# Methods of Mathematical Physics

## —Lecture 4 Taylor and Laurent series—

Lei Du
dulei@dlut.edu.cn
http://faculty.dlut.edu.cn/dulei

School of Mathematical Sciences
Dalian University of Technology

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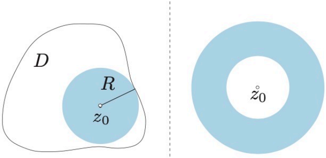
# Contents

# Introduction

In this lecture we will first learn about the fundamental result which says that a holomorphic function $f(z)$ has a power series expansion around any point in the domain D where it lives. See the picture on the left below.




Taylor series: $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ Laurent series: $\sum_{n\in\mathbb{Z}} c_n(z-z_0)^n$

In the second part of this lecture, we will learn about Laurent series, which are like power series, except that negative integer powers of the terms $zz_0$ also occur in the expansion. This will be useful to study functions that are holomorphic in annuli (and in particular punctured discs). See the picture on the right above. They are also useful to classify "singularities", and to evaluate some real integrals.

# Definition

Just like with real series, given a sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers, one can form a new sequence $(s_n)_{n\in\mathbb{N}}$ of its partial sums:

$$\begin{aligned} s_1 &:= a_1, \\ s_2 &:= a_1 + a_2, \\ s_3 &:= a_1 + a_2 + a_3, \\ &\vdots \end{aligned}$$

**Definition**

1. The series $\sum_{n=1}^{\infty} a_n$ converges if $(s_n)_{n\in\mathbb{N}}$ converges, and $\sum_{n=1}^{\infty} a_n := \lim_{n\to\infty} s_n$.
2. The series $\sum_{n=1}^{\infty} a_n$ diverges if $(s_n)_{n\in\mathbb{N}}$ diverges.
3. $\sum_{n=1}^{\infty} a_n$ converges absolutely if the real series $\sum_{n=1}^{\infty} |a_n|$ converges.

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**Definition**

1. The series $\sum\limits_{n=1}^{\infty} a_n$ converges if $(s_n)_{n\in\mathbb{N}}$ converges, and $\sum\limits_{n=1}^{\infty} a_n := \lim\limits_{n\to\infty} s_n$.
2. The series $\sum\limits_{n=1}^{\infty} a_n$ diverges if $(s_n)_{n\in\mathbb{N}}$ diverges.
3. $\sum\limits_{n=1}^{\infty} a_n$ converges absolutely if the real series $\sum\limits_{n=1}^{\infty} |a_n|$ converges.

**Theorem**

*A complex sequence converges if and only if the sequences of its real and imaginary parts converge,* $\sum\limits_{n=1}^{\infty} a_n$ *converges* $\Leftrightarrow$ *the real series* $\sum\limits_{n=1}^{\infty} \mathrm{Re}\,(a_n)$ *and* $\sum\limits_{n=1}^{\infty} \mathrm{Im}\,(a_n)$ *converge.*

# Properties of series

Thus the results from real analysis lend themselves for use in testing the convergence of complex series. For example, it is easy to prove the following two facts.

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
2. If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

# Power series and their region of convergence

**Definition**

Let $(c_n)_{n\in\mathbb{N}}$ be a complex sequence (thought of as a sequence of "coefficients"). An expression of the type

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a power series in the complex variable $z$.

# Power series and their region of convergence

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is called a power series in the complex variable $z$.

- All polynomial expressions are power series, with only finitely many nonzero coefficients. Polynomials converge for all $z \in \mathbb{C}$.
- The power series $\sum\limits_{n=0}^{\infty} z^n$ converges whenever $|z| < 1$, but diverges if $|z| \geq 1$.

# Power series and their region of convergence

A fundamental question is: For what values of $z \in \mathbb{C}$ does the power series $\sum_{n=0}^{\infty} c_n z^n$ converge?

The following result gives the answer to this question.

**Theorem**

*For $\sum_{n=0}^{\infty} c_n z^n$, exactly one of the following hold:*

1. *Either it is absolutely convergent for all $z \in \mathbb{C}$.*
2. *Or there is a unique nonnegative real number $R$ such that*
   - $\sum_{n=0}^{\infty} c_n z^n$ *is absolutely convergent for all $z \in \mathbb{C}$ with $|z| < R$, and*
   - $\sum_{n=0}^{\infty} c_n z^n$ *is divergent for all $z \in \mathbb{C}$ with $|z| > R$.*

(The unique $R > 0$ in the above theorem is called the radius of convergence of the power series, and if the power series converges for all $z \in \mathbb{C}$, we say that the power series has an infinite radius of convergence, and write "$R = \infty$".)

The calculation of the radius of convergence is facilitated in some cases by the following result.

**Theorem**

*Consider the power series*

$$\sum_{n=0}^{\infty} c_n z^n.$$

*If* $\rho := \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$ *exists, then*

- *the radius of convergence is* $1/\rho$*, i.e.,* $R = 1/\rho$ *if* $\rho \neq 0$*.*
- *the radius of convergence is infinite, i.e.,* $R = \infty$ *if* $\rho = 0$*.*

# Power series and their region of convergence

**Theorem**

*Consider the power series* $\sum_{n=0}^{\infty} c_n x^n$. *If* $L := \lim_{n\to\infty} \sqrt[n]{|c_n|}$ *exists, then*

- *the radius of convergence is* $1/L$ *if* $L \neq 0$.
- *the radius of convergence is infinite if* $L = 0$.

# Power series are holomorphic

We have seen that polynomials are power series with an infinite radius of convergence. They are of course also holomorphic there. More generally, that a power series

$$f(z) := \sum_{n=0}^{\infty} c_n z^n$$

that converges for $|z| < R$ is actually holomorphic there, and for $|z| < R$, there holds that $f'(z) = \frac{d}{dz}\left(c_0 + c_1 z + c_2 z^2 + \cdots\right) = c_1 + 2c_2 z + 3c_3 z^2 + \cdots = \sum_{n=1}^{\infty} c_n n z^{n-1}$.

**Theorem**

*Let* $R > 0$ *and* $f(z) := \sum_{n=0}^{\infty} c_n z^n$ *converge for* $|z| < R$. *Then* $f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1}$ *for* $|z| < R$.

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**Theorem**

*Let $R > 0$ and $f(z) := \sum_{n=0}^{\infty} c_n z^n$ converge for $|z| < R$. Then $f'(z) = \sum_{n=1}^{\infty} n c_n z^{n-1}$ for $|z| < R$.*

By a repeated application of the previous result, we have the following.

**Corollary**

*Let $R > 0$ and let $f(z) := \sum_{n=0}^{\infty} c_n z^n$ converge for $|z| < R$. Then for $k \geq 1$,*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)c_n z^{n-k} \text{ for } |z| < R.$$

*In particular, for $n \geq 0$, $c_n = \frac{1}{n!} f^{(n)}(0)$.*

# Power series are holomorphic

There is nothing special about taking power series centered at 0. One can also consider

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n ,$$

where $z_0$ is a fixed complex number. The following results follow immediately from previous theorems.

**Corollary**

*For $\sum_{n=0}^{\infty} c_n (z - z_0)^n$, exactly one of the following hold:*

1. *Either it is absolutely convergent for all $z \in \mathbb{C}$.*
2. *Or there is a unique nonnegative real number $R$ such that*
   - $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ *is absolutely convergent for $|z - z_0| < R$, and*
   - $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ *is divergent for $|z - z_0| > R$.*

# Power series are holomorphic

**Corollary**

*Let* $z_0 \in \mathbb{C}, R > 0$ *and* $f(z) := \sum_{n=0}^{\infty} c_n (z - z_0)^n$ *converge for* $|z - z_0| < R$. *Then*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)c_n (z - z_0)^{n-k} \text{ for } |z - z_0| < R, k \geq 1.$$

*In particular, for* $n \geq 0, c_n = \frac{1}{n!} f^{(n)}(z_0)$.

# Taylor series

We have seen in the last section that complex power series

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

are holomorphic in their region of convergence $|z - z_0| < R$, where $R$ is the radius of convergence. In this section, we will show that conversely, if $f$ is holomorphic in the disc $|z - z_0| < R$, then

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \text{ whenever } |z - z_0| < R,$$

where the coefficients can be determined from the $f$. Thus every holomorphic function $f$ defined in a domain $D$ possesses a power series expansion in a disc around any point $z_0 \in D$.

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where the coefficients can be determined from the $f$. Thus every holomorphic function $f$ defined in a domain $D$ possesses a power series expansion in a disc around any point $z_0 \in D$.

**Theorem**

*If $f$ is holomorphic in $D(z_0, R) := \{z \in \mathbb{C} : |z - z_0| < R\}$, then*
*$f(z) = c_0 + c_1 (z - z_0) + c_2 (z - z_0)^2 + c_3 (z - z_0)^3 + \cdots$ for $z \in D(z_0, R)$, where for $n \geq 0$,*

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

*and $C$ is the circular path with center $z_0$ and radius $r$, where $0 < r < R$ traversed in the anticlockwise direction.*

# Taylor Series

**Corollary (Taylor Series)**

*If*

1. $D$ *be a domain,*
2. $f : D \to \mathbb{C}$ *is holomorphic, and*
3. $z_0 \in D$,

*then*

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots, \quad |z - z_0| < R,$$

*where $R$ is the radius of the largest open disk with center $z_0$ contained in $D$. Also,*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

*where $C$ is the circular path with center $z_0$ and radius $r$, where $0 < r < R$ traversed in the anticlockwise direction.*

# Laurent Series

Laurent series generalize Taylor series. Indeed, while a Taylor series

$$\sum_{n=0}^{\infty} c_n (z - z_0)^n$$

has nongenative powers of the term $z - z_0$, and converges in a disc, a Laurent series is an expression of the type

$$\sum_{n\in\mathbb{Z}} c_n (z - z_0)^n = \cdots + c_{-1} (z - z_0)^{-1} + c_0 + c_1 (z - z_0)^1 + \cdots$$

which has negative powers of $z - z_0$ too.

# Laurent Series

For example, we know that for all $z \in \mathbb{C}$,

$$\exp z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

and so for $z \neq 0$, we have the "Laurent series expansion"

$$\exp\frac{1}{z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{3!} + \cdots .$$

Note that $\exp(1/z)$ is holomorphic in $\mathbb{C}\backslash\{0\}$, which is a degenerate annulus centered at 0 with inner radius $r = 0$ and outer radius $R = +\infty$!

# Laurent Series

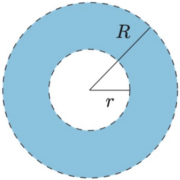
Let us first define what we mean by the convergence of $\sum_{n\in\mathbb{Z}} c_n (z-z_0)^n$.

**Definition**

The Laurent series $\sum_{n\in\mathbb{Z}} c_n (z-z_0)^n$ converges (for $z$ ) if $\sum_{n=1}^{\infty} c_{-n} (z-z_0)^{-n}$ converges and $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ converges. If $\sum_{n\in\mathbb{Z}} c_n (z-z_0)^n$ converges, then we write

$$\sum_{n\in\mathbb{Z}} c_n (z-z_0)^n = \sum_{n=1}^{\infty} c_{-n} (z-z_0)^{-n} + \sum_{n=0}^{\infty} c_n (z-z_0)^n$$

and call this the sum of the Laurent series.



# For what $z$ does $\sum\limits_{n\in\mathbb{Z}} c_n (z-z_0)^n$ converge?

- For $\sum\limits_{n=0}^{\infty} c_n (z-z_0)^n$, there is some $R$ such that it converges for $|z-z_0|<R$ and diverges for $|z-z_0|>R$.
- What about the series $\sum\limits_{n=1}^{\infty} c_{-n} (z-z_0)^{-n}$?

  Set $w := (z-z_0)^{-1}$. $\sum\limits_{n=1}^{\infty} c_{-n} w^n$ also converges for $|w|<\widetilde{R}$ and diverges for $|w|>\widetilde{R}$.

  Then $\sum\limits_{n=1}^{\infty} c_{-n} w^n$ converges when $1/|z-z_0|<\widetilde{R}$, that is for $|z-z_0|>1/\widetilde{R}=: r$, and diverges for $|z-z_0|<r$.

# For what $z$ does $\sum_{n\in\mathbb{Z}} c_n (z-z_0)^n$ converge?

- For $\sum_{n=0}^{\infty} c_n (z-z_0)^n$, there is some $R$ such that it converges for $|z-z_0| < R$ and diverges for $|z-z_0| > R$.
- What about the series $\sum_{n=1}^{\infty} c_{-n} (z-z_0)^{-n}$?

  Set $w := (z-z_0)^{-1}$. $\sum_{n=1}^{\infty} c_{-n} w^n$ also converges for $|w| < \widetilde{R}$ and diverges for $|w| > \widetilde{R}$.

  Then $\sum_{n=1}^{\infty} c_{-n} w^n$ converges when $1/|z-z_0| < \widetilde{R}$, that is for $|z-z_0| > 1/\widetilde{R} =: r$, and diverges for $|z-z_0| < r$.

Hence the Laurent series converges in the annulus $\{z \in \mathbb{C} : r < |z-z_0| < R\}$ and diverges if either $|z-z_0| < r$ or $|z-z_0| > R$. We will also see that

1. Laurent series "converge" in an annulus $\{z \in \mathbb{C} : r < |z-z_0| < R\}$ with center $z_0$ and gives a holomorphic function there, and
2. conversely, if we have a holomorphic function in an annulus with center $z_0$ and it has singularities that lie in the "hole" inside the annulus, then the function has a Laurent series expansion in the annulus.

# Is it holomorphic in the annulus where it converges?

1. $z \mapsto \sum_{n=0}^{\infty} c_n (z-z_0)^n$ is holomorphic in $\{z \in \mathbb{C} : |z-z_0| < R\}$ and so in particular, also in $\{z \in \mathbb{C} : r < |z-z_0| < R\}$.
2. The map

$$w \xrightarrow{g} \sum_{n=1}^{\infty} c_{-n} w^n$$

is holomorphic in $\{w \in \mathbb{C} : |w| < \widetilde{R}\}$. Also the mapping $z \xrightarrow{f} (z-z_0)^{-1} : \mathbb{C} \setminus \{z_0\} \to \mathbb{C}$ is holomorphic. So their composition $g \circ f$ is holomorphic in $\{z \in \mathbb{C} : |z-z_0| > r\}$, that is,

$$z \xrightarrow{g \circ f} \sum_{n=1}^{\infty} c_{-n} (z-z_0)^{-n}$$

is holomorphic in $\{z \in \mathbb{C} : r < |z-z_0|\}$, and so also in particular in the annulus $\{z \in \mathbb{C} : r < |z-z_0| < R\}$.

# Is it holomorphic in the annulus where it converges?

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2. The map

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is holomorphic in $\{w \in \mathbb{C} : |w| < \widetilde{R}\}$. Also the mapping $z \xmapsto{f} (z-z_0)^{-1} : \mathbb{C} \setminus \{z_0\} \to \mathbb{C}$ is holomorphic. So their composition $g \circ f$ is holomorphic in $\{z \in \mathbb{C} : |z-z_0| > r\}$, that is,

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is holomorphic in $\{z \in \mathbb{C} : r < |z-z_0|\}$, and so also in particular in the annulus $\{z \in \mathbb{C} : r < |z-z_0| < R\}$.

Summarizing, we have learnt that any Laurent series $\sum_{n \in \mathbb{Z}} c_n (z-z_0)^n$ converges in an annulus $\{z \in \mathbb{C} : r < |z-z_0| < R\}$ for some $r, R$, and the map $z \mapsto \sum_{n \in \mathbb{Z}} c_n (z-z_0)^n$ is holomorphic in $\{z \in \mathbb{C} : r < |z-z_0| < R\}$.

# A simple example

**Example**

For what $z \in \mathbb{C}$ does the Laurent series

$$\cdots + \frac{1}{8z^3} + \frac{1}{4z^2} + \frac{1}{2z} + 1 + z + z^2 + z^3 + \cdots$$

converge? We have:

1. $1 + z + z^2 + z^3 + \cdots$ converges for $|z| < 1$, and it diverges for $|z| > 1$.
2. $\frac{1}{2z} + \frac{1}{4z^3} + \frac{1}{8z^5} + \cdots$ converges for $\left|\frac{1}{2z}\right| < 1$ and diverges for $\left|\frac{1}{2z}\right| > 1$, that is, it converges for $|z| > 1/2$ and diverges for $|z| < 1/2$.

Hence the given Laurent series converges when $|z| < 1$ and $|z| > 1/2$, that is, it converges inside the annulus $\{z \in \mathbb{C} : 1/2 < |z| < 1\}$, and it diverges when $|z| > 1$ or when $|z| < 1/2$.

# Is it holomorphic in the annulus where it converges?

That conversely, a function holomorphic in an annulus has a Laurent series expansion is the content of the following theorem.

**Theorem**

*If f is holomorphic in* $\mathbb{A} := \{z \in \mathbb{C} : r < |z - z_0| < R\}$ *then*

$$f(z) = \sum_{n \in \mathbb{Z}} c_n (z - z_0)^n \qquad \text{for } z \in \mathbb{A} \tag{1}$$

*where*

1. $c_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$,
2. $C$ *is the circular path given by* $C(t) = z_0 + \rho \exp(it), t \in [0, 2\pi]$,
3. $\rho$ *is any number such that* $r < \rho < R$.

*Moreover, the coefficients are unique in (1).*

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3. *$\rho$ is any number such that $r < \rho < R$.*

*Moreover, the coefficients are unique in (1).*

Note that the uniqueness of coefficients is valid only if we consider a particular fixed annulus. It can happen that the same function has different Laurent expansions, but valid in different annuli.