Methods of Mathematical Physics

— Lecture 5 Singularities, Residue Theory, Argument Principle and Conformal Mappings —

Lei Du
dulei@dlut.edu.cn
http://faculty.dlut.edu.cn/dulei

School of Mathematical Sciences
Dalian University of Technology

April, 2023

Contents

Zeros

In this lecture, we introduce certain basic results on the "zeros" of an analytic function. But, first we have the following definition:

Definition

A zero of an analytic function $f(z)$ is any value of z for which $f(z)$ vanishes.

Zeros

In this lecture, we introduce certain basic results on the "zeros" of an analytic function. But, first we have the following definition:

Definition

A zero of an analytic function $f(z)$ is any value of z for which $f(z)$ vanishes.

Let $f(z)$ be analytic in a domain D and let a be a point of D. Then $f(z)$ can be expanded as a Taylor series about $z = a$ in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \text{ where } a_n = \frac{f^{(n)}(a)}{n!}.$$

If $f(a) = 0$, i.e., a is a zero of $f(z)$, we have $a_0 = 0$.

Zeros

In this lecture, we introduce certain basic results on the "zeros" of an analytic function. But, first we have the following definition:

Definition

A zero of an analytic function $f(z)$ is any value of z for which $f(z)$ vanishes.

Let $f(z)$ be analytic in a domain D and let a be a point of D. Then $f(z)$ can be expanded as a Taylor series about $z = a$ in the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n, \text{ where } a_n = \frac{f^{(n)}(a)}{n!}.$$

If $f(a) = 0$, i.e., a is a zero of $f(z)$, we have $a_0 = 0$.

It may also happen that more of the coefficients a_n vanish. If $a_n = 0$ for $n < m$, but $a_m \neq 0$, then we have

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} a_{n+m} (z-a)^{m+n} = (z-a)^m \sum_{n=0}^{\infty} a_{n+m} (z-a)^n \\ &= (z-a)^m \phi(z), \end{aligned}$$

where $\phi(z) = \sum\limits_{n=0}^{\infty} a_{n+m}(z-a)^n$ is analytic within the region of convergence of Taylor's expansion of $f(z)$ and $\phi(a) \neq 0$.

Zeros

In such case, we say that $f(z)$ has a zero of order m at $z = a$. A zero of order one is said to be a simple zero. If a is a zero of $f(z)$ of order m, then we have

$$f(a) = 0, \quad f'(a) = \cdots = f^{(m-1)}(a) = 0,$$

but $f^{(m)}(a) \neq 0$. This is obvious from Taylor's expansion formula.

Zeros

In such case, we say that $f(z)$ has a zero of order m at $z = a$. A zero of order one is said to be a simple zero. If a is a zero of $f(z)$ of order m, then we have

$$f(a) = 0, \quad f'(a) = \cdots = f^{(m-1)}(a) = 0,$$

but $f^{(m)}(a) \neq 0$. This is obvious from Taylor's expansion formula. Now, we have an important result.

Theorem

Zeros are isolated points.

Zeros

In such case, we say that $f(z)$ has a zero of order m at $z = a$. A zero of order one is said to be a simple zero. If a is a zero of $f(z)$ of order m, then we have

$$f(a) = 0, \quad f'(a) = \cdots = f^{(m-1)}(a) = 0,$$

but $f^{(m)}(a) \neq 0$. This is obvious from Taylor's expansion formula. Now, we have an important result.

Theorem

Zeros are isolated points.

Proof.

Let $f(z)$ be analytic in a domain D. Then we show that, unless $f(z)$ is identically zero, there exists a neighborhood of each point in D throughout which the function has no zero, except possibly at the point itself. Suppose that $f(z)$ has a zero of order m at a. Then, as above,

$$f(z) = (z-a)^m \sum_{n=0}^{\infty} a_{m+n}(z-a)^n = (z-a)^m \phi(z). \tag{1}$$

Now, we have

$$\phi(z) = \sum_{n=0}^{\infty} a_{m+n}(z-a)^n \text{ and } \phi(a) = a_m \neq 0.$$

Zeros

Since the series in (1) is uniformly convergent and each term of the series is continuous at a, it follows that $\phi(z)$, being a sum function, is also continuous at a. Hence, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z - a| < \delta \Longrightarrow |\phi(z) - \phi(a)| < \epsilon. \tag{2}$$

Take $\epsilon = \frac{|a_m|}{2}$ and let δ_0 be the corresponding value of δ. Then (2) gives

$$|z - a| < \delta_0 \Longrightarrow |\phi(z) - a_m| = |\phi(z) - \phi(a)| < \frac{1}{2}|a_m|. \tag{3}$$

It follows that $\phi(a) \neq 0$ at any point in the neighborhood $|z - a| < \delta_0$. For, if $\phi(z) = 0$, then the equality (3) is contradicted. The argument remains valid when $m = 0$. In this case, the two functions ϕ and f are equal and $f(a) \neq 0$. □

Definitions

In this section, we introduce the functions which are analytic at all points of a bounded domain except at a finite number of points. Such exceptional points are called singular points or singularities.

Definitions

In this section, we introduce the functions which are analytic at all points of a bounded domain except at a finite number of points. Such exceptional points are called singular points or singularities.

Definition (Singular Points)

A singular point of a function $f(z)$ is the point at which the function creases to be analytic.

For example, the function $f(z) = \frac{1}{z-1}$ has a singularity at $z = 1$.

Definitions

In this section, we introduce the functions which are analytic at all points of a bounded domain except at a finite number of points. Such exceptional points are called singular points or singularities.

Definition (Singular Points)

A singular point of a function $f(z)$ is the point at which the function creases to be analytic.

For example, the function $f(z) = \frac{1}{z-1}$ has a singularity at $z = 1$.

Definition (Isolated Singularities)

A point a is said to be an isolated singularity of function $f(z)$ if $f(z)$ is analytic at each point in some neighborhood $|z - a| < \delta$ of a, except at the point a itself. Otherwise, it is called non-isolated.

Examples:

1. The function $f(z) = \frac{z+1}{z(z^2+2)}$ possesses three isolated singular points $z = 0, z = \sqrt{2}i$ and $z = -\sqrt{2}i$.
2. The function $\ln z$ has a singularity at the origin, but it is not isolated since every neighborhood of zero contains points on the negative real axis where $\ln z$ ceases to be analytic.

Definitions

Suppose f has an isolated singularity at $z = a$.

Definition (Removable Singularities)

If there a function g, analytic at a and such that $f(z) = g(z)$ for all x in some deleted neighborhood of a, we say that f has a removable singularity at a i.e., if the value of f is connected at the point $z = a$, it becomes analytic there.

Definition (Poles)

If, for $z = a, f(z)$ can be written as $f(z) = \frac{\phi(z)}{\psi(z)}$ where ϕ and ψ are analytic at $a, \phi(a) \neq 0$, and $\psi(a) = 0$, we say that f has a pole at a. In other words, if ψ has a zero of order m at a, we say that f has a pole of order m.

Definition (Essential Singularities)

If f has neither a removable singularity nor a pole at a, we say that f has an essential singularity at a.

Removable Singularities

Let $z = a$ be an isolated singularity of a function $f(z)$. Since the singularity is isolated, there exists a deleted neighborhood N_a defined by

$$0 < |z - a| < \delta$$

in which $f(z)$ is analytic. Then, by Laurent's theorem, we can expand $f(z)$ in a series of non-negative and negative powers of $(z - a)$ in N_a. Thus, with suitable definitions of a_n and b_n in the region N_a, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}.$$

The part $b_n(z-a)^{-n}$ of Laurent's series is called the principal part of $f(z)$ at $z = a$.

Removable Singularities

Let $z = a$ be an isolated singularity of a function $f(z)$. Since the singularity is isolated, there exists a deleted neighborhood N_a defined by

$$0 < |z - a| < \delta$$

in which $f(z)$ is analytic. Then, by Laurent's theorem, we can expand $f(z)$ in a series of non-negative and negative powers of $(z - a)$ in N_a. Thus, with suitable definitions of a_n and b_n in the region N_a, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}.$$

The part $b_n(z-a)^{-n}$ of Laurent's series is called the principal part of $f(z)$ at $z = a$.

Now, there arise three distinct possibilities:

1. **Removable Singularity**. If the principal part of $f(z)$ at $z = a$ consists of no terms, then a is said to be a removable singularity of $f(z)$.
   Alternative Definition. A singularity $z = a$ is called a removable singularity of $f(z)$ if $\lim_{z \to a} f(z)$ exists finitely.

Removable Singularities

Let $z = a$ be an isolated singularity of a function $f(z)$. Since the singularity is isolated, there exists a deleted neighborhood N_a defined by

$$0 < |z - a| < \delta$$

in which $f(z)$ is analytic. Then, by Laurent's theorem, we can expand $f(z)$ in a series of non-negative and negative powers of $(z - a)$ in N_a. Thus, with suitable definitions of a_n and b_n in the region N_a, we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} b_n(z-a)^{-n}.$$

The part $b_n(z - a)^{-n}$ of Laurent's series is called the principal part of $f(z)$ at $z = a$.

Now, there arise three distinct possibilities:

1. **Removable Singularity**. If the principal part of $f(z)$ at $z = a$ consists of no terms, then a is said to be a removable singularity of $f(z)$.
   Alternative Definition. A singularity $z = a$ is called a removable singularity of $f(z)$ if $\lim_{z \to a} f(z)$ exists finitely.
   For example, the function $f(z) = \frac{\sin z}{z}$ has a removable singularity at $z = 0$ since
   $$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$
   has no term containing negative powers of z. However, this singularity can be removed and the function be made analytic by defining $\frac{\sin z}{z} = 1$ at $z = 0$.

Poles

2. **Pole**. If the principal part of a function $f(z)$ at $z = a$ consists of a finite number of terms, say m, we say that a is a pole of order m of $f(z)$. For example, if b_m is the last coefficient that does not vanish, then we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots + \frac{b_m}{(z-a)^m}.$$

Poles of order 1 and 2 are called, respectively, single and double poles.
Alternate Definition. If there exists a positive integer m such that

$$\lim_{z\to a} (z-a)^m f(z) = b \neq 0, \text{ but } \lim_{z\to a} (z-a)^{m+1} f(z) = 0,$$

then $z = a$ is called a pole of order m.

Examples:

- Let $f(z) = \frac{1}{(z-1)^2(z-3)^5}$. Then $z = 1$ is a pole of order 2 and $z = 3$ is a pole of order 5.
- $\csc^2 z$ has an infinite number of double poles.

Isolated Essential Singularities

3. **Isolated Essential Singularity**. If the principal part of $f(z)$ at $z = a$ contains an infinite number of terms, then a is called an isolated essential singularity of $f(z)$. In such a case

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n},$$

the last series being convergent for all values of z in $|z-a| < \delta$ except at $z = a$.
Alternate Definition. If there exists no finite value of m such that

$$\lim_{z \to a} (z-a)^m f(z) = b = \text{ a finite non-zero constant,}$$

then $z = a$ is called an isolated essential singularity.

Examples:

- $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots$ has an isolated essential singularity at $z = 0$.
- The function $f(z) = (z-3)\sin\frac{1}{z+2}$ has Laurent's expansion $f(z) = 1 - \frac{5}{z+2} - \frac{1}{6(z+2)^2} + \frac{5}{6(z+2)^3} + \frac{1}{120(z+2)^4} - \cdots$. Thus $z = -2$ is an essential singularity of $f(z)$.

Isolated Essential Singularities

We must take utmost care while classifying a given point a as an isolated essential singularity of a function $f(z)$ on the basis of Laurent's expansion of $f(z)$ in which the series of negative powers of $z - a$ does not terminate. It is important to bear in mind that the series should be convergent for all values of z in $|z - a| < \delta$, except at $z = a$, for some $\delta > 0$.

Isolated Essential Singularities

We must take utmost care while classifying a given point a as an isolated essential singularity of a function $f(z)$ on the basis of Laurent's expansion of $f(z)$ in which the series of negative powers of $z - a$ does not terminate. It is important to bear in mind that the series should be convergent for all values of z in $|z - a| < \delta$, except at $z = a$, for some $\delta > 0$.

For example, the following series contains an infinite number of terms in the principal part

$$\sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} z^{-n}. \tag{4}$$

But on this ground alone, we should not declare that $z = 0$ is an isolated essential singularity of the sum-function of the series (4). We must also test whether the series (4) converges in some deleted neighborhood of the origin, say, $0 < |z| < \delta$.

Isolated Essential Singularities

We must take utmost care while classifying a given point a as an isolated essential singularity of a function $f(z)$ on the basis of Laurent's expansion of $f(z)$ in which the series of negative powers of $z-a$ does not terminate. It is important to bear in mind that the series should be convergent for all values of z in $|z-a|<\delta$, except at $z=a$, for some $\delta>0$.

For example, the following series contains an infinite number of terms in the principal part

$$\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}+\sum_{n=1}^{\infty}z^{-n}. \tag{4}$$

But on this ground alone, we should not declare that $z=0$ is an isolated essential singularity of the sum-function of the series (4). We must also test whether the series (4) converges in some deleted neighborhood of the origin, say, $0<|z|<\delta$.

Since

$$\frac{1}{2-z}=\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}},\quad \frac{1}{z-1}=\sum_{n=1}^{\infty}z^{-n},$$

and the first series converges for $|z|<2$, while the second series converges for $|z|>1$. Thus, the domain of convergence of the series (4) is the annular region $1<|z|<2$, but it is not a neighborhood of the origin. Indeed, the sum-function of (4) in $1<|z|<2$ is given by

$$f(z)=\frac{1}{z-1}+\frac{1}{2-z}=\frac{1}{3z-2-z^2}.$$

$f(z)$ is a function of which the only singularities are the simple poles at $z=1$ and $z=2$.

Isolated Essential Singularities

Let us consider another example of the series

$$f(z) = \sum_{n=1}^{\infty} (z-1)^{-n}, \tag{5}$$

which gives an impression, at first sight, that the point $z = 1$ is an isolated essential singularity of the sum-function of the series (5). The crux of the matter is that the series converges for $|z-1| > 1$ and this does not define a neighborhood of 1 .

Indeed, the sum-function of (5), in the domain of its convergence, is $\frac{1}{z-2}$, which is analytic at $z = 1$ and of which the only singularity is the simple pole at $z = 2$.

Classification of singularities via limits

Suppose z_0 is an isolated singularity of $f(z)$. Then

z_0 *is removable*	$\Leftrightarrow$	$\lim_{z \to z_0} (z - z_0) f(z) = 0.$
z_0 *is a pole*	$\Leftrightarrow$	$(a)\ \neg\left(\lim_{z \to z_0} (z - z_0) f(z) = 0 \right)$ *and* $(b)\ \exists n \in \mathbb{N}$ *such that* $\lim_{z \to z_0} (z - z_0)^{n+1} f(z) = 0.$ (*The smallest such n is called the* order *of the pole z_0 of f.*)
z_0 *is essential*	$\Leftrightarrow$	$\forall n \in \mathbb{N}\ \neg\left(\lim_{z \to z_0} (z - z_0)^n f(z) = 0 \right).$

(Here $\neg$ is the symbol for negation, to be read as "it is not the case that".)

Classification via Laurent coefficients

Let

1. z_0 be an isolated singularity of $f(z)$, and
2. $f(z) = \sum_{n \in \mathbb{Z}} c_n (z - z_0)^n$ for $0 < |z - z_0| < R$, for some $R > 0$.

Then

z_0 *is removable*	$\Leftrightarrow$	*For all* $n < 0$, $c_n = 0$
z_0 *is a pole*	$\Leftrightarrow$	*There exists an* $m \in \mathbb{N}$ *such that* (a) $c_{-m} \neq 0$ *and* (b) *for all* $n < -m$, $c_n = 0$ *Then the order of the pole* z_0 *is* m.
z_0 *is essential*	$\Leftrightarrow$	*There are infinitely many negative indices* n *such that* $c_n \neq 0$.

Introduction

The inspiration behind this section is the desire to obtain possible values for the integrals $\int_C f(z)dz$, where f is analytic inside the closed curve C and on C, except for a inside C.

- If f has a removable singularity at a, then it is clear that the integral will be zero.
- If $z = a$ is a pole or an essential singularity, then the answer is not always zero, but can be found with little difficulty.

In this section, we show the very surprising fact that Cauchy's residue theorem yields a very elegant and simple method for evaluation of such integrals.

The Residues at Singularities

We know that, in the neighborhood of an isolated essential singularity $z = a$, a single-valued analytic function $f(z)$ can be expanded in Laurent's series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}.$$

Thus the coefficient b_1, which is called the residue of $f(z)$ at $z = a$, is given by the formula

$$b_1 = \frac{1}{2\pi i} \int_\gamma f(z) dz,$$

where γ is any circle with center at a, which includes singularities of $f(z)$. We denote the residue of $f(z)$ at $z = a$ by

$$\mathrm{Res}_{z=a} f(z) \quad \text{or} \quad \mathrm{Res}(f, z_0).$$

If $z = a$ is a single pole, then we also have

$$b_1 = \lim_{z \to \alpha} (z-a) f(z).$$

The Residues at Singularities

A more general definition of the "residue" of a function $f(z)$ at a point is the following.

If $z = a$ is the only singularity of an analytic function $f(z)$ inside a closed contour C and

$$\frac{1}{2\pi i}\int_C f(z)dz$$

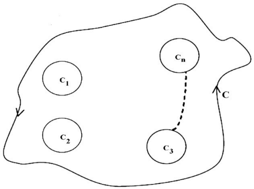
has a definite value, then the value is called the residue of $f(z)$ at $z = a$.

If C includes a finite number of singularities of $f(z)$ which is an analytic function elsewhere inside and on C, then the sum of the residues at singularities is given by

$$\frac{1}{2\pi i}\int_C f(z)dz.$$

The Residues at Singularities

If $f(z)$ is analytic in a multiply connected region bounded by and including the contours C and $C_1, C_2, \cdots, C_n$ contained within C as shown in the following figure.



then the sum of the residues of $f(z)$ at the included essential singularities is easily seen to be given by

$$\frac{1}{2\pi i}\left[\int_C f(z)dz - \sum_{r=1}^{n}\int_{C_r} f(z)dz\right].$$

Calculation of Residues in Some Special Cases

1. **Residues at Simple Poles**:
   When $f(z) = \frac{\phi(z)}{\psi(z)}$, *where* $\phi(a) \neq 0$ *and* $\psi(z)$ *has a simple zero at* $z = a$.
   Since $\psi(z)$ has a simple zero at $z = a, \psi(a) = 0$, but $\psi'(a) \neq 0$. Then it is evident that $f(z)$ has a simple pole at $z = a$. Therefore, we have

$$\begin{aligned}\mathrm{Res}_{z=a} f(z) &= \mathrm{Res}_{z=a} \frac{\phi(z)}{\psi(z)} = \lim_{z \to a} (z - a) \frac{\phi(z)}{\psi(z)} \\ &= \lim_{z \to a} \frac{\phi(z)}{\frac{\psi(z) - \psi(a)}{z - a}} = \frac{\phi(a)}{\psi'(a)}.\end{aligned}$$

Calculation of Residues in Some Special Cases

1. **Residues at Simple Poles**:
   When $f(z) = \frac{\phi(z)}{\psi(z)}$, *where* $\phi(a) \neq 0$ *and* $\psi(z)$ *has a simple zero at* $z = a$.
   Since $\psi(z)$ has a simple zero at $z = a, \psi(a) = 0$, but $\psi'(a) \neq 0$. Then it is evident that $f(z)$ has a simple pole at $z = a$. Therefore, we have

$$\mathrm{Res}_{z=a} f(z) = \mathrm{Res}_{z=a} \frac{\phi(z)}{\psi(z)} = \lim_{z \to a} (z-a) \frac{\phi(z)}{\psi(z)}$$
$$= \lim_{z \to a} \frac{\phi(z)}{\frac{\psi(z)-\psi(a)}{z-a}} = \frac{\phi(a)}{\psi'(a)}.$$

2. **Residues of Poles of Order Greater than Unity**:
   When $f(z)$ *has a pole of order* $m(m > 1)$ *at* $z = a$.
   Laurent's expansion of $f(z)$ in the neighborhood of the point $z = a$ is given by

$$f(z) = \frac{b_m}{(z-a)^m} + \ldots + \frac{b_1}{z-a} + a_0 + a_1(z-a) + \cdots.$$

   Hence we have

$$(z-a)^m f(z) = b_m + b_{m-1}(z-a) + \cdots + b_1(z-a)^{m-1} + a_0(z-a)^m + \cdots$$

   Differentiating both sides with respect to $z, (m-1)$ times, we have

$$D^{m-1}(z-a)^m f(z) = (m-1)! b_1 + m(m-1) \cdots 2 a_0 (z-a) + \cdots.$$

Calculation of Residues in Some Special Cases

Taking the limit as $z \to a$, we have

$$\begin{aligned}(m-1)!b_1 &= \lim_{z\to a}\left[D^{m-1}\left\{(z-a)^m f(z)\right\}\right] \\ &= \lim_{z\to a}\left[\phi^{(m-1)}(z)\right] \\ &= \phi^{(m-1)}(a),\end{aligned}$$

where $f(z) = \frac{\phi(z)}{(z-a)^m}$. Hence we have

$$\mathrm{Res}_{z=a} f(z) = b_1 = \frac{\phi^{(m-1)}(a)}{(m-1)!}$$

In particular, if $\frac{\phi(z)}{(z-a)^2}$, then we have

$$\mathrm{Res}_{z=a} f(z) = \phi'(a).$$

If $\frac{\phi(z)}{(z-a)^3}$, then we have

$$\mathrm{Res}_{z=a} f(z) = \frac{\phi''(a)}{2!}$$

and so on.

Calculation of Residues in Some Special Cases

3. **Another Method**:
Since residue at $z = a$ is the coefficient of $\frac{1}{z-a}$ in Laurent's expansion of $f(z)$, it follows that the residue is the coefficient of $1/t$ in the expansion of $f(a+t)$ as a power series where t is considered sufficiently small.
When $f(z) = \frac{\phi(z)}{z\psi(z)}$, *where the numerator and the denominator have no common factor while* $\psi(0) \neq 0$.

In this case, $f(z)$ has a simple pole at the origin, due to the factor $\frac{1}{z}$, and $f(z)$ also has a number of simple poles arising from the zeros of $\psi(z)$. Hence we have

$$\mathrm{Res}_{z=0} f(z) = \frac{\phi(0)}{\psi(0)}.$$

Suppose that $z = a_m$ is a simple pole of $\frac{1}{\psi(z)}$. Then we have

$$\begin{aligned}\mathrm{Res}_{z=a_m} f(z) &= \lim_{z \to a_m} \left[(z - a_m) \frac{\phi(z)}{z\psi(z)} \right] \\ &= \frac{\phi(a_m)}{a_m \psi'(a_m)}\end{aligned}$$

provided $a_m \neq 0$.

Definition: Residues at Infinity

The definition of residue can be extended to include the point at infinity. If $f(z)$ is analytic or has an isolated essential singularity at infinity and C is a circle enclosing within it all other singularities of $f(z)$ in the finite regions of the z-plane, then the residue at infinity is defined by

$$\frac{1}{2\pi i}\int_C f(z)dz$$

where the integral is taken round C in the negative sense (clockwise direction), provided that this integral has a definite value.

If we take the integral round C in an anti-clockwise direction, then the residue at infinity is $-\frac{1}{2\pi i}\int_C f(z)dz$.

Calculation: Residues at Infinity

By means of the substitution $z = \frac{1}{Z}$, the integral defining the residue at infinity takes the form

$$\frac{1}{2\pi i}\int\left\{-f\left(\frac{1}{Z}\right)\right\}\frac{dZ}{Z^2}$$

taken in a counterclockwise direction round a sufficiently small circle with center at the origin. It follows that, if

$$\lim_{Z\to 0}\left\{-f\left(\frac{1}{Z}\right)Z^{-1}\right\} \text{ or } \lim_{z\to\infty}\{-zf(z)\}$$

has a definite value, then that value is the residue of $f(z)$ at infinity.

Some Residue Theorems

Theorem (Cauchy's Residue Theorem)

If $f(z)$ is regular, except at a finite number of poles $z_0, z_2, \cdots, z_n$ within a closed contour C where its residues are $R_1, R_2, \cdots, R_n$, respectively, and continuous on the boundary C, then

$$\int_C f(z)dz = 2\pi i\,(R_1 + R_2 + \cdots + R_n)$$

or

$$\int_C f(z)dz = 2\pi i \text{ (sum of residues at the poles within) } C.$$

Some Residue Theorems

Theorem (Cauchy's Residue Theorem)

If $f(z)$ is regular, except at a finite number of poles $z_0, z_2, \cdots, z_n$ within a closed contour C where its residues are $R_1, R_2, \cdots, R_n$, respectively, and continuous on the boundary C, then

$$\int_C f(z)dz = 2\pi i (R_1 + R_2 + \cdots + R_n)$$

or

$$\int_C f(z)dz = 2\pi i \text{ (sum of residues at the poles within) } C.$$

Theorem

If a single-valued function has only a finite number of singularities, then the sum of residues at these singularities, including the residue at infinity, is zero.

Two Useful Theorems

Before proceeding to the evaluation of definite integrals, we prove two useful theorems.

Theorem

If C is the arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z-a| = r$ and $\lim_{z \to a} (z-a) f(z) = A$, then

$$\lim_{r \to 0} \int_C f(z) dz = iA \left(\theta_2 - \theta_1 \right).$$

Two Useful Theorems

Before proceeding to the evaluation of definite integrals, we prove two useful theorems.

Theorem

If C is the arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z-a| = r$ and $\lim_{z \to a}(z-a)f(z) = A$, then

$$\lim_{r \to 0} \int_C f(z)dz = iA\left(\theta_2 - \theta_1\right).$$

Proof.

Since $\lim_{z \to a}(z-a)f(z) = A$, it follows that, for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that

$$|(z-a)f(z) - A| < \epsilon$$

whenever $|z-a| < \delta$. But $|z-a| = r$ and so we may take $r < \delta$. Then $|(z-a)f(z) - A| < \epsilon$ on the arc C. Therefore, we have

$$(z-a)f(z) = A + \eta(z),$$

where $|\eta(z)| < \epsilon$, and so

$$f(z) = \frac{A + \eta(z)}{z-a}.$$

Then, putting $z - a = re^{i\theta}$, we have

Two Useful Theorems

$$\int_C f(z)dz = \int_C \frac{A+\eta(z)}{z-a}dz = \int_{\theta_1}^{\theta_2} \frac{(A+\eta(z))re^{i\theta}\,id\theta}{re^{i\theta}}$$
$$= (\theta_2-\theta_1)\,iA + (\theta_2-\theta_1)\,i\eta(z)$$

so that

$$\left|\int_C f(z)dz - iA\,(\theta_2-\theta_1)\right| = (\theta_2-\theta_1)\,|i\eta(z)| < (\theta_2-\theta_1)\,\epsilon \to 0$$

as $\epsilon \to 0$ and $\epsilon \to 0$ as $z \to a$ and $z \to a$ as $r \to 0$. Hence we have

$$\lim_{r\to 0}\int_C f(z)dz = iA\,(\theta_2-\theta_1)\,.$$

Two Useful Theorems

$$\int_C f(z)dz = \int_C \frac{A+\eta(z)}{z-a}dz = \int_{\theta_1}^{\theta_2} \frac{(A+\eta(z))re^{i\theta}\,id\theta}{re^{i\theta}}$$
$$= (\theta_2 - \theta_1)\,iA + (\theta_2 - \theta_1)\,i\eta(z)$$

so that

$$\left|\int_C f(z)dz - iA\,(\theta_2 - \theta_1)\right| = (\theta_2 - \theta_1)\,|i\eta(z)| < (\theta_2 - \theta_1)\,\epsilon \to 0$$

as $\epsilon \to 0$ and $\epsilon \to 0$ as $z \to a$ and $z \to a$ as $r \to 0$. Hence we have

$$\lim_{r\to 0} \int_C f(z)dz = iA\,(\theta_2 - \theta_1)\,.$$

In particular, if $z = a$ is a simple pole of $f(z)$, then A is the residue of $f(z)$ at $z = a$. Thus, if C is a small circle $|z-a| = r$, then we have $\theta_2 - \theta_1 = 2\pi$ and

$$\int_C f(z)dz = 2\pi iA.$$

Particularly, if $(z-a)f(z) \to 0$ as $z \to a$, then we have

$$\int_C f(z)d(z) \to 0 \text{ as } r \to 0.$$

Two Useful Theorems

Theorem

If C is the arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z| = R$ and $\lim_{R\to\infty} zf(z) = A$, then

$$\lim_{R\to\infty} \int_C f(z)dz = i(\theta_2 - \theta_1) A.$$

Two Useful Theorems

Theorem

If C is the arc $\theta_1 \leq \theta \leq \theta_2$ of the circle $|z| = R$ and $\lim_{R\to\infty} zf(z) = A$, then

$$\lim_{R\to\infty} \int_C f(z)dz = i(\theta_2 - \theta_1)A.$$

Proof.

Since $\lim_{R\to\infty} zf(z) = A$, it follows that, for any $\epsilon > 0$, we can choose R so large that

$$|zf(z) - A| < \epsilon \text{ on the arc } C, \text{ or } zf(z) - A = \eta, \text{ where } |\eta| < \epsilon, \text{ or } zf(z) = A + \eta.$$

Therefore, putting $z = Re^{i\theta}$, we have

$$\int_C f(z)dz = \int_C \frac{A+\eta}{z}dz = \int_{\theta_1}^{\theta_2} \frac{(A+\eta)Re^{i\theta}id\theta}{Re^{i\theta}} = Ai(\theta_2 - \theta_1) + \eta i(\theta_2 - \theta_1).$$

Letting $\epsilon \to 0$ and, consequently, $R \to \infty$, we have $\lim_{R\to\infty} \int_C f(z)dz = Ai(\theta_2 - \theta_1)$. In particular, if $zf(z) \to 0$ as $R \to \infty$, then we have $\int_C f(z)dz \to 0$ as $R \to \infty$. □

Type I: Integrals of the form $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$

Let us consider the integrals of the form

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta, \tag{6}$$

where the integrand is a rational function of $\sin\theta$ and $\cos\theta$. The basic idea here is to convert a real trigonometric integral of form (6) into a complex integral, where the contour C is the unit circle $|z| = 1$ centered at the origin. Writing $z = e^{i\theta}$, we have $dz = ie^{i\theta}d\theta$ or $\frac{dz}{iz} = d\theta$ and $\frac{1}{2}(z + z^{-1}) = \cos\theta$, $\frac{1}{2i}(z - z^{-1}) = \sin\theta$ and so

$$\begin{aligned}\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta &= \int_C f\left(\frac{1}{2}(z + z^{-1}), \frac{1}{2i}(z - z^{-1})\right)\frac{dz}{iz}\\ &= \int_C \phi(z)dz,\end{aligned}$$

where C is the unit circle $|z| = 1$. It is evident that $\phi(z)$ is a rational function of z. Hence, by Cauchy's residue theorem, we have

$$\int_C \phi(z)dz = 2\pi i \sum R_C,$$

where $\sum R_C$ is the sum of the residue of the function $\phi(z)$ at its poles inside C.

Type 2: Integrals of the form $\int_{-\infty}^{\infty} f(x)dx$

Suppose $y = f(x)$ is a real function that is defined and continuous on the interval $[0, \infty)$. The improper integral $I_1 = \int_0^{\infty} f(x)dx$ is defined as the limit

$$I_1 = \int_0^{\infty} f(x)dx = \lim_{R\to\infty} \int_0^R f(x)dx. \tag{7}$$

If the limit exists, the integral I_1 is said to be convergent; otherwise, it is divergent. The improper integral $I_2 = \int_{-\infty}^{0} f(x)dx$ is defined similarly:

$$I_2 = \int_{-\infty}^{0} f(x)dx = \lim_{R\to\infty} \int_{-R}^0 f(x)dx. \tag{8}$$

Finally, if f is continuous on $(-\infty, \infty)$, then $\int_{-\infty}^{\infty} f(x)dx$ is defined to be

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_0^{\infty} f(x)dx = I_1 + I_2 \tag{9}$$

provided both integrals I_1 and I_2 are convergent. If either one, I_1 or I_2, is divergent, then $\int_{-\infty}^{\infty} f(x)dx$ is divergent.

It is important to remember that the right-hand side of (9) is not the same as

$$\lim_{R\to\infty} \left[\int_{-R}^0 f(x)dx + \int_0^R f(x)dx\right] = \lim_{R\to\infty} \int_{-R}^R f(x)dx. \tag{10}$$

Type 2: Integrals of the form $\int_{-\infty}^{\infty} f(x)dx$

For the integral $\int_{-\infty}^{\infty} f(x)dx$ to be convergent, the limits (7) and (8) must exist independently of one another. But, in the event that we know that an improper integral $\int_{-\infty}^{\infty} f(x)dx$ converges, we can then evaluate it by means of the single limiting process given in (10):

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx. \tag{11}$$

The limit in (11), if it exists, is called the Cauchy principal value (P.V.) of the integral and is written

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx. \tag{12}$$

On the other hand, the symmetric limit in (11) may exist even though the improper integral $\int_{-\infty}^{\infty} f(x)dx$ is divergent. For example, the integral $\int_{-\infty}^{\infty} xdx$ is divergent since $\lim\limits_{R\to\infty}\int_{0}^{R} xdx = \lim\limits_{R\to\infty}\frac{1}{2}R^2 = \infty$. However, (11) gives

$$\lim_{R\to\infty}\int_{-R}^{R} xdx = \lim_{R\to\infty}\frac{1}{2}\left[R^2-(-R)^2\right] = 0,$$

which shows that P.V. $\int_{-\infty}^{\infty} xdx = 0$.

Type 2: Integrals of the form $\int_{-\infty}^{\infty} f(x)dx$

Suppose $f(x)$ is continuous on $(-\infty, \infty)$ and is an even function, that is, $f(-x) = f(x)$. Then

$$\int_{-R}^{0} f(x)dx = \int_{0}^{R} f(x)dx \text{ and } \int_{-R}^{R} f(x)dx = \int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx = 2\int_{0}^{R} f(x)dx.$$

Thus, if the Cauchy principal value (12) exists, then both $\int_0^\infty f(x)dx$ and $\int_{-\infty}^{\infty} f(x)dx$ converge. The values of the integrals are

$$\int_0^\infty f(x)dx = \frac{1}{2} \text{ P.V.} \int_{-\infty}^{\infty} f(x)dx \quad \text{and} \quad \int_{-\infty}^{\infty} f(x)dx = \text{ P.V.} \int_{-\infty}^{\infty} f(x)dx.$$

To evaluate an integral $\int_{-\infty}^{\infty} f(x)dx$, where the rational function $f(x) = p(x)/q(x)$ is continuous on $(-\infty, \infty)$, by residue theory we replace x by the complex variable z and integrate the complex function f over a closed contour C that consists of the interval $[-R, R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z) = p(z)/q(z)$ in the upper half-plane $\mathrm{Im}(z) > 0$. By Cauchy's Residue Theorem, we have

$$\int_C f(z)dz = \int_{C_R} f(z)dz + \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\left(f(z), z_k\right)$$

where $z_k, k = 1, 2, \ldots, n$ denotes poles in the upper half-plane. If we can show that the integral $\int_{C_R} f(z)dz \to 0$ as $R \to \infty$, then we have

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\left(f(z), z_k\right).$$

Type 2: Integrals of the form $\int_{-\infty}^{\infty} f(x)dx$

It is often tedious to have to show that the contour integral along C_R approaches zero as $R \to \infty$. Sufficient conditions under which this behavior is always true are summarized in the next theorem.

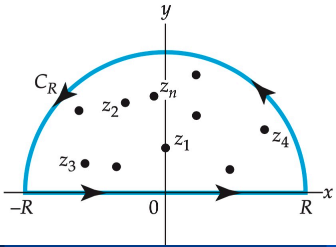
Theorem

Let $f(z) = \frac{p(z)}{q(z)}$, where $p(z)$ and $q(z)$ are polynomials such that

1. *$q(z) = 0$ has no real roots;*
2. *the degree of $p(z)$ is at least two less than that of $q(z)$ so that $\lim\limits_{|z|\to\infty} zf(z) = 0$.*

Then we have

$$P.V. \int_{-\infty}^{\infty} \frac{p(x)}{q(x)} dx = 2\pi i \sum_{k=1}^{n} \mathrm{Res}\left(f(z), z_k\right).$$



Theorem (Jordan's Inequality)

If $0 \le \theta \le \frac{\pi}{2}$, *then*

$$\frac{2\theta}{\pi} \le \sin\theta \le \theta.$$

Theorem (Jordan's Inequality)

If $0 \le \theta \le \frac{\pi}{2}$, *then*

$$\frac{2\theta}{\pi} \le \sin\theta \le \theta.$$

Proof.

Since $\cos\theta$ decreases steadily as θ increases from 0 to $\pi/2$, the mean ordinate of the graph of $y = \cos\theta$ between $\theta = 0$ to θ is

$$\frac{1}{\theta}\int_0^\theta \cos\theta d\theta = \frac{\sin\theta}{\theta}.$$

When $\theta = 0$, the ordinate is $\cos 0 = 1$ and, when $\theta = \frac{\pi}{2}$, the mean ordinate is equal to $\frac{2}{\pi}$. It follows that, when $0 \le \theta \le \frac{\pi}{2}$, the mean ordinate lies between 1 and $\frac{2}{\pi}$, that is,

$$\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1,$$

or

$$\frac{2\theta}{\pi} \le \sin\theta \le \theta.$$

$\square$

Theorem (Jordan's Lemma)

If

1. $f(z) \to 0$ *as* $|z| \to \infty$ *uniformly for* $0 \le \arg z \le \pi$;
2. $f(z)$ *is meromorphic in the upper half-plane, then*

$$\lim_{R\to\infty} \int_{C_R} e^{i\alpha z} f(z) dz = 0 \quad (\alpha > 0),$$

where C_R *denote the semi-circle* $|z| = R$ *and* $\mathrm{Im}(z) > 0$.

Proof.

Assume that $f(z)$ has no singularities on C_R for a sufficiently large value of R. Since $\lim\limits_{R\to\infty} f(z) = 0$, it follows that, for any $\epsilon > 0$, $|f(z)| < \epsilon$ when $|z| = R \ge R_0$ where $R_0 > 0$.
Let C_R denote any semi-circle with radius $R \ge R_0$. From $|f(z)| < \epsilon$, $z = Re^{i\theta}$ and Jordan's inequality, we have

$$\left|\int_{C_R} e^{i\alpha z} f(z) dz\right| \le \int_{C_R} \left|e^{i\alpha z}\right| |f(z)||dz| < \epsilon \int_{C_R} \left|e^{i\alpha z}\right| |dz| = \epsilon \int_0^{\pi} \left|e^{i\alpha(R\cos\theta + iR\sin\theta)}\right| \left|Rie^{i\theta} d\theta\right|$$

$$= \epsilon \int_0^{\pi} e^{-\alpha R \sin\theta} R d\theta \le 2\epsilon R \int_0^{\pi/2} e^{-\alpha R \frac{2\theta}{\pi}} d\theta = \frac{\epsilon\pi}{\alpha}\left(1 - e^{-\alpha R}\right) < \frac{\epsilon\pi}{\alpha}.$$

Type 3: Integrals of the Forms $\int_{-\infty}^{\infty} f(x)\cos\alpha x dx$ and $\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$

Because improper integrals of the form $\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$ are encountered in applications of Fourier analysis, they often are referred to as Fourier integrals. Fourier integrals appear as the real and imaginary parts in the improper integral $\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx$. In view of Euler's formula $e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$, where α is a positive real number, we can write

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = \int_{-\infty}^{\infty} f(x)\cos\alpha x dx + i\int_{-\infty}^{\infty} f(x)\sin\alpha x dx \tag{13}$$

whenever both integrals on the right-hand side converge.

Type 3: Integrals of the Forms $\int_{-\infty}^{\infty} f(x)\cos\alpha x dx$ and $\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$

Because improper integrals of the form $\int_{-\infty}^{\infty} f(x)\sin\alpha x dx$ are encountered in applications of Fourier analysis, they often are referred to as Fourier integrals. Fourier integrals appear as the real and imaginary parts in the improper integral $\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx$. In view of Euler's formula $e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$, where α is a positive real number, we can write

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = \int_{-\infty}^{\infty} f(x)\cos\alpha x dx + i\int_{-\infty}^{\infty} f(x)\sin\alpha x dx \tag{13}$$

whenever both integrals on the right-hand side converge.

Suppose $f(x) = p(x)/q(x)$ is a continuous rational function on $(-\infty, \infty)$. Then both Fourier integrals in (13) can be evaluated at the same time by the complex integral $\int_C f(z)e^{i\alpha z}dz$, where $\alpha > 0$, the contour C consists of the interval $[-R, R]$ on the real axis and a semicircular contour C_R with radius large enough to enclose the poles of $f(z)$ in the upper-half plane.

Theorem

Let $f(z) = \frac{p(z)}{q(z)}$, where $p(z), q(z)$ are polynomials and the degree of $q(z)$ exceeds that of $p(z)$ and $q(z) = 0$ has no real roots. Let $\alpha > 0$. Then

$$\int_{-\infty}^{\infty} f(x)e^{i\alpha x}dx = 2\pi i\sum_{k=1}^{n} \mathrm{Res}\left(f(z)e^{i\alpha z}, z_k\right),$$

where $\sum_{k=1}^{n} \mathrm{Res}\left(f(z)e^{i\alpha z}, z_k\right)$ denotes the sum of the residues of $e^{i\alpha z}f(z)$ at its poles in the upper half-plane.

Type 4: Poles on the Real Axis

If the integrand has simple poles on the real axis, we have the following theorem:

Theorem

Let $f(z) = \frac{p(z)}{q(z)}$, where $p(z), q(z)$ are polynomials and $q(z)$ has only non-repeated real roots, that is, $f(z)$ has only simple poles on the real axis. Let $\alpha > 0$ and let the degree of $q(z)$ exceed that of $p(z)$. Then

$$P.V. \int_{-\infty}^{\infty} e^{i\alpha x} f(x)dx = 2\pi i \sum_{k=1}^{p} \operatorname{Res}(a_k) + \pi i \sum_{k=1}^{q} \operatorname{Res}(b_k)$$

where $a_1, a_2, \cdots, a_p$ are the zeros of $q(z)$ in the region $Im\ z > 0$ and $b_1, b_2, \cdots, b_q$ its zeros in the real axis, where by Res(c) we mean the residue of $e^{i\alpha z} f(z)$ at c.

Type 4: Poles on the Real Axis

The Indenting Method is useful when the integrand has simple poles on the real axis. In such cases, we follow the procedure known as indenting at a point. We exclude the poles on the real axis by enclosing them with a semi-circle of small radii.

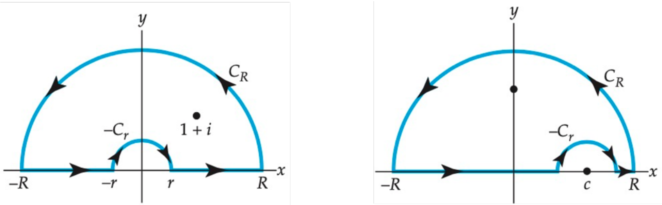


Figure: Two illustrated figures

Logarithmic Derivative

In the previous section we saw how to compute integrals via residues, but in many applications we actually do not have an explicit expression for a function that we need to integrate (or this expression is very complicated). However, it may still be possible to compute the value of a function at any given point. In this situation we cannot immediately apply the Residue Theorem because we don't know where the singularities are. Of course, we could use numerical integration to compute integrals over any path, but computationally this task could be very resource intensive. But if we do know the singularities, we can compute the residues numerically by computing a finite number of the integrals over small circles around these singularities. And after that we can apply the residue theorem to compute the integral over any closed path very effectively: we just sum up the residues inside this path. The argument principle that we study below, in particular, addresses this question. We start by introducing the logarithmic derivative.

Suppose we have a differentiable function f. Differentiating $\mathcal{L}og\ f$ (where $\mathcal{L}og$ is some branch of the logarithm) gives $\frac{f'}{f}$, which is one good reason to call this quotient the logarithmic derivative of f. It has some remarkable properties, one of which we would like to discuss here.

Logarithmic Derivative

Suppose that f is holomorphic in a region G and f has (finitely many) zeros $z_1, \ldots, z_j$ of multiplicities $n_1, \ldots, n_j$, respectively. Then we can express f as

$$f(z) = (z - z_1)^{n_1} \cdots (z - z_j)^{n_j}\, g(z),$$

where g is also holomorphic in G and never zero. Let's compute the logarithmic derivative of f and have:

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{n_1 (z - z_1)^{n_1 - 1} (z - z_2)^{n_2} \cdots (z - z_j)^{n_j} g(z) + \cdots + (z - z_1)^{n_1} \cdots (z - z_j)^{n_j} g'(z)}{(z - z_1)^{n_1} \cdots (z - z_j)^{n_j} g(z)} \\ &= \frac{n_1}{z - z_1} + \frac{n_2}{z - z_2} + \cdots + \frac{n_j}{z - z_j} + \frac{g'(z)}{g(z)}\end{aligned}$$

Something similar happens if f has finitely many poles in G. If $p_1, \ldots, p_k$ are all the poles of f in G with order $m_1, \ldots, m_k$, respectively, then the logarithmic derivative of f can be expressed as

$$\frac{f'(z)}{f(z)} = -\frac{m_1}{z - p_1} - \frac{m_2}{z - p_2} - \cdots - \frac{m_k}{z - p_k} + \frac{g'(z)}{g(z)}$$

where g is a function without poles in G. Naturally, we can combine the expressions for zeros and poles, and have the following theorem.

Argument Principle

Theorem (Argument principle)

If $f(z)$ is analytic within and on a closed contour C except at a finite number of poles and has no zero on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P,$$

where N is the number of zeros and P the number of poles inside C (zeros and poles are counted according to their order or multiplicities).

Argument Principle

Theorem (Argument principle)

If $f(z)$ is analytic within and on a closed contour C except at a finite number of poles and has no zero on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P,$$

where N is the number of zeros and P the number of poles inside C (zeros and poles are counted according to their order or multiplicities).

No reference is made in the proof of the theorem to any arguments of complex quantities, why is this Theorem called the Argument Principle? But in point of fact there is a relation between the number $N-P$ in theorem and $\arg(f(z))$. More precisely,

$$N - P = \frac{1}{2\pi}[\text{change in } \arg(f(z)) \text{ as } z \text{ traverses } C \text{ once in the positive direction}].$$

Argument Principle

Theorem (Argument principle)

If $f(z)$ is analytic within and on a closed contour C except at a finite number of poles and has no zero on C, then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P,$$

where N is the number of zeros and P the number of poles inside C (zeros and poles are counted according to their order or multiplicities).

No reference is made in the proof of the theorem to any arguments of complex quantities, why is this Theorem called the Argument Principle? But in point of fact there is a relation between the number $N-P$ in theorem and $\arg(f(z))$. More precisely,

$$N - P = \frac{1}{2\pi}[\text{change in } \arg(f(z)) \text{ as } z \text{ traverses } C \text{ once in the positive direction}].$$

This principle can be easily verified using the simple function $f(z) = z^2$ and the unit circle $|z| = 1$ as the simple closed contour C in the z-plane. Because the function f has a zero of multiplicity 2 within C and no poles, we have $N - P = 2$. Now, if C is parametrized by $z = e^{i\theta}, 0 \leq \theta \leq 2\pi$, then its image C' in the w-plane under the mapping $w = z^2$ is $w = e^{i2\theta}, 0 \leq \theta \leq 2\pi$, which is the unit circle $|w| = 1$. As z traverses C once starting at $z = 1(\theta = 0)$ and finishing at $z = 1(\theta = 2\pi)$, we see $\arg(f(z)) = \arg(w) = 2\theta$ increases from 0 to 4π. Put another way, w traverses around the circle $|w| = 1$ twice. Thus, $\frac{1}{2\pi}$ [change in $\arg(f(z))$ as z traverses C once in the positive direction] $= \frac{1}{2\pi}[4\pi - 0] = 2$.

Rouché's Theorem

A nice application of the argument principle is a famous theorem due to E. Rouché (1832–1910).The theorem is helpful in determining the number of zeros of an analytic function.

Theorem (Rouché's Theorem)

Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C and let

$$|g(z)| < |f(z)|$$

on C. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

Rouché's Theorem

A nice application of the argument principle is a famous theorem due to E. Rouché (1832–1910).The theorem is helpful in determining the number of zeros of an analytic function.

Theorem (Rouché's Theorem)

Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed curve C and let

$$|g(z)| < |f(z)|$$

on C. Then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

Proof.

Since $|f(z)| > |g(z)|$ on C and $|g(z)| \geq 0$, thus, on $C, |f(z)| > 0$, i.e., $f(z) \neq 0$. We can also infer that, on $C, f(z) + g(z)$ cannot vanish. For this, suppose, if possible, that, for some z on $C, f(z) + g(z) = 0$. Then at that point $f(z) = -g(z)$ and hence $|f(z)| = |g(z)|$. Then, contradicting the hypothesis that $|f(z)| > |g(z)|$, it follows that neither $f(z)$ nor $f(z) + g(z)$ has a zero on C.

Let N and M be the number of zeros of $f(z)$ and $f(z) + g(z)$, respectively, inside C. Then, by the argument principle, we have

$$2\pi N = \Delta_C \arg f(z)$$

and

$$2\pi M = \Delta_C \arg\{f(z) + g(z)\} = \Delta_C \arg\{f(z)\} + \Delta_C \arg\left\{1 + \frac{g(z)}{f(z)}\right\}.$$

Rouché's Theorem

Now, this theorem will be established if we can show that

$$\Delta_C \arg\left\{1+\frac{g(z)}{f(z)}\right\}=0.$$

Since $|g(z)|<|f(z)|$ on C, the transformation $w=1+\frac{g(z)}{f(z)}$ gives points in the w-plane interior to the circle with center $w=1$ and radius of unity. If we write $w=\rho e^{i\theta}$, then $\phi=\arg\left\{1+\frac{g(z)}{f(z)}\right\}$ must lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. It follows that $\arg\left\{1+\frac{g(z)}{f(z)}\right\}$ must return to its original value as z describes C. Since $\arg\left\{1+\frac{g(z)}{f(z)}\right\}$ cannot increase or decrease by a multiple of 2π, we conclude that

$$\Delta_C \arg\left[1+\frac{g(z)}{f(z)}\right]=0.$$

Thus we have

$$2\pi N=2\pi M,$$

whence $N=M$. □

Applications of Rouché's Theorem

Corollary (The Fundamental Theorem of Algebra)

Every polynomial of degree n has n zeros.

Applications of Rouché's Theorem

Corollary (The Fundamental Theorem of Algebra)

Every polynomial of degree n has n zeros.

Proof.

Consider the polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \ldots. + a_n z^n,$$

where $a_n \neq 0$. Observe that $a_n z^n$ has n roots, all at the origin, if $a_n \neq 0$. Let us write

$$f(z) = a_n z^n, \quad g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}.$$

If C denotes the circle with center at the origin and radius $R > 1$, then, on C, we have

$|f(z)| = |a_n| R^n$ and $|g(z)| \leq |a_0| + |a_1| R + \cdots + |a_{n-1}| R^{n-1} \leq (|a_0| + |a_1| + \cdots + |a_{n-1}|) R^{n-1}$.

Hence $|g(z)| < |f(z)|$ on C if

$$R > \frac{|a_0| + |a_1| + |a_2| + \cdots + |a_{n-1}|}{|a_n|}.$$

Since R is arbitrary, if R is sufficiently large, then, by Rouché's theorem, $f(z) + g(z)$, i.e., $P(z)$ has the same number of zeros inside $|z| = R$ as $f(z)$, i.e., $a_n z^n$. □

Theorem (Maximum Modulus Principle)

Let $f(z)$ be analytic within and on a single closed contour C. Then $|f(z)|$ reaches its maximum value on C and not inside C, unless $f(z)$ is a constant. In other words, if M is the maximum value of $|f(z)|$ on and within C, then unless f is constant, $|f(z)| < M$ for every point z within C.

Applications of Rouché's Theorem

Theorem (Maximum Modulus Principle)

Let $f(z)$ be analytic within and on a single closed contour C. Then $|f(z)|$ reaches its maximum value on C and not inside C, unless $f(z)$ is a constant. In other words, if M is the maximum value of $|f(z)|$ on and within C, then unless f is constant, $|f(z)| < M$ for every point z within C.

Theorem (Minimum Modulus Principle)

Let $f(z)$ be analytic inside and on a closed contour C and let $f(z) \neq 0$ inside C. Then $|f(z)|$ attains its minimum value on C and not inside C. In other words, if m is the minimum value of $|f(z)|$ inside and on C, then unless f is constant $|f(z)| > m$ for all points z inside C.

Mappings or Transformations

If $w = f(z)$ is an analytic function of z defined in a domain D of the z-plane, then, to every value of z in D, there corresponds a unique value of w, called the image of the said value of z, which we may represent in another complex plane called the w-plane.

We then say that the points of the domain D in the z-plane are mapped into corresponding points of the w-plane and the set of points of the w-plane which are images of the points of D forms the map of D under the transformation $w = f(z)$. Some information about the function can, however, be displayed by representing sets of corresponding points $z = x + iy$ and $w = u + iv$ on their respective planes. The defining equations are

$$u = u(x, y), \quad v = v(x, y). \tag{14}$$

The correspondence defined by equations (14) between the points in the z-plane and w-plane is called a mapping or transformation of points in the z-plane into points of the w-plane by the function f. The corresponding sets of points in the two planes are called images of each other. The equations (14) are called transformations. If, to each point of the z-plane, there corresponds one and only one point of the w-plane and, conversely, we say that the correspondence is one-to-one.

The main objective of this section is to study a certain especially important and interesting class of mappings, viz. those that preserve the angle between any two differentiable arcs under certain simple conditions, the so-called conformal mappings.

Riemann's Mapping Theorem

Theorem (Riemann's Mapping Theorem)

If D is a simply connected domain in the z-plane, which is neither the z-plane itself nor the extended z-plane, then there is a simple function $f(z)$ such that $w = f(z)$ maps D onto the disk $|w| < 1$.

Jacobian of Transformations

In general, the transformation

$$w = f(z) \quad \text{i.e., } u = u(x, y), v = v(x, y),$$

where $z = x + iy$ and $w = u + iv$, maps a closed region D of the z-plane into a closed region D' of the w-plane. If u, v are continuously differentiable,

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial x},$$

is called the Jacobian of the transformation.

In particular, if $w = f(z) = u + iv$ is an analytic function, then using the Cauchy-Riemann equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$, we have

$$\begin{aligned} J = \frac{\partial(u, v)}{\partial(x, y)} &= \frac{\partial u}{\partial x}\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}\left(-\frac{\partial v}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ &= \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right|^2 = \left|\frac{\partial w}{\partial x}\right|^2 = \left|\frac{\partial w}{\partial z}\right|^2 = |f'(z)|^2, \end{aligned}$$

since $\frac{\partial w}{\partial z} = \frac{\partial w}{\partial x}$. Thus $\frac{\partial(u,v)}{\partial(x,y)} = |f'(z)|^2$ if $f(z)$ is analytic.

Conformal Mappings

A conformal mapping is a mapping that preserves angle between any oriented curves both in magnitude and in sense.

Let the transformation

$$u = u(x, y), \quad v = v(x, y)$$

map a point $P(x_0, y_0)$ of the z-plane to a point $P'(u_0, v_0)$ of the w-plane and let the curves C_1, C_2 intersecting at $z_0 = (x_0, y_0)$ be mapped, respectively, into the curves Γ_1 and Γ_2 at $w_0 = (u_0, v_0)$.

Conformal transformation. If the transformation is such that the angle between C_1 and C_2 at $z_0 = (x_0, y_0)$ is equal both in magnitude and sense to the angle between Γ_1 and Γ_2 at $w_0 = (u_0, v_0)$ as shown below, then it is said to be conformal at $z_0 = (x_0, y_0)$.

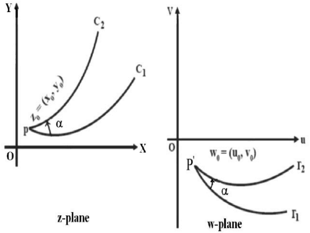


Figure: Conformal transformation.

Isogonal transformation. If the transformation preserves the magnitudes of the angles but not necessarily the sense as shown below, then it is said to be isogonal.

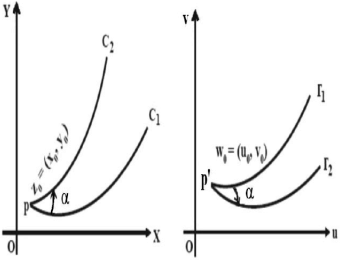


Figure: Isogonal transformation.

Sufficient Condition for Conformal Mappings

Theorem

Let $f(z)$ be an analytic function of z in a domain D of the z plane and let $f'(z) \neq 0$ in D. Then the mapping $w = f(z)$ is conformal at all points of D.

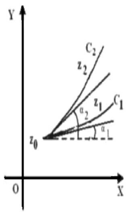
Sufficient Condition for Conformal Mappings

Theorem

Let $f(z)$ be an analytic function of z in a domain D of the z plane and let $f'(z) \neq 0$ in D. Then the mapping $w = f(z)$ is conformal at all points of D.

Proof.

Let $w = f(z)$ be an analytic function of z in a domain D of the z plane and z_0 be an interior point of D. Further, suppose that C_1 and C_2 be two continuous curves passing through z_0 of the z-plane. Suppose that these curves have definite tangents at z_0 making angles α_1 and α_2, respectively, with the real axis. Let z_1 and z_2 be the points on the curves C_1 and C_2, respectively, at the same distance r from the point z_0 where r is small (See figure below).



Sufficient Condition for Conformal Mappings

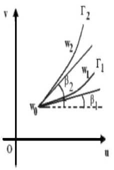
Then we can write

$$z_1 - z_0 = re^{i\theta_1}, \quad z_2 - z_0 = re^{i\theta_2}.$$

As $r \to 0$, the line $z_1 z_0$ and $z_2 z_0$ will tend to the tangents to the curves C_1 and C_2 at z_0, and as $r \to 0$, we have

$$\theta_1 \to \alpha_1, \quad \theta_2 \to \alpha_2.$$

Let w_0, w_1 and w_2 be the images in the w-plane of the points z_0, z_1 and z_2, respectively in the z-plane. Now, as a point moves from z_0 to z_1 along C_1, the image point moves along Γ_1 in the w-plane from w_0 to w_1. Similarly, as a point moves from z_0 to z_2 along C_2, the image point moves along Γ_2 from w_0 to w_2 as shown in figure below.



Suppose that tangents at w_0 to the curves Γ_1 and Γ_2 make angles β_1 and β_2 with the real axis and let $w_1 - w_0 = \rho_1 e^{i\phi_1}, w_2 - w_0 = \rho_2 e^{i\phi_2}$, where $\phi_1 \to \beta_1$ as $\rho_1 \to 0$ and $\phi_2 \to \beta_2$ as $\rho_2 \to 0$.

Sufficient Condition for Conformal Mappings

Since $f(z)$ is analytic, we have

$$f'(z) = \lim_{z_1 \to z_0} \frac{f(z_1) - f(z_0)}{z_1 - z_0} = \lim_{z_1 \to z_0} \frac{w_1 - w_0}{z_1 - z_0} = \lim_{z_1 \to z_0} \frac{\rho_1 e^{i\phi_1}}{re^{i\theta_1}} = \lim_{z_1 \to z_0} \frac{\rho_1}{r} e^{i(\phi_1 - \theta_1)}.$$

Since $f'(z_0) \neq 0$, we may write $f'(z_0) = R_0 e^{i\theta_0}$. It follows that

$$\lim_{z_1 \to z_0} \frac{\rho_1}{r} e^{i(\phi_1 - \theta_1)} = R_0 e^{i\theta_0}.$$

Equating modulus and amplitude on both sides, we have

$$R_0 = \lim_{z_1 \to z_0} \frac{\rho_1}{r} \text{ and } \lim_{z_1 \to z_0} (\phi_1 - \theta_1) = \theta_0 \,(\text{ or } \beta_1 = \alpha_1 + \theta_0).$$

Thus the curve Γ_1 has a definite tangent at w_0 making an angle $\alpha_1 + \theta_0$ with the real axis. Similarly, we can show that Γ_2 has a definite tangent at w_0 making an angle $\alpha_2 + \theta_0$ with the real axis, i.e., $\beta_2 = \alpha_2 + \theta_0$. Consequently, the angle between the tangents at w_0 to the curves Γ_1 and Γ_2, i.e.,

$$\beta_2 - \beta_1 = (\alpha_2 + \theta_0) - (\alpha_1 + \theta_0) = \alpha_2 - \alpha_1,$$

which is the same as the angle between the tangents to C_1 and C_2 at z_0. Also, the angle between the curves has the same sense. Therefore, the transformation $w = f(z)$ is conformal. □

Geometrical Interpretation

We have seen that

$$\lim_{z_1 \to z_0} \frac{\rho_1}{r} e^{i(\phi_1 - \theta_1)} = R_0 e^{i\theta_0}.$$

Therefore, we have $\lim \frac{\rho_1}{r} = R_0 = |f'(z_0)|$ and $\theta_0 = \lim (\phi_1 - \theta_1) = \beta_1 - \alpha_1$, i.e., $\beta_1 = \alpha_1 + \theta_0$, where $\theta_0 = \arg f'(z_0)$. Thus, if $f(z)$ is analytic at z_0 and $f'(z_0) \neq 0$, then under the transformation $w = f(z)$, the tangent at z_0 to any curve C is rotated through an angle $\arg f'(z)$ subject to a magnification

$$\lim \frac{\rho_1}{r} = R_0 = \left| f'(z_0) \right|.$$

Since $f'(z_0)$ is unique, it follows that the magnification R_0 is the same in all the directions through the point z_0, but it varies from point to point. Due to the angle-preserving property of this transformation, any small figure in one plane corresponds to an approximately same figure in the other plane. Thus, to obtain a figure at w_0 corresponding to a figure at z_0, we rotate it through an angle $\arg f'(z_0)$ and subject it to the magnification $|f'(z_0)|$.

Remarks on Conformal Mappings

1. As we know that if ζ an analytic function of w and w is an analytic function of z, then ζ is an analytic function of z. It follows that, if a region of the the z-plane is mapped conformally onto a region of the w-plane and this in its turn is mapped conformally into a region of the ζ-plane, then the mapping of the region of the z-plane directly to the corresponding region of the ζ-plane will also be conformal. In symbols, if $w = f(z)$ maps a domain D conformally to the domain D' and $\zeta = g(w)$ maps the later domain D' conformally to the domain D'', then the mapping $\zeta = g(f(z))$ from D to D'' will also be conformal.
2. There are some transformations which preserve angles between pairs of curves only in magnitude, but, not in sense. Consider, for example, the transformation:

$$\zeta = f(\bar{z}),$$

where f defines a conformal mapping. We observe that $\zeta = f(\bar{z})$ is the combination of the two transformations

$$(a) w = \bar{z}, \quad (b) \zeta = f(w)$$

and while the transformation (a) maps every point into its reflection in the real axis and consequently conserves angle in magnitude, but reverses them in sense, (b) conserves both sense and magnitude. Thus $\zeta = f(\bar{z})$ represents a transformation which is isogonal, but not conformal.

Sufficient Condition for Conformal Mappings

Corollary

A small arc in the z-plane through the point z_0 is magnified in the ratio $|f'(z_0)| : 1$ in the w-plane under the transformation $w = f(z)$ where $f(z)$ is an analytic function and $f'(z) \neq 0$.

Sufficient Condition for Conformal Mappings

Corollary

A small arc in the z-plane through the point z_0 is magnified in the ratio $|f'(z_0)| : 1$ in the w-plane under the transformation $w = f(z)$ where $f(z)$ is an analytic function and $f'(z) \neq 0$.

Proof.

Let ds denote an element of an arc in the z-plane and $d\sigma$ be the corresponding arc in the w-plane given by the transformation $w = f(z)$, where $f(z)$ is analytic and $f'(z) \neq 0$. We then have

$$ds^2 = dx^2 + dy^2, \quad d\sigma^2 = du^2 + dv^2.$$

But we have $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy$. Then, by the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$, it follows that

$$\begin{aligned} d\sigma^2 &= (\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy)^2 + (\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy)^2 = (\frac{\partial u}{\partial x}dx - \frac{\partial v}{\partial x}dy)^2 + (\frac{\partial v}{\partial x}dx + \frac{\partial u}{\partial x}dy)^2 \\ &= [(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2](dx^2 + dy^2) = |\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}|^2 ds^2 = |\frac{\partial w}{\partial x}|^2 ds^2 = |\frac{\partial w}{\partial z}|^2 ds^2 = |f'(z)|^2 ds^2. \end{aligned}$$

Hence $d\sigma = |f'(z)|ds$. Thus a small arc passing through a point z_0 in the z-plane is magnified in the ratio $|f'(z_0)| : 1$ in the w-plane under a conformal transformation $w = f(z)$. □

Superficial Magnification

Let D denote a closed domain in the z-plane and D' be the corresponding closed domain in the w-plane. If A denotes the area of D', then we have

$$A = \iint_D dudv = \iint_D \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dxdy$$

by a well-known theorem for change of variables in a double integral where $\frac{\partial(u,v)}{\partial(x,y)}$ is the Jacobian of the transformation $w = u + iv = f(x + iy)$. Now, by using the Cauchy-Riemann equations, we have

$$\begin{aligned} \frac{\partial(u,v)}{\partial(x,y)} &= \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial y} - \frac{\partial v}{\partial x} \cdot \frac{\partial u}{\partial y} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \\ &= \left| \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \right|^2 = \left| f'(z) \right|^2 \end{aligned}$$

and so

$$A = \iint_D \left| f'(z) \right|^2 dxdy.$$

Some Elementary Transformations

Some elementary transformations such as translation, rotation, magnification, inversion, and others are discussed. In such cases, it is convenient to consider the mapping as a transformation in just one plane.

1. Translation $w = z + \alpha$;
2. Rotation $w = e^{i\theta} z$, where θ is real;
3. Magnification $w = rz$ $(r > 0)$;
4. Inversion $w = \frac{1}{z}$.

The transformation $w = \beta z + \alpha$: an example

Example (1)

Consider the linear transformation

$$w = (1 + i)z + 2 - i$$

and determine the region in the w-plane into which the rectangular region bounded by the lines $x = 0, x = 1$, and $y = 2$ in the z-plane is mapped under this transformation.

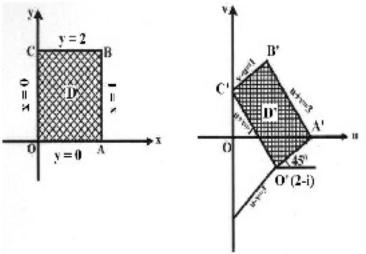
The transformation $w = \beta z + \alpha$: an example

Example (1)

Consider the linear transformation

$$w = (1 + i)z + 2 - i$$

and determine the region in the w-plane into which the rectangular region bounded by the lines $x = 0, x = 1$, and $y = 2$ in the z-plane is mapped under this transformation.



Linear transformation $w = (1 + i)z + 2 - i$.

The transformation $w = \beta z + \alpha$: an example

Proof.

Writing $z = x + iy$ and $w = u + iv$, we have

$$u + iv = (1 + i)(x + iy) + 2 - i = (x - y + 2) + i(x + y - 1).$$

Therefore, we have

$$u = x - y + 2, \quad v = x + y - 1.$$

Thus the line $x = 0$ is mapped into $u = -y + 2, v = y - 1$ or into $u + v = 1$. The line $y = 0$ is mapped into $u = x + 2, v = x - 1$ or into $u - v = 3$. The line $x = 1$ is mapped into $u = -y + 3, v = y$ or into $u + v = 3$.
Finally, the line $y = 2$ is mapped into $u = x, v = x + 1$ or into $v - u = 1$. Hence the given rectangular region D in the z-plane is mapped into the rectangular region D' bounded by the lines $u + v = 1, u - v = 3, u + v = 3$ and $v - u = 1$ in the w-plane. □

Example (2)

Find the image of the infinite strips

$$(a)\frac{1}{4} < y < \frac{1}{2}, \qquad (b)0 < y < \frac{1}{2}$$

under the transformation $w = \frac{1}{z}$. Show the region graphically.

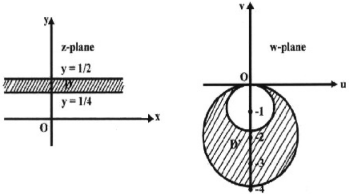
The transformation $w = \frac{1}{z}$: an example

Example (2)

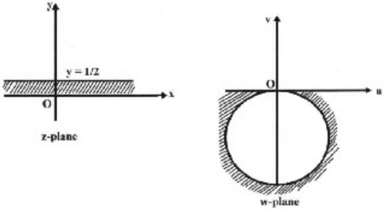
Find the image of the infinite strips

$$(a)\frac{1}{4} < y < \frac{1}{2}, \qquad (b)0 < y < \frac{1}{2}$$

under the transformation $w = \frac{1}{z}$. Show the region graphically.



(a) Inversion of the region $\frac{1}{4} < y < \frac{1}{2}$.



(b) Inversion of the region $0 < y < \frac{1}{2}$.

The transformation $w = \frac{1}{z}$: an example

Proof.

Writing $w = u + iv, z = x + iy$, we have $u + iv = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$ and so $u = \frac{x}{x^2+y^2}$, $v = \frac{-y}{x^2+y^2}$, and so $\frac{u}{v} = -\frac{x}{y}$ or $x = -\frac{uy}{v}$ and $v = -\frac{y}{\frac{u^2y^2}{v^2}+y^2} = -\frac{v^2}{(u^2+v^2)y}$ or $y = -\frac{v}{u^2+v^2}$.

(a) If $y > \frac{1}{4}$, then $-\frac{v}{u^2+v^2} > \frac{1}{4}$, that is, $u^2 + v^2 + 4v < 0$ or $u^2 + (v+2)^2 < 4$. If $y < \frac{1}{2}$, then $-\frac{v}{u^2+v^2} < \frac{1}{2}$ or $u^2 + (v+1)^2 > 1$. Finally, $\frac{1}{4} < y < \frac{1}{2}$ imply $u^2 + (v+2)^2 < 4$ and $u^2 + (v+1)^2 > 1$. Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region D' in the w-plane which is bounded by the two circles $u^2 + (v+2)^2 = 4, \quad u^2 + (v+1)^2 = 1$, such that the region is exterior to the circle $u^2 + (v+1)^2 = 1$ and interior to the circle $u^2 + (v+2)^2 = 4$.

(b) When $y = 0, v = 0$. Also, when $y > 0, v < 0$ since $v = -\frac{y}{x^2+y^2}$ and the line $y = \frac{1}{2}$ is transformed into the circle $u^2 + (v+1)^2 = 1$. From the figure in (a), it is evident that the circle gets bigger as y diminishes from $\frac{1}{2}$ to 0. Hence the infinite strip $0 < y < \frac{1}{2}$ is mapped into the region exterior to the circle $u^2 + (v+1)^2 = 1$ in the lower half-plane, i.e., into the region $u^2 + (v+1)^2 > 1$ and $v < 0$. $\square$

Linear Fractional Transformation

The transformation T defined by

$$w = T(z) = \frac{az + b}{cz + d} \tag{15}$$

where a, b, c, d are complex constants. Linear fractional transformation such that $ad - bc \neq 0$ and w, z are complex variables, is called a bilinear transformation or möbius transformation. The constant $ad - bc$ is called the determinant of the transformation. The transformation (15) is said to be normalized if $ad - bc = 1$. The term "bilinear" is justified by the fact that the transformation (15) can be written in the form:

$$cwz + dw - az - b = 0$$

which is linear in both w and z.

We assume that the determinant of the transformation $ad - bc \neq 0$. For otherwise, $\frac{a}{c} = \frac{b}{d}$ and the function of z in (15) becomes either a constant or meaningless.

Linear Fractional Transformation

The transformation T defined by

$$w = T(z) = \frac{az+b}{cz+d} \tag{15}$$

where a, b, c, d are complex constants. Linear fractional transformation such that $ad - bc \neq 0$ and w, z are complex variables, is called a bilinear transformation or möbius transformation. The constant $ad - bc$ is called the determinant of the transformation. The transformation (15) is said to be normalized if $ad - bc = 1$. The term "bilinear" is justified by the fact that the transformation (15) can be written in the form:

$$cwz + dw - az - b = 0$$

which is linear in both w and z.

We assume that the determinant of the transformation $ad - bc \neq 0$. For otherwise, $\frac{a}{c} = \frac{b}{d}$ and the function of z in (15) becomes either a constant or meaningless.

Inverse Transformation. If $w = T(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$, is a linear transformation, then the inverse transformation T^{-1} is defined by

$$z = T^{-1}(w) = \frac{-dw+b}{cw-a}.$$

The determinant of this transformation is $(-d)(-a) = -bc = ad - bc$, which is the same as that of (15).

Consider the linear fractional transformation

$$w = T(z) = \frac{az+b}{cz+d} \quad (ad - bc \neq 0).$$

1. If $c \neq 0$, then $w = \frac{a}{c} + \frac{bc-ad}{c^2\left(z+\frac{d}{c}\right)}$. This is the same as the mapping obtained by a superimposition of the successive mappings:
   (a) $\xi = z + \frac{d}{c}$;
   (b) $\eta = \frac{1}{\xi}$;
   (c) $\zeta = \frac{bc-ad}{c^2}\eta$;
   (d) $w = \zeta + \frac{a}{c}$.

   These relations show that a linear fractional transformation is the resultant of a translation given by (a), the inversion in the real axis and unit circle given by (b), a rotation and magnification given by (c) and then, finally, a translation again given by (d).
2. If $c = 0$, then $w = \frac{c}{d}z + \frac{b}{d}$ provided $d \neq 0$. Writing $\xi = \frac{a}{d}z$ and $w = \xi + \frac{b}{d}$, it is evident that the given transformation is the resultant of magnification, rotation, and translation. Hence inversion fails in this case and we require only one translation.

Remarks: Translation, rotation, magnification, and inversion are special types of linear fractional transformations.

Condition for One-to-One Correspondence. Let w_1 and w_2 be the points corresponding to z_1 and z_2 given by (15) and then

$$w_1 = \frac{az_1 + b}{cz_1 + d}, \quad w_2 = \frac{az_2 + b}{cz_2 + d},$$

respectively. Then we have

$$w_2 - w_1 = \frac{az_2 + b}{cz_2 + d} - \frac{az_1 + b}{cz_1 + d} = \frac{(ad - bc)(z_2 - z_1)}{(cz_1 + d)(cz_2 + d)}.$$

Hence $w_2 - w_1 = 0$ if $ad - bc = 0$. Thus $w_2 = w_1$. It follows that w is a constant if $ad - bc = 0$ provided $z_1 \neq -d/c$ or $z_2 \neq -d/c$. If $ad - bc = 0$ and either $z_1 \neq -d/c$ or $z_2 \neq -d/c$, then w becomes meaningless. Hence $ad - bc \neq 0$ is the necessary condition for the linear fractional transformation T to set up a one-to-one correspondence between the points of the closed z-plane and closed w-plane.

Product of Two Linear Fractional Transformations

Consider transformations T_1 and T_2 defined by

$$\xi = T_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad (a_1 d_1 - b_1 c_1 \neq 0) \tag{16}$$

and

$$w = T_2(\xi) = \frac{a_2 \xi + b_2}{c_2 \xi + d_2} \quad (a_2 d_2 - b_2 c_2 \neq 0)\,. \tag{17}$$

Now, we establish a transformation which sets up a one-to-one correspondence between the points of z-plane and w-plane by the relation $w = T_2\left(T_1(z)\right)$. By (16) and (17), we have

$$T_2\left(T_1(z)\right) = T_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) = \frac{a_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) + b_2}{c_2\left(\frac{a_1 z + b_1}{c_1 z + d_1}\right) + d_2} = \frac{\left(a_2 d_1 + b_2 c_1\right) z + \left(a_2 b_1 + b_2 d_1\right)}{\left(c_2 a_1 + d_2 c_1\right) z + \left(c_2 b_1 + d_2 d_1\right)}.$$

Hence the transformation above can be written as

$$w = T_2\left(T_1(z)\right) = \frac{\alpha z + \beta}{\gamma z + \delta},$$

where $\alpha = a_2 a_1 + b_2 c_1, \beta = a_2 b_1 + d_2 d_1, \gamma = c_2 a_1 + d_2 c_1$ and $\delta = c_2 b_1 + d_2 d_1$.

Cross-Ratios

If z_1, z_2, z_3, z_4 are distinct points taken in this order then the cross-ratio of these points is defined as follows:

$$\frac{z_1 - z_2}{z_2 - z_3} \Big/ \frac{z_4 - z_1}{z_3 - z_4} \text{ or } \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)},$$

which is denoted by (z_1, z_2, z_3, z_4).

From four points z_1, z_2, z_3 and z_4 lying in the z-plane, we can obtain different cross-ratios according to the order in which the points are taken. Since the four points can permute themselves in 4!, i.e., 24 ways, there will be 24 cross-ratios, but as a matter of fact there will be only six distinct cross-ratios.

$$(z_1, z_2, z_3, z_4) = (z_2, z_1, z_4, z_3) = (z_3, z_4, z_1, z_2) = (z_4, z_3, z_2, z_1), \tag{18}$$

$$(z_1, z_2, z_4, z_3) = (z_2, z_1, z_3, z_4) = (z_3, z_4, z_2, z_1) = (z_4, z_3, z_1, z_2), \tag{19}$$

$$(z_1, z_3, z_2, z_4) = (z_2, z_4, z_1, z_3) = (z_3, z_1, z_4, z_2) = (z_4, z_2, z_3, z_1), \tag{20}$$

$$(z_1, z_3, z_4, z_2) = (z_2, z_4, z_3, z_1) = (z_3, z_1, z_2, z_4) = (z_4, z_2, z_1, z_3), \tag{21}$$

$$(z_1, z_4, z_2, z_3) = (z_2, z_3, z_1, z_4) = (z_3, z_2, z_4, z_1) = (z_4, z_1, z_3, z_2), \tag{22}$$

$$(z_1, z_4, z_3, z_2) = (z_2, z_3, z_4, z_1) = (z_3, z_2, z_1, z_4) = (z_4, z_1, z_2, z_3). \tag{23}$$

We can verify that if λ denotes any one of the ratios in (18), then the ratios in (19), (20), (21), (22) and (23) are

$$\frac{\lambda}{\lambda - 1}, \quad 1 - \lambda, \quad \frac{\lambda - 1}{\lambda}, \quad \frac{1}{1 - \lambda}, \quad \frac{1}{\lambda}$$

respectively.

Cross-Ratios

Remark: Let z_2, z_3, z_4 be any three distinct points in the extended complex plane. First, we suppose that none of these points is ∞. Now consider the linear fractional transformation

$$w = T(z) = \frac{(z - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z)}. \tag{24}$$

It can be easily seen that this transformation mapped the points z_2, z_3, z_4 into the points $0, 1, \infty$, respectively. Further, the image of any point z_1 under this transformation is

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} \quad \text{or} \quad (z_1, z_2, z_3, z_4).$$

Thus we may define the cross-ratio (z_1, z_2, z_3, z_4) of the four points z_1, z_2, z_3, z_4 as the image of z_1 under the linear fractional transformation (24) which maps z_2, z_3, z_4 into $0, 1, \infty$, respectively.

Next, suppose that z_2, z_3 or $z_4 = \infty$. Then the transformation (24) reduces to

$$\frac{z_3 - z_4}{z - z_4}, \quad \frac{z - z_2}{z - z_4}, \quad \frac{z - z_2}{z_3 - z_2}$$

in that order, respectively.

Preservation of Cross-Ratio under Linear Fractional Transformation

Theorem

Cross-ratio are invariant under a linear fractional transformation.

Theorem

Cross-ratio are invariant under a linear fractional transformation.

Proof.

Let w_1, w_2, w_3, w_4 be the images of the four distinct points z_1, z_2, z_3, z_4 in the z-plane under a linear fractional transformation $w = T(z) = \frac{az+b}{cz+d}$, $(ad - bc \neq 0)$. Then we prove that $(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$. In fact, we have $w_1 = \frac{az_i+b}{cz_i+d}$, $i = 1, 2, 3, 4$. Then, we have $w_1 - w_2 = \frac{az_1+b}{cz_1+d} - \frac{az_2+b}{cz_2+d} = \frac{(ad-bc)(z_1-z_2)}{(cz_1+d)(cz_2+d)}$. Similarly, we have $w_3 - w_4 = \frac{(ad-bc)(z_3-z_4)}{(cz_3+d)(cz_4+d)}$. Therefore, it follows that

$$(w_1 - w_2)(w_3 - w_4) = \frac{(ad - bc)^2(z_1 - z_2)(z_3 - z_4)}{(cz_1 + d)(cz_2 + d)(cz_3 + d)(cz_4 + d)}. \tag{25}$$

Similarly, we have

$$(w_2 - w_3)(w_4 - w_1) = \frac{(ad - bc)^2(z_2 - z_3)(z_4 - z_1)}{(cz_1 + d)(cz_2 + d)(cz_3 + d)(cz_4 + d)}. \tag{26}$$

Dividing (25) by (26), we have $\frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$. □

Theorem

The cross-ratio (z_1, z_2, z_3, z_4) is real if and only if the four points z_1, z_2, z_3, z_4 lie an a circle or a straight line.

Theorem

The cross-ratio (z_1, z_2, z_3, z_4) is real if and only if the four points z_1, z_2, z_3, z_4 lie an a circle or a straight line.

Proof.

By the definitions, we have $(z_1, z_2, z_3, z_4) = (\frac{z_1 - z_2}{z_2 - z_3})/(\frac{z_4 - z_1}{z_3 - z_4})$. Therefore, we have

$$\arg(z_1, z_2, z_3, z_4) = \arg(\frac{z_1 - z_2}{z_2 - z_3}) - \arg(\frac{z_4 - z_1}{z_3 - z_4}). \tag{27}$$

Evidently, the difference of the angles on the R.H.S. of (27) is 0 or $\pm\pi$ depending on the relative position of the point z_1, z_2, z_3, z_4 if and only if these four points are concyclic.
In case, $\arg(z_1, z_2, z_3, z_4) = 0$ or $\pm\pi$ the cross-ratio (z_1, z_2, z_3, z_4) is purely real. Hence the cross-ratio is real if and only if the four points lie on a circle. □

Preservation of Cross-Ratio under Linear Fractional Transformation

Theorem

Determine the linear fractional transformation which transforms three distinct points into three specified distinct points.

Preservation of Cross-Ratio under Linear Fractional Transformation

Theorem

Determine the linear fractional transformation which transforms three distinct points into three specified distinct points.

Proof.

Proof. Suppose that we are required to transform the three distinct points into the three specified points w_1, w_2, w_3 and all these numbers are finite. When z becomes w, then we have to find a relation between w and z. Consider the cross-ratios (w_1, w_2, w_3, w_4) and (z_1, z_2, z_3, z_4), we know $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$ or

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}. \tag{28}$$

Thus, when solved for w, (28) gives the required linear fractional transformation. We can write (28) in the form

$$\begin{aligned} &(w - w_1)(w_2 - w_3)(z_1 - z_2)(z_3 - z) \\ &= (z - z_1)(z_1 - z_2)(w_1 - w_2)(w_3 - w). \end{aligned} \tag{29}$$

Now, if $z = z_1$, then the R.H.S. of the equation (29) vanishes and so $w = w_1$. Again, if $z = z_3$, then the L.H.S. of the equation (29) vanishes and consequently $w = w_3$. Finally, if $z = z_2$, then the equation (29) reduces to $w = w_2$. Thus (28) is the required linear fractional transformation. The uniqueness of the transformation is obvious. □

Examples of Linear Fractional Transformations

Example (3)

Find the linear fractional transformation which maps the points $z_1 = 2, z_2 = i, z_3 = -2$ into the points $w_1 = 1, w_2 = i$ and $w_3 = -1$, respectively.

Examples of Linear Fractional Transformations

Example (3)

Find the linear fractional transformation which maps the points $z_1 = 2, z_2 = i, z_3 = -2$ into the points $w_1 = 1, w_2 = i$ and $w_3 = -1$, respectively.

Proof.

Let the required linear fractional transformation be $(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$, i.e.,

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}. \tag{30}$$

Substituting the given values in (30), we obtain

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-2)(i+2)}{(2-i)(-2-z)}$$

$\Longrightarrow \cdots \Longrightarrow$

$$w = \frac{3z + 2i}{iz + 6},$$

which is the required transformation. □

Examples of Linear Fractional Transformations

Example (4)

Find the linear fractional transformation which maps $0, 1$ and ∞ into 1, i and -1, respectively.

Examples of Linear Fractional Transformations

Example (4)

Find the linear fractional transformation which maps $0, 1$ and ∞ into 1, i and -1, respectively.

Proof.

Let the required transformation be given by

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} \tag{31}$$

Substituting the given values in (31), we obtain

$$\frac{(w-1)(i+1)}{(1-i)(-1-w)} = \frac{(z-0)(1-\infty)}{(0-1)(\infty - z)}$$

or

$$\frac{(w-1)(i+1)}{(w+1)(i-1)} = z$$

$\Longrightarrow \cdots \Longrightarrow$

$$w = \frac{(i-1)z + (i+1)}{(i-1)z + (i+1)},$$

which is the required transformation. $\square$

Examples of Linear Fractional Transformations

Example (5)

Find a linear fractional transformation which transforms the unit circle $|z| = 1$ into the real axis in such a way that the points $1, i, -1$ are mapped into $0, 1, \infty$ respectively. Into what regions are the interior and exterior of the circle mapped?

Examples of Linear Fractional Transformations

Example (5)

Find a linear fractional transformation which transforms the unit circle $|z| = 1$ into the real axis in such a way that the points $1, i, -1$ are mapped into $0, 1, \infty$ respectively. Into what regions are the interior and exterior of the circle mapped?

Proof.

We note that the points $z_1 = 1, z_2 = i$ and $z_3 = -1$ lie on the circle $|z| = 1$ and the points $w_1 = 0, w_2 = 1, w_3 = \infty$ lie on the real axis of the w-plane. Now, suppose that the required transformation is

$$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}. \tag{32}$$

Substituting the given values in (32), we obtain $\frac{(w-0)(1-\infty)}{(0-1)(\infty-w)} = \frac{(z-1)(I+1)}{(1-i)(-1-z)}$, then $w = \frac{i(1-z)}{1+z}$.

Thus the inverse transformation is $z = -\frac{w-i}{w+i}$ and so $|z| = \left|\frac{w-i}{w+i}\right|$.

Hence, the unit circle $|z| = 1$ is transformed into $1 = \left|\frac{w-i}{w+i}\right|$ or $|w - i| = |w + i|$, which is the axis of the w-plane since it is the locus of the point equidistant from $w = i$ and $w = -i$, and the center $z = 0$ of the circle $|z| = 1$ is transformed into the point $w = i$ in the w-plane. Thus the interior of the circle $|z| = 1$ is mapped into the upper half-plane. Again, since $w = -i$ corresponds to $z = \infty$, it follows that the exterior of the circle $|z| = 1$ is transformed into the lower half of the w-plane. □

Preservation of the Family of Circles and Straight Lines

Theorem

Every linear fractional transformation maps circles or straight lines into circles or straight lines.

Theorem

Every linear fractional transformation maps circles or straight lines into circles or straight lines.

Proof.

The equation of a circle in the z-plane may be written as

$$Az\bar{z} + Bz + \bar{B}\bar{z} + C = 0, \tag{33}$$

where A, C are real and B is a complex constant such that $B\bar{B} > AC$. If $A = 0$, then the equation (33) represents a straight line. Let $w = \frac{az+b}{cz+d}$ or

$$z = \frac{-dw + b}{cw - a} \tag{34}$$

be any linear fractional transformation. Then (34) transforms (33) into the form

$$A\left(\frac{-dw+b}{cw-a}\right)\left(\frac{-\bar{d}\bar{w}+\bar{b}}{\bar{c}\bar{w}-\bar{a}}\right) + B\left(\frac{-dw+b}{cw-a}\right) + \bar{B}\left(\frac{-\bar{d}\bar{w}+\bar{b}}{\bar{c}\bar{w}-\bar{a}}\right) + C = 0$$

$\Longrightarrow \cdots \Longrightarrow$

$$\alpha w\bar{w} + \beta w + \bar{\beta}\bar{w} + \gamma = 0, \tag{35}$$

where

$$\begin{aligned}
\alpha &= Ad\bar{d} - Bd\bar{c} + \bar{B}\bar{d}c + Cc\bar{c}, \\
\beta &= -A\bar{b}d + B\bar{a}d + \bar{B}\bar{b}c - Cc\bar{a}, \\
\gamma &= Ab\bar{b} - Bb\bar{a} - \bar{B}\bar{b}a + Ca\bar{a}.
\end{aligned}$$

Since $\alpha = \bar{\alpha}$ and $\gamma = \bar{\gamma}$, it follows that α and γ are real and the coefficients of w and $\bar{w}$ are conjugate complex numbers. Also, we have

$$\begin{aligned}
\beta\bar{\beta} - \alpha\gamma &= B\bar{B}(a\bar{a}d\bar{d} + b\bar{b} - c\bar{c} - d\bar{c}\bar{b}a - \bar{d}cb\bar{a}) \\
&\quad - AC(a\bar{a}d\bar{d} + b\bar{b}c\bar{c} - d\bar{c}\bar{b}a - \bar{d}cb\bar{a}) \\
&= (B\bar{B} - AC)(bc - ad)(\bar{b}\bar{c} - \bar{a}\bar{d}) \\
&= (B\bar{B} - AC)|b - c - ad|^2 \geq 0
\end{aligned}$$

since $B\bar{B} > AC$. Hence (35) represents a circle or a straight line. $\square$

Preservation of inverse points with respect to a circle

Theorem

Every linear fractional transformation transforms two points which are inverse points with respect to a circle into two points which are inverse points with respect to the transformed circle.

Preservation of inverse points with respect to a circle

Theorem

Every linear fractional transformation transforms two points which are inverse points with respect to a circle into two points which are inverse points with respect to the transformed circle.

Proof.

Let z, z' be the inverse points with respect to the circle $Az\bar{z} + Bz + \bar{B}\bar{z} + C = 0$, where A, C are real and B is the complex constant. By the definition of inverse points with respect to a circle,

$$Az'\bar{z} + Bz' + \bar{B}\bar{z} + C = 0. \tag{36}$$

Let w and w' be the transforms of z and z', respectively, under the linear fractional transformation $w = \frac{az+b}{cz+d}$. Then we have $z = \frac{dw-b}{-cw+a}$, $z' = \frac{dw'-b}{-cw'+a}$. By Theorem above, the transformed circle is

$$\alpha w\bar{w} + \beta w + \bar{\beta}\bar{w} + \gamma = 0, \tag{37}$$

where $\alpha, \beta, \bar{\beta}$ and γ have the values gives in that theorem. If we substitute the values of z and z' in the condition (36) is becomes

$$\alpha w'\bar{w} + \beta w' + \bar{\beta}\bar{w} + \gamma = 0. \tag{38}$$

Hence (38) shows that w, w' are the inverse points with respect to transformed circle (37). □

Special Linear Fractional Transformations

Recall that Theorems above state that under linear fractional transformation, circles and straight lines are mapped into circles and straight lines and that the inverse points are mapped into inverse points. In this regard, we have taken the straight lines as a special class of circles and the inverse point of a given point with respect to a straight lines is simply the reflection of that point in the straight line. The points 0 and ∞ may be regarded as the inverse points with respect to the circle $|w| = k$. Consequently, the image points of 0 and ∞ must be the inverse points with respect to a circle or straight line in the z-plane as the case may be. For an illustration, if we wish to find the linear fractional transformation

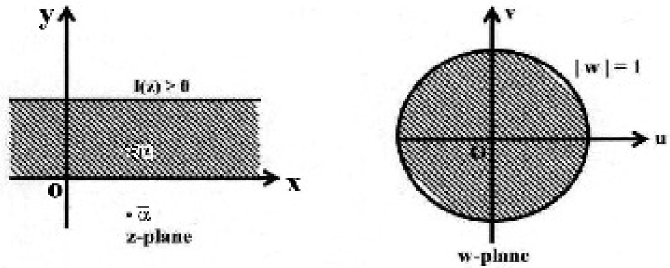
$$w = \frac{az + b}{cz + d},$$

which transforms the real axis $y = 0$ in the z-plane onto the unit circle $|w| = 1$, the points in the z-plane corresponding to $w = 0$ and $w = \infty$ are, respectively, $z = -\frac{b}{a}$ and $z = -\frac{d}{c}$. Therefore, these two points in the z-plane must be the inverse points with respect to $y = 0$. Thus, if we write $-\frac{b}{a} = \alpha$, then $-\frac{d}{c} = \bar{\alpha}$.

Special Linear Fractional Transformations: I

Theorem

Find all the linear fractional transformations which map the half-plane $\mathrm{Im}(z) \geq 0$ *onto the unit circular disc* $|w| \leq 1$.



Special Linear Fractional Transformations: I

Theorem

Find all the linear fractional transformations which map the half-plane $\mathrm{Im}(z) \geq 0$ *onto the unit circular disc* $|w| \leq 1$.

Proof.

Proof. Let the required linear fractional transformation be $w = \frac{az+b}{cz+d} \quad (ad - bc \neq 0)$, which may be written as

$$w = \frac{a}{c} \cdot \frac{z + (\frac{b}{a})}{z + (\frac{d}{c})}. \tag{39}$$

First, we note from (39) that $c \neq 0$, otherwise the point at infinity will correspond.
If $w = 0$, then (39) corresponds $z = -b/a$ and, if $w = \infty$, then (39) corresponds $z = -d/c$.
Then the points $w = 0$ and $w = \infty$, which are the inverse points with respect to the unit circle $|w| = 1$, correspond to the points $z = -b/a$ and $z = -d/c$ in the z-plane. We know that the transformation (39) transforms a straight line of the z-plane into a circle and points symmetrical about the line transform into inverse points of the circle of w-plane. Therefore, the points $z, \bar{z}$ symmetrical about the real axis, i.e., $\mathrm{Im}(z) = 0$ will correspond to $w = 0$ and $w = \infty$, respectively. Hence we may write $-\frac{b}{a} = \alpha$ and $-\frac{d}{c} = \bar{\alpha}$. Then (39) reduces to

$$w = \frac{a}{c} \cdot \frac{z - \alpha}{z - \bar{\alpha}}.$$

Since the real axis $\mathrm{Im}(z) = 0$ is to be transformed into the unit circle $|w| = 1$, the point $z = 0$ on the real axis must correspond to a point on the unit circle $|w| = 1$ so that

$$1 = |w| = \left|\frac{a}{c}\right| \left|\frac{0-\alpha}{0-\bar{\alpha}}\right| \quad \Longrightarrow \quad \left|\frac{a}{c}\right| = 1.$$

since $|\alpha| = |\alpha|$. Hence we may write $\frac{a}{c} = e^{i\lambda}$, where λ is real. Accordingly, the required transformation is

$$w = e^{i\lambda}\frac{z-\alpha}{z-\bar{\alpha}}. \tag{40}$$

Evidently, $z = \alpha$ corresponds to $w = 0$, which is an interior point of the circle $|w| = 1$, and the point $z = \alpha$ must be a point of the upper half-plane, that is, $\mathrm{Im}(\alpha) > 0$. With this restriction, (40) is the required transformation.

Verification of the linear fractional transformation

It can be easily seen that the transformation (40) maps the upper half-plane $\mathrm{Im(z)} \geq 0$ onto the unit circular disc $|w| \leq 1$ provided $I(\alpha) > 0$. For, we have

$$\begin{aligned} w\bar{w} - 1 &= e^{i\lambda}\frac{z-\alpha}{z-\bar{\alpha}} \cdot e^{-i\lambda}\frac{\bar{z}-\bar{\alpha}}{z-\alpha} - 1 \\ &= \frac{z\bar{z} - z\bar{\alpha} - \bar{z}\alpha + \alpha\bar{\alpha} - z\bar{z} + \bar{z}\bar{\alpha} - z\alpha - \alpha\bar{\alpha}}{(z-\bar{\alpha})(\bar{z}-\alpha)} \\ &= \frac{(z-\bar{z})(\alpha-\bar{\alpha})}{(z-\bar{\alpha})(\bar{z}-\alpha)} \\ &= \frac{2i\mathrm{Im(z)} \cdot 2\mathrm{iIm}(\alpha)}{|z-\bar{\alpha}|^2} \end{aligned}$$

and so

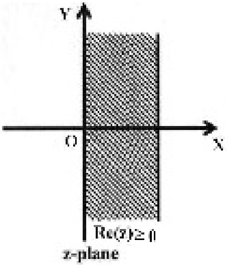
$$w\tilde{w} - 1 = -4\frac{\mathrm{Im}(z)\,\mathrm{Im}(z)}{|z-\bar{\alpha}|^2}. \tag{41}$$

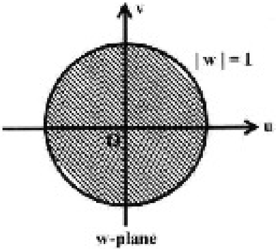
Since $\mathrm{Im}(\alpha) > 0, (41)$ shows that $\mathrm{Im(z)} = 0$ is mapped onto $w\bar{w} - 1 = 0$, i.e., onto $|w|^2 = 1$, i.e., onto $|w| = 1$ and $\mathrm{Im(z)} > 0$ is mapped onto $w\hat{w} - 1 < 0$, i.e., onto $|w| < 1$. Hence $\mathrm{Im(z)} \geq 0$ is transformed onto $|w| \leq 1$.

Special Linear Fractional Transformations: II

Theorem

Find the linear fractional transformation which transforms the halfplane $\mathrm{Re}(z) \geq 0$ *onto the unit circular disc* $|w| \leq 1$.





Special Linear Fractional Transformations: II

Theorem

Find the linear fractional transformation which transforms the halfplane $\mathrm{Re(z)} \geq 0$ *onto the unit circular disc* $|w| \leq 1$.

Proof.

Let the required linear fractional transformation be $w = \frac{az+b}{cz+d} \quad (ad - bc \neq 0)$, which may be written as

$$w = \frac{a}{c} \cdot \frac{z + \left(\frac{b}{a}\right)}{z + \left(\frac{d}{c}\right)} \tag{42}$$

Note that $c \neq 0$, otherwise, the points at infinity would correspond. Also, the points $w = 0$ and $w = \infty$, which are the inverse points with respect to the unit circle $|w| = 1$, correspond to $z = -\frac{b}{a}$ and $z = -\frac{d}{c}$ in the z-plane. By the same arguments as given in previous theorem, these points must be symmetrical with respect to the imaginary axis $\mathrm{Re(z)} = 0$. In this case, the points z and $-\bar{z}$ symmetrical about the imaginary axis $\mathrm{Re(z)} = 0$ will correspond to $w = 0$ and $w = \infty$, the inverse points of the circle $|w| = 1$. Hence we may write $-\frac{b}{a} = \alpha$, $-\frac{d}{c} = \bar{\alpha}$.
Then (42) may be written as

$$w = \frac{a}{c} \cdot \frac{z - \alpha}{z + \bar{\alpha}}.$$

□

Determination of $\frac{a}{c}$

We make use of the fact that any point on $\mathrm{Re(z)} = 0$ must correspond to a point on the unit circle $|w| = 1$ so that

$$1 = |w| = |\frac{a}{c}||\frac{0-\alpha}{0-\bar{\alpha}}| \quad \Longrightarrow \quad \left|\frac{a}{c}\right| = 1$$

since $|\alpha| = |\bar{\alpha}|$. Hence we may write $\frac{a}{c} = e^{i\lambda}$, where λ is real. Thus the required transformation is

$$w = e^{i\lambda}\frac{z-\alpha}{z+\bar{\alpha}} \tag{43}$$

Evidently, $z = \alpha$ corresponds to $w = 0$, which is the center of the circle $|w| = 1$. Hence $z = \alpha$ must be a point of the right half-plane, i.e., $\mathrm{Re}(\alpha) = 0$. With this restriction, (43) is the required transformation.

Verification of the linear fractional transformation

It can be easily seen that the transformation (43) maps the right half-plane $\mathrm{Re(z)} \geq 0$ onto the unit circular disc $|w| \leq 1$ provided $\mathrm{Re}(\alpha) > 0$. For, we have

$$
\begin{aligned}
w\bar{w} - 1 &= e^{i\lambda}\frac{z-\alpha}{z-\bar{\alpha}} \cdot e^{-i\lambda}\frac{\bar{z}-\bar{\alpha}}{\bar{z}+\alpha} - 1 \\
&= \frac{z\bar{z} - z\bar{\alpha} - \bar{z}\alpha + \alpha\bar{\alpha} + z\bar{z} - z\alpha - \bar{z}\bar{\alpha} - \alpha\bar{\alpha}}{(z+\bar{\alpha})(\bar{z}+\alpha)} \\
&= -\frac{(z+\bar{z})(\alpha+\bar{\alpha})}{|z+\bar{\alpha}|^2} \\
&= -\frac{2\mathrm{Re(z)} \cdot 2\mathrm{Re}(\alpha)}{|z+\bar{\alpha}|^2}
\end{aligned}
$$

and so

$$
1 - |w|^2 = -4\frac{\mathrm{Re(z)}\mathrm{Re}(\alpha)}{|z+\bar{\alpha}|^2}. \tag{44}
$$

Since $\mathrm{Re}(\alpha) > 0$, (44) shows that $\mathrm{Re(z)} = 0$ is transformed onto $1 - |w|^2 = 0$, i.e., onto $|w| = 1$ and $\mathrm{Re(z)} > 0$ is transformed onto $1 - |w|^2 > 0$, i.e., onto $|w| < 1$. Hence $\mathrm{Re(z)} \geq 0$ is transformed onto $|w| \leq 1$.

Special Linear Fractional Transformations: III

Theorem

Find all the linear fractional transformations which transform the unit circle $|z| \leq 1$ *onto the unit circular disc* $|w| \leq 1$.

Special Linear Fractional Transformations: III

Theorem

Find all the linear fractional transformations which transform the unit circle $|z| \leq 1$ onto the unit circular disc $|w| \leq 1$.

Proof.

Let the required transformation be $w = \frac{az+b}{cz+d} \quad (ad - bc \neq 0)$, which may be written as

$$w = \frac{a}{c} \cdot \frac{z + (\frac{b}{a})}{z + (\frac{d}{c})}. \tag{45}$$

Clearly, $c \neq 0$, otherwise, the points at infinity in the two planes would correspond. Hence the points $w = 0$ and $w = \infty$, which are the inverse points with respect to the circle $|w| = 1$ correspond to $z = -\frac{b}{a}$ and $z = -\frac{d}{c}$, respectively.
Therefore, the points $-\frac{b}{a}$ and $-\frac{d}{c}$ must be the inverse points with respect to the circle $|z| = 1$ and so we may write $-\frac{b}{a} = \alpha$, $-\frac{d}{c} = \frac{1}{\alpha}$.
Then (45) may be written as

$$w = \frac{a}{c} \cdot \frac{z - \alpha}{z - (1/\bar{\alpha})} = \frac{a\bar{\alpha}}{c} \cdot \frac{z - \alpha}{\bar{\alpha}z - 1}. \tag{46}$$

Determination of $\frac{a}{c}$

To find $\frac{a}{c}$, we may use the fact that any point on $|z| = 1$ must correspond to a point on $|w| = 1$. In particular, the point $z = 1$ on the boundary of $|z| = 1$ must correspond to a point on the boundary of $|w| = 1$. Hence, putting $z = 1$ in (46), we have

$$1 = |w| = \left|\frac{a\bar{z}}{c}\right| \cdot \left|\frac{1-\alpha}{\bar{\alpha}-1}\right| = \left|\frac{a\bar{\alpha}}{c}\right|$$

since $|1-\alpha| = |\bar{\alpha}-1|$. Hence we may write $\frac{a\bar{\alpha}}{c} = e^{i\lambda}$, where λ is real. Thus the transformation may be written as

$$w = e^{i\lambda}\frac{z-\alpha}{\bar{\alpha}z-1}. \tag{47}$$

Also, $z = \alpha$ gives $w = 0$ by (47). Hence α must be an interior point of the circle $|z| = 1$, i.e., $|\alpha| < 1$. With this restriction, (47) is the desired transformation.

Verification of the linear fractional transformation

It can be easily verified that the transformation (47) maps $|z| \le 1$ onto $|w| \le 1$. For, if $|z| = 1$; i.e., $z\bar{z} = 1$, then (47) gives

$$|w| = \left|e^{i\lambda}\right| \frac{z-\alpha}{\bar{\alpha}z-1}| = |\frac{z-\alpha}{\bar{\alpha}z-z\bar{z}}| \implies |w| = \frac{a}{|z|}\left|\frac{z-\alpha}{\bar{z}-\tilde{\alpha}}\right| - 1$$

since $|z| = 1$ and $|z-\alpha| = |\bar{z}-\bar{\alpha}|$. Now, we examine the correspondence between the interiors and exterior of the circle $|z| = 1$ and $|w| = 1$. Now, we have

$$\begin{aligned} w\bar{w} - 1 &= e^{i\lambda}\frac{z-\alpha}{\bar{\alpha}z-1} \cdot e^{-i\lambda}\frac{\bar{z}-\bar{\alpha}}{\alpha\bar{z}-1} - 1 \\ &= \frac{z\bar{z} - z\bar{\alpha} - \bar{z}\alpha + \alpha\bar{\alpha} - \alpha\bar{\alpha}\cdot z\bar{z} + z\bar{\alpha} + \alpha\bar{z} - 1}{(\bar{\alpha}z-1)(\alpha\bar{z}-1} \\ &= -\frac{(z+\bar{z}-1)(1-\alpha+\bar{\alpha})}{|\bar{\alpha}z-1|^2} \end{aligned}$$

and so

$$|w|^2 - 1 = \frac{\left(|z|^2-1\right)\left(1-|\alpha|^2\right)}{|\bar{\alpha}z-1|^2}. \tag{48}$$

Now, (48) shows that the transformation (47) maps $|z| < 1$ on $|w| < 1$ or on $|w| > 1$ conformally if $|\alpha| < 1$ or $|\alpha| > 1$. Hence it shows that the transformation (47) maps $|z| \le 1$ onto $|w| \le 1$ provided $|\alpha| < 1$.

Special Linear Fractional Transformations: IV

Theorem

Find the general linear fractional transformation which maps the circular disc $|z| \leq \rho$ onto the circular disc $|w| \leq \rho'$ and show that it can be put in the form

$$w = \rho\rho' e^{i\lambda} \frac{z - a}{\bar{\alpha} z - \rho^2}. \quad \{|\alpha| < \rho\}$$

Special Linear Fractional Transformations: IV

Theorem

Find the general linear fractional transformation which maps the circular disc $|z| \leq \rho$ onto the circular disc $|w| \leq \rho'$ and show that it can be put in the form

$$w = \rho\rho' e^{i\lambda} \frac{z - a}{\bar{\alpha}z - \rho^2}. \quad \{|\alpha| < \rho\}$$

Proof.

Let the required transformation be $w = \frac{az+b}{cz+d}$ $(ad - bc \neq 0)$, which can be put in the form of

$$w = \frac{a}{c} \cdot \frac{z + (\frac{b}{a})}{z + (\frac{d}{c})}. \tag{49}$$

Clearly, the point $w = 0$ and $w = \infty$ correspond to $z = -\frac{b}{a}$ and $z = -\frac{d}{c}$, respectively. Again, since the point 0 and ∞ are the inverse points with respect to the circle $|w| = \rho'$, the points $-\frac{b}{a}$ and $-\frac{d}{c}$ must be the inverse points with respect to the circle $|z| = \rho$ and so we may write $-\frac{b}{a} = \alpha$, $-\frac{d}{c} = \frac{\rho^2}{\bar{\alpha}}$ $(|\alpha| < \rho)$. Then the transformation (49) becomes

$$w = \frac{a}{c} \cdot \frac{z - \alpha}{z - \rho/\bar{\alpha})} = \frac{a\bar{\alpha}}{c} \cdot \frac{z - \alpha}{\bar{\alpha}z - \rho^2}. \tag{50}$$

Clearly, (50) satisfies the condition $|z| \leq \rho$ and $|w| \leq \rho'$. Hence, for $|z| = \rho$, i.e., $z\bar{z} = \rho^2$, we must have $|w| = \rho'$. Then, from (50), it follows that

$$\rho' = |w| = \left|\frac{a\bar{\alpha}}{c}\right| \cdot \left|\frac{z-\alpha}{\bar{\alpha}z - z\bar{z}}\right| = \left|\frac{a\bar{\alpha}}{c}\right| \left|\frac{z-\alpha}{\bar{\alpha}-\bar{z}}\right| \left|\frac{1}{z}\right| = \left|\frac{a\bar{\alpha}}{c}\right| \cdot \frac{1}{\rho}$$

since $|z-\alpha| = \overline{|z-\alpha|} = |\bar{z}-\bar{\alpha}| = |\bar{\alpha}-\bar{z}|$ and $|z| = \rho$ or

$$\left|\frac{a\bar{\alpha}}{c}\right| = \rho\rho'.$$

Hence we may write $\frac{a\hat{a}}{c} = \rho\rho' e^{i\lambda}$, where λ is real. Putting these in (50), we get

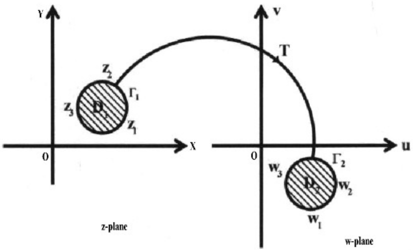
$$w = \rho\rho' e^{i\lambda} \frac{z-\alpha}{\bar{\alpha}z - \rho^2},$$

which is the desired transformation under the restriction $|\alpha| < \rho$.

Let Γ_1 and Γ_2 be any two circles or straight lines, and D_1 be one of the domains in which the complex plane is divided by Γ_1 and D_2 be one of the two domains with boundary Γ_2, so that D_1 is either a half-plane, the interior of a circle, or the exterior of a circle, and the same is true for D_2. Then we can find a linear fractional transformation $T : D_1 \to D_2$ by the following rule:

Let us choose any three distinct points z_1, z_2, z_3 on Γ_1 and let an observer moving along Γ_1 in the direction from z_1 to z_3 through z_2 find the domain D_1 on his left. Next, we choose a triple of distinct points w_1, w_2, w_3 and Γ_2 such that an observer moving along Γ_2 in the direction from w_1 to w_3 through w_2 also finds the domain D_2 on his left. However, the points w_1, w_2, w_3 are arbitrary. Then, the required linear fractional transformation can be given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)\,.$$



Special Linear Fractional Transformations: V

Theorem

Find a linear fractional transformation of the upper half-plane $\mathrm{Im}(z) > 0$ *onto the interior* $|w| < 1$ *of the unit circle* $|w| = 1$.

Special Linear Fractional Transformations: V

Theorem

Find a linear fractional transformation of the upper half-plane $\mathrm{Im}(z) > 0$ *onto the interior* $|w| < 1$ *of the unit circle* $|w| = 1$.

Proof.

Let us choose three points $z_1 = -1, z_2 = 0, z_3 = 1$ on the real axis so that the upper half $\mathrm{Im}(z) > 0$ is on the left of an observer moving along the real axis in the direction from z_1 to z_3 through z_2. We then choose three points w_1, w_2, w_3 on the circle $|w| = 1$ in the direction from w_1 to w_3 through w_2. For the sake o f simplicity, we choose $w_1 = 1, w_2 = -1$, and $w_3 = -i$. Then the desired linear fractional transformation is given by

$$(w, 1, -1, -i) = (z, -1, 0, 1)$$

or

$$\frac{(w-1)(-1+i)}{(1+1)(-i-w)} = \frac{(z+1)(0-1)}{(-1-0)(1-z)}.$$

Solving for w, we have

$$w = \frac{3-4i}{5} \cdot \frac{z + \frac{1-2i}{5}}{z + \frac{1+2i}{5}}.$$

□

Special transformation I: $w = z^2$

Writing $z = x + iy$ and $w = u + iv$, and substituting these values in the given transformation, we have

$$u + iv = (x + iy)^2 = x^2 - y^2 + 2ixy \implies u = x^2 - y^2, \quad v = 2xy.$$

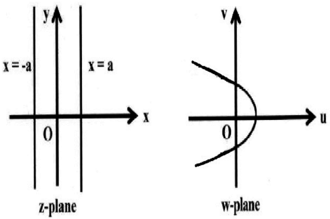
Case I: Consider the image of the line $x = a$. The line $x = a$ is transformed into the curve

$$u = a^2 - y^2, \quad v = 2ay$$

where y is regarded as a parameter. Eliminating y, we have

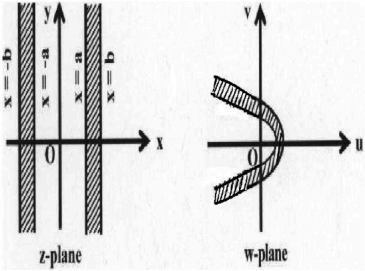
$$u = a^2 - \frac{v^2}{4a^2} \quad \text{or} \quad v^2 = -4a^2(u - a^2),$$

which is a parabola in the w-plane with its vertex at $(a^2, 0)$, focus at the origin, and symmetrical about the real axis. Similarly, the line $x = a$ is also transformed into the same parabola.

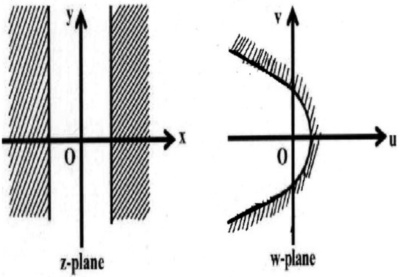


Special transformation I: $w = z^2$

Case II: Show that the strip between the lines $x = a$ and $x = b$ in the z-plane corresponds to the domain lying between the two parabola in the w-plane, where $a, b > 0$ and $b > a$.



(a) Transformed image of the strip $a \leq x \leq b$.

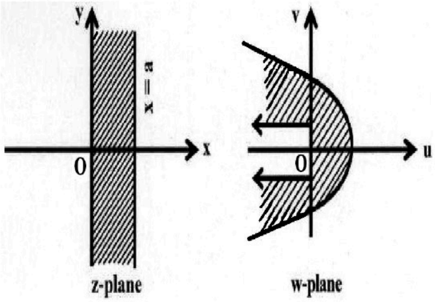


(b) Transformed image of the strip $a \leq x < \infty$.

Special transformation I: $w = z^2$

Case III: Show that the domain in the z-plane between the lines $x = 0$ and $x = a$ is mapped upon the interior of the parabola in the w-plane with a slit along the negative real axis.

Consider $x = a$ and make a tend to zero. We can easily see that the domain $0 < x < a$ is conformally mapped onto the whole interior of the parabola $v^2 = -4a^2(u - a^2)$ with a slit along the negative real axis from $-\infty$ to 0.



Remark: It is evident that the half-plane $\mathrm{Im}(z) > 0$ is mapped on the whole w-plane cut along the positive real axis from 0 to infinity. Similarly, the half-plane $\mathrm{Im}(z) < 0$ is also mapped on the same cut plane.

Special transformation I: $w = z^2$

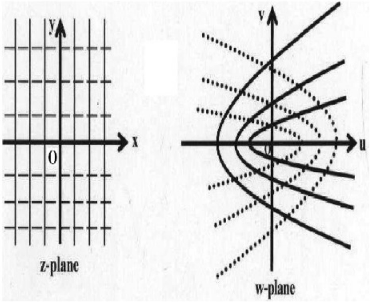
Case IV: Observe that the two families of parabolas

$$v^2 = -4a^2(u - a^2), v^2 = 4c^2(u + c^2)$$

correspond to two orthogonal families of the straight lines

$$x = \pm a, y = \pm c$$

for varying values of the parameters a and c for an orthogonal family or net.



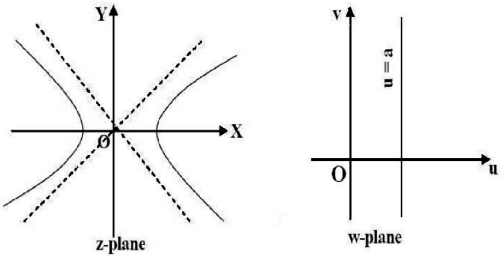
Special transformation II: $z = \sqrt{w}$

The transformation $z = \sqrt{w}$ is the inverse transformation of $w = z^2$ in which we considered the images of the straight lines $x = a$ and $y = c$ in the z-plane. We now consider the images of the lines $u = a, v = c$ in the w-plane under the same transformation. Since $z = x + iy$ and $w = u + iv$, we have $(x + iy)^2 = u + iv$. From this equation, we obtain

$$x^2 - y^2 = u, 2xy = v.$$

Case I: Consider the image of the region $a \leq \mathrm{Re}(w) \leq b(b > a > 0)$. The straight line $u = a > 0$ is transformed into the rectangular hyperbola $x^2 - y^2 = a$ whose asymptotes lie along the lines $y = \pm x$ and show transverse and conjugate axes lie respectively along the z-axis.

It is evident that either branch of the hyperbola is the complete transform of the line and depends on the particular branch of $\sqrt{w}$ chosen.

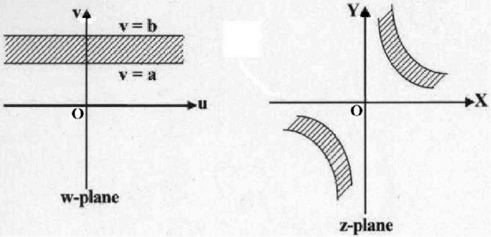


Special transformation II: $z = \sqrt{w}$

Case II: Consider the image of the region

$$a \leq \mathrm{Im}(z) \leq b \quad (b > a > 0),$$

i.e., the image of the infinite strip between the lines $v = a$ and $v = b$ in the w-plane.

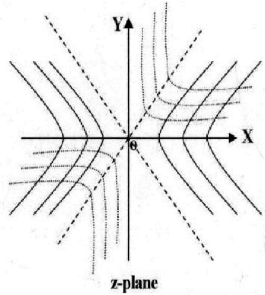


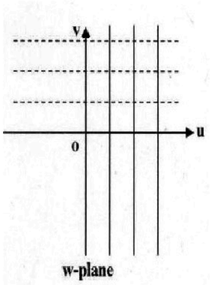
Special transformation II: $z = \sqrt{w}$

Case III: It is well known that the two families of rectangular hyperbolas discussed in Cases I and II form an orthogonal system. We have also observed that the two families are the transforms of the families of straight lines parallel to u-axis and v-axis, respectively. Now, consider the region bounded by

$$u = -a, u = -b, v = -c, v = -d,$$

where $b > a > 0$ and $d > c > 0$.





Special transformation III: $w = e^z$

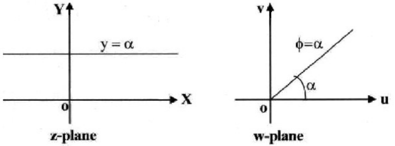
Observe that this transformation is conformal for every value of z since $\frac{dw}{dz} = e^z \neq 0$ anywhere on the z-plane. Writing $z = x + iy$ and $w = Re^{i\phi}$, the given transformation can be written as

$$Re^{i\phi} = e^{x+iy} = e^x \cdot e^{iy} \quad \Longrightarrow \quad R = e^x, \quad \phi = y.$$

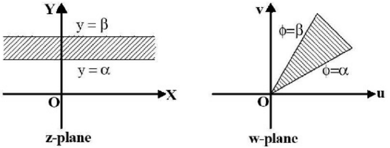
Consider the image of the line $y = \alpha$ $(0 < \alpha < 2\phi)$ in the z-plane. When $e^x (= R)$ increases from 0 to ∞ monotonically as x takes values from $-\infty$ to ∞, the line $y = \alpha$ in the z-plane is mapped onto the ray including the origin in the w-plane. Moreover, it can easily be shown that infinite strip bounded by the lines $y = \alpha$ and $y = \beta$, i.e., the region

$$\alpha < \mathrm{Im}(z) < \beta \quad (\beta - \alpha < \pi)$$

is transformed into the wedge shaded region in the w-plane bounded by the radial lines $\phi = \alpha$ and $\phi = \beta$.



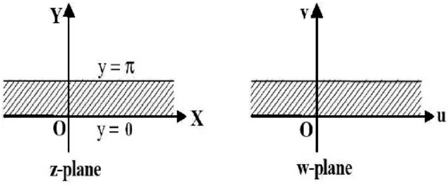
(a) Transformed image of the line $y = \alpha$.



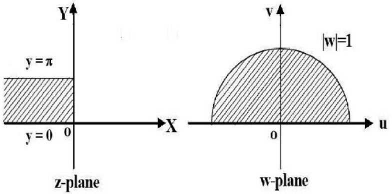
(b) Transformed image of the strip $\alpha \leq y \leq \beta$.

Special transformation III: $w = e^z$

In particular, the infinite strip bounded by the lines $y = 0$ and $y = \pi$ in the z-plane is mapped into the upper half of the w-plane. Also, the infinite strip on the negative side of z-axis; i.e., $0 \le y \le \pi$, $-\infty \le x \le 0$ is mapped on the unit semi-circle $|w| = 1$, $0 \le \phi \le \pi$ in the upper half-plane.



(a) $0 \le y \le \pi$.



(b) $0 \le y \le \pi$, $-\infty \le x \le 0$.