Methods of Mathematical Physics

— Lecture 6 — PDEs: Introduction

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Contents

In general, the equations of mathematical physics include partial differential equatíons (PDEs), ordinary differential equations (ODEs), integral equations and integral-differential equations which are presented from Physics, Mechanics, Astronomy, Chemistry, Biology and Engineering. However, PDEs are main contents and also main topics of our study.

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In this section we will learn some basic definitions concerning partial differential equations and their solutions. In particular, we will define notions of:

- linear vs. nonlinear (semilinear, quasilinear, and fully nonlinear)
- order
- scalar PDEs vs. systems
- homogeneous vs. inhomogeneous
- what exactly mean by a solution to a PDE
- general solutions, arbitrary functions, and auxiliary conditions
- initial value problems (IVP) and boundary value problems (BVP)
- well-posedness: existence, uniqueness, and stability.

Notation

We use standard notation for partial derivatives; for example, if $u(x, y)$ is a function of two variables, then

$$u_x = \frac{\partial u}{\partial x} = \partial_x u, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$$

with the understanding that these partial derivatives are also functions of (x, y). Spatial independent variables will typically be denoted as x, y, z or $x_1, \ldots, x_n$. Each of these variables takes on real values (i.e., real numbers). We often denote an n-tuple (or vector) of independent spatial variables as x. Time will be denoted by t. For PDEs (as opposed to ODEs) there will always be more than one independent variable. We will most often use u to denote the unknown function, i.e., the dependent variable. So, for example, in purely spatial variables we would be dealing with $u(x, y), u(x, y, z)$, or more generally $u(\mathbf{x})$. When time is also relevant, we will deal with $u(x, t), u(x, y, t), u(x, y, z, t)$, or more generally $u(\mathbf{x}, t)$.

Definition of a Partial Differential Equation (PDE)

Definition

A partial differential equation (PDE) is an equation which relates an unknown function u and its partial derivatives together with independent variables. In general, it can be written as

$$F(\text{independent variables, } u, \text{ partial derivatives of } u) = 0,$$

for some function F which captures the structure of the PDE.

For example, in two independent variables a PDE involving only first-order partial derivatives is described by

$$F(x, y, u, u_x, u_y) = 0$$

where $F : \mathbb{R}^5 \to \mathbb{R}$. Laplace's equation is a particular PDE which in two independent variables is associated with $F(u_{xx}, u_{yy}) = u_{xx} + u_{yy} = 0$.

Definition of a Solution to a PDE

Definition

A solution (more precisely, a classical solution) to a PDE in a domain $\Omega \subset \mathbb{R}^N$ (where N is the number of independent variables) is a sufficiently smooth function $u(\mathbf{x})$ which satisfies the defining equation F for all values of the independent variables in Ω.

Remark: If the highest derivatives occurring in the PDE are of order k, then by sufficiently smooth we mean C^k in all the variables.

Thus a solution to $F(x, y, u, u_x, u_y) = 0$ on some domain $\Omega \subset \mathbb{R}^2$ is a C^1 function $u(x, y)$ such that for every $(x, y) \in \Omega$, $F(x, y, u(x, y), u_x(x, y), u_y(x, y)) \equiv 0$.

Examples:

- $u(x, y) = \sin(3x - 2y)$ is a solution to $2u_x + 3u_y = 0$ on $(x, y) \in \mathbb{R}^2$;
- $u(x, t) = \frac{x}{t}$ is a solution to $u_t + uu_x = 0$, for $x \in \mathbb{R}$ and $t > 0$.

For most PDEs, it is impossible to guess a function which satisfies the equation. Indeed, for all but trivial examples, PDEs cannot be simply "solved" by direct integration.

Definition of the Order of a PDE

Definition

We define the order of a PDE to be the order of the highest derivative which appears in the equation.

For example, $u_{xx} + 2uu_{xy} + u_{yy} = e^y$ is a second order equation, and $u_{xxy} + xu_{yy} + 8u = 7y$ is an equation of third order.

Operators

Definition (Operator)

The mathematical operational rule by acting a function generates another function.

Examples:

- $\mathcal{L}(u) = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^3 u}{\partial y^3}$, where $\mathcal{L} = \frac{\partial}{\partial x} + \frac{\partial^2}{\partial x \partial y} + \frac{\partial^3}{\partial y^3}$ is called a differential operator.
- $\mathcal{L}(u) = \int_a^b u(x, \tau) F(\tau, y) d\tau$, a and b are constants, where $\mathcal{L}$ is an integral operator.

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- $\mathcal{L}(u) = \int_a^b u(x,\tau)F(\tau,y)d\tau$, a and b are constants, where $\mathcal{L}$ is an integral operator.

If operators A and B acting any functions u of a set, can generate the same result, i.e., Thus, we have $A(u) = B(u)$, then A and B are called equivalent operators and denoted by $A = B$. The sum of two differential operators A and B is that $(A+B)(u) = A(u) + B(u)$, where u is a function. The product of two operators A and B is the operator whose action to a function is the same with the action of B and A in sequence, namely $AB(u) = A(B(u))$.

Differential operators satisfy the following four properties:

1. commutative law of addition: $A + B = B + A$;
2. associative law of addition: $(A + B) + C = A + (B + C)$;
3. associative law of multiplication: $(AB)C = A(BC)$;
4. distributive law of multiplication to addition: $A(B + C) = AB + AC$;

Operators

Except above results, in general, the following commutative law of multiplication

$$AB = BA$$

is not true. If all coefficients of differential operators are constants, the commutative of multiplication law is valid.

For example, let

$$A = \frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y}, \quad B = \frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y} \quad (xy \not\equiv 0),$$

then

$$\begin{aligned} B(u) &= \frac{\partial^2 u}{\partial y^2} - y\frac{\partial u}{\partial y}, \\ AB(u) &= \left(\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 u}{\partial y^2} - y\frac{\partial u}{\partial y}\right) \\ &= \frac{\partial^4 u}{\partial x^2 \partial y^2} - y\frac{\partial^3 u}{\partial x^2 \partial y} + x\frac{\partial^3 u}{\partial y^3} - xy\frac{\partial^2 u}{\partial y^2} - x\frac{\partial u}{\partial y}. \end{aligned}$$

But

$$\begin{aligned} BA(u) &= \left(\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y}\right)\left(\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y}\right) \\ &= \frac{\partial^4 u}{\partial y^2 \partial x^2} + x\frac{\partial^3 u}{\partial y^3} - y\frac{\partial^3 u}{\partial y \partial x^2} - xy\frac{\partial^2 u}{\partial y^2}, \end{aligned}$$

thus $AB(u) \neq BA(u)$.

Definition of a Linear and Nonlinear PDE

A central dichotomy for PDEs is whether they are linear or not (nonlinear). To this end, let us write the PDE in the following form: *All terms containing u and its derivatives = all terms involving only the independent variables*. We write the left-hand side as $\mathcal{L}(u)$, thinking of it as some (differential) operator $\mathcal{L}$ operating on the function u. Some examples of specific operators are $\mathcal{L}(u)=u_x+u_y$ and $\mathcal{L}(u)=u_{xx}+xu_{yy}+uu_x$. Thus any PDE for $u(\mathbf{x})$ can be written as

$$\mathcal{L}(u)=f(\mathbf{x}), \tag{1}$$

for some operator $\mathcal{L}$ and some function f.

Definition (linear & nonlinear)

We say the PDE is linear if $\mathcal{L}$ is linear in u. That is,

$$\mathcal{L}\left(u_1+u_2\right)=\mathcal{L}\left(u_1\right)+\mathcal{L}\left(u_2\right)$$

and

$$\mathcal{L}\left(cu_1\right)=c\mathcal{L}\left(u_1\right).$$

Otherwise, we say the PDE is nonlinear.

Definition (homogeneous & inhomogeneous)

If $f\equiv 0$ in (1), then we say the PDE is homogeneous. Otherwise, it is inhomogeneous.

Definition of a Semilinear, Quasilinear, and Fully Nonlinear PDE

There are different types of nonlinearities in a PDE and these have a profound effect on their complexity. We divide them into three groups: semilinear, quasilinear, and fully nonlinear.

Definition

A PDE of order k is called:

- semilinear if all occurrences of derivatives of order k appear with a coefficient which only depends on the independent variables,
- quasilinear if all occurrences of derivatives of order k appear with a coefficient which only depends on the independent variables, u, and its derivatives of order strictly less than k,
- fully nonlinear if it is not quasilinear.

By definition, we have the strict inclusions:

$$\text{linear PDEs} \subset \text{semilinear PDEs} \subset \text{quasilinear PDEs}.$$

Definition of a Semilinear, Quasilinear, and Fully Nonlinear PDE

For first-order PDEs in two independent variables x and y, linear means the PDE can be written in the form

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = c_1(x,y)u + c_2(x,y),$$

for some functions a, b, c_1, c_2 of x and y. Semilinear means the PDE can be written in the form

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) = c(x,y,u),$$

for some functions a and b of x and y, and a function c of x, y, and u. Quasilinear means that the PDE can be written in the form

$$a(x,y,u)u_x(x,y) + b(x,y,u)u_y(x,y) = c(x,y,u),$$

for some functions a, b, and c of x, y, and u. **Note that in all cases, the coefficient functions a, b, and c need not be linear in their arguments**.

So for example,

$$\begin{aligned} (xy)u_x + e^y u_y + (\sin x)u &= x^3 y^4 && \text{is linear;} \\ (xy)u_x + e^y u_y + (\sin x)u &= u^2 && \text{is semilinear;} \\ uu_x + u_y &= 0 && \text{is quasilinear;} \\ (u_x)^2 + (u_y)^2 &= 1 && \text{is fully nonlinear.} \end{aligned}$$

Note that we are not concerned with nonlinearities in the independent variables. For example, the first example which is linear has nonlinear terms in the independent variables but is linear with respect to the dependent variables (u and its partial derivatives).

The Principle of Superposition for Linear PDEs

Linear PDEs share a property, known as the **principle of superposition**.

- For homogeneous linear PDEs we can formulate the property as follows: If u_1 and u_2 are two solutions to $\mathcal{L}(u) = 0$ and $a, b \in \mathbb{R}$, then $au_1 + bu_2$ is also a solution to $\mathcal{L}(u) = 0$.
- For inhomogeneous linear PDEs in the form $\mathcal{L}(u) = f$: If u_1 is a solution to the inhomogeneous linear PDE and u_2 is any solution to the associated homogeneous $\mathrm{PDE}\,\mathcal{L}(u) = 0$, then $u_1 + u_2$ is also a solution to the inhomogeneous linear PDE. More generally, if u_1 is a solution to $\mathcal{L}(u) = f_1$ and u_2 a solution to $\mathcal{L}(u) = f_2$, then $au_1 + bu_2$ is a solution to $\mathcal{L}(u) = af_1 + bf_2$.

Scalar vs. Systems

So far we have dealt with PDEs for a scalar unknown u; these equations will be the focus of this course. However, analogous equations for vector-valued functions (more than one function) are also ubiquitous, important, and, in general, very difficult to solve. We call such equations **systems of partial differential equations**, as opposed to scalar PDEs. As with systems of linear algebraic equations, the difficulty here lies in that the partial differential equations are coupled, and one cannot simply solve separately the scalar equations for each unknown function. For example,

$$\begin{cases} u_x + v u_y = 0 \\ u v_x + v_y = v \end{cases}$$

is an example of a system of two equations for unknown functions $u(x, y)$, $v(x, y)$ in two independent variables.

Famous examples of systems of PDEs are the linear Maxwell equations and the quasilinear Euler and Navier-Stokes equations.

General Solutions and Arbitrary Functions

Recall from studying ordinary differential equations the notion of a **general solution vs. a particular solution**. An ODE has an infinite number of solutions and this infinite class of solutions (called the general solution) is parametrized via **arbitrary constants**.

PDEs will also have an infinite number of solutions but they will now be parametrized via **arbitrary functions**. We illustrate with a few examples as follows.

General Solutions and Arbitrary Functions

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Examples:

- Find the general solution on the full domain $\mathbb{R}^2$ for $u(x, y)$ solving

$$u_x = 0.$$

  $u(x, y) = f(y)$ for any function f of one variable is the general solution. If we want to solve the same PDE but in three independent variables (i.e., solve for $u(x, y, z)$), then the general solution would be $u(x, y, z) = f(y, z)$, for any function f of two variables.

- Find the general solution on the full domain $\mathbb{R}^2$ for $u(x, y)$ solving

$$u_{xx} = 0$$

  The PDE tells us that the x derivative of $u_x(x, y)$ must be 0 . Hence, following the logic of the previous example, we find $u_x = f(y)$. Integrating in x gives $u(x, y) = f(y)x + g(y)$, since any "constant" in the integration needs only to be constant in x (not necessarily in y). Thus the general solution for $u(x, y)$ to the $u_{xx} = 0$ is $u(x, y) = f(y)x + g(y)$, for any two functions f and g of one variable.

Auxiliary Conditions: Boundary and Initial Conditions

PDEs will often be supplemented by an auxiliary condition wherein we specify, in some subset of the domain, the value of the solution u and/or its partial derivatives. Our hope is that enforcing this auxiliary condition on the general solution will yield a unique solution.

There are two natural classes of auxiliary conditions. They lead, respectively, to initial value problems (IVPs) and boundary value problems (BVPs).
Two famous examples are, respectively, the initial value problems for the wave and the diffusion (heat) equation in one space dimension:

$$\textbf{IVP Wave: } \begin{cases} u_{tt} = c^2 u_{xx} & \text{for } -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) & \text{for } -\infty < x < \infty \end{cases},$$

$$\textbf{IVP Diffusion: } \begin{cases} u_t = c^2 u_{xx} & \text{for } -\infty < x < \infty, t > 0, \\ u(x,0) = f(x) & \text{for } -\infty < x < \infty \end{cases}.$$

A famous example of boundary value problem is the so-called Dirichlet problem for the Laplacian where in $2\mathrm{D}$:

$$\textbf{BVP Laplace: } \begin{cases} u_{xx} + u_{yy} = 0 & \text{for } (x,y) \in \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}.$$

Definition of a Well-Posed Problem

Definition

We say a PDE with one or more auxiliary conditions constitutes a well-posed problem if the following three conditions hold:

1. **Existence**: for a given choice of auxiliary condition(s) wherein the data is chosen from some function class, there exists a solution to the PDE which satisfies the auxiliary condition(s).
2. **Uniqueness**: there is, in fact, only one such solution.
3. **Stability**: if we perturb slightly the auxiliary condition, then the resulting unique solution does not change much. That is, small changes in the auxiliary condition(s) lead only to small changes in the solution.

Explicit vs. Nonexplicit Solutions, No Solutions

Researchers spend much time on the "so-called" big three: the wave, diffusion, and Laplace equations. Within certain domains, these equations can be explicitly solved; that is, we can find explicit formulas for their solutions in terms of either known functions or, possibly, infinite sums of known functions (power series). However, it is very important to note that most PDEs of any scientific interest cannot be explicitly solved in terms of known functions, or even in terms of possibly infinite sums of known functions (power series).

Does a PDE (with no additional constraint) involving smooth functions of u, its derivatives, and the independent variables always have a solution in a neighborhood of a point in its domain? The answer is no but this is hardly obvious. In fact, in 1958 Hans Lewy[1] provided for the first time a startling example of a simple linear PDE with no solution.

[1] Hans Lewy (1904–1988) was a German-born American mathematician, known for his work on PDEs and on complex variables theory. The relevant paper here is "An example of a smooth linear partial differential equation without solution", Annals of Mathematics 66 (1957), no. 1.

Approximate Solutions via the Computer

Physicists, engineers, computer and data scientists, chemists, biologists, economists, etc., often encounter or derive PDEs for which no explicit solution formula exists. They are usually not interested in the above-mentioned "abstract" mathematical questions. Rather, they seek approximations to the solution. In many cases, this is now possible due to the computational power of modern computers. There are classes of general methods for numerically solving PDEs, and many numerical software packages are well developed and widely used by these practitioners. However, for many nonlinear PDEs, one often needs to develop tailor-made numerical methods designed for the specific PDE. This, undoubtedly, requires a mathematical understanding of the PDE's structure, and this is the focus of many applied mathematicians working in the fields of **numerical analysis and scientific computation**.

Typical PDEs

Three basic linear second order partial differential equations:

1. Wave equation: $u_{tt} - c^2 (u_{xx} + u_{yy} + u_{zz}) = 0$;
2. Heat conduction equation: $u_t - k (u_{xx} + u_{yy} + u_{zz}) = 0$;
3. Laplace equation: $u_{xx} + u_{yy} + u_{zz} = 0$.

These typical equations represent conservative evolution processes, dissipative time-dependent evolution processes and stationary processes respectively, which are main equations of our discussion.

String Oscillation

Consider a tight string with length-l fixed two end points. The problem is to determine motion equation by which we can describe the displacement $u(x,t)$ of the string at any t-moment by initial perturbation. For this reason, we give the following assumptions.

1. The string is soft and elastic, and the tension direction is the same with its tangent.
2. Arbitrary small section is not stretched, and then the tension is constant by Hooke law.
3. Weight/tension < 0.1 (very small).
4. Displacement/length $< 0.1 \Leftrightarrow \frac{\max |u|}{l} < 0.1$.
5. The slope on any point after displacement < 0.1.
6. The string only have transverse vibration.

String Oscillation

Consider a micro-element on the string. Suppose T is the tension on two end points as Figure 1.

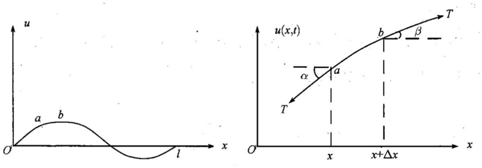

Figure: 1. A micro-element on the string

The force acting on the micro-element along vertical direction is

$$T \sin\beta - T \sin\alpha.$$

By Newton's second law, resultant force equals the product of mass and acceleration, then

$$T \sin\beta - T \sin\alpha = \rho \Delta s \cdot u_{tt},$$

where ρ-density, Δs-arc length of this small section after displacement. Since the slope is very small (< 0.1), we have $\Delta s \approx \Delta x$. Because $\alpha < 0.1$, and $\beta < 0.1$, then

$$\sin\alpha \approx \tan\alpha, \quad \sin\beta \approx \tan\beta,$$

and then

$$\tan\beta - \tan\alpha = \frac{\rho \Delta x}{T} u_{tt}. \tag{2}$$

String Oscillation

By calculation, at t-time we have

$$\tan\alpha \approx (u_x)|_x \left(\approx \frac{\Delta u}{\Delta x}\bigg|_x\right), \quad \tan\beta \approx \frac{\Delta u}{\Delta x}\bigg|_{x+\Delta x} \approx \left(u_x|_{x+\Delta x}\right).$$

Then from (2), we have

$$\frac{1}{\Delta x}\left[(u_x)|_{x+\Delta x} - (u_x)|_x\right] = \frac{\rho}{T}u_{tt}.$$

Letting $\Delta x \to 0^+$ and taking limit, we obtain one-dimensional wave equation

$$u_{tt} = c^2 u_{xx},$$

where $c^2 = \frac{T}{\rho}$. If there is an action of an outer force F, then we obtain the equation

$$u_{tt} = c^2 u_{xx} + f,$$

where $f = \frac{F}{\rho}$.

Membrane Oscillation

Assumptions:

1. Membrane is soft and elastic, and its tension is located in the tangent plane of the membrane.
2. Any small element is not stretched and the tension is constant by Hooke law.
3. Weight of membrane / tension < 0.1.
4. Displacement/ (minimum diameter of membrane) < 0.1.
5. Slope < 0.1.
6. There only is transverse vibration.

Consider a very small element by assumptions, its area $\approx \Delta x \cdot \Delta y$ (Figure 2).

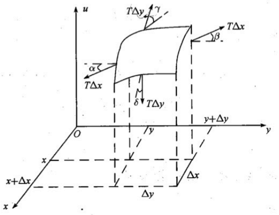

Figure: 2. A micro-element on the membrane

Membrane Oscillation

If $T =$ tension/(unit length), then the forces acting on each side are $T\Delta x$ and $T\Delta y$. The force along vertical direction is

$$T\Delta x \sin\beta - T\Delta x \sin\alpha + T\Delta y \sin\delta - T\Delta y \sin\gamma.$$

Since the slope < 0.1, sine of these angles $\approx$ tangent of these angles, we have the resultant force

$$T\Delta x(\tan\beta - \tan\alpha) + T\Delta y(\tan\delta - \tan\gamma).$$

By Newton's second law, we have

$$T\Delta x(\tan\beta - \tan\alpha) + T\Delta y(\tan\delta - \tan\gamma) = \rho\Delta A \cdot u_{tt}, \tag{3}$$

where $\rho =$ mass/(unit area) – density, $\Delta A \approx \Delta x \cdot \Delta y$ – the area of this element, u_{tt} – acceleration of some point in this region.
By calculation, we have

$$\begin{aligned} &\tan\alpha \approx u_y(x_1, y), \quad \tan\beta \approx u_y(x_1, y + \Delta y), \\ &\tan\delta \approx u_x(x + \Delta x, y_1), \quad \tan\gamma \approx u_x(x, y_1), \end{aligned}$$

where $x_1 \in [x, x + \Delta x], \quad y_1 \in [y, y + \Delta y]$.
Substituting these values for equality (3), we obtain

$$T\Delta x[u_y(x_1, y + \Delta y) - u_y(x_1, y)] + T\Delta y[u_x(x + \Delta x, y_1) - u_x(x, y_1)] = \rho\Delta x \cdot \Delta y \cdot u_{tt}.$$

Membrane Oscillation

Divide above form by $\rho\Delta x\Delta y$ and we get

$$\frac{T}{\rho}\left[\frac{u_y\left(x_1, y+\Delta y\right)-u_y\left(x_1, y\right)}{\Delta y}+\frac{u_x\left(x+\Delta x, y_1\right)-u_x\left(x, y_1\right)}{\Delta x}\right]=u_{tt}.$$

Letting $\Delta x \rightarrow 0, \Delta y \rightarrow 0$, and taking limit, we obtain

$$u_{tt}=c^2\left(u_{xx}+u_{yy}\right),$$

where $c^2=T/\rho$. This equation is called two-dimensional wave equation. If there is an action of outer force F on unit area of membrane, then we get the equation

$$u_{tt}=c^2\left(u_{xx}+u_{yy}\right)+f,$$

where $f=F/\rho$.

Heat conduction in solid

Consider some region D^* surrounded by closed surface B^*. Let $u(x, y, z, t)$ denote the temperature at time t on the point (x, y, z). If the temperature $\neq$ constant, then quantity of heat flows to lower temperature from higher temperature place. By Fourier law, the rate of change on heat flux directs ratio to the gradient of temperature. Therefore, in the body with identical property, the velocity of heat flux can be given

$$v = -K \text{ grad } u,$$

where $K =$ constant, is called heat conduction rate.
Assume D is the region surrounded by arbitrary closed surface B in D^*, then all heat quantity q which is flowed out from D in unit time is

$$q = \iint_B v_n ds,$$

where $v_n = \boldsymbol{v} \cdot \boldsymbol{n}$ is the component of $\boldsymbol{v}$ along outer unit normal $\boldsymbol{n}$ of B. Applying the Gauss theorem (or divergence theorem), we obtain

$$\begin{aligned} q &= \iint_B v_n ds = \iiint_D \mathrm{div}(-K \cdot \mathrm{grad}\, u) dxdydz \\ &= -K \iiint_D \Delta u dxdydz \quad (\Delta u\text{–Laplace operator }). \end{aligned}$$

Heat conduction in solid

But on the other hand, the total heat quantity Q in D is

$$Q = \iiint_D \sigma \rho u dxdydz,$$

where ρ–density, σ–Epecific heat.
Assume that the operation on differentiation and integration can be commutated, then the minus rate of change q^* is

$$q^* = -\frac{\partial}{\partial t} Q = -\iiint_D \sigma \rho \frac{\partial u}{\partial t} dxdydz,$$

which must be equal to q, namely,

$$-\iiint_D \sigma \rho u_t dxdydz = -K \iiint_D \Delta u dxdydz,$$

or

$$\iiint_D (\sigma \rho u_t - K \Delta u)\, dxdydz = 0.$$

Since the D is arbitrary in D^* and the integrand is continuous, we obtain

$$f = \sigma \rho u_t - K \Delta u \equiv 0$$

and

$$u_t = k \Delta u,$$

where $k = K/\sigma\rho$. This equation is called heat conduction equation.

Gravitation Potential

In this section, we will derive the famous equation: Laplace equation. Consider two particles P and Q with masses m and M respectively (Figure 3).

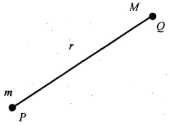

Figure: 3. Gravitation potential of two particles

Assume $r = \overline{PQ}$, distance between P and Q, by the Newton's gravitation law, we get

$$F = G\frac{mM}{r^2},$$

where G is the constant of gravitation, generally, $G = 1$. Then $F = \frac{mM}{r^2}$.
Let $\boldsymbol{r}$ denote the vector $\overrightarrow{PQ}$, then the action force $\boldsymbol{F}$ of the particle P with mass m to the particle Q with unit mass can be written as

$$\boldsymbol{F} = \frac{-m\boldsymbol{r}}{r^3} = \nabla\left(\frac{m}{r}\right),$$

which is called the intensity of field generated by this force $\boldsymbol{F}$.

Gravitation Potential

Assume that some particle with unit mass is attracted by particle P with mass m, then the work W of the action by force $\boldsymbol{F}$ moving this particle from infinite point to the point Q is

$$W = \int_{\infty}^{r} \boldsymbol{F} \cdot d\boldsymbol{r} = \int_{\infty}^{r} \nabla \left(\frac{m}{r} \right) \cdot d\boldsymbol{r} = \frac{m}{r},$$

and $-W$ is called the potential generated at point Q by the action of the particle P, which can be represented as

$$V = -\frac{m}{r}.$$

Then, the intensity of field at point Q is that

$$\boldsymbol{F} = \nabla \left(\frac{m}{r} \right) = -\nabla V.$$

Now, consider n-particles with masses $m_1, m_2, \cdots, m_n$ respectively and distances $r_1, r_2, \cdots, r_n$ respectively from point Q, then the attraction force. $\boldsymbol{F}$ of these particles system to particle Q with unit mass is

$$\boldsymbol{F} = \sum_{k=1}^{n} \nabla \frac{m_k}{r_k} = \nabla \sum_{k=1}^{n} \frac{m_k}{r_k},$$

and the work W by these forces acting particle Q with unit mass is

$$W = \int_{\infty}^{r} \boldsymbol{F} \cdot d\boldsymbol{r} = \sum_{k=1}^{n} \frac{m_k}{r_k} = -V.$$

Gravitation Potential

This potential satisfies

$$\Delta V = \nabla^2 V = -\nabla^2 \sum_{k=1}^{n} \frac{m_k}{r_k} = -\sum_{k=1}^{n} \nabla^2 \frac{m_k}{r_k} = 0, \quad r_k > 0.$$

If the distribution is continuous in some volume R (Figure 4), then we obtain

$$V(x, y, z) = \iiint_R \frac{\rho(\xi, \eta, \zeta)}{r} dR,$$

where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$ and the point Q is located out volume R. Thus, we have

$$\Delta V = \nabla^2 V = 0,$$

which is called Laplace equation or potential equation.

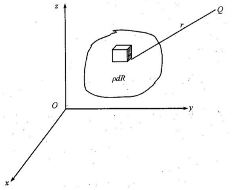

Figure: 4. Gravitation potential of a particle and mass

Linear second order PDEs with two variables

General form of the second order linear equations is

$$\sum_{i,j=1}^{n} A_{ij} u_{x_i x_j} + \sum_{i=1}^{n} B_i u_{x_i} + Fu = G, \tag{4}$$

where $A_{ij} = A_{ji}$ and A_{ij}, B_i, F and G are real functions in $\mathbf{R}^n$ (or $A_{ij}, B_i, F, G \in C(D^n) \subset C(\mathbf{R}^n)$), $\forall X = (x_1, \cdots, x_n) \in D \subset \mathbf{R}^n$. Consider unknown function $u(x, y)$ with two variables (x, y). Then, (4) can be reduced to following form:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G, \tag{5}$$

where $A^2 + B^2 + C^2 > 0$.

Assume u and all coefficients belong to $C^{(2)}(D)$. Let $\Delta = B^2 - 4AC$, which is called discriminant. The classification of the second order equations (5) is determined based on the sign of the Δ as follows.

1. (5) is called hyperbolic type if $\Delta > 0$, at point (x_0, y_0) or in $D \in \mathbf{R}^2$.
2. (5) is called parabolic type if $\Delta = 0$, at point (x_0, y_0) or in $D \in \mathbf{R}^2$.
3. (5) is called elliptic type if $\Delta < 0$, at point (x_0, y_0) or in $D \in \mathbf{R}^2$.

Linear second order PDEs with two variables

We will reduce (5) to standard (or normal) form by the transformation of variables. Let new variables (ξ, η) be given in terms of the formula

$$\begin{cases} \xi = \xi(x, y), \\ \eta = \eta(x, y), \end{cases}$$

where $\xi, \eta \in C^2(D)$ and function determinant (Jacobi)

$$J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0 \text{ in } \bar{D}.$$

If $J \neq 0$, then we have inverse transformation $(\xi(x, y), \eta(x, y)) \Leftrightarrow (x(\xi, \eta), y(\xi, \eta))$, and then $u = u(\xi(x, y), \eta(x, y))$,

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x, \quad u_y = u_\xi \xi_y + u_\eta \eta_y, \\ u_{xx} &= \left(u_\xi \xi_x + u_\eta \eta_x\right)'_x = \left(u_\xi \xi_x\right)'_x + \left(u_\eta \eta_x\right)'_x = \left[\left(u_\xi\right)'_x \xi_x + u_\xi \left(\xi_{xx}\right)\right] + \left[\left(u_\xi\right)'_x \eta_x + u_\eta \eta_{xx}\right] \\ &= \left(u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x\right) \xi_x + u_\xi \xi_{xx} + \left(u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x\right) \eta_x + u_\eta \eta_{xx} \\ &= u_{\xi\xi} \xi_x^2 + 2u_{\xi\eta} \xi_x \eta_x + u_{\eta\eta} \eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}, \\ u_{xy} &= u_{\xi\xi} \xi_x \xi_y + u_{\xi\eta} \left(\xi_x \eta_y + \xi_y \eta_x\right) + u_{\eta\eta} \eta_x \eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}, \\ u_{yy} &= u_{\xi\xi} \xi_y^2 + 2u_{\xi\eta} \xi_y \eta_y + u_{\eta\eta} \eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}. \end{aligned} \tag{6}$$

Substituting (6) for (5), we obtain the new equation with variables (ξ, η):

$$A^* u_{\xi\xi} + B^* u_{\xi\eta} + C^* u_{\eta\eta} + D^* u_\xi + E^* u_\eta + F^* u = G^* \tag{7}$$

Linear second order PDEs with two variables

where

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2,\\
B^* &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y,\\
C^* &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2,\\
D^* &= A\xi_{xx} + B\xi_{xy} + C\xi_{yy} + D\xi_x + E\xi_y,\\
E^* &= A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y,\\
F^* &= F,\\
G^* &= G,
\end{aligned} \tag{8}$$

and

$$B^{*2} - 4A^*C^* = J^2(B^2 - 4AC). \tag{9}$$

Namely, under inverse transformation, the type of equations isn't changed.

Simplification and standard forms

In this section, applying above invertible transformation, we reduce (5) to standard type. Assume $ABC \neq 0$, choose some (ξ, η) such that $A^* = 0$ or $C^* = 0$, by (8), we have

$$A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0 \text{ or } A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0.$$

Let $\varphi(x, y)$ denote $\xi(x, y)$ or $\eta(x, y)$, then

$$A\varphi_x^2 + B\varphi_x\varphi_y + C\varphi_y^2 = 0. \tag{10}$$

We can prove that the partial differential equation (10) is equivalent to the following ordinary differential equation

$$Ady^2 - Bdxdy + Cdx^2 = 0. \tag{11}$$

(The equation (11) is called characteristic equation, and its solutions are called characteristic lines or characteristics.)

In fact, If $\varphi(x, y) = C$ is the solution of (10), then $\varphi(x, y) = C$ is an integral curve in plane, then

$$d\varphi = \varphi_x dx + \varphi_y dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y}. \tag{12}$$

Substituting (12) for (10), we have

$$A\left(\frac{\varphi_x}{\varphi_y}\right)^2 + B\left(\frac{\varphi_x}{\varphi_y}\right) + C = A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0.$$

Simplification and standard forms

Namely, $Ady^2 - Bdxdy + Cdx^2 = 0$. Clearly, vice versa. From (11) we have

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{B \pm \sqrt{\Delta}}{2A}.$$

Since the ordinary differential equations

$$\frac{dy}{dx} = \frac{B + \sqrt{\Delta}}{2A} \text{ and } \frac{dy}{dx} = \frac{B - \sqrt{\Delta}}{2A} \tag{13}$$

are linear first order equations, we can obtain their solutions.

(A) Hyperbolic type. For $\Delta > 0$, take $\begin{cases} \xi = \varphi_1(x, y), \\ \eta = \varphi_2(x, y). \end{cases}$

In fact, from (13), we have two different solutions $\xi = \varphi_1(x,\ y) = C_1, \eta = \varphi_2(x, y) = C_2$ such that $A^* = C^* = 0\,(B^* \neq 0)$ and equation (7) is reduced to standard equation

$$u_{\xi\eta} = H_1\left(\xi, \eta, u, u_\xi, u_\eta\right), \tag{14}$$

which is called the first normal type.

Let $\begin{cases} \alpha = \xi + \eta, \\ \beta = \xi - \eta, \end{cases}$ $J = \begin{vmatrix} \alpha_\xi & \alpha_\eta \\ \beta_\xi & \beta_\eta \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$. The (14) is changed to

$$u_{\alpha\alpha} - u_{\beta\beta} = H_2\left(\alpha, \beta, u, u_\alpha, u_\beta\right)$$

which is called the second normal type.

Simplification and standard forms

(B) Parabolic type. For $\Delta = 0$, take $\xi = \varphi(x, y)$ (or $\eta = \varphi(x, y)$).
In this case, there is only one ordinary differential equation and one solution $\xi = \varphi(x, y) = C$, and then $A^* = 0\,(C^* \neq 0)\,, B^* = 0, B^2 = 4AC, B = \pm 2\sqrt{AC}$. In fact,

$$\frac{dy}{dx} = -\frac{\varphi_x}{\varphi_y} = \frac{B}{2A} = -\frac{\xi_x}{\xi_y},$$

$$\begin{aligned}
A^* &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
&= A\xi_x^2 + 2\sqrt{AC}\xi_x\xi_y + C\xi_y^2 = \left(\sqrt{A}\xi_x + \sqrt{C}\xi_y\right)^2 = 0, \\
B^* &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y \\
&= 2\left(\sqrt{A}\xi_x + \sqrt{C}\xi_y\right)\left(\sqrt{A}\eta_x + \sqrt{C}\eta_x\right) = 0.
\end{aligned}$$

Then equation (7) is reduced to the following type:

$$u_{\eta\eta} = H\left(\xi, \eta, u, u_\xi, u_\eta\right).$$

This is parabolic standard type.
If let $\eta = \varphi(x, y)$, then we have another standard type:

$$u_{\xi\xi} = H^*\left(\xi, \eta, u, u_\xi, u_\eta\right).$$

Simplification and standard forms

(C) Elliptic type. For $\Delta < 0$, in this case, there are not real value solutions. But there are two conjugate complex solutions:

$$\Phi(x, y) = \varphi(x, y) + \mathrm{i}\psi(x, y),$$
$$\Psi(x, y) = \varphi(x, y) - \mathrm{i}\psi(x, y),$$

where $\varphi(x, y)$ and $\varphi(x, y)$ are real functions.

Let $\begin{cases} \xi = \varphi(x, y), \\ \eta = \psi(x, y), \end{cases}$ show that $A^* = C^* \neq 0$ and $B^* = 0$, and (7) is reduced to the standard type as follows:

$$u_{\xi\xi} + u_{\eta\eta} = H\left(\xi, \eta, u, u_\xi, u_\eta\right).$$

Examples

Example (1)

$$x^2 u_{xx} + 2xyu_{xy} + y^2 u_{yy} = 0 \quad (xy \neq 0). \tag{15}$$

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Answer.

1. $\Delta = B^2 - 4AC = 4x^2y^2 - 4x^2y^2 = 0$. (15) is parabolic type.
2. Characteristic equation:

$$x^2 dy^2 - 2xydxdy + y^2 dx^2 = 0,$$

$$\frac{dy}{dx} = \frac{y}{x} \Rightarrow \ln\frac{y}{x} = c_0, \quad \frac{y}{x} = c.$$

Let

$$\begin{cases} \xi = \frac{y}{x} \\ \eta = y \end{cases} \Rightarrow J = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = -\frac{y}{x^2} \neq 0 \quad (xy \neq 0). \tag{16}$$

Substituting (16) for (15), we have $\eta^2 u_{\eta\eta} = 0$, then $u_{\eta\eta} = 0$ is the normal type. □

Examples

Example (2)

$$y^2 u_{xx} - x^2 u_{yy} = 0 \quad (xy \neq 0). \tag{17}$$

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$$y^2 u_{xx} - x^2 u_{yy} = 0 \quad (xy \neq 0). \tag{17}$$

Answer.

1. $\Delta = B^2 - 4AC = 4x^2y^2 > 0$. (17) is hyperbolic type..
2. Characteristic equation:

$$y^2 dy^2 - x^2 dx^2 = 0 \Rightarrow \frac{dy}{dx} = \pm\frac{x}{y},$$

$$\left.\begin{array}{r} \left(\frac{dy}{dx}\right)_1 = \frac{x}{y} \\ \left(\frac{dy}{dx}\right)_2 = \frac{-x}{y} \end{array}\right\} \Rightarrow \left\{ \begin{array}{l} \frac{1}{2}y^2 - \frac{1}{2}x^2 = c_1 \\ \frac{1}{2}y^2 + \frac{1}{2}x^2 = c_2 \end{array}\right.$$

Let

$$\left\{ \begin{array}{l} \xi = \frac{1}{2}\left(y^2 - x^2\right), \\ \eta = \frac{1}{2}\left(y^2 + x^2\right) \end{array}\right. \qquad \Rightarrow J \neq 0. \tag{18}$$

Substituting (18) for (17), we have

$$u_{\xi\eta} = \frac{\eta}{2\left(\xi^2 - \eta^2\right)} u_\xi - \frac{\xi}{2\left(\xi^2 - \eta^2\right)} u_\eta .$$

Examples

Example (3)

Find general solution of the equation

$$u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0. \tag{19}$$

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$$u_{xx} - 2\sin x u_{xy} - \cos^2 x u_{yy} - \cos x u_y = 0. \tag{19}$$

Answer.

1. $\Delta = 4\sin^2 x + 4\cos^2 x = 4 > 0$. (19) is hyperbolic type.
2. Characteristic equation is $dy^2 + 2\sin x dxdy - \cos^2 x dx^2 = 0$, then

$$\left(\frac{dy}{dx}\right)_1 = -\sin x + 1, \quad \left(\frac{dy}{dx}\right)_2 = -\sin x - 1. \Rightarrow \begin{cases} x - y + \cos x = c_1 \\ x + y - \cos x = c_2 \end{cases}$$

Let

$$\begin{cases} \xi = x - y + \cos x, \\ \eta = x + y - \cos x. \end{cases} \tag{20}$$

Substituting (20) for (19), we have $u_{\xi\eta} = 0 \Rightarrow u = \varphi(\eta) + \psi(\xi)$, then
$u(x, y) = \varphi(x + y - \cos x) + \psi(x - y + \cos x)$.

□

D'Alembert formula of Cauchy problem for string oscillation

Example (Homogeneous)

Consider Cauchy problem of string oscillation equation:

$$u_{tt} - a^2 u_{xx} = 0, -\infty < x < +\infty, \quad t > 0, \tag{21}$$

$$u(x,0) = \varphi(x), -\infty < x < +\infty, \tag{22}$$

$$u_t(x,0) = \psi(x), -\infty < x < +\infty. \tag{23}$$

D'Alembert formula of Cauchy problem for string oscillation

Example (Homogeneous)

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$$u_{tt} - a^2 u_{xx} = 0, -\infty < x < +\infty, \quad t > 0, \tag{21}$$
$$u(x,0) = \varphi(x), -\infty < x < +\infty, \tag{22}$$
$$u_t(x,0) = \psi(x), -\infty < x < +\infty. \tag{23}$$

Answer.

1. $\Delta = B^2 - 4AC = 4a^2 > 0$, (21) is hyperbolic type.
2. Characteristic equation is $dx^2 - a^2 dt^2 = 0$. Then $\frac{dx}{dt} = \pm a$, $\begin{cases} \left(\frac{dx}{dt}\right)_1 = a, \\ \left(\frac{dx}{dt}\right)_2 = -a. \end{cases}$

   Characteristic lines are $x - at = c_1$ and $x + at = c_2$.
   Let $\begin{cases} \xi = x - at, \\ \eta = x + at. \end{cases}$ Substituting ξ, η for (21), we have $u_{\xi\eta} = 0$. Integrating this equation on η and ξ, we have $u = f_1(\xi) + f_2(\eta)$. Namely,

   $$u(x,t) = f_1(x - at) + f_2(x + at), \tag{24}$$

   where f_1 and f_2 are any second order continuously differentiable functions in $(-\infty, +\infty)$.

Applying (22), (23), we get

$$f_1(x) + f_2(x) = \varphi(x), \tag{25}$$

$$a\left[-f_1'(x) + f_2'(x)\right] = \psi(x). \tag{26}$$

By (26), we derive that

$$-f_1(x) + f_2(x) = \frac{1}{a}\int_{x_0}^{x} \psi(t)\,dt + c. \tag{27}$$

Using (25), (27), we get

$$f_1(x) = \frac{1}{2}\varphi(x) - \frac{1}{2a}\int_{x_0}^{x}\psi(\lambda)d\lambda - \frac{c}{2}, \tag{28}$$

$$f_2(x) = \frac{1}{2}\varphi(x) + \frac{1}{2a}\int_{x_0}^{x}\psi(\lambda)d\lambda + \frac{c}{2}. \tag{29}$$

Substituting (28) and (29) for (24), we obtain the solution of Cauchy problem $(21) \sim (23)$: $u(x,t) = \frac{1}{2}[\varphi(x-at) + \varphi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at}\psi(\tau)d\tau$, which is called D'Alembert formula, where $\varphi \in C^2, \psi \in C^1$ are arbitrary. □

Example (1)

$$u_{tt} = c^2 u_{xx}, \tag{30}$$
$$u(x,0) = \sin x, \tag{31}$$
$$u_t(x,0) = \cos x. \tag{32}$$

By D'Alembert formula, we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\sin(x+ct) + \sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos\tau d\tau \\ &= \sin x \cos ct + \frac{1}{2c}\sin\tau\Big|_{x-ct}^{x+ct} \\ &= \sin x \cos ct + \frac{1}{2c}[\sin(x+ct) - \sin(x-ct)] \\ &= \sin x \cos ct + \frac{1}{c}\cos x \sin ct. \end{aligned}$$

□

Example (2)

Solve the following Cauchy problem:

$$u_{xx} + 2u_{xy} - 3u_{yy} = 0, \tag{33}$$
$$u(x, 0) = 3x^2, \tag{34}$$
$$u_y(x, 0) = 0. \tag{35}$$

Answer.

$\Delta = 4 + 12 = 16 > 0$, then the equation (33) is hyperbolic. Characteristic equation is $dy^2 - 2dxdy - 3dx^2 = 0$. Then $\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} - 3 = 0$, $\Rightarrow \left(\frac{dy}{dx}\right)_1 = -1$, $\left(\frac{dy}{dx}\right)_2 = 3$, $\Rightarrow \left\{\begin{array}{l} x + y = C_1, \\ 3x - y = C_2. \end{array}\right.$ $\Rightarrow$ Let $\left\{\begin{array}{l} \xi = x + y, \\ \eta = 3x - y. \end{array}\right.$ Substituting (ξ, η) for (33), we derive $u_{\xi\eta} = 0$.
Then

$$u = \varphi(\xi) + \psi(\eta) = \varphi(x + y) + \psi(3x - y). \tag{36}$$

By $(34) \sim (35)$, we have

$$u(x, 0) = \phi(x) + \psi(3x) = 3x^2, \tag{37}$$
$$u_y(x, 0) = \phi'(x) - \psi'(3x) = 0. \tag{38}$$

By (38), we get

$$\varphi(x) - \frac{1}{3}\psi(3x) = C. \tag{39}$$

By $(37), (39)$, we obtain

$$\varphi(x) = \frac{3}{4}\left(x^2 + C\right), \quad \psi(3x) = \frac{3}{4}\left(3x^2 - C\right)$$

or

$$\psi(\eta) = \frac{\eta^2}{4} - \frac{3}{4}C, \quad \varphi(\xi) = \frac{3}{4}\left(\xi^2 + C\right).$$

Thus, by (36), we obtain

$$\begin{aligned} u(x, y) &= \varphi(x+y) + \psi(3x-y) \\ &= \frac{3}{4}\left[(x+y)^2 + C\right] + \frac{1}{4}(3x-y)^2 - \frac{3}{4}C \\ &= 3x^2 + y^2. \end{aligned}$$

□

Duhamel's Principle

Consider solving the inhomogeneous problem

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + f(x,t) \quad (-\infty < x < +\infty, t > 0) \tag{40}$$

$$u|_{t=0} = 0, \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \quad (-\infty < x < +\infty) \tag{41}$$

Theorem (Duhamel's Principle)

Denote $\Omega(x,t,\tau)$ be the solution of the corresponding homogeneous problem

$$\begin{cases} \frac{\partial^2 \Omega}{\partial t^2} = a^2 \frac{\partial^2 \Omega}{\partial x^2} & (-\infty < x < +\infty, t > \tau) \\ \Omega|_{t=\tau} = 0, \left.\frac{\partial \Omega}{\partial t}\right|_{t=\tau} = f(x,\tau) & (-\infty < x < +\infty) \end{cases}$$

then the solution to (40) and (41) is

$$u(x,t) = \int_0^t \Omega(x,t,\tau) d\tau.$$

Example (General case)

$$u_{tt} - a^2 u_{xx} = f(x,t), -\infty < x < +\infty, \quad t > 0, \tag{42}$$
$$u(x,0) = \varphi(x), -\infty < x < +\infty, \tag{43}$$
$$u_t(x,0) = \psi(x), -\infty < x < +\infty. \tag{44}$$

In this sectionn we discuss the method of separating variables which can be used for solving some definite problems on finite region for partial differential equations.
Consider general second order homogeneous equation with two-variables:

$$a(x,y)u_{xx}+c(x,y)u_{yy}+d(x,y)u_x+e(x,y)u_y+f(x,y)u=0, \tag{45}$$

which belongs to one type among the following normal types:

1. for $ac<0$-hyperbolic type;
2. for $ac=0$ and $|a|+|c|>0$-parabolic type;
3. for $ac>0$ elliptic type.

Assume

$$u(x,y)=X(x)Y(x)\not\equiv 0, \tag{46}$$

where $X(x)$ and $Y(y)$ are the functions with individual variable x or y respectively.
Substituting (46) for (45), we have

$$aX''Y+cXY''+dX'Y+eXY'+fXY=0. \tag{47}$$

Separation of variables

Suppose $a(x,y) = a_1(x)$, $c(x,y) = b_1(y)$, $d(x,y) = a_2(x)$, $e(x,y) = b_2(y)$, $f(x,y) = a_3(x) + b_3(y)$. Dividing equality (47) by $X(x)Y(y)$, we have

$$\left(a_1\frac{X''}{X} + a_2\frac{X'}{X} + a_3\right) = -\left(b_1\frac{Y''}{Y} + b_2\frac{Y'}{Y} + b_3\right). \tag{48}$$

Noting that the left end of (48) only depends on x and the right end of (48) only depends on y, thus

$$\frac{d}{dx}\left(a_1\frac{X''}{X} + a_2\frac{X'}{X} + a_3\right) = 0, \tag{49}$$

and

$$a_1\frac{X''}{X} + a_2\frac{X'}{X} + a_3 = \lambda, \tag{50}$$

and then

$$b_1\frac{Y''}{Y} + b_2\frac{Y'}{Y} + b_3 = -\lambda, \tag{51}$$

where λ is a number, which is called separation constant. Equalities (50) and (51) can be written as follows:

$$a_1X'' + a_2X' + (a_3 - \lambda)\,X = 0, \tag{52}$$

and

$$b_1Y'' + b_2Y' + (b_3 + \lambda)\,Y = 0. \tag{53}$$

It is clear that if $X(x)$ and $Y(y)$ are the solutions of (52) and (53) respectively, then $u(x,y) = X(x)Y(y)$ is the solution of (45).

The process by separation of variables for solving mixed problem on string oscillation

Now, we show the process by separating variables of solving the mixed problem on string oscillation:

$$u_{tt} - a^2 u_{xx} = 0, \quad 0 < x < l, t > 0, \tag{54}$$
$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant l, \tag{55}$$
$$u_t(x,0) = g(x), \quad 0 \leqslant x \leqslant l, \tag{56}$$
$$u(0,t) = 0, \quad t \geqslant 0, \tag{57}$$
$$u(l,t) = 0, \quad t \geqslant 0. \tag{58}$$

The process by separation of variables for solving mixed problem on string oscillation

Suppose

$$u(x,t) = X(x)T(t). \tag{59}$$

Substituting (59) for (54), we have $XT'' = a^2X''T$. As $XT \neq 0$, we get

$$\frac{X''}{X} = \frac{T''}{a^2T} = -\lambda \tag{60}$$

where $\lambda =$ constant. Thus

$$X'' + \lambda X = 0, \tag{61}$$

$$T'' + \lambda a^2 T = 0. \tag{62}$$

From (57) and (59), we obtain $u(0,t) = X(0)T(t) = 0$. Since $T(t) \not\equiv 0, \forall t \geqslant 0$, we have $X(0) = 0$. In the same way, $X(l) = 0$.

The process by separation of variables for solving mixed problem on string oscillation

We consider the following boundary value problem for solving $X(x)$:

$$X'' + \lambda X = 0, \tag{63}$$
$$X(0) = 0, \tag{64}$$
$$X(l) = 0. \tag{65}$$

Next, we will find non-trivial (non-zero) solutions of (63) $\sim$ (65), which may be depended on λ.

1. For $\lambda < 0$, the general solution of (63):

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x},$$

where c_1 and c_2 are any constants.
By boundary conditions (64) and (65), we have $\begin{cases} c_1 + c_2 = 0, \\ c_1 e^{\sqrt{-\lambda}l} + c_2 e^{-\sqrt{\lambda}l} = 0. \end{cases}$ Since $\begin{vmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}l} & e^{-\sqrt{-\lambda}l} \end{vmatrix} \neq 0$, we get $c_1 = c_2 = 0$. Namely, $X(x) \equiv 0$, there is not non-trivial solution for (63)$\sim$(65).

The process by separation of variables for solving mixed problem on string oscillation

2. For $\lambda = 0$, then general solution $X(x) = c_1 x + c_2$. y (64) and (65), we get

$$\begin{cases} c_2 = 0 \\ c_1 l + c_2 = 0. \end{cases}$$

Then $c_1 = c_2 = 0, X(x) \equiv 0$. There isn't non-trivial solution yet for (63) $\sim$ (65).

3. For $\lambda > 0$, the general solution $X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. By (64), we have $c_1 = 0$, and by (65), we have $c_2 \sin\sqrt{\lambda}l = 0$. If $c_2 \neq 0$, then $\sin\sqrt{\lambda}l = 0$, and then $\sqrt{\lambda}l = n\pi, n = 1, 2, \cdots$ or $\lambda_n = \left(\frac{n\pi}{l}\right)^2, n = 1, 2, \cdots$. For the infinite values $\left(\frac{n\pi}{l}\right)^2$ of λ_n, there are infinite non-trivial solutions $X_n(x) = \sin\frac{n\pi}{l}x, n = 1, 2, \cdots$.

For $\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2$, the general solutions of the equation (62) is

$$T_n(t) = a_n \cos\frac{an\pi}{l}t + b_n \sin\frac{an\pi}{l}t, \quad n = 1, 2, \cdots, \tag{66}$$

where a_n and b_n are any constants. Therefore

$$u_n(x,t) = X_n(x)T_n(t) = \left(a_n \cos\frac{an\pi}{l}t + b_n \sin\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x.$$

Since the equation (54) is linear and homogeneous, by principle of superposition, we obtain the formal solution:

$$u(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{an\pi}{l}t + b_n \sin\frac{an\pi}{l}t\right)\sin\frac{n\pi}{l}x.$$

The process by separation of variables for solving mixed problem on string oscillation

If

$$f(x)=\sum_{n=1}^{\infty} f_n \sin\frac{n\pi}{l}x,\quad g(x)=\sum_{n=1}^{\infty} g_n \sin\frac{n\pi}{l}x,$$

where f_n and g_n are Fourier coefficients of $f(x)$ and $g(x)$ respectively, then we obtain

$$\begin{aligned} a_n &= f_n = \frac{2}{l}\int_0^l f(\xi)\sin\frac{n\pi}{l}\xi d\xi, \\ b_n &= \frac{l}{an\pi} g_n = \frac{2}{an\pi}\int_0^l g(\xi)\sin\frac{n\pi}{l}\xi d\xi, \quad n=1,2,\cdots . \end{aligned}$$

Applications of the method on separating variables

We continuously introduce some examples for demonstrating the application on separation of variables.

Example (Mixed problem on heat conduction equation)

Consider the following problem:

$$u_t = ku_{xx}, \quad 0 < x < l, \quad t > 0, \tag{67}$$
$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant l, \tag{68}$$
$$u(0,t) = 0, \quad t \geqslant 0, \tag{69}$$
$$u(l,t) = 0, \quad t \geqslant 0. \tag{70}$$

Answer.

Let

$$u(x,t) = X(x)T(t). \tag{71}$$

Substituting (71) for (67), (69), (70), we have

$$X(x)T'(t) = kX''(x)T(t),$$
$$X(0)T(t) = 0, \quad X(l)T(t) = 0 \Rightarrow X(0) = X(l) = 0.$$

Applications of the method on separating variables

Then

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = -\lambda,$$

$$X''(x) + \lambda X(x) = 0, \tag{72}$$

$$T'(t) + \lambda k T(t) = 0. \tag{73}$$

We have the boundary value problem:

$$X''(x) + \lambda X(x) = 0,$$

$$X(0) = 0, \tag{74}$$

$$X(l) = 0. \tag{75}$$

Using the results of above section, we obtain $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ and $X_n(x) = \sin\frac{n\pi}{l}x$, $n = 1, 2, \cdots$. By (73), we have $T_n'(t) + \lambda_n T_n(t) = 0 \quad \Rightarrow \quad T_n(t) = a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt}, \quad n = 1, 2, \cdots$. Then

$$u_n(x,t) = a_n e^{-\left(\frac{n\pi}{l}\right)^2 kt} \sin\frac{n\pi}{l}x, \quad n = 1, 2, \cdots.$$

Assume

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(\frac{n\pi}{l})^2 kt} \sin \frac{n\pi}{l} x.$$

By (68), we get

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{l} x = f(x) \Rightarrow a_n = \frac{2}{l} \int_0^l f(\xi) \sin \frac{n\pi}{l} \xi d\xi, \quad n = 1, 2, \cdots$$

It follows that the solution of mixed problem (67)∼(70) can be expressed as

$$u(x,t) = \sum_{n=1}^{\infty} \left(\frac{2}{l} \int_0^l f(\xi) \sin \frac{n\pi}{l} \xi d\xi \right) e^{-(\frac{n\pi}{l})^2 kt} \sin \frac{n\pi}{l} x.$$

□

Applications of the method on separating variables

Example (The boundary value problem of Laplace equation)

Consider the boundary value problem of Laplace equation:

$$\Delta u = u_{xx} + u_{yy} = 0, \quad 0 < x < a, \quad 0 < y < b, \tag{76}$$
$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant a, \tag{77}$$
$$u(x,b) = 0, \quad 0 \leqslant x \leqslant a, \tag{78}$$
$$u_x(0,y) = 0, \quad 0 \leqslant y \leqslant b, \tag{79}$$
$$u_x(a,y) = 0, \quad 0 \leqslant y \leqslant b. \tag{80}$$

Answer.

Let

$$u(x,y) = X(x)Y(y). \tag{81}$$

Substituting (81) for (76), (79), (80), we have (I) $\begin{cases} X''Y + XY'' = 0, \\ X'(0)Y(y) = 0, \\ X'(a)Y(y) = 0. \end{cases}$

From (I), we get $\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda, \quad X'(0) = X'(a) = 0,$

$$X''(x) + \lambda X(x) = 0, \tag{82}$$
$$Y''(y) - \lambda Y(y) = 0, \tag{83}$$

and the boundary value problem:

$$
\begin{aligned}
&X''(x) + \lambda X(x) = 0, \\
&X'(0) = 0, && (84)\\
&X'(a) = 0. && (85)
\end{aligned}
$$

1. $\lambda < 0, X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$(by (84), (85)) $\Rightarrow \sqrt{-\lambda}\,(c_1 - c_2) = 0, c_1 = c_2$, $\sqrt{-\lambda}\left(c_1 e^{\sqrt{-\lambda}l} - c_2 e^{-\sqrt{-\lambda}l}\right) = 0 \Rightarrow c_1 = c_2 = 0 \Rightarrow X(x) \equiv 0$.
2. $\lambda = 0, X(x) = c_1 x + c_2$ (by (84), (85)) $\Rightarrow c_1 = 0, c_2 \neq 0$ (arbitrary), then $\lambda_0 = 0$, and $X_0(x) = 1$.
3. $\lambda > 0, X(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$ (by (84), (85)) $\Rightarrow \sqrt{\lambda}c_2 = 0, c_2 = 0$, $-\sqrt{\lambda}c_1 \sin\sqrt{\lambda}l = 0$, if $c_1 \neq 0 \Rightarrow \sin\sqrt{\lambda}l = 0$. Then $\lambda_n = \left(\frac{n\pi}{l}\right)^2$, $X_n(x) = \cos\frac{n\pi}{l}x, n = 1, 2, \cdots$. Thus, we obtain $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ $(n = 0, 1, 2, \cdots)$ and $X_n(x) = \cos\frac{n\pi}{l}x, n = 0, 1, 2, \cdots$. Substituting $\lambda_n = \left(\frac{n\pi}{l}\right)^2$ $(n = 0, 1, 2, \cdots)$ for (83), we have

$$
Y_n''(y) - \left(\frac{n\pi}{l}\right)^2 Y_n(y) = 0, \quad n = 0, 1, 2, \cdots.
$$

Applications of the method on separating variables

Then

$$
\begin{aligned}
Y_0(y) &= a_0 y + b_0, \\
Y_n(y) &= a_n e^{\left(\frac{n\pi}{l}\right)y} + b_n e^{-\left(\frac{n\pi}{l}\right)y} \\
&= \bar{a}_n \cosh\left(\frac{n\pi}{l}\right) y + \bar{b}_n \sinh\left(\frac{n\pi}{l}\right) y \\
&= A_n \sinh\left(\frac{n\pi}{l}\right)(y + B_n), \quad n = 1, 2, \cdots,
\end{aligned}
$$

where a_0, b_0, a_n, b_n, or $\bar{a}_n, \bar{b}_n, A_n$ and B_n are arbitrary constants.
By (78) $\Rightarrow u(x, b) = 0$, take $b_0 = -a_0 b, B_n = -b, n = 1, 2, \cdots$. Thus $Y_0(y) = a_0(y - b)$, $Y_n(y) = A_n \sinh\left(\frac{n\pi}{l}\right)(y - b), n = 1, 2, \cdots$. Set

$$
u(x, y) = a_0(y - b) + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{l}\right)(y - b) \cdot \cos\frac{n\pi}{l}x.
$$

By (77), we get
$u(x, 0) = f(x) = -a_0 b + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{-n\pi}{l}\right) b \cdot \cos\frac{n\pi}{l}x = \frac{f_0}{2} + \sum_{n=1}^{\infty} f_n \cos\frac{n\pi}{l}x$, where $f_n = \frac{2}{l}\int_0^l f(\xi)\cos\frac{n\pi}{l}\xi d\xi, n = 0, 1, 2, \cdots$. Then $a_0 = -\frac{f_0}{2b}$, $A_n = -\frac{f_n}{\sinh\left(\frac{n\pi}{l}\right)b}$, $n = 1, 2, \cdots$.

Applications of the method on separating variables

Therefore, we obtain the solution of (76)∼(80) as follows:

$$u(x,y)=\frac{f_0}{2b}(b-y)+\sum_{n=1}^{\infty}\frac{f_n}{\sinh\left(\frac{n\pi}{l}\right)b}\sinh\left(\frac{n\pi}{l}\right)(b-y)\cos\frac{n\pi}{l}x.$$

□

Theorem

There is at most one solution satisfying the following mixed problem:

$$u_{tt} = a^2 u_{xx}, \quad 0 < x < l, t > 0, \tag{86}$$
$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant l, \tag{87}$$
$$u_t(x,0) = g(x), \quad 0 \leqslant x \leqslant l, \tag{88}$$
$$u(0,t) = 0, \quad t \geqslant 0, \tag{89}$$
$$u(l,t) = 0, \quad t \geqslant 0. \tag{90}$$

Uniqueness of the solution for the mixed string oscillation problem

Theorem

There is at most one solution satisfying the following mixed problem:

$$u_{tt} = a^2 u_{xx}, \quad 0 < x < l, t > 0, \tag{86}$$
$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant l, \tag{87}$$
$$u_t(x,0) = g(x), \quad 0 \leqslant x \leqslant l, \tag{88}$$
$$u(0,t) = 0, \quad t \geqslant 0, \tag{89}$$
$$u(l,t) = 0, \quad t \geqslant 0. \tag{90}$$

Proof.

Suppose there are two solutions $u_1(x,t)$ and $u_2(x,t)$ of the problem (86) $\sim$ (90). Let $v(x,t) = u_1(x,t) - u_2(x,t)$, then $v(x,t)$ is the solution of the following problem:

$$v_{tt} = a^2 v_{xx}, \quad 0 < x < l, t > 0, \tag{91}$$
$$v(x,0) = 0, \quad 0 \leqslant x \leqslant l, \tag{92}$$
$$v_t(x,0) = 0, \quad 0 \leqslant x \leqslant l, \tag{93}$$
$$v(0,t) = 0, \quad t \geqslant 0, \tag{94}$$
$$v(l,t) = 0, \quad t \geqslant 0. \tag{95}$$

Now, we prove $v(x,t) \equiv 0$. Consider the function

$$I(t) = \frac{1}{2}\int_0^l \left(a^2 v_x^2 + v_t^2\right) dx, \tag{96}$$

which represents total energy of the string at t-time. Since $v(x,t)$ is twice continuously differentiable, by differentiating with respect to t, in (96), we have

$$\frac{dI}{dt} = \int_0^l \left(a^2 v_x v_{xt} + v_t v_{tt}\right) dx. \tag{97}$$

Integrating by parts, we get

$$\int_0^l a^2 v_x v_{xt} dx = \left[a^2 v_x v_t\right]\Big|_0^l - \int_0^l a^2 v_t v_{xx} dx. \tag{98}$$

By (94), (95), we have $v_t(0,t) = 0$ and $v_t(l,t) = 0$. Thus, from (97) and (98),

$$\frac{dI}{dt} = \int_0^l v_t \left(v_{tt} - a^2 v_{xx}\right) dx. \tag{99}$$

By (91), (99) becomes

$$\frac{dI}{dt} = 0, \quad \forall t \geqslant 0. \tag{100}$$

Then $I(t) = C$ (constant), $\forall t \geqslant 0$. By (92), $v_x(x, 0) = 0$. Noting (93), we get

$$C = I(0) = \frac{1}{2} \int_0^l \left[a^2 v_x^2 + v_t^2 \right] \Big|_{t=0} dx = 0. \tag{101}$$

This means that $I(t) \equiv 0$ and $v_x \equiv 0, v_t \equiv 0$. Therefore $v(x, t) \equiv$ constant, $\forall t \geqslant 0$. By (92), we have $v(x, t) \equiv 0, \forall t \geqslant 0$. Namely $u_1(x, t) \equiv u_2(x, t), \forall t \geqslant 0$, and the Theorem is proved. □

Uniqueness of the solution for the mixed heat conduction

Theorem

There is at most one solution satisfying the following mixed problem:

$$u_t = ku_{xx}, \quad 0 < x < l, t > 0, k > 0, \tag{102}$$
$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant l, \tag{103}$$
$$u(0,t) = 0, \quad t \geqslant 0, \tag{104}$$
$$u(l,t) = 0, \quad t \geqslant 0. \tag{105}$$

Uniqueness of the solution for the mixed heat conduction

Theorem

There is at most one solution satisfying the following mixed problem:

$$u_t = ku_{xx}, \quad 0 < x < l, t > 0, k > 0, \tag{102}$$
$$u(x,0) = f(x), \quad 0 \leqslant x \leqslant l, \tag{103}$$
$$u(0,t) = 0, \quad t \geqslant 0, \tag{104}$$
$$u(l,t) = 0, \quad t \geqslant 0. \tag{105}$$

Proof.

Suppose there are two functions $u_1(x,t)$ and $u_2(x,t)$, which satisfy (102) $\sim$ (105). Then $v(x,t) \equiv u_1(x,t) - u_2(x,t)$ satisfies the problem

$$\begin{aligned} v_t &= kv_{xx}, \quad 0 < x < l, \ t > 0, \\ v(x,0) &= 0, \quad 0 \leqslant x \leqslant l, \\ v(0,t) &= 0, \quad t \geqslant 0, \\ v(l,t) &= 0, \quad t \geqslant 0. \end{aligned} \tag{106}$$

Consider the function

$$J(t) = \frac{1}{2k}\int_0^l v^2 dx. \tag{107}$$

Differentiating $J(t)$ with respect to t, by (106), we have

$$\frac{dJ(t)}{dt} = \frac{1}{k}\int_0^l vv_t dx = \int_0^l vv_{xx} dx.$$

Integrating by part, we get

$$\int_0^l vv_{xx} dx = [vv_x]|_0^l - \int_0^l v_x^2 dx.$$

Since $v(0,t) = v(l,t) = 0$, we have

$$\frac{dJ(t)}{dt} = -\int_0^l v_x^2 dx \leqslant 0.$$

Then $J(t)$ is non-increasing for $t \geqslant 0$.
By $v(x,0) = 0$, we know $J(0) = 0$ and then $J(t) \leqslant 0, \forall t \geqslant 0$. But by (107), we see that $J(t) \geqslant 0, \forall t \geqslant 0$. Then $J(t) \equiv 0, \forall t \geqslant 0$. Since $v(x,t)$ is continuous, by $J(t) \equiv 0$, we obtain $v(x,t) \equiv 0, \forall 0 \leqslant x \leqslant l, t \geqslant 0$. It follows that $u_1(x,t) \equiv u_2(x,t)$ and the Theorem is true. □

Sturm-Liouville problem

Consider the general homogeneous ordinary differential equation with parameter λ :

$$c_1(x)\frac{d^2u}{dx^2} + c_2(x)\frac{du}{dx} + [c_3(x) + \lambda]\, u = 0, \quad c_1(x) \neq 0. \tag{108}$$

Set $p(x) = e^{\int_0^x \frac{c_2(\eta)}{c_1(\eta)} d\eta}$, $q(x) = \frac{c_3(x)}{c_1(x)} p(x)$, $s(x) = \frac{1}{c_1(x)} p(x)$. From (108), we have

$$\frac{d}{dx}\left(p\frac{du}{dx}\right) + [q + \lambda s]u = 0. \tag{109}$$

This equation (109) is called the Sturm-Liouville equation.
Applying the self-adjoint operator

$$L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q, \tag{110}$$

Thus (109) can be written as follows:

$$L(u) + \lambda s(x)u = 0 \tag{111}$$

In above equations, the parameter λ is independent of x, $p(x) \in C^{(1)}[a, b]$, $q(x), s(x) \in C[a, b]$.

Sturm-Liouville problem

If $p(x)$ and $s(x)$ are positive on $[a, b]$ and $-\infty < a < b < +\infty$, then the Sturm-Liouville equation is regular on $[a, b]$. Otherwise, namely $p(a)s(a) = 0$ or $p(b)s(b) = 0$ or $a = -\infty$ or $b = +\infty$, then the Sturm-Liouville equation is singular. The conditions at end points on the Sturm-Liouville equation are given as follows.

1. General boundary conditions

$$\alpha_1 u(a) + \beta_1 u'(a) = 0, \tag{112}$$
$$\alpha_2 u(b) + \beta_2 u'(b) = 0, \tag{113}$$

   where $|\alpha_i| + |\beta_i| > 0, i = 1, 2$.

2. Periodic boundary condition ($p(x) \neq 0$ and $p(a) = p(b)$)

$$u(a) = u(b), \tag{114}$$
$$u'(a) = u'(b). \tag{115}$$

   (109) (or (110)) and (112), (113) as well as (110) (or (109)) and (114),(115) are called the Sturm-Liouville problem.

The value of parameter λ, for which the corresponding Sturm-Liouvill problem has non-trivial solutions, is called eigenvalue and the corresponding non-trivial solutions are called eigenfunctions. Furthermore, the set of all eigenvalues is called spectrum of this problem.

Example (1)

Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$u'' + \lambda u = 0, \quad 0 < x < \pi, \tag{116}$$

$$u(0) = 0, \tag{117}$$

$$u'(\pi) = 0. \tag{118}$$

Example (1)

Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problem:

$$u'' + \lambda u = 0, \quad 0 < x < \pi, \tag{116}$$
$$u(0) = 0, \tag{117}$$
$$u'(\pi) = 0. \tag{118}$$

Answer.

$p(x) = 1, q(x) = 0, s(x) = 1.$

1. $\lambda = 0.$ $u(x) = c_1 x + c_2$. By (117), (118), we have $c_2 = c_1 = 0, u(x) \equiv 0.$
2. $\lambda < 0.$ $u(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$. By (117), (118) $\Rightarrow c_1 + c_2 = 0$ and $\sqrt{-\lambda}\left(c_1 e^{\sqrt{-\lambda}\pi} - c_2 e^{-\sqrt{\lambda}\pi}\right) = 0, \Rightarrow c_1 = c_2 = 0, u(x) =\equiv 0.$
3. $\lambda > 0. u(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. By (117), (118) $\Rightarrow c_1 = 0,\ \sqrt{\lambda}c_2 \cos\sqrt{\lambda}\pi = 0$ if $c_2 \neq 0 \Rightarrow \cos\sqrt{\lambda}\pi = 0 \Rightarrow \lambda_n = \left(n - \frac{1}{2}\right)^2 \quad (n = 1, 2, \cdots)$ are eigenvalues, and $u_n(x) = \sin\left(n - \frac{1}{2}\right)x = \sin\frac{2n-1}{2}x\ (n = 1, 2, \cdots)$ are eigenfunctions.

□

Example (2)

Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problem on Cauchy Euler equation

$$x^2\frac{d^2u}{dx^2}+x\frac{du}{dx}+\lambda u=0,\quad 1<x<e, \tag{119}$$

$$u(1)=0, \tag{120}$$

$$u(e)=0. \tag{121}$$

Example (2)

Find the eigenvalues and eigenfunctions of the following Sturm-Liouville problem on Cauchy Euler equation

$$x^2 \frac{d^2u}{dx^2} + x\frac{du}{dx} + \lambda u = 0, \quad 1 < x < e, \tag{119}$$

$$u(1) = 0, \tag{120}$$

$$u(e) = 0. \tag{121}$$

Answer.

the equation (119) can be written as follows:

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) + \frac{1}{x}\lambda u = 0, \tag{122}$$

where $p(x) = x, q(x) = 0, s(x) = \frac{1}{x}$.

1. $\lambda = 0$. $\frac{d}{dx}\left(x\frac{du}{dx}\right) = 0, x\frac{du}{dx} = c_1, \frac{du}{dx} = \frac{c_1}{x}, u(x) = c_1 \ln x + c_2$. By (120), (121) $\Rightarrow c_2 = c_1 = 0 \Rightarrow u(x) \equiv 0$.

2. $\lambda < 0$. Let $u(x) = x^{\beta}$, then by (119) we have $(\beta^2 + \lambda)x^{\beta} = 0$, $\beta = \pm\sqrt{-\lambda}$, $u(x) = c_1 x^{\sqrt{-\lambda}} + c_2 x^{-\sqrt{-\lambda}}$. By (120), (121) $\Rightarrow c_1 + c_2 = 0$ and $c_1 e^{\sqrt{-\lambda}} + c_2 e^{-\sqrt{-\lambda}} = 0 \Rightarrow c_1 = c_2 = 0, u(x) \equiv 0$.
3. $\lambda > 0$. By (item 2) $\Rightarrow \bar{u}(x) = c_1 x^{i\sqrt{\lambda}} + c_2 x^{-i\sqrt{\lambda}}$. Since $x^{\pm i\sqrt{\lambda}} = e^{\ln x^{\pm i\sqrt{\lambda}}} = e^{\pm i\sqrt{\lambda}\ln x} = \cos\sqrt{\lambda}\ln x \pm i\sin\sqrt{\lambda}\ln x$, we have

$$u(x) = c_1 \cos\sqrt{\lambda}\ln x + c_2 \sin\sqrt{\lambda}\ln x.$$

By (120), $u(1) = 0 \Rightarrow c_1 = 0$; by (121), $u(e) = 0 \Rightarrow c_2 \sin\sqrt{\lambda} = 0$ if $c_2 \neq 0 \Rightarrow \sin\sqrt{\lambda} = 0, \lambda_n = (n\pi)^2$ $(n = 1, 2, \cdots)$ are eigenvalues, and $u_n(x) = \sin(n\pi\ln x)$ $(n = 1, 2, \cdots)$ are eigenfunctions.

□

Eigenvalues and Eigenfunctions of Sturm-Liouville Problem

Example (3)

Periodic Sturm-Liouville problem

$$u'' + \lambda u = 0, \quad -\pi < x < \pi, \tag{123}$$
$$u(-\pi) = u(\pi), \tag{124}$$
$$u'(-\pi) = u'(\pi). \tag{125}$$

Answer.

1. $\lambda < 0$. $u(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$. By (124) $\Rightarrow c_1 e^{-\sqrt{-\lambda}\pi} + c_2 e^{\sqrt{-\lambda}\pi} = c_1 e^{\sqrt{-\lambda}\pi} + c_2 e^{-\sqrt{-\lambda}\pi} \Rightarrow c_1 = c_2 \Rightarrow$ $u(x) = c_1(e^{\sqrt{-\lambda}x} + e^{-\sqrt{\lambda}x})$, by (125) $\Rightarrow c_1 = 0, \Rightarrow u(x) \equiv 0$.
2. $\lambda = 0$. $u(x) = c_1 x + c_2$. By (124) $\Rightarrow c_1 = 0$; by (125), $c_2 \neq 0$ is arbitrary. $\lambda_0 = 0$ is an eigenvalue, $u_0(x) = 1$ is an eigenfunction.
3. $\lambda > 0$. $u(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$. By (124), (125) $\Rightarrow c_1 \cos\sqrt{\lambda}\pi - c_2 \sin\sqrt{\lambda}\pi = c_1 \cos\sqrt{\lambda}\pi + c_2 \sin\sqrt{\lambda}\pi$,

$$\sqrt{\lambda}\left(c_1 \sin\sqrt{\lambda}\pi + c_2 \cos\sqrt{\lambda}\pi\right) = \sqrt{\lambda}\left(-c_1 \sin\sqrt{\lambda}\pi + c_2 \cos\sqrt{\lambda}\pi\right)$$
$$\Rightarrow 2c_2 \sin\sqrt{\lambda}\pi = 0, \quad 2c_1 \sin\sqrt{\lambda}\pi = 0.$$

3. If $c_2 \neq 0, \sin\sqrt{\lambda}\pi = 0, \lambda_n = n^2, n = 1, 2, \cdots$.

   If $c_1 \neq 0, \sin\sqrt{\lambda}\pi = 0, \lambda_n = n^2, n = 1, 2, \cdots$.

   Namely, only if $\lambda_n = n^2, n = 1, 2, \cdots$, then $c_1 \neq 0$ and $c_2 \neq 0$ are arbitrary. Therefore, $\lambda_n = n^2$ $(n = 1, 2, \cdots)$ are eigenvalues, and $u_n^{(1)}(x) = \cos nx$, $u_n^{(2)}(x) = \sin nx$ $(n = 1, 2, \cdots)$ are eigenfunctions.

   In words, the eigenvalues are $\lambda_n = n^2, n = 0, 1, 2, \cdots$, the eigenfunctions are $\{\cos nx\}_{n=0}^{\infty} \cup \{\sin nx\}_{n=1}^{\infty}$ or $\{1\} \cup \{\cos nx\}_{n=1}^{\infty} \cup \{\sin nx\}_{n=1}^{\infty}$. For $\lambda_0 = 0$, there is only one corresponding eigenfunction $u_0(x) = 1$, then $\lambda_0 = 0$ is called simple eigenvalue; for $\lambda_n = n^2, n = 1, 2, \cdots$, there are two linear independent eigenfunctions $\cos nx$ and $\sin nx$, then $\lambda_n = n^2(n = 1, 2, \cdots)$ are called two-repeated eigenvalues.

□

Eigenfunctions

1. Simple eigenvalue: one eigenvalue λ only corresponds to one eigenfunction.
2. Multiple eigenvalue: one eigenvalue λ corresponds to linear independent multi-eigenfunctions.

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Consider the basic properties on the system of eigenfunctions for the Sturm-Liouville problem.

Theorem

Assume $p(x), q(x)$ and $s(x) \in C[a, b]$ and the eigenfunctions $u_j(x)$ and $u_k(x)$ which correspond to eigenvalues λ_j and $\lambda_k (\lambda_j \neq \lambda_k)$ respectively, then $u_j(x)$ and $u_k(x)$ are orthogonal with weight function $s(x)$ on $[a, b]$, namely,

$$\int_a^b s(x) u_j(x) u_k(x) dx = 0.$$

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$$\int_a^b s(x) u_j(x) u_k(x) dx = 0.$$

Proof.

Since $u_j(x), u_k(x)$ are the solutions of the Sturm-Liouville problem corresponding to λ_j and λ_k respectively, we have

$$\frac{d}{dx}\left(p u_j'\right) + \left(q + \lambda_j s\right) u_j = 0, \tag{126}$$

$$\frac{d}{dx}\left(p u_k'\right) + \left(q + \lambda_k s\right) u_k = 0. \tag{127}$$

Eigenfunctions

The result of multiplying (126) by u_k subtracts the result of multiplying (127) by u_j, namely, (126)$\times u_k$-(127)$\times u_j$. We get

$$\begin{aligned}(\lambda_k - \lambda_j)\, s u_j u_k &= u_k \frac{d}{dx}\left(p u_j'\right) - u_j \frac{d}{dx}\left(p u_k'\right) &(128)\\ &= \frac{d}{dx}\left[\left(p u_j'\right) u_k - \left(p u_k'\right) u_j\right]. &(129)\end{aligned}$$

Integrating two-ends of above equality on $[a, b]$, we get

$$\begin{aligned}(\lambda_k - \lambda_j)\int_a^b s u_j u_k dx = &\left[\left(p u_j'\right) u_k - \left(p u_k'\right) u_j\right]\Big|_a^b = \left[p(x)\left(u_k u_j' - u_j u_k'\right)\right]\Big|_a^b\\ =&p(b)\left[u_k(b)u_j'(b) - u_j(b)u_k'(b)\right] - p(a)\left[u_k(a)u_j'(a) - u_j(a)u_k'(a)\right]. \end{aligned} \quad (130)$$

Eigenfunctions

By the boundary conditions of the Sturm-Liouville problem, we have

$$\alpha_1 u_j(a) + \beta_1 u_j'(a) = 0, \tag{131}$$

$$\alpha_1 u_k(a) + \beta_1 u_k'(a) = 0, \tag{132}$$

$$\alpha_2 u_j(b) + \beta_2 u_j'(b) = 0, \tag{133}$$

$$\alpha_2 u_k(b) + \beta_2 u_k'(b) = 0. \tag{134}$$

1. If $\alpha_1 \neq 0$, then (131)$\times u_k'(a)$–(132)$\times u_j'(a)$. We get
   $\alpha_1 \left[u_j(a)u_k'(a) - u_k(a)u_j'(a)\right] = 0 \Rightarrow u_j(a)u_k'(a) - u_k(a)u_j'(a) = 0.$
2. If $\beta_1 \neq 0$, then (131) $\times u_k(a)$–(132) $\times u_j(a)$. We get
   $\beta_1 \left[u_j'(a)u_k(a) - u_k'(a)u_j(a)\right] = 0 \Rightarrow u_k(a)u_j'(a) - u_j(a)u_k'(a) = 0.$

   It follows that $p(a)\left[u_k(a)u_j'(a) - u_j(a)u_k'(a)\right] = 0.$

   In the same way, we can obtain $p(b)\left[u_k(b)u_j'(b) - u_j(b)u_k'(b)\right] = 0.$ Therefore, from (130) and the last two equalities, we have $(\lambda_k - \lambda_j)\int_a^b s(x)u_j(x)u_k(x)dx = 0.$ Since $\lambda_k - \lambda_j \neq 0$, we obtain

$$\int_a^b s(x)u_j(x)u_k(x)dx = 0.$$

Eigenfunctions

Corollary

The system of the eigenfunctions of periodic Sturm-Liouville problem is orthogonal with the weight function $s(x)$ *on* $[a, b]$*. Namely,*

$$\int_a^b s(x)u_k(x)u_j(x)dx = 0,$$

where $u_k(x)$ *and* $u_j(x)$ *are arbitrary eigenfunctions corresponding the eigenvalues* λ_k *and* λ_j $(\lambda_k \neq \lambda_j)$ *respectively.*

Eigenfunctions

Corollary

The system of the eigenfunctions of periodic Sturm-Liouville problem is orthogonal with the weight function $s(x)$ on $[a, b]$. Namely,

$$\int_a^b s(x)u_k(x)u_j(x)dx = 0,$$

where $u_k(x)$ and $u_j(x)$ are arbitrary eigenfunctions corresponding the eigenvalues λ_k and λ_j $(\lambda_k \neq \lambda_j)$ respectively.

Proof.

By the discussion on the previous Theorem, we know that for $\lambda_k \neq \lambda_j$, $(\lambda_k - \lambda_j)\int_a^b s(x)u_k(x)u_j(x)dx = [p(x)(u_k u_j' - u_j u_k')]|_a^b$. Applying the periodic boundary conditions:

$$p(a) = p(b), u_k(a) = u_k(b), u_k'(a) = u_k'(b), u_j(a) = u_j(b) \text{ and } u_j'(a) = u_j'(b),$$

we see that $p(b)[u_k(b)u_j'(b) - u_j(b)u_k'(b)] - p(a)[u_k(a)u_j'(a) - u_j(a)u_k'(a)] = 0$. Then by $\lambda_k - \lambda_j \neq 0$, we get

$$\int_a^b s(x)u_k(x)u_j(x)dx = 0.$$

Eigenfunctions

Theorem

The eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ *of the regular Sturm-Liouville problem are real.*

Eigenfunctions

Theorem

The eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ of the regular Sturm-Liouville problem are real.

Proof.

Suppose a complex eigenvalue $\lambda_j = \alpha + \mathrm{i}\beta$ corresponding eigenfunction is $u_j(x)$. Since the coefficients of the equation are neal, we can see that the conjugate complex number $\bar{\lambda}_j = \alpha - \mathrm{i}\beta$ also is an eigenvalue and the corresponding conjugate function $\bar{u}_j(x)$ is also an eigenfunction. If $\beta \neq 0$, then $\lambda_j \neq \bar{\lambda}_j$, and then by the previous Theorem, we know that u_j and $\bar{u}_j$ are orthogonal with the weight function $s(x)$ on $[a, b]$. Namely,

$$(\lambda_j - \bar{\lambda}_j) \int_a^b s u_j \bar{u}_j dx = 0,$$

then

$$2\beta \int_a^b s(x) \left| u_j(x) \right|^2 dx = 0.$$

Since $s(x) > 0, u_j(x) \not\equiv 0$, we have $\int_a^b s(x) \left| u_j(x) \right|^2 dx > 0$, and $\beta = 0$. This is in contradiction to the above assumption of $\beta \neq 0$. Therefore, λ_j must be real and the Theorem is valid. □

Eigenfunctions

Theorem

For the regular Sturm-Liouville problem, there is a real infinite sequence of eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$, *which satisfies that* $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, *and* $\lim_{n\to\infty} \lambda_n = +\infty$, *and corresponding eigenfunctions* $\{u_n\}_{n=1}^{\infty}$ *construct one completely orthogonal system with the weight* $s(x)$ *on* $[a, b]$. *Namely,*

$$\int_a^b s(x)u_k(x)u_j(x)dx = \begin{cases} 0, & \text{as} \quad k \neq j, \quad \lambda_k \neq \lambda_j \\ M_k, & \text{as} \quad k = j, \quad M_k > 0. \end{cases}$$

Further, any function $f(x)$ *is piecewise smooth on* $[a, b]$ *and satisfies the end point conditions of Sturm-Liouville problem, then* $f(x)$ *can be expanded in an absolutely and uniformly convergent series by the eigenfunction system* $\{u_n(x)\}_{n=1}^{\infty}$ *on* $[a, b]$. *Namely,*

$$f(x) = \sum_{n=1}^{\infty} f_n u_n(x),$$

where

$$f_n = \frac{\int_a^b s(x)f(x)u_n(x)dx}{\int_a^b s(x)u_n^2(x)dx}, \quad n = 1, 2, \cdots.$$

Consider the second order non-homogeneous ordinary differential equation

$$L[u] = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = -f(x), \quad x \in [a, b], \tag{135}$$

and homogeneous boundary conditions

$$\alpha_1 u(a) + \beta_1 u'(a) = 0, \tag{136}$$

$$\alpha_2 u(b) + \beta_2 u'(b) = 0, \tag{137}$$

where $\alpha_i^2 + \beta_i^2 > 0, i = 1, 2,$ and $f(x) \in C[a, b]$.

The boundary value problem of ODE and Green function

Consider the second order non-homogeneous ordinary differential equation

$$L[u] = \frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = -f(x), \quad x \in [a, b], \tag{135}$$

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where $\alpha_i^2 + \beta_i^2 > 0, i = 1, 2,$ and $f(x) \in C[a, b]$.

Definition (Green function)

Let $R = \{(x, \xi) \mid a < x < b, a < \xi < b\}$. A function $G(x, \xi)$ with two variables is called Green function of the problem (135) $\sim$ (137), if $G(x, \xi)$ satisfies that

1. $G(x, \xi) \in C(\bar{R}) \cap C^{(2)}\{\bar{R}\backslash\{(x, \xi) \mid x = \xi, (x, \xi) \in \bar{R}\}\}$;
2. The first derivative of $G(x, \xi)$ on x has jump discontinuity, and $\frac{dG(x,\xi)}{dx}\Big|_{x=\xi^-}^{x=\xi^+} = -\frac{1}{p(\xi)}$;
3. For fixed ξ, except points $x = \xi$, $G(x, \xi)$ for x satisfies corresponding homogeneous equation $LG = 0$ and corresponding homogeneous boundary conditions (136) and (137).

Theorem

If $f(x)$ is continuous in (a, b), then the function

$$u(x) = \int_a^b G(x, \xi) f(\xi) d\xi \tag{138}$$

is the solution of the following problem:

$$\begin{cases} L[u] = -f(x), \quad x \in (a, b), \\ \alpha_1 u(a) + \beta_1 u'(a) = 0, \\ \alpha_2 u(b) + \beta_2 u'(b) = 0. \end{cases}$$

The construction of Green function

In order to construct the Green function, we know that $G(x, \xi)$ satisfies

$$L[G] = 0 \quad (x \neq \xi), \tag{139}$$
$$\alpha_1 G(a, \xi) + \beta_1 G_x(a, \xi) = 0, \tag{140}$$
$$\alpha_2 G(b, \xi) + \beta_2 G_x(b, \xi) = 0. \tag{141}$$

Thus, we first find a non-zero solution $u_1(x)$ such that $L[u_1] = 0$ and $\alpha_1 u_1(a) + \beta_1 u_1'(a) = 0$. Obviously, $c_1 u_1(x)(c_1 \neq 0)$ also satisfies (139) and (140).

In the same way, choose other non-zero solution $u_2(x)$ which is linear independent of $u_1(x)$ such that $L[u_2] = 0$ and $\alpha_2 u_2(b) + \beta_2 u_2'(b) = 0$, and $c_2 u_2(x)(c_2 \neq 0)$ also satisfies (139), (141).

We must emphasize that the requirement that $u_1(x)$ and $u_2(x)$ are linear independent is necessary. In fact, under the condition that there exists only a zero-solution of the following homogeneous problem:

$$L[u] = 0, \tag{142}$$
$$\alpha_1 u(a) + \beta_1 u'(a) = 0, \tag{143}$$
$$\alpha_2 u(b) + \beta_2 u'(b) = 0, \tag{144}$$

if $u_1(x) = cu_2(x) \not\equiv 0(c \neq 0)$, then it is not difficult to see that $u_1(x)$ is a nonzero solution of (142) $\sim$ (144), which is in contradiction to above condition.

The construction of Green function

Chosen Green function $G(x, \xi)$ is as follows:

$$G(x,\xi) = \begin{cases} c_1(\xi)u_1(x), & a \leqslant x < \xi, \\ c_2(\xi)u_2(x), & \xi < x \leqslant b. \end{cases}$$

By the continuity of $G(x, \xi)$ at the point $x = \xi$, we have $c_1(\xi)u_1(\xi) - c_2(\xi)u_2(\xi) = 0$.
By the jump discontinuity of $G_x(x, \xi)$ at the points $x = \xi$, we get

$$\begin{aligned} \frac{dG(x,\xi)}{dx}\bigg|_{x=\xi^-}^{x=\xi^+} &= G_x\left(\xi^+,\xi\right) - G_x\left(\xi^-,\xi\right) \\ &= c_2(\xi)u_2'(\xi) - c_1(\xi)u_1'(\xi) = -\frac{1}{p(\xi)}. \end{aligned}$$

Then, we obtain the following non-homgeneous system of linear algebraic equations

$$\begin{cases} c_1(\xi)u_1(\xi) - c_2(\xi)u_2(\xi) = 0, \\ c_1(\xi)u_1'(\xi) - c_2(\xi)u_2'(\xi) = \frac{1}{p(\xi)}. \end{cases}$$

The determinant of the coefficients is

$$\begin{aligned} \begin{vmatrix} u_1(\xi) & -u_2(\xi) \\ u_1'(\xi) & -u_2'(\xi) \end{vmatrix} &= -\left[u_1(\xi)u_2'(\xi) - u_2(\xi)u_1'(\xi)\right] \\ &\equiv -w(u_1, u_2, \xi) \neq 0. \end{aligned}$$

where $w(u_1, u_2, \xi)$ is called the wornskian determinant.

The construction of Green function

Thus, from above system of algebraic equations, we have

$$c_1(\xi) = \frac{-u_2(\xi)}{p(\xi)w(u_1,u_2,\xi)}, \quad c_2(\xi) = \frac{-u_1(\xi)}{p(\xi)w(u_1,u_2,\xi)}.$$

Therefore, the Green function $G(x,\xi)$ is as follows:

$$G(x,\xi) = \begin{cases} \frac{-u_1(x)u_2(\xi)}{p(\xi)w(u_1,u_2,\xi)}, & a \leqslant x < \xi, \\ \frac{-u_2(x)u_1(\xi)}{p(\xi)w(u_1,u_2,\xi)}, & \xi < x \leqslant b. \end{cases} \tag{145}$$

Now, we show that $p(\xi)w(u_1,u_2,\xi) \equiv C \neq 0$. In fact, since $u_1(x)$ and $u_2(x)$ are the non-zero solutions of $L[u] = 0$, we have

$$\frac{d}{dx}(pu_1') + qu_1 = 0, \tag{146}$$

$$\frac{d}{dx}(pu_2') + qu_2 = 0. \tag{147}$$

(147) $\times u_1-$ (146) $\times u_2 \Rightarrow u_1 \frac{d}{dx}(pu_2') - u_2 \frac{d}{dx}(pu_1' = 0.$
Namely, $\frac{d}{dx}[p(u_1u_2' - u_2u_1')] = 0 \Rightarrow p(u_1u_2' - u_2u_1') = c(\neq 0) \Rightarrow p(\xi)w(u_1,u_2,\xi) = c(\neq 0)$.
Thus

$$G(x,\xi) = \begin{cases} \frac{-u_1(x)u_2(\xi)}{c}, & a \leqslant x < \xi, \\ \frac{-u_2(x)u_1(\xi)}{c}, & \xi < x \leqslant b. \end{cases}$$

The construction of Green function

Theorem

If the corresponding homogeneous boundary value problem (142) $\sim$ (144) only has zero solution, then there exists unique Green function $G(x, \xi)$ of the boundary value problem (135) $\sim$ (137) which is given by (145).

Theorem

Green function of boundary value problem (135 $\sim$ (137) is symmetric, namely $G(x, \xi) = G(\xi, x)$.

Eigenvalue problem and Green function

Consider the non-homogeneous equation with parameter λ:

$$L[u] + \lambda s(x)u = g(x). \tag{148}$$

Assume that $s(x) > 0$ and $g(x)$ is a piecewise continuous function, $G(x,\xi)$ is the Green function of the problem (135) $\sim$ (137), Let $f(x) \equiv \lambda s(x)u(x) - g(x)$. By Theorem (p. 87), we have

$$u(x) = \int_a^b G(x,\xi)f(\xi)d\xi = \int_a^b G(x,\xi)[\lambda s(\xi)u(\xi) - g(\xi)]d\xi.$$

Namely,

$$u(x) = \lambda \int_a^b G(x,\xi)s(\xi)u(\xi)d\xi - \int_a^b G(x,\xi)g(\xi)d\xi. \tag{149}$$

This is a non-homogeneous Fredholm integral equation, which is equivalent to the problem (148)+(136)+(137).

If $s(x) \neq c$, then $G(x,\xi)s(\xi) \neq G(\xi,x)s(x)$, and the kernel of integral equation (149) is not symmetric. We introduce new unknown function $v(x)$ for finding symmetric kernel as follows. Set

$$v(x) = \sqrt{s(x)}u(x). \tag{150}$$

Substituting (150) for (149), we get

$$v(x) = \lambda \int_a^b K(x,\xi)v(\xi)d\xi - \int_a^b K(x,\xi)\frac{g(\xi)}{\sqrt{s(\xi)}}d\xi, \tag{151}$$

where $K(x,\xi) = G(x,\xi)\sqrt{s(x)s(\xi)}$ is a symmetric kernel.

Eigenvalue problem and Green function

According to the theory on the integral equation with symmetric kernel, without proof, we introduce the following results.

Theorem

For the following boundary value problem on nonhomogeneous equation with parameter λ:

$$\frac{d}{dx}\left[p(x)u'(x)\right] + [q(x) + \lambda s(x)]u(x) = g(x), \tag{152}$$

$$\alpha_1 u(a) + \beta_1 u'(a) = 0, \tag{153}$$

$$\alpha_2 u(b) + \beta_2 u'(b) = 0, \tag{154}$$

assume $\{\lambda_n\}_{n=1}^{\infty}$ are eigenvalues and $\{u_n(x)\}_{n=1}^{\infty}$ are corresponding eigenfunctions of the corresponding homogeneous problem, then

1. *if λ is not eigenvalue $\left(\lambda \notin \{\lambda_n\}_{n=1}^{\infty}\right)$, then for any $g(x)$, there exists unique solution $u(x)$ of (152) $\sim$ (154);*
2. *if $\lambda = \lambda_j \in \{\lambda_n\}_{n=1}^{\infty}$ is an eigenvalue, then there exist some solutions of (152) $\sim$ (154), if and only if .*

$$\int_a^b s(x)u_j(x)g(x)dx = 0.$$

where $u_j(x)$ is the corresponding eigenfunction with λ_j.

Gamma function

The gamma function is probably the special function that occurs most frequently in the discussion of problems in physics. At least three different convenient definitions of the gamma function are in common use.

Definition (I: Infinite Limit (Euler))

$$\Gamma(z) \equiv \lim_{n\to\infty} \frac{1\cdot 2\cdot 3\cdots n}{z(z+1)(z+2)\cdots(z+n)} n^z, \quad z \neq 0, -1, -2, -3, \dots .$$

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Definition (II: Definite Integral (Euler))

$$\Gamma(z) \equiv \int_0^\infty e^{-t} t^{z-1} dt, \quad \Re e(z) > 0.$$

The restriction on z is necessary to avoid divergence of the integral. When the gamma function does appear in physical problems, it is often in this form or some variation, such as

$$\Gamma(z) = 2\int_0^\infty e^{-t^2} t^{2z-1} dt, \quad \Re e(z) > 0,$$

or

$$\Gamma(z) = \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{z-1} dt, \quad \Re e(z) > 0.$$

Gamma function

Definition (III: Infinite Product (Weierstrass))

$$\frac{1}{\Gamma(z)} \equiv z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n},$$

where γ is the Euler-Mascheroni constant $\gamma = 0.5772156619 \cdots \left(= \lim_{n\to\infty} \left(\sum_{m=1}^{n} m^{-1} - \ln n\right)\right)$.

Gamma function

- $\Gamma(z+1) = z\Gamma(z)$;
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}, \quad \Gamma(n+\frac{1}{2}) = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}$;
- $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin z\pi}$; (reflection formula)
- $\Gamma(z+1) = \sqrt{2\pi}z^{z+1/2}e^{-z}\left(1 + \frac{1}{12z} + \frac{1}{288z^2} - \frac{139}{51{,}840z^3} + \cdots\right)$. (Stirling's formula)

Introduction of Bessel function

Bessel function

Bessel function is a kind of special functions related to singular Sturm-Liouville problem, and corresponding Bessel equation is posed from some mathematical and physical problems with cylindrical symmetry. In general, Bessel function can not be represented by elementary functions, however, it can be represented in a series form. Standard Bessel equation can be written as follows:

$$x^2y'' + xy' + \left(x^2 - \nu^2\right) y = 0, \tag{155}$$

where ν is a nonnegative real number. Since $x = 0$ is a singular point, by general way, the solution can be represented as

$$y(x) = \sum_{k=0}^{\infty} a_k x^{s+k}, \tag{156}$$

where exponent s is undetermined. Substituting power series (156) for equation (155), we have

$$\begin{aligned}&\left[s(s-1) + s - \nu^2\right] a_0 x^s + \left[(s+1)s + (s+1) - \nu^2\right] a_1 x^{s+1}\\ &\qquad + \sum_{k=2}^{\infty} \left\{\left[(s+k)(s+k) - \nu^2\right] a_k + a_{k-2}\right\} x^{s+k} = 0.\end{aligned} \tag{157}$$

Let the coefficient of x^s be equal to zero, then one can obtain the exponential equation

$$\left(s^2 - \nu^2\right) a_0 = 0. \tag{158}$$

For any $a_0 \neq 0$, from (158) one can derive $s = \pm\nu$. Since the first term is $a_0 x^s$ in (156), obviously, if $\nu > 0$, the solution of the Bessel equation corresponding to $s = \nu$ equals to zero at $x = 0$ and the solution corresponding to $s = -\nu$ must become infinity at $x = 0$.

Bessel function

First, we consider the regular solutions of Bessel equation corresponding to $s = \nu$. Let the coefficient of x^{s+1} be equal to zero in (157), then

$$(2\nu + 1)a_1 = 0. \tag{159}$$

This means $a_1 = 0 (\nu \geqslant 0)$.
Let the coefficients of x^{s+k} be equal to zero in (157), one can obtain recurrence formula

$$a_k = -\frac{a_{k-2}}{k(2\nu + k)}, \quad k = 2, 3, \cdots . \tag{160}$$

By $a_1 = 0$ and (160), we can get $a_{2k-1} = 0 (k = 1, 2, \cdots)$, and

$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k! (\nu + k)(\nu + k - 1) \cdots (\nu + 1)} \quad (k = 1, 2, \cdots). \tag{161}$$

This relation can be written as follows:

$$a_{2k} = \frac{(-1)^k 2^\nu \Gamma(\nu + 1) a_0}{2^{2k+\nu} k! \Gamma(\nu + k + 1)} \quad (k = 1, 2, \cdots), \tag{162}$$

where $\Gamma(z)$ is Gamma function.
Therefore, the regular solution of Bessel equation can be written in the form

$$y(x) = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k 2^\nu \Gamma(\nu + 1)}{2^{2k+\nu} k! \Gamma(\nu + k + 1)} x^{2k+\nu}. \tag{163}$$

Generally, set

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}. \tag{164}$$

Let $J_\nu(x)$ denote the solution defined by the above series which is called the first kind of ν order Bessel function. Thus

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\nu}}{2^{2k+\nu} k! \Gamma(\nu+k+1)}. \tag{165}$$

For determining non-regular solution of Bessel equation, we choose the exponent $s = -\nu$, and by the above same way, one can obtain the similar relation with

$$(-2\nu+1)a_1 = 0.$$

From this, one can get $a_1 = 0$ (first assume $\nu \neq \frac{1}{2}$, for $\nu = \frac{1}{2}$ later specify). By recurrence formula

$$a_k = -\frac{a_{k-2}}{k(k-2\nu)}, \quad k \geqslant 2, \tag{166}$$

we can obtain the non-regular solution

$$J_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k-\nu}}{2^{2k-\nu} k! \Gamma(-\nu+k+1)}. \tag{167}$$

This solution is called first kind of $-\nu$ order Bessel functions.

Bessel function

It is able to prove that J_ν and $J_{-\nu}$ are convergent for all $x \in (-\infty, +\infty)$, and that if ν is not integer, J_ν and $J_{-\nu}$ are linear independent. Therefore, if ν is a non-negative number and non-integer as well, the general solution of the Bessel equation can be written as

$$y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x). \tag{168}$$

If ν is an integer such as $\nu = n$, then from (167), we have

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k-n}}{2^{2k-n} k! \Gamma(-n+k+1)} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n} k! \Gamma(n+k+1)}.$$

This is the result substituting for $k-n$ by k. Then

$$J_{-n}(x) = (-1)^n J_n(x). \tag{169}$$

This shows that J_{-n} and J_n are not linear independent. Thus it is necessary to find the second solution which is independent of J_n. Generally, take a special non-regular solution as

$$Y_\nu(x) = \frac{(\cos \nu\pi) J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}. \tag{170}$$

It is called second kind of ν order Bessel functions. For the case that ν is integer or non-integer, $Y_\nu(x)$ is linear independent of J_n. Then, general solution of Bessel equation can be written as follows:

$$\begin{cases} y(x) = C_1 J_\nu(x) + C_2 J_{-\nu}(x) & \text{as } \nu \text{ is not integer}, \\ y(x) = C_1 J_n(x) + C_2 Y_n(x) & \text{as } \nu = n \text{ is integer}, \\ y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x) & \text{as } \nu \text{ is any real number}. \end{cases} \tag{171}$$

Bessel function

It is necessary to pay attention to that if $\nu \geqslant 0$ or ν is a negative integer, $J_\nu(0)$ is finite, but, if ν is a negative non-integer, $J_\nu(0)$ becomes infinity. In addition, the function $Y_\nu(x)$ becomes infinity at origin and its amplitudes are decreasing as x is increasing. The values of x on which $J_\nu(x) = 0$ or $Y_\nu(x) = 0$ are called zero-points of Bessel functions.

We introduce some useful recurrence formulae:

$$\begin{aligned}
J_{\nu+1}(x) + J_{\nu-1}(x) &= \frac{2\nu}{x} J_\nu(x), \\
\nu J_\nu(x) + x J_\nu'(x) &= x J_{\nu-1}(x), \\
J_{\nu-1}(x) - J_{\nu+1}(x) &= 2J_\nu'(x), \\
\nu J_\nu(x) - x J_\nu'(x) &= x J_{\nu+1}(x), \\
\frac{d}{dx}\left[x^\nu J_\nu(x)\right] &= x^\nu J_{\nu-1}(x), \\
\frac{d}{dx}\left[x^{-\nu} J_\nu(x)\right] &= -x^{-\nu} J_{\nu+1}(x).
\end{aligned}$$

All these formulas are yet valid for $Y_\nu(x)$.

Bessel function

For Bessel functions, there are useful asymptotic expansions as follows.

1. For $|x| > 10$,

$$
\begin{aligned}
J_n(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \Bigg\{ \Bigg[1 &- \frac{\left(4n^2-1\right)\left(4n^2-3^2\right)}{2!(8x)^2} \\
&+ \frac{\left(4n^2-1\right)\left(4n^2-3^2\right)\left(4n^2-5^2\right)\left(4n^2-7^2\right)}{4!(8x)^4} - \cdots \Bigg] \cos\varphi \\
&- \left[\frac{\left(4n^2-1\right)}{8x} - \frac{\left(4n^2-1\right)\left(4n^2-3^2\right)\left(4n^2-5^2\right)}{3!(8x)^3} + \cdots \right] \sin\varphi \Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
Y_n(x) \sim \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \Bigg\{ \Bigg[1 &- \frac{\left(4n^2-1\right)\left(4n^2-3\right)}{2!(8x)^2} - \cdots \Bigg] \sin\varphi \\
&+ \frac{\left(4n^2-1\right)\left(4n^2-3^2\right)\left(4n^2-5^2\right)\left(4n^2-7^2\right)}{4!(8x)^4} - 6.24 \Bigg) \\
&+ \left[\frac{\left(4n^2-1\right)}{8x} - \frac{\left(4n^2-1\right)\left(4n^2-3^2\right)\left(4n^2-5^2\right)}{3!(8x)^3} + \cdots \right] \cos\varphi \Bigg\},
\end{aligned}
$$

where $\varphi = x - \left(n + \frac{1}{2}\right)\frac{\pi}{2}$.

Bessel function

2. For $|x| < 0.1$,

$$J_0(x) \approx 1 - \frac{1}{2}\left(\frac{x}{2}\right)^2,$$

$$J_n(x) \approx \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n, \quad n \geqslant 1.$$

$$Y_0(x) \approx \frac{2}{\pi}\ln\frac{x}{2},$$

$$Y_n(x) \approx -\frac{(n-1)!}{\pi}\left(\frac{2}{x}\right)^n, \quad n = 1, 2, \cdots.$$

3. For $\nu = \pm\frac{1}{2}$,

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \sin x,$$

$$J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi x}\right)^{\frac{1}{2}} \cos x.$$

Bessel function

Bessel functions are orthogonal with the weight function x on $[0, a]$. Namely, if $i \neq j$, then

$$\int_0^a J_n\left(\lambda_{n_i} x\right) J_n\left(\lambda_{n_j} x\right) x dx = 0$$

If $i = j$, then the module N_{n_j} is given by the formula

$$N_{n_j}^2 = \frac{1}{2\lambda_{n_j}^2} \left\{ x^2 \left[\frac{dJ_n\left(\lambda_{n_j} x\right)}{dx} \right]^2 + \left(\lambda_{n_j}^2 x^2 - n^2\right) \left[J_n\left(\lambda_{n_j} x\right)\right]^2 \right\} \Bigg|_0^a$$

where λ_{n_j} is the root of the equation

$$J_n\left(\lambda_{n_j} a\right) + h \frac{dJ_n\left(\lambda_{n_j} a\right)}{dx} = 0.$$

Bessel function

In addition, there are modified Bessel functions concerning closed relation to Bessel functions. Corresponding Bessel equation with parameter λ is

$$x^2y'' + xy' + \left(\lambda^2 x^2 - \nu^2\right) y = 0. \tag{172}$$

Its general solution is

$$y(x) = C_1 J_\nu(\lambda x) + C_2 Y_\nu(\lambda x). \tag{173}$$

If $\lambda = i$, then

$$y(x) = C_1 J_\nu(ix) + C_2 Y_\nu(ix). \tag{174}$$

where $J_\nu(ix)$ can be written as

$$J_\nu(ix) = \sum_{k=0}^{\infty} \frac{(-1)^k (ix)^{2k+\nu}}{2^{2k+\nu} k! \Gamma(\nu + k + 1)} = (i)^\nu I_\nu(x), \tag{175}$$

where

$$I_\nu(x) = \sum_{k=0}^{\infty} \frac{x^{2k+\nu}}{2^{2k+\nu} k! \Gamma(\nu + k + 1)}, \tag{176}$$

which is called the first kind of ν order modified Bessel function. Similarly, we can define $I_{-\nu}(x)$. It is the same as J_ν and $J_{-\nu}$ except integer ν that I_ν and $I_{-\nu}$ are linear independent solutions.

Bessel function

The second kind of ν order modified Bessel function is

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi}, \tag{177}$$

and the corresponding modified Bessel equation is

$$x^2y'' + xy' - \left(x^2 + \nu^2\right)y = 0. \tag{178}$$

Its general solution is

$$y(x) = C_1 I_\nu(x) + C_2 K_\nu(x). \tag{179}$$

Specially, it must be shown that

$$I_\nu(0) = \begin{cases} 1 & \text{if} \quad \nu = 0, \\ 0 & \text{if} \quad \nu > 0, \end{cases} \tag{180}$$

and $\lim\limits_{x\to 0^+} K_\nu(x) = +\infty$.

Singular Sturm-Liouville problem

We consider the singular Sturm-Liouville problem which is related with the Bessel function. Consider transverse virbration equation on circular elastic membrane

$$u_{tt} = c^2(u_{xx} + u_{yy}).$$

Assume u is symmetric on θ, then using the polar coordinates, the above equation can be written as

$$u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r).$$

We discuss the following definite problem

$$\begin{cases} u_{tt} = c^2(u_{rr} + \frac{1}{r}u_r), & 0 < r < 1,\ t > 0, \\ u(r,0) = f(r), & 0 \le r \le 1, \\ u_t(r,0) = 0, & 0 \le r \le 1, \\ u(1,t) = 0, & t \ge 0, \\ \lim\limits_{r\to 0^+} u(r,t) < \infty, & t \ge 0. \end{cases} \tag{181}$$

By separation of variables, let $u(r,t) = R(r)T(t)$. Substituting it for the above wave equation, we have

$$\frac{R'' + R'/r}{R} = \frac{T''}{c^2 T} = -\alpha^2, \tag{182}$$

where α is a positive constant.

Singular Sturm-Liouville problem

Choose the negative sign before α^2 for finding a periodic solution on time. Then, we get two ordinary differential equations

$$rR'' + R' + \alpha^2 rR = 0, \tag{183}$$

$$T'' + \alpha^2 c^2 T = 0. \tag{184}$$

For (183), noting $p(r) = r, p(0) = 0$, and for any $T(t)$, $\lim_{r\to 0+} u(r,t) = \lim_{r\to 0+} R(r)T(t) < \infty$, one can obtain a bounded condition for $R(r)$: $\lim_{r\to 0+} R(r) < \infty$. By the boundary condition $u(1,t) = R(1)T(t) = 0$ $(T(t) \not\equiv 0)$, we get $R(1) = 0$.

Now, solve the following singular Sturm-Liouville problem on $R(r)$:

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \alpha^2 rR = 0, \tag{185}$$

$$R(1) = 0, \tag{186}$$

$$\lim_{r\to 0+} R(r) < \infty. \tag{187}$$

The equation (185) is a zero-order Bessel equation, and its solution is

$$R(r) = CJ_0(\alpha r) + DY_0(\alpha r), \tag{188}$$

where J_0 and Y_0 are first kind and second kind of zero-order Bessel functions respectively.

Singular Sturm-Liouville problem

Since as $r \to 0^+$, $Y_0(ar) \to -\infty$, by (187), we have $D = 0$, and $R(r) = CJ_0(\alpha r)$. By $R(1) = 0$, we get $J_0(\alpha) = 0$. There are infinite positive zero points of this transcendental equation $J_0(\alpha) = 0$:

$$0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots.$$

Substituting $\alpha_n (n = 1, 2, \cdots)$ for (184), we have

$$T_n(t) = A_n \cos \alpha_n ct + B_n \sin \alpha_n ct, \quad n = 1, 2, \cdots.$$

Thus, we obtain special solution of (181):

$$u_n(r, t) = J_0(\alpha_n r)(A_n \cos \alpha_n ct + B_n \sin \alpha_n ct), \quad n = 1, 2, \cdots.$$

Since the equation and boundary conditions of (181) are linear and homogeneous, one can find that the series

$$u(r, t) = \sum_{n=1}^{\infty} J_0(\alpha_n r)(A_n \cos \alpha_n ct + B_n \sin \alpha_n ct), \tag{189}$$

is uniformly convergent and sufficiently differentiable on r and t. Then, this $u(r, t)$ is one solution of (181).

Singular Sturm-Liouville problem

Differentiating (189) on t, we have

$$u_t(r,t) = \sum_{n=1}^{\infty} J_0(\alpha_n r)\left(-A_n\alpha_n c\sin\alpha_n ct + B_n\alpha_n c\cos\alpha_n ct\right)$$

By $u_t(r,0)=0$, then $B_n=0, n=1,2,\cdots$. Thus, we have

$$u(r,t) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r)\cos\alpha_n ct. \tag{190}$$

Furthermore, by $u(r,0)=f(r)$, we get

$$u(r,0) = f(r) \sim \sum_{n=1}^{\infty} A_n J_0(\alpha_n r). \tag{191}$$

Since the eigenfunction system $\{J_0(\alpha_n r)\}_{n=1}^{\infty}$ is completely orthogonal system with the weight function r on $[0,1]$, then if $f(r)$ is piecewise smooth, we can obtain

$$f(r) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r),$$

where

$$A_n = \int_0^1 rf(r)J_0(\alpha_n r)\,dr \Big/ \int_0^1 r\left[J_0(\alpha_n r)\right]^2 dr, \tag{192}$$

and then the $u(r,t)$ given by (190), (192) is the solution of (181).

Legendre function

Legendre function is special case of the hypergeometric function concerning singular Sturm-Liouville problem, which corresponds with the following Legendre equation

$$\left(1-x^2\right) y'' - 2xy' + \nu(\nu+1)y = 0, \tag{193}$$

where ν is a real number.
Since $x = \pm 1$ are singular points, we can assume that the solution of (193) is the power series

$$y(x) = \sum_{k=0}^{\infty} a_k x^k. \tag{194}$$

Differentiating the power series (194) term by term, and substituting the results for (193), we obtain

$$\sum_{k=2}^{\infty} \left\{k(k-1)a_k + [\nu(\nu+1) - (k-1)(k-2)]a_{k-2}\right\} x^{k-2} = 0.$$

Then the coefficients of the power series must satisfy the recurrence formula

$$a_k = \frac{(k-1)(k-2) - \nu(\nu+1)}{k(k-1)} a_{k-2}, \quad k \geqslant 2. \tag{195}$$

According to (195), by a_0 and a_1, one can determine a_{2k} and a_{2k-1} (for $k = 1, 2, \cdots$) respectively. Here a_0 and a_1 are arbitrary constants. Recurrence formula (195) can be written as

$$a_{k+2} = -\frac{(\nu-k)(\nu+k+1)}{(k+1)(k+2)} a_k, \quad k \geqslant 0. \tag{196}$$

Using (196), the solution of Legendre equation (193) is as follows

$$
\begin{aligned}
y(x) &= a_0 p_\nu(x) + a_1 q_\nu(x) \\
=& a_0 \left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k \nu(\nu-2)\cdots(\nu-2k+2)\cdot(\nu+1)(\nu+3)\cdots(\nu+2k-1)}{(2k)!} \cdot x^{2k} \right] \\
&+ a_1 \left[x + \sum_{k=1}^{\infty} \frac{(-1)^k (\nu-1)(\nu-3)\cdots(\nu-2k+1)\cdot(\nu+2)(\nu+4)\cdots(\nu+2k)}{(2k+1)!} \cdot x^{2k+1} \right].
\end{aligned}
$$

It is able to prove that the series $p_\nu(x)$ and $q_\nu(x)$ are convergent for $|x| < 1$ and linear independent each other.

We consider the case for $\nu = n$, where n is a non-negative integer. Then by (196) and $k = n$, we get

$$a_{n+2} = a_{n+4} = \cdots = 0.$$

Then as n is even, series $p_n(x)$ only includes finite terms with last term x^n, namely $p_n(x)$ is a n-th polynomial, but $q_n(x)$ is a infinite series. As n is odd, series $q_n(x)$ only includes finite terms with last term x^n, namely $q_n(x)$ is a n-th polynomial, but $p_n(x)$ is a infinite series. Therefore, for any nonnegative integer, there is only one n-th polynomial between $p_n(x)$ and $q_n(x)$. By suitable normalization, such polynomial can denote $P_n(x)$ which is called the first kind of n-th Legendre polynomial or function and generally it is defined as follows:

$$
P_n(x) = \begin{cases} p_n(x)/p_n(1) & \text{if } n \text{ is even,} \\ q_n(x)/q_n(1) & \text{if } n \text{ is odd.} \end{cases}
$$

Legendre function

The explicit formula of $P_n(x)$ is

$$P_n(x) = \sum_{k=0}^{N} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} \cdot x^{n-2k},$$

where

$$N = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{(n-1)}{2} & \text{if } n \text{ is odd.} \end{cases}$$

$P_l(x)$ $(l = 1, 2, 3, 4)$ is as follows:

$$\begin{aligned} P_0(x) &= 1, \\ P_1(x) &= x, \\ P_2(x) &= \frac{1}{2}\left(3x^2 - 1\right), \\ P_3(x) &= \frac{1}{2}\left(5x^2 - 3x\right), \\ P_4(x) &= \frac{1}{8}\left(35x^4 - 30x^2 + 3\right). \end{aligned}$$

Legendre function

For another infinite series of $p_n(x)$ or $q_n(x)$, by suitable normalization, it can be defined as follows:

$$Q_n(x) = \begin{cases} q_n(x)/q_n(1) & \text{if } n \text{ is even}, \\ p_n(x)/p_n(1) & \text{if } n \text{ is odd}. \end{cases}$$

which is called the second kind of Legendre function.
Initial terms on $Q_n(x)$ can be given by

$$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), \quad Q_1(x) = \frac{x}{2}\ln\left(\frac{1+x}{1-x}\right) - 1.$$

According to the formula of $P_n(x)$ and $Q_n(x)$, we can see that $P_n(x)$ is finite on $[-1,1]$, but $Q_n(x)$ is infinite at $x = \pm 1$. The general solution of the Legendre equation for $\nu = n$ is

$$y(x) = C_1 P_n(x) + C_2 Q_n(x).$$

Generally, the Legendre polynomial can be written as

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n,$$

which is called the Rodrigues formula.

Legendre function

Legendre polynomials satisfy some important recurrence formula:

$$
\begin{aligned}
&(n+1)P_{n+1}(x)-(2n+1)xP_n(x)+nP_{n-1}(x)=0, \quad n\geqslant 1,\\
&\left(x^2-1\right)P_n'(x)=nxP_n(x)-nP_{n-1}(x), \quad n\geqslant 1,\\
&nP_n(x)+P_{n-1}'(x)-xP_n'(x)=0, \quad n\geqslant 1,\\
&P_{n+1}'(x)=xP_n'(x)+(n+1)P_n(x), \quad n\geqslant 0.
\end{aligned}
$$

Another two relations

$$
\begin{aligned}
P_{2n}(-x)&=P_{2n}(x),\\
P_{2n+1}(-x)&=-P_{2n+1}(x).
\end{aligned}
$$

are worth to pay attention, which show that $P_n(x)$ is an even function as n is even, and $P_n(x)$ is an odd function as n is odd.
It is able to prove that Legendre polynomial constructs an orthogonal function system on $[-1,1]$. We have

$$\int_{-1}^{1} P_n(x)P_m(x)dx=0, \quad n\neq m. \tag{197}$$

The module $\|P_n\|$ of $P_n(x)$ is given by

$$\|P_n\|^2=\int_{-1}^{1} P_n^2(x)dx=\frac{2}{2n+1}. \tag{198}$$

Legendre function

Another important equation concerning close relation with Legendre equation (193) is associated Legendre equation

$$\left(1-x^2\right) y'' - 2xy' + \left[n(n+1) - \frac{m^2}{1-x^2}\right] y = 0, \tag{199}$$

where m is an integer number. Although this equation is independent of algebraic sign of integer m, for convenience, the solution corresponding positive value of m taken by us is different from negative value of m. First, consider the case for $m \geqslant 0$. Introduce transformation on the variable

$$y = \left(1-x^2\right)^{\frac{m}{2}} \cdot u, \quad |x| < 1,$$

then associated Legendre equation becomes

$$\left(1-x^2\right) u'' - 2(m+1)xu' + (n-m)(n+m+1)u = 0. \tag{200}$$

Since the equation obtained after m-th differentiating on (193) is the same with (200), we have the general solution of (199):

$$y(x) = \left(1-x^2\right)^{\frac{m}{2}} \frac{d^m Y(x)}{dx^m},$$

where

$$Y(x) = C_1 P_n(x) + C_2 Q_n(x)$$

is the general solution of (193).

Legendre function

A pair of linear independent solutions of (199) is given by associated first kind of Legendre function $P_n^m(x)$ and associated second kind of Legendre function $Q_n^m(x)$, which are defined as follows:

$$P_n^m(x) = \left(1 - x^2\right)^{\frac{m}{2}} \frac{d^m P_n(x)}{dx^m},$$

$$Q_n^m(x) = \left(1 - x^2\right)^{\frac{m}{2}} \frac{d^m Q_n(x)}{dx^m}.$$

It is clear that $P_n^0(x) = P_n(x)$, $Q_n^0(x) = Q_n(x)$ and $P_n^m(x) = 0$ for $m > n$. Now, we can define $P_n^{-m}(x)$ and $Q_n^{-m}(x)$ as follows:

$$P_n^{-m}(x) = (-1)^m \tfrac{(n-m)!}{(n+m)!} P_n^m(x), \qquad m \geqslant 0,$$
$$Q_n^{-m}(x) = (-1)^m \tfrac{(n-m)!}{(n+m)!} Q_n^m(x), \qquad m \geqslant 0.$$

Some initial first kind of associated Legendre functions can be written as

$$P_1^1(x) = \left(1 - x^2\right)^{\frac{1}{2}}, \quad P_2^1(x) = 3x\left(1 - x^2\right)^{\frac{1}{2}}, \quad P_2^2(x) = 3\left(1 - x^2\right).$$

Legendre function

First kind of associated Legendre functions constructs an orthogonal function system on $[-1,1]$, moreover, its orthogonality and modulus can be expressed by the equalities

$$\int_{-1}^{1} P_n^m(x) P_k^m(x) dx = \frac{2(n+m)!}{(2n+1)(n-m)!} \cdot \delta_{nk}, \tag{201}$$

where

$$\delta_{nk} = \begin{cases} 1 & \text{if} \quad n = k, \\ 0 & \text{if } n \neq k. \end{cases}$$

It is easy to see that (201) is similar with (197) and (198). Furthermore, (197) and (198) are special case of (201) for $m = 0$.

It is clear that $P_n^m(x)$ is finite everywhere on $[-1,1]$, but $Q_n^m(x)$ is infinite at end points $x = \pm 1$.

Legendre function

Example (Potential distribution in a sphere)

Find potential distribution $u(r,\theta)$ in the sphere with the centre 0 and the radius a, which is determined by the following problem:

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)=0, \quad 0<r<a, \quad 0<\theta<\pi, \\ u(a,\theta)=f(\theta), \quad 0\leqslant\theta<\pi. \end{cases}$$

Answer.

Let $u(r,\theta)=R(r)\cdot\Phi(\theta)$, then the partial differential equation can be separated as

$$\frac{r^2R''+2rR'}{R}=-\frac{\Phi''+\cot\theta\Phi'}{\Phi}=\lambda,$$

and then

$$r^2R''+2rR'-\lambda R=0, \tag{202}$$

$$\Phi''+\cot\theta\Phi'+\lambda\Phi=0. \tag{203}$$

Legendre function

Let $x = \cos\theta, 0 < \theta \leqslant 2\pi$. (203) becomes

$$\left(1 - x^2\right) \frac{d^2\Phi}{dx^2} - 2x\frac{d\Phi}{dx} + \lambda\Phi = 0, \quad -1 \leqslant x \leqslant 1. \tag{204}$$

This is a Legendre equation. Since the solutions of (204) that are continuous and have continuous derivatives on the closed interval $[-1, 1]$, are the Legendre polynomials $P_n(x)$ corresponding to $\lambda = n(n+1), n = 0, 1, 2, \cdots$, and by the physical meaning of this problem, we take the solution of (203) to be $\Phi_n(\theta) = P_n(\cos\theta)$.

Furthermore, when $\lambda = n(n+1)$, the general solution of the Cauchy-Euler equation (202) is $R_n(r) = C_1 r^n + C_2 r^{-(n+1)}$. By $|u(0, \theta)| < +\infty$, we have $C_2 = 0$. Hence,

$u_n(r, \theta) = A_n r^n P_n(\cos\theta)$, and $u(r, \theta) = \sum\limits_{n=0}^{\infty} A_n r^n P_n(\cos\theta)$.

For $r = a$, $f(\theta) = \sum\limits_{n=0}^{\infty} A_n a^n P_n(\cos\theta)$. By (197) $\sim$ (198), we have

$$A_n = \frac{2n+1}{2a^n} \int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta d\theta.$$

It follows that the solution is $u(r, \theta) = \sum\limits_{n=0}^{\infty} \left(\frac{2n+1}{2} \int_0^{\pi} f(\theta) P_n(\cos\theta) \sin\theta d\theta\right) \left(\frac{r}{a}\right)^n \cdot P_n(\cos\theta)$. □