = \mathbf{b} \text{ is given by } x_j = \frac{\det(B_j)}{\det(A)}$$

Lemma 5: Let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \in \mathbb{R}^2$, the area of the parallelogram with sides $\mathbf{v}, \mathbf{w}$ is

$$A = \left| \det\left(\begin{bmatrix} v_1 & v_2 \\ w_1 & w_2 \end{bmatrix} \right) \right|$$

Def Standard Triple Product: If $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, the scalar triple product $STP(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{x} \bullet (\mathbf{y} \times \mathbf{z})$ is the volume of the parallelepiped with $\mathbf{x}, \mathbf{y}, \mathbf{z}$ as sides

$$V = |STP(\mathbf{x}, \mathbf{y}, \mathbf{z})|$$

Lemma 6: Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$, then

$$STP(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \det\left(\begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix}\right) = \det\left(\begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix}^T\right) = \det\left(\begin{bmatrix} \mathbf{x}^T \\ \mathbf{y}^T \\ \mathbf{z}^T \end{bmatrix}\right)$$

16 Diagonalization and the Eigenvalue

Def Eigenvector: If $A \in M_{n\times n}(\mathbb{F})$ then the vector $\mathbf{x} \neq \mathbf{0}$ is an eigenvector of A if and only if $\exists \lambda \in \mathbb{F}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

Note: λ is an eigenvalue
Note: $(\lambda, \mathbf{x})$ is an eigenpair

Def Eigenvalue Equation: If $A \in M_{n\times n}(\mathbb{F}), \mathbf{x} \in \mathbb{F}^n$, then

$$A\mathbf{x} = \lambda\mathbf{x} \text{ or } (A - \lambda I_n)\mathbf{x} = \mathbf{0}$$

Note: There is an eigenvector iff $A - \lambda I_n$ is not invertible
Note: Thus looking for a λ where $\det(A - \lambda I_n) = 0$

Def Characteristic Polynomial: If $A \in M_{n\times n}(\mathbb{F}), t \in \mathbb{F}$ then the characteristic polynomial is

$$\Delta_A(t) = \det(A - t\lambda)$$

Note: The characteristic equation is $\Delta_A(t) = 0$

Def Eigenspace: If $A \in M_{n\times n}(\mathbb{F}), \lambda_1 \in \mathbb{F}$ is an eigenvalue of A, then the eigenspace is

$$E_{\lambda_1} = N(A - t\lambda)$$

Note: Contains all eigenvectors of λ_1 and $\mathbf{0}$

Def Similar: If $A, B \in M_{n\times n}(\mathbb{F})$, then A is similar to B if $\exists Q \in M_{n\times n}$ such that

$$Q^{-1}AQ = B$$

Note: If A is similar to B, then B is similar to A

Def Similarity Transformation: If $A, Q \in M_{n\times n}(\mathbb{F})$ then the similarity transformation is $T : M_{n\times n} \to M_{n\times n}$ defined by

$$T(A) = Q^{-1}AQ$$

Def Trace: If $A \in M_{n\times n}(\mathbb{F})$ then the trace is the sum of its diagonal entries

$$tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} (A)_{ii}$$

Lemma 1: Let $A, B \in M_{n\times n}(\mathbb{F})$ be similar, then

(i) $\det(A) = \det(B)$

(ii) $tr(A) = tr(B)$

Def Diagonalizable Matrix: If $A \in M_{n\times n}(\mathbb{F})$ and $D \in M_{n\times n}$ is diagonal, then A is diagonalizable if $\exists P \in M_{n\times n}(\mathbb{F})$ such that

$$D = P^{-1}AP$$

Note: A is similar to a diagonal matrix

Lemma 2 Diagonalization I: Let $A \in M_{n\times n}(\mathbb{F})$ have eigenpairs $(\lambda_1, \mathbf{v}_1)\ldots(\lambda_n, \mathbf{v}_n)$ where $\lambda_1 \neq \ldots \neq \lambda_n$. Let $P = (\mathbf{v}_1, \ldots \mathbf{v}_n)$ then P is invertible and

$$P^{-1}AP = D = diag(\lambda_1, \ldots \lambda_n)$$

Lemma 3 Properties of the Characteristic Polynomial I: Let $A \in M_{n\times n}(\mathbb{F})$ have characteristic polynomial $\Delta_A(t) = \det(A - tI_n)$, then

(i) $\Delta_A(t)$ is a n^{th} order polynomial in t

$$\Delta_A(t) = b_0 + b_1 t + \cdots + b_{n-1}t^{n-1} + b_n t^n$$

(ii) $b_n = (-1)^n$

(iii) $b_{n-1} = (-1)^{n-1}tr(A)$

(iv) $b_0 = \det(A)$

Lemma 4 Properties of the Characteristic Polynomial I: Let $A \in M_{n\times n}(\mathbb{C})$ have characteristic polynomial $\Delta_A(t) = \det(A - tI_n)$, with A having eigenvalues $\lambda_1 \ldots \lambda_n$, then

(i)

$$\sum_{i=1}^{n} \lambda_i = tr(A) = (-1)^{n-1}b_{n-1}$$

(ii)

$$\prod_{i=1}^{n} \lambda_i = \det(A) = b_0$$

Corollary 1 of Lemma 4 : Let $A \in M_{n\times n}(\mathbb{F})$, then A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A

Lemma 5: Let $A \in M_{n\times n}(\mathbb{F})$ be similar, then they have the same characteristic polynomials and the same eigenvalues

Def from Lecture: If $P^{-1}AP = D$ (similar), then $D = PAP^{-1}$ and

$$A^n = PDP^{-1}PDP^{-1}\ldots PDP^{-1} = PDI_nDI_n\ldots I_nDP^{-1} = PDD\ldots DP^{-1} = PD^nP^{-1}$$

17 Subspaces, Span and Bases

Def Subspace: A subset $V \in \mathbb{F}^n$ is called a subspace of $\mathbb{F}^n$ to mean that

(i) $\mathbf{0} \in V$

(ii) Closure under addition: $\forall \mathbf{x}, \mathbf{y} \in V, \mathbf{x} + \mathbf{y} \in V$

(iii) Closure under scalar multiplication: $\forall \mathbf{x} \in V, c \in \mathbb{F}, c\mathbf{x} \in V$

Note: $\mathbb{F}^n$ and $\{\mathbf{0}\}$ are trivial subspaces of $\mathbb{F}^n$

Lemma 1 Checking for a Subspace: Let V be a subset of $\mathbb{F}^n$, then V is a subspace if and only if

(i) V is non-empty

(ii) $\forall \mathbf{x}, \mathbf{y} \in V, c \in \mathbb{F}, c\mathbf{x} + \mathbf{y} \in V$

Example 1:

(a) $\mathbb{F}^n$ is a subspace

(b) $\{\mathbf{0}\}$ is a subspace

(c) if $\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p\} \in \mathbb{F}^n$ then $Span(\{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p\})$ is a subspace

(d) Let $A \in M_{n\times n}(\mathbb{F})$, $Col(A)$ is a subspace

(e) Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation, $R(T)$ is a subspace of $\mathbb{F}^m$

(f) Let $A \in M_{m\times n}(\mathbb{F})$, the solution set $A\mathbf{x} = \mathbf{0}$ is a subspace of $\mathbb{F}^n$

(g) Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be a linear transformation, $N(T)$ is a subspace of $\mathbb{F}^n$

(h) Let $A \in M_{n\times n}(\mathbb{F})$ with eigenvalue λ, E_λ is a subspace of $\mathbb{F}^n$

Example 3:

(a) $\mathbb{F}$ has only $\mathbb{F}$ and $\{\mathbf{0}\}$ as subspaces

Def Linear Dependence: $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p$ being linear dependent means that exists $c_1, c_2, \dots c_p$ not all zero such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p = \mathbf{0}$

Note: The trivial linear combination $c_1 = 0, c_2 = 0, \dots c_p = 0$ also makes the $\mathbf{0}$ vector

Def Linear Independence: $\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p$ being linear independent means that there does not exist non-zero $c_1, c_2, \dots c_p$ such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \dots + c_p\mathbf{v}_p = \mathbf{0}$

Def Basis: Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$ be a subset of the subspace $V \in \mathbb{F}^n$. B is a basis means that B is a linearly independent set of vectors which spans V

Lemma 2: Let $\mathbf{0} \in S \subseteq \mathbb{F}^n$ then S is linearly dependent

Lemma 3: Let $S = \{\mathbf{x}\} \subseteq \mathbb{F}^n$, then S is linearly dependent if and only if $\mathbf{x} = \mathbf{0}$

Lemma 4: Let $S = \{\mathbf{x}, \mathbf{y}\} \subseteq \mathbb{F}^n$, then S is linearly dependent if and only if one vector is a multiple of the other

Lemma 5: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\} \subseteq \mathbb{F}^n, A = (\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p \in M_{n \times p})$ with $rank(A) = r$ and pivot columns $q_q, q_2, \ldots q_r$, Let $U = \{\mathbf{v}_{q_1}, \mathbf{v}_{q_2}, \ldots \mathbf{v}_{q_r}\}$, then

(a) S is linearly independent if and only if $r = p$

(b) U is linearly independent

(c) A subset of S that contains U and any other vector from S is linearly dependent

(d) $Span(U) = Span(S)$

Corollary 1 of Lemma 5: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\} \subseteq \mathbb{F}^n$. If $n < p$ then S is linearly dependent

Lemma 6: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\} \subseteq \mathbb{F}^n$ be linearly independent, Let $\mathbf{w} \in \mathbb{F}^n$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p, \mathbf{w}\}$ is linearly dependent if and only if $w \in Span(S)$

Lemma 7: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_k, \ldots \mathbf{v}_p\} \subseteq \mathbb{F}^n$ be linearly independent, then $S \setminus \{\mathbf{v}_k\}$ is linearly independent

Lemma 8: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\} \subset V$ where V is a subspace of $\mathbb{F}^n$, then $Span(S)$ is a subspace of V

Lemma 9: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$ where $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p \in \mathbb{F}^n$, then $Span(S) = \mathbb{F}^n$ if and only if $rank\left(\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_p\end{bmatrix}\right) = n$

Lemma 10: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p\}$ where $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_p \in \mathbb{F}^n$, then if S is a basis for $\mathbb{F}^n$ then S has exactly n vectors ($p = n$)

Lemma 11: Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ for distinct $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n \in \mathbb{F}^n$, then S is linearly independent if and only if $Span(S) = \mathbb{F}^n$

Def Dimension: The number of elements in a basis for $\mathbb{F}^n$ (n) is the dimension or $n-$dimensional

$$dim(\mathbb{F}^n) = n$$

Def Standard Basis: The standard basis for $\mathbb{F}^n$ is the set of n vectors $S = \{\mathbf{e}_1, \mathbf{e}_2, \ldots \mathbf{e}_n\}$

Theorem 1 Unique Representation Theorem: Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n\}$ be a basis for $\mathbb{F}^n$, then $\forall \mathbf{v} \in \mathbb{F}^n$ there exists unique scalars $c_1, c_2, \ldots c_n \in \mathbb{F}$ such that

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$$

Def Coordinates and Components: For a basis of $\mathbb{F}^n$ $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$, with $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \sum_{i=1}^n c_i\mathbf{v}_i \in \mathbb{F}^n$, the coordinate/component vector is

$$[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

Note: This is a if and only if relationship

Lemma 12 Taking Coordinates is a Linear Transformation: Let B be a basis for $\mathbb{F}^n$, then $[]_B : \mathbb{F}^n \to \mathbb{F}^n$ given by $\mathbf{x} \to [\mathbf{x}]_B$ is a linear transformation

Lemma 13: Let $B_1 = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}, B_2 = \{\mathbf{w}_1, \mathbf{w}_2, \dots \mathbf{w}_n\}$ be a bases for $\mathbb{F}^n$, Let $\mathbf{x} \in \mathbb{F}^n$ with $[\mathbf{x}]_{B_1} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, [\mathbf{x}]_{B_2} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, then

$$[\mathbf{x}]_{B_2} = {}_{B_2}[I]_{B_1}[\mathbf{x}]_{B_1} \text{ and } [\mathbf{x}]_{B_1} = {}_{B_1}[I]_{B_2}[\mathbf{x}]_{B_2}$$

where ${}_{B_2}[I]_{B_1} = \begin{bmatrix} [\mathbf{v}_1]_{B_2} & [\mathbf{v}_2]_{B_2} & \dots & [\mathbf{v}_n]_{B_2} \end{bmatrix}$ and ${}_{B_1}[I]_{B_2} = \begin{bmatrix} [\mathbf{w}_1]_{B_1} & [\mathbf{w}_2]_{B_1} & \dots & [\mathbf{w}_n]_{B_1} \end{bmatrix}$

Def Change of Basis (Coordinates) Matrix: The change-of-basis matrix from basis B_1 to basis B_2 is ${}_{B_2}[I]_{B_1}$

Corollary 2: Let $B_1 = \{\mathbf{e}_1, \mathbf{e}_2, \dots \mathbf{e}_n\} = S, B_2 = \{\mathbf{w}_1, \mathbf{w}_2, \dots \mathbf{w}_n\}$ be bases for $\mathbb{F}^n$, Let $\mathbf{x} \in \mathbb{F}^n$ with $[\mathbf{x}]_{B_1} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, [\mathbf{x}]_{B_2} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$, then

$$[\mathbf{x}]_{B_2} = {}_{B_2}[I]_S[\mathbf{x}]_S \text{ and } [\mathbf{x}]_S = {}_S[I]_{B_2}[\mathbf{x}]_{B_2}$$

where ${}_{B_2}[I]_S = \begin{bmatrix} [\mathbf{e}_1]_{B_2} & [\mathbf{e}_2]_{B_2} & \dots & [\mathbf{e}_n]_{B_2} \end{bmatrix}$ and ${}_S[I]_{B_2} = \begin{bmatrix} [\mathbf{w}_1]_S & [\mathbf{w}_2]_S & \dots & [\mathbf{w}_n]_S \end{bmatrix} = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_n \end{bmatrix}$

Corollary 3: The change of basis matrices ${}_{B_1}[I]_{B_2}, {}_{B_2}[I]_{B_1}$ are inverses of each other, that is

$${}_{B_1}[I]_{B_2 B_2}[I]_{B_1} = I_n$$

18 Matrix Representation of a Linear Operator

Def Linear Operator: For a linear transformation $T : \mathbb{F}^n \to \mathbb{F}^m$, T being a linear operator means that $m = n$ such that $T : \mathbb{F}^n \to \mathbb{F}^n$

Def Matrix Representation: For a linear operator T on $\mathbb{F}^n$ with basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$, the matrix representation of T with respect to B is

$$[T]_B = \begin{bmatrix} [T(\mathbf{v}_1)]_B & [T(\mathbf{v}_2)]_B & \dots & [T(\mathbf{v}_n)]_B \end{bmatrix}$$

Lemma 1: Let T be a linear operator on $\mathbb{F}^n$, Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$ be a basis for $\mathbb{F}^n$, if $\mathbf{v} \in \mathbb{F}^n$ then

$$[T(\mathbf{v})]_B = [T]_B[\mathbf{v}]_B$$

Lemma 2: Let T be a linear operator on $\mathbb{F}^n$, Let B_1, B_2 be a bases for $\mathbb{F}^n$, then $[T]_{B_1}$ and $[T]_{B_2}$ are similar, and

$$[T]_{B_2} = {}_{B_2}[I]_{B_1}[T]_{B_1 B_1}[I]_{B_2} = ({}_{B_1}[I]_{B_2})^{-1}[T]_{B_1 B_1}[I]_{B_2}$$

$$[T]_{B_1} = {}_{B_1}[I]_{B_2}[T]_{B_1 B_2}[I]_{B_1} = ({}_{B_2}[I]_{B_1})^{-1}[T]_{B_2 B_2}[I]_{B_1}$$

Corollary 1: Let T be a linear operator on $\mathbb{F}^n$, Let B be a basis for $\mathbb{F}^n$, then $[T]_B$ and $[T]_S$ are similar, and

$$[T]_S = {}_S[I]_B[T]_{BB}[I]_S = ({}_B[I]_S)^{-1}[T]_{BB}[I]_S$$

$$[T]_B = {}_B[I]_S[T]_{SS}[I]_B = ({}_S[I]_B)^{-1}[T]_{SS}[I]_B$$

19 Diagonalization of Linear Operators

Def Linear Operator: For a linear operator T in $\mathbb{F}^n$, the eigenvalue equation

$$T(\mathbf{x}) = \lambda\mathbf{x}$$

where $\mathbf{x}$ is the non-zero eigenvector and $\lambda \in \mathbb{F}$ is the eigenvalue

Lemma 1: Let T be a linear operator on $\mathbb{F}^n$, Let B be a basis for $\mathbb{F}^n$, then $(\lambda, \mathbf{x})$ is an eigenpair of T if and only if $(\lambda, [\mathbf{x}]_B)$ is a eigenpair of $[T]_B$

Def Diagonalizable: For a linear operator T in $\mathbb{F}^n$, T being diagonalizable means that there exists a basis B of $\mathbb{F}^n$ such that $[T]_B$ is a diagonal matrix

Lemma 2: Let T be a linear operator on $\mathbb{F}^n$, then T is diagonalizable if and only if there exists a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_n\}$ of $\mathbb{F}^n$ consisting of eigenvectors of T

Lemma 3: Let T be a linear operator on $\mathbb{F}^n$, Let B be a basis for $\mathbb{F}^n$, then T is diagonalizable if and only if the matrix $[T]_B$ is diagonalizable

Corollary 1: Let $A \in M_{n\times n}(\mathbb{F})$, then A is diagonalizable if and only if there exists a basis of $\mathbb{F}^n$ of eigenvectors of A

Lemma 4: Let $A \in M_{n\times n}(\mathbb{F})$ have eigenpairs $(\lambda_1, \mathbf{v}_1), (\lambda_2, \mathbf{v}_2), \dots (\lambda_m, \mathbf{v}_m)$ for $1 \le m \le n$. If the eigenvalues are all different, then the set $W = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_m\}$ is linearly independent

Def Characteristic Polynomial: For a linear operator T in $\mathbb{F}^n$ and basis B for $\mathbb{F}^n$, the characteristic polynomial of T is

$$\Delta_T(t) = \Delta_{[T]_B}(t)$$

Def Algebraic Multiplicity: The algebraic multiplicity of eigenvalue λ of $A \in M_{n\times n}(\mathbb{F})$ is the highest power of the factor $(t-\lambda)^{a_\lambda}$ that divides the characteristic polynomial, that is

$$(t-\lambda)^{a_\lambda} \mid \Delta_A(t) \text{ but } (t-\lambda)^{a_\lambda+1} \nmid \Delta_A(t)$$

Def Geometric Multiplicity: The geometric multiplicity of eigenvalue λ of $A \in M_{n\times n}(\mathbb{F})$ is the dimension of the eigenspace E_λ, g_λ

Lemma 5: Let λ be an eigenvalue of $A \in M_{n\times n}(\mathbb{F})$, then

$$1 \le g_\lambda \le a_\lambda$$

Lemma 6: Let $A \in M_{n\times n}(\mathbb{F})$ have eigenvalues $\lambda_1, \lambda_2, \dots \lambda_m$ with eigenspaces $E_{\lambda_1}, E_{\lambda_2}, \dots E_{\lambda_m}$ having bases $B_1, B_2, \dots B_m$, then

$$B = B_1 \cup B_2 \cup \dots \cup B_m \text{ is linearly independent}$$

Lemma 7: Let $A \in M_{n\times n}(\mathbb{F})$ have $\Delta_A(t) = (\lambda_1 - t)^{a_{\lambda_1}}(\lambda_2 - t)^{a_{\lambda_2}}\dots(\lambda_m - t)^{a_{\lambda_m}}h(t)$ where $\lambda_1, \lambda_2, \dots \lambda_m$ are eigenvalues of A and $h(t)$ is a polynomial in t with no linear factors, then

$$A \text{ is diagonalizable if and only if both } h(t) = 1 \text{ and } a_{\lambda_i} = g_{\lambda_i} \text{ for each } i = 1, 2, \dots m$$

20 Special Subspaces and Bases

Def Trivial Subspace: $Span(\emptyset) = \{\mathbf{0}\}$ where $\emptyset$ is a basis for $\{\mathbf{0}\}$ with dimension 0

Lemma 1: Let V be a subspace of $\mathbb{F}^n$, then there exist a linearly subset W with $p \leq n$ elements such that

$$Span(W) = V$$

Def Basis: For a subspace U of $\mathbb{F}^n$, the subset W of U being a basis means that

1. $W \subseteq U$
2. W is linearly independent
3. $Span(W) = U$

Lemma 2: Let V be a subspace of $\mathbb{F}^n$, where $U = \{\mathbf{u}_1, \mathbf{u}_2, \dots \mathbf{v}_p\}, W = \{\mathbf{w}_1, \mathbf{w}_2, \dots \mathbf{w}_q\}$ are bases for V, then $p = q$

Def Dimension: For a subspace V of $\mathbb{F}^n$, the dimension $dim(V) = p$ is the number of vectors in a basis for V

Lemma 3 Replacement Theorem: Let V be a subspace of $\mathbb{F}^n$ such that $dim(V) = k > 0$, where $W = \{\mathbf{w}_1, \mathbf{w}_2, \dots \mathbf{w}_q\}$ is a basis for V, then W can be extended to a basis B of $\mathbb{F}^n$

Remark 1: $rank(A) = dim(Col(A))$

Theorem 1 The Dimension Theorem (or Rank-Nullity Theorem): Let $A \in M_{m\times n}(\mathbb{F})$, then

$$n = dim(Col(A)) + dim(N(A))$$

thus

$$n = rank(A) + nullity(A) \text{ and } n = rank(T_A) + nullity(T_A)$$

21 Vector Space

Axioms

(I) Closure under addition: $\forall \mathbf{v}, \mathbf{w} \in V,\ v \oplus \mathbf{w} \in V$

(II) Closure under scalar multiplication: $\forall \mathbf{v} \in V, c \in \mathbb{F},\ c \odot \mathbf{v} \in V$

and eight other axioms need to be satisfied for a vector space

(a) $\forall \mathbf{v}, \mathbf{w} \in V,\ \mathbf{v} \oplus \mathbf{w} = \mathbf{w} \oplus \mathbf{v}$

(b) $\forall \mathbf{v}, \mathbf{w}, \mathbf{z} \in V,\ (\mathbf{v} \oplus \mathbf{w}) \oplus \mathbf{z} = \mathbf{v} \oplus (\mathbf{w} \oplus \mathbf{z})$

(c) $\forall \mathbf{v} \in V,\ \mathbf{0} \oplus \mathbf{v} = \mathbf{v}$

(d) $\forall \mathbf{v} \in V,\ \mathbf{v} \oplus (-\mathbf{v}) = \mathbf{0}$

(e) $\forall \mathbf{v}, \mathbf{w} \in V, c \in \mathbb{F},\ c \odot (\mathbf{v} \oplus \mathbf{w}) = (c \odot \mathbf{v}) \oplus (c \odot \mathbf{w})$

(f) $\forall \mathbf{v} \in V, c, d \in \mathbb{F},\ (c + d) \odot \mathbf{v} = (c \odot \mathbf{v}) \oplus (d \odot \mathbf{v})$

(g) $\forall \mathbf{v} \in V, c, d \in \mathbb{F},\ (c \times d) \odot \mathbf{v} = c \odot (d \odot \mathbf{v})$

(h) $\forall \mathbf{v} \in V, c, d \in \mathbb{F},\ 1 \odot \mathbf{v} = \mathbf{v}$

Def Vector Space: If we are given a set V, a field $\mathbb{F}$, a $\oplus, \odot$, and all axioms hold, this is a vector space over $\mathbb{F}$

Def Linear Combination: For a vector space over $\mathbb{F}$ of $(V, \oplus, \mathbb{F}, \odot)$ with $\mathbf{v}_1, \mathbf{v}_2 \in V, c_1, c_2 \in \mathbb{F}$, then a linear combination is $(c_1 \odot \mathbf{v}_1) \oplus (c_2 \odot \mathbf{v}_2)$

Def Span: For a vector space over $\mathbb{F}$ of $(V, \oplus, \mathbb{F}, \odot)$ with $W = \{\mathbf{w}_1, \mathbf{w}_2, \dots \mathbf{w}_p\} \subset V$, then the set of all linear combinations of the elements of W is

$$Span(W) = \{(c_1 \odot \mathbf{v}_1) \oplus (c_2 \odot \mathbf{v}_2) \oplus \cdots \oplus (c_p \odot \mathbf{v}_p) : c_i \in \mathbb{F}, i = 1, 2, \dots p\}$$

Def Vector Subspace: For a vector space over $\mathbb{F}$ of $(V, \oplus, \mathbb{F}, \odot)$ with a subset U of V, then U being a subspace means that U is a non-empty subset closed under addition and scalar multiplication, thus

1. $U \neq \emptyset$
2. $\forall \mathbf{u}_1, \mathbf{u}_2 \in U,\ \mathbf{u}_1 \oplus \mathbf{u}_2 \in U$
3. $\forall \mathbf{u}_1 \in U, c \in \mathbb{F},\ c \odot \mathbf{u}_1 \in U$

Lemma 1: Let $(V, \oplus, \mathbb{F}, \odot)$ be a vector space over $\mathbb{F}^n$, the zero vector is unique

Lemma 2: Let $(V, \oplus, \mathbb{F}, \odot)$ be a vector space over $\mathbb{F}^n$ with $\mathbf{x} \in V$, the additive inverse $(-\mathbf{x})$ is unique

Lemma 3: Let $(V, \oplus, \mathbb{F}, \odot)$ be a vector space over $\mathbb{F}^n$ with $a \in \mathbb{F}, \mathbf{x} \in V$, then

$$0 \odot \mathbf{x} = \mathbf{0} \text{ and } a \odot \mathbf{0} = \mathbf{0}$$

Lemma 4 The additive inverse: Let $(V, \oplus, \mathbb{F}, \odot)$ be a vector space over $\mathbb{F}^n$ with $\mathbf{x} \in V$, then

$$-\mathbf{x} = (-1) \odot \mathbf{x}$$

Lemma 5 The cancellation identity: Let $(V, \oplus, \mathbb{F}, \odot)$ be a vector space over $\mathbb{F}^n$ with $a \in \mathbb{F}, \mathbf{x} \in V$, if $a \odot \mathbf{x} = \mathbf{0}$, then

$$a = 0 \text{ or } \mathbf{x} = \mathbf{0}$$

Lemma 6: Let $(V, \oplus, \mathbb{F}, \odot)$ be a vector space over $\mathbb{F}^n$ with $W = \{\mathbf{w}_1, \mathbf{w}_2, \dots \mathbf{w}_p\} \subset V$ where $p \geq 1$, then $Span(W)$ is the smallest subspace of V that contains W

Def Linear Dependence: For a vector space over $\mathbb{F}$ of $(V, \oplus, \mathbb{F}, \odot)$ with $W = \{\mathbf{w}_1, \mathbf{w}_2, \dots \mathbf{w}_p\} \subset V$, W being linearly dependent means that $\exists a_i \in \mathbb{F}, i = 1, 2, \dots p \neq 0$ such that

$$(a_1 \odot \mathbf{w}_1) \oplus (a_2 \odot \mathbf{w}_2) \oplus \cdots \oplus (a_p \odot \mathbf{w}_p) = \mathbf{0}$$

Def Basis: For a vector space over $\mathbb{F}$ of $(V, \oplus, \mathbb{F}, \odot)$ with $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p\} \subset V$, B being a basis means that

1. $B \subset V$
2. $Span(B) = V$
3. B is linearly independent

Def Components/Coordinates: For a vector space over $\mathbb{F}$ of $(V, \oplus, \mathbb{F}, \odot)$ with $B = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_p\}$ being a basis for V, the components/coordinates of a vector $\mathbf{v} \in V$ are the scalars such that

$$\mathbf{v} = (a_1 \odot \mathbf{v}_1) \oplus (a_2 \odot \mathbf{v}_2) \oplus \cdots \oplus (a_p \odot \mathbf{v}_p)$$

Note: $[\mathbf{v}]_B = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix}$ is the coordinate vector of $\mathbf{v}$ in B

22 The Rowspace of a Matrix

Def Rowspace: For a $A \in M_{m\times n}(\mathbb{F})$, the rowspace is a vector subspace of $M_{1\times n}(\mathbb{F})$

$$Row(A) = Span(\{\mathbf{A}^1, \mathbf{A}^2, \dots \mathbf{A}^m\})$$

Lemma 1: Let $A \in M_{m\times n}(\mathbb{F})$, if B is performed by elementary row operations on A, then

$$Row(A) = Row(B)$$

Corollary 1: Let $A \in M_{m\times n}(\mathbb{F})$,

$$dim(Row(A)) = rank(A)$$

Lemma 2: Let $A \in M_{m\times n}(\mathbb{F})$, then

$$rank(A) = rank(A^T)$$

23 Matrix Representations of Linear Transformations

Def Linear transformation: For a $T : U \in \mathbb{F}^n \to V \in \mathbb{F}^m$, being a linear transformation means that

1. For all $\mathbf{u}_1, \mathbf{u}_2 \in U$, $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$
2. For all $\mathbf{u} \in U, c \in \mathbb{F}$, $T(c\mathbf{u}) = cT(\mathbf{u})$

Def Matrix Representation: For a $T : U \to V$, with $B_1 = \{\mathbf{u}_1, \mathbf{u}_2, \dots \mathbf{u}_p\}$ being a basis for $U \in \mathbb{F}^n$ and $B_2 = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_q\}$ being a basis for $V \in \mathbb{F}^m$

$${}_{B_2}[T]_{B_1} = \begin{bmatrix} [T(\mathbf{u}_1)]_{B_2} & [T(\mathbf{u}_2)]_{B_2} & \dots & [T(\mathbf{u}_p)]_{B_2} \end{bmatrix}$$

Lemma 1: Let $T : T \to V$, with B_1 being a basis for $U \in \mathbb{F}^n$ and B_2 being a basis for $V \in \mathbb{F}^m$ and ${}_{B_2}[T]_{B_1}$ is the matrix representation of the linear transformation, then for all $\mathbf{x} \in U$

$$[T(\mathbf{x})]_{B_2} = {}_{B_2}[T]_{B_1}[\mathbf{x}]_{B_1}$$