Methods of Mathematical Physics

— Lecture 7 — Fourier & Laplace Transformations

Lei Du
dulei@dlut.edu.cn
http://faculty.dlut.edu.cn/dulei

School of Mathematical Sciences
Dalian University of Technology

May, 2023

Contents

1 **Fourier transformation**

2 Laplace transformation

Fourier integral

The method of Fourier transformation can be used for solving the problems in the unbounded region. Fourier transformation have been evolved from Fourier series on the finite (region) interval. Consider the Fourier series of the function $f(x)$ on $[-l, l]$:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{l} x + b_n \sin \frac{n\pi}{l} x \right), \tag{1}$$

where

$$\begin{cases} a_n = \frac{1}{l} \int_{-l}^{l} f(\xi) \cos \frac{n\pi}{l} \xi d\xi, & n = 0, 1, 2, \cdots \\ b_n = \frac{1}{l} \int_{-l}^{l} f(\xi) \sin \frac{n\pi}{l} \xi d\xi, & n = 1, 2, \cdots \end{cases} \tag{2}$$

Substituting (2) for (1), we have

$$\begin{aligned} f(x) = & \frac{1}{2l} \int_{-l}^{l} f(\xi) d\xi + \sum_{n=1}^{\infty} \frac{1}{l} \left\{ \left(\int_{-l}^{l} f(\xi) \cos \frac{n\pi}{l} \xi d\xi \right) \cos \frac{n\pi}{l} x \right. \\ & \left. + \left(\int_{-l}^{l} f(\xi) \sin \frac{n\pi}{l} \xi d\xi \right) \sin \frac{n\pi}{l} x \right\} \\ = & \frac{1}{2l} \int_{-l}^{l} f(\xi) d\xi + \sum_{n=1}^{\infty} \frac{1}{l} \left\{ \int_{-l}^{l} f(\xi) \cos \frac{n\pi}{l} (\xi - x) d\xi \right\}. \end{aligned} \tag{3}$$

Fourier integral

Assume that $f(x)$ is absolutely integrable in $(-\infty, +\infty)$, namely $\int_{-\infty}^{+\infty} |f(\xi)| d\xi < +\infty$. Then

$$\frac{|a_0|}{2} = \frac{1}{2l} \left| \int_{-l}^{l} f(\xi) d\xi \right| \leqslant \frac{1}{2l} \int_{-\infty}^{+\infty} |f(\xi)| d\xi < +\infty, \text{ and } \lim_{l \to \infty} \frac{1}{2l} \int_{-\infty}^{\infty} |f(\xi)| d\xi = 0.$$

For fixed x, let $l \to \infty$ in formula (2). We get

$$f(x) = \lim_{l \to +\infty} \sum_{n=1}^{\infty} \frac{1}{l} \int_{-l}^{l} f(\xi) \cos \frac{n\pi}{l} (\xi - x) d\xi.$$

If we denote $\alpha_n = \frac{n\pi}{l}, \Delta\alpha_n = \alpha_{n+1} - \alpha_n = \frac{\pi}{l}$, then $f(x)$ can be written as

$$f(x) = \lim_{l \to +\infty} \sum_{n=1}^{\infty} F(\alpha_n) \Delta\alpha_n = \lim_{l \to \infty} \sum_{n=1}^{\infty} F(\alpha_n) \Delta\alpha,$$

where

$$F(\alpha_n) = \frac{1}{\pi} \int_{-l}^{+l} f(\xi) \cos [\alpha_n(\xi - x)] d\xi.$$

If $l \to \infty, \Delta\alpha \to 0$ and then above sum tends to a definite integral. Therefore, we can get

$$f(x) = \int_{0}^{\infty} \left\{ \frac{1}{\pi} \int_{-\infty}^{+\infty} f(\xi) \cos[\alpha(\xi - x)] d\xi \right\} d\alpha. \tag{4}$$

This integral is called the Fourier integral.

Fourier integral transformation

In general, formula (4) can be represented in the complex form. Set

$$\cos\alpha(\xi - x) = \frac{1}{2}\left[e^{\mathrm{i}\alpha(\xi-x)} + e^{-\mathrm{i}\alpha(\xi-x)}\right],$$

then

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_0^{\infty}\int_{-\infty}^{+\infty} f(\xi)\left[e^{\mathrm{i}\alpha(\xi-x)} + e^{-\mathrm{i}\alpha(\xi-x)}\right] d\xi d\alpha \\
&= \frac{1}{2\pi}\left\{\int_0^{\infty}\int_{-\infty}^{+\infty} f(\xi)e^{\mathrm{i}\alpha(\xi-x)} d\xi d\alpha + \int_0^{\infty}\int_{-\infty}^{+\infty} f(\xi)e^{-\mathrm{i}\alpha(\xi-x)} d\xi d\alpha\right\} \\
&= \frac{1}{2\pi}\left\{\int_0^{\infty}\int_{-\infty}^{+\infty} f(\xi)e^{\mathrm{i}\alpha(\xi-x)} d\xi d\alpha + \int_{-\infty}^{0}\int_{-\infty}^{+\infty} f(\xi)e^{\mathrm{i}\alpha(\xi-x)} d\xi d\alpha\right\} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{+\infty} f(\xi)e^{\mathrm{i}\alpha(\xi-x)} d\xi d\alpha \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\xi)e^{\mathrm{i}\alpha\xi} d\xi\right]\cdot e^{-\mathrm{i}\alpha x} d\alpha.
\end{aligned}$$

Fourier integral transformation

Definition

Assume that the function $f(x)$ is piecewise smooth (piecewise continuously derivable) and absolutely integrable in $(-\infty;\infty)$, then the integral

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{i\alpha x} dx \equiv F[f(x)]$$

is called the Fourier integral transformation of $f(x)$, and $f(x)$ is called the Fourier inverse transformation of $F(\alpha)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\alpha) e^{-i\alpha x} d\alpha.$$

Fourier integral transformation

Definition

Assume that the function $f(x)$ is piecewise smooth (piecewise continuously derivable) and absolutely integrable in $(-\infty;\infty)$, then the integral

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{i\alpha x} dx \equiv F[f(x)]$$

is called the Fourier integral transformation of $f(x)$, and $f(x)$ is called the Fourier inverse transformation of $F(\alpha)$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\alpha) e^{-i\alpha x} d\alpha.$$

Theorem (Dirichlet conditions)

Assume that $f(x)$ satisfies the Dirichlet conditions:

1. *$f(x)$ is bounded and absolutely integrable for all $x \in (-\infty, +\infty)$;*
2. *$f(x)$ has at most finite number of extremum points and discontinuities of the first kind.*

Then, for any $x \in (-\infty, \infty)$,

$$\frac{f(x+0) + f(x-0)}{2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(\lambda) e^{i\lambda s} d\lambda \right] e^{-ixs} ds.$$

Fourier integral transformation

Further, let $f(x)$ be an odd function satisfying the Dirichlet condition, then the Fourier transformation becomes the sine Fourier transformation, such that

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(\lambda) \sin(\lambda s) d\lambda \Rightarrow f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_s(s) \sin(xs) ds,$$

and together with this,

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(\lambda) \sin s(x-\lambda) d\lambda \right] ds.$$

If $f(x)$ is an even function satisfying the Dirichlet condition, then the Fourier transformation becomes the cosine Fourier transformation, such that

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(\lambda) \cos(\lambda s) d\lambda \Rightarrow f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(s) \cos(xs) ds,$$

and together with

$$\frac{f(x+0)+f(x-0)}{2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} f(\lambda) \cos s(x-\lambda) d\lambda \right] ds.$$

Fourier integral transformation

Example

Find the Fourier transformation of $f(x) = e^{-|x|}$.

Answer.

$$
\begin{aligned}
F(\alpha) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{\mathrm{i}\alpha\xi} \cdot e^{-|\xi|} d\xi \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \int_{0}^{+\infty} e^{-(1-\mathrm{i}\alpha)\xi} d\xi + \int_{-\infty}^{0} e^{(1+\mathrm{i}\alpha)\xi} \xi d\xi \right\} \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \left. \frac{-e^{-(1-\mathrm{i}\alpha)\xi}}{1-\mathrm{i}\alpha} \right|_{0}^{+\infty} + \left. \frac{e^{(1+\mathrm{i}\alpha)\xi}}{1+\mathrm{i}\alpha} \right|_{-\infty}^{0} \right\} \\
&= \frac{1}{\sqrt{2\pi}} \left\{ \frac{1}{1-\mathrm{i}\alpha} + \frac{1}{1+\mathrm{i}\alpha} \right\} = \frac{1}{\sqrt{2\pi}} \frac{2}{(1+\alpha^2)} = \sqrt{\frac{2}{\pi}} \frac{1}{(1+\alpha^2)}.
\end{aligned}
$$

□

The properties of Fourier transformation

1. The Fourier transformation is linear transformation.
   Assume
$$F[f] = F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\xi) e^{\mathrm{i}\lambda\xi} d\xi,$$
   then for any numbers a and b,
$$F[af(x) + bg(x)] = aF[f] + bF[g].$$

The properties of Fourier transformation

1. The Fourier transformation is linear transformation.
Assume

$$F[f] = F(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\xi) e^{i\lambda\xi} d\xi,$$

then for any numbers a and b,

$$F[af(x) + bg(x)] = aF[f] + bF[g].$$

Proof:

$$\begin{aligned} F[af + bg] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} [af(\xi) + bg(\xi)] e^{i\lambda\xi} d\xi \\ &= a \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\xi) e^{i\lambda\xi} d\xi + b \cdot \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\xi) e^{i\lambda\xi} d\xi \\ &= aF[f] + bF[g]. \end{aligned}$$

The properties of Fourier transformation

2. Displacement theorem. Suppose $F[f]$ is the Fourier transformation of $f(x)$, c is a real constant, then

$$F[f(x-c)] = e^{\mathrm{i}\lambda c}F[f(x)].$$

The properties of Fourier transformation

2. Displacement theorem. Suppose $F[f]$ is the Fourier transformation of $f(x)$, c is a real constant, then

$$F[f(x-c)] = e^{\mathrm{i}\lambda c} F[f(x)].$$

Proof:

$$\begin{aligned} F[f(x-c)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\xi - c) e^{\mathrm{i}\lambda\xi} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\eta) e^{\mathrm{i}\lambda(\eta + c)} d\eta = e^{\mathrm{i}\lambda c} F[f(x)]. \end{aligned}$$

3. Similarity theorem. Assume $F[f(x)] = F(\lambda)$ is the Fourier transformation of $f(x)$, $c \neq 0$ is a constant, then

$$F[f(cx)] = \frac{1}{|c|} F\left(\frac{\lambda}{c}\right).$$

The properties of Fourier transformation

3. Similarity theorem. Assume $F[f(x)] = F(\lambda)$ is the Fourier transformation of $f(x)$, $c \neq 0$ is a constant, then

$$F[f(cx)] = \frac{1}{|c|} F\left(\frac{\lambda}{c}\right).$$

Proof:

$$\begin{aligned} F[f(cx)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(c\xi) e^{i\lambda\xi} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\eta) e^{i\frac{\lambda}{c}\eta} \frac{1}{c} d\eta \quad (\text{ if } c > 0) \end{aligned}$$

or

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int_{\infty}^{-\infty} f(\eta) e^{\mathrm{i}\frac{\lambda}{c}\eta} \frac{1}{c} d\eta \quad (\text{ if } c < 0) \\ =&\frac{1}{\sqrt{2\pi}|c|} \int_{-\infty}^{\infty} f(\eta) e^{\mathrm{i}\frac{\lambda}{c}\eta} d\eta = \frac{1}{|c|} F\left(\frac{\lambda}{c}\right). \end{aligned}$$

The properties of Fourier transformation

4. Differential theorem. Assume $f(x)$ and $f'(x)$ are piecewise smooth and absolutely integral in $(-\infty, +\infty)$, and $\lim\limits_{|x|\to\infty} f(x) = 0$, then

$$F\left[f'(x)\right] = (-i\lambda)F[f(x)].$$

The properties of Fourier transformation

4. Differential theorem. Assume $f(x)$ and $f'(x)$ are piecewise smooth and absolutely integral in $(-\infty, +\infty)$, and $\lim\limits_{|x|\to\infty} f(x) = 0$, then

$$F\left[f'(x)\right] = (-i\lambda)F[f(x)].$$

Proof:

$$\begin{aligned} F\left[f'(x)\right] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f'(\xi) e^{i\lambda\xi} d\xi \\ &= \frac{1}{\sqrt{2\pi}} \left\{ f(\xi) e^{i\lambda\xi} \Big|_{-\infty}^{+\infty} - (i\lambda) \int_{-\infty}^{+\infty} f(\xi) e^{i\lambda\xi} d\xi \right\} \\ &= (-i\lambda)F[f(x)]. \end{aligned}$$

The properties of Fourier transformation

4. Differential theorem. Assume $f(x)$ and $f'(x)$ are piecewise smooth and absolutely integral in $(-\infty,+\infty)$, and $\lim\limits_{|x|\to\infty} f(x)=0$, then

$$F\left[f'(x)\right]=(-i\lambda)F[f(x)].$$

Proof:

$$\begin{aligned}F\left[f'(x)\right]&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}f'(\xi)e^{i\lambda\xi}d\xi\\&=\frac{1}{\sqrt{2\pi}}\left\{f(\xi)e^{i\lambda\xi}\Big|_{-\infty}^{+\infty}-(i\lambda)\int_{-\infty}^{+\infty}f(\xi)e^{i\lambda\xi}d\xi\right\}\\&=(-i\lambda)F[f(x)].\end{aligned}$$

Corollary

Assume $f(x)$ and $f^{(k)}(x)$ $(k=1,2,\cdots n)$ can be operated by the Fourier transformation, and $f^{(k)}(\pm\infty)=0, k=0,1,\cdots n-1$, where $f^{(0)}(x)=f(x)$, then

$$F\left[f^{(n)}(x)\right]=(-i\lambda)^nF[f(x)].$$

The properties of Fourier transformation

5. Assume there exist Fourier transformations of $f(x)$ and $g(x)$, and $F[f(x)] = F(\lambda)$, $F[g(x)] = G(\lambda)$, then
   (i) $F[f * g(x)] = F(\lambda) \cdot G(\lambda)$;
   (ii) $F[f(x) \cdot g(x)] = F * G(\lambda)$.

The properties of Fourier transformation

5. Assume there exist Fourier transformations of $f(x)$ and $g(x)$, and $F[f(x)] = F(\lambda)$, $F[g(x)] = G(\lambda)$, then
   (i) $F[f * g(x)] = F(\lambda) \cdot G(\lambda)$;
   (ii) $F[f(x) \cdot g(x)] = F * G(\lambda)$.

Definition (The convolution and its Fourier transform)

Assume there exist $F[f]$ and $F[g]$, then the integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-\xi) g(\xi) d\xi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(x-\xi) f(\xi) d\xi$$

is called the convolution of $f(x)$ and $g(x)$, and denote $f * g(x)$ or $g * f(x)$.
Similarly, let $F(\lambda) = F[f(x)], G(\lambda) = F[g(x)]$. The integral

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\lambda - s) G(s) ds = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} G(\lambda - s) F(s) ds$$

is called the convolution of $F(\lambda)$ and $G(\lambda)$, and denote $F * G(\lambda)$ or $G * F(\lambda)$.

5. Proof. (i)

$$
\begin{aligned}
F[f*g(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\xi-t)g(t)dt\right]e^{\mathrm{i}\lambda\xi}d\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\eta)e^{\mathrm{i}\lambda(\eta+t)}d\eta\right]g(t)d\eta \\
&= \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\eta)e^{\mathrm{i}\lambda\eta}d\eta\right]\cdot\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(t)e^{\mathrm{i}\lambda t}dt\right] \\
&= F(\lambda)\cdot G(\lambda).
\end{aligned}
$$

The properties of Fourier transformation

5. Proof. (i)

$$
\begin{aligned}
F[f*g(x)] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\xi-t)g(t)dt\right]e^{i\lambda\xi}d\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\eta)e^{i\lambda(\eta+t)}d\eta\right]g(t)d\eta \\
&= \left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\eta)e^{i\lambda\eta}d\eta\right]\cdot\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} g(t)e^{i\lambda t}dt\right] \\
&= F(\lambda)\cdot G(\lambda).
\end{aligned}
$$

Proof. (ii)

$$
\begin{aligned}
F*G(\lambda) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(\lambda-s)G(s)ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\xi)e^{i(\lambda-s)\xi}d\xi\right]G(s)ds \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\xi)\cdot\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} G(s)e^{-is\xi}ds\right]e^{i\lambda\xi}d\xi \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\xi)\cdot g(\xi)e^{i\lambda\xi}d\xi = F[f(x)\cdot g(x)].
\end{aligned}
$$

Application of Fourier integral transformation

Example (1)

Solve the problems

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < +\infty, \quad y > 0, \tag{5}$$

$$u(x,0) = f(x), \quad -\infty < x < +\infty, \tag{6}$$

$$\lim_{|x|\to\infty} u(x,y) = 0, \quad \lim_{|x|\to\infty} u_x(x,y) = 0, \tag{7}$$

$$\lim_{y\to+\infty} |u(x,y)| < +\infty. \tag{8}$$

Application of Fourier integral transformation

Example (1)

Solve the problems

$$u_{xx} + u_{yy} = 0, \quad -\infty < x < +\infty, \quad y > 0, \tag{5}$$

$$u(x,0) = f(x), \quad -\infty < x < +\infty, \tag{6}$$

$$\lim_{|x|\to\infty} u(x,y) = 0, \quad \lim_{|x|\to\infty} u_x(x,y) = 0, \tag{7}$$

$$\lim_{y\to+\infty} |u(x,y)| < +\infty. \tag{8}$$

Answer.

Set $V(\lambda, y) = F[u(x,y)], F(\lambda) = F[f(x)]$. Then, taking the Fourier transformation for (5),

$$F\,[u_{xx} + u_{yy}] = F\,[u_{xx}] + F\,[u_{yy}] = -\lambda^2 F[u] + \frac{d^2}{dy^2} F[u] = -\lambda^2 V + \frac{d^2 V}{dy^2} = 0.$$

By (6), (8), we get

$$F[u(x,0)] = V(\lambda,0) = F[f(x)] = F(\lambda),$$

$$\lim_{y\to\infty} |F[u(x,y)]| = \lim_{y\to\infty} |V(\lambda,y)| < +\infty.$$

Thus,

$$\frac{d^2 V}{dy^2} - \lambda^2 V = 0, \tag{9}$$

$$V(\lambda, 0) = F(\lambda), \tag{10}$$

$$\lim_{y \to \infty} |V(\lambda, y)| < +\infty. \tag{11}$$

Solving (9), we have $V(\lambda, y) = C_1(\lambda)e^{\lambda y} + C_2(\lambda)e^{-\lambda y}$.

By (11), $\begin{cases} \text{if } \lambda > 0, & \text{then } C_1(\lambda) = 0, \\ \text{if } \lambda < 0, & \text{then } C_2(\lambda) = 0, \end{cases}$ or $V(\lambda, y) = \begin{cases} C_2(\lambda)e^{-\lambda y} & \text{if } \lambda > 0 \\ C_1(\lambda)e^{\lambda y} & \text{if } \lambda < 0 \end{cases} = C(\lambda)e^{-|\lambda| y}$.

By (10), we get $C(\lambda) = F(\lambda)$, then $V(\lambda, y) = F(\lambda)e^{-|\lambda| y}$.

Application of Fourier integral transformation

Using the inverse transformation, we have

$$
\begin{aligned}
u(x,y) &= F^{-1}\left[F(\lambda)e^{-|\lambda|y}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\lambda)e^{-|\lambda|y}\cdot e^{-\mathrm{i}\lambda x}d\lambda \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(\xi)e^{\mathrm{i}\lambda\xi}d\xi\right]e^{-|\lambda|y-\mathrm{i}\lambda x}d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\left[\int_{-\infty}^{+\infty} e^{-|\lambda|y+\mathrm{i}(\xi-x)\lambda}d\lambda\right]d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\left[\int_{0}^{\infty} e^{-[y-\mathrm{i}(\xi-x)]\lambda}d\lambda + \int_{-\infty}^{0} e^{[y+\mathrm{i}(\xi-x)]\lambda}d\lambda\right]d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\left[\left.\frac{-e^{-[y-\mathrm{i}(\xi-x)]\lambda}}{y-\mathrm{i}(\xi-x)}\right|_{0}^{\infty} + \left.\frac{e^{[y+\mathrm{i}(\xi-x)]\lambda}}{y+\mathrm{i}(\xi-x)}\right|_{-\infty}^{0}\right]d\xi \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)\left[\frac{1}{y-\mathrm{i}(\xi-x)} + \frac{1}{y+\mathrm{i}(\xi-x)}\right]d\xi \\
&= \frac{y}{\pi}\int_{-\infty}^{+\infty}\frac{f(\xi)}{y^2+(\xi-x)^2}d\xi.
\end{aligned}
$$

Dirac delta function

The Dirac delta function, which is defined to have the properties

$$\delta(x) = 0, \quad x \neq 0, \tag{12}$$

$$f(0) = \int_a^b f(x)\delta(x)dx, \tag{13}$$

where $f(x)$ is any well-behaved function and the integration includes the origin. As a special case of Eq. (12),

$$\int_{-\infty}^{\infty} \delta(x)dx = 1. \tag{14}$$

From Eq. (12), $\delta(x)$ must be an infinitely high, thin spike at $x = 0$, as in the description of an impulsive force or the charge density for a point charge. The problem is that no such function exists, in the usual sense of function.

Dirac delta function

The crucial property in Eq. (12) can be developed rigorously as the limit of a sequence of functions, a distribution. For example, the delta function may be approximated by any of the sequences of functions, Eqs. (15) to (18):

$$\delta_n(x) = \begin{cases} 0, & x < -\frac{1}{2n} \\ n, & -\frac{1}{2n} < x < \frac{1}{2n}, \\ 0, & x > \frac{1}{2n} \end{cases} \tag{15}$$

$$\delta_n(x) = \sqrt{\frac{n}{\pi}} \exp\left(-n^2 x^2\right), \tag{16}$$

$$\delta_n(x) = \frac{n}{\pi} \frac{1}{1 + n^2 x^2}, \tag{17}$$

$$\delta_n(x) = \frac{\sin nx}{\pi x} = \frac{1}{2\pi} \int_{-n}^{n} e^{ixt} dt. \tag{18}$$

While all these sequences cause $\delta(x)$ to have the same properties, they differ somewhat in ease of use for various purposes. Equation (15) is useful in providing a simple derivation of the integral property, Eq. (12). Equation (16) is convenient to differentiate. Its derivatives lead to the Hermite polynomials. Equation (18) is particularly useful in Fourier analysis. In the theory of Fourier series, Eq. (18) often appears (modified) as the Dirichlet kernel:

$$D_n(x) = \frac{1}{2\pi} \frac{\sin\left[\left(n + \frac{1}{2}\right) x\right]}{\sin\left(\frac{1}{2} x\right)}.$$

Properties of $\delta(x)$

- $\delta(-x) = \delta(x)$;
- $\delta(ax) = \delta(x)/|a|,\ a \in \mathbb{R}$;
- $\int\limits_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0)$;
- $\delta(x - a) * f(x) = \int\limits_{-\infty}^{\infty} \delta(x - a) f(x - \xi) d\xi = f(x - a)$;
- $\int\limits_{-\infty}^{\infty} f(x)\delta'(x - x_0)\, dx = -\int\limits_{-\infty}^{\infty} f'(x)\delta(x - x_0)\, dx = -f'(x_0)$;
- $F(\delta(x)) = \sqrt{\frac{1}{2\pi}}$.

Fourier transformations of typical functions

Example (2)

$f(t) = e^{-\alpha|t|}$, with $\alpha > 0$.

Fourier transformations of typical functions

Example (2)

$f(t) = e^{-\alpha|t|}$, with $\alpha > 0$.

Answer.

$$g(\omega) = \sqrt{\frac{1}{2\pi}} \int_{-\infty}^{0} e^{\alpha t + i\omega t} dt + \sqrt{\frac{1}{2\pi}} \int_{0}^{\infty} e^{-\alpha t + i\omega t} dt$$
$$= \sqrt{\frac{1}{2\pi}} \left[\frac{1}{\alpha + i\omega} + \frac{1}{\alpha - i\omega} \right] = \sqrt{\frac{1}{2\pi}} \frac{2\alpha}{\alpha^2 + \omega^2}.$$

Fourier transformations of typical functions

Example (3)

$f(t) = 2\alpha\sqrt{1/2\pi}/\left(\alpha^2 + t^2\right)$, with $\alpha > 0$.

Fourier transformations of typical functions

Example (3)

$f(t) = 2\alpha\sqrt{1/2\pi}/\left(\alpha^2 + t^2\right)$, with $\alpha > 0$.

Answer.

One way to evaluate this transform is by contour integration.

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\alpha e^{i\omega t}}{(t - i\alpha)(t + i\alpha)} dt.$$

The integrand has two poles: $t = i\alpha$ with residue $e^{-\alpha\omega}/i$ and $t = -i\alpha$ with residue $e^{+\alpha\omega}/(-i)$. If $\omega > 0$, our integrand will become negligible on a large semicircle in the upper half-plane. This contour encloses only the pole at $t = i\alpha$, so we get $g(\omega) = \frac{1}{2\pi}(2\pi i)\frac{e^{-\alpha\omega}}{i} \quad (\omega > 0)$.
However, if $\omega < 0$, we must close the contour in the lower half-plane, circling the pole at $t = -i\alpha$ in a clockwise sense (thereby generating a minus sign). This procedure yields $g(\omega) = \frac{1}{2\pi}(-2\pi i)\frac{e^{+\alpha\omega}}{-i} \quad (\omega < 0)$.
If $\omega = 0$, we cannot perform a contour integration on either of the paths, but we then do not need this sophisticated an approach, as we have the elementary integral $g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2\alpha}{t^2+\alpha^2}dt = 1$. In summary, we have

$$g(\omega) = e^{-\alpha|\omega|}.$$

Fourier transformations of typical functions

Example (4)

The Fourier transform of a Gaussian function e^{-at^2}, with $a > 0$.

Fourier transformations of typical functions

Example (4)

The Fourier transform of a Gaussian function e^{-at^2}, with $a > 0$.

Answer.

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-at^2} e^{i\omega t} dt,$$

can be evaluated analytically by completing the square in the exponent,

$$-at^2 + i\omega t = -a\left(t - \frac{i\omega}{2a}\right)^2 - \frac{\omega^2}{4a},$$

which we can check by evaluating the square. Substituting this identity and changing the integration variable from t to $s = t - i\omega/2a$, we obtain (in the limit of large T)

$$g(\omega) = \frac{1}{\sqrt{2\pi}} e^{-\omega^2/4a} \int_{-T-i\omega/2a}^{T-i\omega/2a} e^{-as^2} ds = \frac{1}{\sqrt{2a}} e^{-\omega^2/4a}.$$

Laplace integral transformation

Definition

Assume $|f(t)| \leqslant Me^{s_0 t}\,(0 \leqslant s_0 < s)$ and $f(t)$ is piecewise smooth (denoted by L-(A)), then the integral $F(p) = \int_0^{\infty} f(\tau)e^{-p\tau}d\tau$ is called Laplace integral transformation (denoted by L-T) of $f(t)$ in $(0, +\infty)$, and denote $L[f(t)] = F(p)$; the integral

$$f(t) = \frac{1}{2\pi \mathrm{i}} \int_{s-\mathrm{i}\infty}^{s+\mathrm{i}\infty} F(p)e^{pt}dp$$

is called inverse Laplace transformation of $F(p)$, and denote $L^{-1}[F(p)] = f(t)$.

The properties of Laplace transformation

1. $(L-T)$ is linear transformation

$$L[af(t)+bg(t)] = aL[f(t)] + bL[g(t)],$$

where $f(t)$ and $g(t)$ satisfy L-(A), and a and b are constants.

The properties of Laplace transformation

2. Assume $f(t)$ and $f'(t)$ satisfy L-(A), then

$$L\left[f'(t)\right] = pL[f(t)] - f(0^+).$$

The properties of Laplace transformation

2. Assume $f(t)$ and $f'(t)$ satisfy L-(A), then

$$L\left[f'(t)\right] = pL[f(t)] - f(0^{+}).$$

Proof:

$$\begin{aligned} L\left[f'(t)\right] &= \int_0^{\infty} f'(\tau)e^{-p\tau}\,d\tau \\ &= f(\tau)e^{-p\tau}\Big|_0^{\infty} + \int_0^{\infty} f(\tau)pe^{-p\tau}\,d\tau \\ &= pL[f(t)] - f(0). \end{aligned}$$

The properties of Laplace transformation

2. Assume $f(t)$ and $f'(t)$ satisfy L-(A), then

$$L\left[f'(t)\right] = pL[f(t)] - f(0^+).$$

Proof:

$$\begin{aligned} L\left[f'(t)\right] &= \int_0^{\infty} f'(\tau)e^{-p\tau}\,d\tau \\ &= f(\tau)e^{-p\tau}\big|_0^{\infty} + \int_0^{\infty} f(\tau)pe^{-p\tau}\,d\tau \\ &= pL[f(t)] - f(0). \end{aligned}$$

Furthermore, we have the following corollary.

Corollary

Assume $f(t)$ and $f^{(k)}(t)(k=1,\cdots n)$ satisfy L-(A), then

$$L\left[f^{(n)}(t)\right] = p^n\left(L[f(t)] - \frac{f(0)}{p} - \frac{f'(0)}{p^2} - \cdots - \frac{f^{(n-1)}(0)}{p^n}\right)$$

where $f(0) = f\left(0^+\right), f^{(k)}(0) = f^{(k)}\left(0^+\right), k = 1,\cdots,n-1.$

The properties of Laplace transformation

3. Assume $f(t)$ satisfies L-(A), then

$$\frac{d}{dp}L[f(t)] = L[-tf(t)]$$

and then $\frac{d^nF(p)}{dp^n} = L\left[(-t)^n f(t)\right]$.

The properties of Laplace transformation

3. Assume $f(t)$ satisfies L-(A), then

$$\frac{d}{dp}L[f(t)] = L[-tf(t)]$$

and then $\frac{d^nF(p)}{dp^n} = L\left[(-t)^n f(t)\right]$.

Proof:

$$\begin{aligned}\frac{d}{dp}L[f(t)] &= \frac{d}{dp}\int_0^{\infty} f(\tau)e^{-p\tau}d\tau = \int_0^{\infty} f(\tau)(-\tau)e^{-p\tau}d\tau \\ &= L[-tf(t)].\end{aligned}$$

The properties of Laplace transformation

4. Assume $f(t)$ satisfies L-(A) and $\varphi(t) = \int_0^t f(\tau) d\tau$, then

$$L[\varphi(t)] = \frac{1}{p} L[f(t)] = \frac{1}{p} F(p).$$

The properties of Laplace transformation

4. Assume $f(t)$ satisfies L-(A) and $\varphi(t) = \int_0^t f(\tau)d\tau$, then

$$L[\varphi(t)] = \frac{1}{p}L[f(t)] = \frac{1}{p}F(p).$$

Proof: Since $\varphi'(t) = f(t), \varphi(0) = 0$, then

$$L\left[\varphi'(t)\right] = L[f(t)] = pL[\varphi(t)] - \varphi(0) \Rightarrow L[\varphi(t)] = \frac{1}{p}L[f(t)] = \frac{1}{p}F(p).$$

The properties of Laplace transformation

5. Assume $f(t)$ satisfies L-(A), and $F(p) = L[f(t)], \int_p^{\infty} |F(s)|ds < +\infty$ then

$$\int_p^{\infty} F(s)ds = L\left[\frac{f(t)}{t}\right].$$

The properties of Laplace transformation

5. Assume $f(t)$ satisfies L-(A), and $F(p) = L[f(t)]$, $\int_p^\infty |F(s)|ds < +\infty$ then

$$\int_p^\infty F(s)ds = L\left[\frac{f(t)}{t}\right].$$

Proof:

$$\begin{aligned}\int_p^\infty F(s)ds &= \int_p^\infty \left[\int_0^\infty f(\tau)e^{-s\tau}d\tau\right]ds \\ &= \int_0^\infty f(\tau)\left(\int_p^\infty e^{-s\tau}ds\right)d\tau = \int_0^\infty f(\tau)\left(\frac{-e^{-s\tau}}{\tau}\right)\Big|_p^\infty d\tau \\ &= \int_0^\infty \frac{f(\tau)}{\tau}e^{-p\tau}d\tau = L\left[\frac{f(t)}{t}\right].\end{aligned}$$

The properties of Laplace transformation

6. Delay theorem: Assume $f(t)$ satisfies $\mathrm{L}-(\mathrm{A}), F(p) = L[f(t)], c > 0$, then

$$L[f(t-c)] = e^{-pc}L[f(t)] = e^{-pc}F(p).$$

The properties of Laplace transformation

6. Delay theorem: Assume $f(t)$ satisfies $\mathrm{L}-(\mathrm{A}), F(p)=L[f(t)], c>0$, then

$$L[f(t-c)]=e^{-pc}L[f(t)]=e^{-pc}F(p).$$

Proof:

$$\begin{aligned} L[f(t-c)] &= \int_0^{\infty} f(t-c)e^{-pt}dt, \\ &= \int_0^{\infty} f(\eta)e^{-p(\eta+c)}d\eta = e^{-pc}L[f(t)]. \end{aligned}$$

The properties of Laplace transformation

6. Delay theorem: Assume $f(t)$ satisfies $\mathrm{L}-(\mathrm{A}), F(p)=L[f(t)], c>0$, then

$$L[f(t-c)]=e^{-pc}L[f(t)]=e^{-pc}F(p).$$

Proof:

$$\begin{aligned} L[f(t-c)] &= \int_0^{\infty} f(t-c)e^{-pt}dt, \\ &= \int_0^{\infty} f(\eta)e^{-p(\eta+c)}d\eta = e^{-pc}L[f(t)]. \end{aligned}$$

7. Displacement theorem: Assume $f(t)$ satisfies $\mathrm{L}-(\mathrm{A}), F(p)=L[f(t)]$, then

$$F(p-p_0)=L\left[e^{p_0 t}f(t)\right].$$

The properties of Laplace transformation

6. Delay theorem: Assume $f(t)$ satisfies $\mathrm{L}-(\mathrm{A}), F(p)=L[f(t)], c>0$, then

$$L[f(t-c)]=e^{-pc}L[f(t)]=e^{-pc}F(p).$$

Proof:

$$\begin{aligned} L[f(t-c)] &= \int_0^{\infty} f(t-c)e^{-pt}dt, \\ &= \int_0^{\infty} f(\eta)e^{-p(\eta+c)}d\eta = e^{-pc}L[f(t)]. \end{aligned}$$

7. Displacement theorem: Assume $f(t)$ satisfies $\mathrm{L}-(\mathrm{A}), F(p)=L[f(t)]$, then

$$F(p-p_0)=L\left[e^{p_0 t}f(t)\right].$$

Proof:

$F(p-p_0)=\int_0^{\infty} f(t)e^{-(p-p_0)t}dt=\int_0^{\infty} e^{p_0 t}f(t)e^{-p(t)}dt=L\left[e^{p_0 t}f(t)\right].$

The properties of Laplace transformation

7. Similar theorem: Assume $f(t)$ satisfies L-(A), $a > 0, F(p) = L[f(t)]$, then

$$L[f(at)] = \frac{1}{a} F\left(\frac{p}{a}\right).$$

The properties of Laplace transformation

7. Similar theorem: Assume $f(t)$ satisfies L-(A), $a > 0, F(p) = L[f(t)]$, then

$$L[f(at)] = \frac{1}{a}F\left(\frac{p}{a}\right).$$

Proof:

$$\begin{aligned} L[f(at)] &= \int_0^{\infty} f(at)e^{-pt}dt, \\ &= \int_0^{\infty} f(\eta)e^{-\frac{p}{a}\eta}\frac{1}{a}d\eta = \frac{1}{a}F\left(\frac{p}{a}\right). \end{aligned}$$

The properties of Laplace transformation

Definition

Assume $f(t)$ and $g(t)$ satisfy $\mathrm{L}(\mathrm{A})$, then the integral $\int_0^t f(t-\tau)g(\tau)d\tau$ or $\int_0^t g(t-\tau)f(\tau)d\tau$ is called the convolution of $f(t)$ and $g(t)$, denoted by $f * g(t)$ or $g * f(t)$, and the integral

$$\frac{1}{2\pi \mathrm{i}}\int_{s-\mathrm{i}\infty}^{s+\mathrm{i}\infty} F(p-q)G(q)dq \quad \text{or} \quad \frac{1}{2\pi \mathrm{i}}\int_{s-\mathrm{i}\infty}^{s+\mathrm{i}\infty} G(p-q)F(q)dq$$

is called the convolution of $F(p)$ and $G(p)$, denoted by $F * G(p)$ or $G * F(p)$.

The properties of Laplace transformation

8. Assume that $f(t)$ and $g(t)$ satisfy L-(A), $F(p) = L[f(t)]$, $G(p) = L[g(t)]$, then
   1. $L[f * g(t)] = F(p) \cdot G(p)$;
   2. $L[f(t) \cdot g(t)] = F * G(p)$.

The properties of Laplace transformation

8. Assume that $f(t)$ and $g(t)$ satisfy L-(A), $F(p) = L[f(t)]$, $G(p) = L[g(t)]$, then
   (i) $L[f * g(t)] = F(p) \cdot G(p)$;
   (ii) $L[f(t) \cdot g(t)] = F * G(p)$.

Proof: (i)

$$
\begin{aligned}
L[f * g(t)] &= \int_0^{\infty} \left[\int_0^t f(t-\tau) g(\tau) d\tau \right] e^{-pt} dt = \int_0^{\infty} \left(\int_{\tau}^{\infty} f(t-\tau) e^{-pt} dt \right) g(\tau) d\tau \\
&= \int_0^{\infty} \left(\int_0^{\infty} f(\eta) e^{-p(\eta+\tau)} d\eta \right) g(\tau) d\tau = \left(\int_0^{\infty} f(\eta) e^{-p\eta} d\eta \right) \left(\int_0^{\infty} g(\tau) e^{-p\tau} d\tau \right) \\
&= F(p) \cdot G(p).
\end{aligned}
$$

The properties of Laplace transformation

8. Assume that $f(t)$ and $g(t)$ satisfy L-(A), $F(p) = L[f(t)]$, $G(p) = L[g(t)]$, then
   - (i) $L[f * g(t)] = F(p) \cdot G(p)$;
   - (ii) $L[f(t) \cdot g(t)] = F * G(p)$.

Proof: (i)

$$
\begin{aligned}
L[f * g(t)] &= \int_0^{\infty} \left[\int_0^t f(t-\tau) g(\tau) d\tau\right] e^{-pt} dt = \int_0^{\infty} \left(\int_{\tau}^{\infty} f(t-\tau) e^{-pt} dt\right) g(\tau) d\tau \\
&= \int_0^{\infty} \left(\int_0^{\infty} f(\eta) e^{-p(\eta+\tau)} d\eta\right) g(\tau) d\tau = \left(\int_0^{\infty} f(\eta) e^{-p\eta} d\eta\right) \left(\int_0^{\infty} g(\tau) e^{-p\tau} d\tau\right) \\
&= F(p) \cdot G(p).
\end{aligned}
$$

Proof: (ii)

$$
\begin{aligned}
F * G(p) &= \frac{1}{2\pi \mathrm{i}} \int_{s-\mathrm{i}\infty}^{s+\mathrm{i}\infty} F(p-q) G(q) dq = \frac{1}{2\pi \mathrm{i}} \int_{s-\mathrm{i}\infty}^{s+\mathrm{i}\infty} \left[\int_0^{\infty} f(t) e^{-(p-q)t} dt\right] G(q) dq \\
&= \int_0^{\infty} f(t) \left[\frac{1}{2\pi \mathrm{i}} \int_{s-\mathrm{i}\infty}^{s+\mathrm{i}\infty} G(q) e^{qt} dq\right] e^{-pt} dt = \int_0^{\infty} f(t) g(t) e^{-pt} dt \\
&= L[f(t) \cdot g(t)].
\end{aligned}
$$